# Multibody Dynamics B - Assignment 3

July 4, 2019

## a)

In this assignment we are exploring a double pendulum with passive and active elements. We derive the equations of motion using the systematic approach where only the constraints are needed as well as an algorithm that calculates the accelerations and lambdas. Below we can see the system we are exploring.

1.png

Where

$$\rho = 670kg/m^3 \quad l = 0.6m \quad w = 50mm \quad t = 3mm \quad and \quad g = 9.81m/s^2$$

Now using the systematic approach we find the virtual power for the system above. Firstly we know that the virtual power for the two rigid bodies is

$$\delta P = (F_i - M_{i,j}\ddot{x}_j)\delta\dot{x}_j$$

Furthermore, for the spring we express virtual power equation as

$$\delta P = F_s \Delta\dot{l}$$

Adding this to the power expression for the system we get

$$\delta P = (F_i - M_{i,j}\ddot{x}_j)\delta\dot{x}_j - F_s\Delta\dot{l}$$

Now $\Delta\dot{l}$ can be written as $\Delta\dot{l} = \Delta l_{,j}\dot{x}_i$ and as a virtual power expression this can be written as $\delta\Delta\dot{l} = \Delta l_{,j}\delta\dot{x}_i$, and we can write $\Delta l = C_s(x_i)$. Furthermore adding the constraints of the system we can write our virtual power expression as

$$\delta P = (F_i - M_{i,j}\ddot{x}_j - F_s C_{s,i} - \lambda_k C_{k,i})\delta\dot{x}_j$$

where

$$F_i - M_{i,j}\ddot{x}_j - F_s C_{s,i} - \lambda_k C_{k,i} = 0$$

Are our equations of motion.
This can be written in matrix form as

$$\begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i - C_{s,i}F_s \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}$$

where $F_i = (F_{x1}, F_{y1}, M_1, F_{x2}, F_{y2}, M_2) = (m_1 * g, 0, 0, m_2 * g, 0, 0)$, $M_{ij} = diag(m_1, m_1, I_1, m_2, m_2, I_2)$, where $I_1 = I_2 = \frac{1}{12} * m_1 * l_1^2$ since we assume a slender beam, $\ddot{x}_j = (\ddot{x}_1, \ddot{y}_1, \ddot{\varphi}_1, \ddot{x}_2, \ddot{y}_2, \ddot{\varphi}_2)$, $\lambda_k = (\lambda 1, \lambda 2, \lambda 3, \lambda 4)$ and $k = 1, .., 4$ and $i, j = 1, .., 6$

These differential equations describe the dynamic equilibrium of the system. Now we have the constraint equations, 'the glue', and since our unknowns are the accelerations, $\ddot{x}_j$, we have to differentiate the constraint equations twice with respect to time

$$\dot{C}_k = C_{k,i}\dot{x}_i = 0$$

$$\ddot{C}_k = C_{k,j}\ddot{x}_j + C_{k,ij}\dot{x}_i\dot{x}_j = 0$$

Now we have two rigid bodies and our constraint equations are

$$X_A = x_1 - l_1/2 * cos(\varphi_1)$$

$$Y_A = y_1 - 1_1/2 * sin(\varphi_1)$$

$$X_B = (x_2 - l_2/2 * cos(\varphi_2)) - (x_1 + l_1/2 * cos(\varphi_1))$$

$$Y_B = (y_2 - l_2/2 * sin(\varphi_2)) - (y_1 + l_1/2 * sin(\varphi_1))$$

Since bar 1 is connected to the ground by a cylindrical joint and bar 1 and 2 are connected together by a cylindrical joint. We put these constraints in a zero-delimited form where we have

$$C_k = \begin{pmatrix} x_1 - l_1/2 * cos(\varphi_1) \\ y_1 - 1_1/2 * sin(\varphi_1) \\ (x_2 - l_2/2 * cos(\varphi_2)) - (x_1 + l_1/2 * cos(\varphi_1)) \\ (y_2 - l_2/2 * sin(\varphi_2)) - (y_1 + l_1/2 * sin(\varphi_1)) \end{pmatrix} = \mathbf{0}$$

We use the Matlab, see Apendix A, to derive $C_{k,j}$ and $-C_{k,ij}\dot{x}_i\dot{x}_j$, notice that $C_{k,i} = C_{k,j}^T$, which gives us

$$C_{k,j} = \begin{pmatrix} 1 & 0 & 1/2 * l_1 * sin(\varphi_1) & 0 & 0 & 0 \\ 0 & 1 & -1/2 * l_1 * cos(\varphi_1) & 0 & 0 & 0 \\ -1 & 0 & 1/2 * l_1 * sin(\varphi_1) & 1 & 0 & 1/2 * l_2 * sin(\varphi_2) \\ 0 & -1 & -1/2 * l_1 * cos(\varphi_1) & 0 & 1 & -1/2 * l_2 * cos(\varphi_2) \end{pmatrix}$$

and

$$C_{k,ij}\dot{x}_i\dot{x}_j = \begin{pmatrix} 1/2 * l_1 * \dot{\varphi_1}^2 cos(\varphi_1) \\ 1/2 * l_1 * \dot{\varphi_1}^2 sin(\varphi_1) \\ 1/2 * l_1 * \dot{\varphi_1}^2 cos(\varphi_1) + 1/2 * l_2 * \dot{\varphi_2}^2 cos(\varphi_2) \\ 1/2 * l_1 * \dot{\varphi_1}^2 sin(\varphi_1) + 1/2 * l_2 * \dot{\varphi_2}^2 sin(\varphi_2) \end{pmatrix}$$

Furthermore, we can define $C_S$ as

$$C_s = l_s - l_0 = \sqrt{(X_E - X_D)^2 + (Y_E - Y_D)^2} - l_0$$

where $X_E = x_1 + 1/6 * l_1 * cos(\varphi_1)$, $X_D = -l/2 * l_1$, $Y_E = y1 + 1/6 * l1 * sin(\varphi_1)$ and $Y_D = 0$. This gives us

$$C_s = \sqrt{(x_1 + 1/6 * l_1 * cos(\varphi_1) - (-l/2 * l_1))^2 + (y1 + 1/6 * l1 * sin(\varphi_1) - 0)^2} - l_0$$

where $l_0 = 2/3 * l_1$.

By using Matlab, see appendix A, we get $C_{s,i}$, i.e. the derivative, which is

$$C_{s,i} = \frac{1}{l_s}\begin{pmatrix} x_1 + l_1/6 * cos(\varphi_1) + l_1/2 \\ y_1 + 1_1/6 * sin(\varphi_1) \\ -1/12 * l_1(l_1 * sin(\varphi_1) - 2 * y_1 * cos(\varphi_1) + 2 * x_1 * sin(\varphi_1)) \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

Now $F_s = k_s * C_s$ where $k_s = (15/2)(mg/l)$, and to find the lambdas and accelerations we solve

$$\begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix}^{-1} \begin{pmatrix} F_i - C_{s,i} * k_s * C_s \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}$$

Now from Matlab we get

$$\ddot{x}_j = \begin{pmatrix} 2.1021 m/s^2 \\ 0 m/s^2 \\ -7.0071 rad/s^2 \\ 8.4086 m/s^2 \\ 0 m/s^2 \\ -14.0143 rad/s^2 \end{pmatrix}$$

and

$$\lambda_k = \begin{pmatrix} 0.1056N \\ -0.5915N \\ 0.0845N \\ 0N \end{pmatrix}$$

We see that there is a force in the negative y direction, which makes sense considering there is a spring pulling onto the system. Furthermore, we see that the rotational acceleration for bar 2 is twice that of bar 1, this is due to the spring pulling onto bar 1 and thus also shooting bar 2 in the clockwise direction. Both bars experience acceleration in the x direction, as is expected due to the gravitational field, however they do not exceed the acceleration of gravity, meaning that the system does not become chaotic.

## b)

Now we have a motor in position A and the spring is gone, this gives us an extra constraint in the $C_k$ matrix, while the effects of the spring of course disappear. The constraint is $C_1 = \varphi 1 - \omega t$, following the prescribed motion kinematic constraints, adding this to the constraint equation and again using Matlab to obtain the derivatives, we get

$$C_k = \begin{pmatrix} x_1 - l_1/2 * cos(\varphi_1) \\ y_1 - 1_1/2 * sin(\varphi_1) \\ (x_2 - l_2/2 * cos(\varphi_2)) - (x_1 + l_1/2 * cos(\varphi_1)) \\ (y_2 - l_2/2 * sin(\varphi_2)) - (y_1 + l_1/2 * sin(\varphi_1)) \\ \varphi 1 - \omega * t \end{pmatrix} = \mathbf{0}$$

and thus

$$C_{k,j} = \begin{pmatrix} 1 & 0 & 1/2*l_1*sin(\varphi_1) & 0 & 0 & 0 \\ 0 & 1 & -1/2*l_1*cos(\varphi_1) & 0 & 0 & 0 \\ -1 & 0 & 1/2*l_1*sin(\varphi_1) & 1 & 0 & 1/2*l_2*sin(\varphi_2) \\ 0 & -1 & -1/2*l_1*cos(\varphi_1) & 0 & 1 & -1/2*l_2*cos(\varphi_2) \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Again we have $C_{k,i} = C_{k,j}^T$ and

$$C_{k,ij}\dot{x}_i\dot{x}_j = \begin{pmatrix} 1/2*l_1*\dot{\varphi_1}^2 cos(\varphi_1) \\ 1/2*l_1*\dot{\varphi_1}^2 sin(\varphi_1) \\ 1/2*l_1*\dot{\varphi_1}^2 cos(\varphi_1) + 1/2*l_2*\dot{\varphi_2}^2 cos(\varphi_2) \\ 1/2*l_1*\dot{\varphi_1}^2 sin(\varphi_1) + 1/2*l_2*\dot{\varphi_2}^2 sin(\varphi_2) \\ 0 \end{pmatrix}$$

Now we solve for the new constraints

$$\begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} M_{ij} & C_{k,i} \\ C_{k,j} & 0_{kk} \end{pmatrix}^{-1} \begin{pmatrix} F_i \\ -C_{k,ij}\dot{x}_i\dot{x}_j \end{pmatrix}$$

Using Matlab we get

$$\ddot{x}_j = \begin{pmatrix} 0m/s^2 \\ -11.8435m/s^2 \\ 0rad/s^2 \\ 7.3575m/s^2 \\ -35.5306m/s^2 \\ -24.5250rad/s^2 \end{pmatrix}$$

and

$$\lambda_k = \begin{pmatrix} 0.7394N \\ 2.8567N \\ 0.1479N \\ 2.1425N \\ -0.2662N*m \end{pmatrix}$$

Now the torque that the motor exerts is of course, following the prescribed motion laws;

$$\lambda_5 = \tau = -0.2662N*m$$

and then the power is

$$P = \tau*\omega = (-0.2662)*(-2*\pi) = 1.6725W$$

If we look at the accelerations we can see that they are very high in the y direction for both bars, especially bar 2. This can be expected due to the motor spinning bar 1, which in turn pulls bar 2, resulting in the extremely high accelerations, and in bar 2 case, 3 times as large as bar 1. However, we see that there is no acceleration in the x-direction and no rotational acceleration for bar 1, this is due to the motor constraint, i.e. the motor spins bar 1 at a constant velocity $\omega$. The extremely high accelerations also translate to high joint forces in the vertical direction. The power supplied to the system through the motor is rather small, however our system is neither heavy or big, giving us the conclusion that this power seems reasonable.

## c)

Now we have a system with no motor and no spring, but rather where the system hits a vertical wall positioned horizontally in the origin. This will result in an impact