Machine Learning Tarea #2

Herman Jaramillo Villegas
Universidad de Medellín

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Por favor haga estos ejercicios a mano. Use el computador solo como último recurso y más con el fin de verificar sus cómputos.

We assume the field of complex numbers.

1. Show that for any two vectors $u, v \in \mathbb{C}^n$ $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

2. Show that (see definition in the class notes) Frobenius norm can be written as

$$\|A\|_F = \sqrt{\mathrm{tr}(A^* A)}.$$

   where the **trace** of the matrix B, noted as $\mathrm{tr}(B)$, for any $n \times n$ matrix B, is defined as $\mathrm{tr}(B) = \sum_{i=1}^{n} b_{ii}$. That is, the trace is sum of all the diagonal elements of a square matrix.

3. Prove the Pythagoras theorem for vectors in $\mathbb{R}^n$. That is if u, v are orthogonal, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

4. This problem is useful when studying the support vector machine (SVM) technique. Assume $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$, $\delta \in \mathbb{R}, \delta > 0$.

   (a) Show that the plane $\langle w, x\rangle = b$ is such that $w \neq 0$ is normal to the plane and the distance, from the origin to the plane, is given by

$$d = \frac{b}{\|w\|}.$$

   Note that b could be positive or negative and the distance signed to one or the other side of the origin according to $\mathrm{sign}(b)$. *Hint: Consider $n = 2$ and draw the plane as line away from the origin.*

   (b) Use the previous item to show that, if we have two planes

$$w^T x - b = \pm\delta,$$

   they are parallel and the distance between them is $2\delta/\|w\|$.