Formulation of a Generalized Arithmetic Operator

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Purpose Statement: The purpose of this project was to forumulate a hyperoperator that generalizes the standard arithmetic operators (addition, multiplication, exponentiation, etc.), thus being able to interpolate and extrapolate on them. For further understanding of this goal, consider the following question:
Given

$$a + a + a ... (btimes) = a \cdot b \equiv a[1]b$$
$$\text{and } a \cdot a \cdot a \cdot ... (btimes) = a^b \equiv a[2]b$$

what is $a[1.5]b$?

1 Operator notation and basics

When general operators are needed, I will use the following notations:

1.1 Binary operation

$$\begin{aligned} &a[x]b = c, \text{ where } a, b, c \in \mathbb{C}, \\ &a[1]b \equiv a + b \\ &a[2]b \equiv a \cdot b. \end{aligned}$$

Further operators are formulated below.

Iterated binary operations
Examples:

$$a(+b)^2 \leftrightarrow a + b + b$$
$$a[x]^2 b \leftrightarrow (a[x]b)[x]b$$

We will say that a binary operator applied 0 times is not applied at all:

$$a[x]^0 b = a \tag{1.1.1}$$

1.2 Unary operation

$$[x]a \equiv a[x]a \tag{1.2.1}$$

Iterated unary operations
Iterated unary operations will be denoted by a superscript:
$[x]^n a$ where, for example, $[x]^2 a \leftrightarrow [x][x]a$. A doubly iterated unary operation would be written as

$$([x]^n)^m a$$

and we can expand this expression in the following way:

$$([x]^n)^m a = [x]^n [x]^n ...(\text{m times})...[x]^n a = [x]^{n+n+...(\text{m times})} a = [x]^{n \cdot m} a$$

Thus showing that

$$([x]^n)^m a = [x]^{n \cdot m} a \tag{1.2.2}$$

A unary operator applied 0 times is not applied at all:

$$[x]^0 a = a \tag{1.2.3}$$

This can be proven using eq. 1.1.1 and the axioms of Section 2.

1.3 Compound operation

For compound/multiple operations applied to an operand, operations furthest to the right should be applied first for unary operators.
Examples:

$$[y][x]a = [y]([x]a)$$

1.4 Unary operator iteration

$$\begin{aligned}
&[1]a = a + a \\
&[1]^2 a = (a + a) + (a + a) \\
&\vdots \\
&[1]^n a = a \cdot 2^n \\
\\
&[2]a = a \cdot a \\
&[2]^2 a = (a \cdot a) \cdot (a \cdot a) \\
&\vdots \\
&[2]^n a = a^{2^n}
\end{aligned}$$

Properties: $[x]^m [x]^n a = [x]^{m+n} a$

1.5 Binary operator iteration

$$\begin{aligned}
a(+a)^{b-1} &= a \cdot b \\
a(\cdot a)^{b-1} &= a^b \\
a(^a)^{b-1} &= a^{a^{b-1}}
\end{aligned}$$

The last example is a compound operation (contains more than one instance of a), and it is not obvious how this could be separated into an operand involving a and an operator involving b only (as in the previous two lines. This example is more easily expressed in terms of iterated functions later.

Properties:

$$(a[x]^m b)[x]^n b = a[x]^{m+n} b$$

1.6 Inverses of binary operations

Inverse operations will be denoted by a negative iterand. Inverse operators are defined such that they undo their respective forward operations. This could mean one of several things for binary operators.

Definition 1 $(a[x]b)[x]^{-1}b = a$

Definition 2 $(a[x]b)[x]^{-1}a = b$ or $a[x](b[x]^{-1}a) = b$

Definition 3 $a[x]^{-1}b = a[x](b[x+1](-1))$

Definition 2 can be derived from definition 1 if we assume that [x] is commutative, which it is not in general. Definition 1 results in the following inverses for the first few elementary operators:

$$a[1]^{-1}b = a - b$$
$$a[2]^{-1}b = \frac{a}{b}$$
$$a[3]^{-1}b = a^{1/b}$$

while definition 2 results in the following:

$$a[1]^{-1}b = a - b$$
$$a[2]^{-1}b = \frac{a}{b}$$
$$a[3]^{-1}b = log_a b$$

Definition 3 is only useful if we know how to evaluate $b[x+1](-1)$.

1.7 Inverses of unary operations

Inverses of unary operators are simpler than those of binary operators:

$$[x]^{-1}[x]a = [x][x]^{-1}a = a$$

2 Axiomatic treatment

2.1 The basic axioms

Let us take the following properties of the $[x]$ operators as axioms:

Axiom 1:

$$[x]a = a[x]a \tag{A1}$$

Axiom 2:

$$[x]^n[x]^m a = [x]^{n+m}a \tag{A2}$$

Axiom 3: We found in Section 1.4 that $[1]^n a = a \cdot 2^n$ and $[2]^n a = a^{2^n}$, and so we postulate that

$$[x]^n a = a[x+1]2^n \tag{A3}$$

2.2 Resulting properties

From the definitions of the $[x]$ operators and the three axioms listed above, there are many other properties of the $[x]$ operators which are derived below.

2.2.1 $a[x]b$ for $x = 4$

We can evaluate $a[x]b$ for $x = 4$ when b is a power of 2. For example:

$$[3]^n a = a[4]2^n \tag{2.2.1}$$

Therefore

$$a[4]2 = [3]^1 a = a^a,$$

$$a[4]4 = [3]^2 a = [3][3]a = (a^a)^{(a^a)},$$

etc.

2.2.2

$$(a[x]b)[x]c = a[x](b \cdot c) \tag{2.2.2}$$

Derivation:

$$\begin{aligned}
(a[x]b)[x]c &= [x]^{log_2 c}[x]^{log_2 b}a && \text{(from A3)} \\
&= [x]^{log_2 c + log_2 b}a && \text{(from A2)} \\
&= [x]^{log_2 (b \cdot c)}a \\
&= a[x](b \cdot c) && \text{(from A3)}
\end{aligned}$$

2.2.3 Commutative property

$$(a[x]b)[x]c = (a[x]c)[x]b \tag{2.2.3}$$

Notice that this produces the familiar commutative properties for addition and multiplication if we let a equal their respective identity elements.

Derivation:

$$\begin{aligned} (a[x]b)[x]c &= a[x](b \cdot c) && \text{(from eq. 2.2.2)} \\ &= (a[x]c)[x]b && \text{(from A3)} \end{aligned}$$

2.2.4

$$a[x]^n b = a[x]b^n \tag{2.2.4}$$

Derivation:

$$\begin{aligned} a[x]^n b &= ((a[x]b)[x]b)[x]b... \\ &= ...[x]^{log_2 b}[x]^{log_2 b}[x]^{log_2 b} a && \text{(from A3)} \\ &= [x]^{n \cdot log_2 b} a && \text{(from A2)} \\ &= [x]^{log_2 b^n} a \\ &= a[x]b^n && \text{(from A3)} \end{aligned}$$

2.2.5

$$(a[x]^n b)[x]^m b = a[x]^{n+m} b \tag{2.2.5}$$

Derivation:

$$\begin{aligned} (a[x]^n b)[x]^m b &= (a[x]b^n)[x]b^m && \text{(from eq. 2.2.4)} \\ &= a[x]b^{n+m} && \text{(from eq. 2.2.2)} \\ &= a[x]^{n+m} b && \text{(from eq. 2.2.4)} \end{aligned}$$

2.2.6 Inverse binary operation

$$(a[x]b)[x]^{-1} b = a \tag{2.2.6}$$

Derivation:

$$\begin{aligned} (a[x]b)[x]^{-1} b &= a[x]^{1-1} b && \text{(from eq. 2.2.5)} \\ &= a[x]^0 b \\ &= a \end{aligned}$$

2.2.7 2 as a fixed point

By combining A1 and A3, we can find that

$$[x]^{log_2 a} a = [x+1]a.$$

By letting $a = 2$, we get

$$[x]2 = [x+1]2,$$

indicating that 2 is a fixed point (at least for $x \in \mathbb{Z}$), producing the same result for all unary operators, $[x]$. This explains why

$$2 + 2 = 2 \cdot 2 = 2^2 \tag{2.2.7}$$

and so on.

2.2.8

$$[x]^n a = a[x]^{log_a \prod_{i=0}^{n-1} [x]^i a} a = a[x]\left(\prod_{i=0}^{n-1} [x]^i a\right) \tag{2.2.8}$$

Derivation:

$$\begin{aligned}
a[x]^m a &= (a[x]a)[x]a^{m-1} \\
&= (([x]a)[x]([x]a))[x]a^{m-1-log_a [x]a} \\
&= (([x]^2 a)[x]([x]^2 a))[x]a^{m-1-log_a [x]a - log_a [x]^2 a} \\
&\vdots \\
&= ([x]^n a)[x]a^0 \\
&= [x]^n a
\end{aligned}$$

This final result is reached when

$$\begin{aligned}
m &= 1 + log_a [x]a + log_a [x]^2 a \ldots log_a [x]^{n-1} a \\
&= \sum_{i=1}^{n-1} log_a [x]^i a \\
&= log_a \prod_{i=1}^{n-1} [x]^i a
\end{aligned}$$

2.3 $[x]$ for $x \in \mathbb{C}$

My goal is to find a means of evaluating $[x]a$ for $x \in \mathbb{C}$. Adopting a new axiom may be necessary to do this, as there is still some missing functionality in the current arithmetic system. In general, there are only three ways that it might be possible to relate two given operators, $[x]$ and $[y]$, without adding new functionality:

1. Find some expression, n, such that $[x]^n a = [y]a$ for all a, x, and y.
2. Find some expression, b, such that $[x]b = [y]a$ for all a, x, and y.
3. Find some combination of expressions, n and b, such that $[x]^n b = [y]a$ for all a, x, and y.

2.3.1 Superfunction approach

Given that $[x]^{log_2 a} a = [x+1]a$, we can define $f_1(a) \equiv [x]a$. Iteration of this function $log_2 a$ times yields

$$f_1^{log_2 a}(a) = [x]^{log_2 a} a = [x+1]a.$$

We can then define a new function, $f_2(a) = [x+1]a$ to be iterated $log_2 a$ times:

$$f_2^{log_2 a}(a) = [x+1]^{log_2 a} a = [x+2]a.$$

Iteration of this algorithm results in an expression for $[x+y]a$ in terms of hyper-iterates of the function $f_1(a)$:

$$f_y^{log_2 a}(a) = [x+y]a$$

I might as well set x=0, redefining $f_1(a)$ as $[0]a$, so that $f_y^{log_2 a}(a) = [y]a$.

2.3.2 Different axiom approach

Currently, the operator framework can be defined by A1, A2, and the relation $[x]^{g(a)} a = [x+1]a$, where $g(a) = log_2 a$ currently. In principle, $g(a)$ could be any function of a, so it may be that a different form of $g(a)$ makes the relation $f_y^{g(a)}(a)$ more manageable, though a different g(a) would result in a different progression of operators.

2.3.3 Operator approach

Suppose there is some operator R such that $R^x a = [x]a$. We have thus converted the operator argument, x, to an iteration number, which is more easily evaluated for non-integer values (see 3.1). We can define R_a in terms of a "logarithm" operator $xlog_a$, which is defined by the equation $xlog_a([x]a) = x$. Given that

$$R_a[x]a = R_a R_a^x a = R_a^{x+1} a = [x+1]a,$$

one way to define R_a is by

$$R_a b = [xlog_a(b) + 1]a$$

which reproduces $R_a[x]a = [x+1]a$ but requires us to know the value of $xlog_a a$, which is currently unknown. Even so, this formalism can be used to numerically calculate [x]a for $x \in \mathbb{C}$ using the iteration methods in 3.1.

3 Numerical evaluation

3.1 Fractional iterations of unary [x]

3.1.1 Taylor expansion

(See http://arxiv.org/pdf/hep-th/9707206v2.pdf, pg.31)

$$[x]^n = (1-(1-[x]^{-1}))^{-n}$$

Performing binomial expansion two times yields,

$$\begin{aligned}
[x]^n &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{-n}{i}\binom{i}{j}(-1)^{i+j}[x]^{-j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-n)^{\underline{i}}}{i!}\frac{i!}{j!(i-j)!}(-1)^{i+j}[x]^{-j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i n^{\overline{i}}}{j!(i-j)!}(-1)^{i+j}[x]^{-j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{n^{\overline{i}}}{j!(i-j)!}(-1)^{j}[x]^{-j}
\end{aligned}$$

I expand in terms of negative iterations of [x] because positive iterations of [x] diverge too quickly to be useful for most values of a and n.

3.2 [x] for $x \in \mathbb{C}$

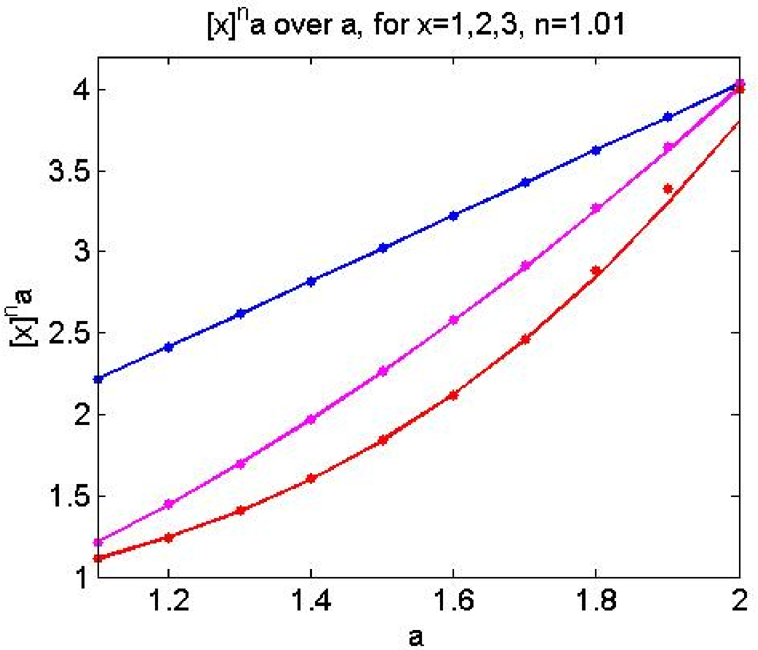
The same methods in 3.1 can be applied to the R operator of 2.3.3 to evaluate [x]a for non-integer x.

$$\begin{aligned}
[x] = R^x &= (1-(1-R))^n \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{x}{i}\binom{i}{j}(-1)^{i+j}R^{j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{x^{\underline{i}}}{i!}\frac{i!}{j!(i-j)!}(-1)^{i+j}R^{j} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{x^{\underline{i}}}{j!(i-j)!}(-1)^{i+j}[j]
\end{aligned}$$

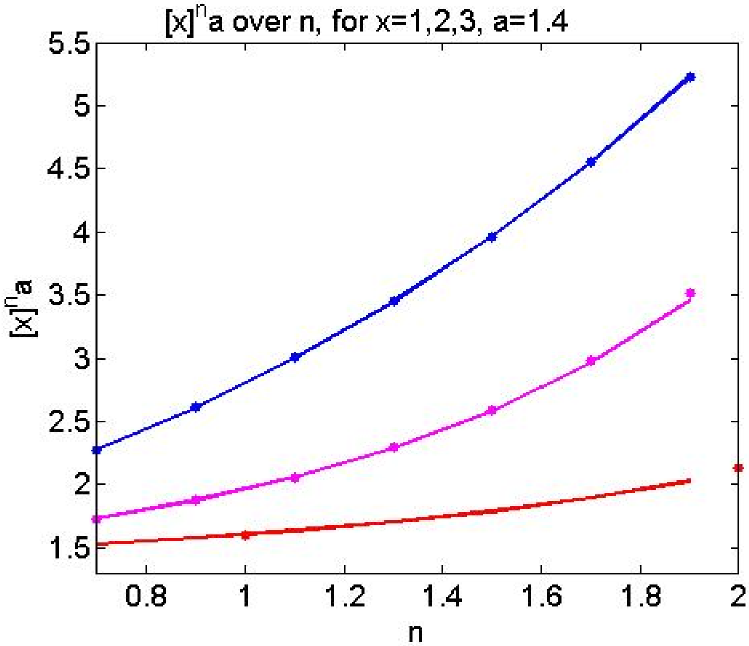
This time I expand in terms of positive iterations, but only because [x] is currently undefined for $x < 0$.

3.3 Plots

Truncating the expansions from sections 3.1.1 and 3.2, the following plots of $[x]^n a$ were produced. Actual/theoretical values are plotted as points and numerical interpolations are plotted as lines.
Below: Plots at fixed x and fixed n over range of a values. Cutoff at 40th term in expansion.



Below: Plots at fixed x and a over range of n values. Cutoff at 40th term in expansion.



Below: Plots at n=1 over range of x and a values. Cutoff after the 4th term in expansion. Left plot shows range inside the interval x=[1,3]. Right plot shows range outside the interval x=[1,3]. Note that calculations for $x > 3$ are poor because I am not yet able to evaluate $[x]a$ for integer values of x greater than 4. Also, from the right plot, it seems that $[0]a = 0$.