Double-dimer condensation and the dP_3 quiver

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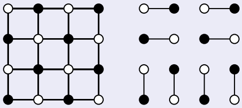
Outline

1. The Dimer Model and Kuo Condensation
2. Main Result: Double-Dimer Condensation
3. Ideas of Proof
4. Application: the dP_3 quiver and the associated cluster algebra

The dimer model

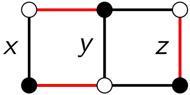
- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$



$$w(M) = xyz$$

- Let $Z^D(G) = \sum_M w(M)$, called the *partition function*.

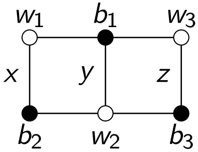
$$Z^D(G) = xyz + x + z$$

The dimer model

Theorem (Kas67)

If G is a bipartite planar graph, there is a matrix K with the property that $Z^D(G) = \det(K)$.

K is a bipartite adjacency matrix, with signs.



$$K = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 1 & -y & 1 \\ x & 1 & 0 \\ 0 & 1 & z \end{pmatrix} \end{array}$$

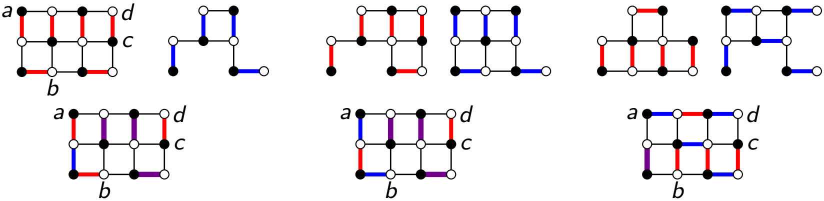
$$\det(K) = xyz + x + z$$

Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\}) = Z^D(G-\{a,b\})Z^D(G-\{c,d\}) + Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



Kuo Condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices $a, b, c,$ and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

Examples of non-bijective proofs:

- Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of Matchings*
- Speyer, *Variations on a theme of Kasteleyn, with Application to the TNN Grassmannian*

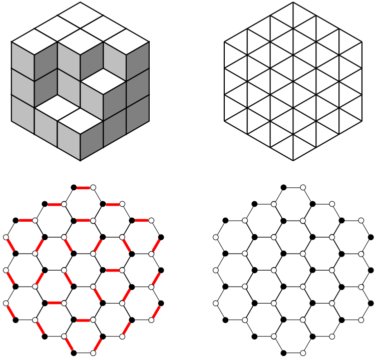
Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

M_i^j is the matrix M with the ith row and the jth column removed.

Applications of Kuo's work

- Tiling enumeration
  New proof of MacMahon's product formula for the generating function for plane partitions that are subsets of an $r \times s \times t$ box.
- Cluster algebras
  (LM17, LM20) Combinatorial interpretation of toric cluster variables for the dP_3 quiver



Main result. An analogue of Kuo's theorem for double-dimer configs.

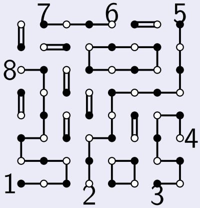
Application #1: Give combinatorial interpretations of toric cluster variables for the dP_3 quiver in the case where the single dimer model was not sufficient (joint with Tri Lai and Gregg Musiker).

Application #2: A problem in Donaldson-Thomas theory and Pandharipande-Thomas theory (joint with Ben Young and Gautam Webb).

Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G.

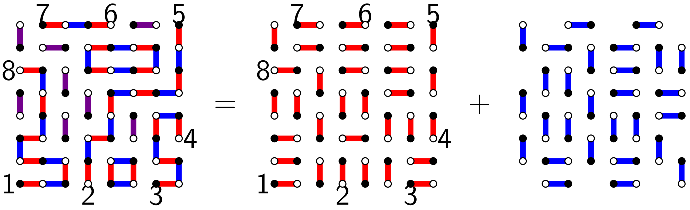
Definition (Double-dimer configuration on $(G, \mathbf{N})$)



Configuration of

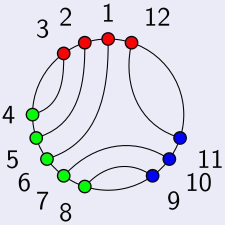
- ℓ disjoint loops
- Doubled edges
- Paths connecting nodes in pairs

Its weight is the product of its edge weights $\times\ 2^{\ell}$

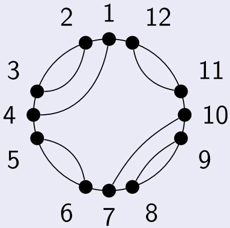


Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite



Not tripartite

We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

Dividing nodes into three sets R, G, and B defines a tripartite pairing.

Main Result

$Z_\sigma^{DD}(G,\mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

Theorem (J.)

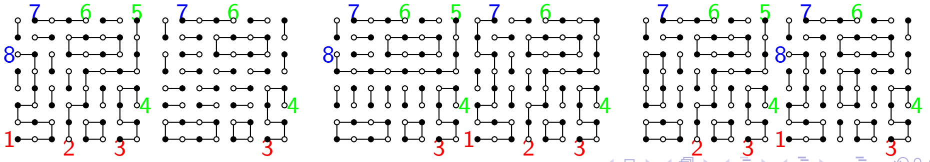
Divide $\mathbf{N}$ into sets R, G, and B and let σ be the corr. tripartite pairing. Let $x, y, w, v \in \mathbf{N}$ such that $x < w \in V_1$ and $y < v \in V_2$. If $\{x, y, w, v\}$ contains at least one node of each RGB color and x, y, w, v appear in cyclic order then

$$Z_\sigma^{DD}(G,\mathbf{N})Z_{\sigma_{xywv}}^{DD}(G,\mathbf{N}-\{x,y,w,v\}) =$$

$$Z_{\sigma_{xy}}^{DD}(G,\mathbf{N}-\{x,y\})Z_{\sigma_{wv}}^{DD}(G,\mathbf{N}-\{w,v\}) + Z_{\sigma_{xv}}^{DD}(G,\mathbf{N}-\{x,v\})Z_{\sigma_{wy}}^{DD}(G,\mathbf{N}-\{w,y\})$$

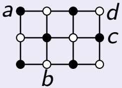
Example.

$$Z_\sigma^{DD}(\mathbf{N})Z_{\sigma_{1258}}^{DD}(\mathbf{N}-1,2,5,8) = Z_{\sigma_{12}}^{DD}(\mathbf{N}-1,2)Z_{\sigma_{58}}^{DD}(\mathbf{N}-5,8) + Z_{\sigma_{18}}^{DD}(\mathbf{N}-1,8)Z_{\sigma_{25}}^{DD}(\mathbf{N}-2,5)$$



Corollaries

Theorem (Kuo04, Theorem 5.1)



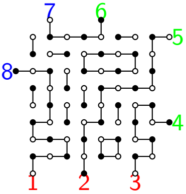
Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\}) = Z^D(G-\{a,b\})Z^D(G-\{c,d\}) + Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

Theorem (J.)

Let $x, y, w, v \in \mathbf{N}$ such that $x < w \in V_1$ and $y < v \in V_2$. If $\{x, y, w, v\}$ contains at least one node of each RGB color and x, y, w, v appear in cyclic order then

$$Z^{DD}_{\sigma}(G, \mathbf{N})Z^{DD}_{\sigma_{xywv}}(G - \{x, y, w, v\}, \mathbf{N} - \{x, y, w, v\}) =$$
$$Z^{DD}_{\sigma_{xy}}(G - \{x, y\}, \mathbf{N} - \{x, y\})Z^{DD}_{\sigma_{wv}}(G - \{w, v\}, \mathbf{N} - \{w, v\}) +$$
$$Z^{DD}_{\sigma_{xv}}(G - \{x, v\}, \mathbf{N} - \{x, v\})Z^{DD}_{\sigma_{wy}}(G - \{w, y\}, \mathbf{N} - \{w, y\})$$



Corollaries

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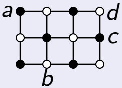

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\}) = Z^D(G-\{a,b\})Z^D(G-\{c,d\}) + Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

Theorem (J.)

Let $x, y, w, v \in \mathbf{N}$ such that $x < w \in V_1$ and $y < v \in V_2$. If $\{x, y, w, v\}$ contains at least one node of each RGB color and x, y, w, v appear in cyclic order then

$$Z_\sigma^{DD}(G,\mathbf{N})Z_{\sigma_{xywv}}^{DD}(G-\{x,y,w,v\},\mathbf{N}-\{x,y,w,v\}) =$$

$$Z_{\sigma_{xy}}^{DD}(G-\{x,y\},\mathbf{N}-\{x,y\})Z_{\sigma_{wv}}^{DD}(G-\{w,v\},\mathbf{N}-\{w,v\}) +$$

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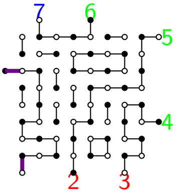
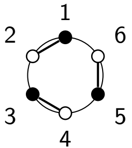


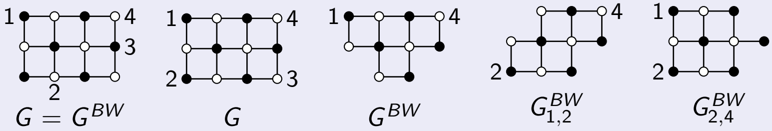


$$\widehat{\Pr}\begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix} = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$

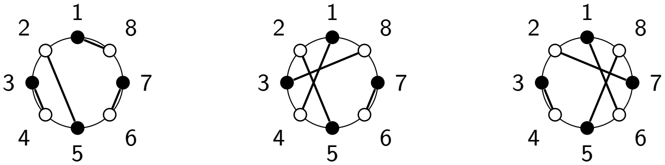
$$\widehat{\Pr}\begin{pmatrix}1 & 3 & 5 & 7\\ 8 & 4 & 2 & 6\end{pmatrix} = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}$$
$$- X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

Definition (KW11a)

$X_{i,j} = \dfrac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$, where $G^{BW} \subseteq G$ only contains nodes that are black and odd or white and even.



- $X_{i,j} = 0$ if i and j have the same parity



$$\widehat{\Pr}\left(\begin{array}{c|c|c|c} 1 & 3 & 5 & 7 \\ 8 & 4 & 2 & 6 \end{array}\right) = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}$$
$$- X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

- Each term in $\widehat{\Pr}(\sigma)$ is of the form

$$X_\tau := \prod_{(i,j)\in\tau} X_{i,j}, \text{ where } \tau \text{ is an odd-even pairing.}$$

- Kenyon and Wilson made a simplifying assumption that all nodes are black and odd or white and even.

Theorem (KW11a, Theorem 1.3)

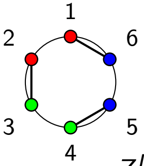
$\widehat{Pr}(\sigma)$ is an integer-coeff homogeneous polynomial in the quantities $X_{i,j}$

Background: Determinant formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]^{i=1,3,\dots,2n-1}_{j=\sigma(1),\sigma(3),\dots,\sigma(2n-1)}.$$



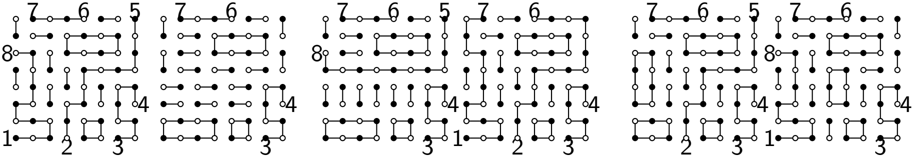
$$\widehat{\Pr}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right) = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

Since $\widehat{\Pr}(\sigma) := \dfrac{Z^{DD}_{\sigma}(G,\mathbf{N})}{(Z^D(G^{BW}))^2}$, the idea of the proof is to combine K-W's matrix with the Desnanot-Jacobi identity:

$$\det(M)\det(M^{i,j}_{i,j}) = \det(M^i_i)\det(M^j_j) - \det(M^j_i)\det(M^i_j)$$

Example

$$Z^{DD}_{\sigma}(\mathbf{N})Z^{DD}_{\sigma_{1258}}(\mathbf{N}-1,2,5,8) = Z^{DD}_{\sigma_{12}}(\mathbf{N}-1,2)Z^{DD}_{\sigma_{58}}(\mathbf{N}-5,8) + Z^{DD}_{\sigma_{18}}(\mathbf{N}-1,8)Z^{DD}_{\sigma_{25}}(\mathbf{N}-2,5)$$

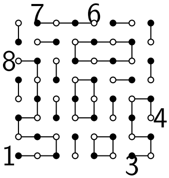


$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}$$

$$\det(M)\det(M^{1,3}_{1,3}) = \det(M^1_1)\det(M^3_3) - \det(M^3_1)\det(M^1_3)$$

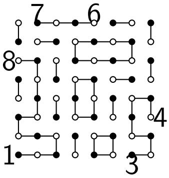
$$\det(M) = \frac{Z^{DD}_{\sigma}(\mathbf{N})}{(Z^{D}(G^{BW}))^2} \qquad ✔$$

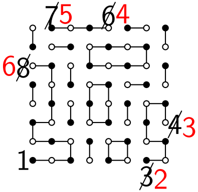
$$\det(M_3^3) \stackrel{?}{=} \frac{Z_{\sigma_2}^{DD}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



- The nodes are not numbered consecutively.

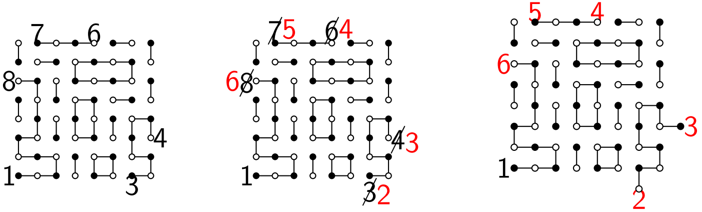
$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$





- Relabel the nodes.
- Node 2 is black and node 3 is white.

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$

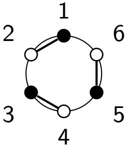


- Add edges of weight 1 to nodes 2 and 3.
- Since $X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$, the K-W matrix for this new graph will have different entries!

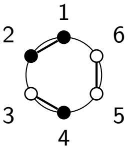
Observation. We need to lift the assumption that the nodes of the graph are black and odd or white and even.

Our Approach

- When the nodes are black and odd or white and even, $G = G^{BW}$, so
  $$X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})} = \frac{Z^D(G_{i,j})}{Z^D(G)}.$$
- Let $Y_{i,j} = \dfrac{Z^D(G_{i,j})}{Z^D(G)}$ and let $\widetilde{\Pr}(\sigma) = \dfrac{Z^{DD}_\sigma(G)}{(Z^D(G))^2}$
- We establish analogues of K-W without their node coloring constraint.

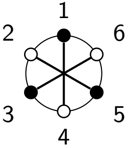


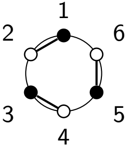
$$\widehat{\Pr}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right) = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$



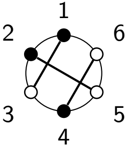
$$\widetilde{\Pr}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right) = Y_{1,3}Y_{2,5}Y_{4,6} + Y_{1,5}Y_{2,6}Y_{4,3}$$

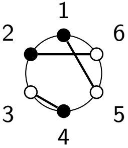
- $X_{i,j} = 0$ if i and j are the same parity
- $Y_{i,j} = 0$ if i and j are the same color





$$\widehat{\Pr}\left(\begin{array}{c|c|c}1&3&5\\2&4&6\end{array}\right) = X_{1,4}X_{2,5}X_{3,6} \;+\; X_{1,2}X_{3,4}X_{5,6}$$





$$\widetilde{\Pr}\left(\begin{array}{c|c|c}1&3&5\\2&4&6\end{array}\right) = Y_{1,3}Y_{2,5}Y_{4,6} \;+\; Y_{1,5}Y_{2,6}Y_{4,3}$$

- Each term in $\widehat{\Pr}(\sigma)$ is of the form
$$X_\tau := \prod_{(i,j)\in\tau} X_{i,j}, \text{ where } \tau \text{ is an odd-even pairing.}$$
- Each term in $\widetilde{\Pr}(\sigma)$ is of the form
$$Y_\rho := \prod_{(i,j)\in\rho} Y_{i,j}, \text{ where } \rho \text{ is an black-white pairing.}$$

A disaster of signs!

Lemma (KW11a, Lemma 3.4)

For odd-even pairings ρ,

$$\mathit{sign}_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\ \mathit{crosses\ of}\ \rho}.$$

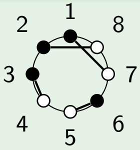
We need a version of this for black-white pairings.

Example ($\mathrm{sign}_{OE}(\rho)$)

If $\rho = \left(\begin{smallmatrix}1 & 3 & 5\\ 6 & 2 & 4\end{smallmatrix}\right)$, then $\mathrm{sign}_{OE}(\rho)$ is the parity of $\left(\frac{6}{2}\ \frac{2}{2}\ \frac{4}{2}\right) = (3\ 1\ 2)$

When ρ is black-white, we define $\mathrm{sign}(\rho)$ similarly.

Example



If $\rho = \left(\begin{smallmatrix}1 & 2 & 3 & 6\\ 7 & 8 & 4 & 5\end{smallmatrix}\right)$, $\mathrm{sign}_{BW}(\rho)$ is the sign of $(3\ 4\ 1\ 2)$.

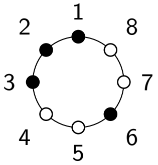
Lemma (KW11a, Lemma 3.4)

For odd-even pairings ρ,

$$sign_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\text{ crosses of }\rho}.$$

Definition

If (i,j) is a pair in a black-white pairing, let $\text{sign}(i,j) = (-1)^{(|i-j|+a_{i,j}-1)/2}$



$a_{7,3} = 1$, so $\text{sign}(7,3) = (-1)^{(|7-3|+1-1)/2} = 1$

$a_{8,3} = 2$, so $\text{sign}(8,3) = (-1)^{(|8-3|+2-1)/2} = -1$

Lemma (J.)

If ρ is a black-white pairing,

$$sign_c(\mathbf{N}) sign_{BW}(\rho) \prod_{(i,j)\in\rho} sign(i,j) = (-1)^{\#\text{ crosses of }\rho}.$$

Determinant Formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\begin{aligned}\widehat{Pr}(\sigma) &= \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]^{i=1,3,\ldots,2n-1}_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)} \\ &= sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored diff}} X_{i,j}]^{i=1,3,\ldots,2n-1}_{j=2,4,\ldots,2n}\end{aligned}$$

Theorem (J.)

When σ is a tripartite pairing,

$$\widetilde{Pr}(\sigma) = sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored differently}} Y_{i,j}]^{i=b_1,b_2,\ldots,b_n}_{j=w_1,w_2,\ldots,w_n}.$$

More general result

Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *Then*

$$\begin{aligned} & sign_{OE}(\sigma) sign_{OE}(\sigma'_{xywv}) Z^{DD}_{\sigma}(G, \mathbf{N}) Z^{DD}_{\sigma_{xywv}}(G, \mathbf{N} - \{x, y, w, v\}) \\ = \; & sign_{OE}(\sigma'_{xy}) sign_{OE}(\sigma'_{wv}) Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\}) \\ & - sign_{OE}(\sigma'_{xv}) sign_{OE}(\sigma'_{wy}) Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\}) \end{aligned}$$

Corollary

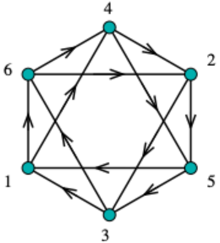
Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

$Z^{DD}_{\sigma}(G, \mathbf{N}) Z^{DD}_{\sigma_{xywv}}(G, \mathbf{N} - \{x, y, w, v\}) =$

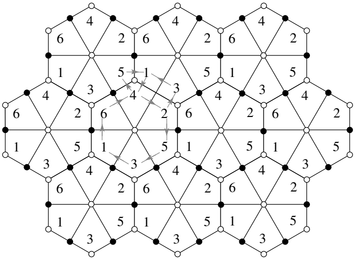
$Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\}) + Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\})$

Application: the dP_3 Quiver

Object of study. The dP_3 *quiver*[1] and its associated *cluster algebra*.



Goal. Understand combinatorial interpretations for *toric cluster variables* obtained from sequences of *mutations*.



Main result. [LMNT, LM17, LM20] In many cases, the Laurent expansion of the toric cluster variables is equal to the partition function for a certain subgraph of the dP_3 *lattice* (with appropriate edge-weights).

[1] The quiver Q associated with the Calabi-Yau threefold complex cone over the third del Pezzo surface of degree 6 ($\mathbb{CP}^2$ blown up at three points).

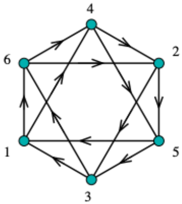
Images shown are Figures 1 and 2 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

Quiver, quiver mutations, and cluster variables

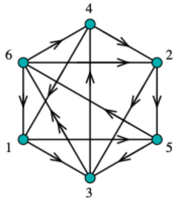
A *quiver* Q is a directed finite graph.

Definition (Mutation at a vertex i)

- For every 2-path $j \to i \to k$, add $j \to k$
- Reverse all arrows incident to i
- Delete 2-cycles



Q



$\mu_1(Q)$

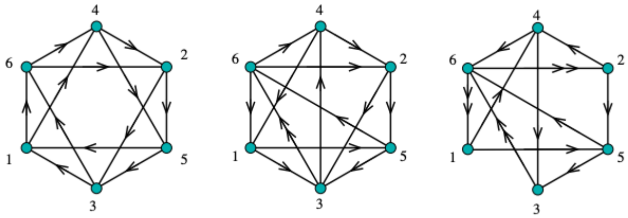
- Define a cluster algebra from a quiver Q by associating a cluster variable x_i to every vertex labeled i.
- When we mutate at vertex i we replace x_i with x_i', where

$$x_i' = \frac{\prod_{i \to j \text{ in } Q} x_j^{a_{i \to j}} + \prod_{j \to i \text{ in } Q} x_j^{b_{j \to i}}}{x_i}$$

- When we mutate at vertex 1 we replace x_1 with $x_1' = \dfrac{x_4 x_6 + x_3 x_5}{x_1}$.

  Now we have the cluster: $\left\{\frac{x_4x_6+x_3x_5}{x_1}, x_2, x_3, \ldots, x_6\right\}$

Quiver, quiver mutations, and cluster variables



Q $\mu_1(Q)$ $\mu_1\mu_4(Q)$

Mutate at 4: replace x_4 with

$$x_4' = \frac{x_3x_6 + x_2x_1'}{x_4} = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4}$$

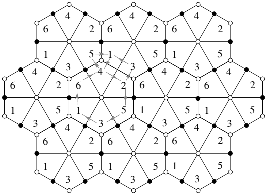
Theorem (FZ02)

Every cluster variable is a Laurent polynomial in $x_1, \ldots, x_n$.

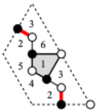
A *toric mutation* is a mutation at a vertex with both in-degree and out-degree 2.

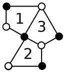
Image shown is Figure 2 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

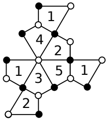
Combinatorial formula for some toric cluster variables



Example. (Z12) Toric cluster variables from the periodic mutation $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ agree with partition functions for subgraphs of the dP_3 lattice with appropriate edge weights (the edge bordering faces i and j has weight $\frac{1}{x_i x_j}$).

 $\frac{x_4x_6+x_3x_5}{x_1}$  $= \left(\frac{1}{x_1^2x_3x_5} + \frac{1}{x_1^2x_4x_6}\right) x_1x_3x_4x_5x_6 = Z^D(G)m(G)$

x_3 in $\mu_3\mu_2\mu_1(Q)$: $\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$ 

x_5 in $\mu_5\mu_4\mu_3\mu_2\mu_1(Q)$: $\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$ 

These subgraphs are Aztec Dragons (see for example CY10).

$\mathbb{Z}^3$ parameterization for toric cluster variables and an algebraic formula

Lai and Musiker (LM17) showed that the set of toric cluster variables is parameterized by $\mathbb{Z}^3$.

Let $z_{i,j,k}$ denote the toric cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$.

Theorem (LM17)

Let $A = \frac{x_3x_5+x_4x_6}{x_1x_2}$, $B = \frac{x_1x_6+x_2x_5}{x_3x_4}$, $C = \frac{x_1x_3+x_2x_4}{x_5x_6}$, $D = \frac{x_1x_3x_6+x_2x_3x_5+x_2x_4x_6}{x_1x_4x_5}$, $E = \frac{x_2x_4x_5+x_1x_3x_5+x_1x_4x_6}{x_2x_3x_6}$. *Then*

$$z_{i,j,k} = x_r A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

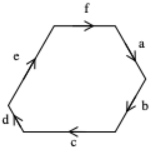
x_r is an initial cluster variable with r depending on $(i-j)$ mod 3 and k mod 2.

Combinatorial interpretation of $z_{i,j,k}$

Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

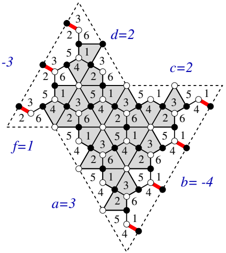
$$(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$$

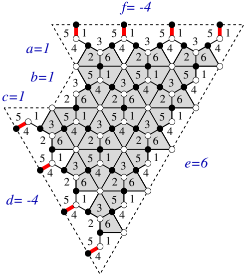
Given a six-tuple $(a,b,c,d,e,f) \in \mathbb{Z}^6$, superimpose the contour $C(a,b,c,d,e,f)$ on the dP_3 lattice.
Magnitude determines length and sign determines direction.

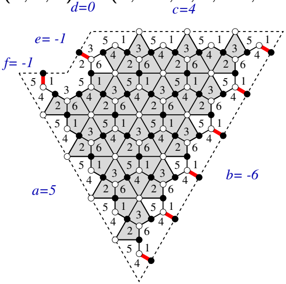


Examples:

$(1,2,1) \to (3,-4,2,2,-3,1)$ $(-2,-2,3) \to (1,1,1,-4,6,-4)$ $(1,2,3) \to (5,-6,4,0,-1,-1)$

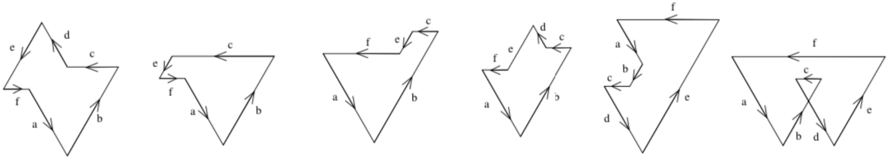






Combinatorial interpretation of $z_{i,j,k}$

Some possible shapes of the contours:



$(+,-,+,+,-,+)$, $(+,-,+,0,-,+)$, $(+,-,+,0,-,-)$, $(+,-,+,+,-,-)$,$(+,+,+,-,+,-)$, $(+,-,+,-,+,-)$

Theorem (LM17)

Let G be the subgraph cut out by the contour $(a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$. As long as $C(a,b,c,d,e,f)$ has no self-intersections,

$$z_{i,j,k} = Z^D(G)m(G)$$

What about when $C(a,b,c,d,e,f)$ is self-intersecting?

Image shown is Figure 12 from T. Lai and G. Musiker, *Beyond Aztec Castles: Toric cascades in the dP_3 Quiver*

Cross-section when k is positive

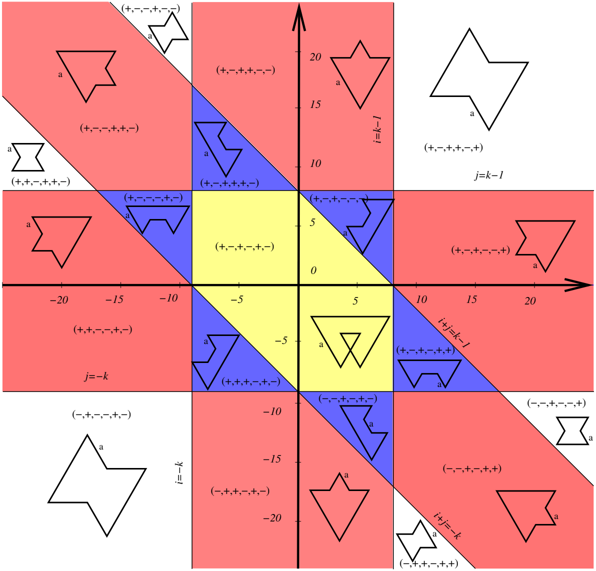
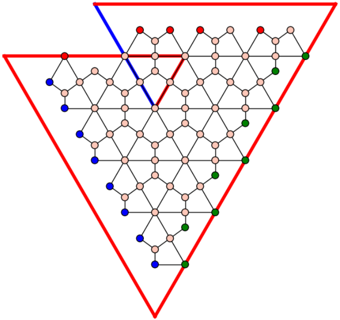


Figure 20 from T. Lai and G. Musiker, *Beyond Aztec Castles: Toric cascades in the dP_3 Quiver*

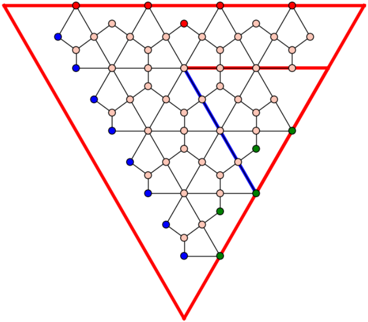
Combinatorial interpretation for self-intersecting contours

Theorem (J.-Lai-Musiker 2020+)

For the dP_3 quiver, we complete the assignment of combinatorial interpretations to toric cluster variables. In particular, for (i, j, k) corresponding to a self-intersecting contour we express $z_{i,j,k}$ as a partition function for a tripartite double-dimer configuration.



$(-1, -2, 4) \rightarrow (2, -1, 3, -5, 6, -4)$



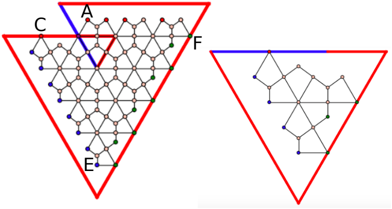
$(1, -2, 4) \rightarrow (2, -3, 5, -5, 4, -2)$

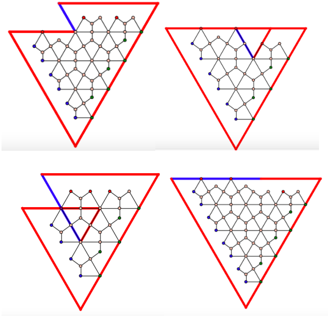
Sketch of proof for self-intersecting contours

Our proof uses a bijection between dimers and double dimers, the dimer interpretations of LM17 as a base case, and then proceeds by induction via double-dimer condensation.

$$z_{-1,-2,4} \cdot z_{0,-2,2} \quad = \quad z_{-1,2,3} \cdot z_{0,-2,3} + z_{-1,-1,3} \cdot z_{0,-3,3}$$

$$\begin{aligned} Z_{\sigma}^{DD}(G,\mathbf{N})Z_{\sigma_5}^{DD}(G-ACEF,\mathbf{N}-ACEF) &= Z_{\sigma_1}^{DD}(G-AC,\mathbf{N}-AC)Z_{\sigma_2}^{DD}(G-EF,\mathbf{N}-EF) \\ &\quad + Z_{\sigma_3}^{DD}(G-CE,\mathbf{N}-CE)Z_{\sigma_4}^{DD}(G-AF,\mathbf{N}-AF) \end{aligned}$$





Thank you for listening, and happy Friday the 13th!

...and now, on to Minecraft!

Image credit: `https://www.grabcraft.com/minecraft/jason-voorhees-friday-the-13th/movie-characters-185`

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