algebra1

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1 Algebra with SymPy: Part I

1.1 Load libraries and define variables to use

1.1.1 Monomials

[1]:
```
from sympy import symbols
x=symbols('x')
x
```

[1]: x

[2]:
```
x**2
```

[2]: x^2

[3]:
```
# Pow, another way to compute powers
from sympy import Pow
Pow(x,2)
```

[3]: x^2

[4]:
```
3*x
```

[4]: $3x$

[5]:
```
# instead of symbols, you can use sympy.abc . See the following example
from sympy.abc import x, y, z,w
f = x**2 + y - z + w
f
```

[5]: $w + x^2 + y - z$

1.1.2 Polynomials

Sum/Substraction

[6]:
```
y = x**2 - x - 6 # sum of monomials
y
```

[6]: $x^2 - x - 6$

[7]:
```python
w = x**3 - x + 8
y+w  # sum
```

[7]:

$x^3 + x^2 - 2x + 2$

[8]:
```python
# yet another way to add
from sympy import Add
Add(y,w)
```

[8]:

$x^3 + x^2 - 2x + 2$

[9]:
```python
y-w  # substraction
```

[9]:

$-x^3 + x^2 - 14$

Multiplication/Factorization

[10]:
```python
z=(x+5)*(x-2)
z
```

[10]:

$(x - 2)(x + 5)$

expand

[11]:
```python
from sympy import expand
zexpand = expand(z)
zexpand
```

[11]:

$x^2 + 3x - 10$

[12]:
```python
display(y*w) # expand is necessary to make the product explicit in powers of the␣
↪independent variable
h = (y*w).expand()
h
```

$(x^2 - x - 6)(x^3 - x + 8)$

[12]:

$x^5 - x^4 - 7x^3 + 9x^2 - 2x - 48$

[13]:
```python
# yet another way to multiply
from sympy import Mul
Mul(y, w)# yet another way to multiply
Mul(y, w).expand()
```

[13]:

$x^5 - x^4 - 7x^3 + 9x^2 - 2x - 48$

[14]:
```python
# combine previous operators
Add(Pow(x, 2), Mul(x, y))
```

[14]:

$x^2 + x(x^2 - x - 6)$

[15]:
```python
# binomial expression
a = symbols('a')
b = symbols('b')
expand((a+b)**5)
```

[15]: $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

factor

[16]:
```python
from sympy import factor
zfactor = factor(zexpand)
zfactor
```

[16]: $(x-2)(x+5)$

[17]:
```python
factor(y)
```

[17]: $(x-3)(x+2)$

[18]:
```python
y.factor()  # yet another way
```

[18]: $(x-3)(x+2)$

[19]:
```python
h.factor()
```

[19]: $(x-3)(x+2)\left(x^3 - x + 8\right)$

Factor over the complex field, roots of a polynomial

[20]:
```python
from sympy import I
factor(x**2 + 1, extension=[I])
```

[20]: $(x-i)(x+i)$

[21]:
```python
h.factor(extension=[I]) # this will not work, why?
```

[21]: $(x-3)(x+2)\left(x^3 - x + 8\right)$

[22]:
```python
h2 = x**3 - x + 8
factor(h2, extension=[I])
```

[22]: $x^3 - x + 8$

Homework 1 : Why `factor()` did not work for the expression `h2` ?

[23]:
```python
from sympy import roots
r = roots(h2)
display(r)
```

```
{-(3*sqrt(1293) + 108)**(1/3)/3 - 1/(3*sqrt(1293) + 108)**(1/3): 1,
 -(-1/2 + sqrt(3)*I/2)*(3*sqrt(1293) + 108)**(1/3)/3 - 1/((-1/2 + sqrt(3)*I/
↪2)*(3*sqrt(1293) + 108)**(1/3)): 1,
 -1/((-1/2 - sqrt(3)*I/2)*(3*sqrt(1293) + 108)**(1/3)) - (-1/2 - sqrt(3)*I/
↪2)*(3*sqrt(1293) + 108)**(1/3)/3: 1}
```

[24]:
```
hfactor = h2.factor(extension=roots(h2))
hfactor
```

[24]:

$$\left(x - \frac{\sqrt[3]{3\sqrt{1293}+108}}{12} - \frac{\sqrt{1293}\sqrt[3]{3\sqrt{1293}+108}}{432} - \frac{5\left(3\sqrt{1293}+108\right)^{\frac{2}{3}}}{432} - \frac{1}{36} - \frac{7}{144\left(3\sqrt{1293}+108\right)^{\frac{2}{3}}} + \frac{}{3\cdot\left(3\sqrt{12\right.}}\right.$$

[25]:
```
from sympy import simplify
rsimp = simplify(r)
rsimp
```

[25]:

$$\left\{-\frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}} - \frac{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}}{3} : 1,\ -\frac{\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}}{3} - \frac{}{\left(-\frac{1}{2}+\frac{\sqrt{}}{2}\right.}\right.$$

[26]:
```
simplify(hfactor) # this actually works
```

[26]:

$x^3 - x + 8$

Homework 2 We would like to come up with an expression such as

$$(x - r_1)(x - r_2)(x - r_3),$$

but with r_i, $i = 1, 2, 3$, as simplified as possible. We will get to that with our nails and learn a bit about the structures on the way. **Hint**: use the simplified roots in the previous line. The answer should look like:abs

$$\left(x + \frac{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}}{3} + \frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}}\right)\left(x + \frac{1}{\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\sqrt[3]{3\sqrt{1293}+108}} + \frac{\left(-\frac{1}{2}\right.}{}\right.$$

At the end please verify by simplifying your product $(x - r_1)(x - r_2)(x - r_3)$. It should produce the original polynomial $x^3 - x + 8$.

Solution is hidden

$$\cdots = \frac{\cdots}{8} + \frac{\left(-\frac{1}{2} \cdots\right.}{\cdots}$$

$$x \quad x + 8$$

Collect Let us consider the following expression, in several variables:

$$xy + xz + x^2yz + yz.$$

For some reason we want to have it written as a polynomial on x. That is,

$$p(x) = yz + x(y + z) + x^2yz.$$

Observe that we collected the terms in x.

[35]:
```
from sympy import symbols, collect
x,y,z = symbols('x y z ')
expr = x*y + x*z + x**2* y*z + y*z
coll = collect(expr, x) # note that the order is from high to low
coll
```

[35]: $x^2yz + x\,(y + z) + yz$

[36]:
```
coll.coeff(x, 1) # extract coefficient for x
```

[36]: $y + z$

[37]:
```
# we now can extract whatever coefficient we want. For example the coefficient␣
↪of x is y+z.
```

1.2 Symplification with radicals

We know **rationalization** from high school. for example

$$\frac{1}{2-\sqrt{2}} = \frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2+\sqrt{2}}{2}.$$

[38]:
```python
from sympy import radsimp, sqrt
radsimp(1/(2 - sqrt(2)))
```

[38]:

$$\frac{\sqrt{2}+2}{2}$$

1.2.1 eval() and srepr()

Evaluates an expression which is defined as a string. We study **diff** (differential) later on **srepr()** do the oppositve. Converts an expression into a string format. This could be useful to share expressions between users that do not want figures or LaTeX symbols.

[39]:
```python
import sympy as sp  # to make exp unique
from sympy import diff # derivative this comes later
a,b,c,s,x = symbols('a,b,c,s,x')
s="a*sp.exp(-b*(x-c)**(2))"
diff(eval(s), x)
```

[39]:

$$-ab\left(-2c+2x\right)e^{-b(-c+x)^2}$$

[40]:
```python
from sympy import srepr # string representation
srepr(diff(eval(s), x))
```

[40]:
```
"Mul(Integer(-1), Symbol('a'), Symbol('b'), Add(Mul(Integer(-1), Integer(2),
Symbol('c')), Mul(Integer(2), Symbol('x'))), exp(Mul(Integer(-1), Symbol('b'),
Pow(Add(Mul(Integer(-1), Symbol('c')), Symbol('x')), Integer(2)))))"
```