1 Introduction

The adjacency matrix A of an (v, k, λ, μ) strongly regular graph G satisfies the equation $A^2 = xA + yI + zJ$ for constants $x, y, z \in \mathbb{R}$. We will say a matrix is three-strongly regular or 3SRG if its adjacency matrix satisfies

$$A^3 = xA + yI + zJ \tag{1}$$

for constants x, y, z. This is equivalent to stating that the number of paths of length 3 between any two points v and U depends only on whether $v = u$, where we denote the number of paths k; $v \neq u$ and v, u are neighbors, where we denote the number of paths λ; and $v \neq u$ and v, u are not neighbors, where we denote the number of paths μ. We say that such a graph on n vertices is 3SRG-(n, k, λ, μ); if it is also r-regular then it is 3SRG-(n, r, k, λ, μ). k, λ, μ are related to the coefficients of 1 as:

$$x = \lambda - \mu$$

$$y = k - \mu$$

$$z = \mu$$

$(n, k, k(a-c)+ck, a(a-c)+k-c+ck, c(a-c)+ck)$ If a matrix is strongly regular, it is also 3SRG:

$$A^3 = A(A^2) = A(xA+yI+zJ) = xA^2+yA+zAJ = x^2A+xyI+xzJ+yA+kJ$$

where $AJ = kJ$ because the graph is regular. Starting with a (n, k, λ, μ) SRG, we obtain an $(n, k, k(\lambda-\mu)+k\mu, \lambda(\lambda-\mu)+k-\mu+k\mu, \mu(\lambda-\mu)+k\mu)$ 3SRG. So the 3SRG graphs are a superset of the SRG graphs. The first example of non-SRG 3SRG graphs are the complete bipartite graphs $K_{m,n}$ for $m \neq n$. It can be easily verified that in this case $A^3 = mnA$, so these are non-regular, $(m+n, 0, mn, 0)$ 3SRGs. There also exist nontrivial regular, non-SRG, 3SRG with $k > 0$. An exhaustive computer search up to order 10, and then a search on transitive graphs up to order 32, revealed many examples:

$$(8, 4, 6, 10, 6)$$

$$(12, 4, 4, 8, 4)x2, (12, 6, 16, 20, 16)$$

$$(14, 10, 66, 73, 68)$$

$$(15, 8, 38, 39, 27)$$

$$(16, 8, 24, 40, 24)$$

$$(18, 8, 38, 39, 18), (18, 9, 36, 45, 36), (18, 12, 90, 99, 90)$$

$$(21, 4, 4, 7, 2)$$

$$(24, 8, 16, 32, 16), (24, 12, 64, 80, 64), (24, 12, 54, 90, 54), (24, 12, 64, 80, 64)$$

$(27,6,6,15,6)\times 2, (27,8,26,27,15), (27,12,60,69,60)\times 3, (27,14,110,111,90), (27,18,210,219,210)\times 3$

$(30,11,56,57,36), (30,12,54,63,54)\times 2, (30,15,108,117,108)\times 2, (30,16,128,144,128), (30,20,272,273,252)$

It is clear that there are two broad classes: 3SRGs where $k = \mu$, and 3SRGs where $k \neq \mu$. We will call the first class balanced and the second imbalanced. There are some simple constraints we can find on the parameters of regular 3SRGs. The number of paths of length 3 starting at a fixed vertex is r^3; pick one of r neighbors three times. We equate r^3 with an expression:

$$r^3 = k + r\lambda + (n - r - 1)\mu \tag{2}$$

where the first term counts paths from a vertex to itself, the second term paths from a vertex to a neighbor, and the third term paths from a vertex to a non-neighbor. This constraint equation holds for all possible regular 3SRGs. There are more trivial constraints : $0 < r < n, k < r(r-1), k$ is even.

2 Spectral Analysis

This section is inspired by chapter 10 of Godsin and Royle. We will now do some spectral analysis. Let A be a (n, r, k, λ, μ)-3SRG. We can write the original equation to which A is a solution:

$$A^3 = (\lambda - \mu)A + (k - \mu)I + \mu J$$

$$A^3 - (\lambda - \mu)A - (k - \mu)I = \mu J$$

Since A is regular, the all ones vector $\vec{j}$ is an eigenvector of A with eigenvalue r, which has multiplicity 1. Since A is symmetric, any eigenvector of A with a different eigenvalue must be orthogonal to $\vec{j}$. Let x be an eigenvector of a with eigenvalue $\theta \neq r$. Then:

$$A^3x - (\lambda - \mu)Ax - (k - \mu)Ix = \mu Jx$$

We have that x is orthogonal to the rows of J, so:

$$A^3x - (\lambda - \mu)Ax - (k - \mu)Ix = 0$$

$$\theta^3 x - (\lambda - \mu)\theta x - (k - \mu)x = 0$$

$$\theta^3 - (\lambda - \mu)\theta - (k - \mu) = 0$$

Thus the other eigenvalues of a are solutions to the cubic $x^3 - (\lambda - \mu)x - (k - \mu) = 0$. Since A is symmetric, its eigenvalues are real and all solutions of that cubic must be real, which means the discriminant of the cubic must be non-negative, and we have:

$$4(\lambda - \mu)^3 - 27(k - \mu)^2 \geq 0$$

$$\lambda \geq \mu$$

are further constraints that hold for all possible parameters. Now let us turn our attention to balanced 3SRGs where $k = \mu$. In this case we have the eigenvalues of A, other than r, are the solutions to $x^3 - (\lambda - \mu)x = 0$. These solutions are $0, \pm\sqrt{\lambda - \mu}$. Let

$$\alpha = \sqrt{\lambda - \mu}$$

$$\beta = -\sqrt{\lambda - \mu}$$

The sum of all eigenvalues with multiplicity is equal to 0, the trace of A. Let m_α, m_β be the multiplicities of α and β respectively. We have

$$m_\alpha \alpha + m_\beta \beta = -r$$

$$(m_\alpha - m_\beta)\alpha = -r$$

Since $r \neq 0$ we have that α must be rational and an integer, and we conclude that for balanced regular 3SRGs:

$$\lambda - \mu \text{ is a perfect square and } (\sqrt{\lambda - \mu})|r$$

Now look at the trace of A^3 instead of A. A^3 has k on the diagonal, so its trace is nk, and its trace is also the sum of the cubed eigenvalues of A, so we have the two equations:

$$m_\alpha \alpha + m_\beta \beta = -r$$

$$m_\alpha \alpha^3 + m_\beta \beta^3 = -r^3 + nk$$

Simplify this to obtain:

$$r\alpha^2 - r^3 + nk = 0$$

Unfortunately this is just the case of 2 where $k = \mu$. We have shown that balanced 3SRGs are integral graphs. This is not true for imbalanced 3SRGs, but experimental data suggests that each imbalanced 3SRG has at least one integer eigenvalue other than r. This is connected somehow to equitable partitions. In SRGs, it is possible to calculate the multiplicities of each eigenvalue through a system of linear equations; this technique has not worked so far on 3SRGs, as the corresponding system of linear equations is insufficiently determined.

Let A be a connected r-regular graph with exactly four distinct eigenvalues. Then A is 3SRG if and only the sum of the eigenvalues is r. Let r, α, β, γ be the four distinct eigenvalues of A, and n be the number of vertices. We can decompose the underlying vector space $\mathbb{F}^n$ as $E_r \oplus E_{\alpha,\beta,\gamma}$. E_r is the one dimensional span of the all-ones vector j, since A is connected. Form the new operator:

$$M = \frac{1}{(r-\alpha)(r-\beta)(r-\gamma)}(A - \alpha I)(A - \beta I)(A - \gamma I)$$

it is clear that M acts like the 0 operator on $E_{\alpha,\beta,\gamma}$, because the terms $(A - xI)$ all commute with each other. On E_r, M acts like the identity:

$$Mj = \frac{(r-\alpha)(r-\beta)(r-\gamma)}{(r-\alpha)(r-\beta)(r-\gamma)}j = j$$

Now since $E_{\alpha,\beta,\gamma}$ is orthogonal to E_r, the operator $\frac{1}{n}J$ also acts like the zero operator on $E_{\alpha,\beta,\gamma}$ and like I on E_r, so we conclude $M = \frac{1}{n}J$. Now we can rearrange the original equation for M:

$$\frac{1}{n}J = \frac{1}{(r-\alpha)(r-\beta)(r-\gamma)}(A-\alpha I)(A-\beta I)(A-\gamma I)$$

$$\frac{(r-\alpha)(r-\beta)(r-\gamma)}{n} = A^3 - (\alpha+\beta+\gamma)A^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)A - I$$

By hypothesis $\alpha+\beta+\gamma = 0$, the A^2 term vanishes and we can write A^3 as a linear combination of A, I, J. Now if $\alpha+\beta+\gamma \neq 0$ and A is a 3SRG, then we can write A^2 as a linear combination A, I, J, implying A is SRG. This is a contradiction because SRGs only have three distinct eigenvalues. This concludes the proof.

3 Equitable Partitions

[Definition of Equitable Partitions]. It appears so far that all 3SRGs on even number of vertices have an equitable partition into two equal sized sets. This implies the adjacency matrix can be written in block form as:

$$A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$$

where B and D are equally regular graphs, and C is a $0,1$-matrix with constant row and column sum. This implies the existence of a non-r integer eigenvalue. The number of such equitable partitions of a 3SRG seems to fluctuate a ls ot, some graphs have many such partitions and some have very few. SRGs with this property were studied in a paper by Jorgenson and Klin, who gave conditions when edge switching the partition produced another SRG. This might also be possible for 3SRGs. In one case a 14-vertex non-SRG 3SRG was produced in this manner, with C equal to the matrix with all rows of the $[7,4]$-Hamming Code. This seems too interesting to be coincidental.
Call the function taking B, C, D to A f. Let K_n be the the graph of the complete matrix on n vertices. There is a correspondence between C where $f(K_n, C, K_n)$ is 3SRG and symmetric BIBDs (v, v, k, k, λ) that satisfy the equation:

$$v^2 - 3v + 2k^2 - 3\lambda + 6k + v\lambda - 3kv - k\lambda = 0$$

[Insert long proof of this] The only such symmetric BIBDs I have found so far are the trivial $(v, v-1, v-2)$-BIBDs and the $(7,4,2)$-BIBD.

4 Further Generalizations

Further questions:
Some type of combinatorial design, can't find it on wikipedia

Hamming code for 14 example, constant weight linear codes?
graph expansion operation: not all are, all so far at least are triple product
edge switching as in paper of jorgenson and klin
vv partition=symmetric equitable partition, highly relevant
not closed under complement
3SRGs unlike SRGS are not distance-regular.
any connected regular graph with 4 distinct eigenvalues is 3srg
Does there exist a non-regular, non-(complete bipartite) 3SRG?
Call a 3SRG balanced if $k = \mu$. When does there exist a balanced non-srg, non-complete bipartite 3SRG on d vertices? When does there exist a non-balanced one? So far balanced are more common than non-balanced.
Why after $d = 16$ does it seem like examples only exist on multiples of 3?

Every A has at least one non-r integer eigenvalue. not true, there is an srg that does not and is 3srg

What happens if we change the equation from $A^3 = xA + yI + zJ$ to $A^n = xA + yI + zJ$?
families of SRGs go along even-SRG, odd-nonSRG 3SRG
5SRG+3SRG, not SRG $\implies$ balanced. balanced 3SRGs can be upgraded to 5SRGs