# Exploiting data for control synthesis
SIDRA2024

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# Contents





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# 1 Assignment 1

In this section, stabilizing controllers are obtained for a time-discrete linear system.

$$x(t+1) = Ax(t) + Bu(t) + d(t) \tag{1}$$

*with matrices*

$$A = \begin{bmatrix} 0.5780 & 0.8492 & 0.4220 & 0.1508 \\ -0.6985 & 0.5780 & 0.6985 & 0.4220 \\ 0.4220 & 0.1508 & 0.5780 & 0.8492 \\ 0.6985 & 0.4220 & -0.6985 & 0.5780 \end{bmatrix} \quad B = \begin{bmatrix} 0.4160 \\ 0.8492 \\ 0.0390 \\ 0.1508 \end{bmatrix}$$

- *Perform experiments on the system using a PE input sequence with* $|u| \leq 1$ *and* $|d| \leq 0.1$ *and design a robust state-feedback controller, as presented in [1].*

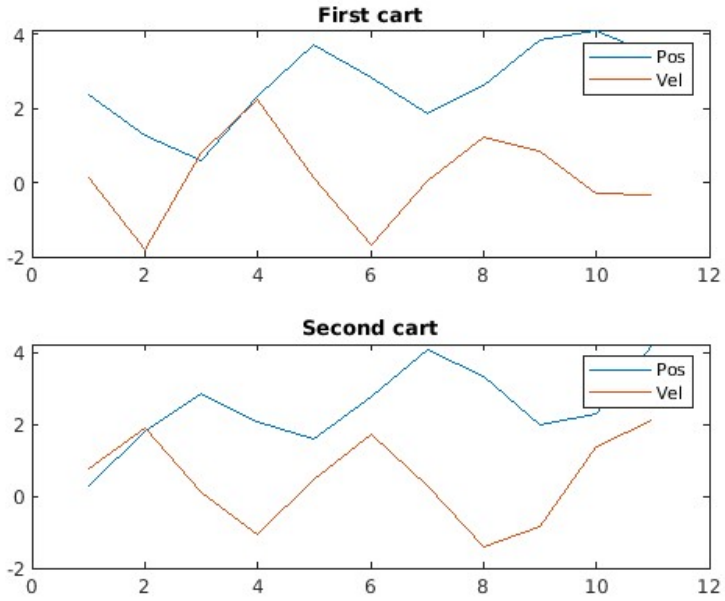


Figure 1: Open loop system

The controller is obtained as the solution of the following optimization problem.

```
cvx_begin sdp
variable Y(T,nx)
variable P(nx,nx) symmetric
[P-eye(nx) X1*Y; Y'*X1' P]>=0;
P==X0*Y
cvx_end
K = U0*Y*inv(P)
```



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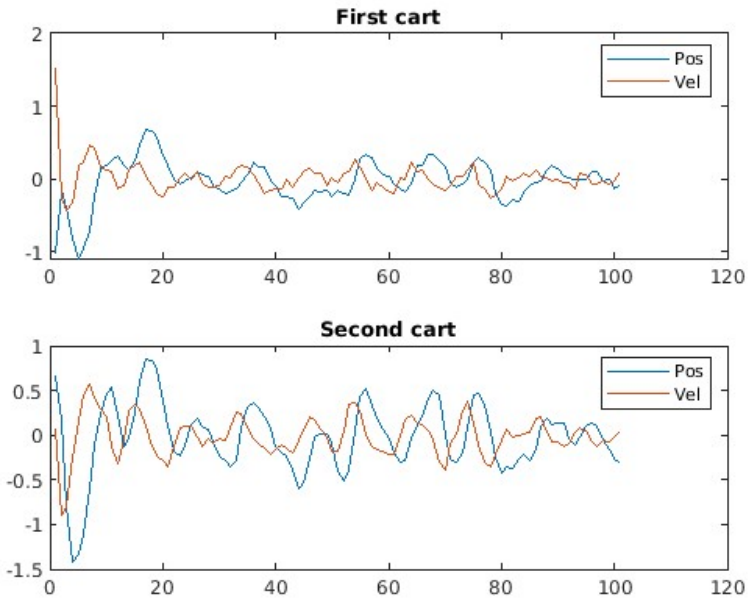


Figure 2: Closed loop system (data-driven pole placement)

The closed loop system exhibits *chattering*.

- *Perform experiments on the system using a PE input sequence with* $|u| \leq 1$ *and* $d \equiv 0$ *and design an infinite-horizon LQR controller with weight matrices* $Q = I_4$ *and* $R = 1$.

An LQR controller of the aforementioned type produces a system trajectory minimizing the cost $J = \sum_{t=0}^{\infty} x^T(t)Qx(t) + u^T(t)Ru(t)$. It can be obtained as the result of the following minimization problem:

```
cvx_begin sdp
variable Y(T,nx)
variable L(nu,nu) symmetric
variable P(nx,nx) symmetric
minimize ( trace(Qx*P) +trace(nu) )
[nu, sqrtm(R)*u*Y; Y'*u'*sqrtm(R)', P] >= 0
[P-eye(nx), X1*Y; Y'*X1', P] >= 0
P==X0*Y
cvx_end
K = U0*Y*inv(P)
```



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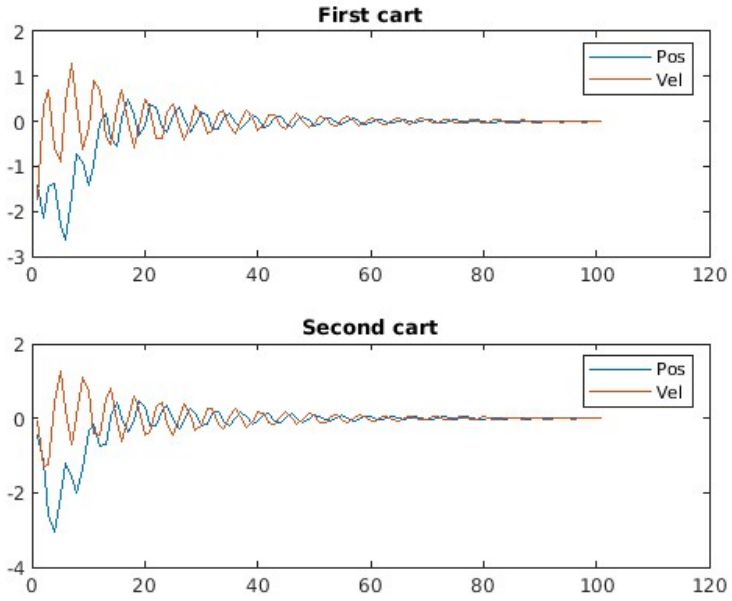


Figure 3: Closed loop system (data-driven LQR)

## 2 Assignment 2

In this section, a regulation problem and a reference tracking problem are tackled using data-driven techniques

*Consider the one-link robot arm, as originally presented in [2]*

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{K_c}{J_2}x_1 - \frac{F_2}{J_2}x_2 + \frac{K_c}{J_2 N_c}x_3 - \frac{mg\nu}{J_2}\cos(x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{K_c}{J_1 N_c}x_1 + \frac{K_c}{J_1 N_c^2}x_3 - \frac{F_1}{J_1}x_4 + \frac{1}{J_1}u
\end{aligned}$$

*where $x_1$ and $x_3$ represent the angular positions of the link and of the actuator shaft, respectively, while $u$ is the torque produced at the actuator axis. The other parameters are given as:*

$$\begin{aligned}
&K_c = 0.4, \quad && F_2 = 0.15, \quad && J_2 = 0.2, \quad && N_c = 2, \\
&F_1 = 0.1, \quad && J_1 = 0.15, \quad && m = 0.4, \quad && g = 9.8, \quad && \nu = 0.1.
\end{aligned}$$



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- *Perform experiments on the system using a $T = 10$ input sequence with $|u| \leq 0.1$. Discretize the continuous-time equations using the approximation rule that you prefer ( an easy choice could be Euler's method, with a step size $h = 0.1$ $s$ ). Collect a dataset and use it to design a stabilizing controller.*

  Consider the nonlinear library of functions ( constituted of just one function ) $Q(x) = cos(x_1)$. Then, consider the nonlinear vector function $Z(x) = [xQ(x)]^T$. This allows the system to be rewritten in the uplifted form:

$$\dot{x} = AZ(x) + Bu$$

  With matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{K_c}{J_2} & \frac{F_2}{J_2} & \frac{K_c}{J_2 N_c} & 0 & -\frac{mg\nu}{J_1} \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{K_c}{J_1 N_c} & 0 & \frac{K_c}{J_1 N_c} & \frac{F_1}{J_1} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_1} \end{bmatrix}$$

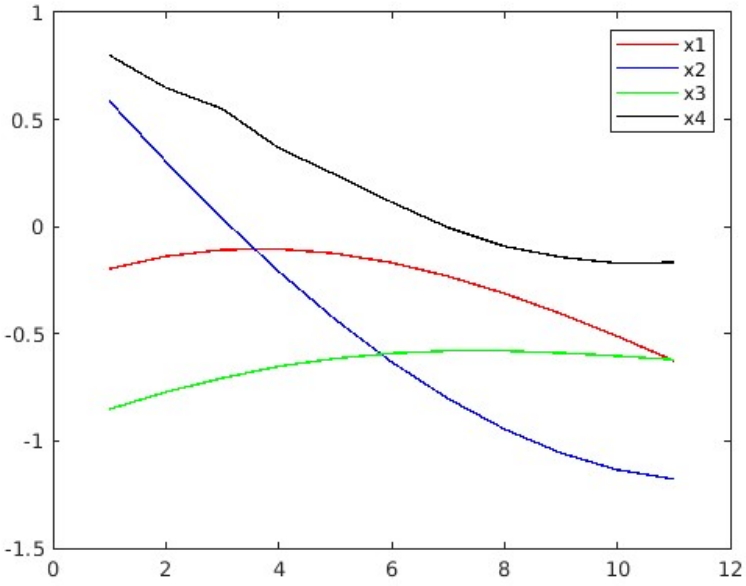


Figure 4: Open-loop robot arm



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The overbound matrix is defined as $RQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
The controller is obtained as the result of the following minimization problem:

```
cvx_begin sdp
variable P1(nx,nx) symmetric
variable Y1(T,nx)
variable G2(T,s-nx)
variable a
P1 >= 0*eye(nx);
a >= 0;
Z0*Y1 == [P1;zeros(s-nx,nx)];
[X1*Y1+(X1*Y1)'+a*eye(nx) X1*G2 P1*RQ;
      (X1*G2)' -eye(s-nx) zeros(s-nx,nx);
      (P1*RQ)' zeros(nx,s-nx) -eye(nx)] <= 0;
Z0*G2 == [zeros(nx,s-nx);eye(s-nx)];
cvx_end
G1 = Y1/P1;
G  = [G1 G2];
K  = U0*G
```

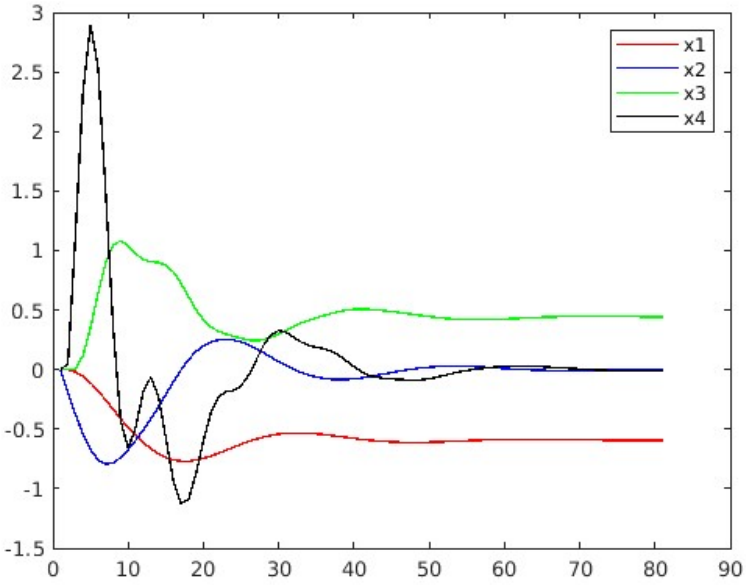


Figure 5: Closed-loop robot arm



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- *Suppose the robot is affected by a constant disturbance* $d = [0.1\ 0.2\ 0.3\ 0.4]^T$*, and that the measurement map is* $y = x_1$*. Design an integral-action controller that tracks the constant reference* $r = \frac{pi}{3}$
  Here the ( already augmented ) system is extended with the controller's own state space, which is a single integrator system driven by the tracking error $e = x_1 - r$.

$$\dot{x} = AZ(x) + Bu + Ed$$
$$\dot{\xi} = h(x) - r$$

  Since we can measure $x_1$, we have that the observation matrix is $C = [1\ 0\ 0\ 0\ 0]$, which gives $h(x) = CZ(x)$. Thus we can rewrite the system into the following form

$$\dot{x} = AZ(x) + Bu + Ed$$
$$\dot{\xi} = CZ(x) - r$$

  Where $E = I_4$.Define the extended state vector $[\ x\ \xi\ ]^T$. With further algebraic manipulation, for which I refer to the original article[2], we reach the form

$$\begin{bmatrix} \dot{x} \\ \dot{xi} \end{bmatrix} = \mathcal{A}Z(x,\xi) + \mathcal{B}u + \mathcal{E}d + \mathcal{I}r$$

  The controller is obtained as the solution of the following optimization problem

```
cvx_begin sdp
variable P1(nx+ny,nx+ny) symmetric
variable Y1(T,nx+ny)
variable G2(T,s-nx)
variable a
P1 >= 0*eye(nx+ny);
a >= 0;
Z0*Y1 == [P1;zeros(s-nx,nx+ny)];
[Z1*Y1+transpose(Z1*Y1)+a*eye(nx+ny) Z1*G2 P1*RQ;
transpose(Z1*G2) -eye(s-nx) zeros(s-nx,r);
transpose(P1*RQ) zeros(r,s-nx) -eye(r)] <= 0;
Z0*G2 == [zeros(nx+ny,s-nx);eye(s-nx)];
M*[Y1 G2] == zeros(1,s+ny);
cvx_end
G1 = Y1/P1;
G  = [G1 G2];
K  = U0*G;
```



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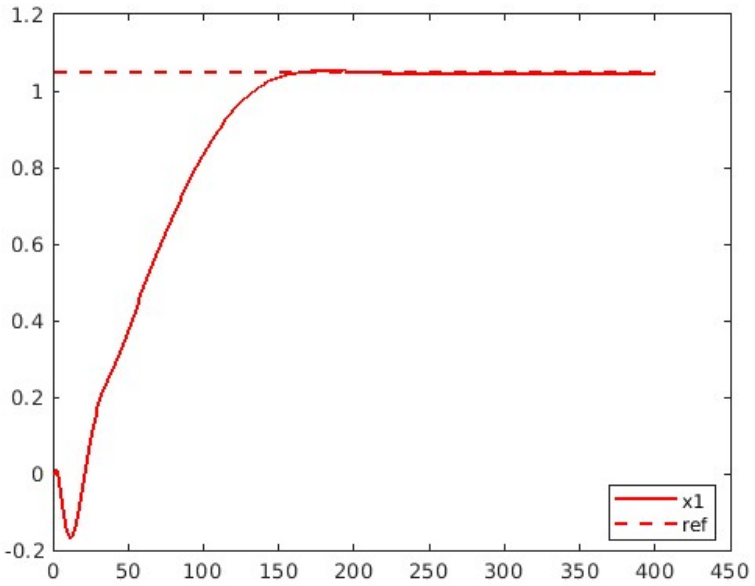


Figure 6: Closed loop system tracking $r$

- *Repeat the previous experiment, but consider the function library* $Q_2(x) = \begin{bmatrix} sin(x_1) \\ cos(x_2) \end{bmatrix}$

  The choice of $Q(x) = cos(x_1)$ comes from necessity. However, nowhere in the state space does the term $sin(x_2)$ appear. This highlights the fact that the choice of library is arbitrary and up to the designer. When considering $Z(x) = \begin{bmatrix} x \\ \xi \\ Q_2(x) \end{bmatrix}$, the system matrices are decorated with an extra column of zero(s)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_c}{J_2} & \frac{F_2}{J_2} & \frac{K_c}{J_2 N_c} & 0 & -\frac{mg\nu}{J_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{K_c}{J_1 N_c} & 0 & \frac{K_c}{J_1 N_c} & \frac{F_1}{J_1} & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  Besides that, the optimization problem is formulated in exactly the same way as before, and the results are checked in simulation



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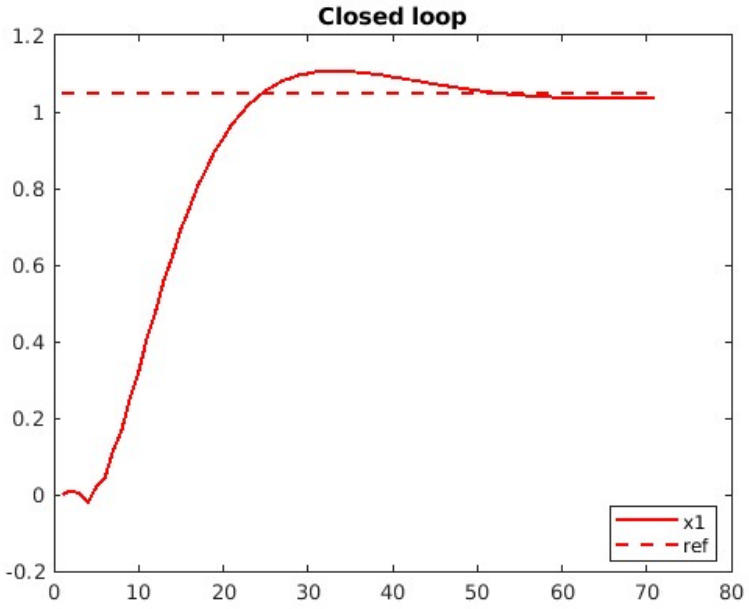


Figure 7: Closed loop system tracking $r$, with modified library

## 3 Extras

*Consider the following nonlinear system*

$$\dot{x_1} = 4.25x_1 + x_2 0.25 u x_1 u$$
$$\dot{x_2} = 6.25x_1 2x_2$$

*It refers to a reaction taking place in a chemical reactor. The two state variables model are, respectively, the deviations from the steady-state output temperature and concentration, while the forcing input represents the coolant flow. Perform $T = 10$ experiments on the system, using an input sequence with magnitude $|u| \leq 1$, and design a stabilizing controller*

When setting $u \equiv 0$, the equations reduce to the linear system $\dot{x} = \begin{bmatrix} 4,25 & 1 \\ -6,25 & 2 \end{bmatrix} x$. Computing the eigenvalues returns the set $\lambda(A) = \{-0.75, 3\}$. The free evolution matrix has a positive real-part eigenvalue, meaning the system is unstable in open loop. This can also be verified by simulation



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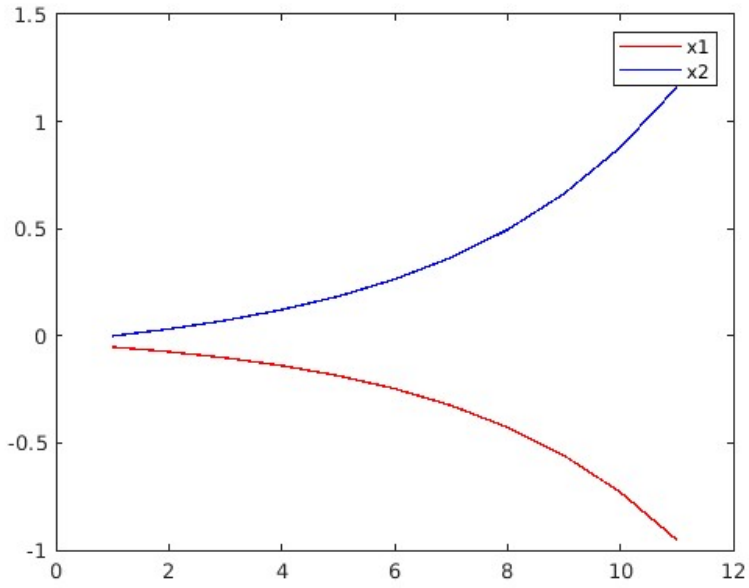


Figure 8: Open loop reactor

By introducing the augmented vector $\xi = \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$ and the library $Q(\xi) = \xi_1\xi_3$, we can rewrite the system in extended form

$$\dot{\xi} = A\xi + Bv$$

with matrices

$$A = \begin{bmatrix} 4.25 & 1 & -0.25 & -1 \\ -6.25 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Where we have exploited the fact that $\dot{\xi} = \dot{u} = v$ to write the controller as a single integrator. The overbound matrix is defined as $RQ = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$. Do note that this is dependant on the initial condition of the system being sufficiently small. In the paper[2], the initial condition was chosen within the region $\mathcal{W} = [\ -0.05\ 0.05] \times \mathrm{R} \times [\ -0.05\ 0.05]$. Again, we obtain the controller as the result of an optimization procedure

```
cvx_begin sdp
variable P1(nx,nx) symmetric
variable Y1(T,nx)
variable G2(T,s-nx)
variable a
P1 >= 0*eye(nx);
a >= 0;
Z0*Y1 == [P1;zeros(s-nx,nx)];
[X1*Y1+(X1*Y1)'+a*eye(nx) X1*G2 P1*RQ;
      (X1*G2)' -eye(s-nx) zeros(s-nx,nx);
```



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```
        (P1*RQ)' zeros(nx,s-nx) -eye(nx)] <= 0;
Z0*G2 == [zeros(nx,s-nx);eye(s-nx)];
cvx_end
G1 = Y1/P1;
G  = [G1 G2];
K  = U0*G ;
```

We see that the controller manages to stabilize the system to the desired equilibrium

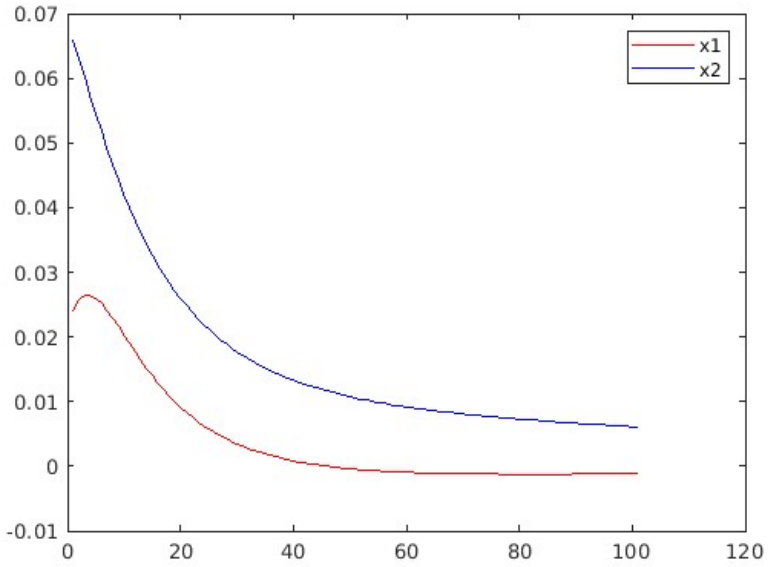


Figure 9: Closed loop reactor

When the system is initialized outside $\mathcal{W}$, the controller fails to stabilize it. For instance, this is what happens when $\xi(0) = \begin{bmatrix} -1.40997503421309 \\ -1.74728260275247 \\ 0 \end{bmatrix}$, which was chosen as a Gaussian extraction



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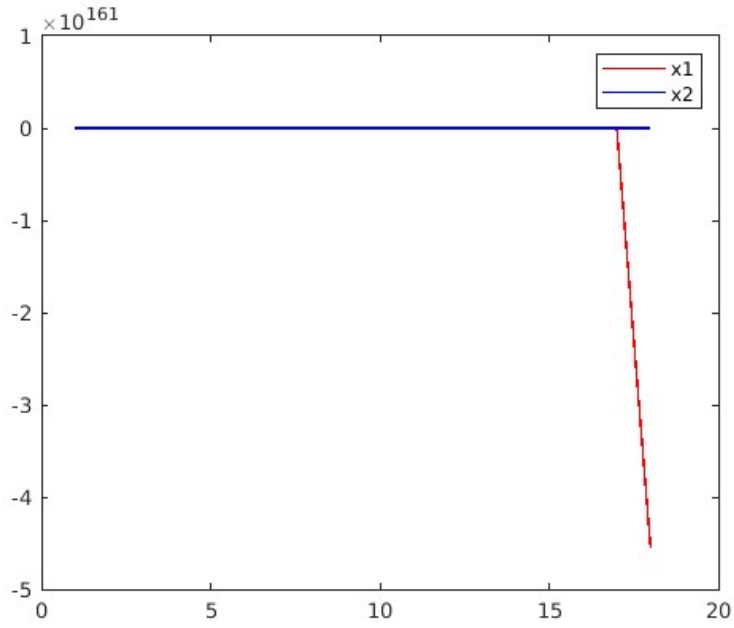


Figure 10: Unstable closed loop system, initialized outside $\mathcal{W}$

## 4 Author's notes

I ran the simulations on Matlab R2024a on a Linux machine. For some reason, cvx wouldn't install at all on the Matlab ROOT, unless i launched it with admin rights using:

```
sudo matlab
```

Even then, the installation would not survive a restart of Matlab. As a result of this, my time-consuming workaround was installing cvx in each simulation run. This is done by downloading cvx from the official website and extracting its content in the workspace folder ( these steps have to be exedcuted just once ), launching matlab with administrator rights, and finally adding the following two lines of code on top of the simulation:

```
addpath(extracted_cvx_folder_path)
cvx_setup
```

The chemical reactor exercise was not required by the homework, however, when I first read the article, I mistakenly understood it to be part of the assignement, and so i decided to include it.

The author's github repo, which is available at `https://github.com/zjhurug/codes-for-contraction-paper`, was very helpful to learn the basics of the syntax of cvx, debug errors in the code, and compare the results. My own github repo, regarding this report, is available at `https://github.com/hizr3/datadriven_sidra2024`.

I am welcoming of feedback, either regarding this homework, or related to my line. At the moment i am working in State Estimation, with a focus on the automotive field.



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## References

[1] Claudio De Persis and Pietro Tesi. Learning controllers for nonlinear systems from data. *Annual Reviews in Control*, page 100915, 2023.

[2] Zhongjie Hu, Claudio De Persis, and Pietro Tesi. Enforcing contraction via data. *arXiv preprint arXiv:2401.07819*, 2024.



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