Analysis

Part I

Basis

n - size of the heap, d - heap arity, h - height of the heap.
$d \geq 2$ is assumed. For $d = 1$ the heap is degenerated, and the operations will take $\Theta(n^2)$ at the worst case.
The amount of items at depth k is at most d^k. Therefore:

$$\begin{aligned} \sum_{k=1}^{h-1} d^k &\leq n \Rightarrow \frac{d^h - 1}{d-1} \leq n \Rightarrow \frac{d^h - 1}{d} \leq n \Rightarrow d^h \leq nd + 1 \Rightarrow \\ h &\leq \log_d(nd+1) \leq \log_d(2nd) = \log_d(2) + \log_d(n) + 1 \leq \log_d(n) + 2 \end{aligned}$$

Analysis of heapifyDown

It takes exactly $d-1$ comparisons to find the minimum child of a node, and 1 comparison to determine if the current node should be swapped with it's minimum child. Therefore d comparisons are done at each iteration, and there are at most k iterations for a node at height k. Therefore for a call on a node of height k, there are at most dk comparisons.

Analysis of arrayToHeap (implemented with heapify)

Let x be the number of leaves in the heap. Then the number of nodes of height 1 is $\lceil \frac{x}{d} \rceil \leq \frac{x}{d} + 1$. The number of nodes of height 2 is $\left\lceil \frac{\lceil \frac{x}{d} \rceil}{d} \right\rceil \leq \frac{\frac{x}{d}+1}{d} + 1 = \frac{x}{d^2} + \frac{1}{d} + 1$. Similarly the number of nodes of height 3 is not greater then $\frac{x}{d^3} + \frac{1}{d^2} + \frac{1}{d} + 1$. Also, obviously $x \leq n$. It's easy to see that the number of nodes of height k is not greater then $\frac{n}{d^k} + 2$. From this, and from the analysis of heapifyDown, , the amount of comparisons done by arrayToHeap is bounded by:

$$\begin{aligned} \mathit{Comparisons} &\leq \sum_{k=1}^{h} dk \left(\frac{n}{d^k} + 2 \right) \overset{*}{\leq} \sum_{k=1}^{h} dk \cdot 2\frac{n}{d^k} = 2nd \sum_{k=1}^{h} \frac{k}{d^k} \leq 2nd \sum_{k=0}^{\infty} \frac{k}{d^k} \\ &= 2nd \frac{d}{(d-1)^2} = 2n \left(\frac{d}{(d-1)} \right)^2 \leq 8n = O(n) \end{aligned}$$

Where the second transition is true for large enough n.
Also, $\Omega(n)$ is an obvious lower bound for the comparisons made by heapify, as at least $n-1$ comparisons are required to find a minimum item in an unordered list.
Therefore arrayToHeap executes $\Theta(n)$ comparisons.

Analysis of DHeapSort (n delete-min calls)

As each delete-min simply calls heapifyDown on the root, then from the analysis of heapifyDown, the number of comparisons is $O(dh) = O(d(\log_d(n)+2)) = O(d\log_d(n)) = O(\frac{d}{\log(d)}\log(n))$. Then the series of n calls to delete-min are executed in $O(\frac{d}{\log(d)}n\log(n))$. This bound is tight at the following scenario shows:
For simplicity, we'll assume the heap's last level is a full, and that d is even (the idea can be adapted to other cases).
We are given a heap where the root has key 1, and of it's d children, the left half have key 2, and the right half have the key 3. For every node that has key 2, all it's descendants have key 2, and for every node that has key 3, all it's descendants have key 3. See Figure 1 for example.
Given this scenario, the first half of the n delete-min calls will make most of the 3 nodes travel all the way from the root to the deepest level (to the left side, instead of the 2 nodes).
In a heap, more than $\frac{n}{2}$ of the nodes are leafs. So in this scenario more than $\frac{n}{4}$ of the nodes are leafs with key 3. When delete-min is called, they are moved to the top, and are heapified down, "dropping" to the left by swapping with nodes with key 2. Few of them become inner nodes before others can become leafs. Let L be the amount of leafs in the heap. $x = \frac{L}{2}$ leafs have key 2, and $x = \frac{L}{2}$ leaves have key 3. The amount of ancestors nodes with key 2 have is bounded:

$$\begin{aligned} \mathit{Ancestors} &= \sum_{k=1}^{h}\left\lceil\frac{x}{d^k}\right\rceil \leq \sum_{k=1}^{h}\left(\frac{x}{d^k}+1\right) = h + x\sum_{k=1}^{h}\frac{1}{d^k} \leq h + x\sum_{k=1}^{\infty}\frac{1}{d^k} \\ &\leq \log_d(n) + 2 + \frac{x}{d-1} \leq \frac{2x}{d-1} \end{aligned}$$

Let J be the amount of nodes with key 3 that end up dropping to the bottom.

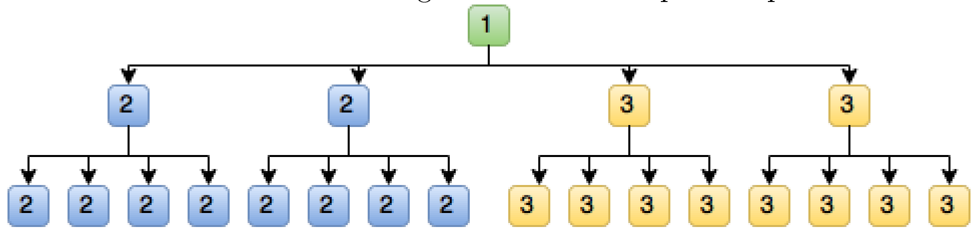
$$J \geq x - \mathit{Ancestors} \geq x - \frac{2x}{d-1} = x\left(1 - \frac{2}{d-1}\right) = x\cdot\frac{d-3}{d-1} \geq \frac{x}{2} = \frac{L}{4} \geq \frac{n}{8} = \Omega(n)$$

Therefore a linear portion of the key-3 nodes will be traveling all the way from the root to the bottom, making $\Theta(d\log_d(n))$ comparisons each, proving the tightness of the bound.
The entire operation will therefore make $\Theta(n)+$ $\Theta(\frac{d}{\log(d)}n\log(n)) = \Theta(\frac{d}{\log(d)}n\log(n))$ comparisons at the worst case.

Measurements Explanation

Assuming the average case has a similar run-time complexity, the measurements can be explained:
As seen in the measurements, the amount of comparisons is linearithmic ($n\log(n)$ relation) in n, as expected.

Figure 1: W.C. Heap Example



The amount of comparisons appears to be lowest when $d = 3$. This can be explained by examining the term $\frac{d}{\log(d)}$. Looking at the function $f(x) = \frac{x}{\log(x)}$, we get $f'(x) = \frac{\log(x)-1}{\log^2(x)}$. Therefore $f'(e) = 0$, and it is clear that $x = e$ is a global minimum for $x \geq 1$. So in theory, an e-heap would have the lowest amount of comparisons. In practice 3 is the closest integer to e, so it yields the lowest amount of comparisons achievable using DHeapSort.

Part II

Analysis of heapifyUp

When heapifyUp is called on an item at depth k, then at most k comparisons and swaps are performed, as the item is only compared to it's parent.

Analysis of n Insert, Decrease-Key calls

The maximum depth of an item in the heap is $h \leq \log_d(n) + 2 = O(\log_d(n))$, and each Insert performs 1 heapifyUp, therefore the amount of comparisons in n calls to Insert is bounded by $O(n \log_d(n)) = O(\frac{n \log(n)}{\log(d)})$

By the exact same reasoning, n calls to Decrease-Key are bounded by $O(n \log_d(n)) = O(\frac{n \log(n)}{\log(d)})$
The bounds are tight, as the following input shows:
n items are inserted with the keys: $2n + 1, 2n, 2n - 1, \ldots, n + 1$, and the Decrease-Key calls are done with $delta = n + 1$, in the order of insertion.
At the insertion phase, each item inserted is the minimum (at that stage), and therefore must be heapified-up from the bottom of the heap to the root. At the decrease phase, each item items changes from the maximum to the minimum, and must be heapified-up from the bottom to the root.
In a heap of size n, more than $\frac{n}{2} = \Theta(n)$ items are leafs. Therefore $\Theta(n)$ of the operations, in both phases, will indeed cost $\Theta(\log_d(n))$ comparisons, showing that the bound is tight.

Therefore the amount of comparisons of in the overall operation is at the worst case $\Theta(\frac{n \log(n)}{\log(d)})$.

Note: if $d \geq n$ then the worst case is $\Theta(n)$, because $\Omega(n)$ is an obvious lower bound.

Note: The TA later changed the requirement to analyze the worst case of a single Decrease-Key call. It's obvious that the W.C. for a single call is $\Theta(\log_d(n)) = \Theta(\frac{\log(n)}{\log(d)})$. We left the full analysis as it didn't seem right to delete it.

Measurements Explanation

Assuming the average case has a similar run-time complexity, the measurements show that for $x = 100$, $x = 1000$, the number of comparisons indeed shrinks as d grows, as expected.
The lower-bound for the best case is $n - 1$ comparisons, in the case where no item changes its position (-1 because the root item never needs to be heapified-up). That bound is indeed achieved for the case where $x = 1$, because if a node's key is higher than it's parent, then decreasing it by 1 will not make them swap. Because the heap is built by insertions and not via arrayToHeap, then if a node and it's parent have the same key, then the parent must have been inserted earlier (because nodes are inserted to the bottom of the heap and are heapified-up). Therefore the parent's key will be decreased earlier, so there isn't a possibility of them swapping.