# Chapter 1

# Extrapolation to zero

In this short chapter we will describe shortly the concept of *extrapolation to zero* and how we can apply it.

## 1.1 Motivation

Let $T : ]0, \varepsilon[ \to \mathbb{R}$ be function and assume that we have

$$T(h) = T_0 + ah^n + O(h^{n+1}). \tag{1.1}$$

for $h \to 0$. We are interested in computing $T_0 = \lim_{h\to 0} T(h)$ up to some desired accurracy. In order to do that we might have to compute $T(h)$ for very small $h$. That might not be feasible since $T(h)$ might be very expensive to compute for small $h$ or even impossible due to numerical instabilities. Hence we would like to somehow accelerate the convergence of $T$ to 0. A nice way to do that is the *Richardson extrapolation scheme* which goes as follows: Let $0 < r < 1$. Plug $rh$ into (1.1). Then we get

$$T(rh) = T_0 + ar^n h^n + O(h^{n+1}). \tag{1.2}$$

Now multiply (1.1) by $r^n$, subtract it from (1.2) and divide the result by $1 - r^n$. Then we get:

$$R(h) = T_0 + O(h^{n+1})$$

where

$$R(h) := \frac{T(rh) - r^n T(h)}{1 - r^n}.$$

Note that $R(h)$ has $O(h^{n+1})$ convergence to $T_0$ while $T(h)$ has $O(h^n)$, i.e. $R(h)$ converges asymptotically faster. But what did we actually do? We took the linear polynomial in $t^n$ which goes through $(rh, T(rh))$ and $(h, T(h))$ and let $R(h)$ be its value at 0, i.e. we interpolated the points and then evaluated the interpolation polynomial outside the interval; hence the term *extrapolation*. This should serve as a motivation for the sequel.

## 1.2 The extrapolation table

We always think of $T$ as arising from some numerical scheme e.g. the trapezoidal rule and then $T_0$ is the integral of some function. Thus we do not require that $T$ is necessarily defined for all values near 0, but only on set which has 0 as an accumulation point. In what follows, we will thus refer to $T$ as a *method* for computing $T_0$.

**Definition 1.1.** Let $T$ be a method for computing $T_0$. We say that $T$ has an asymptotic expansion in $h^p$ up to order $pm$ if there exist constants $\tau_p, \tau_{2p}, \ldots, \tau_{mp} \in \mathbb{R}$ such that

$$T(h) = T_0 + \tau_p h^p + \tau_{2p} h^{2p} + \cdots \tau_{mp} h^{mp} + O(h^{(m+1)p}) \tag{1.3}$$

for $h \to 0$.

Let $(x_1, y_1), \ldots, (x_k, y_k)$ be a collection of points such that $x_1, \ldots, x_k$ are distinct. Then there exists a polynomial $P$ which interpolates the points, i.e. $P(x_i) = y_i$ for all $i$. We say that $P$ is *the interpolation polynomial* for the points if $P$ has the lowest degree among all polynomials which interpolate them. The interpolation polynomial is unique. Let $p > 0$ be an integer and points $(x_1^p, y_1), \ldots, (x_n^p, y_n)$ such that $x_i^p$ are distinct, be given. Let $P$ be the interpolation polynomial for the points. We then call $P(h^p)$ the *interpolation polynomial in $p$* for the points.

Let $T$ be a method with asymptotic expansion in $p$ up to $pm$. The extrapolation process works as follows: We compute $T(h)$ for some points $h_1, h_2, \ldots, h_k$ where $k \leq m$. Then we compute the interpolation polynomial $P$ in $h^p$ which goes through $(h_1, T(h_1)), \ldots, (h_k, T(h_k))$. We then hope that $P(0)$ gives a good approximation $T_0$.

In order to compute $P(0)$ we use the *Neville scheme*. Let $P_{ij}(h^p) \coloneqq P(h^p; h_{i-j+1}^p, h_i^p)$ be the interpolation polynomial in $h^p$ which interpolates $(h_{i-j+1}^p, T(h_{i-j+1}), \ldots, (h_i^p, T(h_i^p))$ and set $T_{ij} \coloneqq P_{ij}(0)$. Then according to the Neville scheme we can compute $T_{ij}$, $j \leq i$, in the following recursive way:

1. $T_{i1} \coloneqq T(h_i)$ for $i = 1, \ldots, k$.
2. $T_{ij} \coloneqq T_{i,j-1} + \dfrac{T_{i,j-1} - T_{i-1,j-1}}{r^p - 1} = \dfrac{r^p T_{i,j-1} - T_{i-1,j-1}}{r^p - 1}$ for $1 < j \leq i$ where $r \coloneqq h_{i-j+1}/h_i$.

If we align $T_{ij}$ to a triangular table, we call that the *extrapolation table*.

## 1.3 Convergence

If we have a numerical method or scheme that has an asymptotic expansion as (1.3), then the error decays polynomially as $h \to 0$. It is known (see e.g. theorem 9.22 in [1]) that $T_{ij}$ has faster polynomial decay of higher degree, as $h \to 0$, then $T$. Let $\varepsilon_k \coloneqq |T_{kk} - T|$. We want to analyze how $\varepsilon_k$ behaves as $k \to +\infty$, i.e. how $\varepsilon_k$ behaves when we increse the number of extrapolation steps. Let $N_n k$ be some measure of the effort needed to compute $T_{kk}$. In what follows we will test numerically the qualitative hypothesis that the error converges exponentially with the computational effort i.e.

$$\varepsilon_k \sim A \exp(-c N_k^q) \tag{1.4}$$

for constants $A, c, q$. Note that if $\varepsilon_k = A \exp(-c N_k^q)$ then $\ln \varepsilon_k = b - c N_k^q$ so in order to test the hypothesis we will do the following: Assume that we have the error $\varepsilon_k$ for $k = 1, \ldots, n$. Then we will compute

$$(b, c, q) \coloneqq \arg\min \left\{ \sum_{k=1}^{n} |\ln \varepsilon_k - (b - c N_k^q)|^2 \right\} \tag{1.5}$$

and see whether the points $(N_k, \ln \varepsilon_k)$ fit well to the graph of $t \mapsto b - ct^q$.

We will also test the hypothesis that the error converges exponentially with the number of extrapolation steps, i.e. whether

$$\varepsilon_k \sim A \exp(-ck^q) \tag{1.6}$$

for constants $A, c, q$.

In order to validate the estimated parameters $b, c, q$ we will do a simple "cross validation" by fitting the model to subsets of the data and see whether the parameters vary a lot. If they vary a lot, we conclude that the fitting is unstable. If they are almost the same we will be more confident in that the model is actually appropriate. The cross validation strategy we will use goes as follows: Suppose that we have done a curve fitting on $(x_k, y_k)$ for $k = 0, 1, \ldots, n$. Then we will do the curve fitting for $(x_{k+3}, y_{k+3}), \ldots, (x_{k+9}, x_{k+9})$ for $k = 0, \ldots, n-6$ and compute the relative variance of the parameters. Let $a_k$, $k = 1, \ldots, m$ be numbers. Then we define their mean value by $\overline{a} := \frac{1}{m} \sum_{k=1}^{m} a_k$, and the relative variance by

$$\frac{1}{m\overline{a}^2} \sum_{k=1}^{m} (a_k - \overline{a})^2.$$

## 1.4 Code

The following Python mehtod computes the extrapolation table for some scheme which has an asymptotic expansion in $h^p$.

```
#sc (Scheme): The scheme to extrapolate
#prob: The problem to apply the scheme to. We assume that sch is an
#      implementation of Scheme which can be applied to prob.
#seq (Sequence): The sequence to use in the extrapolation
#hp (bool): Indicates whether to use high precision arithmetic (true)
#           or standard double precision (false).
#returns: The extrapolation table as a list of lists.
def extrapolate(sc, prob, seq, hp):
  n = len(seq)
  X = [[0 for j in range(i + 1)] for i in range(n)]

  #X[i][j] = T_ij
  for i in range(n):
    X[i][0] = sc.apply(prob, seq[i])
    for j in range(1, i + 1):
      #r = h_{i-j} / h_i = seq[i] / seq[i-j]
      #rp = r^p.
      #Must cast the elements of seq to hp numbers if in hp mode.
      rp = ((mpf('1') * seq[i]) / (mpf('1') * seq[i-j]) if hp else seq[i] / seq[i-j]) ** sc.p
      X[i][j] = (rp * X[i][j-1] - X[i-1][j-1]) / (rp - 1)

  return X
```

# Chapter 2

# Romberg quadrature

## 2.1 The algorithm

Let $f : [a, b] \to \mathbb{R}$ be a function and $I := \int_a^b f(x)dx$. The *trapezoidal rule* is a method for approaching $I$ which works as follows: Let $a = t_0 < t_1 < \cdots < t_n = b$ be a subdivision of $[a, b]$. On each of the intervals $[t_{i-1}, t_i]$ we approximate $\int_{t_{i-1}}^{t_i} f(x)dx$ by the area of a trapezoid with verticies $(t_{i-1}, 0)$, $(t_{i-1}, f(t_{i-1}))$, $(t_i, f(t_i))$, $(t_i, 0)$ i.e. by $\frac{1}{2}(t_i - t_{i-1})(f(t_{i-1}) + f(t_i))$. Hence we approximate $I$ by

$$I = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(x)dx \approx \sum_{i=1}^{n} \frac{1}{2}(t_i - t_{i-1})(f(t_{i-1}) + f(t_i)).$$

If $t_i - t_{i-1} = \frac{1}{n}(b - a) =: h$ for each $i$ then the above estimate becomes

$$I \approx h \left( \frac{1}{2}(f(a) + f(b)) + \sum_{i=1}^{n-1} f(a + ih) \right) \tag{2.1}$$

We define $T_f(h)$ as the right hand side in (2.1).

Let $F : [0, n] \to \mathbb{R}$ be a $2k + 1$ times continuously differentiable function, $n$ a positive integer. Then by Euler's summation formula (see formula 298 in [2]) we have

$$\sum_{i=0}^{n} F(i) = \int_0^n F(x)dx + \frac{1}{2}(F(0) + F(n)) + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}(F^{(2i-1)}(n) - F^{(2i-1)}(0)) + R_k \tag{2.2}$$

where $R_k = \int_0^n P_{2k+1}(x)F^{(2k+1)}(x)dx$, $B_m$ are the *Bernoulli numbers* and $P_m$ the *Bernoulli polynomials*. If let $F(x) := f(a + xh)$ then we get the following asymptotic expansion for the trapezoidal rule:

**Theorem 2.1.** *Let $f : [a, b] \to \mathbb{R}$ be $2k + 1$ times continuously differentiable and $h := (b - a)/n$. Then*

$$T_f(h) = I + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a))h^{2i} + h^{2k+1}R_k(h) \tag{2.3}$$

*where*

$$R_k(h) = \int_a^b P_{k+1}\left(n\frac{x - a}{b - a}\right) f^{(2k+1)}(x)dx. \tag{2.4}$$

The following code is a trivial implementation of the trapezoidal rule. The *TrapezoidalRule* class in an implementation of the abstract class *Scheme* which represents a numerical scheme or method, which has asymptotic expansion in $h^p$. The Scheme class has a method named *apply* which takes in a problem to which the scheme is applied to. The argument $m$ in the apply-method is the number of subintervals that should be used.

```
class TrapezoidalRule(Scheme):
    def __init__(self):
        super(TrapezoidalRule, self).__init__(2)

    def apply(self, inte, m):
        (a,b) = inte.interval
        h = (b - a) / m
        I = 0.5 * (inte.f(a) + inte.f(b))
        for i in range(1, m):
            I += inte.f(a + i * h)

        return I * h
```

Assume that we have computed the value of $T_f(h)$ for $h = h_1, \ldots, h_k$ and we want extrapolate to zero, i.e. we want to compute the value at zero of the interpolation polynomial in $h^2$ for $(h_i^2, T_f(h_i)$, $i = 1, \ldots, k$. Denote by $T_{ij}$ the value at zero of the polynomial in $h^2$ which goes through $(h_{i-j+1}^2, T(h_{i-j+1}), \ldots, (h_i^2, T(h_i))$. The Neville scheme gives us the following algorithm for computing $T_{ij}$, $1 \leq j \leq i \leq k$, recursively:

1. $T_{i1} \coloneqq T_f(h_i)$ for $i = 1, \ldots, k$.
2. $T_{ij} \coloneqq T_{i,j-1} + \frac{T_{i,j-1} - T_{i-1,j-1}}{\left(\frac{h_{i-j+1}}{h_i}\right)^2 - 1}$ for $2 \leq j \leq i$.

## 2.2 Numerical experiments

In this section we are going to apply Romberg quadrature to various functions and also try different sequences. We will analyze how different sequences perform in the sense that we want to measure how many function evaluations we need to attain a prescribed precision.

We will try various functions and the following sequences:

- The harmonic sequence: $a_n = n$, $n \geq 0$.
- The Romberg sequence: $a_n = 2^{n-1}$, $n \geq 1$.
- The Bulirsch sequence: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_{n+2} = 2 \cdot a_n$ for $n \geq 2$. Its first elements are

$$1, 2, 3, 4, 6, 8, 12, 16, 24, 32, \ldots$$

Suppose that we are approximating the integral $I \coloneqq \int_a^b f(x)dx$ using Romberg quadrature. We will use the stepsizes $h_k \coloneqq (b-a)/a_k$ for the extrapolation. Let $T_{ij}$, $i \geq 0$ and $j \leq i$ be the extrapolation table we get and $\varepsilon_k \coloneqq |T_{kk} - I|$ be the error on the diagnoals. Let $N_k$ be the number of function evaluations needed to compute $T_{kk}$. We will use $N_k$ as the measurement of computational effort as mentioned in section 1.3 and we will try to fit the exponential convergence model introduced there. We will also plot the logarithm of the error against the number of extrapolation steps. Note that $N_k = \sum_{i=1}^{k}(a_i + 1)$ where

$(a_i)$ is our sequence, so in case of the Harmonic sequence, we have $N_n = n(n+3)/2 \approx n^2/2$ for $n$ large. Hence if $\varepsilon_n \sim A\exp(-cN_n^q)$ then

$$\varepsilon_n \sim A\exp(-c/2^q n^{2q})$$

for $n$ large. Thus if the error converges exponentially with the number of function evaluations, it will also converge exponentially with the number of extrapolation steps, and the exponent in the latter fitting will be twice the parameter from the former.

If our sequence is the Romberg sequence then $N_k = \sum_{i=1}^{n}(2^{i-1}+1) = 2^k + k - 1 \approx 2^k$ for $k$ large, so if $\varepsilon_k \sim A\exp(-cN_k^q)$ then

$$\varepsilon_k \sim A\exp(-c2^{kq})$$

for $k$ large, which is not exponential convergence. Hence possibilities of having exponential convergence in the number of steps and in the number of evaluations are mutually exclusive for the Romberg sequence.

For the model fitting we will plot the logarithm of the error against the number of function evaluations and the number of extrapolation steps. Then we will try to fit the points on curve of a function of the form $t \mapsto b - ct^q$ and we will report the mean and relative variance of $A \coloneqq e^b$, $c$ and $q$. We will also consider the plot of the base 10 logarithm of the error against the number of function evaluations.

We conduct the experiments in Python 3 and use the high precision arithmetic library mpmath for all the computations. The precision will be set to 500 significant digits so will not have to worry about numerical instabilities.

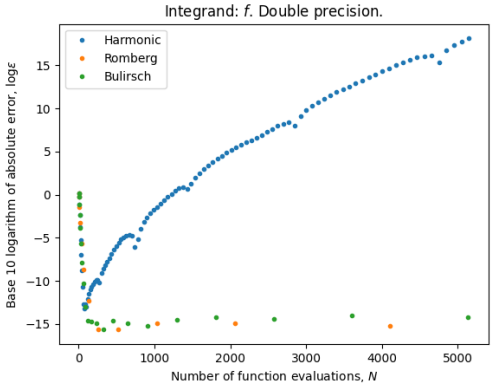
Now we will consider the results of the experiments.

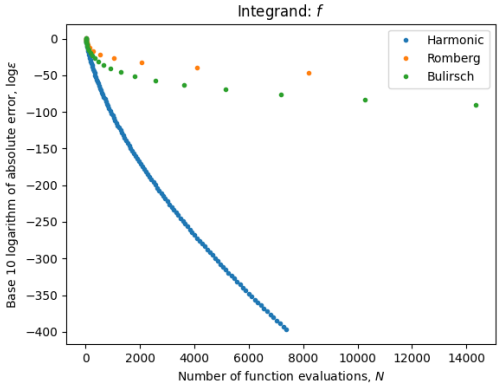
### 2.2.1 Cosine squared

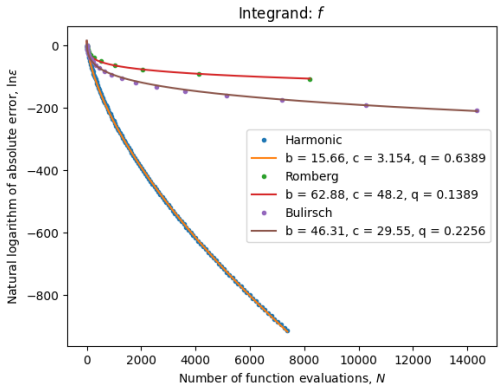
The first function we are going to try is

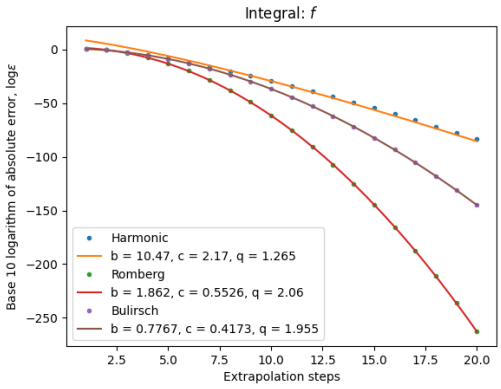
$$f : [0, \pi] \to \mathbb{R}, \quad f(x) \coloneqq \cos^2(x)$$

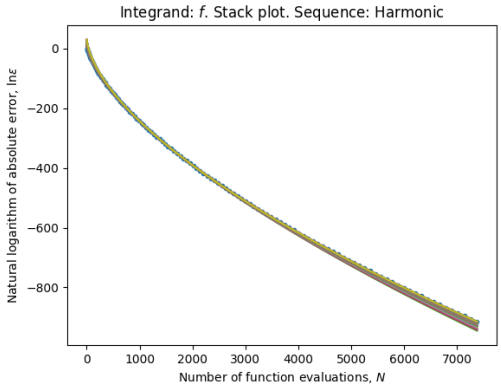
which is entire.

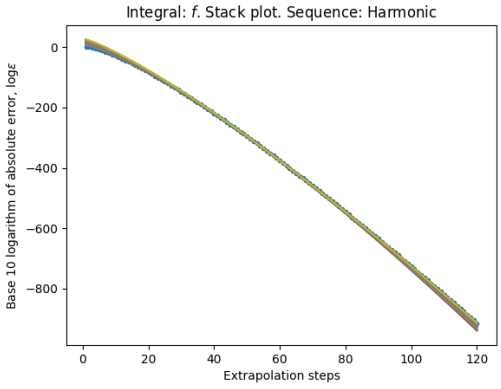


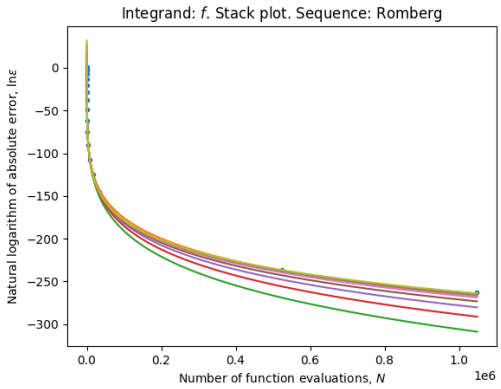



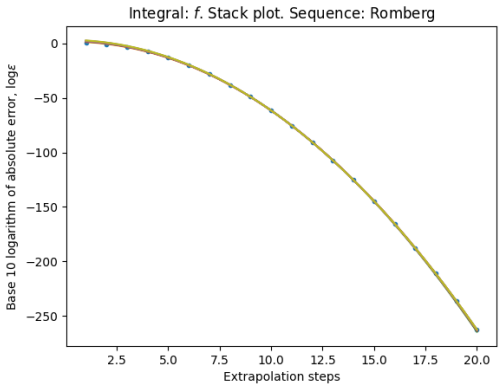

Integrand: $f$
Natural logarithm of absolute error, $\ln\varepsilon$
Number of function evaluations, $N$
Harmonic
b = 15.66, c = 3.154, q = 0.6389
Romberg
b = 62.88, c = 48.2, q = 0.1389
Bulirsch
b = 46.31, c = 29.55, q = 0.2256

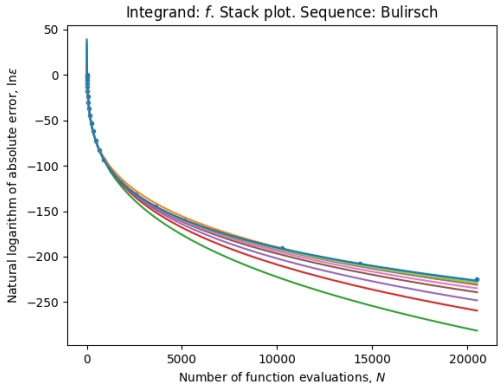

Integral: $f$
Base 10 logarithm of absolute error, $\log\varepsilon$
Extrapolation steps
Harmonic
b = 10.47, c = 2.17, q = 1.265
Romberg
b = 1.862, c = 0.5526, q = 2.06
Bulirsch
b = 0.7767, c = 0.4173, q = 1.955

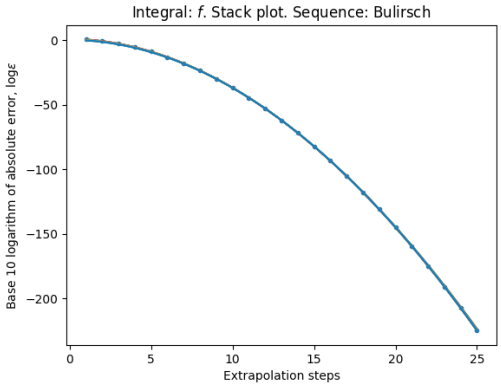

Integrand: $f$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, $\ln\varepsilon$
Number of function evaluations, $N$


Integral: $f$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, $\log\varepsilon$
Extrapolation steps


Integrand: $f$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, $\ln\varepsilon$
Number of function evaluations, $N$
1e6


Integral: $f$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, $\log\varepsilon$
Extrapolation steps


Integrand: $f$. Stack plot. Sequence: Bulirsch
Natural logarithm of absolute error, $\ln\varepsilon$
Number of function evaluations, $N$


Integral: $f$. Stack plot. Sequence: Bulirsch
Base 10 logarithm of absolute error, $\log\varepsilon$
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | $4.426 \cdot 10^{14}$ | 8.674 | 3.524 | 0.01154 | 0.6287 | 0.0004025 |
| Romberg | lin-ln evals-error | $1.853 \cdot 10^{58}$ | 6 | 63.25 | 0.11 | 0.1312 | 0.02905 |
| Bulirsch | lin-ln evals-error | $2.195 \cdot 10^{53}$ | 7.998 | 47.04 | 0.1714 | 0.2034 | 0.04509 |
| Harmonic | lin-ln steps-error | $8.963 \cdot 10^{10}$ | 7.637 | 2.386 | 0.009485 | 1.249 | 0.0002853 |
| Romberg | lin-ln steps-error | 16.31 | 0.1345 | 0.5749 | 0.0008091 | 2.049 | $2.316 \cdot 10^{-5}$ |
| Bulirsch | lin-ln steps-error | 1.658 | 0.06912 | 0.4129 | 0.0006159 | 1.957 | $1.639 \cdot 10^{-5}$ |

We see that the harmonic sequence performes best, then Bulirsch and then Romberg. In standard double precision arithmetic, we get down to machine level precision using Romberg or Bulirsch, but we are like 2 digits from there, using the harmonic sequence.

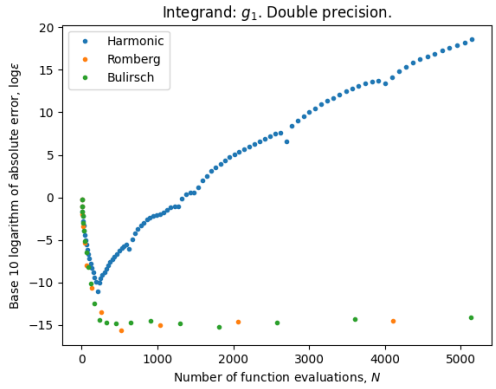
For the Romberg and Bulirsch sequence we clearly have exponential convergence in the number of steps but not in the number of evaluations.

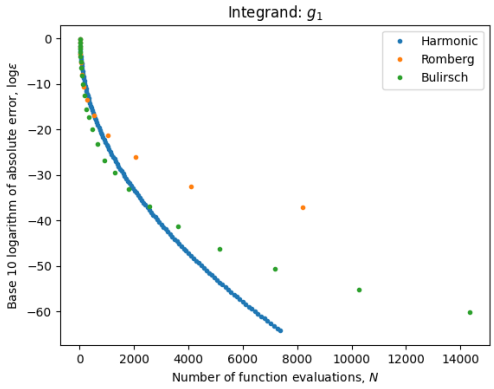
For the harmonic sequence we get unreasonbly large values of $A$ and we can see from the stack plots that the fitting is not so stable.

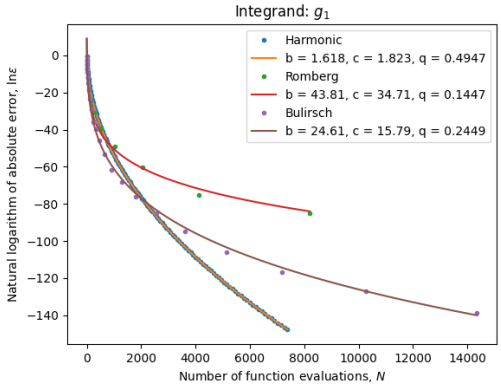
### 2.2.2 Function with poles

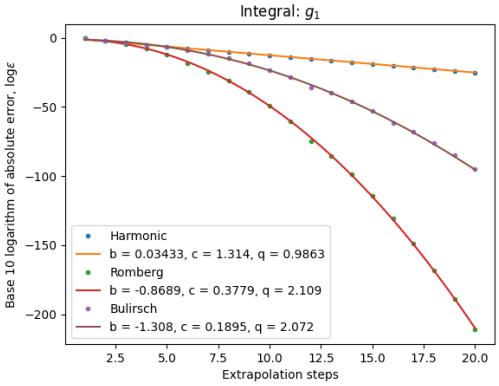
Now we will consider the following function:

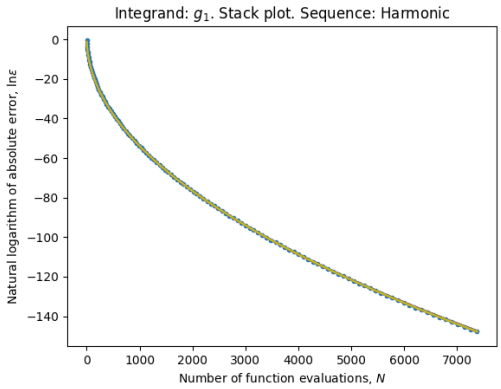
$$g_a : [-1, 1] \to \mathbb{R}, \quad g_a(x) := \frac{1}{a^2 + x^2},\ a > 0$$

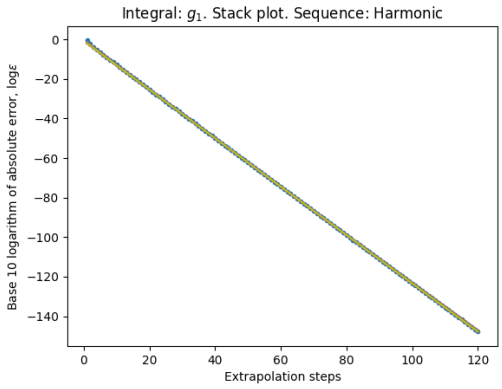


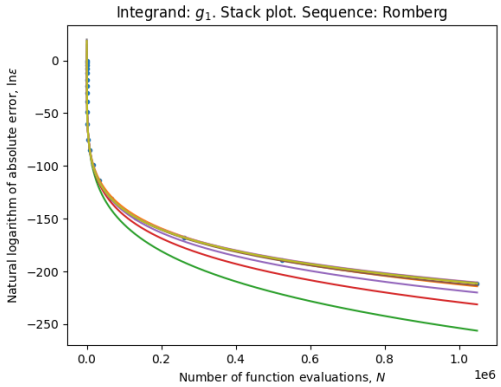



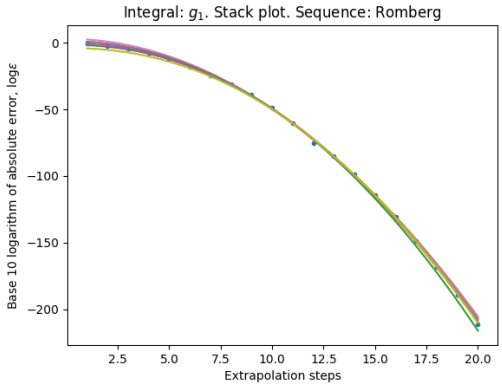



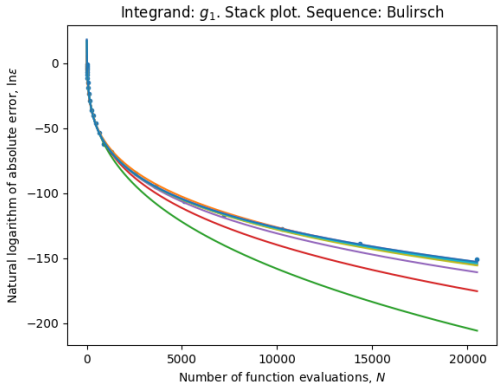



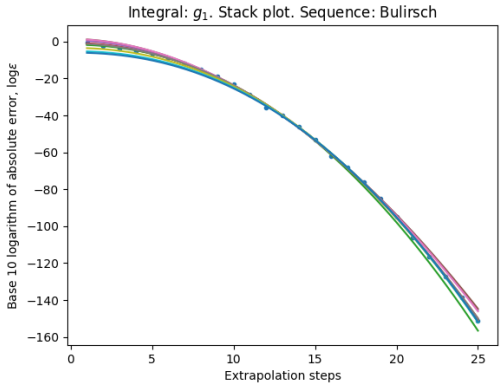


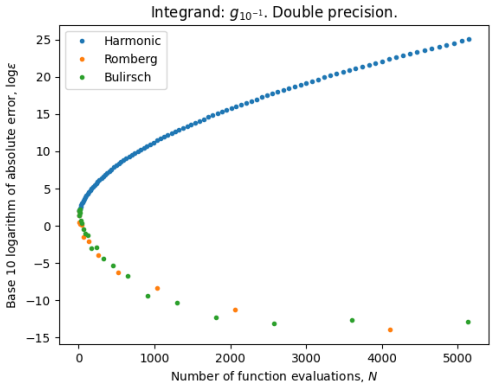



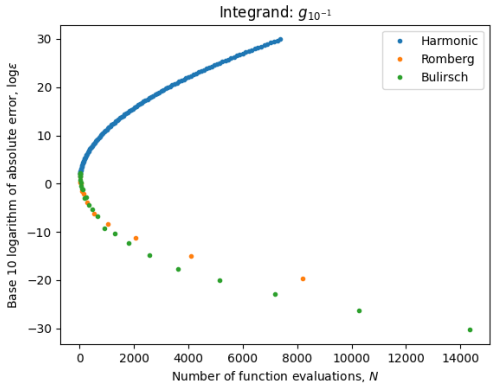



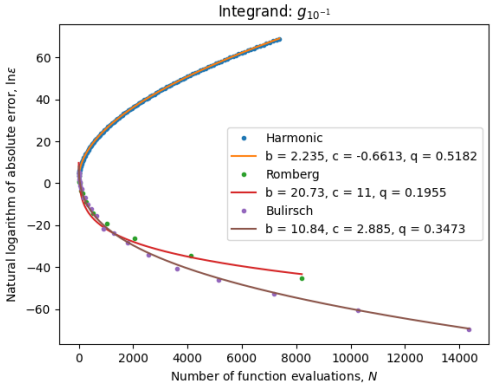

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | 3.771 | 0.02895 | 1.793 | 0.0001375 | 0.4963 | $6.495 \cdot 10^{-6}$ |
| Romberg | lin-ln evals-error | $4.114 \cdot 10^{37}$ | 5.211 | 46.94 | 0.1026 | 0.135 | 0.03489 |
| Bulirsch | lin-ln evals-error | $8.199 \cdot 10^{22}$ | 4.987 | 24.22 | 0.1103 | 0.2236 | 0.04529 |
| Harmonic | lin-ln steps-error | 0.6596 | 0.04548 | 1.277 | 0.0002625 | 0.9918 | $1.093 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 3.747 | 2.921 | 0.4181 | 0.02976 | 2.079 | 0.0008172 |
| Bulirsch | lin-ln steps-error | 1.191 | 1.72 | 0.2048 | 0.105 | 2.062 | 0.00252 |

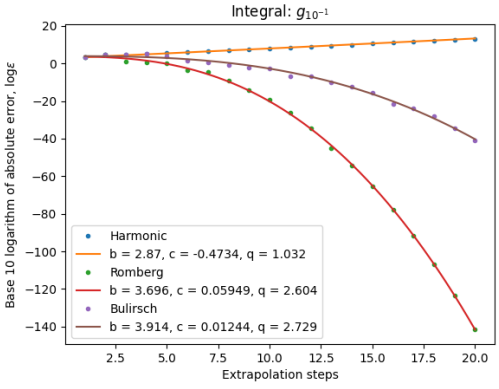
We see that the harmonic sequence performes best, then Bulirsch and then Romberg. In standard double precision arithmetic, we get down to machine level precision using Romberg or Bulirsch, but we are like 5 digits from there, using the harmonic sequence.

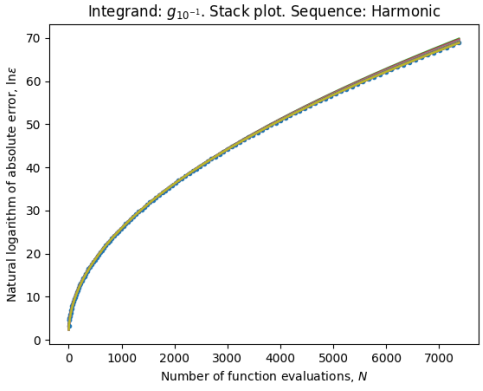
Here we clearly have exponential convergence in the number of evaluations for the Harmonic sequence and hence also in the number of steps. We do not have exponential convergence in the number of evaluations for Romberg and Bulirsch but the model for exponential convergence in the number of steps fits moderately well for those sequences.

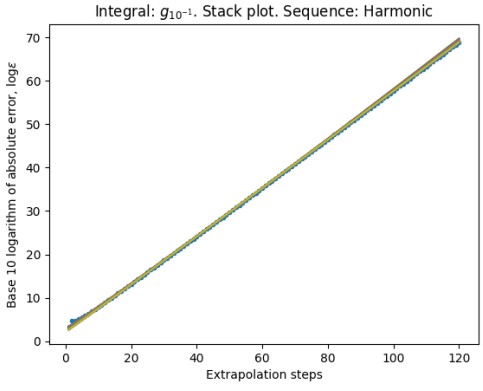


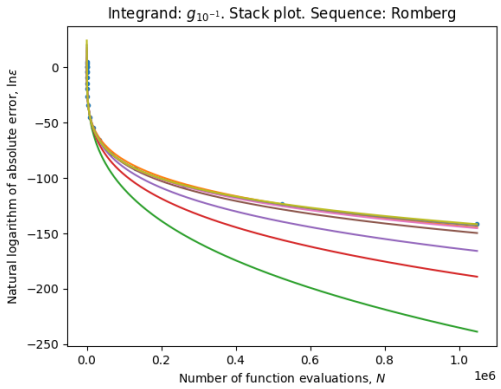



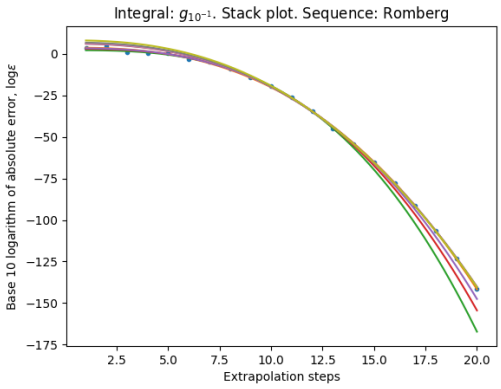



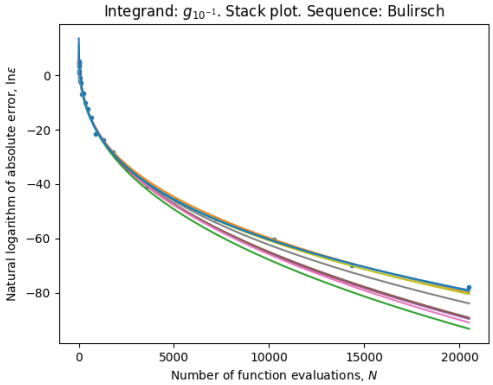



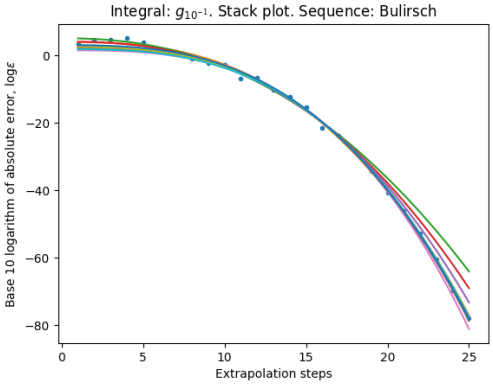





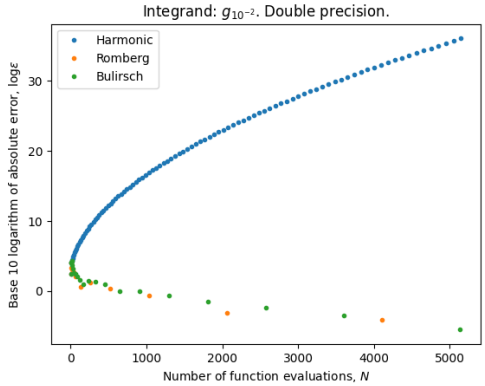



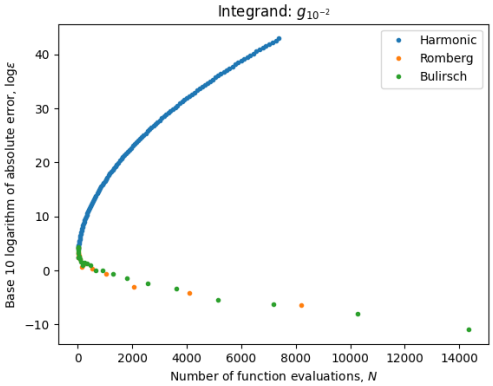



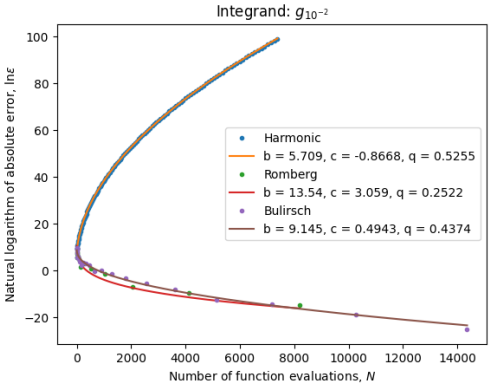



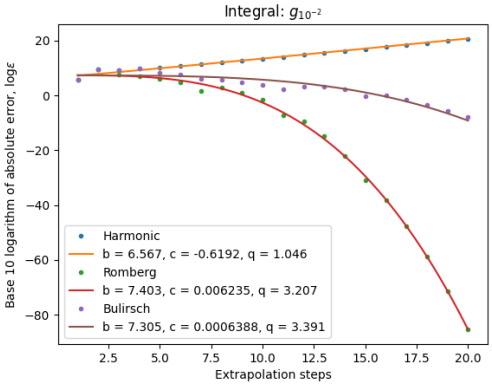

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | 5.231 | 0.0773 | $-0.7178$ | 0.001803 | 0.5101 | $9.026 \cdot 10^{-5}$ |
| Romberg | lin-ln evals-error | $7.533 \cdot 10^{22}$ | 5.999 | 14.35 | 0.3322 | 0.2047 | 0.07855 |
| Bulirsch | lin-ln evals-error | $8.963 \cdot 10^{7}$ | 7.604 | 2.995 | 0.186 | 0.3602 | 0.0129 |
| Harmonic | lin-ln steps-error | 10.97 | 0.05032 | $-0.5086$ | 0.001492 | 1.019 | $6.47 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 727.7 | 2.108 | 0.06624 | 0.1819 | 2.623 | 0.005292 |
| Bulirsch | lin-ln steps-error | 32.46 | 2.026 | 0.01381 | 0.7483 | 2.764 | 0.006506 |

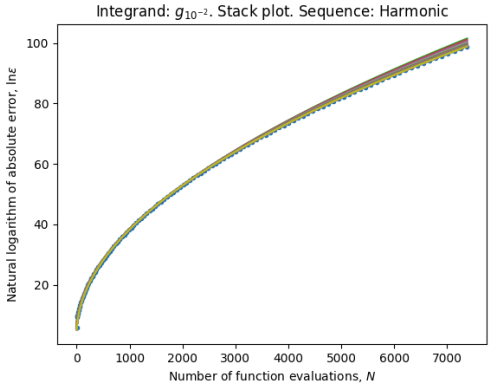
Here we get divergence for the harmonic sequence, but convergence for the other sequences, fastest for Bulirsch. In standard double precision arithmetic, we get down to machine level precision using Romberg or Bulirsch. The model for exponential convergence in number of evaluations does not fit for Bulirsch nor Romberg but it is hard to tell whether we have exponential convergence in the number of steps.

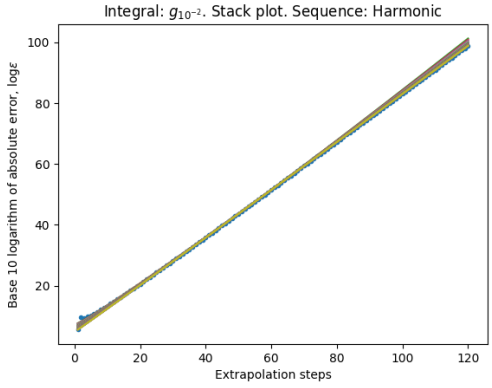


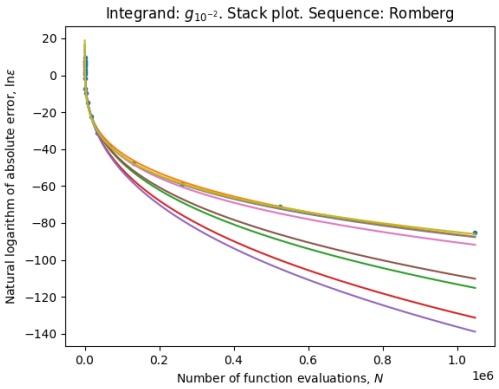



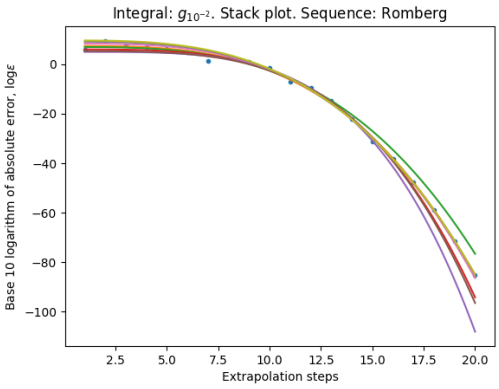

Integrand: $g_{10^{-2}}$
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Harmonic
b = 5.709, c = -0.8668, q = 0.5255
Romberg
b = 13.54, c = 3.059, q = 0.2522
Bulirsch
b = 9.145, c = 0.4943, q = 0.4374

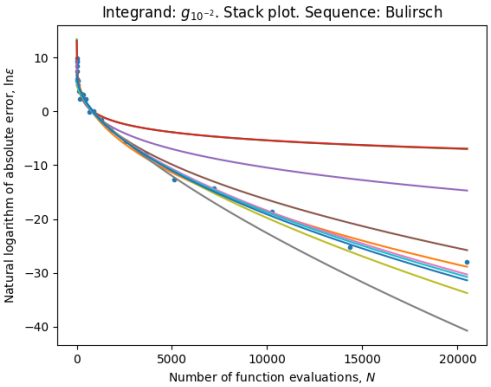

Integral: $g_{10^{-2}}$
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Harmonic
b = 6.567, c = -0.6192, q = 1.046
Romberg
b = 7.403, c = 0.006235, q = 3.207
Bulirsch
b = 7.305, c = 0.0006388, q = 3.391

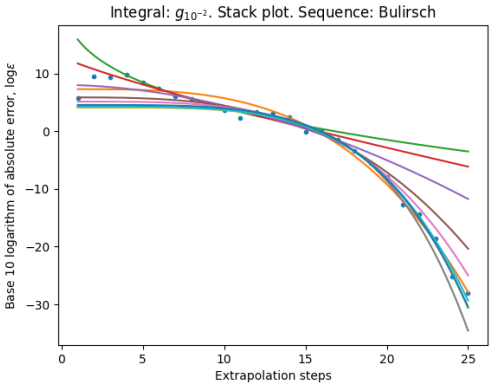

Integrand: $g_{10^{-2}}$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


Integral: $g_{10^{-2}}$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Integrand: $g_{10^{-2}}$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


Integral: $g_{10^{-2}}$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Integrand: $g_{10^{-2}}$. Stack plot. Sequence: Bulirsch
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


Integral: $g_{10^{-2}}$. Stack plot. Sequence: Bulirsch
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | 122.7 | 0.8001 | $-0.9641$ | 0.007205 | 0.5155 | 0.0003682 |
| Romberg | lin-ln evals-error | $8.898 \cdot 10^{10}$ | 5.785 | 3.021 | 0.6343 | 0.2928 | 0.05447 |
| Bulirsch | lin-ln evals-error | . | . | 2876 | 4.874 | 0.4042 | 0.3708 |
| Harmonic | lin-ln steps-error | 323.1 | 0.5753 | $-0.682$ | 0.006766 | 1.029 | 0.0002999 |
| Romberg | lin-ln steps-error | 4384 | 1.505 | 0.005443 | 0.4455 | 3.388 | 0.0103 |
| Bulirsch | lin-ln steps-error | $1.73 \cdot 10^{11}$ | 8 | 1.498 | 6.41 | 3.17 | 0.272 |

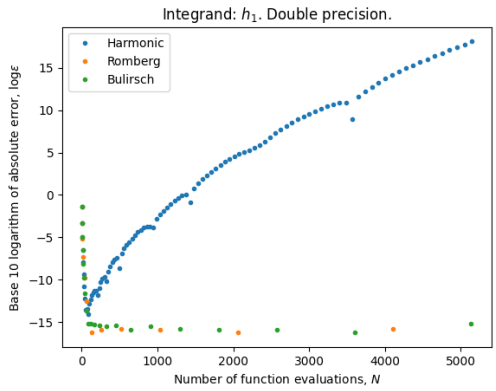
Here the same comments apply as for $a = 10^{-1}$, except that now the Romberg sequence performes better than the Bulirsch sequence and the model fitting is worse.

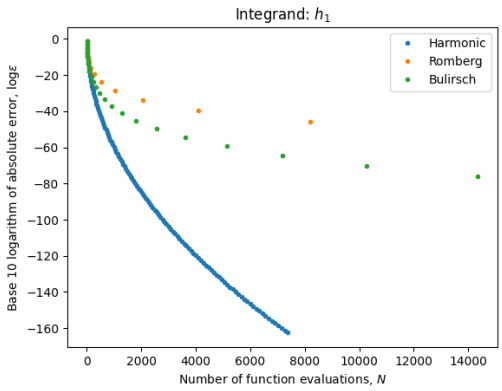
### 2.2.3 Logarithm

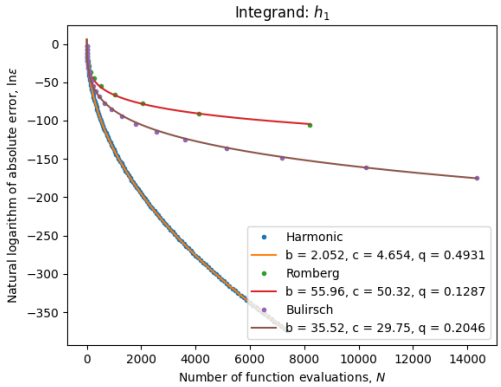
Now we will consider the following function

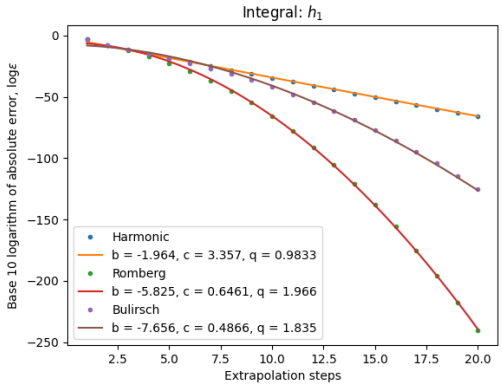
$$h_a : [0, 1] \to \mathbb{R}, \quad h_a(x) := \ln(a + x),\ a > 0.$$

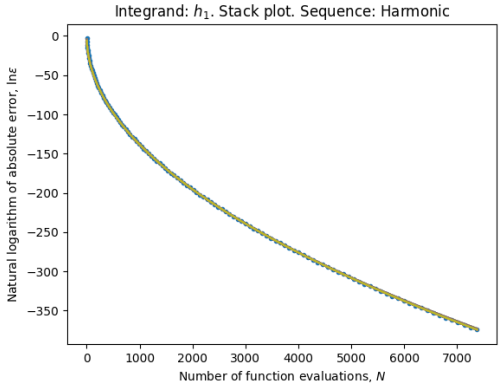
This function is analytic on neighbourhood about the interval but we have a singularity at the horizontal ray from $-a$ to $-\infty$.

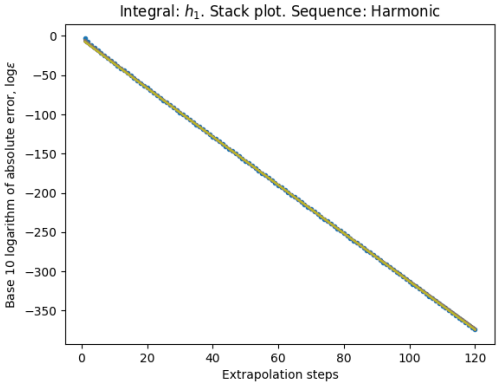


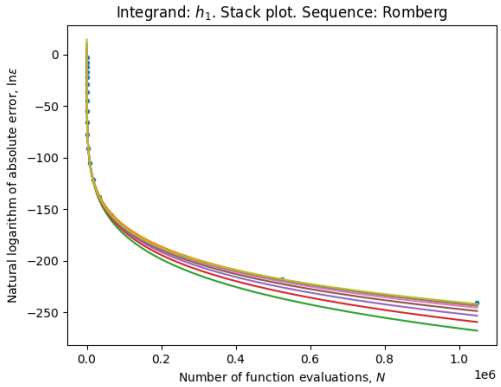



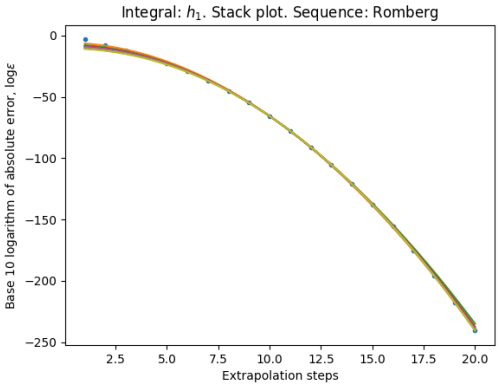



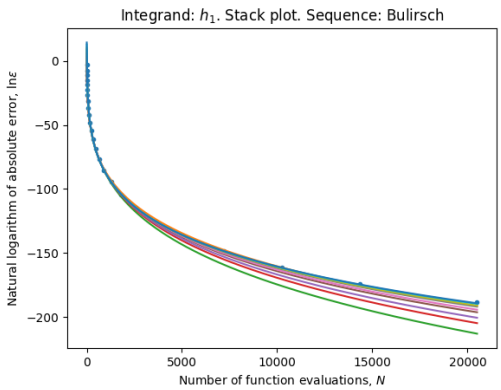



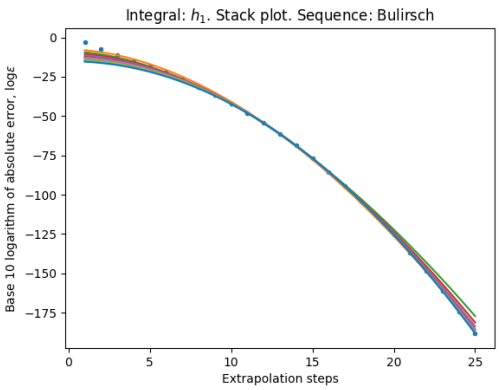


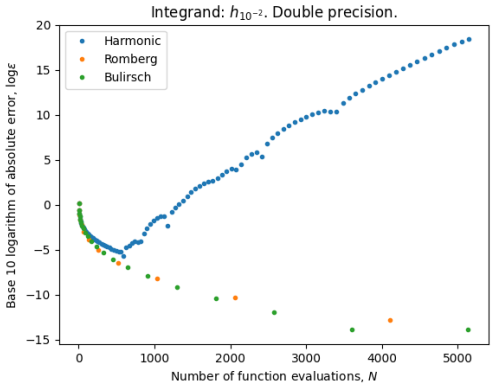



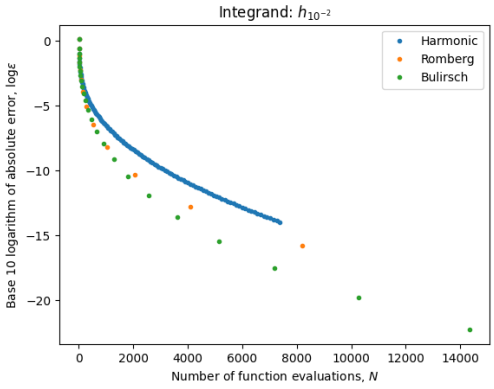



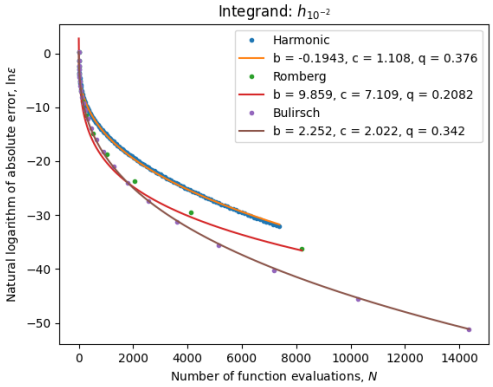

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | 3.072 | 0.3632 | 4.545 | 0.000229 | 0.4954 | $1.085 \cdot 10^{-5}$ |
| Romberg | lin-ln evals-error | $1.538 \cdot 10^{43}$ | 6 | 56.82 | 0.06603 | 0.1269 | 0.0163 |
| Bulirsch | lin-ln evals-error | $1.55 \cdot 10^{31}$ | 7.894 | 36.84 | 0.08131 | 0.1955 | 0.0187 |
| Harmonic | lin-ln steps-error | 0.03919 | 0.5477 | 3.237 | 0.0003865 | 0.9901 | $1.608 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 0.0002063 | 1.214 | 0.5431 | 0.009515 | 2.02 | 0.000263 |
| Bulirsch | lin-ln steps-error | $2.293 \cdot 10^{-5}$ | 3.407 | 0.3579 | 0.06839 | 1.929 | 0.001656 |

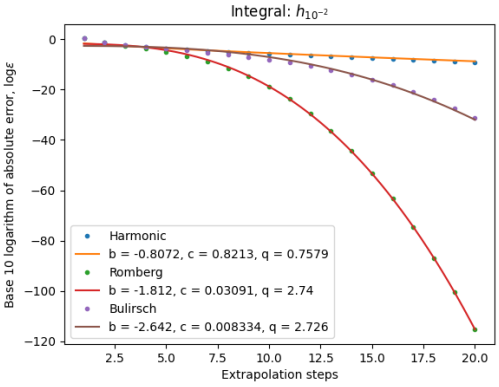
We see that the harmonic sequence performes best, then Bulirsch and then Romberg. In standard double precision arithmetic, we get down to machine level precision using Romberg or Bulirsch, but we are like 2 digits from there, using the harmonic sequence.

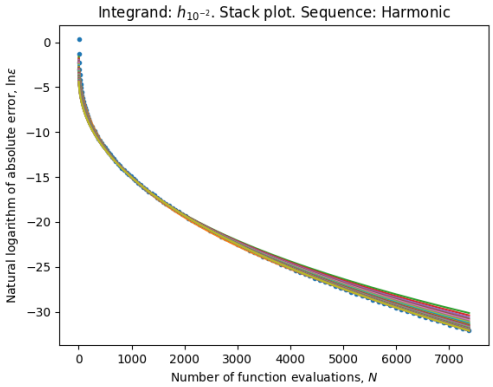
Here we clearly have exponential convergence in the number of evaluations for the harmonic sequence and hence also in the number of steps. For Romberg and Bulirsch we seem to have exponential convergence in the number of steps though the fitting is not as nice as in the case of the harmonic sequence.

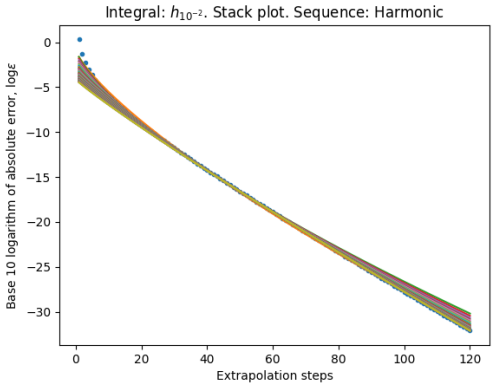


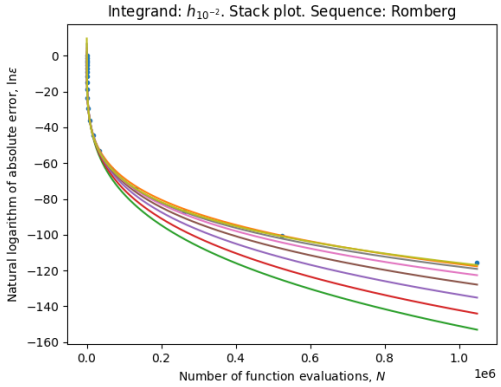



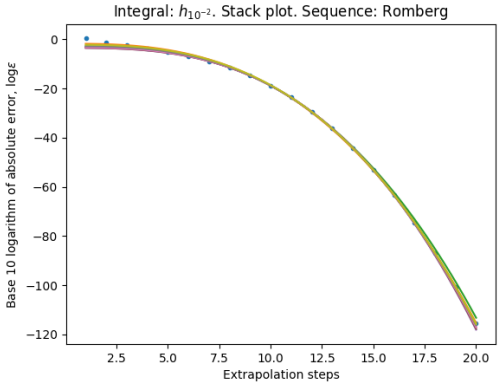



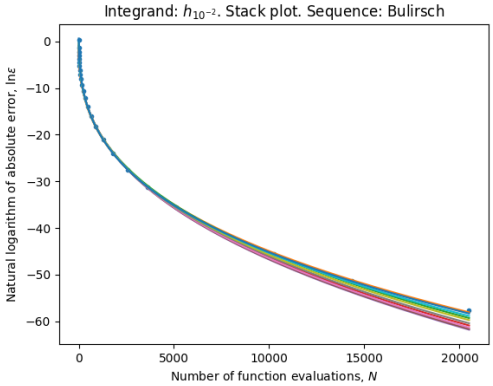



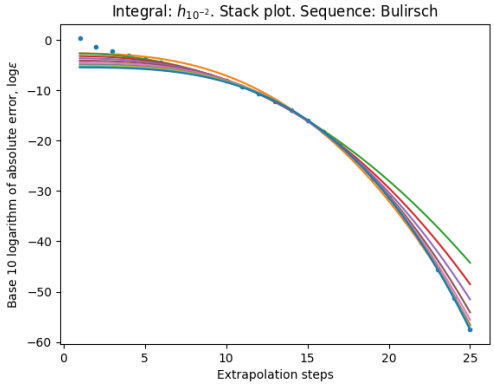





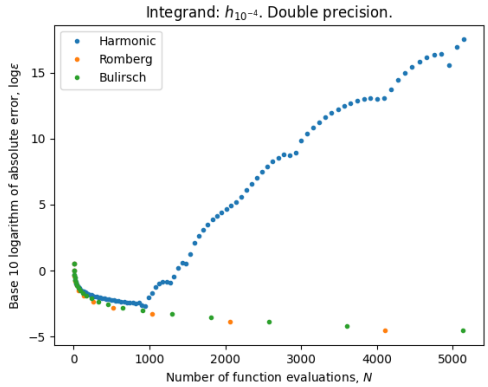



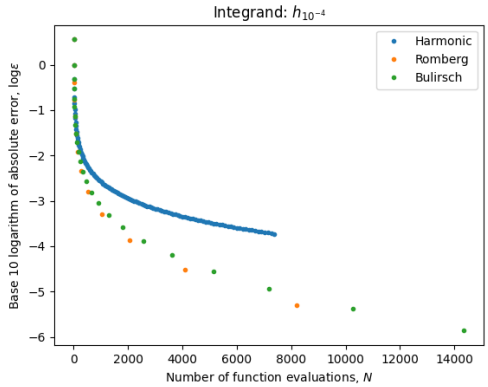



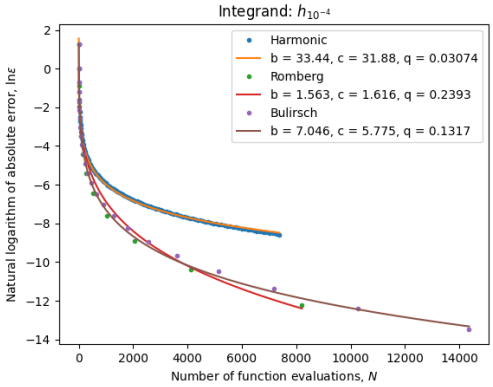

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | 0.1351 | 2.455 | 0.6631 | 0.1084 | 0.4296 | 0.004684 |
| Romberg | lin-ln evals-error | $3.388 \cdot 10^9$ | 5.969 | 6.696 | 0.2315 | 0.2286 | 0.02614 |
| Bulirsch | lin-ln evals-error | 6.859 | 1.956 | 1.626 | 0.0311 | 0.3675 | 0.002076 |
| Harmonic | lin-ln steps-error | 0.07535 | 1.798 | 0.4822 | 0.1106 | 0.8628 | 0.0042 |
| Romberg | lin-ln steps-error | 0.05205 | 0.2618 | 0.02252 | 0.01836 | 2.848 | 0.0002742 |
| Bulirsch | lin-ln steps-error | 0.0214 | 1.016 | 0.00882 | 1.141 | 2.822 | 0.01164 |

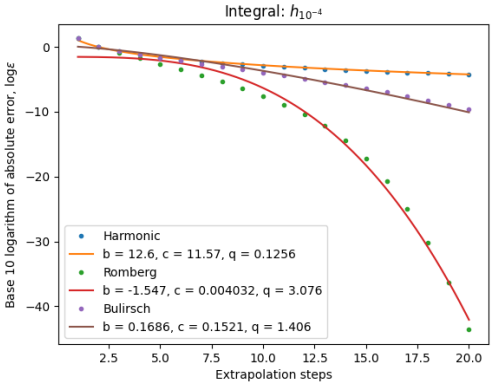
We see that we can not attain high precision using the harmonic sequence and standard double precision. It is hard to tell which sequence performes best in the long run, though we can say that Bulirsch performes better than Romberg.

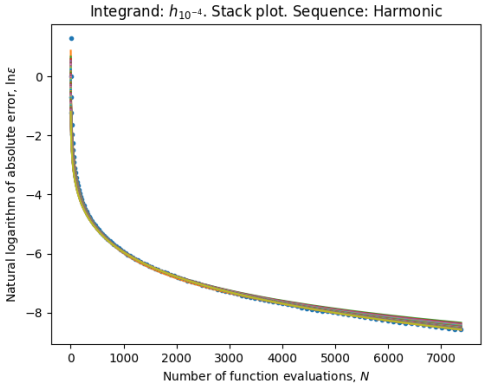
For Romberg, we get a moderately goot fit for exponential convergence in the number of steps. The fitting is quite good for the harmonic sequence but it is quite unstable for Bulirsch.

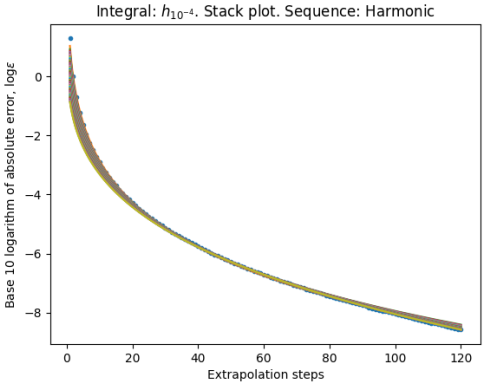


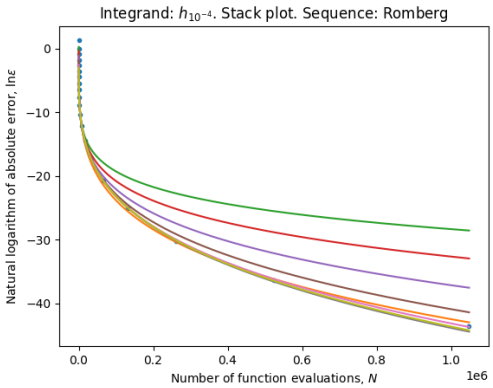



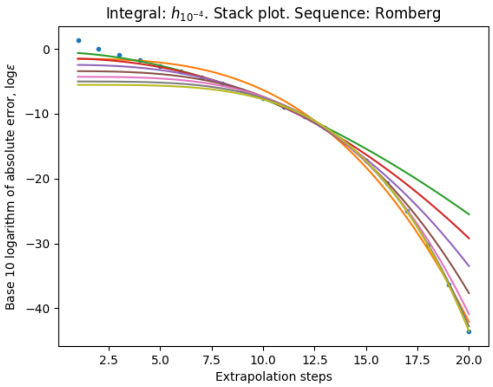

Integrand: $h_{10^{-4}}$
Harmonic
b = 33.44, c = 31.88, q = 0.03074
Romberg
b = 1.563, c = 1.616, q = 0.2393
Bulirsch
b = 7.046, c = 5.775, q = 0.1317
Natural logarithm of absolute error, ln ε
Number of function evaluations, N

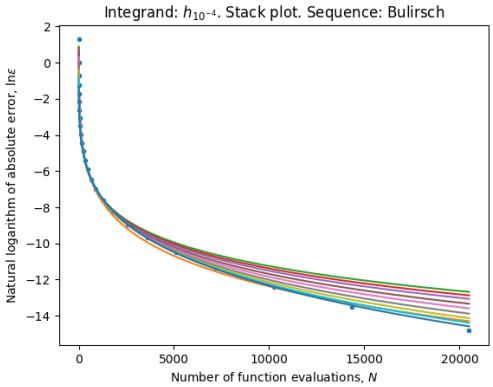

Integral: $h_{10^{-4}}$
Harmonic
b = 12.6, c = 11.57, q = 0.1256
Romberg
b = -1.547, c = 0.004032, q = 3.076
Bulirsch
b = 0.1686, c = 0.1521, q = 1.406
Base 10 logarithm of absolute error, log ε
Extrapolation steps

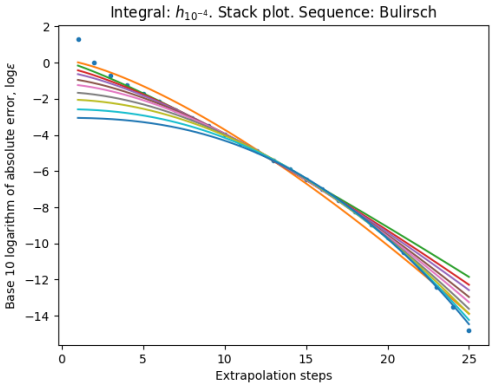

Integrand: $h_{10^{-4}}$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln ε
Number of function evaluations, N


Integral: $h_{10^{-4}}$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log ε
Extrapolation steps


Integrand: $h_{10^{-4}}$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln ε
Number of function evaluations, N
1e6


Integral: $h_{10^{-4}}$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log ε
Extrapolation steps


Integrand: $h_{10^{-4}}$. Stack plot. Sequence: Bulirsch
Natural logarithm of absolute error, ln ε
Number of function evaluations, N


Integral: $h_{10^{-4}}$. Stack plot. Sequence: Bulirsch
Base 10 logarithm of absolute error, log ε
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | $1.328 \cdot 10^{13}$ | 55.82 | 8.14 | 0.5474 | 0.09568 | 0.0936 |
| Romberg | lin-ln evals-error | 106.1 | 5.445 | 2.037 | 0.7267 | 0.2364 | 0.06414 |
| Bulirsch | lin-ln evals-error | $1.888 \cdot 10^{5}$ | 5.399 | 6.394 | 0.3315 | 0.1362 | 0.1274 |
| Harmonic | lin-ln steps-error | $1.395 \cdot 10^{4}$ | 17.13 | 5.261 | 0.2049 | 0.2157 | 0.04738 |
| Romberg | lin-ln steps-error | 0.1362 | 2.091 | 0.02426 | 2.55 | 3.039 | 0.07013 |
| Bulirsch | lin-ln steps-error | 0.4121 | 0.6973 | 0.09479 | 0.8435 | 1.695 | 0.05417 |

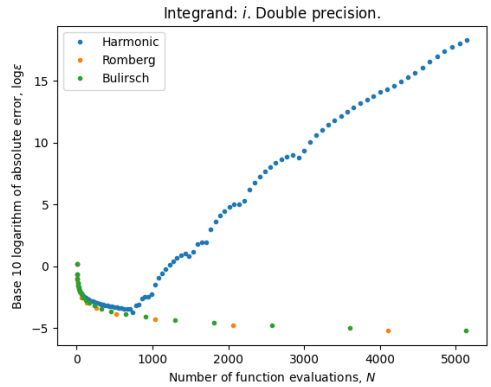
Here again, we do not attain high precision when using the Harmonic sequence in double precision arithmetic. It is hard to say which sequence performes best. We get reasonably good fit for the harmonic sequence but not for Romberg and Bulirsch.

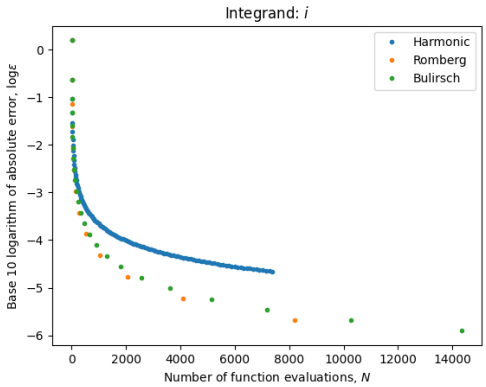
### 2.2.4 Area of half circle

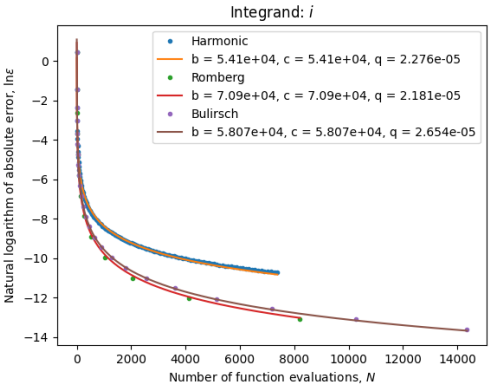
Now we will try the following function:

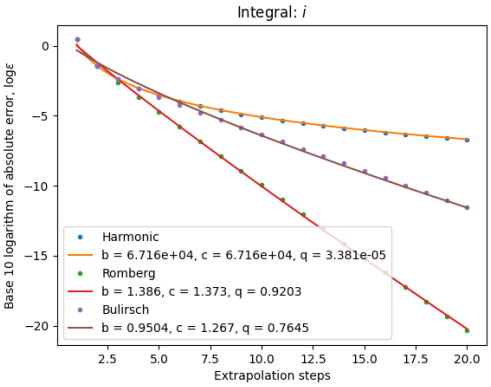
$$i : [-1, 1] \to \mathbb{R}, \quad i(x) := \sqrt{1 - x^2}.$$

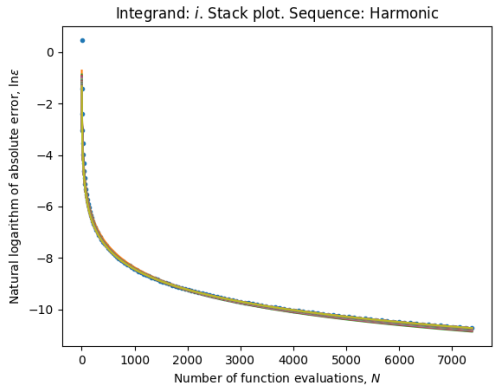
This function is analytic inside the interval of definition but not at the endpoints. Its derivative has singularities at the endpoints.

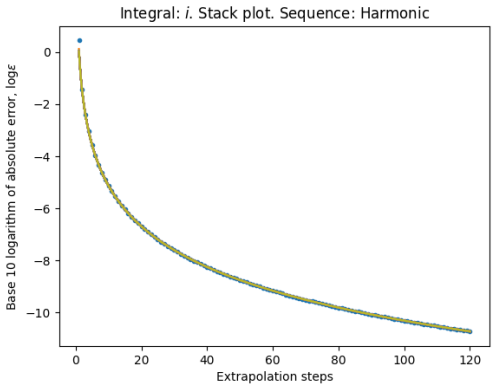


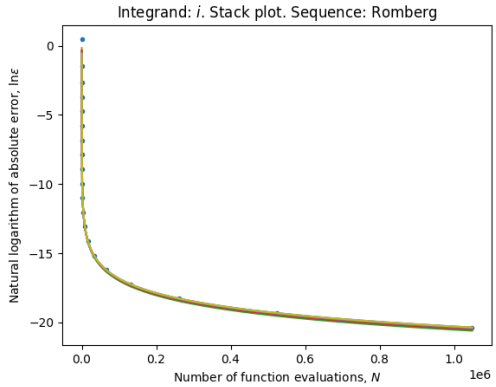



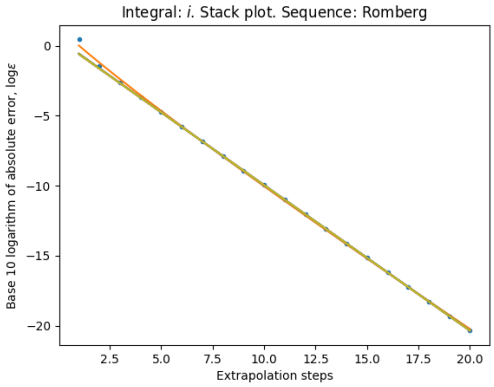



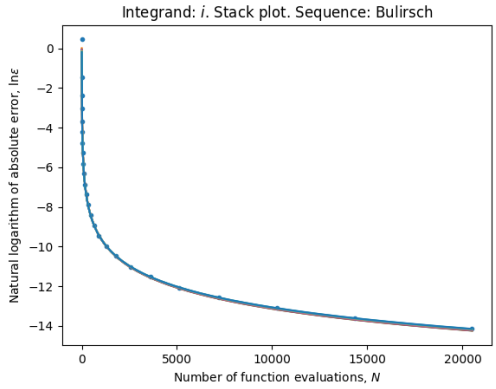



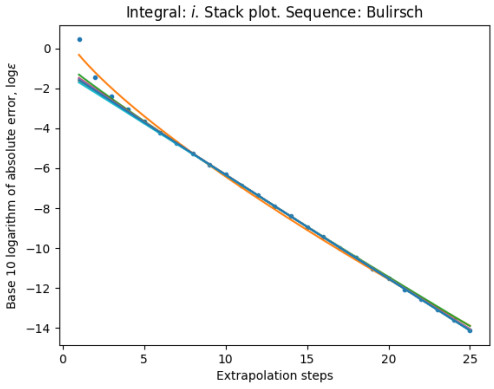



| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | . | . | $5.358 \cdot 10^4$ | 0.003851 | $2.176 \cdot 10^{-5}$ | 0.00346 |
| Romberg | lin-ln evals-error | . | . | $7.348 \cdot 10^4$ | 0.005727 | $2.071 \cdot 10^{-5}$ | 0.005124 |
| Bulirsch | lin-ln evals-error | . | . | $5.763 \cdot 10^4$ | 0.0003158 | $2.647 \cdot 10^{-5}$ | 0.0002409 |
| Harmonic | lin-ln steps-error | . | . | $6.694 \cdot 10^4$ | 0.0002388 | $3.364 \cdot 10^{-5}$ | 0.0002515 |
| Romberg | lin-ln steps-error | 1.581 | 0.0005141 | 1.046 | $5.395 \cdot 10^{-5}$ | 0.9983 | $4.956 \cdot 10^{-6}$ |
| Bulirsch | lin-ln steps-error | 0.375 | 0.03054 | 0.5595 | 0.00836 | 0.9807 | 0.000633 |

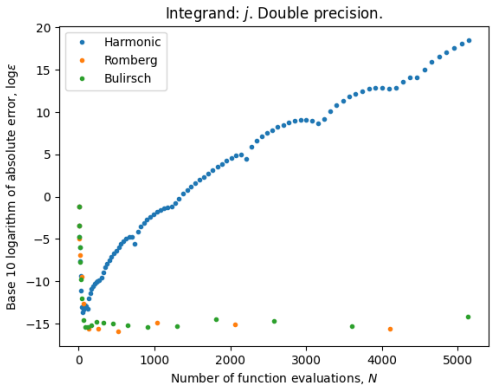
We see that we do not get high precision using double precision arithmetic, independent of sequence. The Romberg and Bulirsch sequence seem to perform similarly well but the harmonic sequence seems to be slowest.

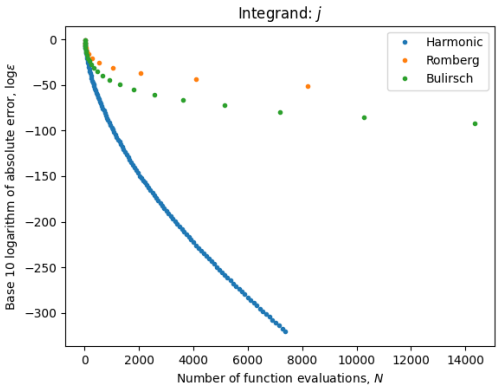
For the harmonic sequence we reject the fitting because we get unreasoable values of the parameters. The model for exponential convergence in the number of steps fits well in the case of the Romberg sequence and Bulirsch.

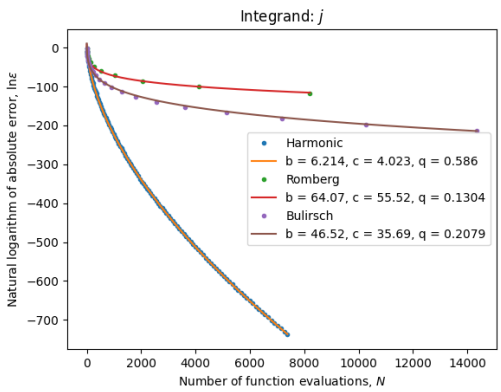
### 2.2.5 Gaussian

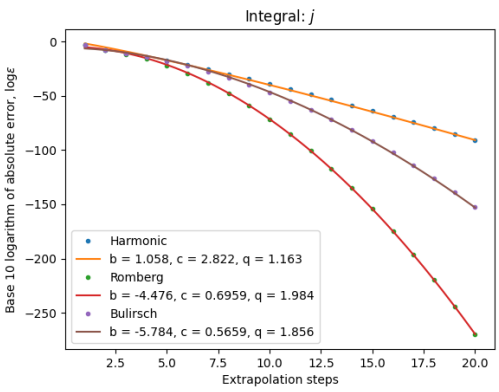
Finally we will consider the Gaussian function

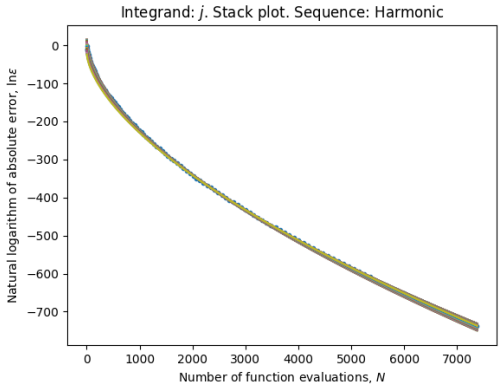
$$j : [0, 1] \to \mathbb{R}, \quad k(x) := \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

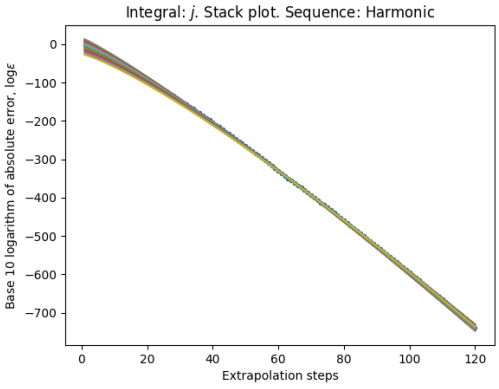


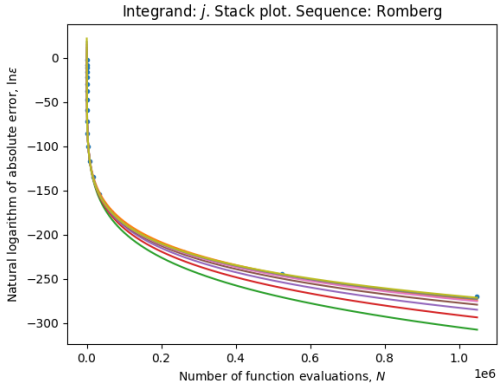



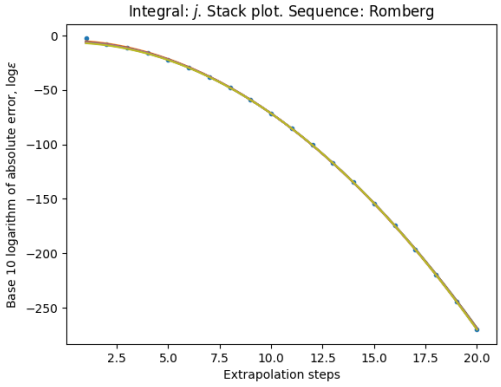



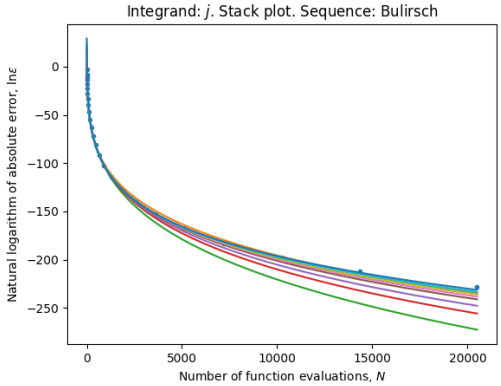



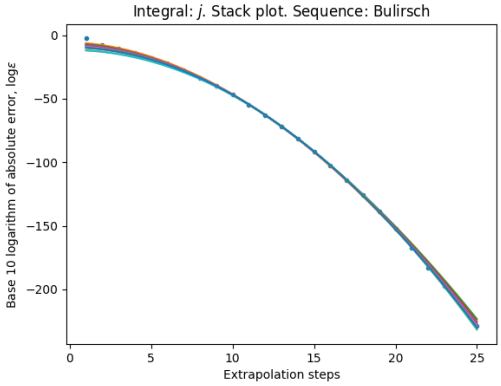






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Harmonic | lin-ln evals-error | $1.037 \cdot 10^{9}$ | 5.523 | 4.257 | 0.02142 | 0.5814 | 0.0007265 |
| Romberg | lin-ln evals-error | $1.398 \cdot 10^{55}$ | 6 | 68.42 | 0.08326 | 0.1248 | 0.02265 |
| Bulirsch | lin-ln evals-error | $1.356 \cdot 10^{51}$ | 8 | 49.55 | 0.1212 | 0.1931 | 0.02978 |
| Harmonic | lin-ln steps-error | $6.922 \cdot 10^{5}$ | 5.362 | 2.939 | 0.02212 | 1.157 | 0.0006628 |
| Romberg | lin-ln steps-error | 0.003928 | 0.1462 | 0.659 | 0.0008259 | 2 | $2.366 \cdot 10^{-5}$ |
| Bulirsch | lin-ln steps-error | 0.0003995 | 1.479 | 0.4704 | 0.02518 | 1.913 | 0.0007081 |

In double precision arithmetic we get down to machine level precision using Romberg or Bulirsch, but we get down to like 2 digits from there, using the harmonic sequence. The harmonic sequence performes best, then Bulirsch and then Romberg.

For the Harmonic sequence it is hard to tell whether we have exponential convergence because we get very large value for $A$. For Romberg we seem to have exponential convergence in the number steps and also for Bulirsch.

| Integrand | Sequence | $b$ | $c$ | $q$ |
|---|---|---|---|---|
| $f$ | Harmonic | 15.66 | 3.1537 | 0.63887 |
| $f$ | Romberg | 62.881 | 48.203 | 0.13887 |
| $f$ | Bulirsch | 46.309 | 29.549 | 0.22556 |
| $g_{10^{-2}}$ | Harmonic | 5.7088 | −0.8668 | 0.52546 |
| $g_{10^{-2}}$ | Romberg | 13.538 | 3.0588 | 0.25221 |
| $g_{10^{-2}}$ | Bulirsch | 9.1445 | 0.49433 | 0.43743 |
| $g_{10^{-1}}$ | Harmonic | 2.2352 | −0.66129 | 0.51817 |
| $g_{10^{-1}}$ | Romberg | 20.725 | 10.998 | 0.19552 |
| $g_{10^{-1}}$ | Bulirsch | 10.844 | 2.8849 | 0.34731 |
| $g_1$ | Harmonic | 1.6178 | 1.823 | 0.49467 |
| $g_1$ | Romberg | 43.807 | 34.715 | 0.14465 |
| $g_1$ | Bulirsch | 24.613 | 15.795 | 0.24492 |
| $h_{10^{-4}}$ | Harmonic | 33.436 | 31.879 | 0.030738 |
| $h_{10^{-4}}$ | Romberg | 1.5634 | 1.6161 | 0.23927 |
| $h_{10^{-4}}$ | Bulirsch | 7.0462 | 5.7755 | 0.13169 |
| $h_{10^{-2}}$ | Harmonic | −0.19426 | 1.1078 | 0.37602 |
| $h_{10^{-2}}$ | Romberg | 9.8594 | 7.1092 | 0.20823 |
| $h_{10^{-2}}$ | Bulirsch | 2.2519 | 2.0217 | 0.34203 |
| $h_1$ | Harmonic | 2.052 | 4.6543 | 0.4931 |
| $h_1$ | Romberg | 55.957 | 50.318 | 0.1287 |
| $h_1$ | Bulirsch | 35.525 | 29.752 | 0.20461 |
| $i$ | Harmonic | 54099 | 54099 | $2.2756 \cdot 10^{-5}$ |
| $i$ | Romberg | 70897 | 70896 | $2.1811 \cdot 10^{-5}$ |
| $i$ | Bulirsch | 58074 | 58073 | $2.6538 \cdot 10^{-5}$ |
| $j$ | Harmonic | 6.2138 | 4.0228 | 0.58595 |
| $j$ | Romberg | 64.075 | 55.521 | 0.13037 |
| $j$ | Bulirsch | 46.521 | 35.69 | 0.20788 |

Table 2.1: Optimal parameters by test case

The values of the optimal parameters in the curve fitting of extrapolation steps against the logarithm of the error are:

| Integrand | Sequence | $b$ | $c$ | $q$ |
|---|---|---|---|---|
| $f$ | Harmonic | 10.466 | 2.1696 | 1.2654 |
| $f$ | Romberg | 1.8624 | 0.55261 | 2.0602 |
| $f$ | Bulirsch | 0.77673 | 0.41734 | 1.9549 |
| $g_{10^{-2}}$ | Harmonic | 6.5675 | −0.61916 | 1.0458 |
| $g_{10^{-2}}$ | Romberg | 7.403 | 0.006235 | 3.2075 |
| $g_{10^{-2}}$ | Bulirsch | 7.3047 | 0.00063882 | 3.3913 |
| $g_{10^{-1}}$ | Harmonic | 2.8699 | −0.47343 | 1.0317 |
| $g_{10^{-1}}$ | Romberg | 3.6964 | 0.059489 | 2.6042 |
| $g_{10^{-1}}$ | Bulirsch | 3.9142 | 0.012441 | 2.7293 |
| $g_1$ | Harmonic | 0.034332 | 1.3144 | 0.98632 |
| $g_1$ | Romberg | −0.86887 | 0.37786 | 2.1086 |
| $g_1$ | Bulirsch | −1.3077 | 0.18952 | 2.0725 |
| $h_{10^{-4}}$ | Harmonic | 12.604 | 11.571 | 0.12559 |
| $h_{10^{-4}}$ | Romberg | −1.5468 | 0.0040324 | 3.0761 |
| $h_{10^{-4}}$ | Bulirsch | 0.16861 | 0.15206 | 1.4061 |
| $h_{10^{-2}}$ | Harmonic | −0.80722 | 0.82135 | 0.75792 |
| $h_{10^{-2}}$ | Romberg | −1.8118 | 0.030909 | 2.74 |
| $h_{10^{-2}}$ | Bulirsch | −2.6424 | 0.0083341 | 2.7259 |
| $h_1$ | Harmonic | −1.9642 | 3.3575 | 0.98328 |
| $h_1$ | Romberg | −5.8254 | 0.64611 | 1.9664 |
| $h_1$ | Bulirsch | −7.6558 | 0.48664 | 1.8348 |
| $i$ | Harmonic | 67160 | 67160 | $3.3808 \cdot 10^{-5}$ |
| $i$ | Romberg | 1.3857 | 1.3726 | 0.92029 |
| $i$ | Bulirsch | 0.95043 | 1.2669 | 0.7645 |
| $j$ | Harmonic | 1.0579 | 2.8215 | 1.1626 |
| $j$ | Romberg | −4.4762 | 0.69593 | 1.9838 |
| $j$ | Bulirsch | −5.7837 | 0.56594 | 1.8564 |

Table 2.2: Optimal parameters by test case

# Chapter 3

# Extrapolation of difference quotients

## 3.1 The algorithm

Let $a \in \mathbb{R}$, $\varepsilon > 0$ and $f : ]a-\varepsilon, a+\varepsilon[ \to \mathbb{R}$ be differentiable at $a$. We are interested in estimating $f'(a)$. Assume that $f$ is $2k+1$ times differentiable at $a$. Then by Taylor's theorem we have

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \cdots + \frac{f^{(2k)}(a)}{(2k)!}h^{2k} + \frac{f^{(2k+1)}(\xi)}{(2k+1)!}h^{2k+1} \tag{3.1}$$

where $a < \xi < a+h$. Now plug $-h$ instead of $h$ in (3.1):

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{2}h^2 - \cdots + \frac{f^{(2k)}(a)}{(2k)!}h^{2k} - \frac{f^{(2k+1)}(\eta)}{(2k+1)!}h^{2k+1} \tag{3.2}$$

where $a-h < \eta < a$. If we subtract (3.2) from (3.1) and divide by $2h$ we get:

$$f'(a) = D_f(h) + \frac{f'''(a)}{3!}h^2 + \cdots + \frac{f^{(2k-1)}(a)}{(2k-1)!}h^{2k-2} + \frac{f^{(2k+1)}(\xi) + f^{(2k+1)}(\eta)}{2\cdot(2k+1)!}h^{2k} \tag{3.3}$$

where

$$D_f(h) \coloneqq \frac{f(a+h) - f(a-h)}{2h} \tag{3.4}$$

is the *symmetric difference quotient* of $f$ at $a$. Note that $\frac{1}{2}(f^{(2k+1)}(\xi) + f^{(2k+1)}(\eta))$ is in the image of $f^{(2k+1)}$ so we can rewrite (3.3) as

$$f'(a) = D_f(h) + \frac{f'''(a)}{3!}h^2 + \cdots + \frac{f^{(2k-1)}(a)}{(2k-1)!}h^{2k-2} + \frac{f^{(2k+1)}(\zeta)}{(2k+1)!}h^{2k} \tag{3.5}$$

where $a-h < \zeta < a+h$. Formula (3.5) tells us that the symmetric difference quotient method has asymptotic expansion in $h^2$ of order $2k-2$ if $f$ is $2k+1$ times differentiable. Thus we can use the following scheme to extrapolate the symmetric difference quotient method:

1. $D_{i1} \coloneqq D_f(h_i)$ for $i = 1, \ldots, k$.

2. $D_{ij} \coloneqq D_{i,j-1} + \frac{D_{i,j-1} - D_{i-1,j-1}}{\left(\frac{h_{i-j+1}}{h_i}\right)^2 - 1}$ for $2 \le j \le i$.

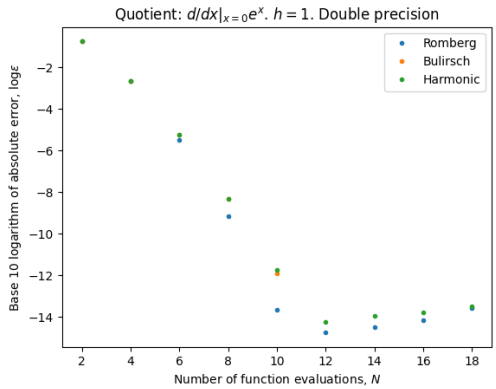
## 3.2 Numerical experiments

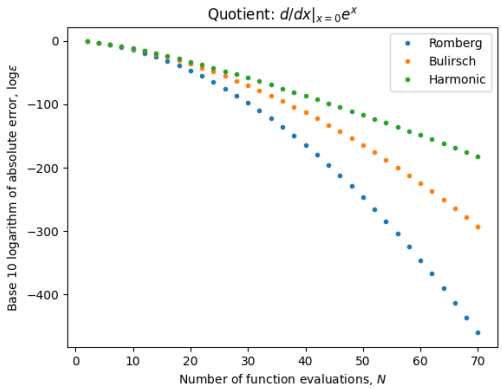
In this section we are going to extrapolate the symmetric difference quotient for approximating the derivative of a function at a given point. Let $h > 0$ be some number, $f : ]a-\varepsilon, a+\varepsilon[ \to \mathbb{R}$ a function differentiable at $a$ and $n_1 < n_2 < \cdots$ a sequence of integers. Let $h_i := h/n_i$. Let $D_{ij}$ be the extrapolation table that we get from extrapolating in $h^2$ using the points $(h_1^2, D_f(h_1)), (h_2^2, D_f(h_2)), \ldots$, as we described in the first chapter. We let $\varepsilon_i := |X_{ii} - f'(a)|$. We want to analyze how $\varepsilon_i$ as $i$ increases and we also want to do similar efficiency analyzis as in the chapter on Romberg quadrature and check whether we have exponential convergence. We will do the computations with precision up to 500 significant digits and also using standard double precision arithmetic.

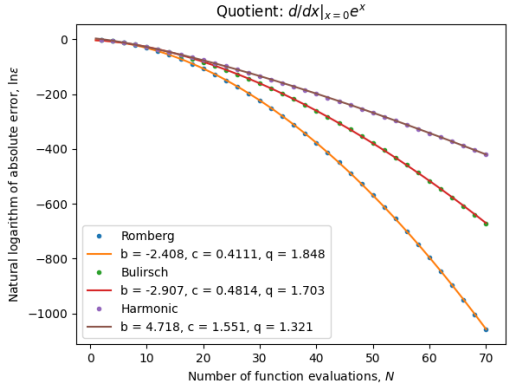
Now we will consider the results of the experiments.

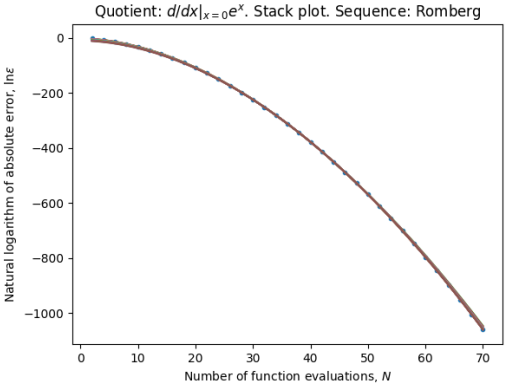
### 3.2.1 The exponential function

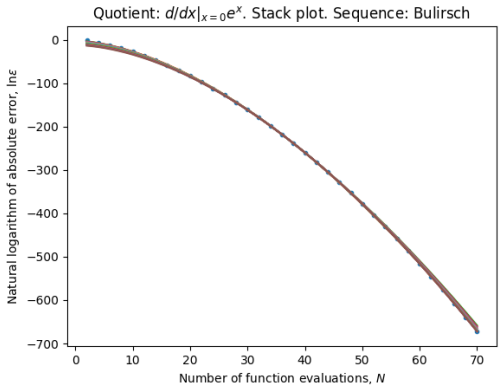
We begin by considering the derivative of the exponential function at zero.

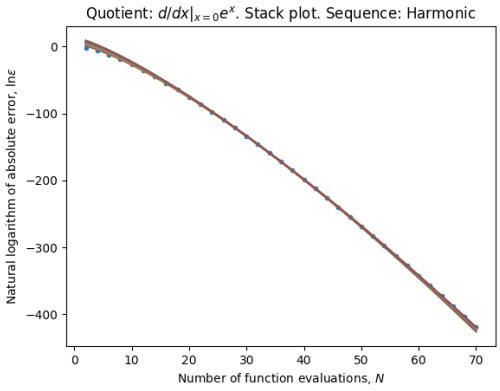




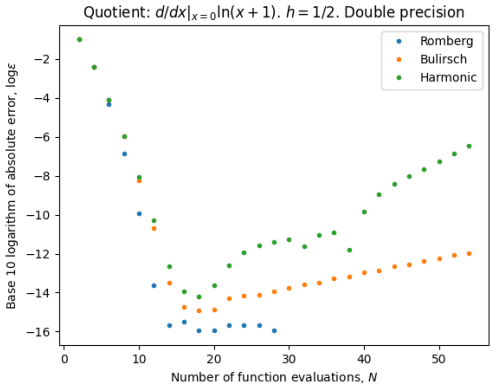

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 0.03687 | 2.584 | 0.4004 | 0.004019 | 1.853 | $6.702 \cdot 10^{-5}$ |
| Bulirsch | lin-ln evals-error | 0.01953 | 3.508 | 0.4544 | 0.0119 | 1.716 | 0.0002258 |
| Harmonic | lin-ln evals-error | $1.122 \cdot 10^{5}$ | 3.293 | 1.732 | 0.007202 | 1.299 | 0.0002359 |

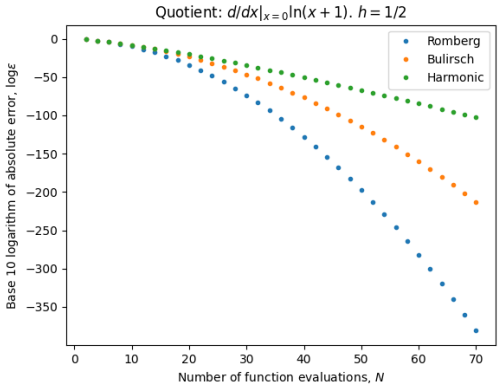
In standard floating point arithmetic, we get down to machine level precision using any sequence. The Romberg sequence works best, then Bulirsch and then the harmonic. The model seems to fit well in all cases.

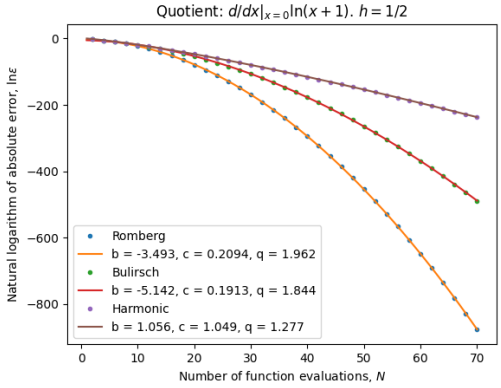
### 3.2.2 Logarithm

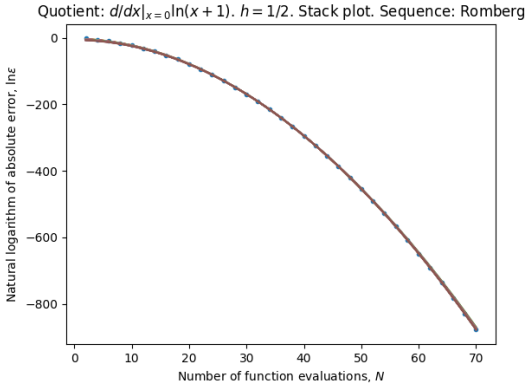
Now we will consider the dervative at zero of the function

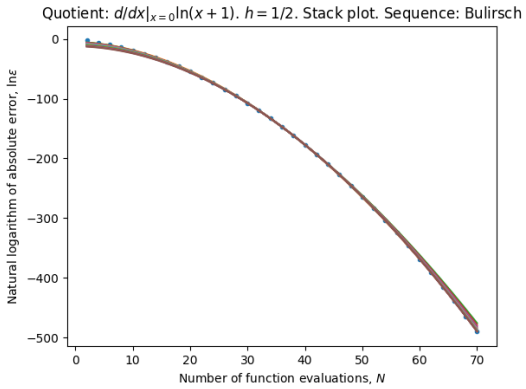
$$g(x) := \ln(x+1).$$

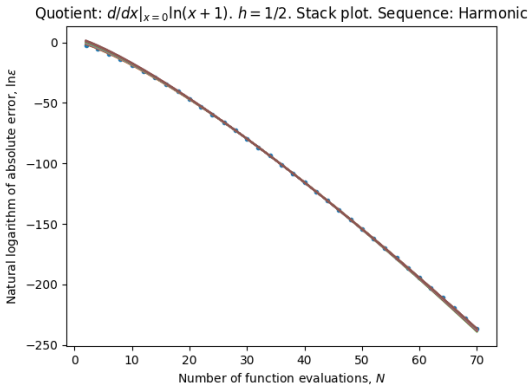




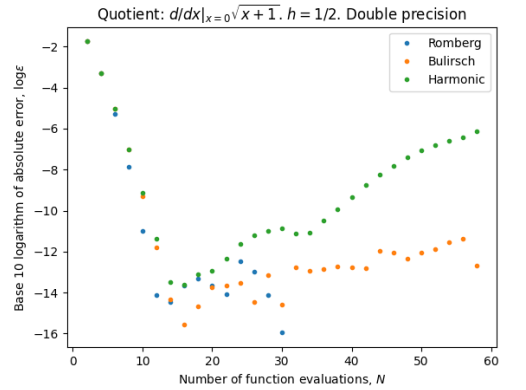

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 0.009098 | 0.9823 | 0.2041 | 0.001354 | 1.967 | $2.052 \cdot 10^{-5}$ |
| Bulirsch | lin-ln evals-error | 0.001179 | 3.144 | 0.1757 | 0.01703 | 1.863 | 0.0002721 |
| Harmonic | lin-ln evals-error | 27.62 | 0.9622 | 1.126 | 0.003487 | 1.263 | 0.0001174 |

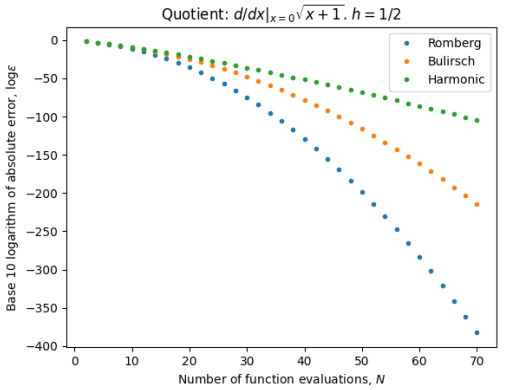
We get down to machine level precision using any sequence, Romberg performes best, then Bulirsch and then the Harmonic one. The model fits well in all cases.

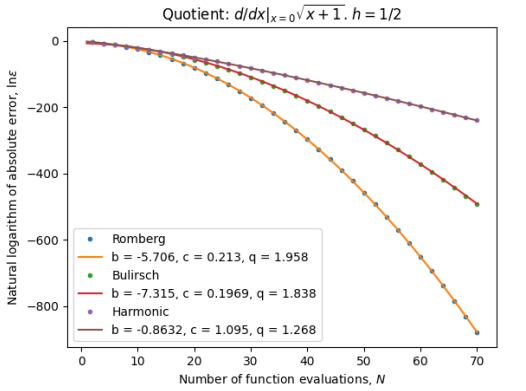
### 3.2.3 Square root

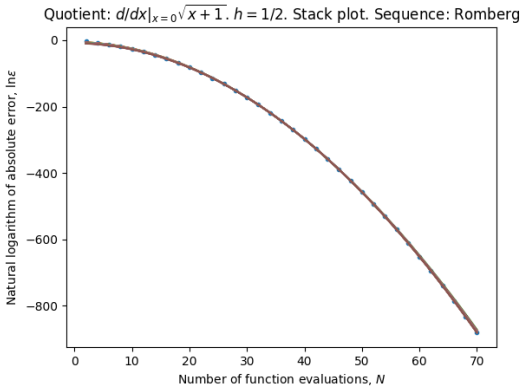
Now we shall consider the derivative at zero of the following function:

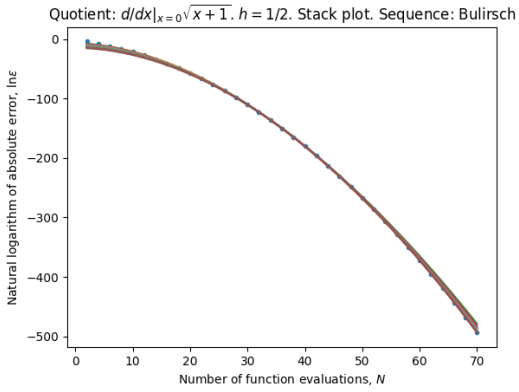
$$h(x) := \sqrt{1+x}$$

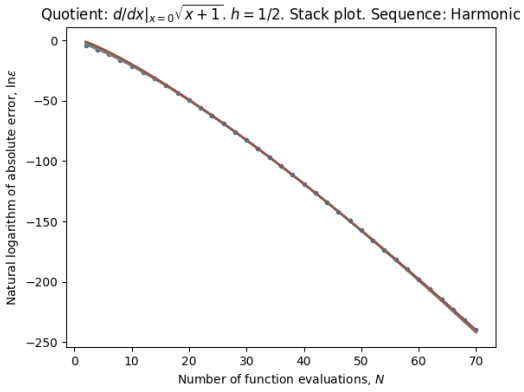





| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 0.0008261 | 1.263 | 0.2064 | 0.001755 | 1.965 | $2.659 \cdot 10^{-5}$ |
| Bulirsch | lin-ln evals-error | 0.0001188 | 3.504 | 0.1792 | 0.01927 | 1.859 | 0.0003075 |
| Harmonic | lin-ln evals-error | 2.598 | 0.8154 | 1.155 | 0.002659 | 1.257 | $8.929 \cdot 10^{-5}$ |

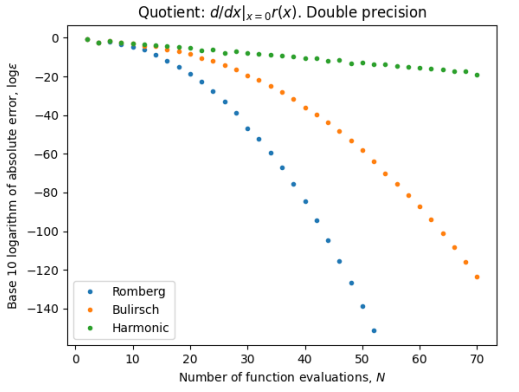
In standard double precision floating point arithmetic we get down to machine level precision using any sequence. The model fits well in all cases.

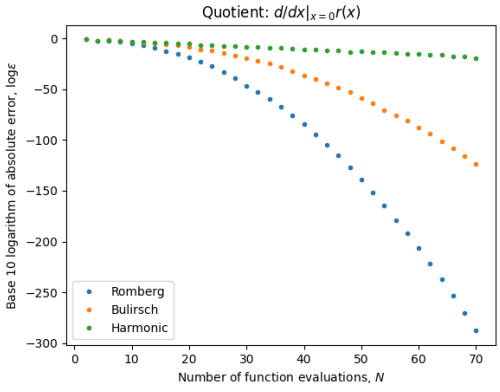
### 3.2.4 Smooth but not analytic function

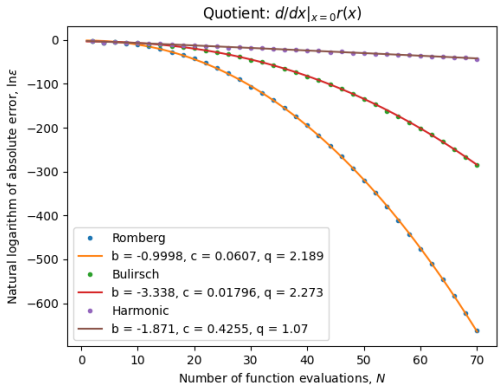
Now we will consider the derivate at zero of the following function:

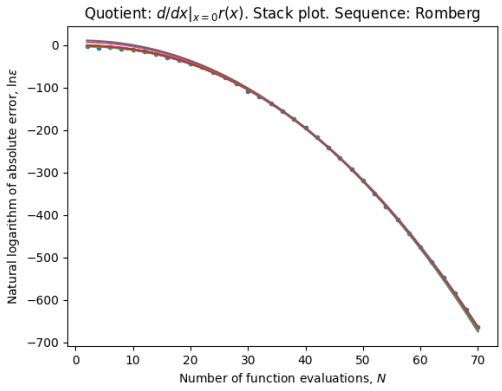
$$r(x) := \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

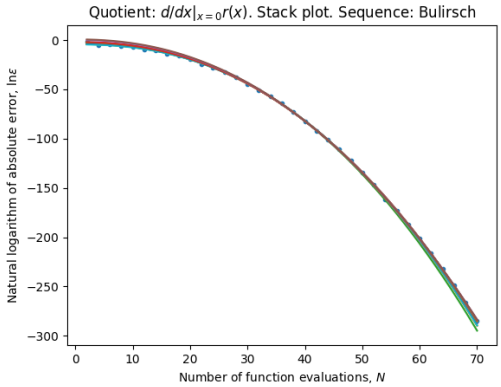
which is smooth but not analytic.

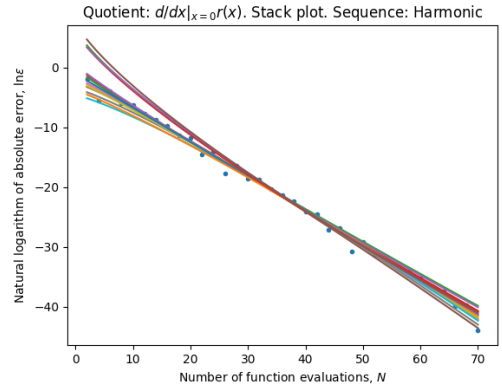





| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 7305 | 12.55 | 0.06014 | 0.02339 | 2.195 | 0.0002104 |
| Bulirsch | lin-ln evals-error | 0.2728 | 4.406 | 0.0178 | 0.03004 | 2.281 | 0.000316 |
| Harmonic | lin-ln evals-error | 287.5 | 4.85 | 0.818 | 0.4753 | 0.9831 | 0.02252 |

Romberg performes best and the harmonic sequence worst.

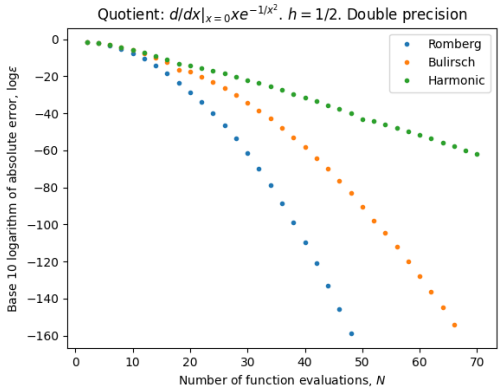
The model seems to fit reasonably well for the Romberg and Bulirsch sequence but the fitting is not so nice for the harmonic sequence.

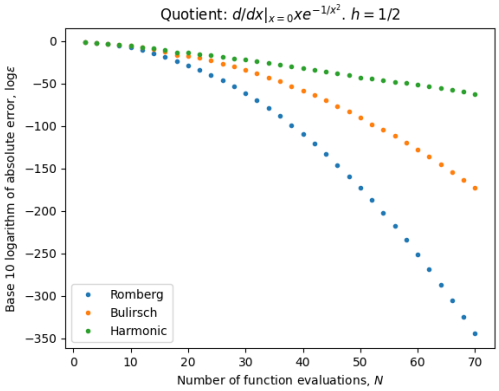
### 3.2.5 Another smooth but not analytic function

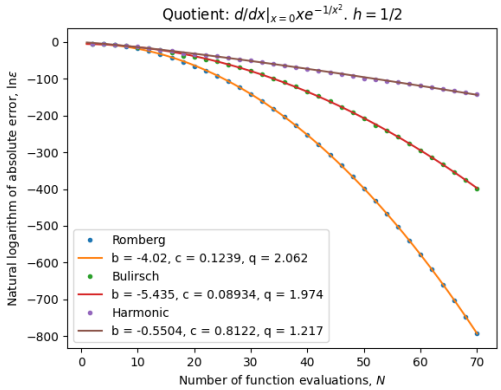
Now we will consider the derivative at zero of the following function:

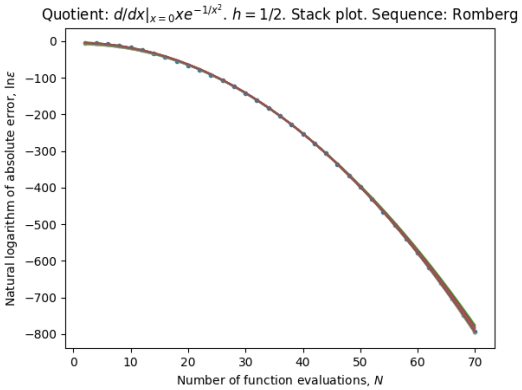
$$i(x) := \begin{cases} xe^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

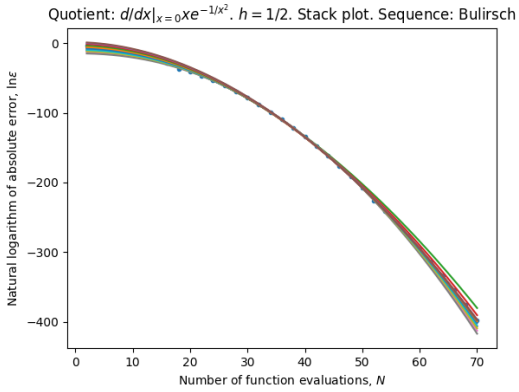
which is smooth but not analytic.

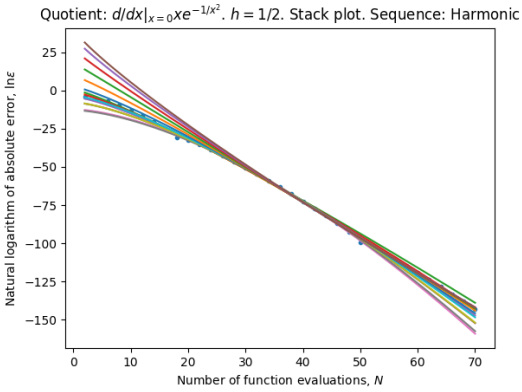



| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 0.009202 | 1.693 | 0.1182 | 0.00696 | 2.073 | $9.612 \cdot 10^{-5}$ |
| Bulirsch | lin-ln evals-error | 0.4381 | 5.98 | 0.07965 | 0.09556 | 2.015 | 0.001606 |
| Harmonic | lin-ln evals-error | $2.061 \cdot 10^{16}$ | 12.84 | 1.483 | 1.04 | 1.216 | 0.03736 |

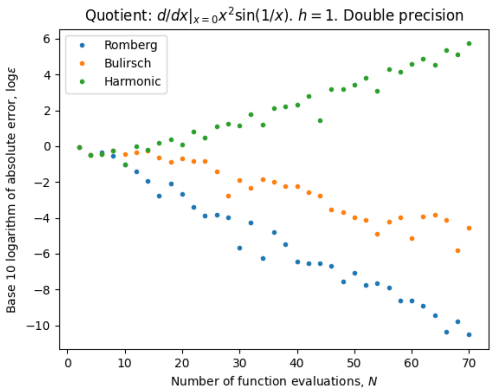
Here Romberg performes best and the harmonic sequence worst.

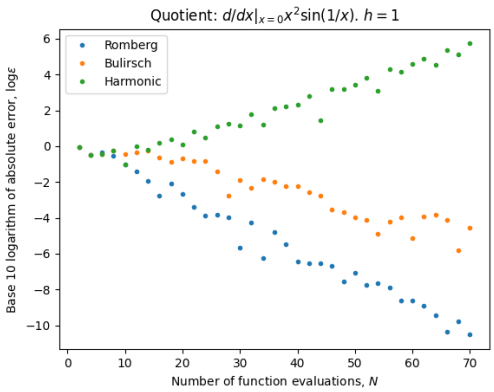
We seem to have nice fit for the Romberg sequence, moderately good for the Bulirsch sequence but not so good for the harmonic sequence.

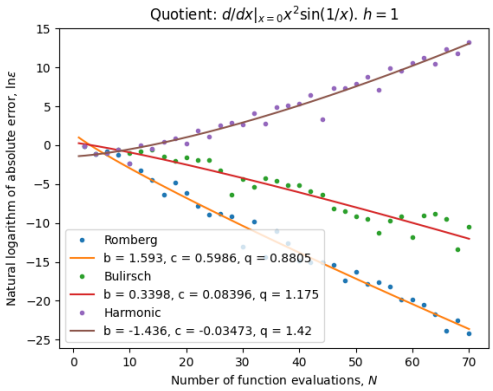
### 3.2.6 Only once differentiable function

Finally we will consider the derivate at zero of the following function which is only once differentiable at that point:

$$j(x) := \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}.$$




| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.99 \cdot 10^{27}$ | 13 | 6.139 | 3.832 | 0.8069 | 0.397 |
| Bulirsch | lin-ln evals-error | . | . | 2316 | 4.918 | 1.24 | 0.4409 |
| Harmonic | lin-ln evals-error | 0.3968 | 1.653 | −0.3656 | 6.72 | 1.307 | 0.09291 |

Here the model simply does not fit. Note that we do not have the asymptotic expansion for the derivate here, since the function is only once differentiable.

The parameters from the fitting are:

| Derivative | Sequence | $b$ | $c$ | $q$ |
|---|---|---|---|---|
| $d/dx\|_{x=0} r(x)$ | Romberg | $-0.99982$ | 0.060703 | 2.189 |
| $d/dx\|_{x=0} r(x)$ | Bulirsch | $-3.3384$ | 0.017962 | 2.2735 |
| $d/dx\|_{x=0} r(x)$ | Harmonic | $-1.8705$ | 0.42555 | 1.0695 |
| $d/dx\|_{x=0} xe^{-1/x^2}$. $h = 1/2$ | Romberg | $-4.0202$ | 0.12391 | 2.0616 |
| $d/dx\|_{x=0} xe^{-1/x^2}$. $h = 1/2$ | Bulirsch | $-5.4352$ | 0.089335 | 1.9741 |
| $d/dx\|_{x=0} xe^{-1/x^2}$. $h = 1/2$ | Harmonic | $-0.55045$ | 0.81216 | 1.2174 |
| $d/dx\|_{x=0} \sin x$. $h = 1/2$ | Romberg | $-5.3663$ | 0.51067 | 1.8066 |
| $d/dx\|_{x=0} \sin x$. $h = 1/2$ | Bulirsch | $-5.1899$ | 0.64124 | 1.651 |
| $d/dx\|_{x=0} \sin x$. $h = 1/2$ | Harmonic | 3.912 | 1.9876 | 1.2878 |
| $d/dx\|_{x=0} \ln(x+1)$. $h = 1/2$ | Romberg | $-3.4927$ | 0.20942 | 1.9619 |
| $d/dx\|_{x=0} \ln(x+1)$. $h = 1/2$ | Bulirsch | $-5.1421$ | 0.19126 | 1.8443 |
| $d/dx\|_{x=0} \ln(x+1)$. $h = 1/2$ | Harmonic | 1.0555 | 1.0494 | 1.2768 |
| $d/dx\|_{x=0} x^2 \sin(1/x)$. $h = 1$ | Romberg | 1.5925 | 0.59863 | 0.88054 |
| $d/dx\|_{x=0} x^2 \sin(1/x)$. $h = 1$ | Bulirsch | 0.33976 | 0.083956 | 1.1751 |
| $d/dx\|_{x=0} x^2 \sin(1/x)$. $h = 1$ | Harmonic | $-1.436$ | $-0.034733$ | 1.4204 |
| $d/dx\|_{x=0} \sqrt{x+1}$. $h = 1/2$ | Romberg | $-5.7063$ | 0.21299 | 1.9582 |
| $d/dx\|_{x=0} \sqrt{x+1}$. $h = 1/2$ | Bulirsch | $-7.315$ | 0.19691 | 1.8379 |
| $d/dx\|_{x=0} \sqrt{x+1}$. $h = 1/2$ | Harmonic | $-0.86323$ | 1.0951 | 1.2682 |
| $d/dx\|_{x=0} e^x$ | Romberg | $-2.408$ | 0.41106 | 1.8479 |
| $d/dx\|_{x=0} e^x$ | Bulirsch | $-2.9068$ | 0.48137 | 1.703 |
| $d/dx\|_{x=0} e^x$ | Harmonic | 4.7175 | 1.5509 | 1.3212 |

Table 3.1: Optimal parameters by test case

Excluding the computation of $d/dx|_{x=0} x^2 \sin 1/x$, the model fits exceptionally. We always get fast convergence except when computing $d/dx|_{x=0} x \sin 1/x$ and extrapolation with the harmonic sequence. Excluding this case, we always get almost down to machine level precision when using double precision arithmetic, using any extrapolation sequence. It is worth noting that $x \sin 1/x$ is only once differentiable at 0 so we do not have the asymptotic expansion for its derivative at 0. The Romberg sequence performes best and the harmonic sequence worst, in all cases.

# Chapter 4

# Initial Value Problems

## 4.1 The explicit midpoint rule

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth mapping and consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(a) = y_a, \quad t \in [a, b]. \tag{4.1}$$

The explicit midpoint methods is a method for computing an approximation to the solution of (4.1), and it goes as follows: Let $n \geq 1$ be an integer and $h \coloneqq (b-a)/2n$. We then define recursively

$$\xi_h(a) \coloneqq y_a, \quad \xi_h(a+h) \coloneqq \xi_h(a) + hf(a, \xi_h(a))$$

and

$$\xi_h(a + (i+1)h) \coloneqq \xi_h(a + (i-1)h) + 2hf(a + ih, \xi(a + ih)).$$

Then $\xi_h$ is an approximate solution to (4.1) defined at $a, a+h, \ldots, b$. We are interested in the value $X_f(h) \coloneqq \xi_h(b)$. It is possible to show that $X_f(h)$ has an asymptotic expansion in $h^2$. We have the following implementation in Python of the explicit midpoint rule for computing $X_f(h)$.

```
class ExplicitMidpointRule(Scheme):

  def __init__(self):
    super(ExplicitMidpointRule, self).__init__(2)

  def apply(self, ivp, n):
    h = (ivp.b - ivp.a) / (2 * n)
    y_sl = ivp.y0
    y_l = ivp.y0 + h * ivp.f(ivp.a, ivp.y0)

    for i in range(1, 2 * n):
      tmp = y_l
      y_l = y_sl + 2 * h * ivp.f(ivp.a + i * h, y_l)
      y_sl = tmp

    return y_l
```

## 4.2 Numerical experiments

In this section we are going to extrapolate the explicit midpoint rule and analyze the convergence of the approximations as we extrapolate more often. Consider the initial value

problem (4.1). Let $n_1 < n_2 < \cdots$ be some sequence of integers and $h_i := (b-a)/n_i$. Let $X_{ij}$ the extrapolation table which we get from extrapolating in $h^2$, using the points $(h_i, X_f(h_i))$. Let $\varepsilon_i := |X_{ii} - y(b)|$ be the absolute error. We are going to do the same convergence and efficency analysis as in the two previous chapters. We will both do the computations using high precision arithmetic with 500 correct digits and also in standard double precision.

In those cases where we do not have an analytic solution to the equations, we computed a reference solution up to 500 significant digits. We did that by using extrapolation with the harmonic sequence and estimating the error as the difference between successive terms in the sequence of approximations.

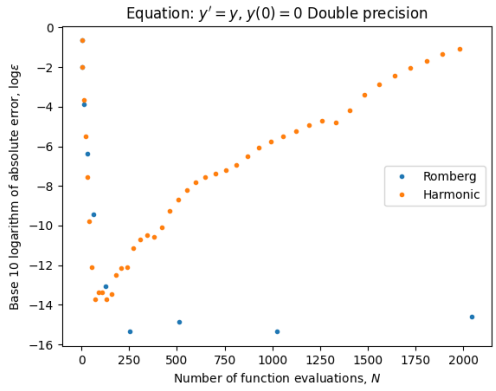
Now we will consider the results of the experiments.

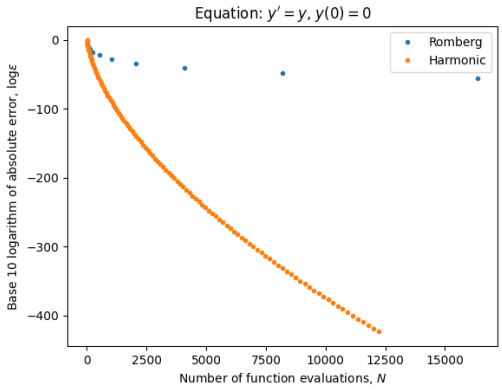
### 4.2.1 Exponential growth

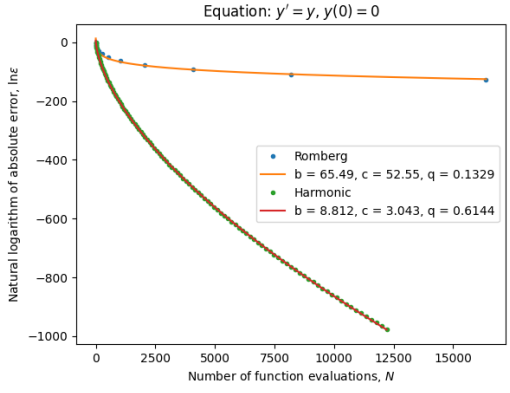
First we will consider the following initial value problem:

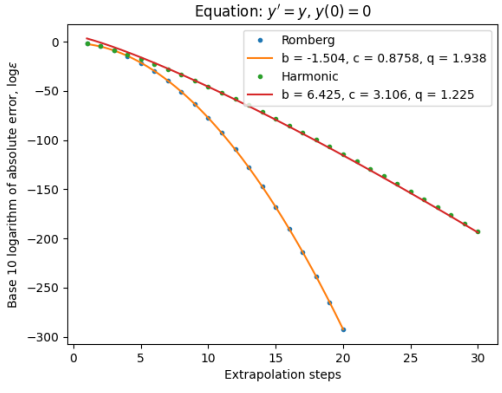
$$y'(x) = y(x), \quad y(0) = 0, \quad x \in [0, 1] \tag{4.2}$$

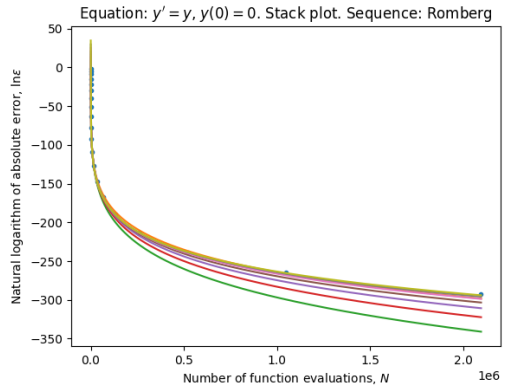
whose solution is the analytic function $y(x) = e^x$.

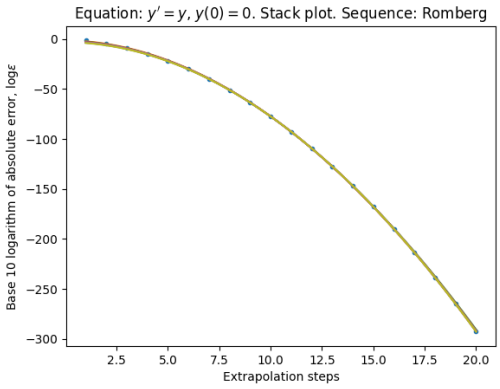


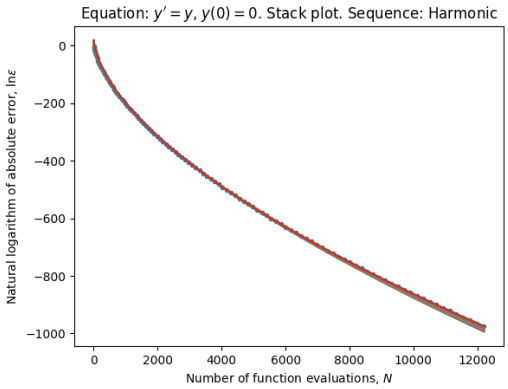



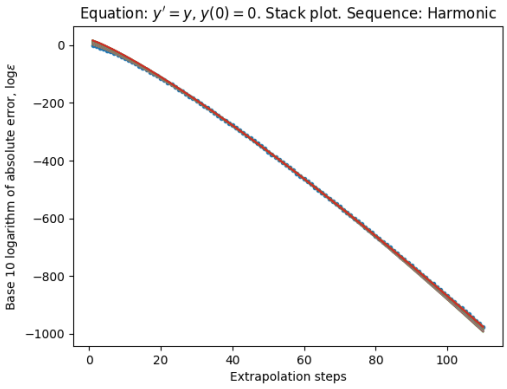






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $6.231 \cdot 10^{65}$ | 6 | 71.92 | 0.1174 | 0.1236 | 0.03122 |
| Harmonic | lin-ln evals-error | $1.204 \cdot 10^{9}$ | 6.365 | 3.264 | 0.005221 | 0.6081 | 0.0001653 |
| Romberg | lin-ln steps-error | 0.0873 | 0.2223 | 0.841 | 0.0009203 | 1.95 | $2.797 \cdot 10^{-5}$ |
| Harmonic | lin-ln steps-error | $3.805 \cdot 10^{7}$ | 6.024 | 3.309 | 0.004502 | 1.214 | 0.0001425 |

The Harmonic sequence performes better. We get down to machine level precision using either sequence in double precision arithmetic.

We clearly have exponential convergence in the number of steps for the Romberg sequence and the fitting for the Harmonic sequence seems nice but we though have very big values for $A$.

### 4.2.2 Logistic curve

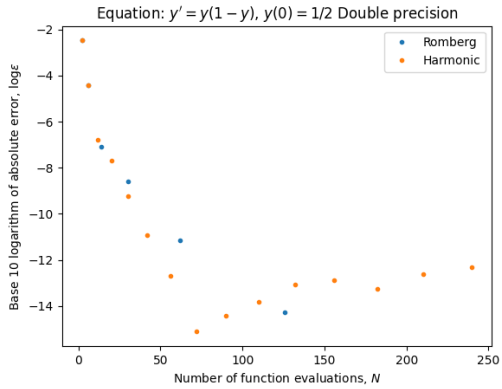
Then we will consider the following initial value problem

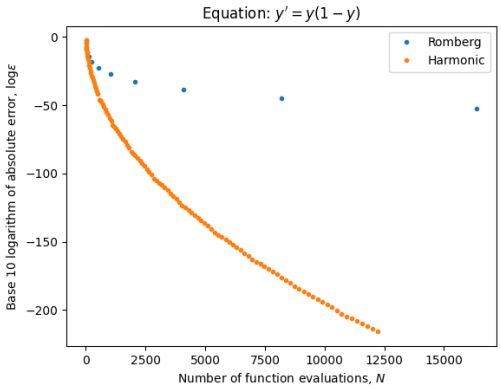
$$y'(x) = y(x)(1 - y(x)), \quad y(0) = 1/2, \quad x \in [0, 1] \tag{4.3}$$

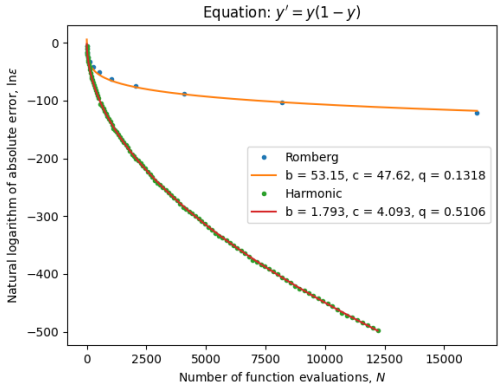
whose solution is the sigmoid function

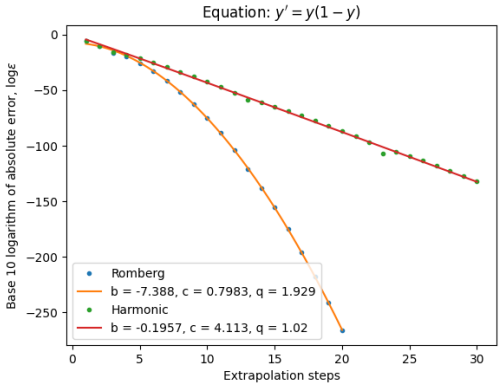
$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

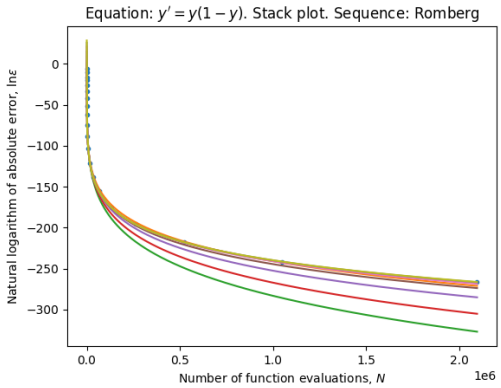
which is analytic.

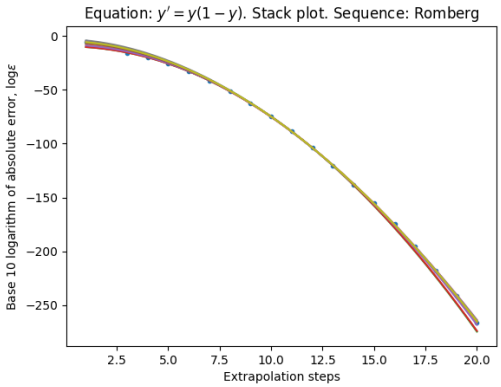

Equation: $y' = y(1-y)$, $y(0) = 1/2$ Double precision
Romberg
Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$

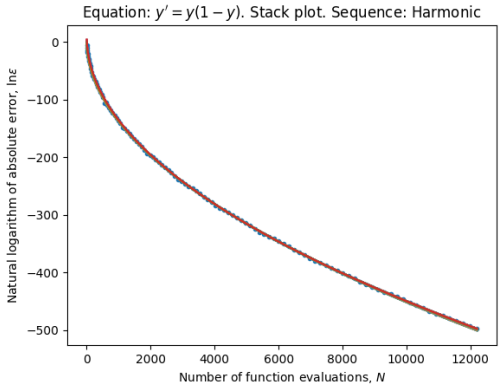

Equation: $y' = y(1-y)$
Romberg
Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$


Equation: $y' = y(1-y)$
Romberg
b = 53.15, c = 47.62, q = 0.1318
Harmonic
b = 1.793, c = 4.093, q = 0.5106
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


Equation: $y' = y(1-y)$
Romberg
b = -7.388, c = 0.7983, q = 1.929
Harmonic
b = -0.1957, c = 4.113, q = 1.02
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Equation: $y' = y(1-y)$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


Equation: $y' = y(1-y)$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Equation: $y' = y(1-y)$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


Equation: $y' = y(1-y)$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $8.848 \cdot 10^{63}$ | 6 | 70.69 | 0.1952 | 0.1226 | 0.0637 |
| Harmonic | lin-ln evals-error | 2510 | 9.384 | 4.249 | 0.00357 | 0.5073 | 0.0001329 |
| Romberg | lin-ln steps-error | 0.01026 | 1.856 | 0.814 | 0.03122 | 1.932 | 0.001091 |
| Harmonic | lin-ln steps-error | 257.8 | 9.223 | 4.256 | 0.003464 | 1.014 | 0.0001294 |

The harmonic sequence performes better and we get down to machine level precisision using either sequence, in double precision.

We seem to have exponential convergence in the number of steps for the Romberg sequence and the models fit well for the harmonic sequence.

### 4.2.3 Tangens

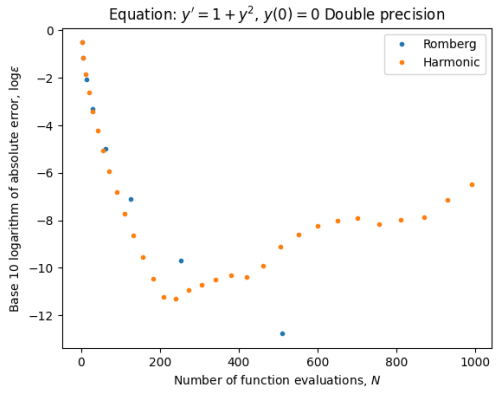
Now we will consider the following equation

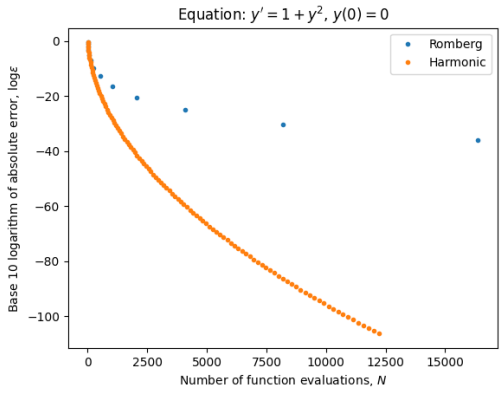
$$y'(x) = 1 + y(x)^2, \quad y(0) = 0, \quad x \in [0, 1] \tag{4.4}$$

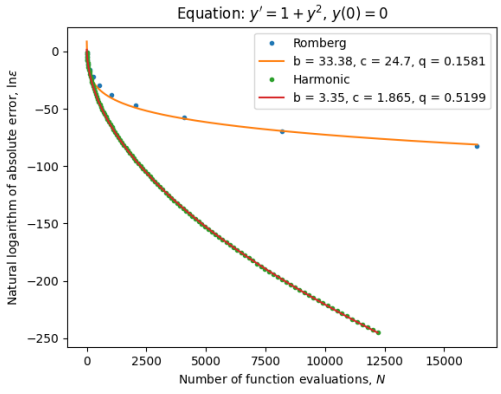
whose solution is

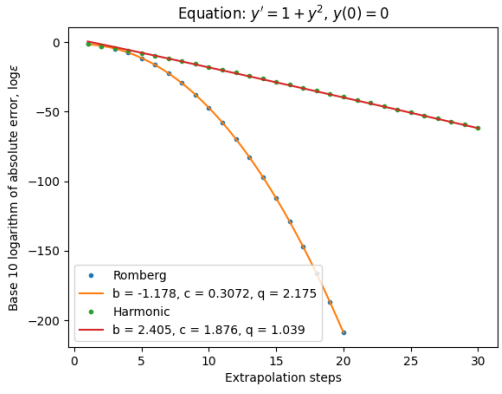
$$y(x) \coloneqq \tan(x)$$

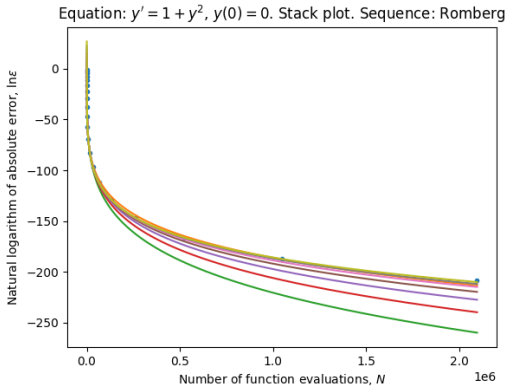
which is meromorphic and we are quite far from singularites.

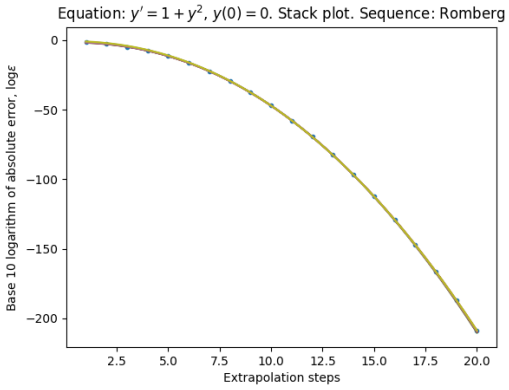


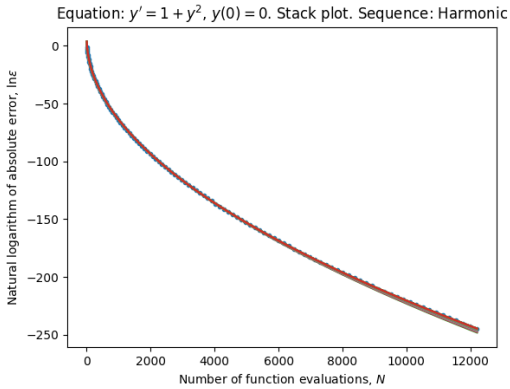



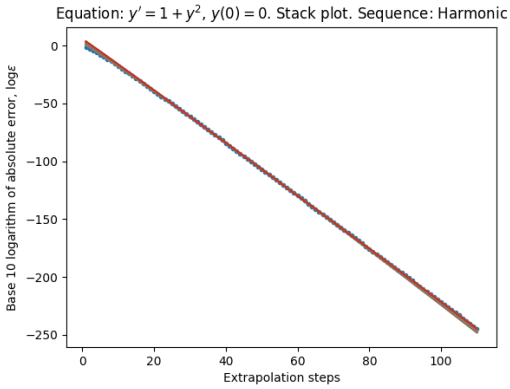






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $9.496 \cdot 10^{36}$ | 6 | 33.25 | 0.1689 | 0.1523 | 0.03648 |
| Harmonic | lin-ln evals-error | 358.8 | 0.5652 | 2.017 | 0.002488 | 0.5127 | 0.000109 |
| Romberg | lin-ln steps-error | 0.3476 | 0.1097 | 0.3039 | 0.001043 | 2.18 | $2.579 \cdot 10^{-5}$ |
| Harmonic | lin-ln steps-error | 121.5 | 0.5373 | 2.021 | 0.002309 | 1.025 | 0.0001013 |

The harmonic sequence performes better and we get down to machine level precision in double precision arithmetic, using either sequence.

Here we clearly have exponential convergence in the number of steps for the Romberg sequence and the fit is also very nice for the harmonic sequence.

### 4.2.4 Equation with singularity

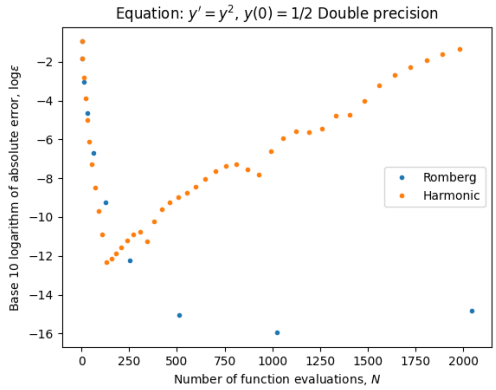
Now we will consider the following initial value problem:

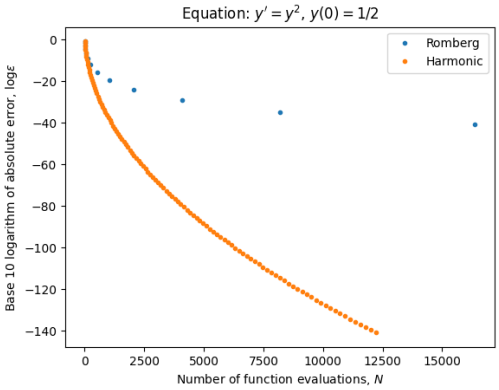
$$y'(t) = y^2(t), \quad y(0) = 1/(1+a), \quad t \in [0, 1] \tag{4.5}$$

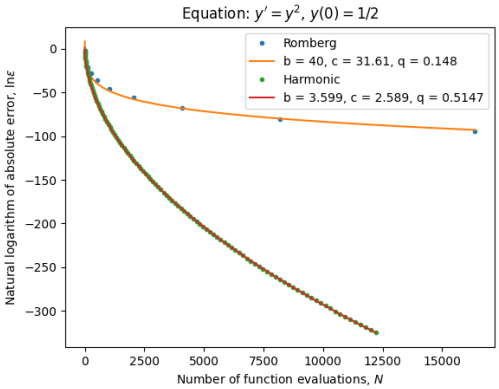
whose solution is

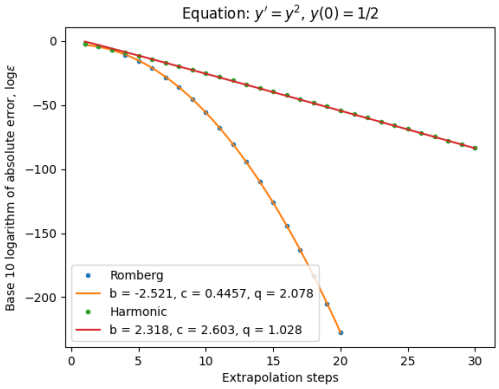
$$y(t) = \frac{1}{1-(t-a)}.$$

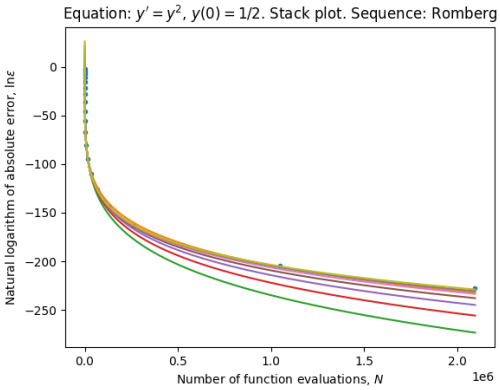
The solution is meromorphic with a pole at $1 + a$.

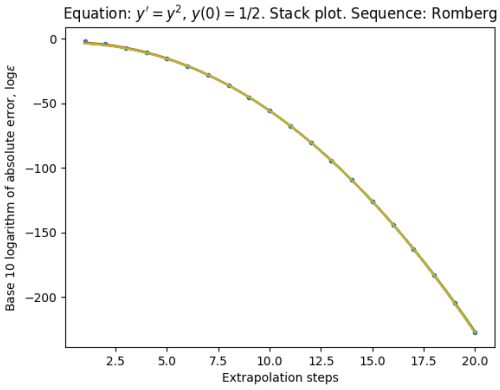

Equation: $y' = y^2$, $y(0) = 1/2$ Double precision
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

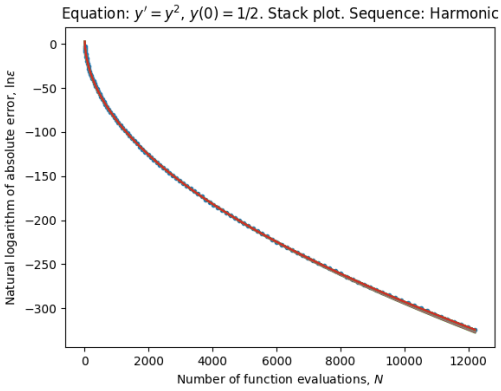

Equation: $y' = y^2$, $y(0) = 1/2$
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

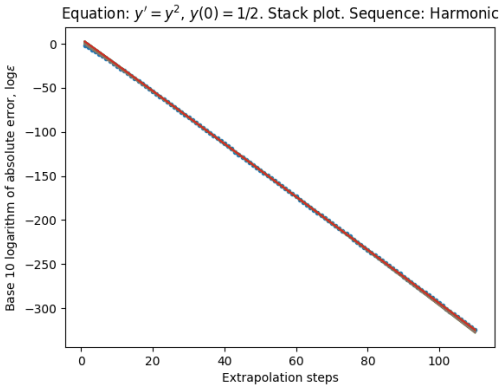

Equation: $y' = y^2$, $y(0) = 1/2$
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Romberg
b = 40, c = 31.61, q = 0.148
Harmonic
b = 3.599, c = 2.589, q = 0.5147


Equation: $y' = y^2$, $y(0) = 1/2$
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Romberg
b = -2.521, c = 0.4457, q = 2.078
Harmonic
b = 2.318, c = 2.603, q = 1.028


Equation: $y' = y^2$, $y(0) = 1/2$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


Equation: $y' = y^2$, $y(0) = 1/2$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Equation: $y' = y^2$, $y(0) = 1/2$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$

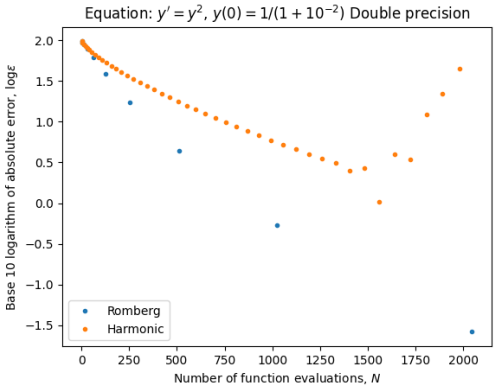

Equation: $y' = y^2$, $y(0) = 1/2$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

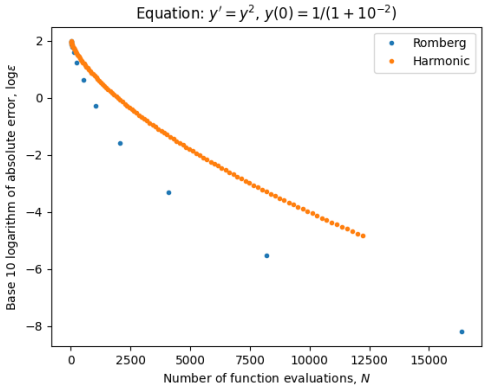
| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.418 \cdot 10^{42}$ | 6 | 42.1 | 0.1387 | 0.141 | 0.03195 |

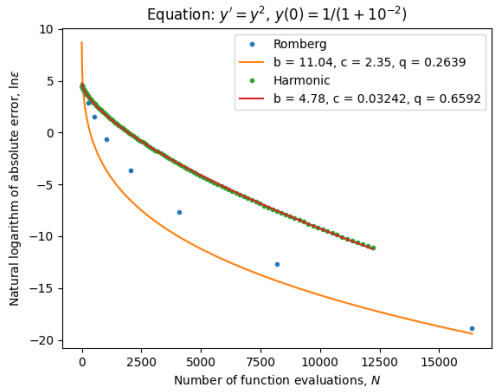
Harmonic lin-ln evals-error 440 0.5196 2.747 0.001256 0.5092 $5.472 \cdot 10^{-5}$ Romberg lin-ln steps-error 0.04186 0.03016 0.4274 0.0002961 2.091 $8.21 \cdot 10^{-6}$ Harmonic lin-ln steps-error 104.3 0.4903 2.752 0.001151 1.018 $5.022 \cdot 10^{-5}$

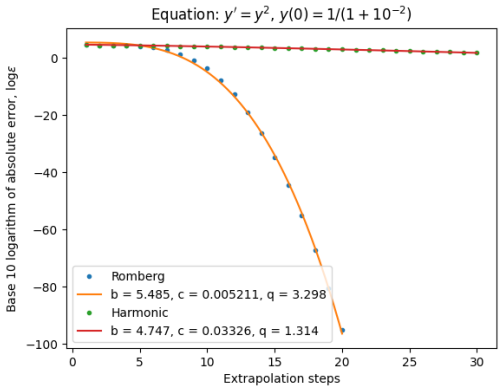
The harmonic sequence performes better and we get almost down to machine level precision using standar double precision floating point arithmetic, with either sequence.

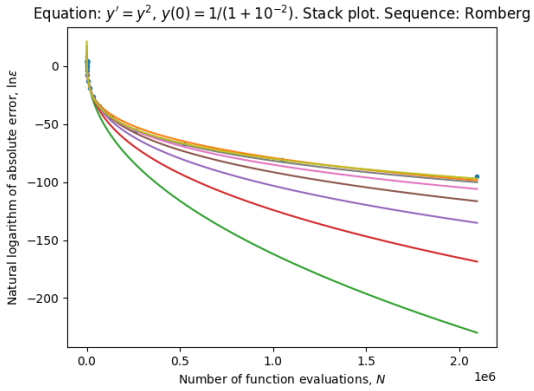
We have very nice fit for the exponential convergence in the number of steps for the Romberg sequence and also for the harmonic sequence.

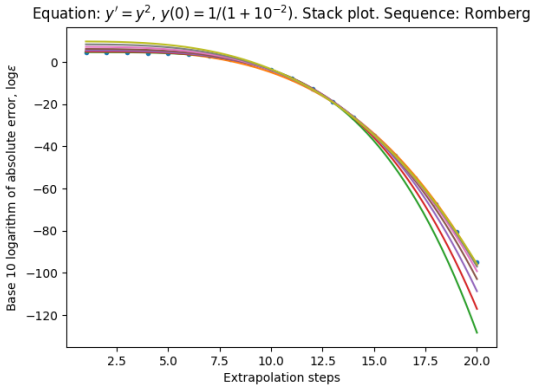


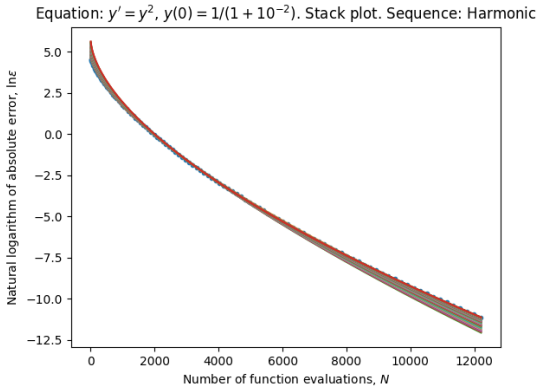



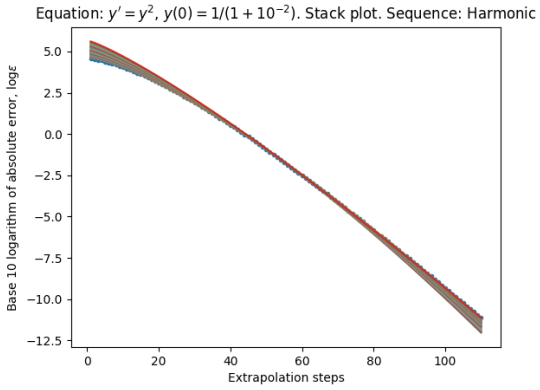



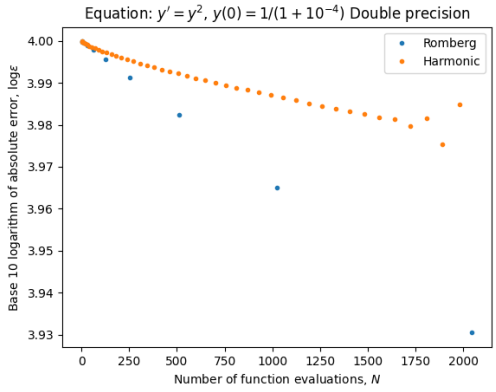


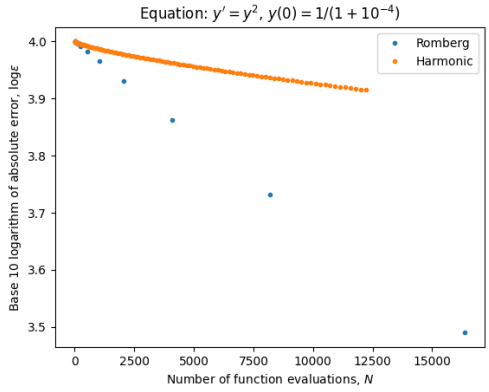



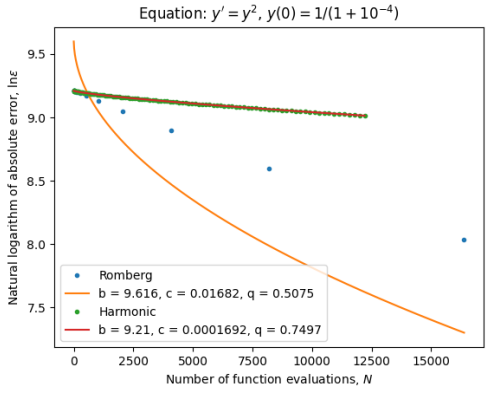

| | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.249 \cdot 10^{12}$ | 5.984 | 2.77 | 0.7656 | 0.3067 | 0.08315 |
| Harmonic | lin-ln evals-error | 173.2 | 0.1158 | 0.03983 | 0.08603 | 0.6452 | 0.002375 |
| Romberg | lin-ln steps-error | 3540 | 2.885 | 0.005212 | 0.6307 | 3.461 | 0.00941 |
| Harmonic | lin-ln steps-error | 165.2 | 0.1095 | 0.04047 | 0.08154 | 1.287 | 0.002248 |

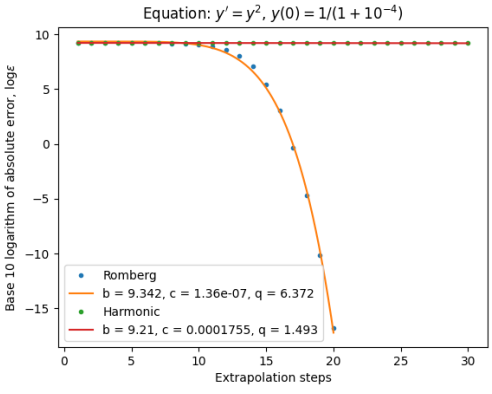
Here Romberg works better and we do not attain as high precisison using the harmonic as when using Romberg, in standard double precision arithmetic.

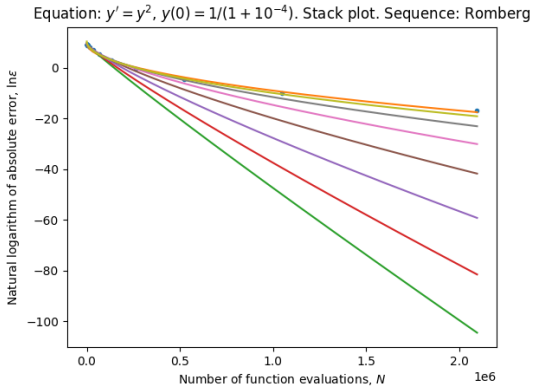
Here the model fits moderately well for Romberg sequence, when considering exponential convergence in the number of steps. We also have moderate fit for the harmonic sequence.

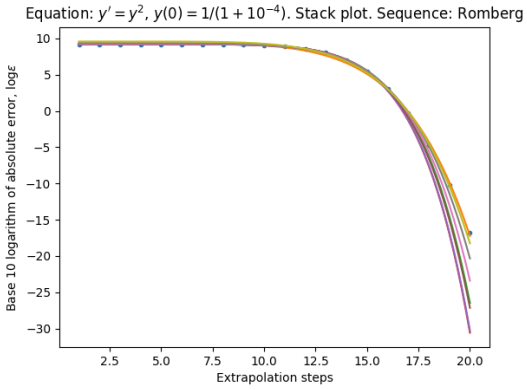


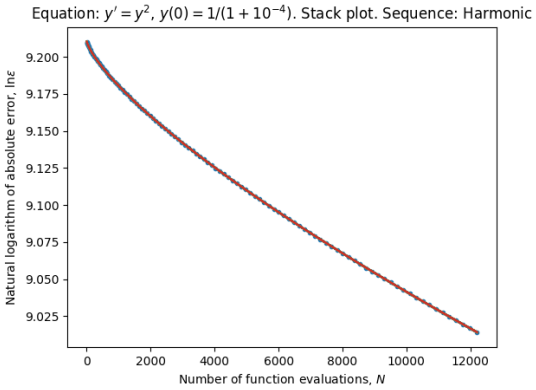



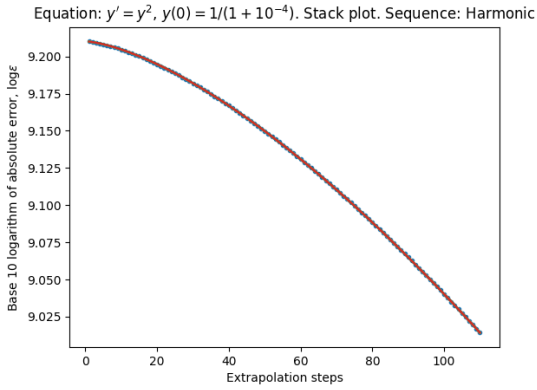






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.462 \cdot 10^4$ | 0.2298 | 0.004335 | 2.285 | 0.7393 | 0.04196 |
| Harmonic | lin-ln evals-error | $1 \cdot 10^4$ | $5.52 \cdot 10^{-9}$ | 0.0001699 | $5.806 \cdot 10^{-5}$ | 0.7494 | $1.194 \cdot 10^{-6}$ |
| Romberg | lin-ln steps-error | $1.087 \cdot 10^4$ | 0.02206 | $1.977 \cdot 10^{-8}$ | 3.201 | 7.604 | 0.006289 |
| Harmonic | lin-ln steps-error | 9997 | $3.302 \cdot 10^{-10}$ | 0.0001749 | $5.29 \cdot 10^{-6}$ | 1.494 | $1.204 \cdot 10^{-7}$ |

Here we clearly do not have exponential convergence in the number of evaluations for the Romberg sequence. We have not so good fit for exponential convergence in the number of steps.

Regarding the harmonic sequence, we note that we have extremely slow convergence, so the $x$ and $y$ values differ by many orders of magnitude, so the results are maybe not so reliable.

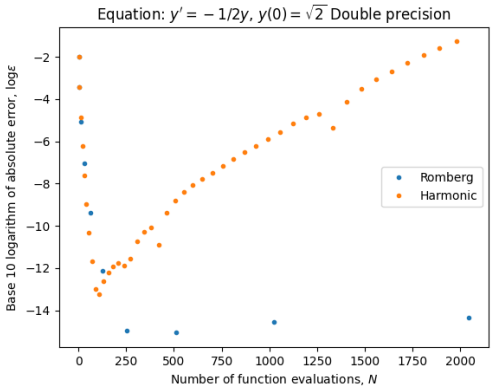
### 4.2.5 Equation with moderate singularity

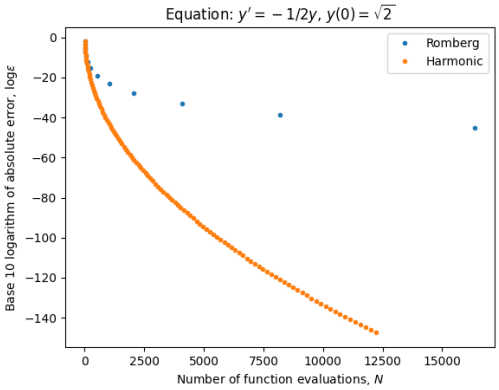
Now we will consider the following initial value problem

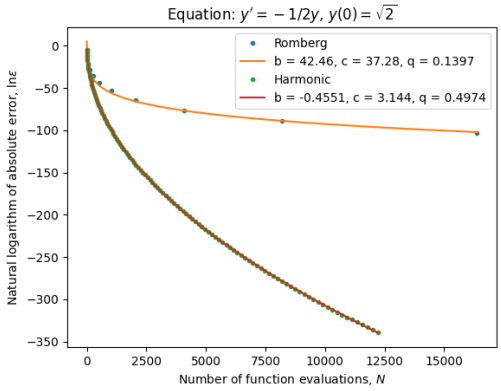
$$y'(t) = -\frac{1}{2y}, \quad y(0) = \sqrt{1+a}, \quad t \in [0, 1] \tag{4.6}$$

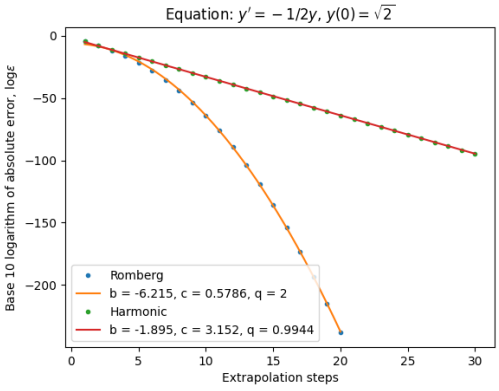
whose solution is

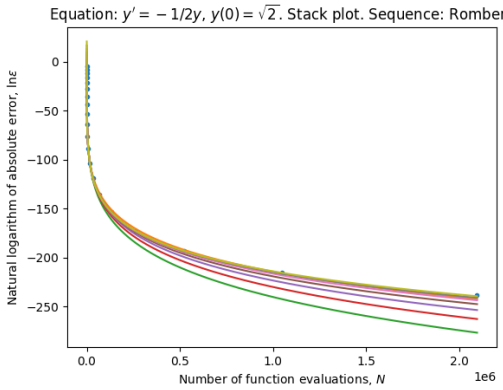
$$y(t) = \sqrt{1 - (t - a)}$$

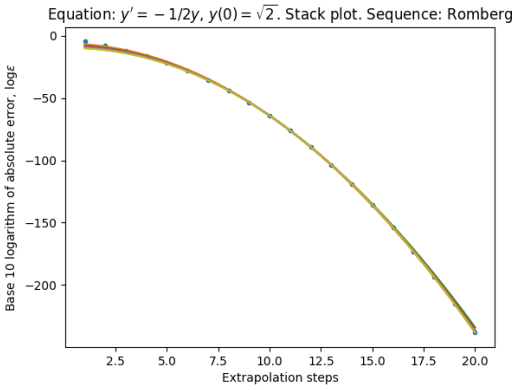

Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$ Double precision
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

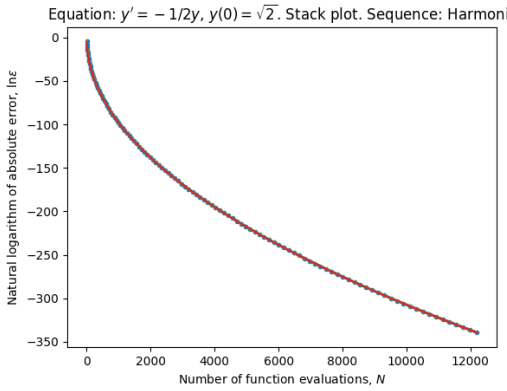

Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

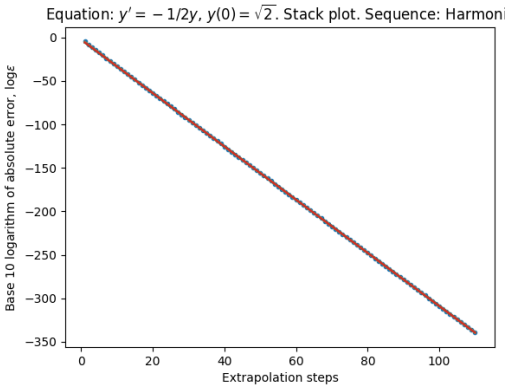

Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Romberg
b = 42.46, c = 37.28, q = 0.1397
Harmonic
b = -0.4551, c = 3.144, q = 0.4974


Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Romberg
b = -6.215, c = 0.5786, q = 2
Harmonic
b = -1.895, c = 3.152, q = 0.9944


Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

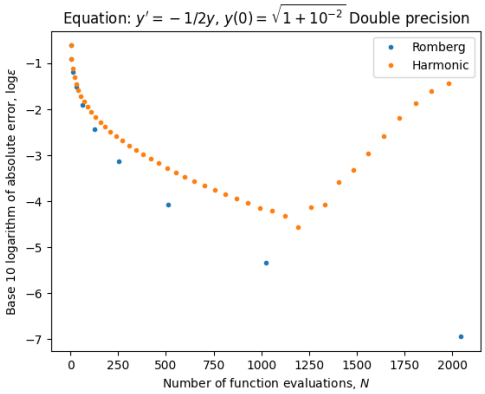

Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$

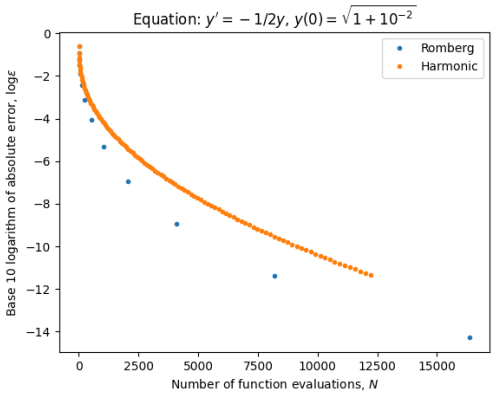

Equation: $y' = -1/2y$, $y(0) = \sqrt{2}$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

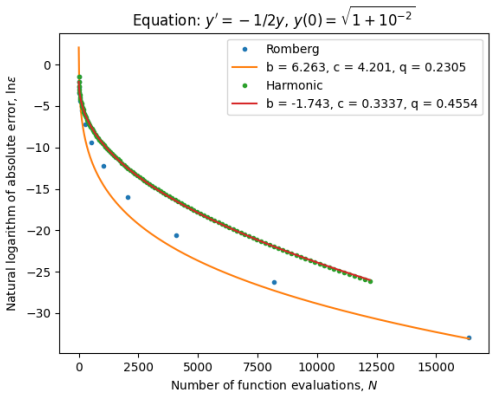
| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $8.335 \cdot 10^{41}$ | 6 | 47.04 | 0.1111 | 0.1343 | 0.02593 |
| Harmonic | lin-ln evals-error | 0.454 | 0.03631 | 3.115 | $3.797 \cdot 10^{-5}$ | 0.4983 | $1.623 \cdot 10^{-6}$ |
| Romberg | lin-ln steps-error | 0.0002771 | 0.7384 | 0.5104 | 0.0058 | 2.038 | 0.0001599 |
| Harmonic | lin-ln steps-error | 0.1009 | 0.0414 | 3.117 | $4.412 \cdot 10^{-5}$ | 0.9965 | $1.9 \cdot 10^{-6}$ |

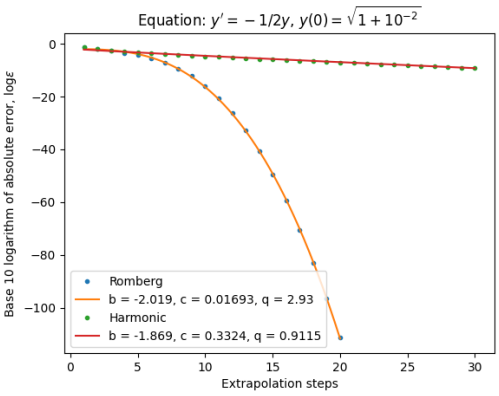
Here the harmonic sequence works better than Romberg and we get down to machine level precision using either sequence, in standard double precision floating point arithmetic.

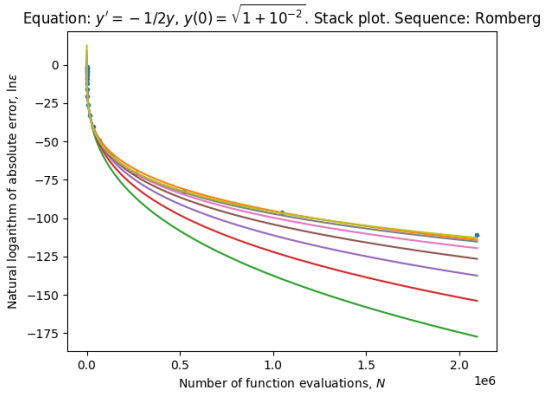
Here we clearly have exponential convergence in the number of steps and evaluations for the harmonic sequence. Regarding Romberg, we seem to have exponential convergence in the number of steps.

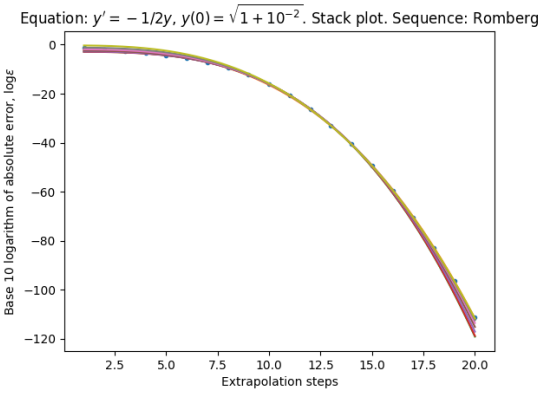


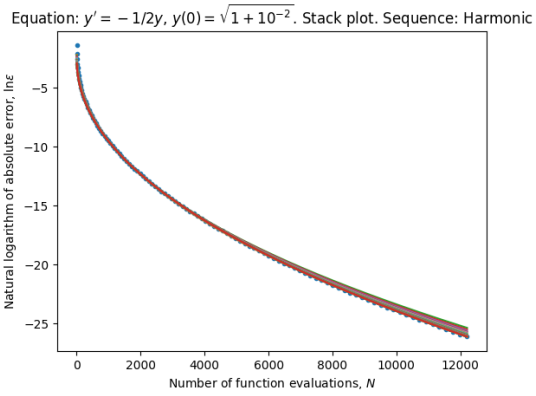



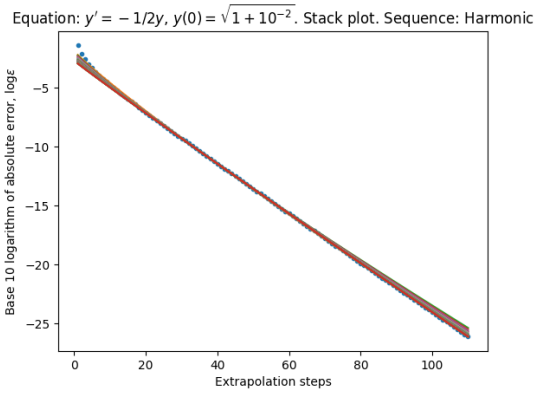



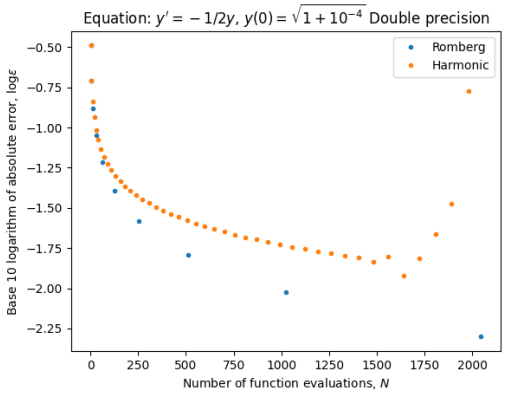


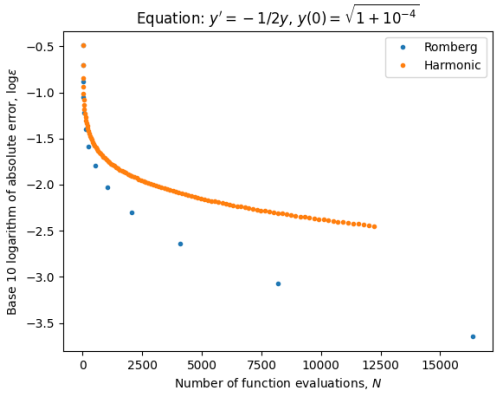



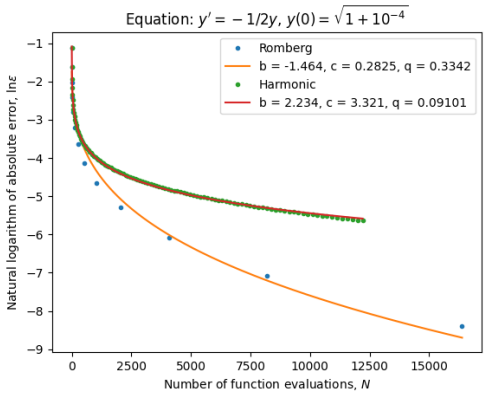

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $3.095 \cdot 10^9$ | 5.978 | 4.521 | 0.4282 | 0.2537 | 0.04639 |
| Harmonic | lin-ln evals-error | 0.09273 | 0.0545 | 0.263 | 0.01142 | 0.4785 | 0.0004549 |
| Romberg | lin-ln steps-error | 0.2097 | 1.241 | 0.01393 | 0.1057 | 3.022 | 0.001297 |
| Harmonic | lin-ln steps-error | 0.08262 | 0.05039 | 0.2624 | 0.01092 | 0.9574 | 0.0004388 |

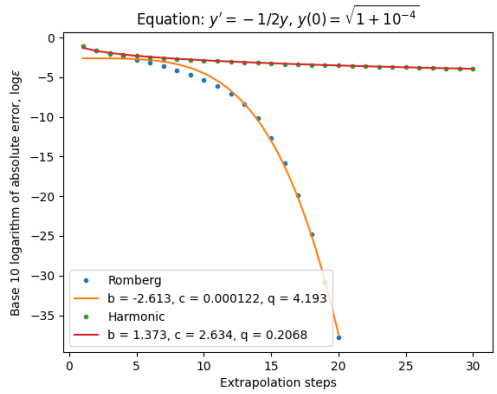
Here Romberg performes better and we do not attain as high precision using the harmonic sequence in standard double precision floating point arithmetic.

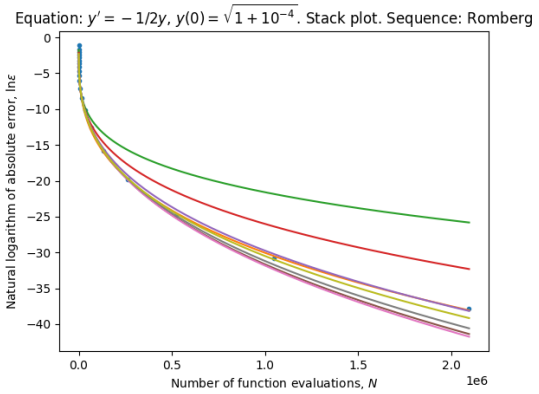
For the Romberg sequence, the model seems to fit moderately well when considering exponential convergence in the number of evaluations. We also have reasonably good fit for the harmonic sequence.

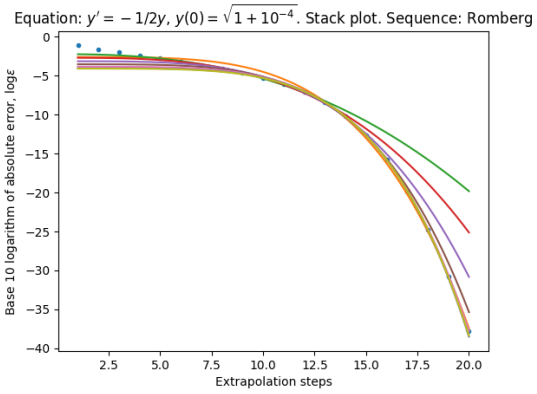


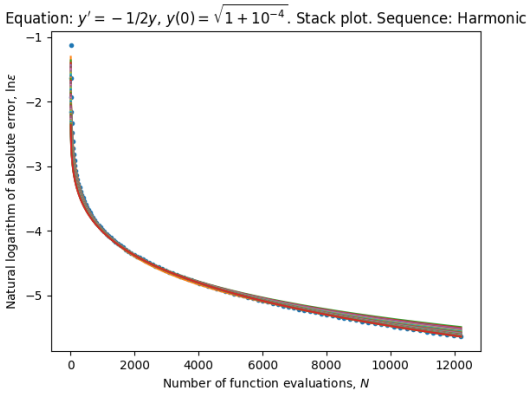



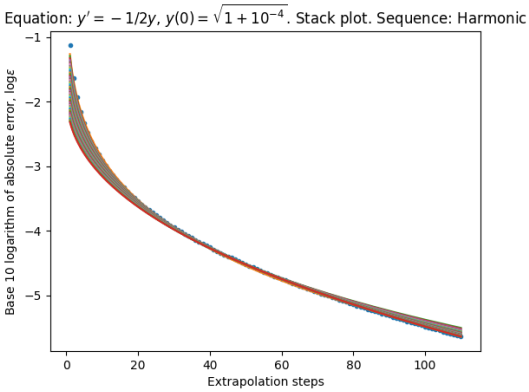






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | 0.1398 | 0.4846 | 0.2489 | 0.4625 | 0.3506 | 0.01881 |
| Harmonic | lin-ln evals-error | 1.282 | 3.403 | 1.258 | 0.3894 | 0.1631 | 0.05472 |
| Romberg | lin-ln steps-error | 0.0429 | 0.5116 | 0.001785 | 3.769 | 4.002 | 0.04408 |
| Harmonic | lin-ln steps-error | 0.8136 | 1.714 | 1.146 | 0.311 | 0.3342 | 0.04634 |

Here, we do not have any clear fit.

### 4.2.6 Circular rotation

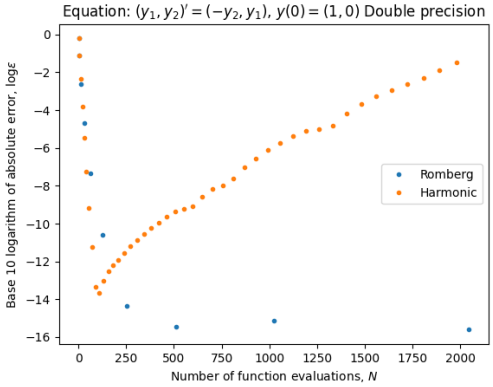
Now we will consider the following system of equations:

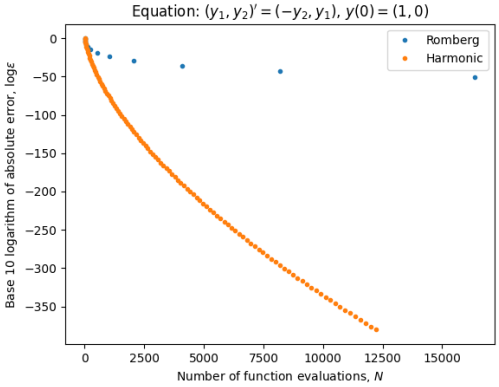
$$(y_1(t), y_2(t))' = (-y_2(t), y_1(t)), \quad y(0) = (1, 0), \quad t \in [0, \pi/2] \tag{4.7}$$

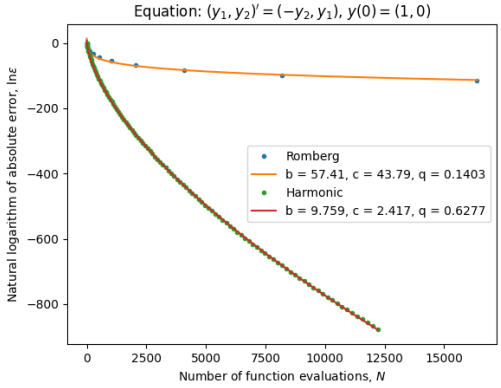
whose solution is

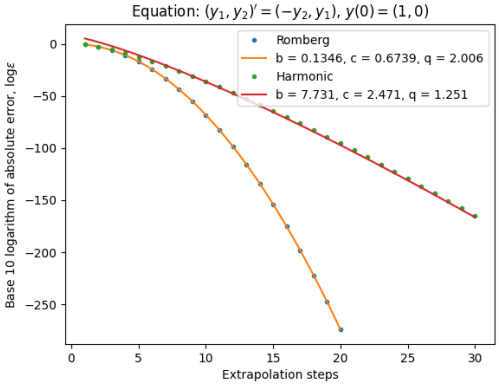
$$(y_1(t), y_2(t)) = (\cos t, \sin t)$$

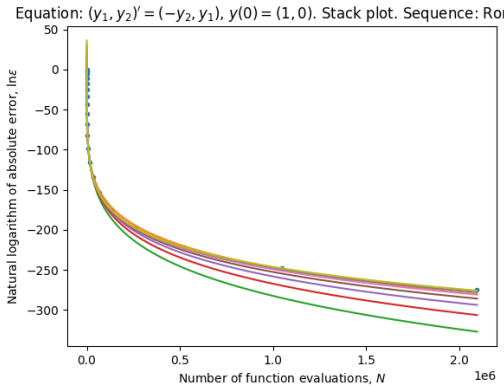
which is entire.

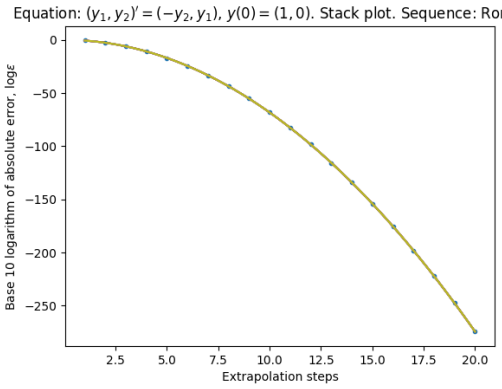

Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$ Double precision
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

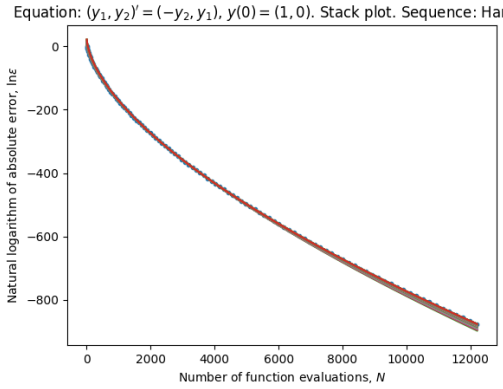

Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

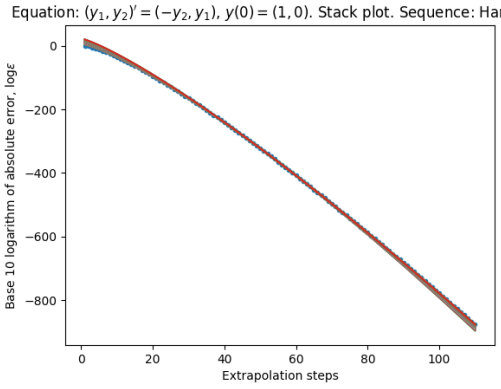

Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Romberg
b = 57.41, c = 43.79, q = 0.1403
Harmonic
b = 9.759, c = 2.417, q = 0.6277


Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Romberg
b = 0.1346, c = 0.6739, q = 2.006
Harmonic
b = 7.731, c = 2.471, q = 1.251


Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$. Stack plot. Sequence: Romberg
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$. Stack plot. Sequence: Romberg
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$. Stack plot. Sequence: Harmonic
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


Equation: $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$. Stack plot. Sequence: Harmonic
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.215 \cdot 10^{60}$ | 6 | 61.05 | 0.1358 | 0.1308 | 0.03453 |
| Harmonic | lin-ln evals-error | $9 \cdot 10^{9}$ | 6.841 | 2.636 | 0.007896 | 0.62 | 0.0002457 |
| Romberg | lin-ln steps-error | 1.083 | 0.000101 | 0.6717 | $4.963 \cdot 10^{-7}$ | 2.007 | $1.48 \cdot 10^{-8}$ |
| Harmonic | lin-ln steps-error | $4.276 \cdot 10^{8}$ | 6.508 | 2.675 | 0.006923 | 1.237 | 0.0002153 |

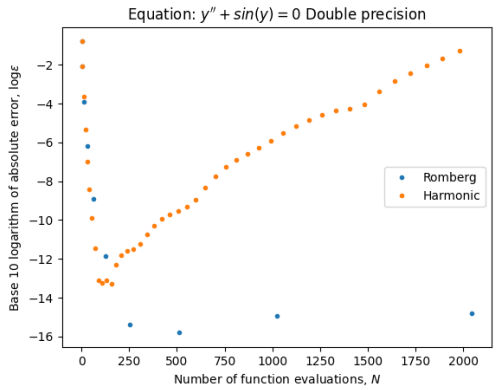
The harmonic sequence works better then Romberg and we get down to machine level precision using either sequence when using standard floating point arithmetic.

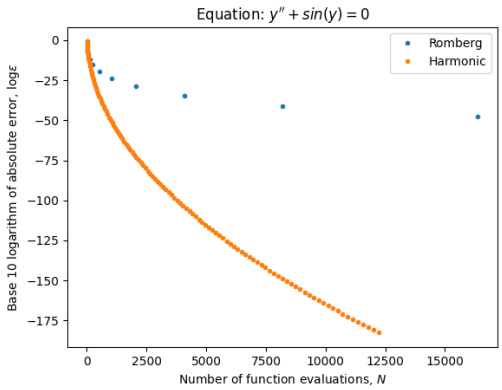
We clearly have exponential convergence in the number of steps for the Romberg sequence. For the harmonic sequence, we seem to have exponential convergence, but though we must note that the mean value of the $A$ coefficient is suspiciously large.

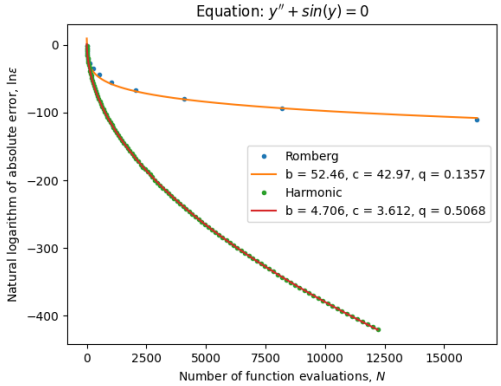
### 4.2.7 Mathematical pendulum

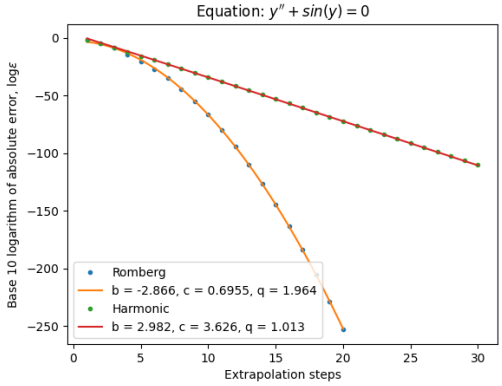
Now we will consider the mathematical pendulum equation:

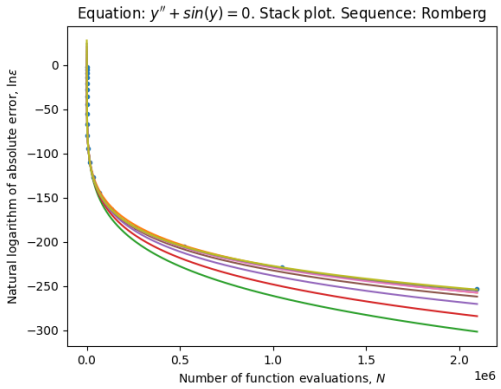
$$y''(t) + \sin y(t) = 0, \quad y(0) = 0,\ y'(0) = 1, \quad t \in [0, 1]. \tag{4.8}$$

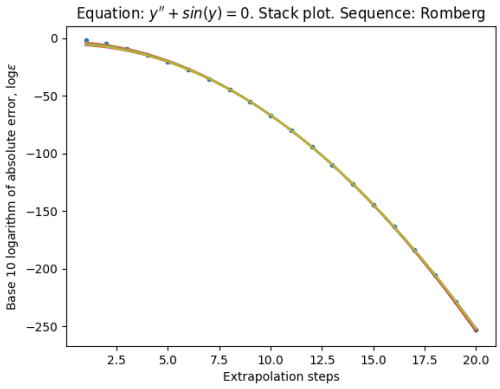


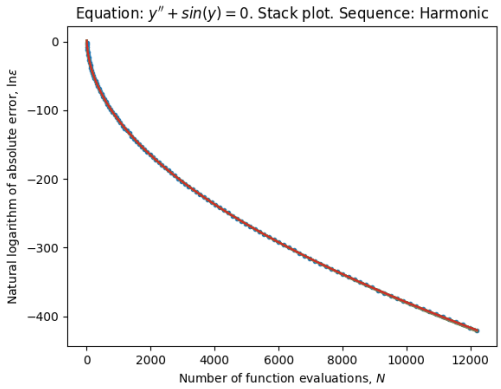



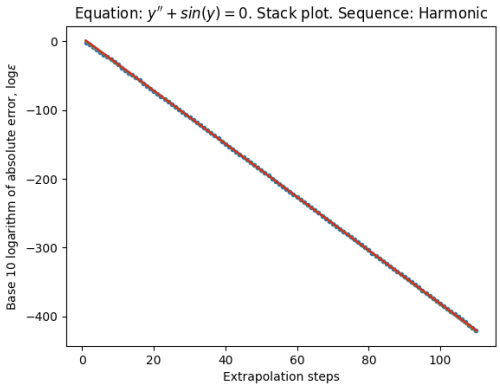






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $3.245 \cdot 10^{52}$ | 6 | 56.9 | 0.1423 | 0.1295 | 0.03859 |
| Harmonic | lin-ln evals-error | 471.5 | 0.2744 | 3.713 | 0.0002618 | 0.5042 | $1.13 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 0.01384 | 0.3333 | 0.6305 | 0.002076 | 1.995 | $6.207 \cdot 10^{-5}$ |
| Harmonic | lin-ln steps-error | 72.54 | 0.2499 | 3.718 | 0.0002297 | 1.008 | $9.918 \cdot 10^{-6}$ |

Here the harmonic sequence works better and we get down to machine level precision in standard double precision floating point arithmetic, using either sequence.

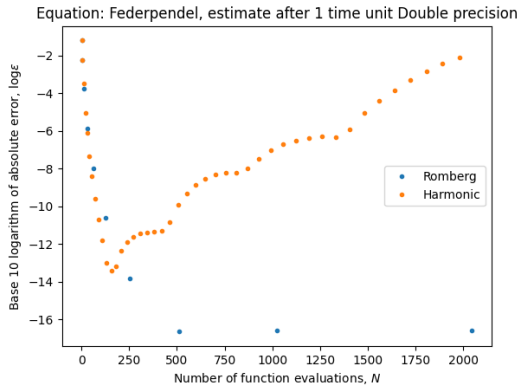
We have nice fit for the exponential convergence in the number of steps for the Romberg sequence. We also have very nice fit for the harmonic sequence.

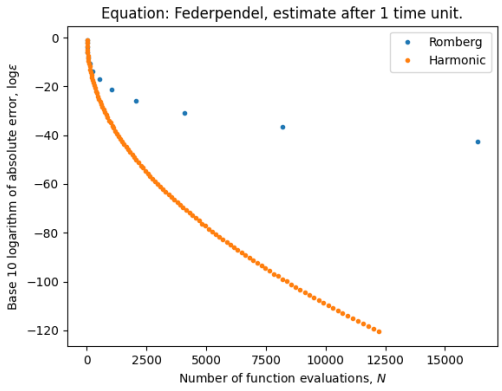
### 4.2.8 Federpendel

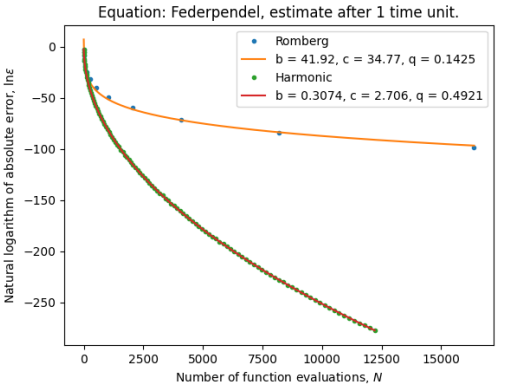
Now we will consider the equation of motion for das Federpendel or the spring pendulum:

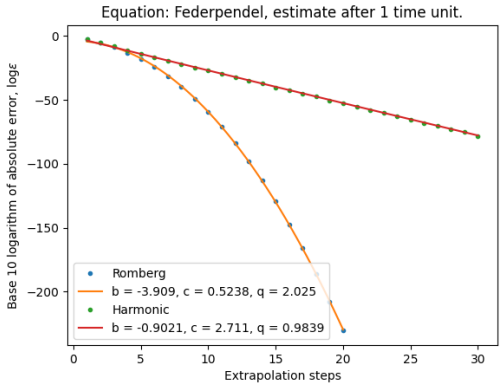
$$\mathbf{p}' = -(|\mathbf{q}| - 1)\frac{\mathbf{q}}{|\mathbf{q}|} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{q}' = \mathbf{p}$$

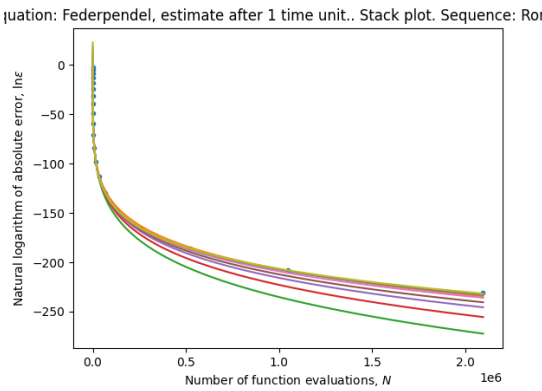
where $\mathbf{p}$ and $\mathbf{q}$ are two dimensional vectors. We will consider it with the initial condition $\mathbf{q}(0) = (1, 0)$ and $\mathbf{p}(0) = (0, 1)$ and try to both estimate the solution at time $t = 1$ and time $t = 2$.

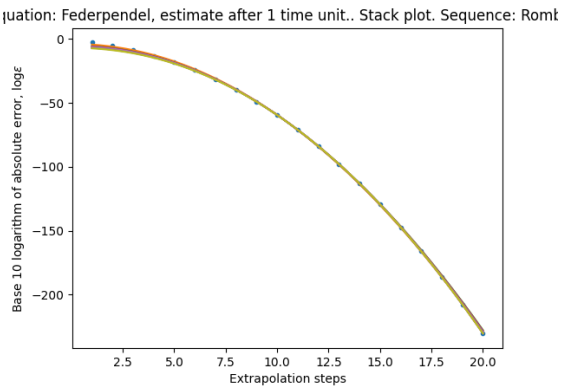

Equation: Federpendel, estimate after 1 time unit Double precision
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

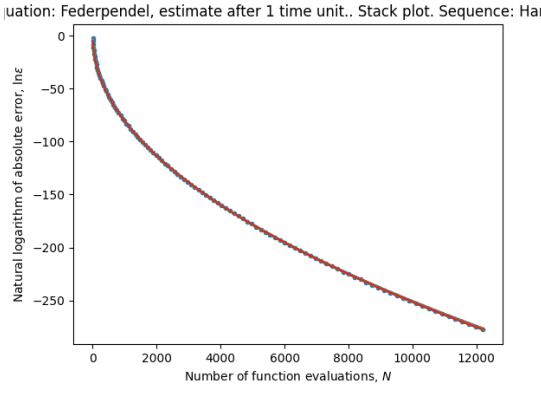

Equation: Federpendel, estimate after 1 time unit.
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

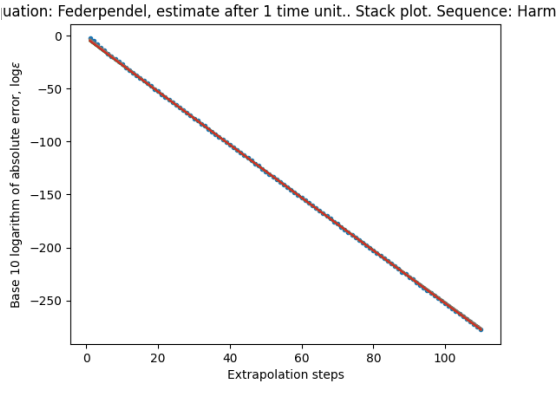

Equation: Federpendel, estimate after 1 time unit.
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Romberg
b = 41.92, c = 34.77, q = 0.1425
Harmonic
b = 0.3074, c = 2.706, q = 0.4921


Equation: Federpendel, estimate after 1 time unit.
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Romberg
b = -3.909, c = 0.5238, q = 2.025
Harmonic
b = -0.9021, c = 2.711, q = 0.9839


uation: Federpendel, estimate after 1 time unit.. Stack plot. Sequence: Romb
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


uation: Federpendel, estimate after 1 time unit.. Stack plot. Sequence: Romb
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

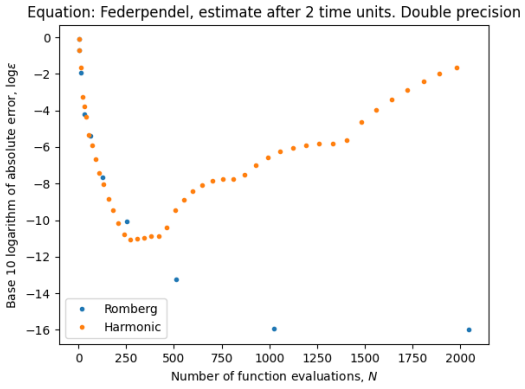

uation: Federpendel, estimate after 1 time unit.. Stack plot. Sequence: Harm
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$

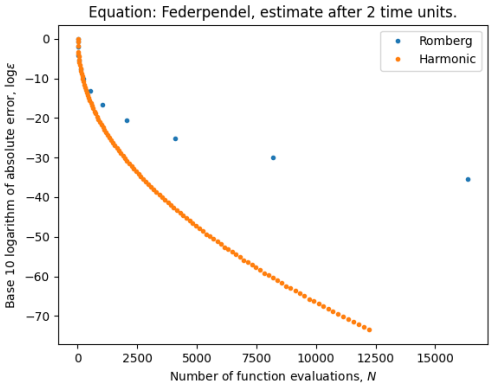

uation: Federpendel, estimate after 1 time unit.. Stack plot. Sequence: Harm
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

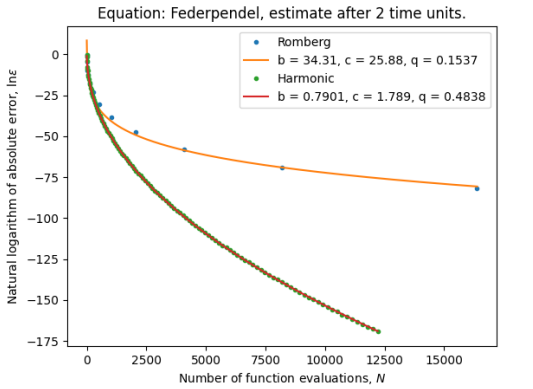
| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $8.241 \cdot 10^{40}$ | 6 | 43.86 | 0.1208 | 0.1376 | 0.02818 |
| Harmonic | lin-ln evals-error | 0.5082 | 0.2486 | 2.612 | 0.0003409 | 0.4955 | $1.457 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 0.002791 | 0.3701 | 0.4615 | 0.003294 | 2.064 | $8.941 \cdot 10^{-5}$ |
| Harmonic | lin-ln steps-error | 0.1465 | 0.2485 | 2.613 | 0.0003471 | 0.9909 | $1.492 \cdot 10^{-5}$ |

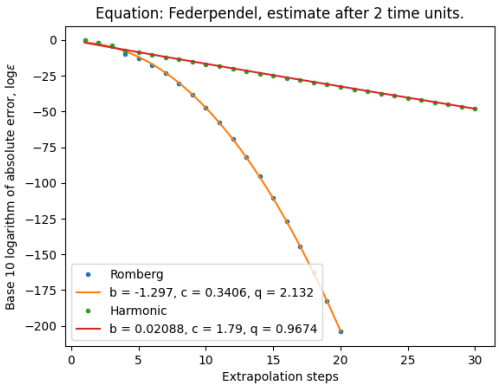
Here the harmonic sequence works better and we get down to machine level precision in standard double precision floating point arithmetic, using either sequence.

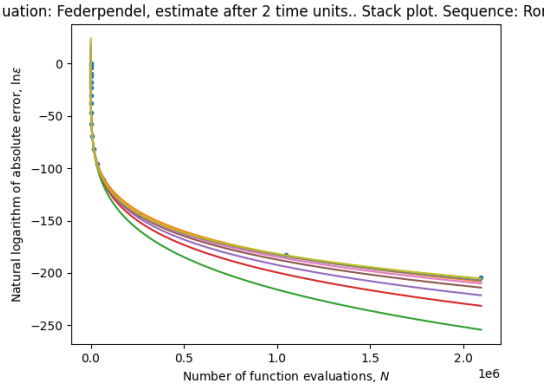
We have nice fit for exponential convergence in the number of steps for the Romberg sequence. We also have nice fit for exponential convergence for the harmonic sequence.

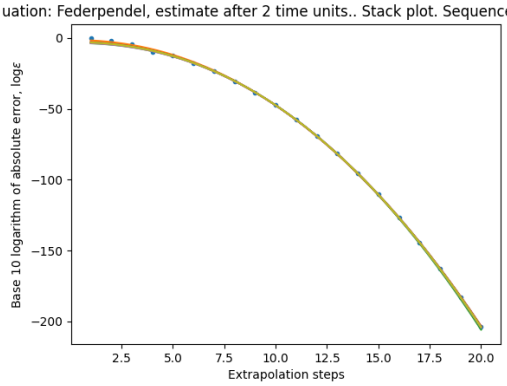


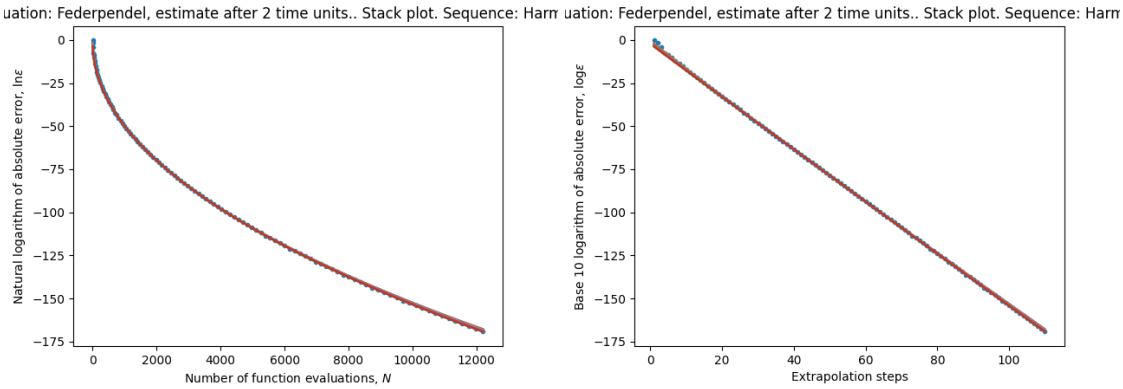






| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $2.335 \cdot 10^{34}$ | 6 | 32.11 | 0.1569 | 0.1519 | 0.03495 |
| Harmonic | lin-ln evals-error | 0.5245 | 0.3886 | 1.643 | 0.001271 | 0.492 | $5.413 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 0.05507 | 0.06164 | 0.2956 | 0.001162 | 2.178 | $3.178 \cdot 10^{-5}$ |
| Harmonic | lin-ln steps-error | 0.2422 | 0.3718 | 1.642 | 0.001252 | 0.984 | $5.36 \cdot 10^{-5}$ |

Here the harmonic sequence also works better and we attain high precision using either sequence in standard double precision floating point arithmetic.

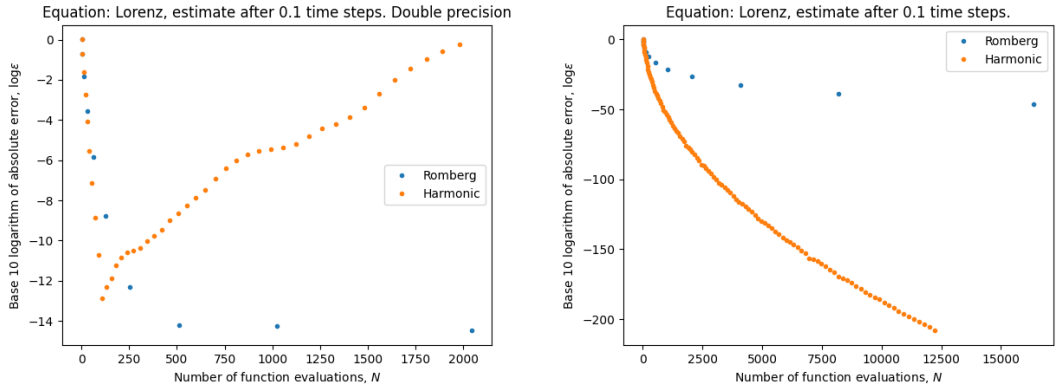
We seem to have very nice fit for exponential convergence in the number of steps for the Romberg sequence. We also have very nice fit for exponential convergence for the harmonic sequence.

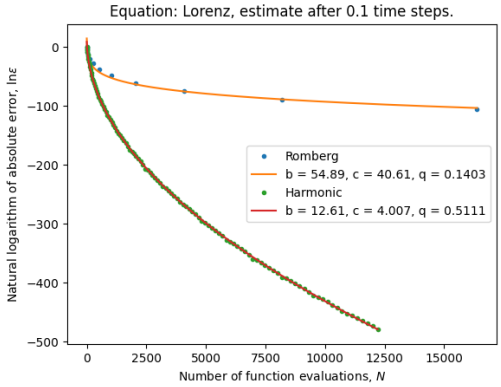
### 4.2.9 Lorenz equations

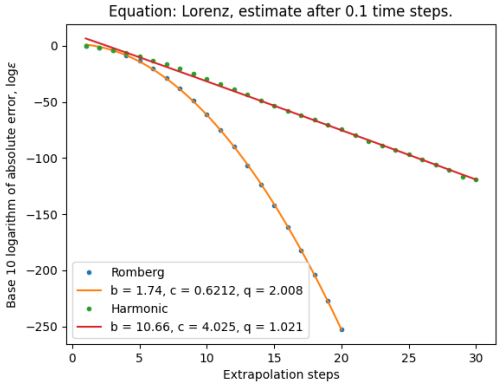
The Lorenz equations are the following system:

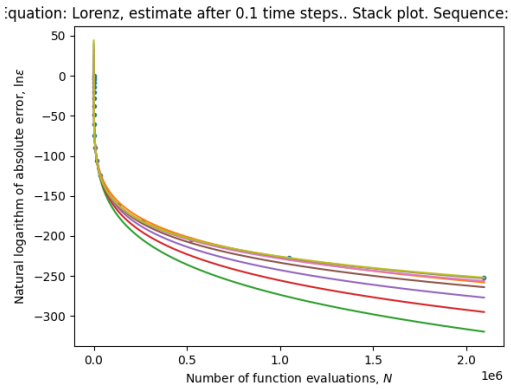
$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = x(\rho - z) - y, \quad \frac{dz}{dt} = xy - \beta z$$

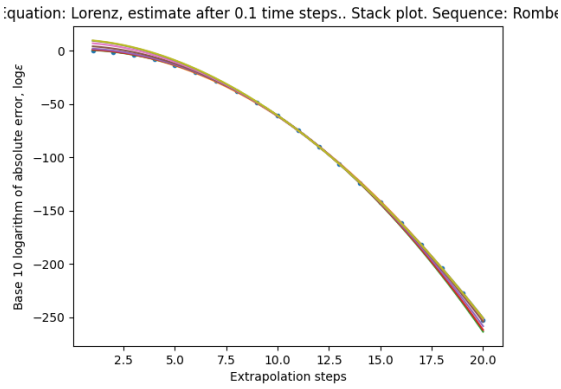
where $\sigma$, $\rho$ and $\beta$ are constants. In our experiment, the constants are set to $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The initial condition we will consider is $(x(0), y(0), z(0)) = (1, 1, 1)$.

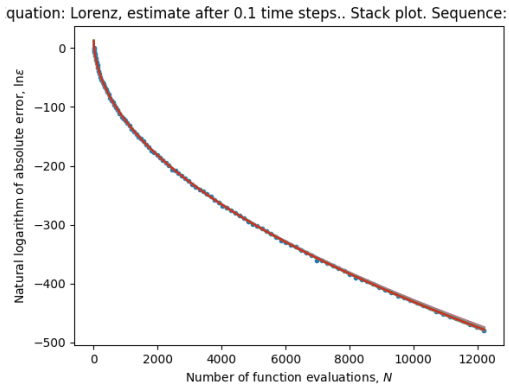


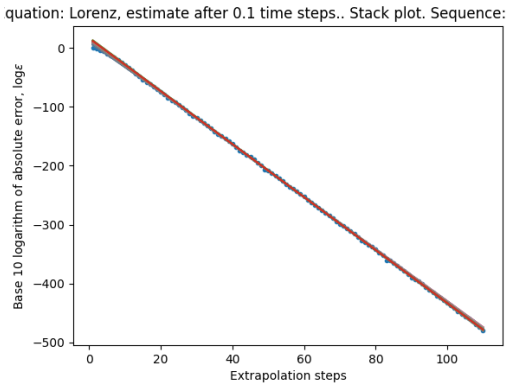


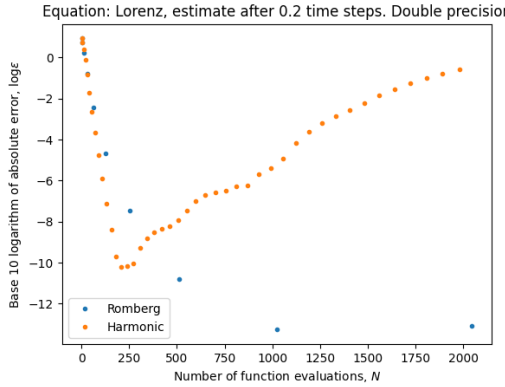



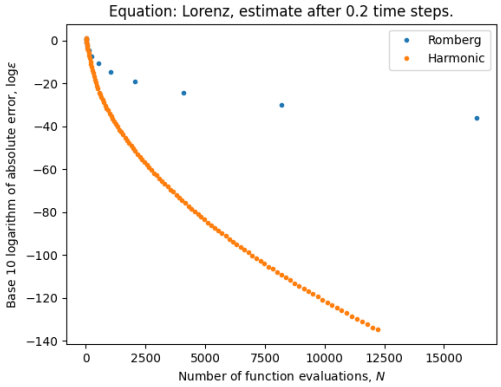



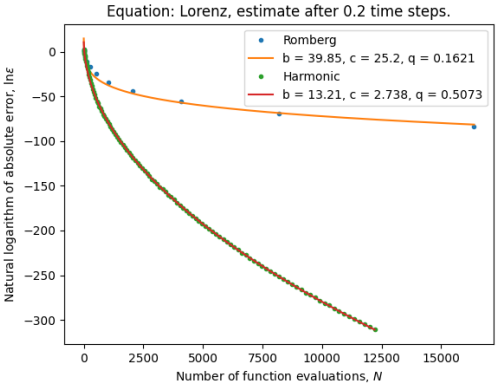

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $1.532 \cdot 10^{73}$ | 6 | 66.42 | 0.2606 | 0.1285 | 0.07315 |
| Harmonic | lin-ln evals-error | $2.639 \cdot 10^{7}$ | 6.828 | 4.214 | 0.001814 | 0.5062 | $7.152 \cdot 10^{-5}$ |
| Romberg | lin-ln steps-error | 8932 | 2.321 | 0.7248 | 0.06447 | 1.976 | 0.001984 |
| Harmonic | lin-ln steps-error | $2.903 \cdot 10^{6}$ | 6.603 | 4.22 | 0.001799 | 1.012 | $7.141 \cdot 10^{-5}$ |

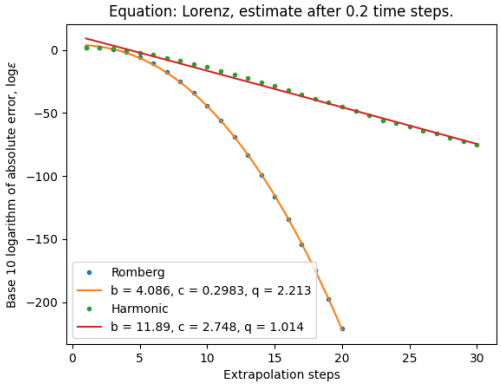
Here the harmonic sequence works better and we get down to machine level precision in standard double precision arithmetic.

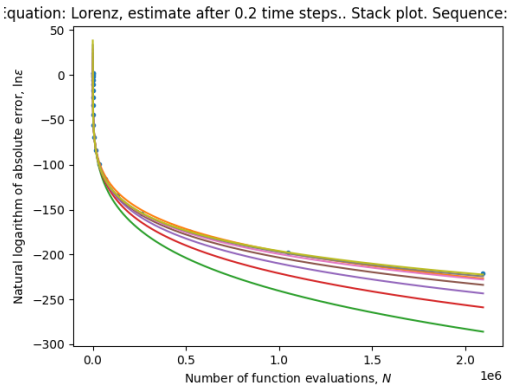
The model for exponential convergence in the number of steps fits moderately well for the Romberg sequence. We get nice fit for exponential convergence for the harmonic sequence.

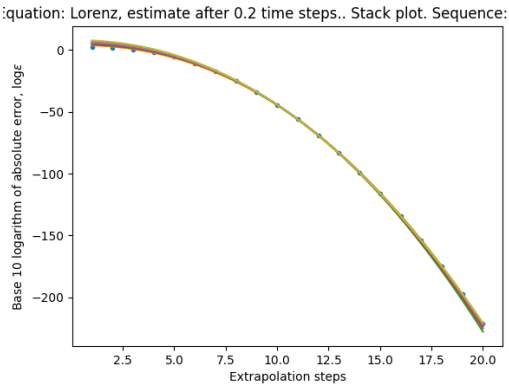

Equation: Lorenz, estimate after 0.2 time steps. Double precision
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

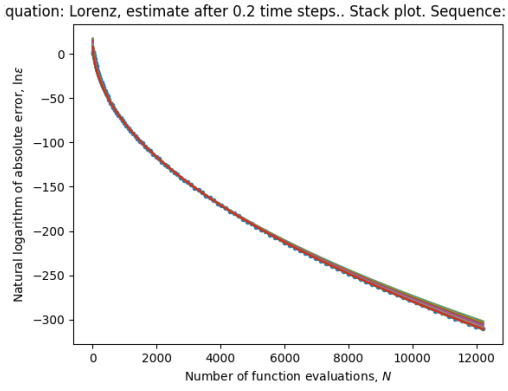

Equation: Lorenz, estimate after 0.2 time steps.
Base 10 logarithm of absolute error, log$\varepsilon$
Number of function evaluations, $N$
Romberg
Harmonic

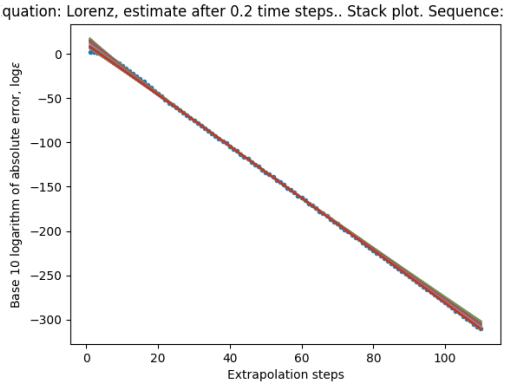

Equation: Lorenz, estimate after 0.2 time steps.
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
Romberg
b = 39.85, c = 25.2, q = 0.1621
Harmonic
b = 13.21, c = 2.738, q = 0.5073


Equation: Lorenz, estimate after 0.2 time steps.
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps
Romberg
b = 4.086, c = 0.2983, q = 2.213
Harmonic
b = 11.89, c = 2.748, q = 1.014


:quation: Lorenz, estimate after 0.2 time steps.. Stack plot. Sequence: Rombe
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$
1e6


:quation: Lorenz, estimate after 0.2 time steps.. Stack plot. Sequence: Rombe
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps


quation: Lorenz, estimate after 0.2 time steps.. Stack plot. Sequence: Harmo
Natural logarithm of absolute error, ln$\varepsilon$
Number of function evaluations, $N$


quation: Lorenz, estimate after 0.2 time steps.. Stack plot. Sequence: Harmo
Base 10 logarithm of absolute error, log$\varepsilon$
Extrapolation steps

| Sequence | Plot | $A$-mean | $A$-var | $c$-mean | $c$-var | $q$-mean | $q$-var |
|---|---|---|---|---|---|---|---|
| Romberg | lin-ln evals-error | $2.009 \cdot 10^{46}$ | 6 | 37.38 | 0.2052 | 0.1527 | 0.04644 |
| Harmonic | lin-ln evals-error | $7.974 \cdot 10^{8}$ | 7.349 | 2.931 | 0.02946 | 0.5011 | 0.001063 |
| Romberg | lin-ln steps-error | 1040 | 1.066 | 0.337 | 0.01327 | 2.18 | 0.0003518 |
| Harmonic | lin-ln steps-error | $1.538 \cdot 10^{8}$ | 7.211 | 2.931 | 0.02863 | 1.002 | 0.001041 |

Here we also get down to machine level precision in standard double precision floating point arithmetic. The harmonic sequence performes better.

The model for exponential convergence in the number of steps fits moderately well for the Romberg sequence. We get moderate fit for exponential convergence for the harmonic sequence.

The values of the optimal parameters in the fitting of the number evaluations against the error, are:

| IVP | Sequence | $b$ | $c$ | $q$ |
|---|---|---|---|---|
| $y' = y,\ y(0) = 0$ | Romberg | 65.485 | 52.551 | 0.13291 |
| $y' = y,\ y(0) = 0$ | Harmonic | 8.8124 | 3.0433 | 0.61436 |
| $y' = y(1-y)$ | Romberg | 53.149 | 47.621 | 0.13177 |
| $y' = y(1-y)$ | Harmonic | 1.7927 | 4.0933 | 0.51064 |
| $y' = 1 + y^2,\ y(0) = 0$ | Romberg | 33.382 | 24.704 | 0.15812 |
| $y' = 1 + y^2,\ y(0) = 0$ | Harmonic | 3.3496 | 1.8647 | 0.51993 |
| $(y_1, y_2)' = (-y_2, y_1),\ y(0) = (1, 0)$ | Romberg | 57.413 | 43.792 | 0.1403 |
| $(y_1, y_2)' = (-y_2, y_1),\ y(0) = (1, 0)$ | Harmonic | 9.7592 | 2.4171 | 0.62765 |
| $y' = y^2,\ y(0) = 1/2$ | Romberg | 39.999 | 31.606 | 0.14796 |
| $y' = y^2,\ y(0) = 1/2$ | Harmonic | 3.5994 | 2.5894 | 0.51472 |
| $y' = y^2,\ y(0) = 1/(1 + 10^{-2})$ | Romberg | 11.039 | 2.3505 | 0.26394 |
| $y' = y^2,\ y(0) = 1/(1 + 10^{-2})$ | Harmonic | 4.7796 | 0.032416 | 0.6592 |
| $y' = y^2,\ y(0) = 1/(1 + 10^{-4})$ | Romberg | 9.6163 | 0.016825 | 0.50749 |
| $y' = y^2,\ y(0) = 1/(1 + 10^{-4})$ | Harmonic | 9.2104 | 0.00016925 | 0.74975 |
| $y' = -1/2y,\ y(0) = \sqrt{2}$ | Romberg | 42.458 | 37.281 | 0.13969 |
| $y' = -1/2y,\ y(0) = \sqrt{2}$ | Harmonic | −0.45512 | 3.1436 | 0.49744 |
| $y' = -1/2y,\ y(0) = \sqrt{1 + 10^{-2}}$ | Romberg | 6.2634 | 4.2006 | 0.23055 |
| $y' = -1/2y,\ y(0) = \sqrt{1 + 10^{-2}}$ | Harmonic | −1.7425 | 0.33371 | 0.45544 |
| $y' = -1/2y,\ y(0) = \sqrt{1 + 10^{-4}}$ | Romberg | −1.4645 | 0.28248 | 0.33417 |
| $y' = -1/2y,\ y(0) = \sqrt{1 + 10^{-4}}$ | Harmonic | 2.2343 | 3.3206 | 0.091009 |
| $y'' + sin(y) = 0$ | Romberg | 52.459 | 42.97 | 0.13575 |
| $y'' + sin(y) = 0$ | Harmonic | 4.7064 | 3.6116 | 0.50678 |
| Federpendel, estimate after 1 time unit. | Romberg | 41.924 | 34.768 | 0.14246 |
| Federpendel, estimate after 1 time unit. | Harmonic | 0.30737 | 2.706 | 0.49211 |
| Federpendel, estimate after 2 time units. | Romberg | 34.307 | 25.884 | 0.15365 |
| Federpendel, estimate after 2 time units. | Harmonic | 0.79006 | 1.789 | 0.48379 |
| Lorenz, estimate after 0.1 time steps. | Romberg | 54.885 | 40.609 | 0.14032 |
| Lorenz, estimate after 0.1 time steps. | Harmonic | 12.615 | 4.0068 | 0.5111 |
| Lorenz, estimate after 0.2 time steps. | Romberg | 39.854 | 25.197 | 0.16206 |
| Lorenz, estimate after 0.2 time steps. | Harmonic | 13.212 | 2.7376 | 0.50732 |

Table 4.1: Optimal parameters by test case

We note that in those cases where the singularities of the solutions are not very close to our time interval, then $q$ is close to 0.5 for the harmonic sequence and close to 0.2 for

the Romberg sequence.

The values of the optimal parameters in the fitting of the number of extrapolation steps against the error, are:

| IVP | Sequence | $b$ | $c$ | $q$ |
|---|---|---|---|---|
| $y' = y$, $y(0) = 0$ | Romberg | $-1.5037$ | 0.87585 | 1.9378 |
| $y' = y$, $y(0) = 0$ | Harmonic | 6.425 | 3.1061 | 1.225 |
| $y' = y(1 - y)$ | Romberg | $-7.388$ | 0.79828 | 1.9293 |
| $y' = y(1 - y)$ | Harmonic | $-0.19572$ | 4.1127 | 1.0204 |
| $y' = 1 + y^2$, $y(0) = 0$ | Romberg | $-1.1782$ | 0.30723 | 2.1752 |
| $y' = 1 + y^2$, $y(0) = 0$ | Harmonic | 2.405 | 1.8763 | 1.0387 |
| $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$ | Romberg | 0.13463 | 0.67385 | 2.0061 |
| $(y_1, y_2)' = (-y_2, y_1)$, $y(0) = (1, 0)$ | Harmonic | 7.7307 | 2.4712 | 1.2513 |
| $y' = y^2$, $y(0) = 1/2$ | Romberg | $-2.521$ | 0.44573 | 2.0777 |
| $y' = y^2$, $y(0) = 1/2$ | Harmonic | 2.3181 | 2.6033 | 1.0284 |
| $y' = y^2$, $y(0) = 1/(1 + 10^{-2})$ | Romberg | 5.4845 | 0.0052111 | 3.2981 |
| $y' = y^2$, $y(0) = 1/(1 + 10^{-2})$ | Harmonic | 4.7474 | 0.033263 | 1.3137 |
| $y' = y^2$, $y(0) = 1/(1 + 10^{-4})$ | Romberg | 9.3422 | $1.3598e - 07$ | 6.3724 |
| $y' = y^2$, $y(0) = 1/(1 + 10^{-4})$ | Harmonic | 9.2101 | 0.00017545 | 1.4929 |
| $y' = -1/2y$, $y(0) = \sqrt{2}$ | Romberg | $-6.2149$ | 0.57862 | 1.9997 |
| $y' = -1/2y$, $y(0) = \sqrt{2}$ | Harmonic | $-1.8945$ | 3.1518 | 0.99437 |
| $y' = -1/2y$, $y(0) = \sqrt{1 + 10^{-2}}$ | Romberg | $-2.0186$ | 0.016933 | 2.9296 |
| $y' = -1/2y$, $y(0) = \sqrt{1 + 10^{-2}}$ | Harmonic | $-1.8689$ | 0.33241 | 0.91148 |
| $y' = -1/2y$, $y(0) = \sqrt{1 + 10^{-4}}$ | Romberg | $-2.613$ | 0.00012201 | 4.1931 |
| $y' = -1/2y$, $y(0) = \sqrt{1 + 10^{-4}}$ | Harmonic | 1.3726 | 2.6336 | 0.20682 |
| $y'' + sin(y) = 0$ | Romberg | $-2.8658$ | 0.69546 | 1.9636 |
| $y'' + sin(y) = 0$ | Harmonic | 2.9822 | 3.6265 | 1.0128 |
| Federpendel, estimate after 1 time unit. | Romberg | $-3.9094$ | 0.52384 | 2.0251 |
| Federpendel, estimate after 1 time unit. | Harmonic | $-0.90208$ | 2.7108 | 0.98385 |
| Federpendel, estimate after 2 time units. | Romberg | $-1.2973$ | 0.34056 | 2.132 |
| Federpendel, estimate after 2 time units. | Harmonic | 0.020877 | 1.79 | 0.96743 |
| Lorenz, estimate after 0.1 time steps. | Romberg | 1.7404 | 0.62119 | 2.0083 |
| Lorenz, estimate after 0.1 time steps. | Harmonic | 10.659 | 4.0254 | 1.0213 |
| Lorenz, estimate after 0.2 time steps. | Romberg | 4.0863 | 0.2983 | 2.2131 |
| Lorenz, estimate after 0.2 time steps. | Harmonic | 11.891 | 2.7477 | 1.014 |

Table 4.2: Optimal parameters by test case

# Bibliography

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[2] Konrad Knopp. *Theorie und Anwendung der unendlichen Reihen.*, Springer Verlag, Berlin, Heidelberg, New York, (5. Auflage) 1964.