Project A

Key Topic: Finite Difference Methods for solving ODEs

In this project, the dynamics of a simple and a double pendulum are studied to test different methods for integrating systems of ODEs.

1 Single pendulum

The equation of motion of a simple pendulum of mass m and length l, making an angle θ to the vertical (Fig. 1(a)), can be written

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - \gamma\frac{d\theta}{dt} \quad , \tag{1}$$

where γ is a damping coefficient. First, re-scale equation (1) by using a convenient time scale, and express the problem as a pair of coupled first-order ODEs. This will give you new *effective* variables for the time t and damping coefficient γ.

To test the numerical methods, consider the case of small oscillations ($\theta \ll 1$) starting from rest ($\theta(t = 0) = \theta_0$). Using the small angle approximation write a program to solve the system of ODEs using the following methods:

- Euler's method.
- The leapfrog method.
- The fourth-order Runge-Kutta method (RK4).

Using your program:

1. Develop a way to test if a given method is stable. Document the method in your report and argue why it is a good test of stability.
2. Using your stability test, evaluate whether the methods are stable for
   (a) the undamped pendulum ($\gamma = 0$) and
   (b) the weakly damped pendulum ($\gamma > 0$) (try $\gamma = 0.2$).
3. In those cases where the method is conditionally stable, find the value of the step length, Δt, at which the method becomes unstable. You may find the critical value of Δt by trial and error (in the Euler case you should be able to find it by stability analysis).

   Which numerical method would you recommend for a general oscillatory problem with damping?

2 Double pendulum

A second pendulum with length l and mass M, which makes an angle φ to the vertical, is added as shown in Fig. 1(b). For small oscillations, with damping, the equations of motion become:

$$ml\frac{d^2\theta}{dt^2} = -(m+M)g\theta + Mg\varphi - \gamma\frac{d\theta}{dt} \tag{2}$$

$$Ml\left(\frac{d^2\theta}{dt^2} + \frac{d^2\varphi}{dt^2}\right) = -Mg\varphi - \gamma\frac{d\theta}{dt} - \gamma\frac{d\varphi}{dt} \tag{3}$$

The equations can be written as the following system of first order differential equations:

$$\frac{d}{d\tilde{t}}\begin{bmatrix}\theta\\ \varphi\\ \omega\\ \nu\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -(R+1) & R & -G & 0\\ (R+1) & -(R+1) & G(1-R^{-1}) & -G/R\end{bmatrix}\begin{bmatrix}\theta\\ \varphi\\ \omega\\ \nu\end{bmatrix} \quad , \tag{4}$$

where $R = M/m$ and $G = \gamma/(m\sqrt{gl})$ and time is in units of $\sqrt{l/g}$.

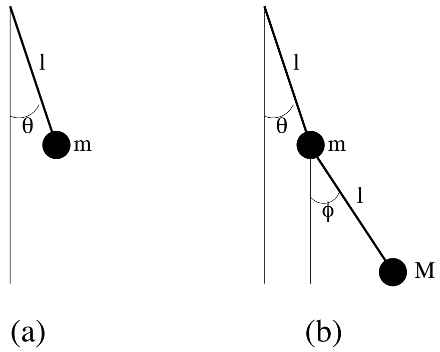

Figure 1: Notation used for (a) the single pendulum and (b) the double pendulum.

1. Write a program to solve the system using your preferred method to solve the single pendulum. Use the initial conditions $\theta = 0.1$ and $\varphi = d\theta/dt = d\varphi/dt = 0$.
2. First, consider the case of no damping and calculate the motion for some different values of R, e.g. $R = 1$, $R = 0.01$ and $R = 100$. Calculate the total energy of the system to check that the solution is stable. Do you need to change the time step as R is changed?
3. Now consider the case of damping when $G = 1$, and calculate the motion for different values of R, e.g. $R = 1$, $R = 0.01$ and $R = 100$. In your report, discuss what is happening to the dynamics of the system as R is varied. What numerical method could you use to ensure stability of a system like this for general values of the parameters?

3 Report Guidelines & Submission

The **report** should not need to be more than **1500 words long** (excluding figure captions and tables). (*If the report is much longer it might count against you as getting information across in a concise manner is of great importance.*) Make sure the approximate number of words is written at the end of the report. It should include brief descriptions of the aims and methods, a summary of the results, a brief discussion, and a conclusion. Your focus should depend on how far you get with the programming. For the single pendulum, your results should include an assessment of the suitability of the three different numerical methods for the problem, an explanation and justification of your stability condition(s), and a discussion of how this applies to the different methods. If you finish the part on the double pendulum, you should discuss its dynamics as a function of the relative masses, as well as the issues involved in ensuring stability of the numerical method. If you do not manage to finish the double pendulum investigation, it is still worth explaining how you would approach it, on the basis of your results from the single pendulum.

Marks will be awarded for: results achieved; understanding shown of numerical methods and physics; organisation and general quality of report; presentation of report (including graphs); and quality of programming. Some additional points to remember: Try to be selective in the choice of results to present. Choose your graphs carefully to illustrate typical results, and to make relevant comparisons. You can also make good use of tables. Integrate graphs and tables with the main text and make sure to give them proper captions that briefly explain what you see without any reference to the main body of the text. You do not need to give a lot of mathematical detail in the report.

Your **source code** should be commented appropriately and easy to read. You don't need to explain the programming itself in the report but should comment on any tricks or strategies used to make the code better or more efficient.

Submission – the report and source code must be submitted to "Computational Physics 2013–14" on Blackboard Learn no later than **12 noon** on **Monday 11th November**. Please submit the report in PDF format only. Please put multiple source code files into a ZIP file before uploading. There will be separate places to submit the report and the code. If you are having technical problems with the submission email me at `rj.kingham@imperial.ac.uk` .