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Learning controllers for nonlinear systems from data

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ABSTRACT

This article provides an overview of a new approach to designing controllers for nonlinear systems using data-driven control. Data-driven control is an important area of research in control theory, and this novel method offers several benefits. It can recreate from a data-centred perspective many of the results available in the model-based case, including local stabilization based on Taylor or polynomial expansion, absolute stabilization, as well as approximate and exact feedback linearization. Moreover, the method is analytically and computationally simple, and permits to infer regions of attraction and invariant sets, also when the data are corrupted by noise.

1. Introduction

Data-driven control is one of the most important research areas in control theory. By data-driven control, we mean those design techniques where the controller is determined by employing data collected from system. This can be done by sequential system identification and model-based control (called the *indirect* approach) or by directly seeking a control law compatible with available data (called the *direct* approach). The interest for data-driven control is related primarily to the difficulties that one may encounter in deriving a mathematical model of the system to control from first-principle laws. Specifically, although we often have information about the structure of the system to control, an exact model of the system is almost always impossible to obtain. In such cases, it is natural to exploit data as an extra source of information.

The history of data-driven control is rich and varied. Regarding indirect methods we can mention all the main approaches to system identification such as prediction error, maximum likelihood, and subspace methods (Ljung, 2010). As for direct methods, notable examples are some forms of adaptive control (Astolfi, 2020), unfalsified control (Safonov & Tsao, 1997) and the VRFT method (Campi & Savaresi, 2006). We refer the interested reader to Bazanella, Campestrini, and Eckhard (2022), Hou and Wang (2013), Pillonetto, Dinuzzo, Chen, De Nicolao, and Ljung (2014) and Recht (2019) for recent surveys.

Despite the vast amount of work done over the years, data-driven control remains an open research question at least for what concerns nonlinear systems. The main issues are concerned with providing theoretical guarantees and computationally tractable algorithms. Specific issues related to the first point include the difficulty to design stabilizing controllers from a finite number of datapoints and to estimate region of attractions and invariant sets for the closed-loop system. Further, data are invariably corrupted by noise, and noise prevents exact identification of the dynamics (whether open-loop or closed-loop dynamics). As a result, the control law should guarantees a certain degree of robustness to such uncertainties and developing robust controllers for nonlinear systems is much harder than in the linear case.

Contribution. The aim of this article is to discuss a recent line of research on data-driven control, introduced in De Persis and Tesi (2020). This line of research rests on two basic elements: behavioural theory or, more specifically, the so-called Fundamental Lemma, and convex programming. The Fundamental Lemma, introduced in Willems, Rapisarda, Markovsky, and De Moor (2005), stipulates that the whole set of trajectories that a linear system can generate can be represented by a finite set of system trajectories provided that such trajectories come from sufficiently excited dynamics. This theory thus gives conditions under which data (in the form of trajectories) offer an alternative representation to parametric modelling. As shown in De Persis and Tesi (2020), this representation turns out to be suitable for controller design in the form of data-dependent semidefinite programs (SPD), hence convex programs.

In De Persis and Tesi (2020), this idea has been developed to solve optimal and robust design for linear system, notably using noisy data of low complexity, as well as to provide local stabilization results for nonlinear systems. These ideas have since been developed in many directions, also by other research groups. In this article we will review some of these developments, focusing on nonlinear systems. As we will see, the proposed method is capable of recreating from a data-centred perspective many of the results available in the model-based case,

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including local stabilization based on Taylor or polynomial expansion, absolute stabilization, as well as approximate and exact feedback linearization. Moreover, the method is analytically and computationally simple, and permits to infer regions of attraction and invariant sets, also when disturbances act on the system during both the data acquisition phase and the execution of the control task.

We will present the main idea and some results in Sections 2–5. In Section 6, we will discuss in more detail the main features of our method, also in relation to other approaches that have been proposed in similar contexts. Finally, we will discuss current limitations and open research questions.

Notation. Throughout the paper, $\succ$ ($\succeq$) and $\prec$ ($\preceq$) denote positive and negative (semi)-definiteness, respectively; $\mathbb{S}^{n\times n}$ denotes the set of $n\times n$ real-valued symmetric matrices; $A^\top$ is the transpose of A. We let $|x|$ denote the 2-norm of a vector x, and let $\|A\|$ be the induced 2-norm of a matrix A. For convenience, we will often write (x,u) to denote the vector $[x^\top\, u^\top]^\top$.

2. Framework, outline of the paper, and preliminary facts

We consider a nonlinear system in the form

$$x^+ = f(x,u) \tag{1}$$

where $x\in\mathbb{R}^n$ is the state and $u\in\mathbb{R}^m$ is the control input, x^+ denotes forward shifting, i.e., $x^+(k)=x(k+1)$, $k\in\mathbb{N}$, f is the drift vector field, which is assumed to be unknown (at least, not known exactly), with $(\overline{x},\overline{u})=(0,0)$ a known equilibrium of interest. We shall assume that $f(x,u)$ is a smooth function. More often than not, its continuity or continuous differentiability will be enough. Let

$$\mathbb{D} := \{x(k),u(k),\ k=0,1,\dots,T\} \tag{2}$$

be a dataset collected from the system with an experiment, meaning that we have a set of state and input samples that satisfy $x(k+1)=f(x(k),u(k))$ for $k=0,\dots,T-1$ where $T>0$ is a user-defined parameter. The problem of interest is to determine, using $\mathbb{D}$, a control law $u=\kappa(x)$ that stabilizes the system around the origin (globally or locally, both cases will be considered).

In this paper we discuss the approach introduced in De Persis and Tesi (2020), considering a variety of scenarios: from the case where f has a generic structure up to specific cases, for example when the dynamics are bilinear or in Lure's form. Other classes of nonlinear systems such as the polynomial ones are out of the scope of this paper and we refer the interested reader to Guo, De Persis, and Tesi (2022a). Specifically, the paper is organized as follows:

1. Sections 3 and 4 tackle the case where f has generic structure. We will discuss both the scenarios where f is unknown (except for norm bounds, Section 3) or it consists of known (but otherwise generic) basis functions (Section 4).
2. Section 5 considers various cases where f has special form, in particular systems in Lur'e form (Section 5.1) and bilinear dynamics (Section 5.2).
3. A discussion on the results is provided in Section 6 along with some extensions, and Section 7 ends the paper with concluding remarks.

2.1. Petersen's lemma

A technical result which is extensively used throughout the paper is the so-called Petersen's lemma (Petersen & Hollot, 1986).

Lemma 1 (*Petersen's Lemma*). *Let $S\in\mathbb{S}^{n\times n}$, $M\in\mathbb{R}^{n\times p}$, $N\in\mathbb{R}^{q\times n}$, $\Delta\in\mathbb{R}^{q\times r}$ be given matrices. Let $\mathcal{D}:=\{D\in\mathbb{R}^{q\times p}: DD^\top\preceq\Delta\Delta^\top\}$. Then,*

$$S+MD^\top N+N^\top DM^\top\prec 0 \quad \forall D\in\mathcal{D}$$

if and only if there exists $\varepsilon>0$ such that

$$S+\varepsilon^{-1}MM^\top+\varepsilon N^\top\Delta\Delta^\top N\prec 0.$$

3. Stabilization via Taylor's expansion

Lyapunov's indirect method is one of the oldest and most popular approaches to model-based controller design for nonlinear systems (LaSalle, 1986, Chapter 7). We describe how to devise a data-based version of this method by first considering the special case of linear systems with input disturbances. Consider the system

$$x^+ = Ax+Bu+Ed \tag{3}$$

where A,B are unknown matrices, $d\in\mathbb{R}^s$ is an unknown signal representing a disturbance, and E is a known matrix that specifies which channel the signal d enters. If such an information is not available, then $E=I_n$ and $s=n$.

Consider the dataset $\mathbb{D}$ in (2), and define

$$U_0 := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1)\end{bmatrix}\in\mathbb{R}^{m\times T}, \tag{4a}$$

$$X_0 := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1)\end{bmatrix}\in\mathbb{R}^{n\times T}, \tag{4b}$$

$$X_1 := \begin{bmatrix} x(1) & x(2) & \cdots & x(T)\end{bmatrix}\in\mathbb{R}^{n\times T}, \tag{4c}$$

$$D_0 := \begin{bmatrix} d(0) & d(1) & \cdots & d(T-1)\end{bmatrix}\in\mathbb{R}^{s\times T}, \tag{4d}$$

and note that $X_1=AX_0+BU_0+ED_0$.

The main idea of the approach introduced in De Persis and Tesi (2020) is to use the data to provide a data-based representation of the closed-loop dynamics which can be turned into a (convex) controller design program.

Lemma 2. *Consider any matrices $K\in\mathbb{R}^{m\times n}$, $G\in\mathbb{R}^{T\times n}$ such that*

$$\begin{bmatrix} K\\ I_n\end{bmatrix}=\begin{bmatrix} U_0\\ X_0\end{bmatrix}G. \tag{5}$$

System (3) with $u=Kx$ results in the closed-loop dynamics $x^+=\Psi x+Ed$ with $\Psi:=(X_1-ED_0)G$.

Proof. The result follows directly from the identity

$$A+BK=\begin{bmatrix} B & A\end{bmatrix}\begin{bmatrix} K\\ I_n\end{bmatrix}=\begin{bmatrix} B & A\end{bmatrix}\begin{bmatrix} U_0\\ X_0\end{bmatrix}G \tag{6}$$

and the relation $X_1=AX_0+BU_0+ED_0$. ■

Lemma 2 has deep implications for controller design. To have stability, all we need is indeed to design G so that $\Psi=(X_1-ED_0)G$ is Schur (all the eigenvalues of Ψ have modulus <1) and then set $K=U_0G$. Since the matrix D_0 is unknown, the idea is to ensure that $(X_1-ED)G$ is Schur for all the matrices $D\in\mathcal{D}$, with $\mathcal{D}$ a set where D_0 is known (or deemed) to belong. The technical tool that allows us to implement this idea is precisely Petersen's lemma, as we formalize in the next result.

Theorem 1. *Consider system (3) along with a dataset $\mathbb{D}$. Let $\mathcal{D}:=\{D\in\mathbb{R}^{s\times T}: DD^\top\preceq\Delta\Delta^\top\}$ where the matrix $\Delta\in\mathbb{R}^{s\times q}$ is chosen by the designer, and suppose $D_0\in\mathcal{D}$. Suppose there exist two matrices $Y\in\mathbb{R}^{T\times n}$, $P\in\mathbb{S}^{n\times n}$, and a scalar $\epsilon>0$ such that*

$$\begin{bmatrix} P-\Omega & (X_1Y)^\top & Y^\top\\ X_1Y & P-\epsilon E\Delta\Delta^\top E^\top & 0_{n\times T}\\ Y & 0_{T\times n} & \epsilon I_T\end{bmatrix}\succ 0 \tag{7a}$$

$$X_0Y=P \tag{7b}$$

with $\Omega\succ 0$ chosen by the designer. Then, the controller $K=U_0YP^{-1}$ renders $A+BK$ Schur, thus renders the closed-loop system (globally) asymptotically stable

Proof. Suppose (7a) holds. A Schur complement gives

$$\begin{bmatrix} P-\Omega & (X_1Y)^\top\\ X_1Y & P\end{bmatrix}-\epsilon^{-1}\begin{bmatrix} Y^\top\\ 0_{n\times T}\end{bmatrix}\begin{bmatrix} Y & 0_{T\times n}\end{bmatrix}$$

$$-\epsilon \begin{bmatrix} 0_{n\times s} \\ E \end{bmatrix} \Delta\Delta^\top \begin{bmatrix} 0_{s\times n} & E^\top \end{bmatrix} \succ 0.$$

By Lemma 1,

$$\begin{bmatrix} P-\Omega & Y^\top (X_1-ED)^\top \\ (X_1-ED)Y & P \end{bmatrix} \succ 0 \quad \forall D\in\mathcal{D}. \tag{8}$$

By applying another Schur complement we get

$$Y^\top (X_1-ED)^\top P^{-1}(X_1-ED)Y - P + \Omega \prec 0 \quad \forall D\in\mathcal{D}. \tag{9}$$

Pre- and post-multiplying both sides for P^{-1} and letting $G = YP^{-1}$, we obtain the Lyapunov's stability condition for $(X_1 - ED)G$, in particular for $(X_1 - ED_0)G$ because $D_0 \in \mathcal{D}$. Finally notice that the constraint $X_0Y = P$ is equivalent to $X_0G = I_n$. This, together with $K = U_0G$, implies (5), meaning that $A + BK = (X_1 - ED_0)G$. This concludes the proof. ■

A few comments are in order. The choice of $\mathcal{D}$ reflects our prior information or guess about d. For instance, if $\delta > 0$ is a known bound on the norm of d, i.e., $|d(i)| \le \delta$ for $i = 0, 1, \ldots, T-1$, we can take $\Delta = \delta\sqrt{T} I_s$, where Δ can be regarded as an energy bound on d. Δ can be made independent of the length T of the dataset by effectively taking into account the instantaneous bound on d (Bisoffi, De Persis, & Tesi, 2021). We discuss this option in Appendix along with a few advantages it offers. Stochastic disturbances can also be accounted for (possibly, with other choices of Δ), see De Persis, Rotulo, and Tesi (2023, Section VI.C). In general, large sets $\mathcal{D}$ make condition $D_0 \in \mathcal{D}$ easier to hold but make (7a) more difficult to satisfy.

The choice of Ω is instead arbitrary in the sense that it does not affect the feasibility of (7a). The reason to have $\Omega \succ 0$ (in fact, also $\Omega = 0$ guarantees stability) is to have information on the decay rate of the unforced dynamics. This will be especially useful in the context of *nonlinear* controller design, as we will discuss shortly.

A final comment regards (5). Condition (5) depends on the matrix $[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}]$ and this matrix reflects how '*rich*' is our dataset. In particular, if $[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}]$ is full row rank then we can parametrize any control loop with data. This fact can be related to a popular result proposed by Jan C. Willems and collaborators, known as the *Fundamental Lemma* (Willems et al., 2005). The Fundamental Lemma asserts that if we excite the dynamics of a linear controllable system with a *persistently exciting* input then, in the absence of noise, the resulting input–output trajectory is as informative as the system's parametric model (i.e., any other trajectory of the system can be generated from the recorded input–output data). In particular, $[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}]$ is full row rank. As shown in Lemma 2, this property allows us to *directly* parametrize all possible control loops, also with noisy data.

The Fundamental Lemma and the role of persistency of excitation for controller design has recently been subject of numerous studies, see e.g. van Waarde, De Persis, Camlibel and Tesi (2020) and van Waarde, Eising, Trentelman and Camlibel (2020). For an extension of these concepts to certain classes of nonlinear systems the interested reader is referred to Alsalti, Lopez and Müller (2023).

3.1. Stabilization via Taylor's expansion

The previous discussion provides a conceptual model that can be readily extended to nonlinear systems. In fact, we see that (3) is also a first-order approximation of (1) where d represents the remainder term in the Taylor's expansion of f. Specifically, assuming that f is a smooth function we can write (1) as

$$x^+ = Ax + Bu + Hr(x,u) \tag{10}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x,u)=(0,0)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x,u)=(0,0)}$$

are unknown and $r(x,u) \in \mathbb{R}^p$ is the unknown remainder containing high-order terms. Like E in (3), H is a known matrix specifying which components of f are actually nonlinear. If this information is not available we simply set $H = I_n$.

The following two results descend from Lemma 2 and Theorem 1.

Lemma 3. *Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{T\times n}$ such that (5) holds. Then, system (1) under the control law $u = Kx$ results in the closed-loop dynamics*

$$x^+ = \Psi x + Hr(x, Kx) \tag{11}$$

where $\Psi := (X_1 - HR_0)G$, and

$$R_0 := \begin{bmatrix} r(0) & r(1) & \ldots & r(T-1) \end{bmatrix} \tag{12}$$

where $r(k) := r(x(k), u(k))$.

Theorem 2. *Consider system (1) along with a dataset $\mathbb{D}$. Let $\mathcal{R} := \{R \in \mathbb{R}^{p\times T} : RR^\top \preceq \Delta\Delta^\top\}$ where the matrix $\Delta \in \mathbb{R}^{p\times q}$ is chosen by the designer, and suppose $R_0 \in \mathcal{R}$. Suppose there exist two matrices $Y \in \mathbb{R}^{T\times n}$, $P \in \mathbb{S}^{n\times n}$, and a scalar $\epsilon > 0$ such that (7) holds, with E replaced by the matrix H in (10). Then, the controller $K = U_0YP^{-1}$ renders the origin a (locally) asymptotically stable equilibrium for the closed-loop system.*

Theorem 2 descends directly from the fact that $A + BK$ is Schur and that $\lim_{x\to 0} \frac{|r(x,Kx)|}{|x|} = 0$, meaning that the remainder term $r(x, Kx)$ goes to zero faster than x.

The choice of $\mathcal{R}$ obeys the same considerations made for $\mathcal{D}$. In particular, a point-wise bound on $|r(x,u)|$ can be obtained from coarse information on f. For instance, a Lipschitz bound on the gradient of the components of f valid on a convex set $\mathcal{P} \subseteq \mathbb{R}^{n+m}$ makes sure that $|r(x,u)| \le c|(x,u)|^2$ for $(x,u) \in \mathcal{P}$ and for some known constant c (Folland, 1990; Guo, De Persis, & Tesi, 2022b), which gives a known bound for $\|R_0\|$ provided that the experiment is conducted in such a way that $r(k) \in \mathcal{P}$, which can be done without loss of generality.

3.1.1. Region of attraction and invariant sets

Different from the linear case, Taylor's expansions lead to local stability result, and it is therefore important to have estimates of the *region of attraction* (RoA), defined as the set of initial states for which the dynamics of the controlled system converge to zero. To this end, the key observation is that Lemma 3 gives an exact description of the closed-loop dynamics, where the only unknown terms are R_0 and $r(x, Kx)$, both related to the remainder. Thus, a bound on the norm of $r(x,u)$ is sufficient to estimate the RoA. Specifically, let $\delta : \mathbb{R}^n \to \mathbb{R}$ be a known function such that

$$|r(x,Kx)| \le \delta(x),\ x\in\mathbb{R}^n \quad \text{and} \quad \lim_{x\to 0}\frac{\delta(x)}{|x|} = 0. \tag{13}$$

(All the derivations which follow can be extended if we replace $\mathbb{R}^n$ in the first condition with a domain of interest $\mathcal{Q} \subset \mathbb{R}^n$.) By the discussion made previously, a bound like $|r(x,u)| \le c|(x,u)|^2$ gives $\delta(x) = c|(x, Kx)|^2$. Domain knowledge can be exploited to derive tighter bounds, as we exemplify later on in Example 1.

With this in mind, suppose that we found a stabilizing controller K with the method described in Theorem 2 and consider the closed-loop system, which has dynamics (11). Let $V(x) = x^\top P^{-1}x$ with P obtained from (7). It is simple to see that $V(x)$ gives a Lyapunov function for the linear part of the dynamics. In particular,

$$\begin{aligned} &V(x^+) - V(x) \\ &= (\Psi x + Hr(x,Kx))^\top P^{-1}(\Psi x + Hr(x,Kx)) - x^\top P^{-1}x \\ &\le -x^\top \Phi x + (2\Psi x + Hr(x,Kx))^\top P^{-1}Hr(x,Kx), \end{aligned}$$

where $\Phi := P^{-1}\Omega P^{-1}$. Recalling that $\Psi = (X_1 - HR_0)G$ and that $\|R_0\| \le \|\Delta\|$ by hypothesis, we obtain

$$V(x^+) - V(x) \le -x^\top \Phi x + g(x, \delta(x)), \tag{14}$$

where

$$g(z,\eta) := r_1(z)|\eta| + r_2(z)|\eta| + r_3|\eta|^2, \tag{15a}$$

$$r_1(z) := 2|(X_1 G z)^\top P^{-1} H|, \tag{15b}$$

$$r_2(z) := 2\|\Delta\|\|H^\top P^{-1} H\||Gz|, \tag{15c}$$

$$r_3 := \|H^\top P^{-1} H\|, \tag{15d}$$

$G = YP^{-1}$ and Y is obtained from (7).

Noting that the right-hand side of (14) involves known quantities, the next result follows at once.

Corollary 1. *Consider the same setting as in Theorem 2, and let $\delta : \mathbb{R}^n \to \mathbb{R}$ be a known function satisfying (13). Let $\mathcal{W} := \{x : -x^\top \Phi x + g(x, \delta(x)) < 0\}$ and $\overline{\mathcal{W}} = \mathcal{W} \cup \{0\}$. (Note that $\mathcal{W}$ is nonempty.) Finally, let $V(x) := x^\top P^{-1} x$ and $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$ with $\gamma > 0$ arbitrary. Then, the origin is an asymptotically stable equilibrium for the closed-loop system, and any set $\mathcal{R}_\gamma$ contained in $\overline{\mathcal{W}}$ is a positively invariant set (the trajectories never escape $\mathcal{R}_\gamma$) and gives an estimate of the RoA.*

3.1.2. Robust invariance sets

In a similar manner, the analysis can be extended to include process disturbances. Consider the system

$$x^+ = f(x, u) + Ed \tag{16}$$

where d is a bounded disturbance with known bound, i.e., $|d(t)| \leq \nu$ for all $t \geq 0$ for some known ν, and E is a known matrix specifying which part of the dynamics is affected by the disturbance. We can thus express the dynamics in Taylor's expansion:

$$x^+ = Ax + Bu + \underbrace{\begin{bmatrix} H & E \end{bmatrix}}_{:=J} \underbrace{\begin{bmatrix} r(x,u) \\ d \end{bmatrix}}_{=: s(x,u,d)}. \tag{17}$$

A stability property that accounts for the presence of disturbances is the following Blanchini (1999):

Definition 1. A set $\mathcal{R}$ is called *robustly positively invariant* (RPI) for the system $\xi^+ = f(\xi, d)$ if for every $\xi(0) \in \mathcal{R}$ and all $d(t) \in \mathcal{I}$, with $\mathcal{I}$ a compact set, the solution is such that $\xi(t) \in \mathcal{R}$ for $t > 0$.

The results of the previous section can be extended to this case with minor modifications. In particular, let

$$S_0 := \begin{bmatrix} s(0) & s(1) & \cdots & s(T-1) \end{bmatrix} \tag{18}$$

be the dataset associated with remainder and disturbance term, where we set $s(k) := s(x(k), u(k), d(k))$ for brevity. We introduce $\mathcal{S} = \{S \in \mathbb{R}^{\upsilon \times T} : SS^\top \preceq \Delta\Delta^\top\}$, where υ is the integer equal to the dimension of $(r(x,u), d)$. Clearly, a bound on S_0 can be determined from individual bounds on $r(x,u)$ and d. The following result thus follows at once:

Theorem 3. *Consider system (16) along with a dataset $\mathbb{D}$. Let $\mathcal{S} := \{S \in \mathbb{S}^{\upsilon \times T} : SS^\top \preceq \Delta\Delta^\top\}$ where the matrix $\Delta \in \mathbb{R}^{\upsilon \times q}$ is chosen by the designer, and suppose $S_0 \in \mathcal{S}$. Suppose there exist two matrices $Y \in \mathbb{R}^{T \times n}$, $P \in \mathbb{S}^{n \times n}$, and a scalar $\epsilon > 0$ such that (7) holds, with E replaced by the matrix J in (17). Then, the controller $K = U_0 Y P^{-1}$ renders the origin a (locally) asymptotically stable equilibrium for the closed-loop system.*

Note that by studying asymptotic stability of the origin we are implicitly assuming that $d = 0$ in the closed-loop system $x^+ = f(x, Kx) + Ed$. Assume now that $d \neq 0$ in the closed-loop system and let ν be a known bound on $|d|$. By proceeding as in (14), we obtain

$$V(x^+) - V(x) \leq -x^\top \Phi x + g(x, (\delta(x), \nu))$$

where $g(z, \eta)$ is defined in (15), with H replaced by J given in (17). It is then straightforward to extend Corollary 1 to robust positive invariance.

Corollary 2. *Consider the same setting as in Theorem 3, and let $\delta : \mathbb{R}^n \to \mathbb{R}$ be a known function satisfying (13), and let ν be a known bound on $|d|$. Let $V(x) = x^\top P^{-1} x$, and define $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$, with $\gamma > 0$ arbitrary. Finally, let $\mathcal{X} := \{x : -x^\top \Phi x + g(x, (\delta(x), \nu)) \leq 0\}$ where $g(z, \eta)$ is defined in (15), with H replaced by J given in (17), and $\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$ where $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$ ($\mathcal{Z}$ is the subset of $\mathcal{R}_\gamma$ for which the Lyapunov difference can be positive). If*

$$V(x) - x^\top \Phi x + g(x, (\delta(x), \nu)) \leq \gamma \quad \forall x \in \mathcal{Z} \tag{19}$$

then $\mathcal{R}_\gamma$ is an RPI set for the closed-loop system.

Proof. Assume $x \in \mathcal{R}_\gamma$. Consider first the case $x \notin \mathcal{Z}$. Since $x \in \mathcal{R}_\gamma$ then $x \notin \mathcal{X}^c$. Hence $x \in \mathcal{X}$, and therefore $V(x^+) - V(x) \leq -x^\top \Phi x + g(x, (\delta(x), \nu)) \leq 0$, implying $x^+ \in \mathcal{R}_\gamma$. Next, consider the case $x \in \mathcal{Z}$. In view of (19), we have $V(x^+) \leq \gamma$, thus $x^+ \in \mathcal{R}_\gamma$. ■

3.2. Example 1

Consider the Euler discretization of an inverted pendulum

$$x_1^+ = x_1 + T_s x_2 \tag{20a}$$

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u + d\,, \tag{20b}$$

where $T_s = 0.1$ is the sampling time, m is the mass to be balanced, ℓ is the distance from the base to the centre of mass of the balanced body, μ is the coefficient of rotational friction, and g is the acceleration due to gravity. The states x_1, x_2 are the angular position and velocity, respectively, u is the applied torque and d an external disturbance. The system has an unstable equilibrium in $(x, u) = (0, 0)$, corresponding to the pendulum upright position, which we want to stabilize. Suppose that the unknown parameters are $m = 1$, $\ell = 1$, $g = 9.8$ and $\mu = 0.01$.

We collect data by running an experiment with input uniformly distributed in $[-0.5, 0.5]$, and with an initial state within the same interval. We consider a disturbance uniformly distributed in $[-0.01, 0.01]$, which gives a reasonable input signal-to-noise ratio around 30 dB. We collect $T = 10$ samples (corresponding to the motion of the pendulum that oscillates around the upright position).

In this example, $E = H = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, and the remainder term is $r(x, u) = r(x) = \frac{T_s g}{\ell}(\sin x_1 - x_1)$. We satisfy (13) by setting $\delta(x) = 2|\sin x_1 - x_1|$, thus over-approximating $r(x)$ by more than 100%. While other choices of $\delta(x)$ are possible, this is an example where it is reasonable to know the structure of the remainder, which comes from physical considerations, namely Lagrange's equations of motion. Accordingly, we set $\Delta = \sqrt{T}\,\mathrm{diag}(c, \nu)$, with c equal to the maximum of $\delta(x)$ over the experimental data and $\nu = 0.01$ bound on the disturbance. Finally, we set $\Omega = I_2$.

The controller design program is feasible and we obtain $K = \begin{bmatrix} -21.9778 & -9.6747 \end{bmatrix}$. An illustration of RoA and RPI sets is given in Fig. 1.

We close this section with some comments regarding T. A main feature of De Persis and Tesi (2020) is not to require large datasets. The analysis shows indeed that even $n + m$ samples may suffice (*cf.* (5)), suggesting that the quality of data weights more than the quantity. (The importance of large datasets becomes apparent when disturbances have suitable statistics and large datasets can help to filter out noise De Persis et al., 2023; Dean, Mania, Matni, Recht, & Tu, 2020). On the other hand, and with specific regard to Taylor's method, choosing T sufficiently small helps to keep the system in a neighbourhood of the equilibrium during the data collection phase, which is needed to keep Δ small enough (recall that Δ is the bound on the remainder).

When dealing with open-loop unstable systems, using small datasets can be advantageous, in particular it may be more convenient to consider multiple short experiments rather than a single but long one, see van Waarde, De Persis et al. (2020) for a detailed discussion on this point. For unstable systems another option is to collect data using

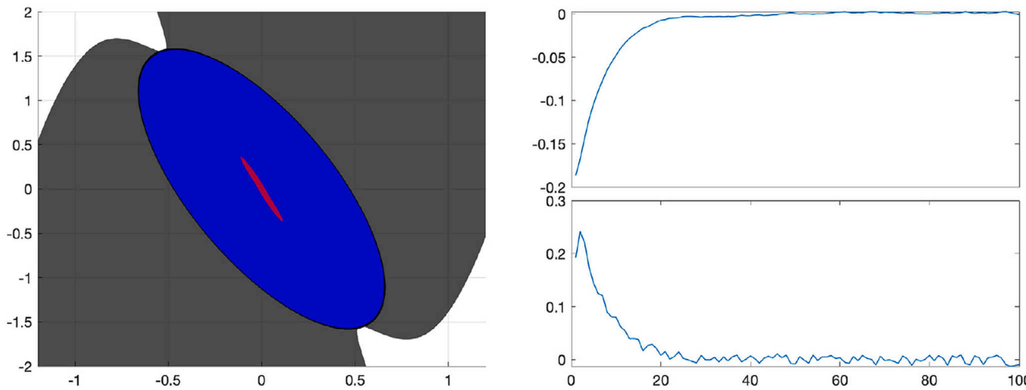

Fig. 1. Results for Example 1. (Left) The grey set represents the set $\mathcal{X}$, the blue set is the RPI set $\mathcal{R}_\gamma$; here, $P^{-1} = \begin{bmatrix} 0.2007 & 0.0587 \\ 0.0587 & 0.0351 \end{bmatrix}$ and $\gamma = 0.0442$. The black set wrapping $\mathcal{R}_\gamma$ is the RoA, which is larger than the RPI set. The red set around the origin is $\mathcal{Z}$. States originating in $\mathcal{Z}$ do not exit $\mathcal{R}_\gamma$. (Right) Closed-loop behaviour starting inside the RPI set (the position is displayed in the top figure). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

a pre-stabilizing controller K superimposed by a sufficiently exciting input. By implementing a pre-stabilizing controller, it is possible to safely conduct experiments that may result in improved controllers or controllers that guarantee larger regions of attraction.

4. Linearly parametrized basis functions

Domain knowledge often gives information about the system to control. This is typically the case with electrical and mechanical systems where first-principles laws (such as Lagrange's equations of motion in Example 1) tell us what "type" of dynamics we are dealing with. In this section, we show how to incorporate this information into controller design and improve over Taylor's method.

To convey the idea in a simple manner, we consider a nonlinear system in the form

$$x^+ = f(x) + Bu + Ed. \tag{21}$$

The general case of dynamics $x^+ = f(x,u)$ is addressed in De Persis et al. (2023, Section V.B), Both $f(x)$ and B are regarded unknown but we now assume that we know a continuous function $Z : \mathbb{R}^n \to \mathbb{R}^S$ such that $f(x) = AZ(x)$ for some matrix $A \in \mathbb{R}^{n\times S}$. Under such an assumption, system (21) reads equivalently as

$$x^+ = AZ(x) + Bu + Ed \tag{22}$$

with A, B unknown.

This setup means that we know a library of functions capable to describe the dynamics of the system. We allow Z to contain terms not present in f, which may arise from an imprecise knowledge of the system dynamics. We will directly consider the case where Z contains *both* linear and nonlinear functions, *i.e.*,

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}, \tag{23}$$

where $Q : \mathbb{R}^n \to \mathbb{R}^{S-n}$ contains only nonlinear functions. The special case where $Z(x) = x$ reduces the analysis to that of linear systems, see Section 3.

The problem of interest is to determine, using $\mathbb{D}$, a control law $u = KZ(x)$ that stabilizes the system around the origin. With respect to Taylor's method, now the control law is *nonlinear*. In particular, the control law implements a linear feedback on $Z(x)$ rather than a generic nonlinear feedback on x, and this is motivated by the specific approach that we want to consider, namely nonlinearity cancellation/minimization. Roughly, the idea is to find K such that the closed-loop system behaves nearly as a linear system, i.e., such that $AZ(x) + Bu \approx Mx$, and we choose $u = KZ(x)$ to cancel out the nonlinearities that are present in $AZ(x)$.

The starting point of the analysis is to provide an analogue of Lemma 3.

Lemma 4. *Consider any matrices $K \in \mathbb{R}^{m\times S}$, $G \in \mathbb{R}^{T\times S}$ satisfying*

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G, \tag{24}$$

where

$$Z_0 := \begin{bmatrix} Z(0) & Z(1) & \cdots & Z(T-1) \end{bmatrix}, \tag{25}$$

and we set $Z(k) := Z(x(k))$. Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T\times n}$. System (22) under the control law $u = KZ(x)$ results in the closed-loop dynamics

$$x^+ = \Psi x + \Xi Q(x) + Ed, \tag{26}$$

where $\Psi := (X_1 - ED_0)G_1$ and $\Xi := (X_1 - ED_0)G_2$.

The proof is analogous to the one of Lemma 3 and is therefore omitted. The main difference between Lemma 3 and this lemma lies in condition (24). In contrast with (5), this new condition involves Z_0 instead of X_0, and this leads to a closed-loop representation that does not involve any unknown remainder term.

We now proceed with the analysis. For clarity, we will discuss noise-free case and noisy case separately.

4.1. Noise-free case and region of attraction

In the noise-free case, the closed-loop dynamics reads $x^+ = \Psi x + \Xi Q(x)$ with $\Psi = X_1 G_1$ and $\Xi = X_1 G_2$. All we need is thus to set a design program which makes Ψ Schur and renders Ξ as small as possible.

Theorem 4. *Consider a nonlinear system as in (22) with $d \equiv 0$ along with the following SDP in the decision variables $P \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, and $G_2 \in \mathbb{R}^{T\times(S-n)}$:*

$$\underset{P,Y_1,G_2}{\text{minimize}} \quad \|X_1 G_2\| \tag{27a}$$

$$\text{subject to} \quad Z_0 Y_1 = \begin{bmatrix} P \\ 0_{(S-n)\times n} \end{bmatrix}, \tag{27b}$$

$$\begin{bmatrix} P & (X_1 Y_1)^\top \\ X_1 Y_1 & P \end{bmatrix} \succ 0, \tag{27c}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(S-n)} \\ I_{S-n} \end{bmatrix}. \tag{27d}$$

The following holds:

(i) If this SDP is feasible and the solution achieves zero cost ($\|X_1 G_2\| = 0$) then the control law $u = KZ(x)$ with

$$K = U_0 \begin{bmatrix} Y_1 P^{-1} & G_2 \end{bmatrix} \tag{28}$$

linearizes the closed-loop dynamics, and renders the origin a globally asymptotically stable equilibrium.

(ii) Assume that

$$\lim_{x\to 0}\frac{|Q(x)|}{|x|}=0\,. \tag{29}$$

If the SDP is feasible then $u = KZ(x)$, with K as in (28), renders the origin an asymptotically stable equilibrium.

Proof. Suppose that (27) is feasible. Let $G_1 = Y_1P^{-1}$, and note that the two constraints (27b) and (27d) together return $Z_0\begin{bmatrix}G_1 & G_2\end{bmatrix} = I_S$. This identity, along with (28), gives (24). By Lemma 4, the closed-loop dynamics satisfies $x^+ = \Psi x+\Xi Q(x)$ with $\Psi = X_1G_1$ and $\Xi = X_1G_2$, and (27c) ensures that Ψ is Schur. This gives the result. ■

Let $V(x) := x^\top P^{-1}x$, and note that

$$V(x^+)-V(x)=\underbrace{(\Psi x+\Xi Q(x))^\top P^{-1}(\Psi x+\Xi Q(x))-x^\top P^{-1}x}_{=:h(x)} \tag{30}$$

where the matrices Ψ,Ξ, and P are all known. Therefore, we immediately obtain the following result.

Corollary 3. *Consider the same setting as in Theorem 4. Let $\mathcal{W} := \{x : h(x) < 0\}$ with $h(x)$ as in (30), and consider the Lyapunov function $V(x) = x^\top P^{-1}x$. Then, any sub-level set $\mathcal{R}_\gamma := \{x : V(x) \le \gamma\}$ of V contained in $\mathcal{W}\cup\{0\}$ is a positively invariant set for the closed-loop system and gives an estimate of the RoA.*

We make a number of comments before moving to discuss the feedback linearization problem and the case of noisy data.

Item (i) of Theorem 4 gives a clear indication of the advantage that we can have by knowing the basis functions of f, namely *global* instead of *local* asymptotic stability. Item (ii) aligns more with Taylor's method but with the substantial difference that it involves a known quantity $Q(x)$ instead of the unknown remainder term $r(x,Kx)$, and this difference turns out to be crucial when estimating the RoA. This is evident once we note that the set $\mathcal{W}$ in Corollary 3 does not involve any over-approximation, as is instead the case with Taylor's method (Corollary 1).

Another interesting feature of this approach is that the design is fully *modular* in the sense that (27) treats linear and nonlinear dynamics separately: the linear part is stabilized via (27c) while the nonlinear part is minimized through (27a); further, the two consistency constraints (27b), (27d) are independent of each other. This has some interesting implications. Rewrite the dynamics as

$$x^+ = \overline{A}x+\hat{A}Q(x)+Bu \tag{31}$$

where we partitioned A as $A = \begin{bmatrix}\overline{A} & \hat{A}\end{bmatrix}$. Since linear and nonlinear design are decoupled, a sufficient condition for (27) to admit a solution is that $\left[\begin{smallmatrix}U_0\\ Z_0\end{smallmatrix}\right]$ is full row rank (data are rich) and $(\overline{A},B)$ is stabilizable. This result is formalized in De Persis et al. (2023, Theorem 5). This is not possible with Taylor's method as it treats the nonlinearity as an uncertainty term, and for this reason the design of the linear part is cast as a *robust* design problem.

We make a final comment on item (ii) of Theorem 4. Condition (29) can be replaced by requiring that $Q(x)$ is differentiable at $x=0$ and satisfies $Q(0)=0$. In fact, in this case $Q(x)$ admits a Taylor's expansion at $x=0$:

$$Q(x)=\underbrace{\left[\frac{\partial Q}{\partial x}\right]_{x=0}}_{=:F}x+r(x),$$

with $r:\mathbb{R}^n\to\mathbb{R}^{S-n}$ a differentiable function of the state such that $\lim_{x\to 0}\frac{|r(x)|}{|x|}=0$. Accordingly, the dynamics can be rewritten as

$$x^+=(\overline{A}+\hat{A}F)x+\hat{A}r(x)+Bu. \tag{32}$$

Thus Theorem 4 becomes applicable with $Q(x)$ replaced by $r(x)$.[1] In fact, this observation shows that item (ii) of Theorem 4 is conceptually analogous to Taylor's method but with the key difference that the nonlinearity is known.

4.2. Feedback linearization

Stabilizing system (21) by cancelling the nonlinear term $X_1G_2Q(x)$ as done in Section 4.1 brings to mind feedback linearization techniques, which combine a nonlinear feedback with a suitable change of coordinates. In this subsection we discuss how feedback linearization can be performed from data using the framework presented in this paper. We sketch the main idea referring the reader to De Persis et al. (2023, Section VII.B) for details.

We consider a single input single output system

$$x^+=f(x,u) \tag{33a}$$

$$y=h(x) \tag{33b}$$

where $u,y\in\mathbb{R}$, $h(0)=0$ and, as before, $f(0,0)=0$. We assume that both the state x and the output y are measured. The system of interest is one for which a normal form – obtained from a change of coordinates – exists. To avoid keeping track of the set where such a normal form is valid, we will work with a normal form that holds globally. For the latter to exist, we assume that the condition

$$\frac{\partial h\circ f_0^i\circ f(x,u)}{\partial u}=0,\forall(x,u)\in\mathbb{R}^{n+1},0\le i\le n-2 \tag{34a}$$

$$\left.\frac{\partial h\circ f_0^{n-1}\circ f(x,u)}{\partial u}\right|_{u=0}=\text{constant}\neq 0,\forall x\in\mathbb{R}^n \tag{34b}$$

where $f_0(x)=f(x,0)$, $f_0^d=\underbrace{f_0\circ f_0\circ\cdots\circ f_0}_{d\ times}$, holds and define

$$\begin{bmatrix}h(x)\\ h\circ f_0(x)\\ \vdots\\ h\circ f_0^{n-1}(x)\end{bmatrix}=:\Phi_0(x) \tag{35}$$

as a continuous global change of coordinates (Isidori, 1995; Monaco & Normand-Cyrot, 1987, p. 11).[2] Under condition (34a), if we set

$$w(k):=\begin{bmatrix}y(k)\\ y(k+1)\\ \vdots\\ y(k+n-1)\end{bmatrix}=\Phi_0(x(k)),$$

system (33) in the coordinates w becomes

$$w(k+1)=\begin{bmatrix}w_2(k)\\ w_3(k)\\ \vdots\\ w_n(k)\\ h\circ f_0^{n-1}\circ f(x(k),u(k))\end{bmatrix},\ y(k)=w_1(k). \tag{36}$$

In model-based control, at this stage one would determine a function $u=\gamma(x,v)$ such that $h\circ f_0^{n-1}\circ f(x,\gamma(x,v))=v$ by the Implicit Function Theorem. This however requires a knowledge of $h\circ f_0^{n-1}\circ f(x,u)$ which we do not have. To overcome this obstacle, we first consider a Taylor expansion of the function $h\circ f_0^{n-1}\circ f(x,u)$ with respect to the variable u at the point $\overline{u}=0$, which in view of (34b) returns

$$h\circ f_0^{n-1}\circ f(x,u)=h\circ f_0^n(x)+bu$$

[1] As an example, consider the inverted pendulum system. If we take $Q(x)=\sin(x_1)$ then condition (29) is not satisfied. Nonetheless, we can replace $Q(x)$ with $r(x)=\sin(x_1)-x_1$.

[2] (i) $\Phi_0(x)$ is invertible, i.e., there exists a function $\Phi_0^{-1}(w)$ such that $\Phi_0^{-1}(\Phi_0(x))=x$ for all $x\in\mathbb{R}^n$, and (ii) both $\Phi_0(x),\Phi_0^{-1}(w)$ are continuous.

where $b := \frac{\partial h \circ f_0^{n-1} \circ f(x,u)}{\partial u}|_{u=0} \neq 0$ for all $x \in \mathbb{R}^n$. Second, in the spirit of this section, we assume that the unknown nonlinearity $h \circ f_0^n(x)$ can be expressed as a linear combination of a known vector $Z(x)$ made of linearly independent functions, that is

$$h \circ f_0^n(x) = a^\top Z(x) \tag{37}$$

for some $a \in \mathbb{R}^S$ and

$$Z(x) = \begin{bmatrix} h(x) \\ Q(x) \end{bmatrix}. \tag{38}$$

Hence, under conditions (34), (37), the system in normal form is

$$w^+ = A_c w + B_c(a^\top Z(x) + bu), \quad y = C_c w$$

where (A_c, B_c, C_c) are matrices in Brunowsky form (Zeitz, 1989).

In what follows, we will design a feedback controller $u = KZ(x)$. Compared to (23), in $Z(x)$ in (38) we are considering $h(x)$ rather than x. This is because we are working with the normal form (36) whose state is w and of which only the first component $w_1 = h(x)$ is available for feedback.

We start defining the matrix of input samples U_0 as in (4a), Z_0 as in (25) and

$$W_0 := \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{39a}$$

$$W_1 := \begin{bmatrix} w(1) & w(2) & \cdots & w(T) \end{bmatrix} \in \mathbb{R}^{n\times T}, \tag{39b}$$

which satisfy the identity $W_1 = A_c W_0 + B_c(a^\top Z_0 + bU_0)$. Note that the matrices W_0, W_1 are known because they consist of output samples.

Corollary 4. *Consider the nonlinear system (33). Assume that the conditions (34), (37) hold and that Φ_0 in (35) is a continuous global change of coordinates. Consider the following SDP in the decision variables $G_1 \in \mathbb{R}^{T\times n}$, $G_2 \in \mathbb{R}^{T\times(S-n)}$ and $k_1 \in \mathbb{R}$:*

$$\underset{G_1,G_2,k_1}{\text{minimize}} \quad \|(W_1 - A_c W_0)G_2\| \tag{40a}$$

$$\text{subject to} \quad Z_0 G_1 = \begin{bmatrix} 1 \\ 0_{(S-1)\times 1} \end{bmatrix}, \tag{40b}$$

$$(W_1 - A_c W_0)G_1 = B_c k_1, \tag{40c}$$

$$k_1 \in (-1,1), \tag{40d}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{1\times(S-1)} \\ I_{S-1} \end{bmatrix}. \tag{40e}$$

(i) If the SDP is feasible and the solution achieves zero cost then $u = KZ(x)$, with $K = U_0 G$, linearizes the closed-loop system and renders the origin a globally asymptotically stable equilibrium.

(ii) If $\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}$ has full row rank, then the SDP is feasible, the solution achieves zero cost and the same conclusion as in (i) holds.

Proof. (i) Conditions (40b), (40e) along with the definition of the controller gain K, show that the identity (24) holds. Thus, the closed-loop system is of the form

$$w^+ = A_c w + B_c(a^\top Z(x) + bKZ(x)) \tag{41a}$$

$$= A_c w + B_c a^\top Z(x) + B_c bU_0 GZ(x) \tag{41b}$$

$$= A_c w + (W_1 - A_c W_0)GZ(x) \tag{41c}$$

$$= A_c w + (W_1 - A_c W_0)G_1 e_{1,n}^\top w \tag{41d}$$

$$= (A_c + B_c k_1 e_{1,n}^\top)w \tag{41e}$$

where $e_{k,p}$ represents the kth standard unit vector in $\mathbb{R}^p$, hence $e_{1,n} = [\,1\ 0\ \dots\ 0\,]^\top \in \mathbb{R}^n$. The equality (41c) follows from the identities $B_c bU_0 G = W_1 G - A_c W_0 G - B_c a^\top Z_0 G$, (40b) and (40e), the equality (41d) from $(W_1 - A_c W_0)G_2 = 0$ and (41e) from (40c). Hence, the controller $u = KZ(x)$ linearizes the closed-loop system. Condition (40d) ensures Schur stability of $w^+ = (A_c + B_c k_1 e_{1,n}^\top)w$. As $x = \Phi^{-1}(w)$, with $\Phi^{-1}(w)$ a continuous function such that $0 = \Phi^{-1}(0)$, the thesis follows.

(ii) Under conditions (34), (37), in the coordinates w system (33a) takes the form $w^+ = A_c w + B_c(a^\top Z(x) + bu)$. As $b \neq 0$, there exists $K = b^{-1}(-a^\top + k_1 e_{1,S}^\top)$, where $k_1 \in (-1,1)$. This results in the closed-loop system $w^+ = (A_c + B_c k_1 e_{1,n}^\top)w$. For such matrix K, let G be such that $\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G$. Hence, (40b), (40e) hold. Thanks to this identity, the closed-loop system $w^+ = A_c w + B_c(a^\top Z(x) + bKZ(x))$ can be written as $w^+ = [A_c + (W_1 - A_c W_0)G_1 e_{1,n}^\top]w + (W_1 - A_c W_0)G_2 Q(x)$, by the same arguments used to obtain (41b)–(41c). By comparing it with $w^+ = (A_c + B_c k_1 e_{1,n}^\top)w$, we conclude that it must hold $((W_1 - A_c W_0)G_1 - B_c k_1)h(x) + (W_1 - A_c W_0)G_2 Q(x) = 0$. By structure of the matrices A_c, B_c and W_0, W_1, the matrix $W_1 - A_c W_0$ is made of all zero entries except for the last row. Hence, the previous identity is equivalent to the scalar identity $(e_{n,n}^\top (W_1 - A_c W_0)G_1 - k_1)h(x) + e_{n,n}^\top (W_1 - A_c W_0)G_2 Q(x) = 0$. By the linear independence of the functions in $Z(x)$, this implies $e_{n,n}^\top (W_1 - A_c W_0)G_1 - k_1 = 0$ and $e_{n,n}^\top (W_1 - A_c W_0)G_2 = 0$. Bearing in mind once again that $W_1 - A_c W_0$ is made of all zero entries except for the last row, we conclude that $(W_1 - A_c W_0)G_1 = B_c k_1$, which is (40c), and $(W_1 - A_c W_0)G_2 = 0$. This ends the proof. ■

The result shows that for systems that are feedback linearizable, under conditions on the richness of data, the approach outlined in Section 4.1 leads to a feasible SDP, hence to a feedback linearizing controller.

4.3. Noisy case and robust invariance

We now turn our attention to extend the results of Section 4.1 to the case of noisy data, which involves minor changes. It is sufficient to note that (42c) below replaces (27c) to account for noise. In fact, (42c) is the same as (7a), corresponding to the robust design of the linear part of the dynamics.

Theorem 5. *Consider a nonlinear system as in (22), with $Q(x)$ satisfying the condition (29), and let $\mathbb{D}$ be a dataset. Let $\mathcal{D} := \{D \in \mathbb{R}^{s\times T} : DD^\top \preceq \Delta\Delta^\top\}$ where the matrix $\Delta \in \mathbb{R}^{s\times q}$ is chosen by the designer, and suppose $D_0 \in \mathcal{D}$. Consider the following program:*

$$\underset{P,Y_1,G_2}{\text{minimize}} \quad \|X_1 G_2\| \tag{42a}$$

$$\text{subject to} \quad (27b), (27d), \tag{42b}$$

$$\begin{bmatrix} P - \Omega & (X_1 Y_1)^\top & Y_1^\top \\ X_1 Y_1 & P - \epsilon E\Delta\Delta^\top E^\top & 0_{n\times T} \\ Y_1 & 0_{T\times n} & \epsilon I_T \end{bmatrix} \succ 0, \tag{42c}$$

where $\Omega \succ 0$ is chosen by the designer.

If the design program is feasible then the control law $u = KZ(x)$ with K given by (28) renders the closed-loop system (locally) asymptotically stable.

The estimate of RPI sets is straightforward. As before, let $V(x) = x^\top P^{-1} x$ and note that

$$V(x^+) - V(x) \le \\ -x^\top \Phi x + (2\Psi x + \Xi Q(x) + Ed)^\top P^{-1}(\Xi Q(x) + Ed), \tag{43}$$

where $\Phi = P^{-1}\Omega P^{-1}$, $\Psi = (X_1 - ED_0)G_1$, and where $\Xi = (X_1 - ED_0)G_2$. Accordingly, we have

$$V(x^+) - V(x) \le \\ \underbrace{-x^\top \Phi x + \ell_1(x) + \ell_2(x) + \ell_3(x) + \ell_4(x)}_{=:\ell(x)} + g(Z(x), v) \tag{44}$$

where g is as in (15), with H replaced by the matrix E, v a known bound on $|d|$, and where

$$\ell_1(x) := (2X_1 G_1 x + X_1 G_2 Q(x))^\top P^{-1} X_1 G_2 Q(x),$$

$$\ell_2(x) := \|\Delta\| |(2X_1 G_1 x + X_1 G_2 Q(x))^\top P^{-1} E| |G_2 Q(x)|,$$

$$\ell_3(x) := \|\Delta\| |2G_1 x + G_2 Q(x)| |E^\top P^{-1} X_1 G_2 Q(x)|,$$

$$\ell_4(x) := \|\Delta\|^2 \|E^\top P^{-1} E\| |2G_1 x + G_2 Q(x)| |G_2 Q(x)|,$$

which are all known quantities.

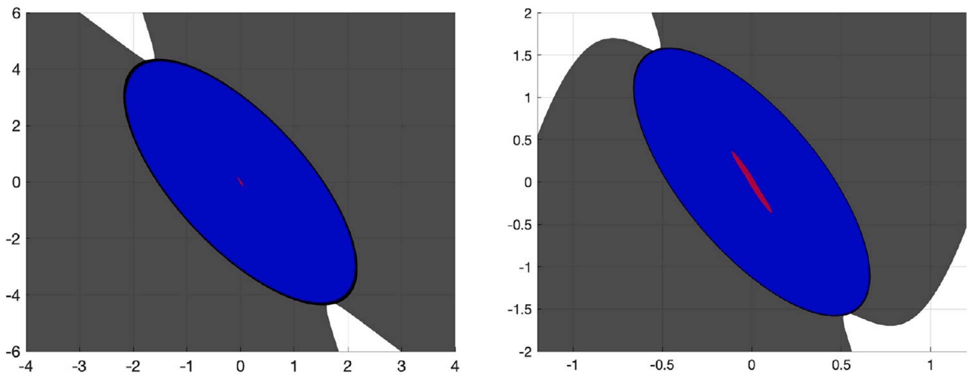

Fig. 2. Results for Example 2. (Left) Method with known basis functions. (Right) Taylor's method. See Fig. 1 for the definition of the various sets displayed.

Corollary 5. *Consider the same setting as in Theorem 5, and let ν be a known bound on $|d|$. Let $V(x) = x^\top P^{-1} x$, and define $\mathcal{R}_\gamma := \{x : V(x) \le \gamma\}$, with $\gamma > 0$ arbitrary. Finally, let $\mathcal{X} := \{x : \ell(x) + g(Z(x), \nu) \le 0\}$ where $g(z,\eta)$ is defined in (15), with H replaced by the matrix E, and $\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$ where $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$ ($\mathcal{Z}$ is the subset of $\mathcal{R}_\gamma$ for which the Lyapunov difference can be positive). If*

$$V(x) + \ell(x) + g(Z(x), \nu) \le \gamma \quad \forall x \in \mathcal{Z} \tag{46}$$

then $\mathcal{R}_\gamma$ is an RPI set for the closed-loop system.

A direct comparison with Corollary 2 shows that the uncertainty term $g(x, (\delta(x), \nu))$ in Taylor's method (recall that $\delta(x)$ is the bound on the remainder and ν is the bound on the disturbance) has now reduced to $g(Z(x), \nu)$, and $g(Z(x), \nu) \approx g(x, \nu)$ whenever $\|X_1 G_2\|$ is sufficiently small, leading to an improved estimate of the RPI sets.

In relation with this aspect, (44) suggests that it might be convenient to *regularize* the objective function in (42) so as to mitigate the effect of the disturbances. As shown in Example 2 below, a convenient choice is the following one:

$$\underset{P,Y_1,G_2}{\text{minimize}} \quad \|X_1 G_2\| + \lambda_1 \|P\| + \lambda_2 \|G_2\| \tag{47a}$$

$$\text{subject to} \quad (27b), (27d), (42c), \tag{47b}$$

with $\lambda_1, \lambda_2 \ge 0$ weighting parameters. Penalizing $\|P\|$ increases the smallest eigenvalue of Φ, while penalizing $\|G_2\|$ decreases the various terms ℓ_i in (44). Note that penalizing $\|P\|$ might increase the terms ℓ_i, but while these quantities depend on P^{-1}, Φ depends on P^{-2}, so penalizing $\|P\|$ can still be advantageous.

4.4. Example 2

Consider the same experimental setting as Example 1. This time we collect $T = 30$ datapoints so as to emphasize that we do not need the experiment to be carried out close to the equilibrium (with 10 samples x_1 never exceeds 12° while 30 samples lead x_1 beyond 45°). We choose $Q(x) = \sin x_1 - x_1$, and solve (47) with $\lambda_1 = \lambda_2 = 0.1$.

The design program is feasible and gives the controller $K = \begin{bmatrix} -23.9436 & -11.4581 & -9.8564 \end{bmatrix}$, which generates the term -9.8564 $(\sin x_1 - x_1)$ that approximately cancels out the nonlinearity. This extra term improves the estimate of RoA and RPI sets, as reported in Fig. 2.

5. Special classes of systems

In the previous sections we have treated fairly general classes of nonlinear systems. In the first case, we have Taylor expanded the nonlinear vector field and treated the remainder as a perturbation. In the second case, we have expressed the nonlinear vector field in terms of a known basis of nonlinear functions. In both cases we could design the controller from data. However, the difference in the priors available for the two cases leads to a noticeable change when estimating the region of attraction. The two methods have in common the design strategy: since the nonlinearities are general and no specific structure on how they affect the dynamics is assumed to be known to the designer, the control stabilizes the linear part and approximately cancels the nonlinearities. This strategy has the additional advantage of providing the designer with the flexibility and power of linear robust control methods, which help dominating the nonlinearities of the system.

On the other hand, we have seen that if more priors are known about the system, for instance, if it is feedback linearizable, then stronger control results are obtainable. In this section, we continue along this line of arguments and see what is achievable for systems for which extra information is available about the nonlinearities. We examine systems that are in Lur'e form and bilinear systems.

5.1. Systems in Lur'e form

In Section 4 we took advantage of a linear parametrization of the vector field f to design feedback controllers that make the closed-loop system dominantly linear by minimizing the nonlinearities. In many practical cases, the vector field f can be separated into a linear part and a nonlinear part that satisfies particular constraints (Yakubovich, Leonov, & Gelig, 2004). For these cases, one can provide a more refined analysis than for a system with a general nonlinear vector field, as we discuss in this section.

We consider system (21) where the vector field $f(x)$ is replaced by $Ax + \varphi(t, Hx)$, namely

$$\begin{aligned} x^+ &= Ax + Bu + Lv + Ed \\ z &= Hx \\ v &= \varphi(t, z) \end{aligned} \tag{48}$$

where A, B, E are defined as before, $\varphi : \mathbb{R}_{\ge 0} \times \mathbb{R}^p \to \mathbb{R}^q$ is a memoryless, possibly time-varying, function in z and $L \in \mathbb{R}^{n\times q}$, $H \in \mathbb{R}^{p\times q}$ are *known* matrices. Here the variable z and the matrix H should not be confused with the one used in (15).

We are interested in functions that satisfy the constraint

$$\begin{bmatrix} z \\ \varphi(t,z) \end{bmatrix}^\top \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^\top & \hat{R} \end{bmatrix} \begin{bmatrix} z \\ \varphi(t,z) \end{bmatrix} \ge 0, \tag{49}$$

which holds for all the pairs $(t, z) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^p$ with $z \in \operatorname{im} H$, where $\hat{Q} \in \mathbb{S}^{p\times p}$, $\hat{S} \in \mathbb{R}^{p\times q}$ and $\hat{R} \prec 0 \in \mathbb{S}^{q\times q}$ are known matrices. Since $\hat{R} \prec 0$ then $z = 0$ implies that $\varphi(t, z) = 0$ for all $t \ge 0$. In the analysis to follow we will make use of the matrices Q, S, R defined as

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} := \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^\top & \hat{R} \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}. \tag{50}$$

For any (x, v), where $v = \varphi(t, Hx)$, it holds that

$$\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \ge 0. \tag{51}$$

Thanks to the specific form of system (48) and the constraint (49), one can aim at asymptotic stabilization taking into account the nonlinearity rather than minimizing it, namely one can aim at uniform global asymptotic stabilization of (48), or, to adopt a term consistent with the literature, at the absolute stabilization of (48) (*cf.* Khalil, 2001, Definition 10.2).

Definition 2. System (48) with $d = 0$ is absolutely stabilizable via linear state-feedback $u = Kx$ if there exists a matrix K such that $\overline{x} = 0$ is a globally uniformly asymptotically stable equilibrium for the closed-loop system

$$\begin{aligned} x^+ &= (A+BK)x + Lv \\ z &= Hx \\ v &= \varphi(t,z) \end{aligned} \tag{52}$$

for any function φ that satisfies the inequality (49). ■

In the definition above we are assuming that d, which is present during the data acquisition phase, is absent during the execution of the control task. We will remark later about the case when $Ed \neq 0$ appears in (52).

As in the previous sections, to carry out the analysis we rely on a data-based representation of the closed-loop system. In Section 4, it was considered the case in which the vector of nonlinearities $Z(x)$ was known, which allows the designer to compute the matrix Z_0 in (25). The nonlinearity $\varphi(t,z)$ considered here satisfies (49), but its precise analytic expression may not be known. What we assume instead is that samples of the nonlinearity can be measured during the experiment, namely that the data matrix

$$V_0 := \begin{bmatrix} v(0) & v(1) & \cdots & v(T-1) \end{bmatrix} \in \mathbb{R}^{q\times T} \tag{53}$$

is available to the designer in addition to the matrices U_0, X_0, X_1 given in (4). Hence, in this section the dataset (2) is extended as

$$\mathbb{D} := \{x(k), u(k), v(k),\ k = 0,1,\dots,T\} \tag{54}$$

where by an abuse of notation we continue to use the same symbol $\mathbb{D}$ as in (2). Then, similarly to Lemma 2, we have the following:

Lemma 5. *Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{T\times n}$ such that (5) holds. System (48) with $u = Kx$ results in the closed-loop dynamics*

$$\begin{aligned} x^+ &= (X_L - ED_0)Gx + Lv + Ed \\ z &= Hx \\ v &= \varphi(t,z) \end{aligned} \tag{55}$$

where

$$X_L := X_1 - LV_0. \tag{56}$$

Having obtained the data-based representation (55), we can consider the absolute stabilization problem for system (48).

Theorem 6. *Consider system (48) along with the dataset $\mathbb{D}$ in (54). Let $\mathcal{D} := \{D \in \mathbb{R}^{s\times T} : DD^\top \preceq \Delta\Delta^\top\}$ where the matrix $\Delta \in \mathbb{R}^{s\times q}$ is chosen by the designer, and suppose $D_0 \in \mathcal{D}$. Suppose that there exist two matrices $Y \in \mathbb{R}^{T\times n}$, $P \in \mathbb{R}^{T\times n}$ and two scalars $\epsilon, \tau > 0$ such that (7b) holds and*

1. ($Q \succeq 0$)

$$\begin{bmatrix} P-\tau\Omega & PS & (X_LY)^\top & PQ^{1/2} & Y^\top \\ S^\top P & -R & L^\top & 0_{q\times n} & 0_{q\times T} \\ X_LY & L & P-\epsilon E\Delta\Delta^\top E^\top & 0_{n\times n} & 0_{n\times T} \\ Q^{1/2}P & 0_{n\times q} & 0_{n\times n} & I_n & 0_{n\times T} \\ Y & 0_{T\times q} & 0_{T\times n} & 0_{T\times n} & \epsilon I_T \end{bmatrix} \succ 0 \tag{57}$$

or

2. ($Q = 0$ or $Q \preceq 0$)

$$\begin{bmatrix} P-\tau\Omega & PS & (X_LY)^\top & Y^\top \\ S^\top P & -R & L^\top & 0_{q\times T} \\ X_LY & L & P-\epsilon E\Delta\Delta^\top E^\top & 0_{n\times T} \\ Y & 0_{T\times q} & 0_{T\times n} & \epsilon I_T \end{bmatrix} \succ 0 \tag{58}$$

with $\Omega \succ 0$ chosen by the designer and X_L defined in (56). Then, the controller $K = U_0YP^{-1}$ renders $\overline{x} = 0$ a globally uniformly asymptotically stable equilibrium of the closed-loop system (52) for any function φ that satisfies the inequality (49).

Proof. 1. Bearing in mind that $Q \succeq 0$, by the Schur complement, condition (57) can be rewritten as

$$\begin{bmatrix} P-\tau\Omega & PS & (X_LY)^\top & PQ^{1/2} \\ \star & -R & L^\top & 0_{q\times n} \\ \star & \star & P & 0_{n\times n} \\ \star & \star & \star & I_n \end{bmatrix} - \epsilon^{-1}\begin{bmatrix} Y^\top \\ 0_{q\times T} \\ 0_{n\times T} \\ 0_{n\times T} \end{bmatrix}\begin{bmatrix} Y & 0_{T\times q} & 0_{T\times n} & 0_{T\times n} \end{bmatrix} - \epsilon\begin{bmatrix} 0_{n\times s} \\ 0_{q\times s} \\ E \\ 0_{n\times s} \end{bmatrix}\Delta\Delta^\top\begin{bmatrix} 0_{s\times n} & 0_{s\times q} & E^\top & 0_{s\times n} \end{bmatrix} \succ 0$$

which, by Lemma 1, is equivalent to

$$\begin{bmatrix} -P+\tau\Omega & PS & -[(X_L-ED)Y]^\top & -PQ^{1/2} \\ \star & R & L^\top & 0_{q\times n} \\ \star & \star & -P & 0_{n\times n} \\ \star & \star & \star & -I_n \end{bmatrix} \prec 0$$

for all $D \in \mathcal{D}$. By left- and right-multiplying by $\mathrm{diag}(P^{-1}, I_q, I_n, I_n)$, performing the change of variable $G := YP^{-1}$ and applying again the Schur complement, the previous inequality is rewritten as

$$\begin{bmatrix} -P^{-1}+\tau\Phi+Q & S & -[(X_L-ED)G]^\top \\ \star & R & L^\top \\ \star & \star & -P \end{bmatrix} \prec 0,$$

where we recall that $\Phi = P^{-1}\Omega P^{-1}$. One more application of the Schur complement yields

$$\begin{bmatrix} \Psi_L(D)^\top P^{-1}\Psi_L(D) - P^{-1} + \tau\Phi & \Psi_L(D)^\top P^{-1}L \\ \star & L^\top P^{-1}L \end{bmatrix} + \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \prec 0 \quad \forall D \in \mathcal{D}$$
$$P \succ 0$$

where $\Psi_L(D) := (X_L - ED)G$. Multiplying by τ^{-1} and setting $\hat{P} := \tau P$, $\hat{\tau} := \tau^{-1}$, the inequality becomes

$$\begin{bmatrix} \Psi_L(D)^\top \hat{P}^{-1}\Psi_L(D) - P^{-1} + \Phi & \Psi_L(D)^\top \hat{P}^{-1}L \\ \star & L^\top \hat{P}^{-1}L \end{bmatrix} + \hat{\tau}\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \prec 0 \quad \forall D \in \mathcal{D}$$
$$\hat{P} \succ 0.$$

As $D_0 \in \mathcal{D}$, the inequality holds in particular for $D = D_0$ and it implies that, for all (x, v),

$$\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} \Psi_L^\top \hat{P}^{-1}\Psi_L - \hat{P}^{-1} + \Phi & \Psi_L^\top \hat{P}^{-1}L \\ \star & L^\top \hat{P}^{-1}L \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \hat{\tau}\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \leq 0$$

where, for the sake of simplicity, we denote $(X_L - ED_0)G$ by Ψ_L instead of $\Psi_L(D_0)$. For $v = \varphi(t,z)$, with $z = Hx$,

$$\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \geq 0 \tag{59}$$

in view of (49) and (50). Condition (7b) and $K = U_0YP^{-1}$ show that $\Psi_L = A + BK$ by Lemma 5. Choose $V(x) = x^\top \hat{P}^{-1}x$. Then, along the trajectories of (52), we have

$$\begin{aligned}&V(x^+) - V(x)\\ &= (\Psi_L x + L\varphi(t,Hx))^\top \hat P^{-1}(\Psi_L x + L\varphi(t,Hx)) - x^\top \hat P^{-1} x\\ &= \begin{bmatrix} x\\ v\end{bmatrix}^\top \begin{bmatrix} \Psi_L^\top \hat P^{-1}\Psi_L - \hat P^{-1} & \Psi_L^\top \hat P^{-1} L\\ L^\top \hat P^{-1}\Psi_L & L^\top \hat P^{-1} L\end{bmatrix}\begin{bmatrix} x\\ v\end{bmatrix} \le -x^\top \Phi x\end{aligned}$$

for all (x,v), which shows the thesis. The proof of 2. is similar and is omitted. ■

Compared to Theorem 5, the last result allows us to obtain a uniform global asymptotic stabilization result versus a local one, provided that the nonlinearity $v=\varphi(t,z)$ satisfies (49) and that the samples (53) are measured during the experiment. For other results on the data-driven absolute stabilization problem we refer the reader to Luppi, De Persis, and Tesi (2022).

Remark 1. Conditions (57) and (58) are indebted to the S-procedure. In fact, the proof underscores that the global uniform stabilization of $x=0$ for system (52) with φ any function that satisfies the inequality (49) is tackled via the condition

$$\begin{aligned}&V(x^+) - V(x) + x^\top \Phi x =\\ &\begin{bmatrix} x\\ v\end{bmatrix}^\top \begin{bmatrix} \Psi_L^\top \hat P^{-1}\Psi_L - \hat P^{-1} + \Phi & \Psi_L^\top \hat P^{-1} L\\ \star & L^\top \hat P^{-1} L\end{bmatrix}\begin{bmatrix} x\\ v\end{bmatrix} \le 0\end{aligned}$$

which is required to hold for all (x,v), that satisfy (59), with $v=\varphi(t,Hx)$. By the lossless S-procedure (Yakubovich et al., 2004, pp. 67–68), under the condition that the constraint (49) is regular, which implies that the constraint (59) is also regular, the condition is equivalent to the existence of $\hat\tau \ge 0$ such that

$$\begin{bmatrix} \Psi_L^\top \hat P^{-1}\Psi_L - \hat P^{-1} + \Phi & \Psi_L^\top \hat P^{-1} L\\ \star & L^\top \hat P^{-1} L\end{bmatrix} + \hat\tau \begin{bmatrix} Q & S\\ S^\top & R\end{bmatrix} \preceq 0.$$

If $\hat\tau = 0$, then $L^\top \hat P^{-1} L \preceq 0$, which is a contradiction, since $\hat P \succ 0$ and $L\neq 0$. Hence, $\hat\tau \neq 0$ without loss of generality, and the condition above is equivalent to

$$\tau>0, \begin{bmatrix} \Psi_L^\top P^{-1}\Psi_L - P^{-1} + \tau\Phi + Q & \Psi_L^\top P^{-1} L + S\\ \star & L^\top P^{-1} L + R\end{bmatrix} \preceq 0$$

where $P=\tau\hat P \succ 0$ and $\tau = \hat\tau^{-1}$. As D_0 in Ψ_L is unknown, instead of the above, one can consider the condition

$$\begin{aligned}&\tau>0\\ &\begin{bmatrix} \Psi_L(D)^\top P^{-1}\Psi_L(D) - P^{-1} + \tau\hat\Phi & \Psi_L(D)^\top P^{-1} L\\ \star & L^\top P^{-1} L\end{bmatrix}\\ &\quad + \begin{bmatrix} Q & S\\ S^\top & R\end{bmatrix} \preceq 0 \quad D\in\mathcal{D}.\end{aligned}$$

Retracing the same steps of the proof of the previous result and applying a nonstrict version of Lemma 1 (Bisoffi, De Persis, & Tesi, 2022, Fact 2), one arrives at nonstrict versions of conditions (57) and (58).

If the disturbance term Ed is present during the execution of the control task, namely, the closed-loop system is

$$\begin{aligned} x^+ &= (A+BK)x + Lv + Ed\\ z &= Hx\\ v &= \varphi(t,z),\end{aligned} \tag{60}$$

then the evolution of $V(x) = x^\top \hat P^{-1} x$, where $\hat P = \tau P$, along the solutions of (60) satisfies

$$\begin{aligned}V(x^+) - V(x) \le -x^\top \Phi x + [2(\Psi_L x + L\varphi(t,Hx))\\ +Ed]^\top \hat P^{-1} Ed.\end{aligned}$$

Before proceeding further, we observe that, although the analytic expression of φ may not be known to the designer, we do have a bound $\varphi(t,Hx)^\top\varphi(t,Hx) \le c x^\top H^\top \hat S \hat S^\top H x$, with $c>0$ known, for all $(t,x)\in \mathbb{R}_{\ge 0}\times\mathbb{R}^n$, thanks to (49).[3] We return to consider the Lyapunov difference above. As in (14) and (15), we have

$$V(x^+) - V(x) \le -x^\top \Phi x + q(x,\nu), \tag{61}$$

where $\nu>0$ is the known bound on d, i.e. $|d(t)|\le\nu$ for all $t\ge 0$, and

$$q(\zeta,\eta) := r_1(\zeta)|\eta| + r_2(\zeta)|\eta| + r_3(\zeta)|\eta| + r_4|\eta|^2, \tag{62a}$$

$$r_1(\zeta) := 2|(X_L G\zeta)^\top \hat P^{-1} E|, \tag{62b}$$

$$r_2(\zeta) := 2\|\Delta\|\|E^\top \hat P^{-1} E\||G\zeta|, \tag{62c}$$

$$r_3(\zeta) := 2\|L^\top \hat P^{-1} E\|\sqrt{c}|\hat S^\top H\zeta|, \tag{62d}$$

$$r_4 := \|E^\top \hat P^{-1} E\|. \tag{62e}$$

Then, the analogue of Corollary 5 can be given.

Corollary 6. *Consider the same setting as in Theorem 6, and let ν be a known bound on $|d|$. Let $V(x) = x^\top \hat P^{-1} x$, where $\hat P^{-1} = \tau^{-1}P^{-1}$, and define $\mathcal{R}_\gamma := \{x : V(x)\le\gamma\}$, with $\gamma>0$ arbitrary. Finally, let $\mathcal{X} := \{x : -x^\top\Phi x + q(x,\nu) < 0\}$ where $q(\zeta,\eta)$ is defined in (62), and $\mathcal{Z} := \mathcal{R}_\gamma\cap\mathcal{X}^c$ where $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$ ($\mathcal{Z}$ is the subset of $\mathcal{R}_\gamma$ for which the Lyapunov difference can be positive). If*

$$V(x) - x^\top\Phi x + q(x,\nu) \le \gamma \quad \forall x\in\mathcal{Z} \tag{63}$$

then $\mathcal{R}_\gamma$ is an RPI set for the closed-loop system (60) for any function φ that satisfies the inequality (49).

As a final remark, the technique adopted in this section can be useful for the design of a stabilizer for systems of the form (21) even when the vector field $f(x)$ is not described by $Ax+\varphi(t,Hx)$. Similar to Section 3.1, we can expand $f(x)$ via Taylor's series and write system (21) as

$$x^+ = Ax + Bu + Lr(x) + Ed$$

where $r(x)$ is the remainder and the matrix L specifies which components of f are actually nonlinear. The (coarse) knowledge of a Lipschitz bound on the gradient of the components of f valid on a convex set $\mathcal{Q}\subseteq\mathbb{R}^n$ makes sure that $r(x)^\top r(x)\le c x^\top x$ for $x\in\mathcal{Q}$ and for some known constant c. Then condition (59), where $Q=cI_n$, $S=0_{n\times n}$ and $R=-I_n$, holds for $v=r(x)$, $z=x$ and $x\in\mathcal{Q}$ and the results of this section are applicable. Theorem 6 will return a controller that makes the origin a locally asymptotically stable equilibrium and for which any Lyapunov sublevel set contained in $\mathcal{Q}$ is an estimate of the RoA. If the conditions of Corollary 6 are satisfied for a sublevel set $\mathcal{R}_\gamma$ contained in $\mathcal{Q}$, then $\mathcal{R}_\gamma$ is an RPI for the closed-loop system. Compared with the results in Section 3.1, here the remainder is explicitly taken into account in the controller design, at the price of more complex conditions.

5.2. Bilinear systems

Another important class of nonlinear systems for which the previous methods can be tailored to account for the nonlinearity is the class of bilinear systems

$$x^+ = Ax + Bu + Lxu + Ed. \tag{64}$$

The importance of these systems is manifold. In the continuous time case, they are universal approximators of general nonlinear systems and they have been extensively considered for the control of nonlinear systems via Carleman linearization. More recently, they have been studied in connection with Koopman operator theory (Fu & You, 2022; Goswami & Paley, 2021). In our setting, they are also relevant from

[3] By (49) $\varphi(t,z)^\top(-\hat R)\varphi(t,z) \le z^\top \hat Q z + z^\top \hat S\varphi(t,z) + \varphi(t,z)^\top \hat S^\top z$. For any $\mu>0$, $z^\top\hat S\varphi(t,z) + \varphi(t,z)^\top\hat S^\top z \le 2\mu\varphi(t,z)^\top\varphi(t,z) + 2\mu^{-1}z^\top\hat S\hat S^\top z$. Set $2\mu < \lambda_{\min}(-\hat R)$. Then, $\varphi(t,z)^\top\varphi(t,z) \le \frac{2\mu^{-1}}{\lambda_{\min}(-\hat R)-2\mu} z^\top\hat S\hat S^\top z$, which implies $\varphi(t,Hx)^\top\varphi(t,Hx) \le \frac{2\mu^{-1}}{\lambda_{\min}(-\hat R)-2\mu} x^\top H^\top \hat S\hat S^\top Hx$.

a methodological point of view because the state dependence of their input vector fields can be dealt with by an ingenious application of Petersen's lemma (Khlebnikov, 2016).

For the sake of simplicity and brevity, we examine here the case in which $u \in \mathbb{R}$ is a scalar, $d = 0$ and $L \in \mathbb{R}^{n\times n}$ is known. All these simplifying assumptions can be relaxed and we refer the interested reader to Bisoffi, De Persis, and Tesi (2020).

We collect the dataset (2) and define the matrices U_0, X_0, X_1 as in (4a)–(4c). We compute the samples

$$v(k) := x(k)u(k), \quad k = 0, 1, \dots, T-1$$

and assemble the matrix

$$V_0 := \begin{bmatrix} v(0) & v(1) & \cdots & v(T-1) \end{bmatrix} \in \mathbb{R}^{q\times T}. \tag{65}$$

The samples $v(k)$ and the matrix V_0 introduced here should not be confused with the homonymous symbols used in Section 5.1.

The following result is easily understood from the analogous results of the previous sections and its proof is omitted.

Lemma 6. *Consider any matrices $K \in \mathbb{R}^{1\times n}$, $G \in \mathbb{R}^{T\times n}$ such that (5) holds. System (64) with $d = 0$ and $u = Kx$ results in the closed-loop dynamics*

$$x^+ = (X_L + LxU_0)Gx \tag{66}$$

where

$$X_L := X_1 - LV_0. \tag{67}$$

Note that the bilinearity of the system is preserved due to the presence of x between L and G in the term $LxGx$. As in the previous sections, a Lyapunov based design of the controller is carried out. In the Lyapunov difference computed along the solutions of (66) the term LxU_0Gx gives rise to cubic terms that we tackle by applying Lemma 1 on a bounded set. This approach has the additional value of providing an explicit estimate of the RoA of the closed-loop system at the design stage, without any further analysis.

Theorem 7. *Consider system (64) with $d = 0$, along with the dataset $\mathbb{D}$ in (2). Suppose that there exist two matrices $Y \in \mathbb{R}^{T\times n}$, $P \in \mathbb{R}^{T\times n}$ and a scalar $\epsilon > 0$ such that (7b) and*

$$\begin{bmatrix} P & 0_{n\times n} & (U_0Y)^\top & (X_LY)^\top \\ \star & \epsilon^{-1}P & 0_{n\times 1} & \epsilon^{-1}PL^\top \\ \star & \star & \epsilon^{-1} & 0_{1\times n} \\ \star & \star & \star & P \end{bmatrix} \succ 0 \tag{68}$$

with X_L defined in (67). Then, the controller $K = U_0YP^{-1}$ renders $\overline{x} = 0$ an asymptotically stable equilibrium of the closed-loop system $x^+ = (A + BK)x + LxKx$ and its RoA contains the set $\mathcal{E}_P := \{x \in \mathbb{R}^n : x^\top P^{-1}x \le 1\}$.

Proof. We left- and right-multiply (68) by $\mathrm{diag}(P^{-1}, \epsilon P^{-1}, 1, I_n)$, set $G := YP^{-1}$ and obtain

$$\begin{bmatrix} P^{-1} & 0_{n\times n} & (U_0G)^\top & (X_LG)^\top \\ \star & \epsilon P^{-1} & 0_{n\times 1} & L^\top \\ \star & \star & \epsilon^{-1} & 0_{1\times n} \\ \star & \star & \star & P \end{bmatrix} \succ 0.$$

Applying the Schur complement twice, this is equivalent to

$$\begin{bmatrix} P^{-1} - \Psi^\top P^{-1}\Psi & \Psi^\top P^{-1}L & (U_0G)^\top & 0_{n\times 1} \\ \star & \epsilon P^{-1} & 0_{n\times 1} & L^\top P^{-1} \\ \star & \star & \epsilon^{-1} & 0 \\ \star & \star & \star & P^{-1} \end{bmatrix} \succ 0$$

where $\Psi := X_LG$. By rearranging rows and columns, we arrive at

$$\begin{bmatrix} P^{-1} - \Psi^\top P^{-1}\Psi & 0_{n\times 1} & \Psi^\top P^{-1}L & (U_0G)^\top \\ \star & P^{-1} & P^{-1}L & 0_{n\times 1} \\ \star & \star & \epsilon P^{-1} & 0_{n\times 1} \\ \star & \star & \star & \epsilon^{-1} \end{bmatrix} \succ 0.$$

Another application of the Schur complement returns the equivalent condition

$$\begin{aligned} &\begin{bmatrix} \Psi^\top P^{-1}\Psi - P^{-1} & 0_{n\times 1} \\ \star & -P^{-1} \end{bmatrix} + \epsilon^{-1} \begin{bmatrix} \Psi^\top P^{-1}LP^{\frac{1}{2}} \\ P^{-1}LP^{\frac{1}{2}} \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} \Psi^\top P^{-1}LP^{\frac{1}{2}} \\ P^{-1}LP^{\frac{1}{2}} \end{bmatrix}^\top + \epsilon \begin{bmatrix} (U_0G)^\top \\ 0_{n\times 1} \end{bmatrix} \begin{bmatrix} (U_0G)^\top \\ 0_{n\times 1} \end{bmatrix}^\top \prec 0 \\ &\qquad\qquad \epsilon > 0,\ P \succ 0. \end{aligned}$$

Note that for each $x \in \mathcal{E}_P$, $|P^{-\frac{1}{2}}x| \le 1$. By Lemma 1, the previous condition is equivalent to

$$\begin{aligned} &\begin{bmatrix} \Psi^\top P^{-1}\Psi - P^{-1} & 0_{n\times 1} \\ \star & -P^{-1} \end{bmatrix} + \begin{bmatrix} (U_0G)^\top \\ 0_{n\times 1} \end{bmatrix} x^\top P^{-\frac{1}{2}} \cdot \\ &\cdot \begin{bmatrix} -P^{\frac{1}{2}}L^\top P^{-1}\Psi & P^{\frac{1}{2}}L^\top P^{-1} \end{bmatrix} + [\star]^\top \prec 0\ \forall x \in \mathcal{E}_P \end{aligned}$$

where

$$[\star]^\top = \begin{bmatrix} -P^{\frac{1}{2}}L^\top P^{-1}\Psi & P^{\frac{1}{2}}L^\top P^{-1} \end{bmatrix}^\top P^{-\frac{1}{2}}x \begin{bmatrix} (U_0G)^\top \\ 0_{n\times 1} \end{bmatrix}^\top$$

which is compactly written as

$$\begin{bmatrix} \mathcal{P} & (U_0G)^\top(Lx)^\top P^{-1} \\ \star & -P^{-1} \end{bmatrix} \prec 0\ \forall x \in \mathcal{E}_P$$

where

$$\begin{aligned} \mathcal{P} := &\ \Psi^\top P^{-1}\Psi - P^{-1} + (U_0G)^\top(Lx)^\top P^{-1}\Psi \\ &+ \Psi^\top P^{-1}LxU_0G \end{aligned}$$

or, by the Schur complement, equivalently as

$$P \succ 0,\ \mathcal{P} + (U_0G)^\top(Lx)^\top P^{-1}(Lx)(U_0G) \prec 0\ \forall x \in \mathcal{E}_P$$

Condition (7b) and the choice $K = U_0G$ allows us to apply Lemma 6. Hence, the closed-loop dynamics $x^+ = (A + BK)x + LxKx$ results in the system $x^+ = \Psi x + LxU_0Gx$, where we recall that we have set $\Psi = X_LG$. Let $V(x) := x^\top P^{-1}x$ and compute the Lyapunov difference along the solutions of the closed-loop dynamics $x^+ = \Psi x + LxU_0Gx$ to obtain

$$\begin{aligned} &\ V(x^+) - V(x) \\ = &\ x^\top \left(\mathcal{P} + (U_0G)^\top(Lx)^\top P^{-1}(Lx)(U_0G)\right) x^{\cdot} \end{aligned}$$

Hence, we conclude that $V(x^+) - V(x) = 0$ for $x = 0$ and $V(x^+) - V(x) < 0$ for all $x \in \mathcal{E}_P \setminus \{0\}$, which proves the thesis. ∎

The result can be strengthened in a few aspects. One is to slightly modify condition (68) to guarantee exponential stability rather than asymptotic stability as done in the previous sections. Another one is to maximize the volume of the ellipsoid $\mathcal{E}_P$ by minimizing the cost function $-\log\det(P)$ subject to constraints (7b), (68). See Bisoffi et al. (2020, Corollary 1) for details.

6. Discussion

6.1. Features of the methods and extensions

Our framework features some nice properties. The first and most evident is the generality of applicability: it can be applied to linear and nonlinear systems, including the case where the vector field has a generic structure. At the same time, our framework is flexible enough to be tailored whenever the system of interest belong to some special class, as detailed in Section 5. Another interesting feature of our framework regards the design complexity, meant as the number of samples needed for controller design as well as the type of design programs. Regarding the first aspect, all the results we presented rest on the fulfilment of certain rank conditions (*cf.* Lemma 2) on the data matrices that can be fulfilled even with a small number of samples, highlighting that the quality of data generally counts more than the quantity. The second aspect refers to the fact that all the design programs we discussed amount to convex programming, which can be easily solved for datasets of moderate size.

To our best knowledge, De Persis and Tesi (2019) and its extended version (De Persis & Tesi, 2020) were the first works to exploit the Fundamental Lemma to explicitly derive data-dependent controller design programs. There are now numerous other studies, also from other research groups, that have produced comparable results. An interesting class of systems that admit a similar treatment are polynomial and rational systems (Ahmadi, Chaudhry, Sindhwani, & Tu, 2023; Guo, De Persis, & Tesi, 2020; Guo et al., 2022a; Luppi, Bisoffi, De Persis, & Tesi, 2023; Nejati, Zhong, Caccamo, & Zamani, 2022; Strässer, Berberich, & Allgöwer, 2021). These classes of systems can actually be treated using specialized design tools such as *sum-of-squares* (SOS) programming, which can reduce conservativeness. Close to the analysis of Section 5.1 is van Waarde and Camlibel (2021), which considers data-dependent SDP for the stabilization of Lur'e systems. Still in the domain of SDP are Guo et al. (2022b) and Martin, Schön, and Allgöwer (2022) that combine Taylor approximation methods with SOS programming. Regarding approximation methods and in connection with the discussion after Corollary 6, we also mention (Cheah, Bhattacharjee, Hemati, & Caverly, 2022), which considers robust state-feedback design for systems in Taylor's form with unknown nonlinearities but known A and B, and the technical tool involved is Yakubovich's S-lemma. In this regard, it is also worth mentioning (van Waarde, Camlibel, & Mesbahi, 2022), which introduces a matrix version of the S-lemma with application to data-driven control. In the category of convex programming, we finally mention the recent works (Dai & Sznaier, 2023; Nortmann & Mylvaganam, 2023; Rotulo, De Persis, & Tesi, 2022; Verhoek, Tòth, & Abbas, 2023) which tackle time-varying systems, and Cetinkaya and Kishida (2021) which proposes a method for stabilizing periodic orbits. In this paper, we decided to present our main results for discrete-time systems. Some of the aforementioned works have been developed for continuous-time systems. While continuous-time systems have their own peculiarities, many of the results presented here can indeed be rephrased for that, see for instance (De Persis et al., 2023).

Taking up the question of sample-complexity, it is worth remarking that the analysis pursued here does not rely on any statistical assumptions about the noise signals. When the noise signals have suitable statistics it is possible to improve the results in the traditional sense of sample complexity. This means that collecting more data can show improved stability and performance properties, as recently popularized in Dean et al. (2020) and Recht (2019). In the present context, this analysis has been tackled in De Persis et al. (2023) which shows that averaging datasets from multiple experiments permits to filter out noise for certain statistical models like Gaussian and uniformly distributed models. Specifically, the analysis shows that the norm of the noise matrix D_0, which is a key component of the design programs, decreases as the number N of experiments increases. This implies that a stabilizing controller is eventually found by taking N sufficiently large. The results are given in probability, which follows by utilizing statistical concentration bounds on the norm of the matrix D_0 (De Persis et al., 2023, Section V-C).

A final extension that is worth mentioning regards the case of measurement noise, namely the case in which we measure $y = x + w$ instead of x where w is a noise signal. For linear systems this problem has been addressed in De Persis and Tesi (2020, Section V-A), where we show that the case of measurement noise can be related to that of process noise, as we consider in this paper. It is actually simple to see that the extension of De Persis and Tesi (2020, Section V-A) to nonlinear systems does not pose any difficulties, at least in the case of linearly parametrized models studied in Section 4, thus whenever we know the "type" of dynamics we are dealing with (*e.g.*, polynomial dynamics). We omit the technical details.

6.2. Connections with other methods

Data-driven control is currently one of most popular topics in control theory and showcases many interesting methods, some of which are substantially different from what has been discussed so far. Among this variety of methods, we mention the approaches based on dual Lyapunov theory (Dai & Sznaier, 2020), adaptive control (Astolfi, 2020), and Koopman linearization (Fu & You, 2022; Kaiser, Kutz, & Brunton, 2021; Lian, Wang, & Jones, 2021). Iterative methods like adaptive control have the intrinsic capability to handle time-varying dynamics and environments, but they typically provide only *asymptotic* results. A boost in this direction may come indeed from recent results on statistical learning methods, as we were mentioning. This is essentially the idea of most of recent iterative methods, usually presented under the heading of *reinforcement learning* (Fazel, Ge, Kakade, & Mesbahi, 2018; Recht, 2019). Koopman theory is interesting especially because it returns a linear approximation of the system, enabling us to use all the tools for controller design that are available for linear systems. However, this theory is not yet systematically developed, and the main difficulty is related to give theoretical guarantees on the quality of the approximation model (Kaiser et al., 2021).

Another interesting research line regards kernel-based methods (Hu, Persis, & Tesi, 2023; Lederer, Umlauft, & Hirche, 2019; Umlauft, Pöhler, & Hirche, 2018; von Rohr, Neumann-Brosig, & Trimpe, 2021). As with Koopman linearization, the idea is to determine a model of the system and then apply some controller design technique. The main difference is that the system model is typical nonlinear. Model error bounds, either probabilistic or deterministic, can be determined (Lederer et al., 2019; Maddalena, Scharnhorst, & Jones, 2021). The challenge is related to the nonlinearities of the system model that are determined by the choice of the kernel functions (typical choices are Gaussian and polynomial kernels). While certain kernels may systematically lead to accurate models, such models need not be "optimal" for controller design in the sense that there might be no clear way to exploit the nonlinearities when designing the controller and all one can do is to try to cancel out such nonlinearities with feedback. The interest for the so-called *control-oriented* identification is motivated precisely by issues of this kind (Formentin & Chiuso, 2021).

The discussion made in the previous paragraph suggests other two observations. The first one is related to the debate regarding indirect (sequential system identification and model-based controller design) versus direct data-driven control. This question seems overemphasized, though. As an example, the Fundamental Lemma can be viewed as a system identification result, but offers an alternative representation to parametric modelling that is suitable for *direct* controller design, as we showed in this paper. In some contexts, differences between indirect and direct control have been however observed. We refer the reader to Dörfler, Coulson and Markovsky (2023) and Krishnan and Pasqualetti (2021) for works addressing this point.

The second observation regards controller design at large. Nonlinear controller design is a difficult topic, even in a model-based context. Taylor and Koopman linearization have received considerable attention because they can be combined with simple and computationally efficient controller design tools. Different from Taylor and Koopman linearization, feedback linearization has produces less results. In particular, existing results either lack stability guarantees (Gadginmath, Krishnan, & Pasqualetti, 2022; Westenbroek et al., 2020), or assume the knowledge of the state coordinates that render the system linearizable (Alsalti, Lopez, Berberich, Allgöwer and Müller, 2023; De Persis et al., 2023; Shenoy, Saradagi, Pasumarthy, & Chellaboina, 2023; Tabuada, Ma, Grizzle, & Ames, 2017; Umlauft & Hirche, 2020), which limits the range of applicability. Needless to say, advances in this direction would provide major contributions for data-driven control, including a better understanding of related techniques like *immersion*, Carleman or Koopman linearization.

Obvious, but anyway important to remark is that this discussion has been restricted to pure stabilization problems. There is a vast and fruitful body of work on data-driven control in other important areas like optimal (Baggio, Katewa, & Pasqualetti, 2019; Dörfler, De Persis and Tesi, 2023; Yuan & Cortés, 2022), predictive (Berberich, Köhler, Müller, & Allgöwer, 2022; Coulson, Lygeros, & Dörfler, 2019, 2022; Hewing, Wabersich, Menner, & Zeilinger, 2020), and networked control (Allibhoy & Cortés, 2020; Baggio, Bassett, & Pasqualetti, 2020). Regarding predictive control and the general domain of *safety*, the results of this paper can be used to obtain an initial safe policy (with a safe RoA) that can be used in iterative learning methods (Hewing et al., 2020).

7. Concluding remarks

Data-driven control is one of the most important topics in modern control theory. We have discussed a line of work that combines ideas from nonparametric modelling and convex optimization. While fruitful, this line of work should be expanded in many directions. A main challenge common to all data-driven control methods is how to deal with input–output data, rather than input-state data. This problem has been studied also in the context of nonlinear systems (Campi & Savaresi, 2006; Fagiano & Novara, 2016; Safonov & Tsao, 1997) but there remain certain difficulties with regard to providing theoretical guarantees, especially finite-sample guarantees, difficulties shared by traditional adaptive control techniques. Assuming that the nonlinearities can be expressed via a dictionary of known functions and under suitable observability properties, an extension of the ideas presented in the paper to input–output data has been recently discussed in Dai, De Persis, Monshizadeh, and Tesi (2023).

On a more general perspective, we believe that the interest towards data-driven control can be an opportunity to rethink nonlinear controller design in the most broad sense, including model-based approaches.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix. Methods for data processing

After Theorem 1 we have briefly remarked on the set $\mathcal{D}$ to which the matrix D_0 belongs. In the case of deterministic disturbances with known instantaneous bound δ, the matrix Δ appearing in the condition $DD^\top \preceq \Delta\Delta^\top$ that defines $\mathcal{D}$ is chosen as $\Delta = \delta\sqrt{T} I_n$. Borrowing ideas from Bisoffi et al. (2021), a different choice of Δ, which is independent of T and has some advantages that are highlighted later on, is possible. For the sake of generality, we present the discussion for the class of nonlinear systems with linearly parametrized basis functions studied in Section 4.

The starting point is a set of matrices of "synthetic" data $\overline{U}_0$, $\overline{X}_0$, $\overline{X}_1$, $\overline{Z}_0$, $\overline{\Delta}$, obtained from an ellipsoidal overapproximation of the set $\mathcal{I}$ of systems' matrices that are consistent with the data measured from system (22). To be precise, let us define the set $\mathcal{I}$ as

$$\mathcal{I} := \bigcap_{i=0}^{T-1} \left\{ (\hat{A}, \hat{B}) : x(i+1) = \hat{A} Z(i) + \hat{B} u(i) + d, |d| \le \delta \right\}$$

where $Z(i) := Z(x(i))$ as in (25). Following Bisoffi et al. (2021), an ellipsoidal over-approximation $\overline{\mathcal{I}}$ of $\mathcal{I}$ is defined as the matrix ellipsoid

$$\overline{\mathcal{I}} = \left\{ (\hat{A}, \hat{B}) : \begin{bmatrix} I & | & \hat{B} & \hat{A} \end{bmatrix} \left[\begin{array}{c|c} \overline{\mathbf{C}} & \overline{\mathbf{B}}^\top \\ \hline \overline{\mathbf{B}} & \overline{\mathbf{A}} \end{array} \right] \left[\begin{array}{c} I \\ \hline \hat{B} \\ \hat{A} \end{array} \right] \le 0 \right\} \tag{A.1}$$

where the matrices $\overline{\mathbf{A}} \in \mathbb{R}^{(m+S)\times(m+S)}$, $\overline{\mathbf{B}} \in \mathbb{R}^{(m+S)\times n}$, $\overline{\mathbf{C}} \in \mathbb{R}^{n\times n}$ are defined as the solution of the optimization problem

$$\text{min.} \qquad -\log\det \overline{\mathbf{A}} \quad (\text{over } \overline{\mathbf{A}}, \overline{\mathbf{B}}, \tau_1, \ldots, \tau_{T-1}) \tag{A.2a}$$

$$\text{s. t.} \qquad \begin{bmatrix} -I_n - \sum_{i=0}^{T-1} \tau_i \gamma_i & \star & \star \\ \overline{\mathbf{B}} - \sum_{i=0}^{T-1} \tau_i \beta_i & \overline{\mathbf{A}} - \sum_{i=0}^{T-1} \tau_i \alpha_i & \star \\ \overline{\mathbf{B}} & 0 & -\overline{\mathbf{A}} \end{bmatrix} \le 0 \tag{A.2b}$$

$$\overline{\mathbf{A}} \succ 0, \quad \tau_i \ge 0 \text{ for } i = 0, 1, \ldots, T-1 \tag{A.2c}$$

and $\overline{\mathbf{C}} := \overline{\mathbf{B}}^\top \overline{\mathbf{A}}^{-1} \overline{\mathbf{B}} - I_n$. In the optimization problem above, for $i = 0, \ldots, T-1$, $\alpha_i, \beta_i, \gamma_i$ are the data-dependent quantities

$$\begin{aligned} \gamma_i &:= -\delta^2 I_n + x(i+1)x(i+1)^\top \in \mathbb{R}^{n\times n}, \\ \beta_i &:= -\begin{bmatrix} u(i) \\ Z(i) \end{bmatrix} x(i+1)^\top \in \mathbb{R}^{(m+S)\times n}, \\ \alpha_i &:= \begin{bmatrix} u(i) \\ Z(i) \end{bmatrix} \begin{bmatrix} u(i) \\ Z(i) \end{bmatrix}^\top \in \mathbb{R}^{(m+S)\times(m+S)}. \end{aligned} \tag{A.3}$$

Based on Bisoffi et al. (2021, Section 5), it can be shown that $\overline{\mathcal{I}}$ is bounded, it overapproximates $\mathcal{I}$, i.e. $\overline{\mathcal{I}} \supseteq \mathcal{I}$, and its size is minimized.

Next, we define the matrices of synthetic data $\overline{X}_1, \overline{Z}_0, \overline{U}_0, \overline{\Delta}$ as those matrices that satisfy the equation

$$\left[\begin{array}{c|c} \overline{\mathbf{C}} & \overline{\mathbf{B}}^\top \\ \hline \overline{\mathbf{B}} & \overline{\mathbf{A}} \end{array} \right] = \left[\begin{array}{c|c} \overline{X}_1 \overline{X}_1^\top - \overline{\Delta}\,\overline{\Delta}^\top & -\overline{X}_1 \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^\top \\ \hline -\begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix} \overline{X}_1^\top & \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix} \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^\top \end{array} \right]. \tag{A.4}$$

A solution to this equation exists and is given by

$$\begin{aligned} \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix} &= \overline{\mathbf{A}}^{1/2} \in \mathbb{R}^{(m+S)\times(m+S)} \\ \overline{X}_1^\top &= -\begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^+ \overline{\mathbf{B}} \in \mathbb{R}^{(m+S)\times n} \\ \overline{\Delta} &= I_n \end{aligned} \tag{A.5}$$

where $\begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^+$ denotes the left inverse of $\begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}$ and to obtain the last identity we used that $\overline{X}_1 \overline{X}_1^\top = \overline{\mathbf{B}}^\top \overline{\mathbf{A}}^{-1} \overline{\mathbf{B}}$ and $\overline{\mathbf{C}} = \overline{\mathbf{B}}^\top \overline{\mathbf{A}}^{-1} \overline{\mathbf{B}} - I_n$. Hence, the over-approximation $\overline{\mathcal{I}}$ in (A.1) can be explicitly written as

$$\overline{\mathcal{I}} = \Bigg\{ (\hat{A}, \hat{B}) : \begin{bmatrix} I & | & \hat{B} & \hat{A} \end{bmatrix} \left[\begin{array}{c|c} \overline{X}_1 \overline{X}_1^\top - \overline{\Delta}\,\overline{\Delta}^\top & -\overline{X}_1 \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^\top \\ \hline -\begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix} \overline{X}_1^\top & \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix} \begin{bmatrix} \overline{U}_0 \\ \overline{Z}_0 \end{bmatrix}^\top \end{array} \right] \left[\begin{array}{c} I \\ \hline \hat{B} \\ \hat{A} \end{array} \right] \le 0 \Bigg\}. \tag{A.6}$$

Following the same arguments of Bisoffi et al. (2022, Section 2.3), the set in (A.6) is equivalently rewritten as

$$\begin{aligned} \overline{\mathcal{I}} = \Big\{ (\hat{A}, \hat{B}) : \overline{X}_1 = \hat{A}\overline{Z}_0 + \hat{B}\overline{U}_0 + \overline{D}, \overline{D} \in \mathbb{R}^{n\times T}, \\ \begin{bmatrix} I & \overline{D} \end{bmatrix} \begin{bmatrix} -\overline{\Delta}\,\overline{\Delta}^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ \overline{D}^\top \end{bmatrix} \le 0 \Big\}, \\ \text{with } \overline{\Delta} = I_n. \end{aligned}$$

Since the ground truth matrices A, B belong to the over-approximation $\overline{\mathcal{I}}$, it is true that

$$\overline{X}_1 = A\overline{Z}_0 + B\overline{U}_0 + \overline{D}_0$$

for some $\overline{D}_0$ such that $\overline{D}_0\,\overline{D}_0^\top \preceq \overline{\Delta}\,\overline{\Delta}^\top = I_n$. Hence, the results discussed in the paper can be given in terms of $\overline{X}_1, \overline{Z}_0, \overline{U}_0, \overline{\Delta}$ instead of X_1, Z_0, U_0, Δ (similar matrices of processed data can be obtained for linear systems and nonlinear systems with state-dependent input matrices). This offers a few advantages. First, the bound Δ is now independent of the length of the dataset T. Second, as the matrices of synthetic data $\overline{X}_1, \overline{Z}_0, \overline{U}_0, \overline{\Delta}$ describe the set $\overline{C}$, they might positively impact the feasibility of the control design problems, as discussed in Bisoffi et al. (2021, Section 5). Finally, as the dimensions of matrices $\overline{X}_1, \overline{Z}_0, \overline{U}_0, \overline{\Delta}$ are also independent of T (in fact, the only depend on the dimension of the state, the control input and the vector $Z(x)$), using synthetic data is convenient when the SDPs presented in the paper are implemented with a dataset of considerable length T.

A second option, detailed in De Persis et al. (2023, Section VI.C), is to perform multiple experiments, each of length T, and then *average* the data resulting from each of the experiments. This permits us to keep T at moderate values and benefit from large datasets. In particular, averaging reduces the noise variance, hence this method is particularly effective for zero-mean disturbances. For important classes of noise such as Gaussian noise, it is actually possible to explicitly upper bound the norm of the disturbance matrix as a function of the number of experiments. This permits us to give high-probability guarantees that a stabilizing controller is found, in line with sample-complexity results available for linear systems (Recht, 2019).

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