

Definition. Let \mathcal{G} be a finite edge-weighted bipartite planar graph with a set \mathbf{N} of $2n$ special vertices called nodes on the outer face labeled $1, 2, \dots, 2n$ in counterclockwise order. A double-dimer configuration on $(\mathcal{G}, \mathbf{N})$ is a configuration of disjoint loops, doubled edges, and simple paths that connect the nodes in pairs. Each configuration is weighted by the product of its edge-weights times 2^ℓ , where ℓ is the number of loops in the configuration.

Definition. Let \mathcal{G}^{BW} be the subgraph of \mathcal{G} formed by deleting the nodes except for the ones that are black and odd or white and even. Let $\mathcal{G}_{i,j}^{BW}$ be defined as \mathcal{G}^{BW} was, but with node i included in $\mathcal{G}_{i,j}^{BW}$ if and only if it was not included in \mathcal{G}^{BW} , and similarly for node j . Let Z^{BW} and $Z_{i,j}^{BW}$ be the weighted sum of dimer configurations of dimer configurations of \mathcal{G}^{BW} and $\mathcal{G}_{i,j}^{BW}$, respectively. Define Z^{WB} and $Z_{i,j}^{WB}$ similarly but with the roles of black and white reversed.

For each planar matching of the nodes σ , let $\Pr(\sigma)$ be the probability that a random double-dimer configuration has pairing σ . Kenyon and Wilson showed that assuming all of the nodes are black and odd or white and even, $\widehat{\Pr}(\sigma) := \Pr(\sigma) \frac{Z^{WB}}{Z^{BW}}$ is an integer coefficient polynomial in the quantities $X_{i,j} := Z_{i,j}^{BW} / Z^{BW}$.

Note. They assume that all odd nodes of \mathcal{G} are black and all even nodes are white.

Definition. If S is a balanced subset of nodes, that is, a subset containing an equal number k of white and black nodes. Let $Z^D(S) = Z^D(\mathcal{G} \setminus S)$ be the weighted sum of dimer covers of $\mathcal{G} \setminus S$. Let $S^c = \mathbf{N} \setminus S$.

Lemma (Lemma 3.1, KW 2006). $Z^D(S)Z^D(S^c)$ is a sum of double-dimer configurations for all connection topologies π for which π connects no element of S to an element of S^c . That is,

$$Z^D(S)Z^D(S^c) = Z^{DD} \sum_{\pi} M_{S,\pi} \Pr(\pi)$$

where $M_{S,\pi}$ is 0 or 1 according to whether π connects nodes in S to S^c or not.

Let G be a graph with node set \mathbf{N} that does not necessarily have the property that odd nodes of \mathcal{G} are black and all even nodes are white. Let $M \subseteq N$ be the set of nodes that are odd and white or even and black. Let \tilde{G} be G with an extra vertex and edge with weight 1 added to each node in M . So all of the nodes in \tilde{G} are black and odd or white and even.

Additionally, assume that G has an equal number of black and white nodes and \tilde{G} has an equal number of black and white nodes. These two assumptions imply that M is a balanced subset. Note that

$$Z^D(\tilde{G} \setminus S) = Z^D(G \setminus (S \cap M^c) \cup (M \cap S^c))$$

Then Lemma 3.1 says that

Corollary (Corollary to Lemma 3.1). Let S be a balanced subset of nodes. $Z^D(G \setminus (S \cap M^c) \cup (M \cap S^c))Z^D(G \setminus (S^c \cap M^c) \cup (M \cap S))$ is a sum of double-dimer

configurations for all connection topologies π for which π connects no element of S to an element of S^c . That is,

$$Z^D(G \setminus (S \cap M^c) \cup (M \cap S^c)) Z^D(G \setminus (S^c \cap M^c) \cup (M \cap S)) = Z^{DD}(G) \sum_{\pi} M_{S,\pi} \Pr(\pi)$$

where $M_{S,\pi}$ is 0 or 1 according to whether π connects nodes in S to S^c or not.

If $T = (S \cap M^c) \cup (M \cap S^c)$, then $S = (T \cap M^c) \cup (M \cap T^c)$, so:

Corollary (Corollary to Lemma 3.1). Let T be a balanced subset of nodes. $Z^D(T)Z^D(T^c)$ is a sum of all connection topologies π for which π connects no elements of $(T \cap M^c) \cup (M \cap T^c)$ to $((T \cap M^c) \cup (M \cap T^c))^c$. That is

$$Z^D(G \setminus T) Z^D(G \setminus T^c) = Z^{DD}(G) \sum_{\pi} M_{(T \cap M^c) \cup (M \cap T^c), \pi} \Pr(\pi)$$

where $M_{(T \cap M^c) \cup (M \cap T^c), \pi}$ is 0 or 1 depending on whether π connects nodes in $(T \cap M^c) \cup (M \cap T^c)$ to $((T \cap M^c) \cup (M \cap T^c))^c$.

Assuming that \mathcal{G} has the property that the odd-numbered nodes are black and the even-numbered nodes are white, Kenyon and Wilson prove the following.

Lemma (Lemma 3.2, KW 2006). Let S be a balanced subset of $\{1, \dots, 2n\}$. Then

$$\frac{Z^D(S)Z^D(S^c)}{(Z^D)^2} = \det[(1_{i,j \in S} + 1_{i,j \notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}]_{j=2,4,\dots,2n}^{i=1,3,\dots,2n-1}$$

If we do not assume that \mathcal{G} has the property that the odd-numbered nodes are black and the even-numbered nodes are white, we have the following conjecture:

Conjecture. Let \mathcal{G} be a bipartite planar graph with set of nodes $\{1, 2, \dots, 2n\}$. Let S be a balanced subset of $\{1, \dots, 2n\}$. Then

$$\frac{Z^D(\mathcal{G} \setminus S)Z^D(\mathcal{G} \setminus S^c)}{(Z^D(\mathcal{G}))^2} = \det \left[(1_{i,j \in S} + 1_{i,j \notin S}) \times \text{sign}(i, j, S) \frac{Z^D(\mathcal{G}_{i,j})}{Z^D(\mathcal{G})} \right]_{j=w_1, w_2, \dots, w_n}^{i=b_1, b_2, \dots, b_n}$$

where b_1, b_2, \dots, b_n are the black nodes of $\{1, 2, \dots, 2n\}$ listed in ascending order, w_1, w_2, \dots, w_n are the white nodes of $\{1, 2, \dots, 2n\}$ listed in ascending order, and $\text{sign}(i, j, S) = 1$ or -1 depending on i, j , and S .

Proof. Let K be a Kasteleyn matrix of \mathcal{G} . Then the signs of the edges \mathcal{G} are so that there are an odd number of $-$ signs if the face has 0 mod 4 edges and an even number of $-$ signs if the face has 2 mod 4 edges. Let $(w_1, b_1), \dots, (w_k, b_k)$ be any noncrossing pairing of the nodes of S , where w_1, \dots, w_k are the white nodes of S and b_1, \dots, b_k are the black nodes of S (such a noncrossing pairing always exists since S is a balanced subset). Adjoin edges e_i of weight W to the outer face connecting w_i to b_i for $1 \leq i \leq k$. (What if two nodes are adjacent, and doing this creates a multigraph?). We claim that we can choose a weight for each e_i that retains the Kasteleyn sign condition.

To see this, first observe that adjoining the edge e_1 creates exactly two new faces, since w_1 and b_1 are on the outer face (i.e. the face bounding the infinite region) of \mathcal{G} . Let p and

q be the two paths from w_1 to b_1 consisting of edges along the outer face of \mathcal{G} . Observe that p and q must both have odd length, since w_1 and b_1 are different colors. There are two different cases, depending on whether the outer face of \mathcal{G} has $0 \bmod 4$ edges or $2 \bmod 4$ edges.

- (1) The outer face of \mathcal{G} has $0 \bmod 4$ edges.

When the outer face of \mathcal{G} has $0 \bmod 4$ edges, one of the paths p or q must have length $4\ell + 1$ for some ℓ and the other path must have length $4j + 3$ for some j . Assume without loss of generality that p has length $4\ell + 1$ for some ℓ . Since the outer face of \mathcal{G} has $0 \bmod 4$ edges, the product of the signs of the edges of the outer face of \mathcal{G} is -1 . This means that either the product of the signs of the edges of p is 1 and the product of the signs of the edges of q is -1 , or the product of the signs of the edges of p is -1 and the product of the signs of the edges of q is 1 . So there are two cases:

- (a) The product of the signs of the edges of p is 1 .

Since the face bounded by the path p and the edge e_1 has $2 \bmod 4$ edges, give e_1 sign 1 . Then since the product of the signs of the edges of q is -1 , the face bounded by the path q and the edge e_1 has $0 \bmod 4$ edges and an odd number of $-$ signs.

- (b) The product of the signs of the edges of p is -1 .

Since the face bounded by the path p and the edge e_1 has $2 \bmod 4$ edges, give e_1 sign -1 . Then since the product of the signs of the edges of q is 1 , the face bounded by the path q and the edge e_1 has $0 \bmod 4$ edges and an odd number of $-$ signs.

- (2) The outer face of \mathcal{G} has $2 \bmod 4$ edges. When the outer face of \mathcal{G} has $2 \bmod 4$ edges, either both p and q have length $1 \bmod 4$ or both p and q have length $3 \bmod 4$. Since the outer face of \mathcal{G} has $2 \bmod 4$ edges, the outer face has an even number of $-$ signs. This means that either both p and q have an even number of $-$ signs or both p and q have an odd number of minus signs. So there are several cases:

- (a) p and q have length $1 \bmod 4$ In this case, both the face bounded by the path p and the edge e_1 and the face bounded by the path q and the edge e_1 have $2 \bmod 4$ edges.

- (i) p and q both have an even number of $-$ signs Give e_1 sign 1 . Then both new faces have an even number of $-$ signs.

- (ii) p and q both have an odd number of $-$ signs Give e_1 sign -1 . Then both new faces have an even number of $-$ signs.

- (b) p and q have length $3 \bmod 4$ In this case, both the face bounded by the path p and the edge e_1 and the face bounded by the path q and the edge e_1 have $0 \bmod 4$ edges.

- (i) p and q both have an even number of $-$ signs Give e_1 sign -1 . Then both new faces have an odd number of $-$ signs.

- (ii) p and q both have an odd number of $-$ signs Give e_1 sign 1 . Then both new faces have an odd number of $-$ signs.

So in all cases we have chosen the sign for e_1 so that the resulting graph still satisfies the Kasteleyn sign condition. By induction, we can add choose a sign for each e_i so that the resulting graph satisfies the Kasteleyn sign condition. Let K_W be the matrix defined as follows: $(K_W)_{s,t} = \text{sign}(e_i)W$ if $(s,t) = (w_i, b_i)$ for some i , and $(K_W)_{s,t} = (K)_{s,t}$ otherwise. By construction, if $K_0 := K_W|_{W=0}$, then $K_0 = K$. Reorder the columns of K_W so that w_1, \dots, w_k are the first k rows and b_1, \dots, b_k are the first k columns. Since K is a Kasteleyn matrix of G and we chose signs of e_i to retain the Kasteleyn sign condition, $Z^D(\mathcal{G} \setminus S) = \pm[W^k] \det(K_W)$. We see that

$$\frac{Z^D(\mathcal{G} \setminus S)}{Z^D(\mathcal{G})} = \frac{[W^k] \det(K_W)}{[W^0] \det K_W}$$

Then

$$\frac{[W^k] \det(K_W)}{[W^0] \det K_W} = \pm \frac{\det K_{\setminus S}}{\det K}$$

how to determine sign here? Not sure why any of this is necessary if I can't keep track of the sign. is $Z^D(\mathcal{G} \setminus S) = \pm \det(K_{\setminus S})$? Maybe not because although both \mathcal{G} and $\mathcal{G} \setminus S$ have Kasteleyn matrices, if we let K be the Kasteleyn matrix for \mathcal{G} , its not necessarily the case that the Kasteleyn matrix for $\mathcal{G} \setminus S$ is $K_{\setminus S}$. □

Conjecture. Let \mathcal{G} be a bipartite planar graph with set of nodes $\{1, 2, \dots, 2n\}$. Assume that G has the property that the nodes alternate between black and white except for in two places, i.e. there are exactly two pairs of consecutive nodes that are both the same color: (s, t) , (u, v) . Assume without loss of generality that s and t are both black and u and v are both white. Additionally assume that there is a white vertex on the outer face of G between s and t and a black vertex on the outer face of G between u and v . Let S be a balanced subset of $\{1, \dots, 2n\}$. Then

$$\frac{Z^D(\mathcal{G} \setminus S) Z^D(\mathcal{G} \setminus S^c)}{(Z^D(\mathcal{G}))^2} = \pm \det \left[(1_{i,j \in S} + 1_{i,j \notin S}) \times \text{sign}(i, j) \frac{Z^D(\mathcal{G}_{i,j})}{Z^D(\mathcal{G})} \right]_{\substack{i=b_1, b_2, \dots, b_n \\ j=w_1, w_2, \dots, w_n}}$$

where b_1, b_2, \dots, b_n are the black nodes of $\{1, 2, \dots, 2n\}$ listed in ascending order, w_1, w_2, \dots, w_n are the white nodes of $\{1, 2, \dots, 2n\}$ listed in ascending order, and $\text{sign}(i, j) = (-1)^{(|b-w|-1)/2}$ if b and w are different party and $\text{sign}(i, j) = (-1)^{|b-w|/2}$ if b and w are the same parity.

Recall that the cross of a pairing is a set of two parts a, c and b, d of ρ such that $a < b < c < d$. Kenyon and Wilson define the parity of an odd-even pairing $\rho = ((1, w_1), (3, w_2), \dots, (2n-1, w_n))$ to be the parity of the permutation $(w_1/2)(w_2/2) \dots (w_n/n)$. Then they prove the following

Lemma (Lemma 3.4, KW 2006). For odd-even pairings ρ ,

$$(-1)^{\text{parity of } \rho} \prod_{(i,j) \in \rho} (-1)^{(|i-j|-1)/2} = (-1)^{\# \text{ crosses of } \rho}$$

Any pairing on $2n$ elements ρ can be written $\rho = ((i_1, j_1), (i_2, j_2), \dots, (i_n, j_n))$ so that $i_k < j_k$ for all k and $i_1 < i_2 < \dots < i_n$. We define $\text{sgn}(\rho)$ to be the parity of the permutation $j_1 j_2 \dots j_n$, when j_1, \dots, j_n are relabeled with the labels $1, \dots, n$ so that $\min_{1 \leq k \leq n} \{j_k\}$ is relabeled $1, \dots, \max_{1 \leq k \leq n} \{j_k\}$ is relabeled n .

Conjecture. For any pairing ρ ,

$$\text{sgn}(\rho) \prod_{(i,j) \in \rho} f(i,j) = (-1)^{\# \text{ crosses of } \rho}$$

where $f(i,j)$ is defined as follows:

$$f(i,j) = \begin{cases} (-1)^{(|i-j|-1)/2} & \text{if } i, j \text{ are different parity} \\ (-1)^{|i-j|/2} & \text{if } i, j \text{ are both odd} \\ (-1)^{(|i-j|-2)/2} & \text{if } i, j \text{ are both even} \end{cases}$$

Note. This conjecture has been checked for all pairings on 4, 6, 8, 10, 12, and 14 elements.

Observe that if σ is an odd-even pairing on $2n$ elements, then σ can be viewed as a map from the odd integers $1, \dots, 2n-1$ to the even integers. A pairing τ can be viewed as a map from the even integers to the integers. If we relabel $\tau(\sigma(1))\tau(\sigma(3)) \dots \tau(\sigma(2n-1))$ so that the smallest number is labeled with 1, the second smallest with 2, and so on, then we have a permutation on n integers that we can find the sign of.

Conjecture. When a graph G with six nodes is colored as follows: Nodes 1, 4, and 5 are black and nodes 2, 3, and 6 are white, then

- $\frac{1|4|5}{2|6|3} \equiv \frac{1|3|5}{2|4|6} - \frac{1|3|5}{2|6|4}$
- $13|24|56 \equiv 12|34|56 - 14|32|56$
- $13|25|46 \equiv 12|34|56 - 12|36|54 - 14|32|56 + 16|32|54$
- $16|24|35 \equiv -16|32|54 + 16|34|52$

Note. Idea for proof of first bullet: for any odd-even pairing ρ , $\langle \rho, LHS \rangle_2 = 2^2$ or 2 , depending on whether 1 and 2 are paired in the odd-even pairing or not. If 1 and 2 are paired in the odd-even pairing, $\langle \rho, RHS \rangle_2 = 2^3 - 2^2 = 2^2$ or $\langle \rho, RHS \rangle_2 = -2^2$. If 1 and 2 are not paired in the odd-even pairing, $\langle \rho, RHS \rangle_2 = 2$ or -2 . Need to figure out signs.