NUMBERS WITH INTEGER COMPLEXITY CLOSE TO THE LOWER BOUND

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Abstract. Define $f(n)$ to be the *integer complexity* of n, the smallest number of ones needed to write n using an arbitrary combination of addition and multiplication. John Selfridge showed that $f(n) \geq 3\log_3 n$ for all n. Define $d(n) = f(n) - 3\log_3 n$; in this paper we present a method for classifying all n with $d(n) < r$ for a given r. From this, we derive a number of consequences. We prove that $f(2^m 3^k) = 2m + 3k$ for $m \leq 31$ with m and k not both zero, and present a method that can, with more computation, prove the same for larger m. We extend a result of Daniel Rawsthorne by finding formulae for the r'th largest number with complexity k, so long as k is sufficiently large relative to r. Defining A_r to be the set of all n with $d(n) < r$, we prove that $A_r(x) = \Theta((\log x)^{\lfloor r \rfloor + 1})$. Finally we prove that the set of all values of d is well-ordered, with order type ω^ω, as was conjectured earlier by Juan Arias de Reyna. We also make some conjectures that suggest a possible approach for proving $f(n) \not\sim 3\log_3 n$.

1. Introduction and motivation

In this paper we consider the notion of *integer complexity*, as was introduced by Mahler and Popken in 1953 in [6], and later popularized by Richard Guy in [3]; it appears as problem F26 in his *Unsolved Problems in Number Theory* [4]. We say the complexity of a natural number n is the smallest number ones needed to write it using any combination of addition and multiplication. For instance, 7 has a complexity of 6, since it can be written using 6 ones as $(1+1+1)(1+1)+1$, but not with any fewer. We will refer to a representation of n that uses the smallest number of ones as *most-efficient*, and we will denote the complexity of n by $f(n)$. Note that we can compute $f(n)$ recursively; $f(1) = 1$, and for $n > 1$,

$$f(n) = \min_{\substack{a,b \in \mathbb{N} \\ a+b=n \text{ or } ab=n}} f(a) + f(b).$$

(For those interested in actually computing f, there is an algorithm slightly faster than the naïve one, due to Srinivas and Shankar and described in [8]; we used this in our own calculations.)

The complexity of n is approximately logarithmic in n. For $n > 1$, we have $f(n) \leq 3\log_2 n$ as can be seen by writing n in binary. John Selfridge showed (as noted in [7]) that we also have $f(n) \geq 3\log_3 n$. In more detail, let us denote by $E(k)$ the largest number writable with k ones. What Selfridge showed is that $E(k)$ is given by the formulae $E(1) = 1$; $E(3k) = 3^k$; $E(3k+2) = 2\cdot 3^k$; and $E(3k+4) = 4\cdot 3^k$. (Note, by the way, that for $k > 1$, we have $E(k+3) = 3E(k)$.) In particular, for any k we have $E(k) \leq 3^{k/3}$, and so for any n we have $f(n) \geq 3\log_3 n$.

Obviously a slightly better lower bound can be obtained from Selfridge's result, namely, $f(n) \geq L(n)$ where $L(n)$ is given by

$$L(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 3k & \text{if} \quad 3^k \leq n < 4 \cdot 3^{k-1} \text{ and } k \neq 0 \\ 3k+2 & \text{if} \quad 2 \cdot 3^k \leq n < 3^{k+1} \\ 3k+4 & \text{if} \quad 4 \cdot 3^k \leq n < 2 \cdot 3^{k-1} \end{cases}$$

Note that $L(n)$ is equal to the smallest number of ones needed to make a number that is at least n, as well as the smallest k such that $E(k) \geq n$. We also have, for $n > 1$, $L(3n) = L(n)+3$. Also observe that, past the initial 1, the ratios of successive values of E are $3/2, 4/3, 3/2, 3/2, 4/3, 3/2, \ldots$; this sequence of ratios will reoccur later.

Not much more about the growth rate of f is known, though better upper bounds have recently been proven. Indeed, it is not even known that $f(n) \not\sim 3\log_3(n)$, a rather weak assertion. We provide a potential approach to this problem in Section 11.

The notion of integer complexity is somewhat similar to the more well-known notion of addition chain length. An addition chain for n of length k is an increasing sequence $(a_0, \ldots, a_k)$, starting with $a_0 = 1$, ending with $a_k = n$, and such that, for $i > 0$, each a_i is the sum of two (possibly equal) previous terms. The length of the shortest addition chain for n is denoted $l(n)$ [9]. Both are ways of measuring the complexity of a given number, and both are approximately logarithmic in n. However they get there a bit differently – in an addition chain, once a number is constructed once, it can then be used repeatedly at no additional cost. This is not allowed in integer complexity; $f(5) = 5$, as every 1 in $1+1+1+1+1$ must be paid for. Instead of allowing free reuse, though, we do allow multiplication. It seems that allowing either multiplication or free reuse is enough to make the resulting complexity measure approximately logarithmic. Interestingly, if we allow both of these by considering $\tau(n)$, the length of the shortest addition-multiplication chain for n, while obviously $\tau(n) = \Omega(\log\log n)$, it is not known that $\tau(n) = O(\log\log n)$, nor is it expected to be true. In fact, the growth rate of τ – specifically, $\tau(n!)$ – is related to the P-vs.-NP problem for Blum-Shub-Smale machines over the complex numbers. If there is no sequence a_n such that $\tau(a_n n!)$ grows only polylogarithmically, then $P \neq NP$ for such machines. [2]

Unfortunately, integer complexity seems to be lacking in nice properties. For instance, it is not monotonic; $f(11) = 8$, but $f(12) = 7$ (via the representation $(1+1+1+1)(1+1+1)$). One case where it does behave nicely is powers of 3; for $n > 0$ we have $f(3^n) \leq 3n$, since we can write 3^n as the product of n copies of $1+1+1$; and by Selfridge's bound above, we do indeed have that $f(3^n) = 3n$, and there is no shorter way of writing 3^n. However, if we attempt to generalize this to, say, powers of 5, this fails; while $f(5) = 5$, and $f(25) = 10$, and $f(125) = 15$, it turns out that $f(5^6) = 29$ instead of the expected 30, as $5^6 = 15626 = 1+(1+1)(1+1)(1+1)(1+1+1)(1+1+1)(1+(1+1)(1+1)(1+1)(1+1+1)(1+1+1)(1+1+1))$.

The corresponding question for powers of 2, whether $f(2^m) = 2m$ for $m > 0$, is open. More generally, is $f(2^m 3^k) = 2m + 3k$, if m and k are not both zero? Jānis Iraids has verified this for $2^m 3^k \leq 10^{12}$ [5], so in particular $f(2^m) = 2m$ for $1 \leq m \leq 39$. Note that while handling general powers of 2 is hard, fixed powers of

2 with varying powers of 3 turn out to be considerably more workable. In Section 5 of this paper we will prove:

Theorem 1.1. *For $m \le 31$ and any k with m and k not both zero, $f(2^m 3^k) = 2m + 3k$.*

Moreover, we will present a general method that can prove this for larger m with more computation.

It is not too hard to prove this for smaller m; from Selfridge's bound we see that this holds for $m \le 2$. Indeed, Selfridge's result is in fact strong enough to show:

Proposition 1.2. *For $m \le 10$ and any k with m and k not both zero, $f(2^m 3^k) = 2m + 3k$.*

Proof. It suffices to show it for $m = 10$. Note that $E(19+3k) = 4 \cdot 3^{5+k} = 972 \cdot 3^k < 1024 \cdot 3^k$; hence $2^{10}3^k$ cannot be made with less than $20 + 3k$ ones. ☐

Define $E_r(k)$ to mean the r-th highest number writable with k ones (where we are zero-indexing, so $E_0(k) = E(k)$). Rawsthorne proved in 1989 in [7] that for $k \ge 6$, we have $E_1(k) = (8/9)E(k)$. From this we can conclude:

Proposition 1.3. *For $m \le 13$ and any k with m and k not both zero, $f(2^m 3^k) = 2m + 3k$.*

Proof. It suffices to show it for $m = 13$. Note that $E_1(25 + 3k) = 32 \cdot 3^{5+k} = 7776 \cdot 3^k < 8192 \cdot 3^k$. Hence $2^{13}3^k$ cannot be written with less than $26 + 3k$ ones, unless it is equal to $E(25 + 3k) = 4 \cdot 3^{7+k}$, which it is not. ☐

One might ask if $f(3n) = f(n)+3$ for $n > 1$, but this is false; 107 is the smallest counterexample, with $f(107) = 16$ but $f(321) = 18$. However, in Section 5 we will show that this is true if $n > E(f(n))/3$. Furthermore in the next section we will prove the following, earlier conjectured by Juan Arias de Reyna:

Theorem 1.4. *For any m, there exists a K such that for any $k \ge K$, we have $f(3^k m) = 3(k - K) + f(3^K m)$.*

Now one approach to proving $f(2^m 3^k) = 2m + 3k$ for larger m might be to directly extend Rawsthorne's result − if one had similar formulae for $E_r(k)$ (at least for k sufficiently large relative to r), one could apply this same technique to push m up further. In fact such formulae exist − in Section 6 we will prove:

Theorem 1.5. *Given $r \ge 0$, and k_0 a congruence class modulo 3, there exists K and h such that for $k \ge K$ with $k \equiv k_0 \pmod 3$, we have $E_r(k) = hE(k)$, where K and h are given by tables that can be found in Section 6.*

(It is also possible to prove these formulae by other methods than the ones in this paper, but we do not have the space here to discuss such techniques.) But as we will later see, regardless of the size of r, these cannot get us past $m = 18$. We would like to consider $E_\omega(k)$, but this is meaningless. A change of viewpoint is needed.

Juan Arias de Reyna in [1] found the correct way of considering these formulae to extend them beyond the finite case. Rather than write $E_r(k) = hE(k)$, instead write $E_r(k)/E(k) = h$. Replace $E_r(k)$ with an arbitrary n, and $E(k)$ with $E(f(n))$. So instead of talking about the highest numbers that can be made with k ones, we can talk about the highest values of $R(n) := n/E(f(n))$ that can occur. It is not obvious that this replacement is justified, i.e. that $f(E_r(k)) = k$ when k is large enough for the appropriate formulae to hold; we will prove this when we derive the formulae for $E_r(k)$. We will then classify the highest values of these that can occur, and use this to prove $f(2^m 3^k) = 2m + 3k$ for higher values of $m \leq 31$ and there is no obvious obstacle to using this for any given m.

For this to really "extend the formulae for E_r into the transfinite range", we would want the set of all values of $R(n)$ to be reverse well-ordered. This turns out to be true:

Theorem 1.6. *The set of all values of $R(n)$ is reverse well-ordered, with order type ω^ω.*

Arias conjectured this in [1] and we will prove this in Sections 8 and 9.

2. Introduction part 2 – defect and integral defect

While E is easy to compute, it is still not the easiest to work with, so instead of $n/E(f(n))$ we will prefer to consider $n/3^{f(n)/3}$. Nothing is lost if we switch to this formulation; $3^{k/3} = cE(k)$ for a constant c depending only on the residue class of k modulo 3 (and whether or not $k = 1$), so if we know what k is modulo 3 it is then easy to convert back and forth between the two notions. Indeed, $n/3^{f(n)/3}$ actually carries more information, as due to the irrationality of $3^{1/3}$ and $3^{2/3}$ we can from this number determine the residue of $f(n)$ modulo 3 and thus determine $R(n)$, whereas the reverse is not possible. (The case $n = 1$ puts a slight snag in this, but it easy to see that if $n/3^{f(n)/3} = 3^{-1/3}$ then we must have $n = 1$, as otherwise we would have $f(n) = 4 \cdot 3^k$ but $n = 4 \cdot 3^{k-1}$, an impossibility. More on this in a moment.)

We will make one more change of viewpoint – instead of working with $n/3^{f(n)/3}$, we will work with $f(n) - 3\log_3(n)$, which we will denote $d(n)$ and call the *defect* of n. This encodes the same information, as they're related by $n/3^{f(n)/3} = 3^{-d(n)/3}$; in particular from it we can deduce both $R(n)$ the residue of $f(n)$ modulo 3. The problem now becomes to classify numbers of small defect – numbers with complexity close to the lower bound.

We will prove in Section 7 is that there are not very many numbers with complexity close to the lower bound:

Theorem 2.1. *Let A_r denote the set of all n with $d(n) < r$, and let $A_r(x)$ denote the number of elements of A_r which are less than x. Then for any $r \geq 0$, we have $A_r(x) = \Theta((\log x)^{\lfloor r \rfloor + 1})$.*

To be concrete about the equivalence between d and R,

$$d(n) = \begin{cases} -3\log_3(R(n)) & \text{if} \quad f(n) \equiv 0 \pmod 3 \\ -3\log_3(R(n)) + d(2) & \text{if} \quad f(n) \equiv 2 \pmod 3 \\ -3\log_3(R(n)) + 2d(2) & \text{if} \quad f(n) \equiv 1 \pmod 3 \text{ and } n \neq 1 \\ 0 & \text{if} \quad n = 1 \end{cases}$$

In fact $d(n)$ encodes enough information that we have:

Lemma 2.2. *If $d(n) = d(m)$ for two numbers n and m, then either $n = m3^k$ for some k or vice versa. Indeed if $d(n) - d(m)$ is any rational number, in particular any integer, the same conclusion holds.*

Proof. If $d(n) - d(m)$ is rational, then $\log_3(n/m)$ is rational; since n/m is rational, the only way this can occur is if $\log_3(n/m)$ is an integer. □

Of course, there is a better lower bound than $3\log_3(n)$, namely $L(n)$, so define $D(n) = f(n) - L(n)$, the *integral defect* of n. We will not typically work with this directly, however, as it is much less convenient and is a lossy notion; for instance, $D(n) = 0$ for all n from 1 to 10. To say $D(n) = 0$ is to say $f(n) = L(n)$, that is, $f(n)$ is the smallest k such that $E(k) \geq n$, or in other words, that $E(f(n))$ is the smallest value of E which is at least n. So more generally to say $D(n) = l$, or $L(n) = f(n) - l$, then, is to say that $E(f(n))$ is the l-th value of E past the smallest one which is at least n. Thus what D actually tells is how many values $E(k)$ there are such that $n \leq E(k) \leq E(f(n))$, it being one less than this number. Since (barring the initial 2/1) the ratios of successive values of E repeat with a period of 3, it follows that $D(n)$ is actually telling us the broad range in which $R(n)$ falls, though the changeover points will depend on the residue of $f(n)$ modulo 3. (Hence if we know $d(n)$, since we can determine $R(n)$ and the residue of $f(n)$ modulo 3, we can determine $D(n)$.)

Equivalently, to say that $D(n) \leq l$, is the same as to say that, n is greater than all numbers m with $f(m) < f(n) - l$. Thus, $D(n) = 0$ is the same as to say that n is greater than all m with $f(m) < f(n)$. Note that D is numerically close to d, since $L(n)$ is close to $3\log_3 n$; for any n, by using the formulae for E and L we can see that $3\log_3 n \leq L(n) < 3\log_3 n + 1 + 2d(2)$, and therefore $D(n) \leq d(n) < D(n) + 1 + 2d(2)$.

Note that if $n = ab$ with $f(n) = f(a) + f(b) - k$, then $d(n) = d(a) + d(b) - k$, and in particular $d(3n) = d(n)$ if and only if $f(3n) = 3 + f(n)$. Similarly note for $n > 1$, if $f(3n) = f(n) + 3 - k$, then $D(3n) = D(n) - k$.

The notion of defect gives us an easy proof of the following conjecture of Arias:

Theorem 2.3. *For any m, there exists a K such that for any $k \geq K$, we have $f(3^k m) = 3(k - K) + f(3^K m)$.*

Proof. For any n, note that $d(3n) \leq d(n)$, with equality if and only if $f(3n) = f(n) + 3$. So the sequence $d(m), d(3m), d(9m), \ldots$ is non-increasing, nonnegative, and can only decrease in integral amounts, hence it must eventually stabilize, which proves the theorem. (The proof could be done with D instead with little modification.) □

Note that this still leaves open

Question 2.4. *In Theorem 2.3, is it possible to get a bound on K based on m?*

3. Main Lemma

Here, we state and prove what will be our primary tool for the rest of the paper. Given any $r > 0$, we can, by applying this lemma repeatedly, put restrictions on what n can satisfy $d(n) < r$.

For any real $r \geq 0$, define A_r to be $\{n \in \mathbb{N} : d(n) < r\}$; define D_r to be $\{d(n) : n \in \mathbb{N}, d(n) < r\}$; and define B_r to consist of those elements of A_r that cannot be written most efficiently as $3m$ for any m (so that A_r consists precisely of elements of B_r times powers of 3). Define $\overline{A}_r$, $\overline{D}_r$, and $\overline{B}_r$ to be like A_r, D_r, and B_r but with nonstrict inequalities.

We will use the main lemma to inductively build up a superset of A_r. Note that as a consequence of Rawsthorne's result that $E_1(k) = (8/9)E(k)$ for $k \geq 6$, we have that for any n, if $R(n) > 8/9$ then $R(n) = 1$. Considering what this yields for

$d(n)$ (depending on n modulo 3), we see that $d(2) = 2 - 3\log_3 2 = 0.107\ldots$ is the smallest non-zero defect, i.e. $A_{d(2)}$ consists only of powers of 3. This is the base case to which our lemma is the inductive step.

(Though we will later derive the previously mentioned formulae for $E_r(k)$ as a consequence of this base case and main lemma, it is actually possible to prove them by an entirely different method which does not use Rawsthorne's result as a base case. Again, however, we do not have the space to discuss such here.)

Finally define T_α to consist of those natural numbers $n < \frac{1}{3^{\frac{1-\alpha}{3}}-1} + 1$ whose only shortest representation is as either 1 or as $(n-1)+1$. Note that for $\alpha < 1$, we have that T_α is a finite set. Then:

Proposition 3.1. *For any $\alpha < 1$ and $k \geq 2$, any element of $B_{(k+1)\alpha}$ can be written most efficiently in one of the following forms:*

(1) A product of an element of $B_{i\alpha}$ and an element of $B_{j\alpha}$ with $i + j = k + 2$ and $2 \leq i, j \leq k$, with defects totalling less than $(k + 1)\alpha$;

(2) An element a of $A_{k\alpha}$ plus a number $b \leq a$ such that

$$d(a) + f(b) < (k + 1)\alpha + 3\log_3 2,$$

possibly times an element of B_α;

(3) Or an element of T_α, possibly times an element of B_α;

And any element of $B_{2\alpha}$ can be written in one of the following forms:

(1) A product of up to three elements of B_α, with defects totalling less than 2α;

(2) An element a of A_α plus a number $b \leq a$ such that

$$d(a) + f(b) < 2\alpha + 3\log_3 2,$$

possibly times an element of B_α;

(3) Or an element of T_α, possibly times an element of B_α.

Proof. Suppose $m \in B_{(k+1)\alpha}$; take a most efficient representation of m, which is either ab, $a + b$, or 1. If $m = 1$, we are done.

Suppose m can be most efficiently written as a product, say as $\prod_{i=1}^r m_i$ (where $r \geq 2$ and each m_i cannot be written most efficiently as a product); so $\sum_{i=1}^r d(m_i) = d(m) < (k + 1)\alpha$. Note that no product of a subset of the m_i can be written most efficiently as 3 times another number, as else we could choose one of the m_i to be 3, contrary to the assumption $m \in B_{(k+1)\alpha}$. If $k \geq 2$, then either there exists an i with $d(m_i) \geq k\alpha$, or else we can partition the $d(m_i)$ into two nonempty sets each with sum less than $k\alpha$; in this case, call their products a and b. Then $d(a) + d(b) < (k + 1)\alpha$, so if we let $(i - 1)\alpha$ be the largest multiple of α at most $d(a)$, then $d(b) < j\alpha$ if we let $j = k + 2 - i$. If such a partition is not possible, then letting a be m_i with $d(m_i) \geq k\alpha$, and b the product of the others, then m is the product of an element of B_α and an element of $B_{(k+1)\alpha}$ which cannot be written as a product; see next paragraph for how to handle this. In the case $k = 1$, either there exists an i with $d(m_i) \geq k\alpha$, or else we can partition the $d(m_i)$ into either two or three nonempty sets each with sum less than α, so either we have the "see next paragraph case", or we have a product of three elements of B_α.

So now we have to determine what elements of $B_{(k+1)\alpha}$ cannot be written in this way, both to complete the multiplication case and to handle the addition case. Suppose $m = a + b$ with $f(m) = f(a) + f(b)$ and b minimal, so $a \geq b$. Then

$$f(a) + f(b) = f(m) < 3\log_3(a + b) + (k + 1)\alpha \leq 3\log_3(2a) + (k + 1)\alpha,$$

so $d(a)+f(b) < (k+1)\alpha+3\log_3 2$. Furthermore, b cannot be written most efficiently as a sum $c+d$ or else we could regroup $a+(c+d)$ as $(a+c)+d$ which would contradict the minimality of b.

If $a \in A_{k\alpha}$, we are done. Otherwise, we have

$$3\log_3 a + k\alpha + f(b) < f(a) + f(b) = f(m) <$$
$$3\log_3(a+b) + (k+1)\alpha \le 3\log_3(2a) + (k+1)\alpha,$$

so $f(b) < 3\log_3 2 + \alpha$; since $\alpha < 1$, we have $f(b) \le 2$ and thus $b \le 2$; by the assumption that b cannot be written most efficiently as a sum, we have $b = 1$. Hence $3\log_3 a + k\alpha + 1 < 3\log_3(a+1) + (k+1)\alpha$; if we solve for a, we find that $m = a+1 \in T_\alpha$; and we may assume that this is the only most efficient way to write m as otherwise it would be covered in one of the other cases. □

Note that while we specified that the representation is most efficient, and included constraints based on the defects of the numbers in the already-known A_r, we don't really need either of these for most purposes, when we are just interested in computing some superset of $A_{(k+1)\alpha}$. We don't even have any easy way of checking whether the resulting reprentations are most-efficient, after all. In particular, in the addition case, note that the requirement that $d(a)+f(b) < (k+1)\alpha+3\log_3 2$ implies the weaker requirement that just $f(b) < (k+1)\alpha + 3\log_3 2$, or just $L(b) < (k+1)\alpha + 3\log_3 2$. We'll define the set of b satisfying $f(b) < x + 3\log_3 2$ to be S_x; note that S_x is always finite.

For $\overline{A}_k$ and $\overline{B}_k$ we have a similar theorem – just use $\overline{B}_r$'s instead of B_r's in the pure product case and use a nonstrict inequality – and the proof is the same except for the strictness of the inequalities. Whether we state theorems in terms of A_r or $\overline{A}_r$ will depend on convenience, but typically the distinction will not matter since $\overline{A}_r \setminus A_r$ can consist of at most a single number times powers of three.

4. Computation and bootstrapping

We will explicitly computing the set A_1. We do this for two reasons. Firstly, we will later compute A_r for larger r, so A_1 is on the way. Secondly, later results in Section 7 and onward will depend on one the structure of A_1. This section accomplishes a sort of bootstrapping – we compute A_1 using $A_{d(2)}$ as our base case, but once we have A_1 it will become our new base case.

Since we want to compute A_1, we will here discuss how we can use the main lemma to compute A_r for small r. The main lemma allows us to inductively construct a superset of A_r, but if we want to determine A_r itself – and if we don't want this computation to blow up very quickly – we'll need some way of then determining the actual complexities of the resulting candidates.

Upper bounds are easy. To find lower bounds on complexities, we will typically use the following technique: Say we want to show that $f(n) \ge k$; since $f(n)$ is always an integer, it suffices to show $f(n) > k-1$, which we do by noting that if $f(n) \le k-1$ held, it would put n in some A_l which we have already determined and know it's not in. In particular, we have:

Lemma 4.1. *Take $\alpha \le 1/2$. Say $d(a) < i\alpha$ and $d(b) < j\alpha$, and let $k+2 = i+j$. Then $f(ab) = f(a) + f(b)$ unless $d(ab) < k\alpha$.*

Proof. Note

$$f(ab) \ge 3\log_3(ab) + k\alpha = 3\log_3 a + 3\log_3 b + (i+j-2)\alpha > f(a) + f(b) - 1$$

so $f(ab) \geq f(a) + f(b)$, done. $\square$

Lemma 4.2. *Take $\alpha \leq d(2)$. Say $d(a) < k\alpha$, then $f(3^m(a+1)) = 3m + f(a) + 1$ unless $d(3^m(a+1)) < k\alpha$.*

Proof. Note

$$f(3^m(a+1)) \geq 3\log_3(a+1) + 3m + k\alpha > f(a) + 3m$$

so $f(3^m(a+1)) \geq 3m + f(a) + 1$, done. $\square$

With these two lemmas in hand, and the base case knowledge of A_α for $\alpha \leq d(2)$, we can pick a step size $\alpha \leq d(2)$ and inductively compute $A_{k\alpha}$. If we know $A_{k\alpha}$, and the complexities of the numbers therein, first we use the main lemma to generate candidates for $A_{(k+1)\alpha}$; then we check which of these are already in $A_{k\alpha}$. For those numbers that are already in, we are done. For those that are not, we can use the above two lemmas to determine their complexities and hence whether they are actually in. This works so long as $k\alpha < 5 - 3\log_3 2 = 3 + d(2)$ (or $k \leq 28$), as then $S_{k\alpha} = \{1\}$.

So for A_r for $r < 3 + d(2)$, we have a method to determine A_r, which is almost an algorithm – unfortunately to turn it into an actual algorithm, we would need some way to represent the data that would allow the required operations, and we have not figured this out. Hence all our computations have been done by hand. This method works for $r \geq 3 + d(2)$ as well, by specially tweaking Lemma 4.2 and the additive case of the main lemma to handle additive constants other than 1, but we will not discuss this here.

It isn't obvious that expressions of the form $a+b$ are ever relevant when $a, b \neq 1$; Rawsthorne's computations in [7] failed to uncover any most-efficient representations $a+b$ with $a, b \neq 1$, for instance. However, such numbers do exist, as Arias and Van de Lune found [4]; they even give a prime p which is most efficiently represented as $6 + (p-6)$ but not $1 + (p-1)$, namely $353942783 = 6 + 353942777$.

In any case, we can then compute that the numbers with defect less than 1 are as follows:

- 3^k of complexity $3k$ for $k \geq 1$
- $2^a 3^k$ of complexity $2a + 3k$ for $a \leq 9$
- $5 \cdot 2^a 3^k$ of complexity $5 + 2a + 3k$ for $a \leq 3$
- $7 \cdot 2^a 3^k$ of complexity $6 + 2a + 3k$ for $a \leq 2$
- $19 \cdot 3^k$ of complexity $9 + 3k$
- $13 \cdot 3^k$ of complexity $8 + 3k$
- $(3^n + 1)3^k$ of complexity $1 + 3n + 3k$ (unless $n = 0$)

(Note also 1 is the only number of defect exactly 1.) The real significance of this list, however, that makes everything in Section 7 and onward work, is:

Corollary 4.3. *For every $\alpha < 1$, we have that B_α is a finite set.*

Note this fails for B_1; the defects of $(3^n + 1)3^k$ approach 1 as n approaches infinity.

5. Further results of computation

Using the above method we have in fact classified all numbers with defect less than $22d(2) = 2.3586\ldots$. This is greater than $2 + 2d(2)$ so this in particular allows us to determine all numbers with integral defect at most one.

Due to its length, we have attached a table of the results separately.

Since determining A_r allows us to put lower bounds on the complexities of any numbers not in it. we have the following:

Lemma 5.1. *Suppose that none of* $2^{n+9}3^k$ *lie in* $A_{nd(2)}$ *for any* k*. Then for any* $m \le n+9$ *and any* k *(with* m *and* k *not both zero),* $f(2^m3^k) = 2m+3k$*.*

Proof. It suffices to show that $f(2^{n+9}3^k) > 2n+3k+17$, but by assumption,

$$f(2^{n+9}3^k) > (n+9)3\log_3 2 + 3k + nd(2) = 2n + 3k + 27\log_3 2 > 2n + 3k + 17,$$

and we are done. □

From our classification, it is straightforward to check that $2^{31}3^k$ does not lie in $A_{22d(2)}$ for any k, so we can conclude

Corollary 5.2. *For* $m \le 31$ *and any* k *with* m *and* k *not both zero,* $f(2^m3^k) = 2m+3k$*.*

as we claimed above. We can also classify the set of n with $D(n) = 0$ as follows:

Corollary 5.3. *The* n *with* $D(n) = 0$ *are precisely those numbers that can be written in one of the following forms:*

- 2^m3^k *with* $m \le 10$
- $2^a(2^b3^l+1)3^k$ *with* $a+b \le 2$*.*

We will show in Section 6 that from this statement we can deduce formulae for the $E_r(k)$ (for k sufficiently large relative to r), and vice versa; it will turn out that the n satisfying $D(n) = 0$ are almost exactly the numbers that can be written as $E_r(k)$ for such k and r. Meanwhile we leave it to the reader to see how Corollary 5.3 implies that $f(2^m3^k) = 2m+3k$ for $m \le 18$ with m and k not both zero. Obviously a similar classification could be made for $D(n) \le 1$, but it is ugly and unenlightening so we have not listed it here.

Our computations also tell us a little about the question of when $f(3n)$ is equal to $f(n)+3$. We see that aside from 1, the number of smallest defect for which $f(3n) \ne f(n)+3$ is 683. We also see that

Corollary 5.4. *If* $D(3n) = 0$*, then* $D(n) = 0$*, and if* $D(3n) = 1$*, then* $D(n) = 1$*. Hence if* $D(n) \le 2$ *and* $n \ne 1$*,* $f(3n) = f(n)+3$*.*

If we rewrite the condition $D(n) \le 2$ as $D(n) < 3$, and observe that $D(n) < 3$ if and only if $R(n) > 1/3$, then can rewrite this conclusion as

Corollary 5.5. *For* $n > 1$*, if* $n > E(f(n))/3$*, then* $f(3n) = f(n)+3$*.*

Knowing A_2 makes the "α to 2α" case of the main lemma unnecessary, but we haven't computed the full ordering on the set of defects less than 2 so it's easier to use the main lemma.

6. Formulae for $E_r(n)$

As mentioned in in the previous section, Corollary 5.3 is equivalent to a series of formulae for $E_r(k)$ that work so long as k is sufficiently large depending on r, and the $E_r(k)$ obtained this way are almost exactly the n satisfying $D(n) = 0$. In this section we prove this. (Though we do not include here our original proof of Corollary 5.3, it is worth noting that our original proof went the other way – it proved Corollary 5.3 by way of these formulae. Hence why we formulate this as an equivalence rather than just stating that the formulae are a corollary.)

Fix k modulo 3. The numbers n with $D(n) = 0$ and $f(n) \equiv k \pmod 3$ are the numbers of lowest defect (or highest R) with $f(n) \equiv k \pmod 3$. If we look at the numbers n with integral defect of 0 and split them up by their complexity modulo 3, we get that n is one of:

- For $f(n) \equiv 0 \pmod 3$:
  - $(2 \cdot 3^m + 1)3^k$ with $R(n) = \frac{2 \cdot 3^m+1}{3^{m+1}}$
  - $2(3^m + 1)3^k$ (for $m \geq 1$) with $R(n) = \frac{2(3^m+1)}{3^{m+1}}$
  - $64 \cdot 3^k$ with $R(n) = 64/81$
  - $512 \cdot 3^k$ with $R(n) = 512/729$
- For $f(n) \equiv 2 \pmod 3$:
  - $(4 \cdot 3^m + 1)3^k$ with $R(n) = \frac{(4 \cdot 3^m+1)}{2 \cdot 3^{m+1}}$
  - $2(2 \cdot 3^m + 1)3^k$ with $R(n) = \frac{2(2 \cdot 3^m+1)}{2 \cdot 3^{m+1}}$
  - $4(3^m + 1)3^k$ (for $m \geq 1$) with $R(n) = \frac{4(3^m+1)}{2 \cdot 3^{m+1}}$
  - $2 \cdot 3^k$ with $R(n) = 1$
  - $128 \cdot 3^k$ with $R(n) = 64/81$
  - $1024 \cdot 3^k$ with $R(n) = 512/729$
- For $f(n) \equiv 1 \pmod 3$, $n \neq 1$:
  - $(3^m + 1)3^k$ (for $m \geq 1$) with $R(n) = \frac{(3^m+1)}{4 \cdot 3^{m-1}}$
  - $32 \cdot 3^k$ with $R(n) = 8/9$
  - $256 \cdot 3^k$ of $R(n) = 64/81$

Note that from the above tables, for each congruence class of k modulo 3, the numbers n with $D(n) = 0$ are precisely those with the ω largest values of R, or ω smallest defects, among m with $f(m) \equiv k \pmod 3$ – this is what will give us the connection between n with $D(n) = 0$, and the formulae for $E_r(k)$, the largest numbers writable with k ones. (Indeed, we expect that more generally, for each congruence class of k modulo 3, the numbers n with $D(n) \leq m$ are precisely those with the ω^m largest values of R; see Section 11 for why.)

Suppose we want to determine the r-th largest number with complexity k. From these tables, we can determine the r-th largest value of R that occurs for numbers with complexity congruent to k modulo 3. Call this value h; then $hE(k)$ is indeed the r-th largest number with complexity k if and only if all $r + 1$ values of R that are at least h and occur for numbers with complexity congruent to k modulo 3, do indeed occur for numbers with complexity k. We also know that if $D(3n) = 0$, then $D(n) = 0$, or in other words, that if $D(n) = 0$ and $k \equiv f(n) \pmod 3$, then n can be written with k ones if and only if $R(n)E(k)$ is an integer. So this take k to be large enough for all of these to be integers; hence we get the following table of resulting formulae for the r-th largest number with complexity k. When these formulae are valid, we have that they are also equal to $E_r(k)$, the r-th largest number writable with k ones, because each of them is greater than $E(k-1)$; hence any n greater than such a number which was writable with k ones would also have to have complexity exactly k (thus finally justifying our claim back in Section 1 that the expression $n/E(f(n))$ is a generalization of $E_r(k)/E(k)$).

As a corollary, we see that $D(n) = 0$ precisely when it can be written as $E_r(k)/3^l$ for some r, k as above and some l. So we have shown:

Theorem 6.1. *Given $r \geq 0$, and k_0 a congruence class modulo 3, there exists K and h such that for $k \geq K$ with $k \equiv k_0 \pmod 3$, we have $E_r(k) = hE(k)$, where K and h are given by the following tables:*

- *For $k \equiv 0 \pmod 3$:*

	r	h	K
	0	1	3
	1	8/9	6
	2	64/81	12
	3	7/9	12
	4	20/27	12
	5	19/27	12
	6	512/729	18
	7	56/81	18
	8	55/81	18
	9	164/243	18
	10	163/243	18
(for $n \geq 6$)	$2n-1$	$2/3+2/3^n$	$3n$
(for $n \geq 6$)	$2n$	$2/3+1/3^n$	$3n$

- *For $k \equiv 2 \pmod 3$:*

	r	h	K
	0	1	2
	1	8/9	8
	2	5/6	8
	3	64/81	14
	4	7/9	14
	5	20/27	14
	6	13/18	14
	7	19/27	14
	8	512/729	20
	9	56/81	20
	10	37/54	20
	11	55/81	20
	12	164/243	20
	13	109/162	20
	14	163/243	20
(for $n \geq 6$)	$3n-3$	$2/3+2/3^n$	$3n+2$
(for $n \geq 6$)	$3n-2$	$2/3+1/(2\cdot 3^{n-1})$	$3n+2$
(for $n \geq 6$)	$3n-1$	$2/3+1/3^n$	$3n+2$

- *For $k \equiv 1 \pmod 3$:*

	r	h	K
	0	1	1
	1	8/9	10
	2	5/6	10
	3	64/81	16
	4	7/9	16
	5	41/54	16
(for $n \geq 4$)	$n+2$	$3/4+1/(4\cdot 3^n)$	$3n+4$

Now we claim that given these formulae, it is actually possible to deduce Corollary 5.3; indeed, this is the route our original proof of Corollary 5.3 took.

Fix k modulo 3, and let $\lambda = \lim_{n\to\infty} h_r$, which is $2/3$ if k is 0 or 2 modulo 3 and $3/4$ if k is 1 modulo 3; i.e., $\lambda = E(k-1)/E(k)$ (save for when $k = 1$ or $k = 2$, which are easily handled). Now suppose n is of one of the forms listed in Corollary 5.3; checking that $D(n) = 0$ is then straightforward. So we just need the converse; suppose $D(n) = 0$ and $f(n) = k$. Then $n > E(k-1)$ and so $R(n) > \lambda$. Hence there exists some r with $R(n) \geq h_r > \lambda$. Thus if we let $N = n3^l$ be such that $k+3l \geq K_r$, we must have $R(N) = R(n) \in \{h_0, \ldots, h_{r-1}\}$. (We have $R(N) = R(n)$ because $D(n) = 0$.) And given this fact it then follows from the above tables that n is of one of the forms listed in Corollary 5.3.

Of note is that the original formulae for $E_0(k)$ and $E_1(k)$ that served as our base cases were both originally proven directly by induction on k, which raises the question of whether the same can be done for general $E_r(k)$ now that the formulae for them are known.

7. Abstract results – Ternary families

In our discussion of concrete computations above, we used a small step size $\alpha \leq d(2)$, and kept our superset of A_r small by using a filtering step. In what follows, we will use a different trick to keep our supersets of A_r from getting too large; we will use step sizes arbitrarily close to 1, and ignore any filtering step.

Define the set of *ternary families* to be the smallest set of functions $\mathbb{Z}_{\geq 0}^k \to \mathbb{N}$ (k varying) satisfying:

- Every "singleton" function $\mathbb{Z}_{\geq 0}^0 \to \mathbb{N}$ is a ternary family.
- Given ternary families $F : \mathbb{Z}_{\geq 0}^k \to \mathbb{N}$ and $G : \mathbb{Z}_{\geq 0}^l \to \mathbb{N}$, the function $(x, y) \mapsto F(x)G(y) : \mathbb{Z}_{\geq 0}^{k+l} \to \mathbb{N}$ is a ternary family. We'll denote this map $F \otimes G$.
- Given a ternary family F and a positive integer c, the function $(x, n) \mapsto F(x)3^n + c$ is a ternary family. We'll denote this map F_c.

We'll refer to the number of arguments of a ternary family F as its *rank* and denote it $\operatorname{rk} F$. Also for a ternary family F we'll define the corresponding *expanded ternary family* to be the function $\tilde{F}(x, n) = F(x)3^n$; we'll use the rank of $\tilde{F}$ synonymously with the rank of F. We'll also refer to the image of an (expanded) ternary family as an (expanded) ternary family when there is no ambiguity.

Proposition 7.1. *Given any $0 < \alpha < 1$, and any $k \geq 1$, we have that $B_{k\alpha}$ is contained in a union of finitely many ternary families of rank at most $k-1$. (As a corollary, we get the same for $A_{k\alpha}$ but with expanded families.)*

Proof. Induct on k using the main lemma, Corollary 4.3, and the finiteness of S_r for any r and T_α. □

And hence:

Corollary 7.2. *For any r, B_r is contained in a union of finitely many ternary families of rank at most $\lfloor r \rfloor$. (Expanded ones for A_r.)*

Proof. Note that $r = (\lfloor r \rfloor + 1)(r/(\lfloor r \rfloor + 1))$ and $r/(\lfloor r \rfloor + 1) < 1$, which proves the claim. □

Now we will prove some bounds on the size of ternary families.

Proposition 7.3. *Let S be the image of a ternary family of rank k. Then $S(x) = O((\log x)^k)$. ($O((\log x)^{k+1}$ for the corresponding expanded family.)*

This is an easy induction from the definition of ternary family.

In fact, we have:

Theorem 7.4. *Let S be the image of a ternary family $F(n_1, \ldots, n_k)$ of rank k. Then $S(x) = \Theta((\log x)^k)$; more specifically,*

$$\frac{1}{k!^2}(\log_3 x)^k \lesssim S(x) \lesssim \frac{1}{k!}(\log_3 x)^k.$$

And if $\tilde{S}$ is the image of $\tilde{F}$,

$$\frac{1}{k!(k+1)!}(\log_3 x)^{k+1} \lesssim \tilde{S}(x) \lesssim \frac{1}{(k+1)!}(\log_3 x)^{k+1}.$$

Proof. We'll consider the expanded-out form of F, as a sum of terms each of the form $c3^{a+\sum_{i\in I} n_i}$ for some $a \ge 0$, $3 \nmid c$, and $I \subseteq [k]$. Consider the set $\mathcal{C}$ of such I that occur and partially order it under inclusion. Note that if we actually take the various $a + \sum_{i\in I} n_i$ for specific n_i, they respect this order except on a finite union of proper "affine subspaces" of the domain (since if the order is violated, we have $\sum_{i\in J} n_i < m$ for some J and m), so exclude these and call this restricted domain X. Indeed, all sufficiently large points lie in X. (By an "affine subspace" of $\mathbb{Z}^k_{\ge 0}$ we mean the intersection of $\mathbb{Z}^k_{\ge 0}$ with an affine subspace of $\mathbb{Q}^k$.)

Then if we let $T(x)$ be the set of all $(n_1, \ldots, n_k)$ such that $F(n_1, \ldots, n_k) \le x$, we have $|T(x) \cap X| \sim \frac{1}{k!}(\log_3 x)^k$. The problem, then, is just to show that this translates to $S(x)$; we will do this by showing that F is at most $k!$-to-1 outside a finite union of proper affine subspaces of the domain.

Note that any ternary family has a constant term (constant term of $() \mapsto m$ is m, constant term of G_c is c, constant term of $G \otimes H$ is product of their constant terms). Observe that for each argument n_i there is a unique minimal element of $\mathcal{C}$ that contains it; we'll call this the *key* of n_i. (If $F = G \otimes H$ with n_i an argument to G, key of n_i is its key in G times constant term of H; if $F(n_1, \ldots, n_k) = G_c(n_1, \ldots, n_k)$ with $i \ne k$, key of n_i is its key in G times 3^{n_k}; if $i = k$ its key is 3^{n_k} times constant term of G.) Furthermore, by a straightforward induction using the above, no two arguments to F have the same key. (If $F : () \mapsto m$ this is trivial. If $F = G \otimes H$, two arguments to same factor, true by inductive hypothesis; if to different factors, their keys are disjoint. If $F = G_c$, two arguments to G, true by inductive hypothesis; if one is n_k, true as one key is just $\{n_k\}$ and the other isn't.)

We don't want any the exponent $a + \sum_{i\in I} n_i$ of any two keys to be equal, but this is again just excluding a finite number of proper affine subspaces, so if Y is the new domain, we still have $|T(x) \cap Y| \sim \frac{1}{k!}(\log_3 x)^k$.

Suppose $(n_1, \ldots, n_k) \in Y$; we claim that if we know $F(n_1, \ldots, n_k)$ and the total order $(n_1, \ldots, n_k)$ induces on the keys by looking at their exponents (by assumption, it makes no two of them equal), we can reconstruct $(n_1, \ldots, n_k)$, meaning that F is at most $k!$-to-1 on Y.

To determine $(n_1, \ldots, n_k)$, we can now use the following algorithm:

(1) Initialize the set Φ of known arguments to $\emptyset$, and the remaining number N to $F(n_1, \ldots, n_k)$.
(2) Repeat the following until $\Phi = [n]$:

(a) Determine all terms that can be determined from the variables currently in Φ; subtract the resulting numbers from N.
(b) Pick the lowest key whose corresponding n_i does not already have $i \in \Phi$; say it's $c3^{a+\sum_{i\in I} n_i}$ not already picked. All but one argument in I, say n_j, will already be in Φ and have values recorded. We can determine this n_j via $a+\sum_{i\in I} n_i = v_3(N)$, where $v_3(N)$ is the highest power of 3 dividing N; so add j to Φ and mark down the above stated value for n_j.

So to finish the proof we must justify the claims made in the statement of the algorithm and show that it yields the only possible value for $(n_1, \ldots, n_k)$. That there will only ever be one $j \in I$ not already in Φ follows because I is a key, say the key of n_j; then for each other $i \in I$, there must be a smaller (in the partial order) J that contains it, or else I would be the key of both n_i and n_j; and since we are assuming $(n_1, \ldots, n_k) \in Y$, any such variables have keys less than I in the total order, and hence (by construction) have already been determined and added to Φ.

So it just remains to show that we must indeed have $a+\sum_{i\in I} n_i = v_3(N)$ above, that we are forced to determine n_j thus. This amounts to showing that N is a sum of terms of the form $c3^A$, where $3 \nmid c$ and each A is greater than $a+\sum_{i\in I} n_i$ except for the occurrence of such from I itself. But if we had such a term, and it were a key, then by assumption it would have been already determined and subtracted off; and if it weren't a key, then since $(n_1, \ldots, n_k) \in Y$, its keys would be even smaller, and would have been already determined, and so it would have been subtracted off too.

Hence F is at most $k!$-to-1 on Y and the theorem follows.

A similar argument shows that $\tilde{F}$ is also at most $k!$-to-1 on Y (though it's technically not the same Y), so it holds for expanded families as well. □

So we have:

Theorem 7.5. *For any r, let $k = \lfloor r \rfloor$; then $B_r(x) = \Theta((\log x)^k)$, $A_r(x) = \Theta((\log x)^{k+1})$.*

Proof. The upper bound follows immediately from the above lemmas. For the lower bound, note that B_k contains the ternary family

$$F(n_1, \ldots, n_k) = (\ldots((3 \cdot 3^{n_1} + 1)3^{n_2} + 1)\ldots)3^{n_k} + 1.$$

(None of these are multiples of three, and each have complexity at most $3(1 + n_1 + \ldots + n_k) + k$ and so have defect less than k.) Hence the lower bound follows from the above lemmas as well. (Although checking the lower bound for this particular family is a lot easier than proving Lemma 7.4 in general!) □

(Note that barring $r = 0$, for which B_r is empty but $\overline{B}_r = \{3\}$, the same holds for $\overline{B}_r$ and $\overline{A}_r$, as $\overline{B}_r \setminus B_r$ is finite by Theorem 2.3.)

So while it's a long way from proving $f(n) \not\approx 3\log_3 n$, at least we can prove

Corollary 7.6. *There exist numbers of arbitrarily large defect.*

8. Well-ordering of defects

In this section we now prove that, as Juan Arias de Reyna previously conjectured, the set of defects is well-ordered, with order type ω^ω. (His original conjecture actually took a slightly different form; more on that in the next section.)

Define the *leading coefficient* (denoted LC) of a ternary family $F(n_1, \ldots, n_k)$ to be the limit $\lim_{n_1,\ldots,n_k \to \infty} F(n_1, \ldots, n_k)/3^{\sum n_i}$. Note also that this can also be determined recursively; leading coefficient of a constant is itself, $LC(F \otimes G) = LC(F)LC(G)$, and $LC(F_c) = LC(G)$. (Hence in particular $LC(F)$ is always finite and nonzero.) From this recursion it follows that $F(n_1, \ldots, n_k)$ is always at least $3^{\sum n_i} LC(F)$. Define the *base complexity* (denoted BC) of a ternary family by a similar recursion; base complexity of a singleton n is $f(n)$, $BC(F \otimes G) = BC(F) + BC(G)$, and $BC(F_c) = BC(F) + f(c)$. As written this is actually not well-defined, but we can make it so by simply taking the smallest possible value if there's any ambiguity. Note that for any ternary family F and any $n_1, \ldots, n_k$, we have $f(F(n_1, \ldots, n_k)) \le BC(F) + 3\sum n_i$. Finally define the *obvious defect upper bound*, UB, of a ternary family by $UB(F) = BC(F) - 3\log_3 LC(F)$. Note that this is, in fact, an upper bound on the defect of any number in the image of F. Also note $BC(F) \ge \operatorname{rk} F + f(LC(F))$ and hence $UB(F) \ge \operatorname{rk} F + d(LC(F))$; this can be proven by induction, using a decomposition of F such that BC always adds.

Proposition 8.1. *Let S be the image of any (expanded) ternary family of rank k. Then the set of defects of S is well-ordered, with order type less than ω^{k+1}.*

Proof. Take a ternary family F of rank k. For any $n_1, \ldots, n_k$, consider the difference between the actual complexity $f(F(n_1, \ldots, n_k))$ and the upper bound of $BC(F) + 3\sum n_i$. This difference can take on only finitely many values, as it cannot be more than $UB(F)$, since otherwise the defect would be less than 0; the same applies to the expanded version. Hence we can split up $\mathbb{Z}^k_{\ge 0}$ into a union of finitely many sets, on each of which $d(F(n_1, \ldots, n_k))$ is given by $BC(F) + 3\sum n_i - 3\log_3(F(n_1, \ldots, n_k))$ minus some constant. It suffices to show that each of these sets of defects is well-ordered with order type at most ω^k, as the natural sum of finitely many ω^k's is certainly less than ω^{k+1}. And for this it suffices to show that the image of $d_F : (n_1, \ldots, n_k) \mapsto 3\sum n_i - 3\log_3(F(n_1, \ldots, n_k))$ is well-ordered, as adding constants doesn't change the order. (Note, by the way, that $\lim_{x\to\infty} d_F(x) = -3\log_3 LC(F)$ by definition of LC.)

In fact, it suffices to show that d_F is monotonic; $\mathbb{Z}^k_{\ge 0}$ is a well-partial order (in the strong sense of that term), so any totally-ordered image of it under a monotonic function is well-ordered. Futhermore the resulting well-order must have order type at most ω^k, as if we actually pull it back to a total order on $\mathbb{Z}^k_{\ge 0}$ (breaking ties between points with equal d_F by looking at lexicographic order), it extends the original partial order, and the natural product of k ω's is in fact ω^k.

To prove d_F monotonic, we induct on F. For singleton functions it is trivial, and as $d_{F\otimes G} = d_F + d_G$, if it is true for two families it is true for their product. Finally, say $G(x, n) = F_c(x, n) = F(x)3^n + c$; if d_F is monotonic, clearly d_G is monotonic in x, so we need only check monotonicity in n. This holds because the inequality $3n - 3\log_3(F(x)3^n + c) < 3(n + 1) - 3\log_3(F(x)3^{n+1} + c)$ is equivalent to the inequality $c < 3c$. Hence d_F is monotonic – in fact, strictly so – and we are done. ∎

In fact, we also have:

Proposition 8.2. *Let S be the image of any ternary family of rank k. Then the set of defects of S has order type at least ω^k. (Hence also for expanded families.)*

Proof. The proof will imitate that of Theorem 7.4; unfortunately it will be slightly more complicated, due to the necessity of working from the left rather than the right.

Note first that using d_F from the proof of Proposition 8.1, it suffices to show that the image of $\mathbb{Z}_{\geq 0}^k$ under d_F has order type at least ω^k, since the image of d_F can as described above be written as a finite union of translates of subsets of $d(S)$, so if $d(S)$ had order type less than ω^k, the image of d_F would have order type at most the natural sum of finitely many ordinals each less than ω^k. So we will exhibit a subset of $\mathbb{Z}_{\geq 0}^k$ whose image under d_F has order type ω^k.

Once again write $F(n_1, \ldots, n_k) = \sum c3^{a+\sum_{i\in I} n_i}$ with $a \geq 0$, $3 \nmid c$, and $I \subseteq [n]$, so that $d_F(n_1, \ldots, n_k) = -3\log_3(\sum c3^{a-\sum_{i\notin I} n_i})$. Once again let $\mathcal{C}$ be the set of I that occur, partially ordered under inclusion. Since we are now working from the left, for each argument n_i we'll look at its *anti-key* – the unique maximal set in $\mathcal{C}$ that doesn't include it. (If $F = G \otimes H$ with n_i a variable in G, this is its anti-key in G times the leading term of H; if $F = G_c$ with $i \neq k$, this is its anti-key in G times 3^{n_k}; and if $i = k$, this is the leading term of G. Where the leading term of F has all arguments in its exponent, and so the leading term of $() \mapsto m$ is m, that of $G \otimes H$ is that of G times that of H, and that of G_c is that of G.) Note that the coefficient c of the leading term is not what we defined earlier as the leading coefficient, which is $c3^a$. As with keys, no two arguments have the same anti-key, by a straightforward induction. (If $F : () \mapsto m$, this is trivial; if $F = G \otimes H$, both arguments to same factor, true by inductive hypothesis; if to different factors, each is included in the other's anti-key; if $F = G_c$, one of the arguments is n_k, then the other's anti-key includes n_k; if neither is n_k, true by inductive hypothesis.)

This time, we will let X be the set of $(n_1, \ldots, n_k)$ that yield an order on the values $\lfloor \log_3 c \rfloor + a - \sum_{i\notin I} n_i$ (equivalently on $\lfloor \log_3 c \rfloor + a + \sum_{i\in I} n_i$) that respects the partial order on the terms; the extra $\lfloor \log_3 c \rfloor$ is because we are now working from the left rather than the right – we care where the ternary representation of c starts rather than where it ends. As before, X consists of just removing finitely many "proper affine subspaces" from the domain. Indeed, once again, the stronger statement that all sufficiently large points lie in X remains true. (Recall, by an "affine subspace" of $\mathbb{Z}_{\geq 0}^k$ we mean the intersection of $\mathbb{Z}_{\geq 0}^k$ with an affine subspace of $\mathbb{Q}^k$.)

Now pick a total order $\preceq$ on the anti-keys extending their partial order (there is at least one such). Let Y be the subset of X on which for any anti-key $T_1 = c_1 3^{a_1 - \sum_{i\notin I_1} n_i}$ and any other term $T_2 = c_2 3^{a_2 - \sum_{i\notin I_1} n_i}$ with $T_2 \subseteq T_3 \preceq T_1$ for some anti-key T_3, we have $a_1 - \sum_{i\notin I_1} > \lfloor \log_3 c_2 \rfloor + a_2 - \sum_{i\notin I_2}$ (i.e. the ternary representation of T_2 is fully to the right of that of T_1). (Again, note that last time we only needed to prevent collisions of keys; working from the left we are required to additionally prevent collisions of anti-keys with other terms, because terms extend to the left, not the right.)

Put an order $\preceq$ on the arguments given by $\preceq$ on their anti-keys. What we now need is that, firstly, on Y, the d_F order is lexicographic, if we read the arguments in decreasing order with respect to $\preceq$; and secondly, Y is actually big enough to contain an ω^k.

So relabel $n_1, \ldots, n_k$ as $m_1, \ldots, m_k$ with $m_k \preceq \ldots \preceq m_1$. Now say we have $(n'_1, \ldots, n'_k)$ and $(n''_1, \ldots, n''_k)$ which we relabel with $m'_1, \ldots, m'_k$ and $m''_1, \ldots, m''_k$, and say $m'_j = m''_j$ for $j < i$ and $m'_i > m''_i$. Then looking at $d_F(n'_1, \ldots, n'_k)$ and $d_F(n''_1, \ldots, n''_k)$, all terms involving only $m_1, \ldots, m_{i-1}$ are the same so we can subtract these off. Consider then the anti-key of m_i, which is $c3^{a-\sum_{i\notin I} n_i}$; any other term which has not been subtracted off involves one of $m_i, \ldots, m_k$, and so by definition of Y, its ternary representation lies entirely to the right of the anti-key of m_i, and so it will not be relevant for comparison. Thus it's purely a matter of the anti-key of m_i; however all the non-m_i variables in $-\sum_{i\notin I} n_i$ are the same between the two points. So since $m'_i > m''_i$ we get $\sum c3^{a-\sum_{i\notin I} n'_i} < \sum c3^{a-\sum_{i\notin I} n''_i}$ and hence $d_F(n'_1, \ldots, n'_k) > d_F(n''_1, \ldots, n''_k)$.

Finally we just need to show that Y is "large enough", i.e., that if we choose $m_1, \ldots, m_k$ in order, one by one, to yield a point $(n_1, \ldots, n_k)$ in Y, that at each step there are infinitely many choices. We don't need to worry about it not being in X since all sufficiently large points are in X; we just need to take care of the additional restriction to be in Y.

So suppose we've already picked $m_1, \ldots, m_{i-1}$; we need to show that there are infinitely many possible m_i that will cause $m_1, \ldots, m_i$ to satisfy the restrictions involving only $m_1, \ldots, m_i$. Note that the anti-key for any m_i can only exclude the variables $m_1, \ldots, m_i$. By assumption, we need not check any of the restrictions where the anti-key of m_i is the term T_1 (using T_1 and T_2 from before), as T_2 would involve variables not among $m_1, \ldots, m_i$. And we need not check the terms excluding the variables $m_1, \ldots, m_i$ which are not the anti-key of m_i, as those not excluding m_i have already been checked, and those that do can (if the restrictions on the anti-key are satisfied) will follow automatically from the assumption that the point is in X. So say the anti-key of m_i is I and say J is the anti-key of some m_j with $j < i$; we just need to check there are infinitely many m_i that will satisfy this one restriction. But this one restriction is just that, for some constant b, we have $b - \sum_{l\notin I} m_l > -\sum_{l\notin J} m_l$, i.e., $\sum_{l\notin I} m_i - \sum_{l\notin J} m_l > b$. But all the variables in this save for m_i are $m_1, \ldots, m_{i-1}$, which have already been determined; and m_i appears only in the former sum. Hence we need only pick m_i large enough (based on $m_1, \ldots, m_{i-1}$).

Hence for each $m_1, \ldots, m_{i-1}$, if we just pick m_i sufficiently large based on $m_1, \ldots, m_{i-1}$, no restrictions will be violated. Hence the order type of $d_F(Y)$ is ω^k, and we are done. □

Corollary 8.3. *No ternary family of rank $k+1$ can be covered by any finite union of expanded ternary families of rank k; hence in particular B_{k+1} cannot be covered by finitely many expanded ternary families of rank k.*

Hence we have:

Theorem 8.4. *For any r, the set D_r is well-ordered, with order type at least $\omega^{\lfloor r\rfloor}$ and less than $\omega^{\lfloor r\rfloor+1}$.*

Proof. Exactly the same as the proof of Theorem 7.5, with the same example providing the lower bound. Note once again that it is much easier to prove the lower bound in this particular case than in general – indeed the family used to provide a lower bound has an order type of exactly $\omega^{\lfloor r\rfloor}$. □

Corollary 8.5. *The set of all defects is well-ordered with order type ω^ω.*

When r is an integer, we can actually pin things down a bit more:

Theorem 8.6. *Let k be an integer; then the order type of $\overline{D}_k$ is exactly ω^k, unless $k = 1$, in which case it's $\omega + 1$.*

Proof. If it were any larger than $\omega^k + 1$, it would contain a copy of ω^k, strictly bounded above by a non-maximum element of $\overline{D}_k$, hence an element of $\overline{D}_k$ less than k. So there would be a copy of ω^k bounded away from k, contradicting Lemma 8.4. Similarly, if it has order type equal to $\omega^k + 1$, the maximum element must be k itself, or else we again get a copy of ω^k bounded away from k. So the order type is ω^k unless k is itself a defect, which happens only when $k = 1$. □

Note that we can use Lemma 8.4 or Lemma 7.5 to prove that in certain nice cases, "most" elements of a ternary family must have complexity equal to the obvious upper bound.

Corollary 8.7. *Suppose F is a ternary family of rank k with $UB(f) < k + 1$. Then the set S of $m = F(n_1, \ldots, n_k)$ with $f(m) < BC(F) + 3\sum n_i$ has $S(x) = O((\log x)^{k-1})$, and has its defects have order type less than ω^k. Furthermore the defects of F have order type exactly ω^k.*

Proof. Any defect of a number in S must be at most $UB(f) - 1 < k$, so $S \subseteq A_r$ for some $r < k$, and the first result is immediate. To see that the defects of F have order type exactly ω^k, let T be the image of F minus S, and note that since the defects of F have order type at least ω^k, and the defects of S have order type less than ω^k, the defects of T must have order type at least ω^k; however from the proof of Proposition 8.1 we know it also has order tpe at most ω^k. So combining S and T, the order type of the defects of F is less than $\omega^k 2$; but since the defects of T are confinal in it, with order type ω^k, the overall order type must be ω^k. □

9. Alternate forms of well-ordering and some conjectures of Arias

Corollary 8.5 was previously conjectured by Juan Arias de Reyna in [1] in a slightly different form. What we have proved is that the set $\{n/3^{f(n)/3} : n \in \mathbb{N}\}$ is reverse well-ordered, with reverse order type ω^ω. It may be more natural to discuss $R(n)$ rather than $n/3^{f(n)/3}$, but it is easy to translate between these if we know $f(n)$ modulo 3. Rather than consider the set of all defects, we will separately consider the sets $\{n/3^{f(n)/3} : n \equiv 0 \pmod 3\}$, $\{n/3^{f(n)/3} : n \equiv 2 \pmod 3\}$, $\{n/3^{f(n)/3} : n \equiv 1 \pmod 3, n \neq 1\}$. (We exclude 1 for simplicity because it does not follow the pattern of other n congruent to 1 modulo 3.)

These three sets are all reverse well-ordered, with reverse order type at most ω^ω; to see that each has reverse order type exactly ω^ω, consider the ternary families

$$F_k(n_1, \ldots, n_k) = (\ldots((3 \cdot 3^{n_1} + 1)3^{n_2} + 1)\ldots)3^{n_k} + 1$$
$$G_k(n_1, \ldots, n_k) = (\ldots((2 \cdot 3^{n_1} + 1)3^{n_2} + 1)\ldots)3^{n_k} + 1$$
$$H_k(n_1, \ldots, n_k) = (\ldots((4 \cdot 3^{n_1} + 1)3^{n_2} + 1)\ldots)3^{n_k} + 1$$

all of which provide an ω^k's worth of defects meeting the obvious upper bound due to Corollary 8.7.

We can multiply each of these sets by the appropriate constants to see that each of the three sets $\{R(n) : n \equiv 0 \pmod 3\}$, $\{R(n) : n \equiv 2 \pmod 3\}$, $\{R(n) : n \equiv 1$

(mod 3), $n \neq 1\}$ are reverse well-ordered with reverse order type ω^ω. Or by using different constants, we could put $3^{\lfloor f(n)/3 \rfloor}$ in the denominator instead of $E(f(n))$, which is the form Arias originally conjectured it in. (Although his actual original conjecture was slightly stronger and would have implied that $f(3n) = f(n) + 3$ for all $n > 1$, which is false; removing that aspect leaves what we have proved here.)

Indeed, knowing this we can even recombine the three (together with 1) to say that $\{R(n) : n \in \mathbb{N}\}$ (or $\{n/3^{\lfloor f(n)/3 \rfloor} : n \in \mathbb{N}\}$) is reverse well-ordered with reverse order type ω^ω. Since it's the union of finitely many reverse well-ordered sets, it too is reverse well-ordered, with reverse order type at least ω^ω. To see that it is exactly ω^ω, observe that if it were any larger, then some proper final segment of it would have reverse order type ω^ω; but this would decompose into a union of proper final segments of the four sets making it up (all of them are coinitial in it as they all get arbitrarily close to 0), implying that ω^ω was at most the natural sum of finitely many ordinals less than it, which is false.

Now let $A = \{R(n) : n \equiv 0 \pmod 3\}$, let $B = \{R(n) : n \equiv 1 \pmod 3, n \neq 1\}$, let $C = \{R(n) : n \equiv 2 \pmod 3\}$, and let a_α, b_α, c_α denote the α'th element from the top (0-indexed, so $a_0 = b_0 = c_0 = 1$) in A, B, C respectively for $\alpha < \omega^\omega$. Arias also made a conjecture which, reformulated slightly, states that for any $\beta < \omega^\omega$,

$$\lim_{n\to\infty} a_{\omega\beta+n} = (2/3)c_\beta$$
$$\lim_{n\to\infty} b_{\omega\beta+n} = (3/4)a_\beta$$
$$\lim_{n\to\infty} c_{\omega\beta+n} = (2/3)b_\beta$$

Looking at our actual data for this, it is easy to see the reason this holds for small defect; take the first of these statements as an example. For every $c_\beta = R(n)$, we get an infinite family of numbers $n3^k + 1$, as well as a collection of infinite families of numbers $p(q3^k + 1)$ for $pq = n$, $f(p) + f(q) = f(n)$; since these families occur in the same order as the original n, it is the limit of $R(n3^k + 1)$ and the other families – together with finitely many other things that don't affect the limit – that appear on the left hand side, and (for examples so far, anyway) this is always eventually the obvious upper bound for sufficiently large k, and everything matches up, with a factor of 2/3 or 3/4 to account for the change of modulo 3 complexity. However, it is not at all clear that this pattern will hold up once we leave the realm of products and +1's, which we know eventually happens. So the conjecture is somewhat explained, but far from proved.

10. "Cut-off" ternary families and the structure of A_r

We know that each B_r can be covered by finitely many ternary families of rank $\lfloor r \rfloor$, but we suspect a slightly stronger statement is true:

Conjecture 10.1. *B_r is the union of finitely many ternary families of rank at most $\lfloor r \rfloor$.*

More generally, we conjecture the following slightly stronger statement:

Conjecture 10.2. *Let F be a ternary family of rank k and let*

$$R = \sup_{x \in \mathbb{Z}^k_{\geq 0}} d(F(x)).$$

Pick $r < R$ and let S consist of all the elements in the image of F with $d(n) < r$. Then S is the union of finitely many ternary families of rank less than k.

We cannot yet prove this but we can prove something close. Let us consider what happens when we "cut off" a ternary family by restricting to those elements with less than a fixed defect. First let us define the notion of an *aligned subfamily* of a ternary family. Given a ternary family F, we can partially order its arguments by comparing their keys. (In Section 8 we instead compared anti-keys; it is easy to check that this yields the reverse partial order.) If we consider an argument with maximal key and fix it, it is easy to check that the resulting function G on the remaining arguments is again a ternary family; we will call this a direct aligned subfamily of the original family. More generally we will say G is an aligned subfamily of F if it can be obtained from F by repeatedly taking direct aligned subfamilies (equivalently, by fixing a set of arguments which is upwardly closed in the above order). A proper aligned subfamily of F will mean one other than F itself (and hence of lower rank than F).

(We say "aligned subfamily" rather than "subfamily" so that someone may later define "subfamily" in a way that allows, e.g., $(3^n+1)(3^m+1)$ to be a subfamily of $(((3^a+1)3^b+1)3^c+1)$, since the former can be obtained from the latter by restricting to $a=c$. Note that we we did not use any notion of subfamily in the conjecture above – the example just given shows it would be false if we used aligned subfamilies, and while a broader notion of subfamily would be appropriate, we have not yet been able to sensibly define one.)

With this definition, we can show:

Proposition 10.3. *Let F be a ternary family of rank k and let*

$$R = \sup_{x \in \mathbb{Z}_{\geq 0}^k} d_F(x) = -3 \log_3 LC(F).$$

Pick $r < R$ and let S consist of all those $F(x)$ with $d_F(n) < r$. Then S can be covered by finitely many proper aligned subfamilies of F.

Proof. Consider the "anti-key complements", $N_i = \sum_{i \notin T_i} n_j$ where T_i is the anti-key of n_i. First observe that there is some C such that if $N_i > C$ for every i then $d_F(n_1, \ldots, n_k) \geq r$. To see this, note that every nonconstant term in the expanded-out form of $3^{-\sum n_i} F(n_1, \ldots, n_k)$ is at most a constant times some anti-key, so for any $\varepsilon > 0$ we can pick C such that if $N_i > C$ for every i then $3^{-\sum n_i} F(n_1, \ldots, n_k) < LC(F) + \varepsilon$, which is equivalent to the statement above.

Hence S can be covered by the finitely many sets $N_i \leq C$; now we show each of these can be covered by finitely many proper aligned subfamilies. For any anti-key T_i, take a maximal anti-key $T_{i'}$ containing it. Then we have $n_{i'} \leq N_{i'} \leq N_i$, so each set $N_i \leq C$ can be covered by the finitely many sets $n_{i'} = 0, 1, \ldots, C$. But since $n_{i'}$ has maximal anti-key, all of these are proper aligned subfamilies, and we are done. □

And now we can actually strengthen the statement slightly, to be even closer:

Corollary 10.4. *Let F, k, R, r, S as in Proposition 10.3. Then S is the union of finitely many proper aligned subfamilies of F.*

Proof. Induct on k. If $k = 0$, this is trivial. Otherwise, cover S with finitely many direct aligned subfamilies of F by Proposition 10.3 (obviously any aligned subfamily of F can be covered by a direct aligned subfamily). For any one of these subfamilies G, we have that there is some C such that, after rearranging coordinates,

$G(x) = F(C, x)$ and $d_G(x) + 3C = d_F(C, x)$. So if we let T be the image of G, then $T \cap S$ consists of those $G(x)$ with $d_G(x) < r - 3C$. So if $r - 3c \geq -3\log_3 LC(G)$, then $T \subseteq S$; whereas if $r - 3c < -3\log_3 LC(G)$, then by the inductive hypothesis, $T \cap S$ can be written as a union of images of finitely many proper aligned subfamilies of G. Either way, we are done. □

The same reasoning shows that if we had written Conjecture 10.2 with "can be covered by" instead of "is the union of", the latter (and hence Conjecture 10.1) would still be deducible from it. (In fact, this would be slightly easier, as while there is an offset between d_F and d_G, there is none between $d \circ F$ and $d \circ G$). Also note that the statement would remain true if were to define S with nonstrict inequalities, since the values of d_F are well-ordered, and also that one can easily make analogous statement for expanded families.

The reason we cannot actually prove Conjecture 10.2 is that we cannot currently prove any statements about when the complexity of some $F(n_1, \ldots, n_k)$ will be different than usual. We expect this should not happen often but it's not currently clear how to formalize this.

11. Conjectures on pinning down growth rate and order type

So at this point we have shown that for real $r \geq 0$ with $k = \lfloor r \rfloor$, we have that $\overline{B}_r(x) = \Theta((\log x)^k)$, while $\overline{A}_r(x) = \Theta((\log x)^{k+1})$, and the order type of $\overline{D}_r$ is at least ω^k, while less than ω^{k+1}. However, so far the only case when we've pinned down any of these exactly for arbitrarily large r is $\overline{D}_r$ when r is an integer. Nonetheless, we have strong suspicions about what the correct growth rates should be for all r, and what the correct and order types should be for many r, so in this section we state and justify those conjectures. In the process we will also state various other related conjectures we have. We will also show that if made uniform, our conjecture about the growth rate of $\overline{A}_r(x)$ would in fact imply that $f(n) \not\sim 3\log_3 n$, a statement that remains unproven.

(If you prefer A_r and B_r to $\overline{A}_r$ and $\overline{B}_r$, note that as long as $r \geq 1$, $B_r(x) = \Theta(\log x)$ and $A_r(x) = \Theta((\log x)^2)$ while $(\overline{B}_r \setminus B_r)(x) = O(1)$ and $(\overline{A}_r \setminus A_r)(x) = O(\log x)$ so $\overline{B}_r(x) \sim B_r(x)$ and $\overline{A}_r(x) \sim A_r(x)$; while if $r < 1$, we completely understand what happens. Regarding $\overline{D}_r$, see below.)

For $0 \leq s < 1$ we will use $D_1(s)$ to denote the number of elements of D_1 which are less than or equal to s; as we know from Corollary 4.3, this is always finite for $s < 1$. To actually compute this number for a given s, note that the elements of D_1, listed in order, begin with 0, $d(2)$, $2d(2)$, $3d(2)$, $4d(2)$, $5d(2)$, $d(5)$, $6d(2)$, $d(7)$, $d(5)+d(2)$, $7d(2)$, $d(7)+d(2)$, $d(5)+2d(2)$, $8d(2)$, $d(7)+2d(2)$, $d(5)+3d(2)$, $d(19)$, $9d(2)$, $d(13)$, after which they are given by $1 - 3\log_3(1 + 3^{-k})$ for $k \geq 4$.

Then we conjecture:

Conjecture 11.1. *Suppose $r = k + s$ where k an integer and $s \in D_1$. Then the order type of $\overline{D}_r$ is $\omega^k D_1(s)$, unless $r = 1$, in which case it is $\omega + 1$.*

We make no statement about what happens if r is not of this form. (Note that except when $r \leq 1$, this will be the order type of D_r as well, as in no other case can a number of this form be a defect, by Corollary 5.3 and Lemma 2.2.)

For the precise growth rate of $\overline{B}_r(x)$ and $\overline{A}_r(x)$, we make the following conjecture when r is an integer:

Conjecture 11.2. *For $k \geq 0$ an integer,*

$$\overline{B}_k(x) \sim \frac{(k+1)^{k-1}}{k!^2}(\log_3 x)^k,$$

$$\overline{A}_k(x) \sim \frac{(k+1)^{k-2}}{k!^2}(\log_3 x)^{k+1}.$$

(We actually have a conjecture for when r is arbitrary, but this is considerably more complicated so we will state it later.)

Before we justify these ideas let us note that if we could prove Conjecture 11.2 with a uniformity condition – getting actual bounds rather than just asympotics – we could prove that $f(n) \not\approx 3\log_3 n$: The asymptotics would have to be turned into actual bounds, but only a slight strenghthening (since $\frac{(k+1)^{k-2}}{k!^2} \leq \frac{k^{k-2}}{(k-1)!^2}$) is needed:

Proposition 11.3. *Let $\alpha = 0.3178\ldots$ be the smallest positive solution to $2\alpha - \alpha\log\alpha = 1$, and assume that there is a β with $0 < \beta < \alpha$ and functions $P(x,k)$ and $Q(x,k)$ such that:*

(1) $P(x,k)$ is increasing in x and k and $\log P(x, \beta\log x + 1) = o(x)$.
(2) $Q(x,k)$ is increasing in x and k and $(\log x)Q(x, \beta\log x + 1) = o(x)$.
(3) For sufficiently large x we have for $k \geq 1$,

$$A_k(x) - A_{k-1}(x) \leq P(x,k)\frac{k^k}{k!^2}(\log x)^{k+1} + Q(x,k).$$

Then there exist arbitrarily large n such that $f(n) \geq (\frac{3}{\log 3} + \beta)\log n = (3 + \beta\log 3)\log_3 n$; in particular, $f(n) \not\approx 3\log_3 n$.

Note that the conditions on P and Q allow us quite a bit of wiggle room, as does the fact that we only need to suppose that this bounds $A_k(x) - A_{k-1}(x)$ rather than $A_k(x)$ itself.

Proof. For sufficiently large x, for all $k \geq 1$,

$$A_k(x) \leq \sum_{i=0}^{k} P(x,i)\frac{i^i}{i!^2}(\log x)^{i+1} + \sum_{i=0}^{k} Q(x,i).$$

Using the fact that P and Q are increasing, and folding in the $i = 0$ term, we obtain for $k \geq 2$

$$A_k(x) \leq \sum_{i=1}^{k} 2P(x,k)\frac{i^i}{i!^2}(\log x)^{i+1} + (k+1)Q(x,k).$$

We will use here the version of Stirling's formula, valid for $n \geq 1$, that $n! = (2\pi)^{1/2}\frac{n^n\sqrt{n}}{e^n}e^{\frac{\theta(n)}{12n}}$, where $\theta(n)$ is a function satisfying $0 \leq \theta(n) \leq 1$ for all n.

We get that

$$\begin{aligned} A_k(x) &\leq 2P(x,k)\sum_{i=1}^{k}\frac{(e^2\log x)^i\log x}{i^i} + (k+1)Q(x,k) \\ &= 2P(x,k)(\log x)\sum_{i=1}^{k}\frac{(e^2\log x)^i}{i^i} + (k+1)Q(x,k). \end{aligned}$$

Now, assume that $f(n) \le (\frac{3}{\log 3} + \beta) \log n$ for all sufficiently large n. Set $t(x) = \lceil \beta \log x \rceil$. Then by our initial assumption, we have that for sufficiently large x

$$x \le A_{t(x)}(x) \le 2P(x, t(x))(\log x) \sum_{i=1}^{t(x)} \frac{(e^2 \log x)^i}{i^i} + (t(x)+1)Q(x, t(x)).$$

Now, consider the function $x \mapsto \frac{C^x}{x^x}$. This function is increasing with respect to x as long as $0 < x \le C/e$. So since $\alpha < e$, for any i in the range we care about, $\frac{(e^2 \log x)^i}{i^i} \le \frac{(e^2 \log x)^{i+1}}{(i+1)^{i+1}}$. So for sufficiently large x,

$$x \le 2P(x, t(x))t(x) \frac{(e^2 \log x)^{t(x)}}{t(x)^{t(x)}} + (t(x)+1)Q(x, t(x))$$

and thus we have

$$x \le 2P(x, \beta \log x + 1)(\beta \log x + 1) \frac{(e^2 \log x)^{\beta \log x + 1}}{(\beta \log x)^{\beta \log x}} + \\ (\beta \log x + 2)Q(x, \beta \log x + 1).$$

Now, by assumption the Q term is $o(x)$, so we must have for sufficiently large x

$$x \le 3P(x, \beta \log x + 1)(\beta \log x + 1) \frac{(e^2 \log x)^{\beta \log x + 1}}{(\beta \log x)^{\beta \log x}}$$

Taking logs of both sides, we obtain:

$$\log x \le \log 3 + \log P(x, \beta \log x + 1) + \log(\beta \log x + 1) + \\ (\beta \log x + 1)(2 + \log \log x) - (\beta \log x)(\log(\beta \log x))$$

and hence

$$\log x \le \log 3 + \log P(x, \beta \log x + 1) + \log 2 + \log b + 2 \log \log x \\ + 2\beta \log x + 2 \log \log x - (\beta \log \beta) \log x$$

which is a contradiction for x sufficiently large so long as $2\beta - \beta \log \beta < 1$. $\square$

Now to the ideas behind these conjectures. The main idea here is that given $r \ge 0$ and letting $k = \lfloor r \rfloor$ and $s = r - k$, the "bulk" of $\overline{A}_r$ should consist of what we will call the *primary ternary families* of rank k. We will define the set of primary ternary families to be the smallest set of ternary families such that:

(1) If $n \in B_1$, the singleton family $() \mapsto n$ is primary.
(2) If F and G are primary ternary families such that $f(LC(F)LC(G)) = f(LC(F)) + f(LC(G))$; $LC(F)LC(G) \in A_1$; and neither F nor G is the singleton family $() \mapsto 3$; then $F \otimes G$ is primary.
(3) If F is a primary ternary family, then so is F_1.

So for a primary ternary family F, we have $LC(F) \in A_1$ and, by a straightforward induction, $UB(F) \le \text{rk } F + d(LC(F))$, which forces $UB(f) = \text{rk } F + d(LC(F))$. In particular, note that primary ternary families satisfy the conditions of Corollary 8.7. Our idea that these should form the "bulk" of A_r can be formalized as follows:

Conjecture 11.4. *If $r \geq 0$, $k = \lfloor r \rfloor$, $s = r - k$, then $\overline{B}_r$ can be covered by (or perhaps is even the union of) finitely many ternary families of rank at most $k-1$, together with the primary ternary families F of rank k such that $d(LC(F)) \leq s$.*

Certainly these families are all contained in $\overline{B}_r$, which shows that in the case that $s \in D_1$, the order type of $\overline{D}_r$ is at least $\omega^k D_1(s)$ (since $\overline{D}_r$ contains a copy of ω^k with supremum $r + s'$ for each $s' \leq s$ in D_1), and if Conjecture 11.4 is true then so is Conjecture 11.1 (since the union of finitely many copies of ω^k, all with the same supremum, is again a copy of ω^k).

Here's another consequence about order. Let ind r, for $r \neq 1$ a defect, denote the ordinal corresponding to r when we partition the set of defects (1 excluded) based on complexity modulo 3. Then we noted in Section 6 that for any n, we have $D(n) = 0$ if and only if ind $d(n) < \omega$. We conjecture more generally:

Conjecture 11.5. *For any whole number k, we have $D(n) \leq k$ if and only if* ind $n < \omega^{k+1}$.

Why should this be true? Under the above hypotheses, let us locate the first copy of ω^k to occur for a given congruence class of complexity modulo 3. These will be the primary ternary families with leading coefficients 3, 2, and 4, though which one corresponds to which congruence class varies, since the resulting complexity (for those inputs where the obvious upper bound is met) would be $3+k$, $2+k$, and $4+k$, respectively. But note that for n with $f(n) \equiv 0 \pmod 3$, the integral defect $D(n)$ increments when $d = 1 + d(2), 2 + 2d(2), 3, 4 + d(2), 5 + 2d(2), 6, \ldots$; for $f(n) \equiv 2 \pmod 3$, it increments when $d = 1 + 2d(2), 2, 3 + d(2), 4 + 2d(2), 5, 6 + d(2), \ldots$; and for $n \neq 1, f(n) \equiv 1 \pmod 3$, it increments when $d = 1, 2 + d(2), 3 + 2d(2), 4, 5 + d(2), 6 + 2d(2), \ldots$. Knowing this it is easy to check that for $f(n)$ fixed modulo 3, the integral defect $D(n)$ increments precisely when $d(n)$ hits $UB(F)$, for F chosen from the families above to have complexity congruent to $f(n)$ modulo 3 (when the obvious upper bound is satisfied). So among numbers with complexity fixed modulo 3, the point where $D(n)$ first equals k would be precisely right after the first occurrence of an ω^k. Note also that our computations (discussed in Section 5) show that this is true for $k = 1$ as well as for $k = 0$; indeed, so far out computations support Conjecture 11.4 entirely.

Pinning down what the growth rate of A_r should be will take a bit more work. Assuming Conjecture 11.4, it will only increment when r passes a number of the form $k + s$, where k an integer and $s \in D_1$. The question then is just how much the primary ternary families with obvious upper bound $k + s$ actually collectively contribute.

Suppose we consider each primary ternary family as a labeled rooted tree, as follows:

(1) The singleton family $() \mapsto n$ is represented by a single node, labeled with an n; unless $n = 3$, in which case we use a 1 instead.
(2) The tree for F_1 is made by taking the tree for F and adding a node labeled with a 1 as the new root.
(3) The tree for $F \otimes G$ is made by taking the trees for F and G, deleting the roots, disjoint unioning the results, and then adding a new root, labeled with the product of the old roots.

Note that if a tree T corresponds to a family F, then number of nodes in the tree is $1 + \text{rk } F$, and the product of the labels is $LC(F)$ if all leaf 1's are replaced by 3's. We will speak of the "rank of T" interchangeably with the "rank of F". We can actually set up a useful correspondence between the inputs of F and the non-root nodes of T recursively as follows: If F is given by $F() = m$, there is nothing to associate. If F is given by $F(n_1, \ldots, n_k) = G(n_1, \ldots, n_l)H(n_{l+1}, \ldots, n_k)$, we will associate to n_i whichever node corresponded to it in G or H. If F is given by $F(n_1, \ldots, n_k) = G(n_1, \ldots, n_{k-1})3^{n_k} + C$, we associate to each of $n_1, \ldots, n_{k-1}$ whichever node corresponded to it in G, and to n_k we associate the root of the tree for G (previously unassigned due to being a root).

Unsurprisingly, if F and G have isomorphic trees, they are essentially the same. (Here an "isomorphism" of these trees is assumed to preserve both root and labelling.) More precisely, if we have rank k families F, G with trees T, R respectively, and an isomorphism $\sigma : T \to R$, then allowing σ to act on the inputs to F, G via the above correspondence, we have $F(n_{\sigma(1)}, \ldots, n_{\sigma(k)}) = G(n_1, \ldots, n_k)$. The proof is a straightforward induction on k so we omit it.

Strictly speaking, we should speak of trees generating ternary families rather than vice versa, as we don't know that distinct (or rather, non-isomorphic) trees yield distinct families. In fact we hypothesise that not only do non-isomorphic trees yield distinct families (or rather, families that remain distinct after permutation of inputs), but that given two non-isomorphic trees of rank k, the intersection S of the images of the resulting families is "small" in the sense that $S(x) = O((\log x)^{k-1})$, and thus in consdering overall growth rate we do not need to worry about overlap between families from non-isomorphic trees. Or perhaps we can make a stronger statement, such as that if the corresponding families are F and G, then the set of x such that $F(x) \in S$ could be covered by finitely many hyperplanes (and obviously similarly with G). (We actually have further conjectures along these lines – Conjecture 10.2 could actually be related, for instance – but we do not have the space to discuss this further here.)

We further conjecture the following refinement of Theorem 7.4:

Conjecture 11.6. *Let T be a tree corresponding to a primary ternary family F. Then outside of a finite union of proper affine subspaces of the domain, F is $|\mathrm{Aut}\ T|$-to-1; consequently, if S is the image of F, and $\tilde{S}$ the expanded version,*

$$S(x) \sim \frac{1}{k!|\mathrm{Aut}\ T|}(\log_3 x)^k,$$

$$\tilde{S}(x) \sim \frac{1}{k!|\mathrm{Aut}\ T|}(\log_3 x)^{k+1}.$$

We showed in the proof of Theorem 7.4 the relation between the size of the fibers of F and the growth rate of $S(x)$ and $\tilde{S}(x)$; the question is, why should most of the fibers be of size $|\mathrm{Aut}\ T|$? We can actually show that most of them will be at least of size $|\mathrm{Aut}\ T|$, which proves the following refinement of Theorem 7.4:

Proposition 11.7. *With T, F, S, and $\tilde{S}$ as above, and letting N be the number of topological sorts of T,*

$$\frac{1}{Nk!}(\log_3 x)^k \lesssim S(x) \lesssim \frac{1}{k!|\mathrm{Aut}\ T|}(\log_3 x)^k,$$

$$\frac{1}{N(k+1)!}(\log_3 x)^{k+1} \lesssim \tilde{S}(x) \lesssim \frac{1}{(k+1)!|\mathrm{Aut}\ T|}(\log_3 x)^{k+1}$$

Proof. It suffices to show that the family $F(n_1, \ldots, n_k)$ is, outside a finite union of proper affine subspaces of the domain, at least $|\mathrm{Aut}\ T|$-to-1 and at most N-to-1.

We've actually already proved the upper bound on the size of the fibers – it is easy to check that with our variable-vertex correspondence, T minus the root is actually the Hasse diagram for the partial order on the variables induced by the partial order on the keys. Thus N, the number of topological sorts on T, is also the number of total orders extending the partial order on the keys. So the claim that F is usually at most N-to-1 is just what we showed in the proof of Theorem 7.4.

The lower bound on the fibers holds simply because for any $\sigma \in \mathrm{Aut}\ T$, if we let σ act on the coordinates via our variable-vertex correspondence, then as noted above we have $F(n_{\sigma(1)}, \ldots, n_{\sigma(k)}) = F(n_1, \ldots, n_k)$, and hence F is at least $|\mathrm{Aut}\ T|$-to-1 whenever all the coordinates are distinct.

□

As for why we expect that F will usually be $|\mathrm{Aut}\ G|$-to-1 instead of merely at least such, the intuition is similar to our notion of "small intersection" above; just as it seems two families should not have substantial overlap unless they are isomorphic, so it seems that two forms of the same family (meaning, two total extensions of the partial order on the keys, or anti-keys) should not have substantial overlap unless they are related by an automorphism. Note that we can make similar statements for more general ternary families, but we must be more careful with how we define the corresponding tree, and we must allow automorphisms to ignore powers of 3 when matching certain labels.

Continuing, observe that if we want to consider all rank-k trees corresponding to primary ternary families F with $UB(F) = k + s$, then in fact any rooted tree on $k + 1$ nodes, labeled with numbers not divisible by 3 such that their product is a most-efficient representation of a number with defect in $s + \mathbb{Z}$, will do. Knowing this, and assuming the conjectures above, the problem of figuring out just what constant is contributed by the primary ternary ternary families F with $\mathrm{rk}\ F = k$ and $LC(F) = n$ reduces to counting the number of isomorphism classes of rooted trees on $k + 1$ nodes with labels most-efficiently multiplying to n (ignoring powers of 3), where each class is weighted inversely by its number of automorphisms. But when isomorphism classes are weighted inversely by their number of automorphisms, the result is just the total number of such objects, divided by $(k+1)!$. And we know how to count rooted trees on $k + 1$ vertices; by Cayley's formula, this is just $(k + 1)^k$. Meanwhile, since the labelling is independent of the tree structure, we can just count that separately and multiply.

So now we need to know how to most-efficiently represent elements of B_1 (and 1) as products. For 1, there is nothing to check. For those elements of B_1 not of the form $3^m + 1$ for $k \geq 4$, it is easily checked that all these can be most-efficiently represented by decomposition into primes. It remains to show that for $m \geq 4$, the number $3^m + 1$ can never be most-efficiently represented as a product. To see this, note that if $m \geq 4$ and $3^m + 1 = ab$ is a most-efficient representation, the factors a and b would have defects less than 1, and neither could be powers of 3; so $d(a), d(b) \geq d(2)$ and hence $d(a), d(b) < 1 - d(2)$. But $B_{1-d(2)} = \{3, 2, 4, 8, 16, 32, 5, 64, 7, 10, 128, 14, 20, 256\}$ and no product of two of these can be written as $3^m + 1$ for $k \geq 4$. Thus for our purposes, numbers of the form $3^m + 1$ for $k \geq 4$ are effectively "prime". Hence if $n = 3^b p_1^{a_1} \ldots p_l^{a_l}$ where each p_i is either a non-3 prime, or of the form 3^m+1 for $m \ge 4$, we get a total of $\prod_{i=1}^{l} \binom{k+a_i}{a_i}$ possible labellings.

Unpacking all this leads us at last to the following conjecture for the asympotic growth rate of $\overline{A}_r$, in general:

Conjecture 11.8. *Given $r \ge 0$, let $k = \lfloor r \rfloor$ and $s = r-k$. Then $\overline{B}_r(x) \sim C\frac{(k+1)^{k-1}}{k!^2}(\log_3 x)^k$ and $\overline{A}_r(x) \sim C\frac{(k+1)^{k-2}}{k!^2}(\log_3 x)^{k+1}$ where C is determined as follows:*

(1) For $0 \le s < d(2)$, take $C = 1$.
(2) For $d(2) \le s < 2d(2)$, take $C = k+2$.
(3) For $2d(2) \le s < 3d(2)$, take $C = \binom{k+3}{2}$.
(4) For $3d(2) \le s < 4d(2)$, take $C = \binom{k+4}{3}$.
(5) For $4d(2) \le s < 5d(2)$, take $C = \binom{k+5}{4}$.
(6) For $5d(2) \le s < d(5)$, take $C = \binom{k+6}{5}$.
(7) For $d(5) \le s < 6d(2)$, take $C = \binom{k+6}{5} + k + 1$.
(8) For $6d(2) \le s < d(7)$, take $C = \binom{k+7}{6} + k + 1$.
(9) For $d(7) \le s < d(5)+d(2)$, take $C = \binom{k+7}{6} + 2(k+1)$.
(10) For $d(5)+d(2) \le s < 7d(2)$, take $C = \binom{k+7}{6} + (k+1)(k+3)$.
(11) For $7d(2) \le s < d(7)+d(2)$, take $C = \binom{k+8}{7} + (k+1)(k+3)$.
(12) For $d(7)+d(2) \le s < d(5)+2d(2)$, take $C = \binom{k+8}{7} + 2(k+1)(k+2)$.
(13) For $d(5)+2d(2) \le s < 8d(2)$, take $C = \binom{k+8}{7} + (k+1)(\binom{k+3}{2} + k + 2)$.
(14) For $8d(2) \le s < d(7)+2d(2)$, take $C = \binom{k+9}{8} + (k+1)(\binom{k+3}{2} + k + 2)$.
(15) For $d(7)+2d(2) \le s < d(5)+3d(2)$, take $C = \binom{k+9}{8} + 2(k+1)\binom{k+3}{2}$.
(16) For $d(5)+3d(2) \le s < d(19)$, take $C = \binom{k+9}{8} + (k+1)(\binom{k+4}{3} + \binom{k+3}{2})$.
(17) For $d(19) \le s < 9d(2)$, take $C = \binom{k+9}{8} + (k+1)(\binom{k+4}{3} + \binom{k+3}{2} + 1)$.
(18) For $9d(2) \le s < d(13)$, take $C = \binom{k+10}{9} + (k+1)(\binom{k+4}{3} + \binom{k+3}{2} + 1)$.
(19) For $d(13) \le s < 1$, take $C = \binom{k+10}{9} + (k+1)(\binom{k+4}{3} + \binom{k+3}{2} + N - 1)$ where $N = \lfloor -\log_3(3^{\frac{1-s}{3}} - 1) \rfloor$.

Here then is the generalization of Conjecture 11.2 we promised earlier. Note that similar ideas could presumaby be applied to other notions of defect, although one may need to alter the definition of primary ternary families F to allow the more general $D(LC(F)) = 0$ rather than the current $d(LC(F)) < 1$. However even without doing that we can see that, if we define $\tilde{A}_k$ to be the set of numbers n with $D(n) \le k$, the above ideas imply:

Conjecture 11.9. $\tilde{A}_k(x) \sim \overline{A}_{k+1+2d(2)}(x) \sim \binom{k+4}{2}\frac{(k+2)^{k-1}}{(k+1)!^2}(\log_3 x)^{k+1}$

Unfortunately we do not at present seem to have any way to prove the crucial Conjecture 11.4; our tools do not seem to have enough resolution for such a thing.

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