Example 2.5.1. Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Definition. The notation $\lim_{x \to a} f(x) = \infty$ means the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (on either side of a) but not equal to a. Similarly, $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made as large negative as we like for all values of x sufficiently close to a, but not equal to a. This does not mean that the limit exists. It just expresses the particular way in which the limit does not exist.

Example 2.5.2. Let

$$g(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 2 \\ \frac{1}{x-2} & \text{if } x > 2. \end{cases}$$

Find $\lim_{x \to 2^-} g(x)$, $\lim_{x \to 2^+} g(x)$, and $\lim_{x \to 2} g(x)$, if they exist.

Definition. A function $f(x)$ has a vertical asymptote at $x = a$ if at least one of the following conditions is true: $\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = \infty$, or $\lim_{x \to a^+} f(x) = -\infty$.

Definition. The limit of $f(x)$ as x approaches infinity is the value L (if one exists) that $f(x)$ approaches when x is taken to be as large as desired (in the positive direction), and we write $\lim_{x \to \infty} f(x) = L$. Similarly, $\lim_{x \to -\infty} f(x)$ is the value (if one exists) that $f(x)$ approaches when x is taken to be as large as possible in the negative direction.

Example 2.5.3. (a) $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$, and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

(b) $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$

(c) $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to\infty} \ln(x) = \infty$

Note. If n is a positive integer and k is a constant then $\lim_{x\to\infty} \frac{k}{x^n} = 0$ and $\lim_{x\to-\infty} \frac{k}{x^n} = 0$

Example 2.5.4. Compute $\lim_{x\to\infty} f(x)$ for $f(x) = x^{10} - 2x^3 + 5x - 1$.

Note. If $f(x)$ is a polynomial, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are uniquely determined by the leading term of $f(x)$. (Recall that the leading term of a polynomial is the term containing the highest power on x). In other words, if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} a_n x^n \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} a_n x^n$$

Example 2.5.5. Compute $\displaystyle\lim_{x\to\infty} \frac{x^2+3}{2x^2+2x+4}$

Definition. A function $f(x)$ has a horizontal asymptote at $y = c$ if at least one of the following is true: $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$

Example 2.5.6. Let $f(x) = \dfrac{x-1}{x-x^2}$.

(a) Find the domain of $f(x)$. Write your answer in interval notation.

(b) Is $f(x)$ continuous at $x = 1$? Explain.

(c) Find a function $g(x)$ such that $f(x) = g(x)$ for all $x \neq 1$, and g is continuous at $x = 1$.

(d) Sketch a graph of $f(x)$. (Hint: First sketch a graph of $g(x)$, and then think about how the graph of f should differ from the graph of g).

(e) Find all vertical and horizontal asymptotes of $f(x)$. Use limits to justify your answers.

Example 2.5.7. Find all horizontal and vertical asymptotes of $f(t) = \dfrac{3}{e^t - 2}$.