Optimal Control of Make-to-Stock Systems

Xuefeng Gao
Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong S.A.R., China, xfgao@se.cuhk.edu.hk.

Junfei Huang
Department of Decision Sciences and Managerial Economics, CUHK Business School, The Chinese University of Hong Kong, Shatin, Hong Kong S.A.R., China, junfeih@cuhk.edu.hk.

We consider multi-class make-to-stock production/inventory systems in which the manager makes three decisions including pricing, outsourcing and scheduling to maximize the long-run average profit. For a sequence of systems in the heavy-traffic regime, with linear or strictly convex holding/waiting cost functions, we propose a sequence of policies and establish its asymptotic optimality. Our proof combines the lower bound approach and a thorough steady-state analysis of the systems. We also establish general results on the existence and tightness of the stationary distributions of the state processes under a more general family of polices.

Key words: Make-to-stock, heavy traffic, steady-state analysis, asymptotically optimal control

---

1. Introduction We consider the dynamic control of a make-to-stock production/inventory system, in which multiple types of products are produced and stored in inventory to satisfy customer demand. Customers are price-sensitive in the sense that the demand arrival processes depend on the prices of the products. A customer's demand is satisfied if the desired product is available in the inventory; otherwise, the customer waits in a queue or the manager may decide to outsource the new order if too many orders are waiting. The cost of outsourcing a customer order is fixed. Products held in the inventory incur inventory holding costs, and customers waiting for products incur waiting costs. The system manager's objective is to maximize the long-run average profit by dynamically making three types of decisions: pricing, outsourcing (whether to outsource a new order or not) and scheduling (which product to prioritize in the production process).

The above system, with *linear* holding/waiting cost functions, was recently proposed and analyzed in Ata and Barjesteh (2019), which generalized the classical model in Wein (1992) by incorporating pricing and outsourcing decisions. The problem of finding the optimal control policy for the system is analytically intractable in general. Hence, Ata and Barjesteh (2019) considered the heavy-traffic regime (see e.g. Harrison (1988)) where both the system capacity and demand are large with the server utilization close to one, and they conducted an approximate analysis. They formulated and analyzed the corresponding Brownian control problem (BCP), i.e., the limiting control problem associated with Brownian motions. Then they interpreted the solutions of the BCP in the context of the original control problem for the make-to-stock system. Ata and Barjesteh (2019) illustrated the effectiveness of their policy via simulation studies, but they did not analyze their proposed policy directly for the original system.

In this paper, we consider the above system with both *linear* and *strictly convex* holding/waiting cost functions. We propose a dynamic control policy and *theoretically* prove its asymptotic optimality in the heavy-traffic regime. We also establish some general results on the existence and tightness of stationary distributions for multi-class make-to-stock systems under a more general family of policies. These steady-state results do not depend on the holding/waiting cost functions, hence are of independent interests and may be used to analyze multi-class make-to-stock systems under other scheduling policies. Despite being closely related to Ata and Barjesteh (2019), our paper is significantly different from theirs in several aspects.

First, our focus is to propose a policy and prove its asymptotic optimality for multi-class make-to-stock systems modeled by multi-dimensional continuous-time Markov chains with discrete state spaces. In contrast, their focus is to solve the Bellman equation for the BCP, and use the continuous state solution to inform a policy. They do not prove the asymptotic optimality of their policy. As such, we must employ new methods that are totally different from theirs. Specifically, we develop a new method by combining the lower bound approach with steady-state State Space Collapse (SSC) to prove the asymptotic optimality of our proposed policy for the multi-class make-to-stock system. To the best of our knowledge, our paper is the first one to use such a method to analyze a production/inventory system. The lower bound approach (also known as the verification theorem) is a standard approach to analyze BCPs with continuous states, see e.g. Harrison and Taksar (1983), Dai and Yao (2013). We adapt it to analyze our make-to-stock systems with discrete states. Depending on whether products are held in inventory or customer orders are waiting, the system states can be either positive or negative, and this fact brings several difficulties to the steady-state analysis of the underlying multi-dimensional Markov chains. To prove the existence of stationary distributions, we use the Foster-Lyapunov criteria (see e.g., Meyn and Tweedie (1993)), and to establish the tightness of stationary distributions, we adopt the approach in Gamarnik and Zeevi (2006) to obtain tail abounds for stationary distributions. Despite using standard frameworks, due to the fact that the system states can be either positive or negative, the difficulties come in the form of carefully designing appropriate Lyapunov functions, which require us to explore the details of the proposed policies. In order to prove the steady-state SSC, we apply the framework of Bramson (1998) to establish uniform attractions of the hydrodynamic limits, and the difficulty mainly lies in a detailed analysis of different scenarios for the systems under the proposed policy.

Second, our proposed policy has some important differences compared to Ata and Barjesteh (2019). When the holding/waiting cost functions are linear, our proposed policy is almost identical to their policy. However, there is an important difference in the outsourcing decision, where we will not outsource an order if the inventory level of the only class to be outsourced exceeds the safety stock. Our outsourcing decision is intuitive but cannot be derived from the corresponding BCP, and this difference appears to be essential to the construction of Lyapunov functions in the steady-state analysis. When the holding/waiting cost functions are strictly convex, the pricing and outsourcing policies share similar structures as in the linear cost setting. However, the scheduling policy is different and it extends the generalized $c\mu$ rule (van Mieghem (1995)) to make-to-stock systems with both positive and negative states.

Third, we consider both linear and strictly convex holding/waiting cost functions, while Ata and Barjesteh (2019) only consider linear holding/waiting cost functions. There are many applications that motivate convex holding/waiting costs, see e.g. Dai and Yao (2013), Mandelbaum and Stolyar (2004), van Mieghem (1995). Mathematically, the generalization from linear to strictly convex cost functions for our problem is non-trivial, as additional technical difficulties arise from the analysis of certain free boundary problems. In Ata and Barjesteh (2019), the authors derive the Bellman equation for the BCP, which consists of an ODE with several free boundary conditions. They solve this free-boundary ODE explicitly via solving a Riccati equation, relying on the linear cost structure. In contrast, we need to consider more general free-boundary ODEs to incorporate both linear and strictly convex holding/waiting cost functions. The solution to the new ODEs plays an important role in determining the parameters of our proposed policy and establishing its asymptotic optimality. We contribute to the literature by proving the existence of a unique smooth solution to this new class of free-boundary ODEs.

Our contributions. To summarize, although the contribution of modelling is credited to Ata and Barjesteh (2019) and our proposed policy is inspired by theirs, we still make the following significant contributions:

1. We propose a policy for a multi-class make-to-stock system with linear or strictly convex holding/waiting cost functions, and establish the asymptotic optimality of the policy.

2. We establish general results on the existence and tightness of the stationary distributions of the inventory processes. These results do not rely on the structures of holding/waiting cost functions, and hence may be applicable to analyze other policies.

3. Our proof illustrates how to combine the lower bound approach with steady-state SSC to analyze discrete-state production/inventory systems with both positive and negative states.

1.1. Literature Review We survey relevant studies and explain the difference between our work and the existing literature. For a comprehensive literature review on make-to-stock systems, see Ata and Barjesteh (2019).

The earlier work on the scheduling of multi-class make-to-stock systems includes Zheng and Zipkin (1990). It is generally difficult to derive exact solutions for the optimal dynamic scheduling policy for multi-class make-to-stock systems, hence Wein (1992) consider the heavy-traffic regime and solve a related BCP to propose a scheduling policy for the systems. These papers do not consider pricing and outsourcing decisions as in our work. Dynamic pricing has been studied in Xu and Chao (2009) for a make-to-stock production system selling a single-type product. They obtain optimal pricing and production control policy for the system.

In a broader context, our work is related to asymptotic analysis of production/inventory systems. See e.g. Plambeck and Ward (2006), Reiman and Wang (2015) and the recent survey paper Goldberg et al. (2021). Reiman and Wang (2015) study multi-product assemble-to-order (ATO) systems and propose a policy that is asymptotically optimal when the lead time grows. Our paper considers the heavy-traffic regime where demand and system capacity both grow and is more related to Plambeck and Ward (2006). Plambeck and Ward (2006) study the optimal control of a high-volume ATO system with the objective of maximizing expected infinite horizon discounted profit. They prove asymptotic optimality of their proposed policy in the heavy-traffic regime. Our work differs from them in that we consider different systems and we focus on long-run average profit as the objective. This in turn leads to a different method of analysis where steady state analysis plays an important role in our work but not in theirs.

The outsourcing decision and scheduling decision are related to the admission control and dynamic scheduling in queueing systems. Concerning admission controls, Plambeck et al. (2001) show that for a multi-class single-server queueing system, to asymptotically achieve a desired bound on the throughput, it is enough to reject one class of customers. Ward and Kumar (2008) develop asymptotically optimal admission control of a $GI/GI/1$ queue with impatient customers in heavy traffic. Our model is different from theirs. Concerning dynamic scheduling, for a single-server queue with a convex waiting cost function, van Mieghem (1995) established the asymptotic optimality of the generalized-$c\mu$ rule; Mandelbaum and Stolyar (2004) generalized the work of van Mieghem (1995) to the parallel-server setting. In our paper, the scheduling policy also has a form of the generalized-$c\mu$ rule when the state cost functions are convex. However, different from the model in van Mieghem (1995) where the states are nonnegative, the system states in our model can be either positive or negative, and this complicates the analyses. The long-run average objective we consider is also related to ergodic control of queueing systems. For instance, Budhiraja et al. (2011) prove that near optimal control policies in an associated diffusion control problem can be used to construct asymptotically optimal rate control policies for the original single class queueing networks, Huang and Gurvich (2018) show that the service rate derived from an intuitive Brownian control problem is universally nearly optimal for a single-server queueing system and Arapostathis et al. (2015) consider the scheduling problem in a many-server queue. Although the lower bound approach has also been applied in the latter two papers, we combine the approach with a thorough steady state analysis and state-space collapse in stationarity to analyze a complex make-to-stock production/inventory system.

Organization. The rest of the paper is organized as follows. We describe the model in Section 2, and introduce the heavy traffic framework in Section 3. We introduce the proposed policy and prove the main results in Section 4. The asymptotic optimality of the proposed policy, and general results about the steady states are established in Section 5. The paper is concluded in Section 6. We leave the proofs of all propositions and lemmas in the appendix.

Notation. All vectors are understood to be column vectors. For $K \in \mathbb{N}$, let $\mathbb{D}^K$ be the space of all $\mathbb{R}^K$-valued functions that are right continuous on $[0,\infty)$ and have left limits on $(0,\infty)$, equipped with the Skorohod J_1-topology. All the stochastic processes are assumed to have sample paths in $\mathbb{D}^K$ for an appropriate K. We use $\Rightarrow$ to denote weak convergence. We use $A \subset B$ to mean that A is a strict subset of B. For a sequence of functions $f^n(\cdot) \in \mathbb{D}^K$, $f^n(t) \to f(t)$ u.o.c. as $n \to \infty$ means that $f^n(t)$ uniformly converges to $f(t)$ on compact sets.

2. A Multi-Class Make-to-Stock System We consider a make-to-stock system as in Ata and Barjesteh (2019), adopting their notation and terminologies with slight modifications to accommodate our analysis. The system sells K types of products to customers. Each customer order needs one product and a customer order is class k if a type k product is needed. In the following, we will use customer order and order interchangeably. Denote by $\mathcal{K} := \{1, \ldots, K\}$.

Let $E(\cdot) = \{(E_k(t)); k \in \mathcal{K}, t \geq 0\}$ be the arrival process of orders, with $E_k(t)$ being the number of class k orders arrived by time t. We assume the process $E_k(\cdot) = \{E_k(t); t \geq 0\}$ is a non-homogeneous Poisson process with rate to be controlled, that is,

$$E_k(t) = N_k\left(\int_0^t \lambda_k(s)ds\right),$$

where $N_k(\cdot) = \{N_k(t); t \geq 0\}$ is a unit rate Poisson process. The arrival rate vector $\lambda(t) = (\lambda_k(t); k \in \mathcal{K})$ is to be controlled, and can choose values from a set $\mathcal{L} \subset \mathbb{R}_+^K$. The manager can control the arrival rate $\lambda(t)$ by controlling the price vector $p(t) = (p_k(t); k \in \mathcal{K})$, which can choose values from another set $\mathcal{P} \subset \mathbb{R}_+^K$. Here $p_k(t)$ is the unit price of product k at time t. Assume that there is a non-negative demand function $\Lambda : \mathcal{P} \to \mathcal{L}$ so that $\lambda(t) = \Lambda(p(t))$ for $t \geq 0$, and a unique inverse demand function $\Lambda^{-1} : \mathcal{L} \to \mathcal{P}$ such that $p(t) = \Lambda^{-1}(\lambda(t))$, then there is a one-to-one relationship between $p(t) \in \mathcal{P}$ and $\lambda(t) \in \mathcal{L}$. As a result, in the following we consider the control of the arrival rates $\lambda(\cdot)$.

Customer orders may be outsourced to avoid long waiting times. Denote by $O(\cdot) = (O_k(\cdot); k \in \mathcal{K})$ the K-dimensional outsourcing process, with $O_k(t)$ being the number of class k orders outsourced up to time t. Introduce a random variable ξ_{ki} for the ith class k order to indicate whether that order is outsourced ($\xi_{ki} = 1$) or not ($\xi_{ki} = 0$). Then

$$O_k(t) = \sum_{i=1}^{E_k(t)} \xi_{ki}. \tag{1}$$

The actual number of class k orders accepted by time t is then

$$A_k(t) = E_k(t) - O_k(t).$$

The production time of each class k product is assumed to be exponential with rate μ_k. For $k \in \mathcal{K}$, let $S_k(t)$ denote the number of class k products manufactured until time t if the system were to continuously work on class k products up to time t. Then for $k \in \mathcal{K}$, $S_k(\cdot) = \{S_k(t) : t \geq 0\}$ is a Poisson process with rate μ_k. It is assumed that $N_k(\cdot), k \in \mathcal{K}$, and $S_k(\cdot), k \in \mathcal{K}$, are independent. Let $T_k(t)$ denote the cumulative amount of time the system devoted to class k products until time t, then the number of class k products produced by time t is

$$F_k(t) = S_k(T_k(t)).$$

The manager makes dynamic scheduling decisions by determining $T(\cdot) = (T_k(\cdot); k \in \mathcal{K})$.

The system manager controls the system by deciding the arrival rate, outsourcing new orders and scheduling capacity to maximize the long-run average profit. A dynamic control policy is denoted by $\psi = (\lambda, O, T)$. Under a control policy ψ, denote by $Q_k(t, \psi)$ the inventory level of class k products at time t, for $k \in \mathcal{K}$. As the policy will be clear from the context, we will drop the notation ψ for brevity. If $Q_k(t) \geq 0$, it denotes the number of class k products in inventory; if $Q_k(t) < 0$, its absolute value denotes the number of class k orders waiting. Then the dynamics of the inventory process $Q_k(\cdot) = \{Q_k(t) : t \geq 0\}$ is

$$Q_k(t) = Q_k(0) + F_k(t) - E_k(t) + O_k(t).$$

We also introduce $C(t)$ to indicate the product class that is currently in service. That is, $C(t) = k, k \in \mathcal{K}$, means the system is producing a class k product at time t, and $C(t) = 0$ means the system is idle. The system state process is $\mathfrak{X}(\cdot) = \{\mathfrak{X}(t); t \geq 0\}$ with $\mathfrak{X}(t) = (Q_1(t), \ldots, Q_K(t), C(t))$.

Introduce the process $I(\cdot) = \{I(t); t \geq 0\}$, with $I(t) = t - \sum_{k \in \mathcal{K}} T_k(t)$ for $t \geq 0$, which is the cumulative idle time of the system by time t.

Definition 1. A control policy $\psi = (\lambda, O, T)$ is said to be *feasible* if it is non-anticipating with respect to $\mathfrak{X}$, $\lambda(t) \in \mathcal{L}$ for $t \geq 0$, and
1. $I(\cdot), T(\cdot), O(\cdot)$ are non-decreasing with $I(0) = T(0) = O(0) = 0$,
2. $I(\cdot), T(\cdot)$ are continuous.

We will focus on feasible Markov control policies. To describe Markov control policies, we introduce process $\xi(\cdot) = (\xi_k(\cdot), k \in \mathcal{K})$ with $\xi_k(\cdot) = \{\xi_k(t); t \geq 0\}$, in which $\xi_k(s-)$ indicates the outsourcing decision for a virtual new class k order arriving at time s: if $\xi_k(s-) = 1$ and a class k order arrives at s, then that order is outsourced; if $\xi_k(s-) = 0$, that order is accepted. Here, we use $\xi_k(s-)$ to indicate that the decision is made with the information before time s. Denote by τ_{ki} the arrival epoch of the ith class k order. Then $\xi_{ki} = \xi_k(\tau_{ki}-)$, and from (1) we have

$$O_k(t) = \int_0^t \xi_k(s-) dE_k(s). \tag{2}$$

From (2), controlling $O_k(\cdot)$ is by controlling the process $\xi_k(\cdot)$.

Definition 2. A policy $\psi = (\lambda, O, T)$ is called a *Markov control policy* if $(\lambda(t), \xi(t))$ can be represented as a measurable function, from $\mathbb{R}^{K+1}$ to $\mathbb{R}^{2K}$, of $\mathfrak{X}(t)$. For notational brevity, in the following, we use $(\lambda(\cdot), \xi(\cdot))$ to denote the corresponding measurable function and hence $(\lambda(t), \xi(t)) = (\lambda(\mathfrak{X}(t)), \xi(\mathfrak{X}(t)))$. Denote by Π the set of all feasible Markov control policies.

Note that although a policy $\psi = (\lambda, O, T)$ has three parts, we only assume $(\lambda(t), \xi(t))$ are functions of $\mathfrak{X}(t)$. This is because $T_k(t) = \int_0^t 1_{\{C(s)=k\}} ds$ for $k \in \mathcal{K}$. From this, controlling $T(\cdot)$ is equivalent to controlling $C(\cdot) = \{C(t); t \geq 0\}$, which is part of the system state process $\mathfrak{X}(\cdot)$.

Under a Markov control policy, one can verify that the process $\mathfrak{X}(\cdot)$ is a continuous-time Markov chain with countable state space $S = \mathbb{Z}^K \times (\mathcal{K} \cup \{0\})$. Denote by ν the distribution of $\mathfrak{X}(0)$, the initial state of $\mathfrak{X}(\cdot)$.

Denote by δ_k the variable cost of producing a class k product, and a vector $\delta = (\delta_k)_{k \in \mathcal{K}}$. Then the profit generated by a class k order arriving at time s is $p_k(s-) - \delta_k$ (recall that $p(s) = \Lambda^{-1}(\lambda(s))$). Some orders are outsourced and we assume each outsourced class-k order incurs a cost $\vartheta_k > 0$. Then the total cost of outsourcing by time t is $\sum_{k \in \mathcal{K}} \vartheta_k O_k(t)$. Hence, the cumulative profit till time t is

$$\int_0^t (p(s-) - \delta) \cdot dE(s) - \sum_{k \in \mathcal{K}} \vartheta_k O_k(t) = \int_0^t (\Lambda^{-1}(\lambda(s-)) - \delta) \cdot dE(s) - \sum_{k \in \mathcal{K}} \vartheta_k O_k(t).$$

The state cost function of class k, $q_k : \mathbb{R} \to \mathbb{R}_+$, comprises inventory holding and customer waiting costs. We assume that for $k \in \mathcal{K}$, $q_k(x)$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$, with $q_k(0) = 0$. We will consider two types of state cost functions: 1) for $k \in \mathcal{K}$, $q_k(x)$ is strictly convex; 2) for $k \in \mathcal{K}$, $q_k(x)$ is linear on $(-\infty, 0]$ and linear on $[0, \infty)$.

Then the state cost till time t is $\int_0^t \sum_{k \in \mathcal{K}} q_k(Q_k(s)) ds$. Hence, the expected cumulative profit process associated with initial distribution ν and policy $\psi = (\lambda, O, T)$ is

$$V(t, \nu, \psi) = \mathbb{E}_\nu \left[\int_0^t \left(\Lambda^{-1}(\lambda(s-)) - \delta \right) \cdot dE(s) - \int_0^t \sum_{k \in \mathcal{K}} q_k(Q_k(s)) ds - \sum_{k \in \mathcal{K}} \vartheta_k O_k(t) \right].$$

The system manager seeks to find a feasible Markov control policy to maximize the long-run average profit, that is

$$\max_{\psi \in \Pi} \limsup_{t \to \infty} \frac{1}{t} V(t, \nu, \psi). \tag{3}$$

Define the profit rate function π as follows:

$$\pi(x) = x'(\Lambda^{-1}(x) - \delta), \quad x \in \mathcal{L},$$

where $\delta = (\delta_k)$ and recall that δ_k is the variable cost of manufacturing a class k product. One can verify that $\int_0^t (p(s-) - \delta) \cdot d(E(s) - \int_0^s \lambda(u)du)$ is a martingale. Together with the fact that $\lambda(\cdot)$ (because $\lambda(\cdot) = \lambda(\mathfrak{X}(\cdot))$) has only countable jumps, one has

$$\mathbb{E}_\nu \left[\int_0^t (p(s-) - \delta) \cdot dE(s) \right] = \mathbb{E}_\nu \left[\int_0^t \pi(\lambda(s)) ds \right].$$

Hence

$$V(t, \nu, \psi) = \mathbb{E}_\nu \left[\int_0^t \pi(\lambda(s)) ds - \int_0^t \sum_{k \in \mathcal{K}} q_k(Q_k(s)) ds - \sum_{k \in \mathcal{K}} \vartheta_k O_k(t) \right].$$

Following Ata and Barjesteh (2019), we also consider the static planning problem

$$\text{maximize} \quad \pi(\lambda) \quad \text{subject to } \lambda \in \mathcal{L}. \tag{4}$$

We assume $\pi(\cdot)$ is twice continuously differentiable on $\mathcal{L}$ and the problem (4) has a unique optimal solution $\lambda^* \in \text{interior}(\mathcal{L})$ with $\sum_{k \in \mathcal{K}} \lambda_k^* / \mu_k = 1$. That is, ignoring the randomness in the system, the profit maximizing demand rate from the problem (4) puts the system to be critically loaded. Note that since $\mathcal{L} \subset \mathbb{R}_+^K$, we have $\lambda_k^* > 0$ for $k \in \mathcal{K}$. We also assume the Hessian matrix of π, $\nabla^2 \pi(\cdot)$, is continuous at the point λ^*, and $\nabla^2 \pi(\lambda^*)$ is negative definite.

3. Heavy Traffic Framework Problem (3) is difficult to solve. Ata and Barjesteh (2019) solved a related BCP and proposed a policy for the linear holding/waiting cost case. We will adopt the same heavy traffic framework with slight modifications. Different from Ata and Barjesteh (2019), in which the effectiveness of their policy is illustrated numerically but not rigorously analyzed for the discrete-state make-to-stock systems, our purpose is to establish the asymptotic optimality of a feasible Markov control policy for the discrete-state systems with either linear or strictly convex holding/waiting cost functions. See Assumption 1 below for more details.

We consider a sequence of make-to-stock systems as above, under an asymptotic framework known as the *conventional heavy traffic* regime. The systems are indexed by $n \in \mathbb{N}$. The relevant parameters and processes in the nth system will be appended a superscript n. For example, the control in the nth system is denoted by $\psi^n = (\lambda^n, O^n, T^n)$, in which λ^n is the arrival rate process to the nth system, T_k^n records the time allocated to producing class k products, and O_k^n denotes the number of class k orders outsourced. The outsourcing process of class k orders $O_k^n(t) = \int_0^t \xi_k^n(s-) dE_k^n(s)$, hence the control of O^n is via ξ^n. We follow Ata and Barjesteh (2019) and consider the arrival rate $\lambda^n(t)$ of the form:

$$\lambda^n(t) = n\lambda^* + \sqrt{n}\zeta^n(t),$$

with $\zeta^n : \mathbb{R}_+ \to \mathbb{R}^K$. The control of $\lambda^n(\cdot)$ is via $\zeta^n(\cdot)$ and hence we will write the control as $\psi^n = (\zeta^n, O^n, T^n)$. Slightly different from Ata and Barjesteh (2019), we assume the service rate $\mu_k^n = n\mu_k$. This is mainly for notational simplicity, and does not change the results. The system state process in the nth system is denoted by $\mathfrak{X}^n(\cdot) = \{\mathfrak{X}^n(t) := (Q_1^n(t), \ldots, Q_K^n(t), C^n(t)); t \geq 0\}$, whose initial state $\mathfrak{X}^n(0)$ follows distribution ν^n. We will focus on feasible Markov control policies, that is, $(\zeta^n(t), \xi^n(t))$ can be represented as a measurable function of $\mathfrak{X}^n(t)$.

Define the diffusion-scaled processes $\widetilde{Q}^n(\cdot) = \{(\widetilde{Q}_1^n(t), \ldots, \widetilde{Q}_K^n(t)); t \geq 0\}$ and $\widetilde{O}^n(\cdot) = \{(\widetilde{O}_1^n(t), \ldots, \widetilde{O}_K^n(t)); t \geq 0\}$ with

$$\widetilde{Q}^n(t) = \frac{Q^n(t)}{\sqrt{n}}, \quad \text{and} \quad \widetilde{O}^n(t) = \frac{O^n(t)}{\sqrt{n}}.$$

Let $\widetilde{\mathfrak{X}}^n(t) = (\widetilde{Q}^n(t), C^n(t))$, where $C^n(t)$ is unscaled and takes values in $\mathcal{K} \cup \{0\}$. Under a feasible Markov control policy ψ^n, the process $\widetilde{\mathfrak{X}}^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n = \frac{1}{\sqrt{n}}\mathbb{Z}^K \times (\mathcal{K} \cup \{0\})$. Let $\widetilde{\nu}^n$ be the distribution of the initial state $\widetilde{\mathfrak{X}}^n(0)$.

Define $\rho_k = \lambda_k^*/\mu_k$ to be the nominal workload of class k orders for $k\in\mathcal{K}$. Let

$$\widetilde{X}_k^n(t) = \frac{S_k^n(T_k^n(t)) - \mu_k^n T_k^n(t)}{\sqrt{n}} - \frac{N_k(\int_0^t \lambda_k^n(s)ds) - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}}$$

and

$$\widetilde{Y}_k^n(t) = \sqrt{n}(\rho_k t - T_k^n(t)).$$

Then the dynamics of $\widetilde{Q}^n$ under a control policy ψ^n is

$$\begin{aligned}\widetilde{Q}_k^n(t) =& \widetilde{Q}_k^n(0) + \frac{S_k^n(T_k^n(t))}{\sqrt{n}} - \frac{E_k^n(t)}{\sqrt{n}} + \tilde{O}_k^n(t)\\ =& \widetilde{Q}_k^n(0) + \frac{S_k^n(T_k^n(t)) - \mu_k^n T_k^n(t)}{\sqrt{n}} - \frac{N_k(\int_0^t \lambda_k^n(s)ds) - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}} + \tilde{O}_k^n(t)\\ &+ \frac{\mu_k^n(T_k^n(t) - \rho_k t) + n\lambda_k^* t - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}}\\ =& \widetilde{Q}_k^n(0) + \widetilde{X}_k^n(t) - \mu_k \widetilde{Y}_k^n(t) - \int_0^t \zeta_k^n(s)ds + \tilde{O}_k^n(t).\end{aligned} \tag{5}$$

We also define the one-dimensional nominal workload process $\widetilde{W}^n(\cdot) = \{\widetilde{W}^n(t); t\geq 0\}$ with

$$\widetilde{W}^n(t) := \sum_{k\in\mathcal{K}} \frac{\widetilde{Q}_k^n(t)}{\mu_k}, \quad \text{for } t\geq 0. \tag{6}$$

Define $\widetilde{I}^n(t) := \sum_{k\in\mathcal{K}} \widetilde{Y}_k^n(t)$. Using $\sum_{k\in\mathcal{K}} \rho_k = 1$, one has $\widetilde{I}^n(t) = \sqrt{n}(t - \sum_{k\in\mathcal{K}} T_k^n(t)) = \sqrt{n} I^n(t)$, where $I^n(\cdot)$ is the cumulative idle time process. Then from (5) and (6), for $t\geq 0$,

$$\widetilde{W}^n(t) = \widetilde{W}^n(0) + \sum_{k\in\mathcal{K}} \frac{\widetilde{X}_k^n(t)}{\mu_k} - \widetilde{I}^n(t) - \int_0^t \sum_{k\in\mathcal{K}} \frac{\zeta_k^n(s)}{\mu_k} ds + \sum_{k\in\mathcal{K}} \frac{\tilde{O}_k^n(t)}{\mu_k}. \tag{7}$$

Following Ata and Barjesteh (2019), we assume $\Lambda^n(x) = n\Lambda(x)$, then $(\Lambda^n)^{-1}(nx) = \Lambda^{-1}(x)$. The variable cost of manufacturing $\delta = (\delta_k)$ is assumed to be independent of n and hence will not be scaled. Then the resulted profit rate function $\pi^n(nx) = n\pi(x)$. The outsourcing cost rate ϑ_k^n is assumed to vary with n: $\vartheta_k^n = \frac{r_k}{\sqrt{n}}$, where r_k is a given constant for each k.

We assume that the state cost function $q_k^n(x)$ in the nth system is given by

$$q_k^n(x) = g_k(x/\sqrt{n}).$$

Here the functions $g_k(\cdot): \mathbb{R}\to\mathbb{R}_+$ are assumed to be one of the following two types:

Assumption 1 **(State cost functions)**. 1. *Strictly convex: for each $k\in\mathcal{K}$, g_k is strictly convex, decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$, with $g_k(0) = 0$. Furthermore, $g_k(\cdot)$ is a sub-polynomial function, that is, there exist constants $m\in\mathbb{N}$ and $c>0$ such that*

$$g_k(x) \leq c(1+|x|^m), \quad \textit{for } x\in\mathbb{R}.$$

2. *Linear: there exist positive constants $h_k, p_k > 0$, $k\in\mathcal{K}$, such that for $k\in\mathcal{K}$ and $x\in\mathbb{R}$,*

$$g_k(x) = \begin{cases} h_k x, & x\geq 0,\\ -p_k x, & x<0.\end{cases}$$

We assume that there is a unique class minimizing $h_k\mu_k$ and a unique class maximizing $p_k\mu_k$.

Then the expected cumulative profit process associated with policy ψ^n is

$$\begin{aligned} V^n(t,\nu^n,\psi^n) =& \mathbb{E}_{\nu^n}\left[\int_0^t \big((\Lambda^n)^{-1}(\lambda^n(s))-\delta\big)\cdot dE^n(s) - \sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds - \sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right] \\ =& \mathbb{E}_{\nu^n}\left[\int_0^t n\pi(\lambda^*+\zeta^n(s)/\sqrt{n})ds - \sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds - \sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right]. \end{aligned}$$

Here, $\mathbb{E}_{\nu^n}$ denotes the expectation conditioned on that the initial state $\mathfrak{X}^n(0)$ follows ν^n.

From (4), $n\pi(\lambda^*)t$ serves as an upper bound on the cumulative profit process $V^n(t,\nu^n,\psi^n)$. Maximizing $V^n(t,\nu^n,\psi^n)$ is equivalent to minimizing the deviation of $V^n(t,\nu^n,\psi^n)$ from $n\pi(\lambda^*)t$, which is

$$\begin{aligned} \widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n) =& n\pi(\lambda^*)t - V^n(t,\nu^n,\psi^n) \\ =& \mathbb{E}_{\nu^n}\left[\int_0^t n\cdot(\pi(\lambda^*)-\pi(\lambda^*+\zeta^n(s)/\sqrt{n}))ds + \sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds + \sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right] \\ =& \mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^t n\cdot(\pi(\lambda^*)-\pi(\lambda^*+\zeta^n(s)/\sqrt{n}))ds + \sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds + \sum_{k\in\mathcal{K}} r_k\widetilde{O}_k^n(t)\right]. \end{aligned}$$

where $\mathbb{E}_{\widetilde{\nu}^n}$ denotes the expectation conditioned on that $\widetilde{\mathfrak{X}}^n(0)$ follows $\widetilde{\nu}^n$.

Define

$$c^n(y) := n(\pi(\lambda^*)-\pi(\lambda^*+y/\sqrt{n})) \geq 0, \quad y\in\mathbb{R}^K. \tag{8}$$

Then we have

$$\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n) = \mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^t c^n(\zeta^n(s))ds + \sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds + \sum_{k\in\mathcal{K}} r_k\widetilde{O}_k^n(t)\right]. \tag{9}$$

In the following, we will call $\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n)$ the expected loss (due to randomness and the controls) under an initial state distribution $\widetilde{\nu}^n$ and a control ψ^n. The long-run average expected loss for a given control policy ψ^n is defined by

$$\widetilde{V}^n(\widetilde{\nu}^n,\psi^n) := \liminf_{t\to\infty}\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n). \tag{10}$$

We will focus on feasible Markov control policies. For technical reasons, we also assume that the arrival rate cannot change in the order of n to simplify the analysis.

Definition 3. A sequence of polices $\{\psi^n\}$ is called *asymptotically admissible* if for each n, $\psi^n\in\Pi^n$, that is, ψ^n is a feasible Markov control policy of the nth system, and there exists a sequence of nonnegative numbers $\{a_n\}$ with $\lim_{n\to\infty} a_n = 0$, such that $\|\zeta^n(t)/\sqrt{n}\|\leq a_n$ almost surely for all $t\geq 0$ and $n\geq 1$.

Denote by $\bar{\Pi}$ the set of sequences of control policies that are asymptotically admissible.

Definition 4 (Asymptotic optimality). A sequence of polices $\{\psi_*^n\}$ is *asymptotically optimal* if it is asymptotically admissible, and for any other sequence of policies $\{\psi^n\}\in\bar{\Pi}$,

$$\liminf_{n\to\infty}\widetilde{V}^n(\widetilde{\nu}^n,\psi^n) \geq \liminf_{n\to\infty}\widetilde{V}^n(\widetilde{\nu}^n,\psi_*^n) \tag{11}$$

for any sequence of initial distributions $\{\widetilde{\nu}^n\}$.

4. Main Results In this section, we propose a sequence of control policies $\{\psi_*^n\}$ and establish its asymptotic optimality. We first introduce an Ordinary Differential Equation (ODE) in Section 4.1, which will help us identify several parameters of the policy. We then describe the proposed policy and state our main result (Theorem 1) in Section 4.2.

4.1. An ODE For each $w \in \mathbb{R}$, consider the following minimization problem:

$$\begin{aligned} h(w) := \min_{x \in \mathbb{R}^K} \quad & \sum_{k \in \mathcal{K}} g_k(x_k) \\ \text{s.t.} \quad & \sum_{k \in \mathcal{K}} \frac{x_k}{\mu_k} = w. \end{aligned} \tag{12}$$

Denote by $x^* = (x_1^*, \ldots, x_K^*)$ an optimal solution if it exists. It is clear that x^* depends on w. The following existence and uniqueness results are standard, see van Mieghem (1995).

LEMMA 1. *For state cost functions g_k, $k \in \mathcal{K}$, satisfying Assumption 1 and any fixed $w \in \mathbb{R}$, there exists a unique optimal solution x^* to (12). Introduce the lifting function $\Delta(\cdot)$ by $\Delta(w) := x^*$. Then the function $\Delta(w) = (\Delta_1(w), \ldots, \Delta_K(w))$ is well defined, and*

$$h(w) = \sum_{k \in \mathcal{K}} g_k(\Delta_k(w)). \tag{13}$$

Define a column vector $m = (m_k)_{k \in \mathcal{K}}$ where $m_k = 1/\mu_k$ for each k, and let $H = -\nabla^2 \pi(\lambda^*)/2$. Recall that $\nabla^2 \pi(\lambda^*)$ is the Hessian matrix of π at λ^* and is assumed to be negative definite, hence H is positive definite. Define the effective outsourcing cost κ as follows:

$$i^* = \operatorname{argmin}\left\{\frac{r_k}{m_k} : k \in \mathcal{K}\right\} \quad \text{and} \quad \kappa = \frac{r_{i^*}}{m_{i^*}}. \tag{14}$$

If there are multiple i^* minimizing $\frac{r_k}{m_k}$, we specify i^* to be the minimum index.

The following lemma identifies some parameters that we will use in the proposed policies. Its proof can be found in Section E. To facilitate the presentation, we define

$$c(u) := \min\{x' H x : m' x = u, x \in \mathbb{R}^K\} = \frac{1}{m' H^{-1} m} u^2, \quad \text{for } u \in \mathbb{R}, \tag{15}$$

where the last equality is due to Lemma 1 of Ata and Barjesteh (2019).

LEMMA 2. *Let $\sigma^2 = \sum_{k \in \mathcal{K}} \frac{2\lambda_k^*}{\mu_k^2}$. Then there exist unique constants $l_* < 0 < u_*, \gamma^* > 0$ and a unique real-valued function $\Phi \in C^2(\mathbb{R})$ satisfying the differential equation:*

$$\frac{\sigma^2}{2} \Phi''(x) - \frac{m' H^{-1} m}{4} (\Phi'(x))^2 + h(x) = \gamma^*, \quad x \in (l_*, u_*), \tag{16}$$

and $\Phi'(x) \in [-\kappa, 0]$ for all $x \in \mathbb{R}$, with $\Phi'(x) = -\kappa$ for $x \le l_*$, $\Phi'(x) = 0$ for $x \ge u_*$, and $\Phi''(x) = 0$ for $x \notin (l_*, u_*)$. As a consequence, there exists a positive constant C such that for any $u, x \in \mathbb{R}$:

$$\frac{\sigma^2}{2} \Phi''(x) - u \Phi'(x) + c(u) + h(x) \ge \gamma^*, \tag{17}$$

$$-\kappa \le \Phi'(x) \le 0, \quad |\Phi''(x)| \le C. \tag{18}$$

In addition, $\Phi'''(x)$ exists almost everywhere and $|\Phi'''(x)| \le C$ whenever it exists.

It is clear that (16) is essentially a first-order ODE for the unknown function $\Phi'(\cdot)$. Such an equation falls in the class of Riccati equations which are first-order ODEs that are quadratic in the unknown functions. Note that γ^* and the boundary points l_* and u_* of the ODE (16) are also unknown and need to be determined, hence we call the ODE a free boundary ODE as in Dai and Yao (2013). If h is derived from the linear state cost functions g_k, then Ata and Barjesteh (2019) provided the closed-form expressions of Φ and the constants l_*, u_*, γ^* (see Section 6.3 there). However, if h is derived from the strictly convex state cost functions g_k, to the best of our knowledge, closed-form expressions are unavailable for the function Φ or the constants l_*, u_*, γ^*.

4.2. The Proposed Policies We propose a sequence of feasible Markov control policies $\{\psi_*^n\}$ with ψ_*^n applied to the nth system. In the three parts of the policy ψ_*^n, the arrival rate control and the outsourcing decisions have the same structure for both types of state cost functions, while the scheduling decision depends on whether the state cost functions are linear or strictly convex. Following Wein (1992) and Ata and Barjesteh (2019), we adopt the concept of *safety stock* from the inventory management literature, and denote by a non-negative integer α_k the safety stock for class k. We say that class k is *in danger of being backordered* if the inventory is less than the parameter α_k, and these parameters α_k are assumed to be independent of n and are usually calibrated by numerical experiments.

The details of the proposed policy ψ_*^n are as follows:

---

1. Arrival rates: Given the nominal workload process $\widetilde{W}^n(t)$ in (6), Φ, l_* and u_* in Lemma 2, the proposed arrival rate vector is $\lambda_*^n(t) = n\lambda^* + \sqrt{n}\cdot\zeta_*^n(t)$ with

$$\zeta_*^n(t) = \frac{H^{-1}m}{2}\Phi'\left(l_* \vee (\widetilde{W}^n(t)\wedge u_*)\right). \tag{19}$$

2. Outsourcing: if $\widetilde{W}^n(t) \le l_* < 0$ and $Q_{i^*}^n(t) \le \alpha_{i^*}$, outsource new class i^* orders at time t; otherwise, do not outsource any new order.

3. Scheduling: First identify $\mathcal{C}^n(t)$, the set of candidate classes at time t: if $\sum_{k\in\mathcal{K}} \frac{(Q_k^n(t))^+}{\mu_k} > u^n = \sqrt{n}u_*$, let $\mathcal{C}^n(t) = \{k\in\mathcal{K} : Q_k^n(t) < \alpha_k\}$; otherwise, let $\mathcal{C}^n(t) = \mathcal{K}$. The scheduling decision depends on the structure of the state cost functions:

(a) Strictly convex: when the system is ready to produce a new product, it will work on the class $\arg\min_{k\in\mathcal{C}^n(t)} g_k'(\widetilde{Q}_k^n(t))\mu_k$.

(b) Linear: let $\mathcal{N}^n(t) = \{k \in \mathcal{C}^n(t) : Q_k^n(t) < \alpha_k\}$ and $\mathcal{P}^n(t) = \{k\in\mathcal{C}^n(t) : Q_k^n(t) \ge \alpha_k\}$. When the system is ready to produce a new product, if $\mathcal{N}^n(t) \neq \emptyset$, it will work on the class $\arg\max_{k\in\mathcal{N}^n(t)} p_k\mu_k$; otherwise, it will work on the class $\arg\min_{k\in\mathcal{P}^n(t)} h_k\mu_k$.

We will append a subscript $*$ to processes under the proposed policy ψ_*^n. For example, $\widetilde{\mathfrak{X}}_*^n(\cdot) = \{\widetilde{\mathfrak{X}}_*^n(t); t\ge 0\}$ is the (scaled) system state process under the control policy ψ_*^n.

Remark 1. In the case of strictly convex state cost functions, if $\widetilde{Q}_k^n(t) < 0$, then $g_k'(\widetilde{Q}_k^n(t)) < 0$; if $\widetilde{Q}_k^n(t) > 0$, then $g_k'(\widetilde{Q}_k^n(t)) > 0$. As a result, for both types of state cost functions, *if the inventory of class k reaches its safety stock, that is, $Q_k^n \ge \alpha_k$, and there exists another class j such that $Q_j^n(t) < 0$, then the system will not produce new class k products.* This observation will be used frequently in the proofs.

Remark 2 (Safety stocks). Note that $\mathcal{C}^n(t) = \emptyset$ means that there is no candidate class at time t, which implies that the system will become idle if it finishes producing a product at time t. This happens if and only if $\sum_{k\in\mathcal{K}} \frac{(Q_k^n(t))^+}{\mu_k} > u^n = \sqrt{n}u_*$ and $Q_k^n \ge \alpha_k$ for all $k \in \mathcal{K}$. Intuitively, it means when all classes have enough products on hand (more than the safety stock) and the nominal workload is too large (above a threshold), then there is no risk of being out of stock and hence the system should stop working. Note that the system would keep working if at least one class has an inventory level lower than the safety stock, even if the nominal workload is too large. For the setting with linear state cost functions, if there are not enough products on hand for some classes (i.e., less than the corresponding safety stocks), then the system will give priority to these classes. These observations are consistent with the usual strategy in inventory management to reduce the risk of running out of stock.

Remark 3 (Difference from Ata and Barjesteh (2019)). For the setting with linear state cost functions, our policy is almost identical to the one in Ata and Barjesteh (2019), except that in the outsourcing part, we have the additional requirement $Q_{i^*}^n(t) \le \alpha_{i^*}$. This requirement is intuitive because if class i^* has many products on hand (higher than the safety stock), it is better to reduce its inventory level and hence not outsource its orders. This additional requirement is essential in establishing the general results of the existence and tightness of the stationary distributions of inventory processes. It is unclear how to get it from the BCP.

Remark 4 (Consistency). If $\alpha_k = 0$ for all $k \in \mathcal{K}$, then the scheduling policies are consistent for both cases, because for the linear state cost functions, b_k and h_k are the corresponding derivatives of the state cost rate functions. Note that the scheduling component of the policy in the setting with strictly convex holding/waiting cost functions is a form of the generalized-$c\mu$ rule (van Mieghem (1995)). If there exists a class k such that $\alpha_k > 0$, then the policies can be different: when there are customers waiting, in the linear cost case the system may work on a class with positive inventory if this class has the largest $p_k\mu_k$ in $\mathcal{N}^n(t)$ (and the queue length of this class satisfies $0 < Q_k^n(t) < \alpha_k$), while in the strictly convex case, the system will always work on a class with customers waiting, as discussed in Remark 1.

Our main result is the following theorem.

Theorem 1 **(Asymptotic optimality)**. *Suppose Assumption 1 holds. Then the sequence of policies $\{\psi_*^n\}$ is asymptotically optimal.*

In view of (11), the theorem claims that for any initial distribution $\widetilde{\nu}^n$, the long-run average deviation of profit from the upper bound of the nth system is asymptotically minimized under policy ψ_*^n; this is equivalent to that the long-run average profit of the nth system is asymptotically maximized under the policy ψ_*^n, for any initial distribution $\widetilde{\nu}^n$.

5. Analyses of the Policies

We now analyze systems under the proposed policies and prove Theorem 1. In Section 5.1, we conduct steady state analyses, and in Section 5.2, we illustrate how to adapt the lower bound approach to the discrete-state make-to-stock systems.

5.1. Steady-State Analysis

In this subsection, we will prove results for the existence of steady states, and the tightness of the sequence of stationary distributions, for systems under more general policies. These results might be used to analyze multi-class make-to-stock systems under other scheduling policies; hence, we believe they are of independent interests.

To this end, introduce a set $\Psi \subset \bar{\Pi}$, where each element of Ψ is a sequence of control policies that are asymptotically admissible (see Definition 3). We use $\{\psi_\diamond^n\}$ to denote a generic element of Ψ, with the nth policy $\psi_\diamond^n$ for the nth system satisfying:

---

1. Arrival rates: the arrival rate vector $\lambda^n(t) = n\lambda^* + \sqrt{n}\cdot\zeta^n(t)$.
2. Outsourcing: fix a class $i^\diamond \in \mathcal{K}$ and a constant $l_\diamond < 0$. If $\widetilde{W}^n(t) \le l_\diamond$ and $Q_{i^\diamond}^n(t) \le \alpha_{i^\diamond}$, outsource new class $i^\diamond$ orders at time t; otherwise, do not outsource any new order.
3. Scheduling: there exists $u_\diamond > 0$ such that if $\sum_{k\in\mathcal{K}} \frac{(Q_k^n(t))^+}{\mu_k} > \sqrt{n}u_\diamond$, then the system will not work on any class k such that $Q_k^n(t) \ge \alpha_k$; the system will not work on a class if the inventory of that class reaches its safety stock and there exists another class with customers waiting.

It is easy to verify that the sequence of the proposed policies $\{\psi_*^n\}$ is one element of Φ: the condition for the outsourcing part can be verified with $l_\diamond = l_*$, $i^\diamond = i^*$; and the condition for the scheduling part can be verified with $u_\diamond = u_*$, the definition of $\mathcal{C}^n(t)$ and from Remarks 1.

Fix an element of Ψ, that is, a sequence of policies $\{\psi_\diamond^n\}$. Consider a sequence of systems, with the nth system $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$ under the policy $\psi_\diamond^n$, we can prove the following two propositions.

Proposition 1 **(Existence of steady states)**. *Under the policy $\psi_\diamond^n$, there exists a stationary distribution for the Markov process $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$ for all sufficiently large n.*

We do not claim the uniqueness of the stationary distribution. Denote $\widetilde{\mathfrak{X}}_\diamond^n(\infty)$ the random vector that follows a stationary distribution. The next result establishes the tightness of $\{\widetilde{\mathfrak{X}}_\diamond^n(\infty)\}$.

Proposition 2 **(Tightness of stationary distributions)**. *The sequence of random vectors $\{\widetilde{\mathfrak{X}}_\diamond^n(\infty)\}$ is tight.*

5.2. Proof of Theorem 1 We show (Theorem 2) in Section 5.2.1 that the constant γ^* is a lower bound for the long-run average expected loss under any asymptotically admissible sequence of policies. Then in Section 5.2.2, we verify that the lower bound γ^* is achieved under the sequence of proposed policies $\{\psi_*^n\}$.

The following lemma plays an essential role in both the proofs of Theorems 2 and 1. We use $\triangle\mathbb{Y}(t) = \mathbb{Y}(t) - \mathbb{Y}(t-)$ to denote the jump of a process $\mathbb{Y}$ at time t, and we write

$$\widetilde{\mathbb{X}}^n(t) := \sum_{k\in\mathcal{K}} \frac{\widetilde{X}_k^n(t)}{\mu_k} \quad \text{and} \quad \widetilde{\mathbb{O}}^n(t) := \sum_{k\in\mathcal{K}} \frac{\widetilde{O}_k^n(t)}{\mu_k}. \tag{20}$$

The idea is to apply Ito's formula, with the function Φ in Lemma 2, to the semimartingale $\widetilde{W}^n$ in (7), take Taylor expansion of Φ and use the stationarity of $\widetilde{W}^n(\cdot)$. The proof is deferred to the end of this section.

LEMMA 3. *Suppose there exists a stationary distribution $\widetilde{\pi}^n$ for the Markov chain $\widetilde{\mathfrak{X}}^n(\cdot)$ under an admissible policy ψ^n and assume $\widetilde{\mathfrak{X}}^n(0)$ follow $\widetilde{\pi}^n$. For Φ in Lemma 2, we have*

$$0 = \mathbb{E}[\Phi(\widetilde{W}^n(t))] - \mathbb{E}[\Phi(\widetilde{W}^n(0))] = \Psi_1(t,\widetilde{\pi}^n,\psi^n) + \Psi_2(t,\widetilde{\pi}^n,\psi^n) + \Psi_3(t,\widetilde{\pi}^n,\psi^n), \tag{21}$$

where

$$\Psi_1(t,\widetilde{\pi}^n,\psi^n) = \mathbb{E}\left[\int_0^t \left(-\sum_{k\in\mathcal{K}} \frac{\zeta_k^n(s)}{\mu_k}\Phi'(\widetilde{W}^n(s-)) + \sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k^2}\Phi''(\widetilde{W}^n(s-))\right) ds\right], \tag{22}$$

$$\Psi_2(t,\widetilde{\pi}^n,\psi^n) = \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{\mathbb{O}}^n(s) - \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{I}^n(s), \tag{23}$$

$$\begin{aligned}\Psi_3(t,\widetilde{\pi}^n,\psi^n) = &\sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2} \times \mathbb{E}\left[\int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))\left(\frac{\zeta_k^n(s)}{\sqrt{n}}ds - \frac{\mu_k}{\sqrt{n}}d\widetilde{Y}_k^n(s)\right)\right] \\ &- \mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right)\right] \\ &+ \mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0} \frac{1}{6}\Phi'''(\Delta_{\frac{1}{\sqrt{n}}}(\widetilde{W}^n(s)))(\triangle\widetilde{W}^n(s))^3\right].\end{aligned} \tag{24}$$

5.2.1. A Lower Bound We prove a strong version of lower bound. For this, let

$$\widetilde{V}^n := \inf_{\widetilde{\nu}^n,\psi^n\in\Pi^n} \widetilde{V}^n(\widetilde{\nu}^n,\psi^n). \tag{25}$$

In the above, $\widetilde{\nu}^n$, the distribution of $\widetilde{\mathfrak{X}}^n(0)$, can be any distribution supported on $\widetilde{S}^n$.

THEOREM 2 **(Lower Bound)**. *We have*

$$\liminf_{n\to\infty} \widetilde{V}^n \ge \gamma^*.$$

From Theorem 2, for any sequence of initial distributions $\{\widetilde{\nu}^n\}$ and asymptotically admissible sequence of control policies $\{\psi^n\}$, we have

$$\liminf_{n\to\infty} \widetilde{V}^n(\widetilde{\nu}^n,\psi^n) \ge \liminf_{n\to\infty} \widetilde{V}^n \ge \gamma^*.$$

As a result, γ^* serves as a lower bound for the long-run average expected loss under any asymptotically admissible sequence of control policies $\{\psi^n\}$ and initial distributions $\{\widetilde{\nu}^n\}$.

Proof of Theorem 2. For notational simplicity, we denote by C generic constants that are independent of n, although the value of C may differ from line to line. Recall $\widetilde{V}^n$ defined in (25). Consider two sets: $A_1 = \{n : \widetilde{V}^n > \gamma^* + 1\}$ and $A_2 = \{n : \widetilde{V}^n \le \gamma^* + 1\}$. If A_2 is finite, then the conclusion holds. Otherwise, it is enough to consider $\liminf_{n \in A_2 : n \to \infty} \widetilde{V}^n$. In the following, we focus on this lim inf and for notational simplicity, we omit $n \in A_2$ and will always assume $n \in A_2$. From the definition of $\widetilde{V}^n$, for any $\epsilon \in (0,1)$, there exists $\widetilde{\nu}^n$ and ψ^n such that

$$\widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \le \widetilde{V}^n + \epsilon \le \gamma^* + 2, \quad \text{for all } n \text{ sufficiently large.} \tag{26}$$

The following lemma ensures us that we can always assume $\widetilde{\nu}^n$ to be a stationary distribution.

Lemma 4. *Suppose (26) holds. Then there exists a stationary distribution $\widetilde{\pi}^n$ for the Markov chain $\widetilde{\mathfrak{X}}^n(\cdot)$ under the policy ψ^n for all sufficiently large n and*

$$\widetilde{V}^n(\widetilde{\pi}^n, \psi^n) \le \widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \le \widetilde{V}^n + \epsilon \le \gamma^* + 2. \tag{27}$$

The proof of this result is deferred to Appendix A. In the following, we will assume that $\widetilde{\mathfrak{X}}^n(0)$ follows a stationary distribution $\widetilde{\pi}^n$ so that the systems start from stationarity. For notational simplicity we use $\mathbb{E}$ to represent $\mathbb{E}_{\widetilde{\pi}^n}$ in the analysis below.

Our goal is to show

$$\liminf_{n \to \infty} \widetilde{V}^n(\widetilde{\pi}^n, \psi^n) \ge \gamma^*, \tag{28}$$

which then yields $\liminf_{n\to\infty} \widetilde{V}^n + \epsilon \ge \gamma^*$ by (27). Then Theorem 2 holds because ϵ is arbitrary.

Note that because the existence of the stationary distribution $\tilde{\pi}^n$, from (21), we have

$$0 = \Psi_1(t, \widetilde{\pi}^n, \psi^n) + \Psi_2(t, \widetilde{\pi}^n, \psi^n) + \Psi_3(t, \widetilde{\pi}^n, \psi^n).$$

We first analyze the term $\Psi_1(t, \widetilde{\pi}^n, \psi^n)$. Since Φ satisfies the condition (17), if we denote by $\bar{c}(\zeta) = \zeta' H \zeta$ for $\zeta \in \mathbb{R}^K$, then from (15), $c(u) \le \bar{c}(\zeta^n(s))$ for $u = \sum_{k\in\mathcal{K}} \frac{\zeta_k^n(s)}{\mu_k} = m'\zeta^n(s)$. Using (17) with this u, for each x and $s \ge 0$,

$$-\sum_{k\in\mathcal{K}} \frac{\zeta_k^n(s)}{\mu_k} \Phi'(x) + \sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k^2} \Phi''(x) + \bar{c}(\zeta^n(s)) + h(x) \ge \gamma^*.$$

Since $\widetilde{W}^n$ has only finite jumps on $[0,t]$, we can then infer that

$$\begin{aligned}
\Psi_1(t, \widetilde{\pi}^n, \psi^n) &\ge \gamma^* t - \mathbb{E}\left[\int_0^t \bar{c}(\zeta^n(s))ds + \int_0^t h(\widetilde{W}^n(s))ds\right] \\
&\ge \gamma^* t - \mathbb{E}\left[\int_0^t \bar{c}(\zeta^n(s))ds + \int_0^t \sum_{k\in\mathcal{K}} g_k(\widetilde{Q}_k^n(s))ds\right],
\end{aligned}$$

where in the second inequality we have used the facts that $\widetilde{W}^n(t) = \sum_{k\in\mathcal{K}} \frac{\widetilde{Q}_k^n(t)}{\mu_k}$ and that if $\sum_{k\in\mathcal{K}} q_k/\mu_k = x$, then $\sum_{k\in\mathcal{K}} g_k(q_k) \ge h(x)$ from the definition of $h(x)$ in (12).

We next analyze the term $\Psi_2(t, \widetilde{\pi}^n, \psi^n)$. Using (18) and the fact that $\widetilde{I}^n$ is nondecreasing, we obtain for each t,

$$\Psi_2(t, \widetilde{\pi}^n, \psi^n) \ge -\kappa \widetilde{\mathbb{O}}^n(t) \ge -\sum_{k\in\mathcal{K}} r_k \widetilde{O}_k^n(t),$$

where the last inequality is from the definition of κ in (14) and the definition of $\widetilde{\mathbb{O}}^n$ in (20).

Finally we analyze the term $\Psi_3(t, \widetilde{\pi}^n, \psi^n)$. Using the fact that the magnitude of jumps of $\widetilde{X}_k^n(s)$ is $\frac{1}{\sqrt{n}}$, we get

$$\mathbb{E}\left| \sum_{s\le t : |\triangle \widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right)\right| \le \mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{C}{2n\mu_k^2} O_k^n(t)\right] \le \mathbb{E}\left[\frac{C}{\sqrt{n}} \widetilde{\mathbb{O}}^n(t)\right].$$

Since $|\Phi'''(x)|\le C$, we have

$$\mathbb{E}\left|\sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0}\frac{1}{6}\Phi'''(\Delta_{\frac{1}{\sqrt{n}}}(\widetilde{W}^n(s)))(\triangle\widetilde{W}^n(s))^3\right|\le\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0}\frac{C}{6}|\triangle\widetilde{W}^n(s)|^3\right]$$
$$\le\frac{C}{6\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t\lambda_k^n(s)ds+\mu_k^nT_k^n(t))]}{n},$$

where in the last inequality we use the definition of $\widetilde{W}^n$ in (7) and

$$\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0}|\triangle\widetilde{W}^n(s)|^3\right]\le\left(\frac{1}{\sqrt{n}}\right)^3\cdot\sum_{k\in\mathcal{K}}\mathbb{E}\left[N_k\left(\int_0^t\lambda_k^n(u)du\right)+S_k^n(T_k^n(t))\right]$$
$$=\frac{1}{\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t\lambda_k^n(s)ds+\mu_k^nT_k^n(t))]}{n}.$$

Therefore, using $|\Phi''(x)|\le C$, we deduce from (24) that

$$|\Psi_3(t,\widetilde{\pi}^n,\psi^n)|\le\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n)+\mathbb{E}\left[\frac{C}{2\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right],\tag{29}$$

where

$$\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n)=\left|\mathbb{E}\left[\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\int_0^t\Phi''(\widetilde{W}^n(s-))\frac{1}{2\sqrt{n}}d\widetilde{Y}_k^n(s)\right]\right|$$
$$+\mathbb{E}\left[\int_0^t\sum_{k\in\mathcal{K}}\frac{C}{\mu_k^2}\frac{|\zeta_k^n(s)|}{2\sqrt{n}}ds\right]+\frac{C}{6\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t\lambda_k^n(s)ds+\mu_k^nT_k^n(t))]}{n}.$$

Recall from (9) that

$$\widetilde{V}^n(t,\widetilde{\pi}^n,\psi)=\mathbb{E}\left[\int_0^tc^n(\zeta^n(s))ds+\sum_{k\in\mathcal{K}}\int_0^tg_k(\widetilde{Q}_k^n(s))ds+\sum_{k\in\mathcal{K}}r_k\widetilde{O}_k^n(t)\right],\tag{30}$$

where $c^n(\zeta)=n(\pi(\lambda^*)-\pi(\lambda^*+\zeta/\sqrt{n}))$. Hence, we obtain

$$\mathbb{E}\left[\frac{C}{2\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right]\le\frac{C}{2\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n).$$

It follows that

$$|\Psi_3(t,\widetilde{\pi}^n,\psi^n)|\le\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n)+\frac{C}{\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n).$$

On combining these estimates, we can deduce from (21) that

$$\left(1+\frac{C}{\sqrt{n}}\right)\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n)-\gamma^*t\ge\mathbb{E}\left[\int_0^t(c^n(\zeta^n(s))-\bar{c}(\zeta^n(s)))ds\right]-\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n).\tag{31}$$

We next analyze the terms on the right-hand-side of the above inequality. Using Taylor's theorem and the fact that $\nabla\pi(\lambda^*)=0$, we obtain

$$c^n(\zeta)=n(\pi(\lambda^*)-\pi(\lambda^*+\zeta/\sqrt{n}))=-\zeta'\left(\int_0^1\nabla^2\pi(\lambda^*+\theta\zeta/\sqrt{n})(1-\theta)d\theta\right)\zeta$$
$$=\zeta'H\zeta-\zeta'\left(\int_0^1[\nabla^2\pi(\lambda^*+\theta\zeta/\sqrt{n})-\nabla^2\pi(\lambda^*)](1-\theta)d\theta\right)\zeta,$$

where we recall that $H := -\nabla^2\pi(\lambda^*)/2$. Since $\nabla^2\pi(\cdot)$ is continuous at λ^* and $|\zeta_k^n(s)/\sqrt{n}| \le a_n$ with $a_n \to 0$ as $n \to \infty$, we have for $\bar{c}(\zeta) = \zeta' H \zeta$ and any $\epsilon > 0$,

$$|c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s))| \le \int_0^1 \|\nabla^2\pi(\lambda^* + \theta\zeta^n(s)/\sqrt{n}) - \nabla^2\pi(\lambda^*)\|(1-\theta)d\theta \cdot |\zeta^n(s)|^2 \le \epsilon\|\zeta^n(s)\|^2, \tag{32}$$

for all sufficiently large n. Thus, we have

$$\left|\int_0^t (c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s)))ds\right| \le \epsilon \cdot \int_0^t \|\zeta^n(s)\|^2 ds.$$

Denote the smallest eigenvalue of $H = -\nabla^2\pi(\lambda^*)/2$ by λ_H. We know $\lambda_H > 0$ since H is positive definite. Then (32) implies that

$$c^n(\zeta^n(s)) \ge (\lambda_H - \epsilon)\|\zeta^n(s)\|^2, \quad \text{for } 0 < \epsilon < \lambda_H. \tag{33}$$

By (30), this leads to

$$\mathbb{E}\left[\left|\int_0^t (c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s)))ds\right|\right] \le \frac{\epsilon}{\lambda_H - \epsilon}\widetilde{V}^n(t, \widetilde{\pi}^n, \psi^n). \tag{34}$$

Next, we study the term $\hat{\Psi}_3(t, \widetilde{\pi}^n, \psi^n)$. We show below that

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t}\hat{\Psi}_3(t, \widetilde{\pi}^n, \psi^n) = 0. \tag{35}$$

To this end, we can show the following lemma, the proof of which is deferred to Appendix A.

LEMMA 5. *Assume $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$ for all sufficiently large n, and (27) holds. Then*

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\frac{1}{2\sqrt{n}}\int_0^t \Phi''(\widetilde{W}^n(s-))d\sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\widetilde{Y}_k^n(s)\right] = 0, \tag{36}$$

where we recall that $\widetilde{Y}_k^n(t) = \sqrt{n}(\rho_k t - T_k^n(t))$ for $k \in \mathcal{K}$ and $t \ge 0$.

In addition, by assumption $|\zeta_k^n(s)/\sqrt{n}| \le a_n \to 0$. Hence, we have

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t \sum_{k\in\mathcal{K}} \frac{C}{\mu_k^2}\frac{|\zeta_k^n(s)|}{2\sqrt{n}}ds\right] = 0.$$

Furthermore, note that $\lambda_k^n(s) = n\lambda^* + \sqrt{n}\zeta_k^n(s)$ with $|\zeta_k^n(s)/\sqrt{n}| \le a_n \to 0$, $\mu_k^n = n\mu_k$ and $T_k^n(t) \le t$, we obtain $0 \le \mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t \lambda_k^n(s)ds + \mu_k^n T_k^n(t))]/n \le \lambda_k^* t + a_n t + \mu_k t$. Hence,

$$\lim_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t}\frac{C}{6\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t \lambda_k^n(s)ds + \mu_k^n T_k^n(t))]}{n} = 0.$$

It then follows that (35) holds.

Therefore, we can infer from (31), (34) and (35) that

$$\liminf_{n\to\infty} \liminf_{t\to\infty} \frac{1}{t}\left[\widetilde{V}^n(t, \widetilde{\pi}_n, \psi^n)\left(1 + \frac{\epsilon}{\lambda_H - \epsilon} + \frac{C}{2\sqrt{n}}\right) - \gamma^* t\right] \ge 0,$$

which implies $\liminf_{n\to\infty} \liminf_{t\to\infty} \frac{1}{t}\widetilde{V}^n(t, \widetilde{\pi}_n, \psi^n)\left(1 + \frac{\epsilon}{\lambda_H - \epsilon}\right) \ge \gamma^*$. Sending $\epsilon \to 0+$ yields $\liminf_{n\to\infty} \widetilde{V}^n(\widetilde{\pi}_n, \psi^n) \ge \gamma^*$. This proves (28). Hence, we have proved the desired result. □

5.2.2. Proof of Theorem 1 In this section, we prove that the lower bound γ^* is achieved asymptotically under the sequence of proposed policies $\{\psi_*^n\}$. The following proposition will be essential and its proof can be found in Appendix D. Recall that $\widetilde{Q}_{k*}^n(\infty)$ and $\widetilde{W}_*^n(\infty)$ follow the stationary distributions of $\widetilde{Q}_{k*}^n(\cdot)$ and $\widetilde{W}_*^n(\cdot)$ respectively, under the control policy ψ_*^n.

PROPOSITION 3 **(State-Space Collapse).** *For systems under the proposed policies* $\{\psi_*^n\}$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}_{k*}^n(\infty)) - h\left(l_* \vee (\widetilde{W}_*^n(\infty)\wedge u_*)\right)\right|\right] = 0.$$

Recall the uniqueness of the lifting function $\Delta(\cdot)$ in Lemma 1. From the above proposition, one can expect that $\widetilde{Q}_{k*}^n(\infty)$ is close to $\Delta_k(l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*))$ in L_1-norm, that is, the K-dimensional inventory levels $\widetilde{Q}_*^n(\infty)$ are close to functions of the one-dimensional nominal workload $\widetilde{W}_*^n(\infty)$. Furthermore, because $l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*)$ is bounded, the scaled inventory levels $\widetilde{Q}_{k*}^n(\infty), k\in\mathcal{K}$, are also expected to be bounded asymptotically, even though we only outsource class i^* orders.

Now we prove Theorem 1. We first show that for any stationary distribution $\widetilde{\pi}_*^n$ of $\widetilde{\mathfrak{X}}_*^n(\cdot)$,

$$\lim_{n\to\infty} \widetilde{V}^n(\widetilde{\pi}_*^n, \psi_*^n) = \gamma^*. \tag{37}$$

Suppose the initial distribution of $\widetilde{\mathfrak{X}}_*^n(\cdot)$ is $\widetilde{\pi}_*^n$, and in the following we use $\mathbb{E}$ to denote $\mathbb{E}_{\widetilde{\pi}_*^n}$. From Lemma 3, we have

$$0 = \Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n) + \Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n) + \Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n).$$

We first compute $\Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n)$ under the policy ψ_*^n. For the function Φ in Lemma 2, one can readily verify, using the expression of ζ_*^n in (19) and $\bar{c}(\zeta) = \zeta' H\zeta$, that $-\sum_{k\in\mathcal{K}} \frac{\zeta_{k*}^n(s)}{\mu_k}\Phi'(x) + \sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k^2}\Phi''(x) + \bar{c}(\zeta_*^n(s)) + h\left(l_*\vee(x\wedge u_*)\right) = \gamma^*$ for $x\le l_*$ and $x\ge u_*$. Hence

$$-\sum_{k\in\mathcal{K}} \frac{\zeta_{k*}^n(s)}{\mu_k}\Phi'(\widetilde{W}_*^n(s)) + \sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k^2}\Phi''(\widetilde{W}_*^n(s)) + \bar{c}(\zeta_*^n(s)) + h\left(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*)\right) = \gamma^*,$$

where we recall from the proof of Theorem 2 that $\bar{c}(\zeta) = \zeta' H\zeta$ for $\zeta\in\mathbb{R}^K$ and $\bar{c}(\cdot)$ serves as a quadratic approximation of the function $c^n(\cdot)$ given in (8). Then we can deduce from (22) that

$$\Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n) = \gamma^* t - \mathbb{E}\left[\int_0^t \bar{c}(\zeta_*^n(s))ds + \int_0^t h\left(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*)\right)ds\right].$$

Next we show that under the policy ψ_*^n, we have

$$\Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n) = -\sum_{k\in\mathcal{K}} r_k \widetilde{O}_{k*}^n(t), \tag{38}$$

Under the policy ψ_*^n, when $\widetilde{\mathbb{O}}_*^n$ jumps (i.e., an order is outsourced), we have $\widetilde{W}_*^n(t)\le l_*$; On the other hand, when $\widetilde{I}_*^n(s)$ increases (i.e., the system stops production and becomes idle), we have $\sum_{k\in\mathcal{K}} \frac{(Q_{k*}^n(t))^+}{\mu_k} \ge \sqrt{n}u_*$ and $Q_{k*}^n(t)\ge \alpha_k > 0$ for all $k\in\mathcal{K}$, and hence $W_*^n(t) = \sum_{k\in\mathcal{K}} \frac{Q_{k*}^n(t)}{\mu_k} \ge \sqrt{n}u_*$. Also note that $\Phi'(x) = -\kappa$ for $x\le l_*$, and $\Phi'(x) = 0$ for $x\ge u_*$. Hence,

$$\Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n) = \mathbb{E}\left[\int_0^t \Phi'(\widetilde{W}_*^n(s-))d\widetilde{\mathbb{O}}_*^n(s) - \int_0^t \Phi'(\widetilde{W}_*^n(s-))d\widetilde{I}_*^n(s)\right] = -\kappa\cdot\mathbb{E}[\widetilde{\mathbb{O}}_*^n(t)].$$

In addition, note that under the proposed policy ψ_*^n, only class i^* orders will be outsourced. Using the definition of $\kappa = \frac{r_{i^*}}{m_{i^*}}$ we have

$$\kappa\cdot\widetilde{\mathbb{O}}_*^n(t) = \kappa\cdot\widetilde{O}_{i^**}^n(t)m_{i^*} = r_{i^*}\widetilde{O}_{i^**}^n(t) = \sum_{k\in\mathcal{K}} r_k\widetilde{O}_{k*}^n(t).$$

Hence we obtain (38).

Finally, for $\Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n)$, as in (29) we have

$$|\Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n)| \leq \hat{\Psi}_3(t,\widetilde{\pi}_*^n,\psi_*^n) + \frac{C}{\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n).$$

On combining these estimates, we can deduce from (21) that

$$\begin{aligned}\left(1-\frac{C}{\sqrt{n}}\right)\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n) - \gamma^* t \leq \mathbb{E}\left[\int_0^t (c^n(\zeta_*^n(s)) - \bar{c}(\zeta_*^n(s)))ds\right] + \hat{\Psi}_3(t,\widetilde{\pi}_*^n,\psi_*^n) \\ + \mathbb{E}\left[\left|\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_{k*}^n(s))ds - \int_0^t h(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*))ds\right|\right]\end{aligned}$$

For the last term in the above inequality, we apply Proposition 3 to obtain

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\left|\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_{k*}^n(s))ds - \int_0^t h(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*))ds\right|\right] = 0.$$

For the other terms, we can control them similarly as in the proof of Theorem 2 (indeed, ζ_*^n is bounded now, which even simplifies the arguments). Then

$$\lim_{n\to\infty}\widetilde{V}^n(\widetilde{\pi}_*^n,\psi_*^n) = \lim_{n\to\infty}\liminf_{t\to\infty}\frac{1}{t}\left[\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n)\right] \leq \gamma^*.$$

On combining with the lower bound result in Theorem 2, we obtain (37).

Finally, for n large enough, under ψ_*^n and any initial distribution $\widetilde{\nu}^n$, there is a stationary distribution $\widetilde{\pi}_*^n$ of $\widetilde{\mathfrak{X}}_*^n(\cdot)$ such that

$$\widetilde{V}^n(\widetilde{\nu}^n,\psi_*^n) = \liminf_{t\to\infty}\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi_*^n) = \widetilde{V}^n(\widetilde{\pi}_*^n,\psi_*^n) = \gamma^*.$$

The proof is therefore complete. □

Proof of Lemma 3. We apply Ito's formula to the semimartingale $\widetilde{W}^n$ in (7) with the function Φ in Lemma 2 to obtain

$$\begin{aligned}\Phi(\widetilde{W}^n(t)) =& \Phi(\widetilde{W}^n(0)) + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s) + \sum_{s\leq t:|\triangle\widetilde{W}^n(s)|>0}\left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{W}^n(s))^2\right) \\ &+ \sum_{s\leq t:|\triangle\widetilde{W}^n(s)|>0}\left(\triangle\Phi(\widetilde{W}^n(s)) - \Phi'(\widetilde{W}^n(s-))\triangle\widetilde{W}^n(s) - \frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{W}^n(s))^2\right).\end{aligned} \tag{39}$$

From the dynamics of $\widetilde{W}^n$ in (7),

$$\begin{aligned}\sum_{s\leq t:|\triangle\widetilde{W}^n(s)|>0}\left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{W}^n(s))^2\right) =& \sum_{s\leq t:|\triangle\widetilde{\mathbb{X}}^n(s)|>0}\left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right) \\ &- \sum_{s\leq t:|\triangle\widetilde{\mathbb{O}}^n(s)|>0}\left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right).\end{aligned}$$

This equality holds because the epochs at which the process $\widetilde{W}^n$ jumps constitute a subset of those at which $\widetilde{\mathbb{X}}^n$ jumps, and when there is an arrival of customer order which is outsourced (i.e., $\widetilde{\mathbb{O}}^n$ jumps up), the process $\widetilde{\mathbb{X}}^n$ jumps but $\widetilde{W}^n$ does not.

Additionally, using Taylor expansion, we have

$$\triangle\Phi(\widetilde{W}^n(s)) - \Phi'(\widetilde{W}^n(s-))\triangle\widetilde{W}^n(s) - \frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{W}^n(s))^2 = \frac{1}{6}\Phi'''(\Delta_{\frac{1}{\sqrt{n}}}(\widetilde{W}^n(s)))(\triangle\widetilde{W}^n(s))^3,$$

where $\Delta_{\frac{1}{\sqrt{n}}}(\widetilde{W}^n(s)) \in (\min\{\widetilde{W}^n(s), \widetilde{W}^n(s-)\}, \max\{\widetilde{W}^n(s), \widetilde{W}^n(s-)\})$. Hence, from (39)

$$\begin{aligned}\Phi(\widetilde{W}^n(t)) =& \Phi(\widetilde{W}^n(0)) + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s) + \sum_{s\le t:|\triangle\widetilde{\mathbb{X}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right) \\ &- \sum_{s\le t:|\triangle\widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right) + \sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0} \frac{1}{6}\Phi'''(\Delta_{\frac{1}{\sqrt{n}}}(\widetilde{W}^n(s)))(\triangle\widetilde{W}^n(s))^3.\end{aligned} \tag{40}$$

From the dynamics of $\widetilde{W}^n$ in (7), and using the martingale property of $\widetilde{\mathbb{X}}^n$ and the boundedness of Φ' in (18), we can readily infer that

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s)\right] \\ &= \mathbb{E}\left[-\int_0^t \sum_{k\in\mathcal{K}} \frac{\zeta_k(s)}{\mu_k}\Phi'(\widetilde{W}^n(s-))ds + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{\mathbb{O}}^n(s) - \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{I}^n(s)\right].\end{aligned}$$

Moreover, since $\widetilde{\mathbb{X}}^n(t)$ is a linear combination of $\widetilde{X}_k^n(t)$, where the jump magnitude of $\widetilde{X}_k^n$ is $1/\sqrt{n}$ whenever there is a jump, and Φ'' is bounded from (18), we then have

$$\begin{aligned}&\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{\mathbb{X}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right)\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2}\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{X}_k^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{X}_k^n(s))^2\right)\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2} \times \frac{1}{n}\mathbb{E}\left[\int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))dN_k\left(\int_0^s \lambda_k^n(u)du\right) + \int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))dS_k^n(T_k^n(s))\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2} \times \mathbb{E}\left[\int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))\left(\frac{\lambda_k^n(s)+n\lambda_k^*}{n}ds - \frac{\mu_k}{\sqrt{n}}d\widetilde{Y}_k^n(s)\right)\right].\end{aligned}$$

Recall that the system is assumed to start from stationarity. Hence we can take expectation on both sides of (40), use the stationarity of $\widetilde{W}^n(\cdot)$, and $\frac{\lambda_k^n(s)+n\lambda_k^*}{n} = 2\lambda_k^* + \frac{1}{\sqrt{n}}\zeta_k^n(s)$ to obtain (21). The proof is complete. ☐

6. Conclusions and Future Research In this paper, we consider the optimal control of a multi-class make-to-stock system where the manager makes pricing, outsourcing, and scheduling decisions to maximize the long-run average profit. We propose a policy and establish the asymptotic optimality of the proposed policies in the heavy-traffic regime.

For future research, it would be interesting to extend our current work to a network setting. For that, one may focus on network structures satisfying the complete resource pooling conditions, as in Mandelbaum and Stolyar (2004). When constructing Lyapunov functions, we use the property that the machine would not be idle when some orders are outsourced. In a network, some servers' capacity may be wasted when some orders are outsourced, which would complicate the analysis. One may restrict the holding/waiting cost functions to be strictly convex as in Mandelbaum and Stolyar (2004), and that might ensure that there is a negligible likelihood of simultaneously wasting capacity and outsourcing orders.

Appendix. Proofs

A. Proofs of Auxiliary Lemmas Within the Proof of Theorem 2

A.1. Proof of Lemma 4 To prove Lemma 4, we can first infer from (26) and the definition of $\widetilde{V}^n(\widetilde{\nu}^n,\psi^n)$ in (10) that under policy ψ^n, there exists a sequence $\{t_i\}$ such that

$$\lim_{t_i\to\infty}\frac{\widetilde{V}^n(t_i,\widetilde{\nu}^n,\psi^n)}{t_i}=\liminf_{t\to\infty}\frac{\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n)}{t}\le\gamma^*+2. \tag{41}$$

For both strictly convex and linear functions $g_k, k\in\mathcal{K}$, there exists a constant $c>0$ such that $g_k(x)>c|x|$, for $|x|>1$. As a result, from (41), there exists a constant C (independent of n) such that $\limsup_{t_i\to\infty}\frac{\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^{t_i}\sum_{k\in\mathcal{K}}|\widetilde{Q}_k^n(s)|ds\right]}{t_i}\le C$. By Markov inequality, this implies for $\epsilon>0$,

$$\limsup_{t_i\to\infty}\frac{1}{t_i}\int_0^{t_i}\mathbb{P}_{\widetilde{\nu}^n}\left(\sum_{k\in\mathcal{K}}|\widetilde{Q}_k^n(s)|\ge C/\epsilon\right)ds\le\epsilon. \tag{42}$$

Hence if we define $H_{t_i}(\cdot):=\frac{1}{t_i}\int_0^{t_i}\mathbb{P}_{\widetilde{\nu}^n}(\widetilde{\mathfrak{X}}^n(s)\in\cdot)ds$, then from (42) and the definition of $\widetilde{\mathfrak{X}}^n(t)=(\widetilde{Q}_1^n(t),\dots,\widetilde{Q}_K^n(t),C(t))$, the sequence of probability measures $\{H_{t_i}:t_i>0\}$ is tight. Note that $\widetilde{\mathfrak{X}}^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n=\frac{1}{\sqrt{n}}\mathbb{Z}^n\times(\mathcal{K}\cup\{0\})$, we can assume that every function is continuous on $\widetilde{S}^n$ (endowed with discrete topology), and it follows that $\widetilde{\mathfrak{X}}^n(\cdot)$ satisfies the Feller property, i.e., $\mathbb{E}[f(\widetilde{\mathfrak{X}}^n(t)|\widetilde{\mathfrak{X}}^n(0)=\mathfrak{x}]$ is a bounded and continuous function of $\mathfrak{x}$ for all t whenever f is bounded and continuous. We can then follow the proof of the Krylov-Bogoliubov theorem (Prato 2006, Theorem 7.1) and infer that there exists a stationary distribution $\widetilde{\pi}^n$ for $\widetilde{\mathfrak{X}}^n(\cdot)$ and moreover, a subsequence of probability measures $\{H_{t_i}:t_i>0\}$ converges weakly to this invariant measure $\widetilde{\pi}^n$ as $t_i\to\infty$. For notational simplicity, in the following we use $\{t_i\}$ to denote this further subsequence. From (9) and the property of stationary distributions we can infer that

$$\begin{aligned}\widetilde{V}^n(\widetilde{\pi}^n,\psi^n)=&\liminf_{t\to\infty}\frac{1}{t}\mathbb{E}_{\widetilde{\pi}^n}\left[\int_0^t c^n(\zeta^n(\widetilde{Q}^n(s)))ds+\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}r_k\int_0^t\xi_k(s-)\lambda_k^n(s)ds\right]\\ =&\mathbb{E}_{\widetilde{\pi}^n}\left[c^n(\zeta^n(\widetilde{Q}^n(0)))+\sum_{k\in\mathcal{K}}g_k(\widetilde{Q}_k^n(0))+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}r_k\xi_k(\widetilde{\mathfrak{X}}^n(0))\lambda_k^n(\widetilde{\mathfrak{X}}^n(0))\right].\end{aligned}$$

In the first equality above we use $\widetilde{O}_k^n(t)=\frac{1}{\sqrt{n}}\int_0^t\xi_k(s-)dE_k^n(s)=\frac{1}{\sqrt{n}}\int_0^t\xi_k(s-)\lambda_k^n(s)ds+\frac{1}{\sqrt{n}}\int_0^t\xi_k(s-)d(E_k^n(s)-\int_0^s\lambda_k^n(u)du)$ and the fact that $\frac{1}{\sqrt{n}}\int_0^t\xi_k(s-)d(E_k^n(s)-\int_0^s\lambda_k^n(u)du)$ is a martingale. From $H_{t_i}\Rightarrow\widetilde{\pi}^n$ and the Fatou's lemma, we have

$$\begin{aligned}\widetilde{V}^n(\widetilde{\pi}^n,\psi^n)\le&\liminf_{t_i\to\infty}\frac{1}{t_i}\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^{t_i}c^n(\zeta^n(\widetilde{Q}^n(s)))ds+\sum_{k\in\mathcal{K}}\int_0^{t_i}g_k(\widetilde{Q}_k^n(s))ds+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}r_k\int_0^t\xi_k(s-)\lambda_k^n(s)ds\right]\\ =&\widetilde{V}^n(\widetilde{\nu}^n,\psi^n)\le\widetilde{V}^n+\epsilon\le\gamma^*+2,\end{aligned}$$

where the last display is due to (26). Hence, we have proved (27). □

A.2. Proof of Lemma 5 Note that the integral in (36) is a Lebesgue integral and $\widetilde{W}^n$ can only have finite jumps almost surely, it is enough to prove

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\frac{1}{2\sqrt{n}}\int_0^t\Phi''(\widetilde{W}^n(s))d\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\widetilde{Y}_k^n(s)\right]=0.$$

We now first show that for each $k\in\mathcal{K}$,

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t\Phi''(\widetilde{W}^n(s))dT_k^n(s)\right]=\mathbb{E}\left[\int_0^1\Phi''(\widetilde{W}^n(s))dT_k^n(s)\right], \tag{43}$$

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t\Phi''(\widetilde{W}^n(s))d(\rho_k s)\right]=\mathbb{E}\left[\int_0^1\Phi''(\widetilde{W}^n(s))d(\rho_k s)\right]. \tag{44}$$

To see this, using the facts that $|\Phi''(x)| \le C$ and $T_k^n(t) - T_k^n(s) \le t-s$ for $0 \le s \le t$, we have

$$\left| \mathbb{E}\left[\frac{1}{t}\int_0^{\lfloor t \rfloor} \Phi''(\widetilde{W}^n(s))dT_k^n(s) \right] - \mathbb{E}\left[\frac{1}{t}\int_0^{t} \Phi''(\widetilde{W}^n(s))dT_k^n(s) \right] \right| \le \frac{C(t-\lfloor t \rfloor)}{t} \le \frac{C}{t},$$

where $\lfloor t \rfloor$ denotes the floor operator. Because $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, we have $\mathbb{E}[\int_0^{\lfloor t \rfloor} \Phi''(\widetilde{W}^n(s))dT_k^n(s)] = \lfloor t \rfloor \cdot \mathbb{E}[\int_0^1 \Phi''(\widetilde{W}^n(s))dT_k^n(s)]$. Then (43) readily follows. Similarly we can obtain (44). Hence, using the definition of $\widetilde{Y}_k^n$ we have

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\frac{1}{\sqrt{n}}\int_0^t \Phi''(\widetilde{W}^n(s-))d\widetilde{Y}_k^n(s) \right] = \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))d(\rho_k s) \right] - \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))dT_k^n(s) \right].$$

Second, we show that the process $\{T_k^n(t) - \rho_k t : t\in[0,1]\}$ converges weakly to zero on $C[0,1]$ as $n\to\infty$. Because $\rho_k \le 1$ and $T_k^n(t) - T_k^n(s) \le t-s$ for $0\le s\le t\le 1$, $\{T_k^n(t)-\rho_k t : t\in[0,1]\}$ is tight on $C[0,1]$. Thus it is enough to prove $T_k^n(t) - \rho_k t \Rightarrow 0$ for each $t\in[0,1]$, which is true if

$$\mathbb{E}[|T_k^n(t) - \rho_k t|] \to 0, \quad \text{as } n\to\infty. \tag{45}$$

To prove (45), note that $T_k^n(t) - \rho_k t = -\widetilde{Y}_k^n(t)/\sqrt{n}$. Then from (5) we can compute

$$\mathbb{E}\left[|T_k^n(t) - \rho_k t|\right] \le \frac{1}{\mu}\mathbb{E}\left[\left|\frac{\widetilde{Q}_k^n(t)}{\sqrt{n}}\right| + \left|\frac{\widetilde{Q}_k^n(0)}{\sqrt{n}}\right| + \left|\frac{\widetilde{X}_k^n(t)}{\sqrt{n}}\right| + \left|\int_0^t \frac{\zeta_k^n(s)}{\sqrt{n}}ds\right| + \left|\frac{\widetilde{O}_k^n(t)}{\sqrt{n}}\right| \right].$$

The first, second, and fifth terms on the right-hand side of the above inequality all converge to 0 as $n\to\infty$ because $\widetilde{V}^n(\widetilde{\pi}^n,\psi^n) < \gamma^*$ by our assumption. The fourth term converges to 0 because $|\frac{\zeta^n(s)}{\sqrt{n}}| \le a_n \to 0$. The third term converges to 0 because

$$\mathbb{E}\left[\left|\frac{\widetilde{X}_k^n(t)}{\sqrt{n}}\right|\right] \le \mathbb{E}\left[\sup_{0\le u\le 1}\left|\frac{S_k^n(u)-\mu_k^n u}{n}\right|\right] + \mathbb{E}\left[\sup_{0\le u\le 1}\left|\frac{N_k(\int_0^u \lambda_k^n(s)ds) - \int_0^u \lambda_k^n(s)ds}{n}\right|\right] \to 0.$$

Thus, we obtain (45). To complete the proof, we also need the following lemma.

Lemma 6. *Assume $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$ for all sufficiently large n and (27) holds. Then the sequence $\{\widetilde{W}^n(\cdot): n\ge 1\}$ is tight.*

By Lemma 6, for any subsequence, there exists a further subsequence $\{n_i\}$ of $\{n\}$ such that $\widetilde{W}^{n_i} \Rightarrow \widetilde{W}$ on $D[0,1]$ for some $\widetilde{W}$. For this subsequence $\{n_i\}$, we also have $T_k^{n_i}(\cdot) \Rightarrow \rho_k\cdot$ as $n_i\to\infty$. Since $T_k^{n_i}(\cdot)$ is non-decreasing with $T_k^{n_i}(1)\le 1$, and Φ'' is continuous and bounded, we can apply Lemma 8.3 of Dai and Dai (1999) and the dominated convergence theorem to obtain

$$\lim_{n_i\to\infty} \mathbb{E}\int_0^1 \Phi''(\widetilde{W}^{n_i}(s))dT_k^{n_i}(s) = \mathbb{E}\int_0^1 \Phi''(\widetilde{W}(s))d(\rho_k s) = \lim_{n_i\to\infty}\mathbb{E}\int_0^1 \Phi''(\widetilde{W}^{n_i}(s))d(\rho_k s).$$

It follows that (36) holds for the subsequence $\{n_i\}$. Because (36) holds for any subsequence, it also holds for the whole sequence. □

Proof of Lemma 6. We use the tightness criteria in (Jacod and Shiryaev 2013, Chapter VI. Theorem 4.1), and verify the three conditions there. From (27), $\mathbb{P}(|\widetilde{Q}^n(0)| > C/\epsilon) \le \epsilon$ for all n sufficiently large. Hence the sequence $\{\widetilde{Q}^n(0)\}$ is tight, and Condition (i), i.e., tightness of $\{\widetilde{W}^n(0)\}$ holds. To verify Conditions (ii) and (iii), it suffices to check the following two conditions: there exists a sequence of bounded numbers $\{C_n\}$ such that for all n sufficiently large and for any $0\le s<r<t$,

$$\mathbb{E}[|\widetilde{W}^n(t) - \widetilde{W}^n(s)|] \le C_n(t-s), \tag{46}$$

$$\mathbb{E}\left[|\widetilde{W}^n(r) - \widetilde{W}^n(s)| \times |\widetilde{W}^n(t) - \widetilde{W}^n(r)|\right] \le C_n(t-s)^2. \tag{47}$$

We first verify (46). From the dynamics of $\widetilde{W}^n$ in (7) we have

$$\widetilde{W}^n(t)-\widetilde{W}^n(s)=\sum_{k\in\mathcal{K}}\frac{\widetilde{X}_k^n(t)-\widetilde{X}_k^n(s)}{\mu_k}-(\widetilde{I}^n(t)-\widetilde{I}^n(s))-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du+\sum_{k\in\mathcal{K}}\frac{\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)}{\mu_k}.$$

Since $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, it follows that $\mathbb{E}[\widetilde{W}^n(t)-\widetilde{W}^n(s)]=0$. Also note that $\mathbb{E}[\sum_{k\in\mathcal{K}}\frac{\widetilde{X}_k^n(t)-\widetilde{X}_k^n(s)}{\mu_k}]=0$. As a result,

$$\mathbb{E}[\widetilde{I}^n(t)-\widetilde{I}^n(s)]=\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]+\sum_{k\in\mathcal{K}}\frac{\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]}{\mu_k}.$$

Because the system is stationary when $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, we can then obtain $\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]=a_k^n(t-s)$, where a_k^n are non-negative and bounded (because of (27)). Similarly, we have $\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]=b^n\cdot(t-s)$, where $\{b^n\}$ is a sequence of bounded numbers. To see the boundedness, we can compute that

$$\left|\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]\right|\le\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|}{\mu_k}du\right]\le(t-s)\sqrt{\frac{\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|^2}{\mu_k^2}du\right]}{t-s}}\le C(t-s),$$

where C is a generic constant, the second inequality above follows from Cauchy-Schwarz inequality and the last inequality is due to (33) and (27). From these estimates, we then infer that

$$\begin{aligned}\mathbb{E}[|\widetilde{W}^n(t)-\widetilde{W}^n(s)|]&\le\mathbb{E}\left[\sum_{k\in\mathcal{K}}\frac{|\widetilde{X}_k^n(t)-\widetilde{X}_k^n(s)|}{\mu_k}\right]+2\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|}{\mu_k}du\right]+2\sum_{k\in\mathcal{K}}\frac{\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]}{\mu_k}\\&\le C_n\cdot(t-s),\end{aligned}\tag{48}$$

where the sequence of real numbers $\{C_n\}$ is bounded. Hence, we have proved (46).

Next, we prove (47). Let $\mathcal{F}_r^n$ be the natural filtration generated by $\mathfrak{X}^n$ till time r. Then

$$\begin{aligned}\mathbb{E}\left[|\widetilde{W}^n(r)-\widetilde{W}^n(s)|\times|\widetilde{W}^n(t)-\widetilde{W}^n(r)|\right]&=\mathbb{E}\left[|\widetilde{W}^n(r)-\widetilde{W}^n(s)|\times\mathbb{E}\left[|\widetilde{W}^n(t)-\widetilde{W}^n(r)|\Big|\mathcal{F}_r^n\right]\right]\\&\le\mathbb{E}\left[|\widetilde{W}^n(r)-\widetilde{W}^n(s)|\times C_n(t-r)\right]\\&\le C_n^2(t-r)(r-s)\le C_n^2(t-s)^2,\end{aligned}$$

where the first inequality follows from (48). Hence, the tightness of $\{\widetilde{W}^n(\cdot):n\ge1\}$ follows. $\square$

B. Proof of Proposition 1 We use the Foster-Lyapunov criteria, see e.g., (Meyn and Tweedie 1993, Theorem 4.5). Under the policy $\psi_\diamond^n$, the process $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n=\frac{1}{\sqrt{n}}\mathbb{Z}^n\times(\mathcal{K}\cup\{0\})$. In addition, it satisfies the Feller property, by using a similar argument as in the proof of Lemma 4. Write $G^n=\left(G_{x,x'}^n\right)_{x,x'\in\widetilde{S}^n}$ for the generator matrix of $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$. The Foster-Lyapunov criteria states that if there exist a function $V:\widetilde{S}^n\to\mathbb{R}_+$ where $V(x)\to\infty$ as $|x|\to\infty$, and a constant $r>0$ such that

$$G^nV(x):=\sum_{x'\in\widetilde{S}^n}G_{x,x'}^n(V(x')-V(x))\le-1,\quad\text{for }x\in S\text{ with }\sum_{k\in\mathcal{K}}\frac{|x_k|}{\mu_k}>r,\tag{49}$$

then there exists a stationary distribution for the Markov chain $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$.

We next construct the Lyapunov function V which satisfies the drift inequality in (49). Let Υ be a positive constant such that

$$\Upsilon\cdot\frac{\lambda_i^*}{\mu_i}\ge\sum_{k\ne i}\frac{\lambda_k^*}{\mu_k}+1,\quad\text{for all }i\in\mathcal{K}.\tag{50}$$

This is feasible by picking $\Upsilon \geq \frac{2}{\min_k\left\{\frac{\lambda_k^*}{\mu_k}\right\}} - 1$. Now for $x = (x_1, \dots, x_K, \xi) \in S$ we define

$$
\begin{aligned}
V_0(x) &= 1_{\left\{\xi \in \{k \in \mathcal{K} : x_k \geq \alpha_k/\sqrt{n}\}\right\}}, && (51)\\
V_1(x) &= \Upsilon \cdot \sum_{k \in \mathcal{K}} \frac{(x_k - \alpha_k/\sqrt{n})^+}{\mu_k} + \sum_{k \in \mathcal{K}} \frac{(x_k - \alpha_k/\sqrt{n})^-}{\mu_k}, && (52)\\
V(x) &= V_0(x) + V_1(x), && (53)
\end{aligned}
$$

where we recall that ξ denotes the index of the customer class that the system is working on. Pick $r > 2\max\{|u_\diamond|, |l_\diamond|\}$. We verify (49) by considering $x \in \widetilde{S}^n$ with $\sum_{k \in \mathcal{K}} \frac{|x_k|}{\mu_k} > r$ and discussing several cases separately. We use e_k to denote a vector in which all elements are zeros except for the kth element, which equals one.

Case (1): $V_0(x) = 1$ for $x = (x_1, \dots, x_K, \xi)$, i.e., the inventory level of the product class in service exceeds the safety-stock level. We study the following two subcases.

Case (1a): $\sum_{k \in \mathcal{K}} \frac{x_k^+}{\mu_k} > u_\diamond$. In this case, consider that the Markov chain $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$ transits to a new state $x' = (x_1', \dots, x_K', \xi')$ from the current state $x = (x_1, \dots, x_K, \xi)$. If the state transition is due to an order arrival, then $V_0(x') \leq V_0(x)$ by the definition of $V_0(\cdot)$; on the other hand, if the state transition is due to a production completion, then x_ξ will increase by one, and as a result the indicator function $V_0(x') = 0$ and $\xi' \neq \xi$ since the system will not work on a class with $x_k' \geq \alpha_k/\sqrt{n}$ when $\sum_{k \in \mathcal{K}} \frac{(x_k')^+}{\mu_k} > u_\diamond$ according to the policy $\psi_\diamond^n$. Hence, we have

$$
\begin{aligned}
G^n V(x) &= \sum_{k \in \mathcal{K}} \lambda_k^n(x) \times \left(V(x - \frac{e_k}{\sqrt{n}}) - V(x)\right) + \mu_\xi^n \left(V\left(x + \frac{e_\xi}{\sqrt{n}} + e_{K+1}(\xi' - \xi)\right) - V(x)\right)\\
&\leq \sum_{k \in \mathcal{K}} \lambda_k^n(x) \times \left(V_1(x - \frac{e_k}{\sqrt{n}}) - V_1(x)\right) + \mu_\xi^n \left(-1 + V_1\left(x + \frac{e_\xi}{\sqrt{n}}\right) - V_1(x)\right)\\
&\leq \sum_{k \in \mathcal{K}} \frac{\lambda_k^n(x)}{\mu_k \sqrt{n}} + \Upsilon \frac{\mu_\xi^n}{\sqrt{n}} - \mu_\xi^n = \sum_{k \in \mathcal{K}} \frac{\sqrt{n}\lambda_k^* + \zeta_k(x)}{\mu_k} + \Upsilon \sqrt{n}\mu_\xi - n\mu_\xi \leq -1,
\end{aligned}
$$

for all sufficiently large n. In the above we use $\lambda_k^n(x) = n\lambda_k^* + \sqrt{n}\zeta_k^n(x)$ where $|\zeta_k^n(x)/\sqrt{n}| \leq a_n \to 0$, and $\mu_k^n = n\mu_k$ for each $k \in \mathcal{K}$.

Case (1b): $\sum_{k \in \mathcal{K}} \frac{x_k^+}{\mu_k} \leq u_\diamond$. In this case, since $\sum_{k \in \mathcal{K}} \frac{|x_k|}{\mu_k} > r$, then $\sum_{k \in \mathcal{K}} \frac{x_k^-}{\mu_k} = \sum_{k \in \mathcal{K}} \frac{|x_k|}{\mu_k} - \sum_{k \in \mathcal{K}} \frac{x_k^+}{\mu_k} > r - u_\diamond > 0$. For policy $\psi_\diamond^n$, when there is a class with customers waiting, the system would not start to produce a new product from a class with an inventory level higher than the corresponding safety stock. Hence if the state transition is due to a production completion, then for the new state $x' = (x_1', \dots, x_K', \xi')$, we still have $\xi' \neq \xi$, and $V_0(x') = 0$. Following a similar computation as in Case (1a), we obtain that $G^n V(x) \leq -1$ for all sufficiently large n.

Case (2): $V_0(x) = 0$ for $x = (x_1, \dots, x_K, \xi)$. In this case, from the definition of V_0 in (51) we know that there is either no product in service ($\xi = 0$) or the class index ξ of the product in service satisfies $x_\xi < \alpha_\xi/\sqrt{n}$. Suppose the system state x transits to x' at the next event, which can be either an order arrival or a production completion. We argue that $V_0(x') = 0$ according to the policy $\psi_\diamond^n$. To see this, we first consider the case where the state transition is due to an order arrival. Note that if the system is occupied before an order arrival, the arrival of any class k will not change ξ (the class in service) and will reduce x_k by $1/\sqrt{n}$, so V_0 remains at zero at the new state x'. If the system is idle before the arrival, then $x_k \geq 0$ for each k (i.e., no order is waiting) and we have $\sum_{k \in \mathcal{K}} \frac{x_k^+}{\mu_k} = \sum_{k \in \mathcal{K}} \frac{|x_k|}{\mu_k} > r$. According to the scheduling part of the policy $\psi_\diamond^n$, after the new arrival, the system only chooses from the classes with $x_k' < \alpha_k/\sqrt{n}$, hence we also have $V_0(x') = 0$ by (51). Next we consider the case where the state transition is due to a production completion. Since $\sum_{k \in \mathcal{K}} \frac{|x_k|}{\mu_k} > r$, we obtain that either $\sum_{k \in \mathcal{K}} \frac{x_k^+}{\mu_k} > r/2 > u_\diamond$ or $\sum_{k \in \mathcal{K}} \frac{x_k^-}{\mu_k} > r/2 > |l_\diamond|$. In the former case, we have $V_0(x') = 0$ from the previous discussion. In the latter case, there is at least one class with customers waiting. By the scheduling part of the policy, the system will not work on classes with inventory level higher than the corresponding safety stock level α_k. Hence we also have $V_0(x')=0$. In summary, the indicator function V_0 does not change its value when there is a state transition from x to x' if $V_0(x)=0$.

We next consider two subcases in order to compute $G^nV(x)$ when $\sum_{k\in\mathcal{K}}\frac{|x_k|}{\mu_k}>r$.

Case (2a): $x_k\geq\alpha_k/\sqrt{n}$ for all k. Then the system is idle under the proposed policy. Hence only an order arrival can change the state and using the definition of V in (53) we have

$$\begin{aligned}
G^nV(x) &= \sum_{k:x_k\geq\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)\\
&= \sum_{k:x_k>\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)+\sum_{k:x_k=\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)\\
&= -\Upsilon\sum_{k:x_k>\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}+\sum_{k:x_k=\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\\
&\leq -\sqrt{n}+\sum_{k\in\mathcal{K}}\frac{a_n}{\mu_k}\leq -1,\quad \text{provided that } n \text{ is sufficiently large.}
\end{aligned}$$

In the last inequality we use the fact that $\lambda_k^n(x)=n\lambda_k^*+\sqrt{n}\zeta_k^n(x)$ where $|\zeta_k^n(x)/\sqrt{n}|\leq a_n\to 0$, and the inequality $\Upsilon\frac{\lambda_i^*}{\mu_i}\geq\sum_{k\neq i}\frac{\lambda_k^*}{\mu_k}+1$ for all $i\in\mathcal{K}$.

Case (2b): There exists a class k_0 such that $x_{k_0}<\alpha_{k_0}/\sqrt{n}$ and the system is working on this class. We consider the following two situations.

i). If $x_k\leq\alpha_k/\sqrt{n}$ for all k, then the arrivals of class $i^\diamond$ orders are outsourced according to the policy. This is because if $\sum_{k\in\mathcal{K}}\frac{|x_k|}{\mu_k}>r$ and $x_k\leq\alpha_k/\sqrt{n}$ for all $k\in\mathcal{K}$, we must have $\sum_{k\in\mathcal{K}}\frac{x_k}{\mu_k}<l_\diamond$. Hence using (53) and the fact that $\sum_{k\in\mathcal{K}}\frac{\lambda_k^*}{\mu_k}=1$, we can compute

$$\begin{aligned}
G^nV(x) &= \mu_{k_0}^n\left(V_1(x+\frac{e_{k_0}}{\sqrt{n}})-V_1(x)\right)+\sum_{k\neq i^\diamond}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)\\
&= -\left(\sqrt{n}-\sum_{k\neq i^\diamond}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\right)\leq -\frac{\lambda_{i^\diamond}^*}{\mu_{i^\diamond}}\sqrt{n}+\sum_{k\neq i^\diamond}\frac{a_n}{\mu_k}\leq -1,\quad \text{for } n \text{ sufficiently large.}
\end{aligned}$$

ii). Otherwise, there exist a class k_1 such that $x_{k_1}>\alpha_{k_1}/\sqrt{n}$. Let $\underline{\mathcal{K}}(x)=\mathcal{K}$ if class $i^\diamond$ orders should not be outsourced, and $\underline{\mathcal{K}}(x)=\mathcal{K}\backslash\{i^\diamond\}$ if class $i^\diamond$ orders should be outsourced. Because $x_{k_1}>\alpha_{k_1}/\sqrt{n}$, $k_1\in\underline{\mathcal{K}}(x)$. We can obtain from (50) and (53) that

$$\begin{aligned}
G^nV(x) &= \mu_{k_0}^n\left(V_1(x+\frac{e_{k_0}}{\sqrt{n}})-V_1(x)\right)+\sum_{k\in\underline{\mathcal{K}}(x)}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)\\
&\leq \sum_{k\in\underline{\mathcal{K}}(x):x_k>\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)+\sum_{k\in\underline{\mathcal{K}}(x):x_k\leq\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\left(V_1(x-\frac{e_k}{\sqrt{n}})-V_1(x)\right)\\
&\leq -\Upsilon\sum_{k\in\underline{\mathcal{K}}(x):x_k>\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}+\sum_{k\in\underline{\mathcal{K}}(x):x_k\leq\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\leq -\sqrt{n}+\Upsilon\sum_{k\in\mathcal{K}}\frac{a_n}{\mu_k}\leq -1,
\end{aligned}$$

provided that n is sufficiently large. The proof is therefore complete. □

C. Proof of Proposition 2 To prove the tightness of $\{\widetilde{\mathfrak{X}}_\diamond^n(\infty)\}$, we will analyze the hydrodynamic-scaled processes. To this end, we first introduce the hydrodynamic-scaled processes under the proposed policy $\psi_\diamond^n$ as follows. For the simplicity of the presentation, we omit the subscript $\diamond$ notation in defining these processes. For $k\in\mathcal{K}$ and $t\geq 0$, define

$$\begin{aligned}
&\bar{S}_k^n(t)=\frac{S_k^n(t/\sqrt{n})}{\sqrt{n}},\quad \bar{N}_k^n(t)=\frac{N_k(\sqrt{n}t)}{\sqrt{n}},\quad \bar{T}_k^n(t)=\sqrt{n}T_k^n(t/\sqrt{n}),\\
&\bar{Q}_k^n(t)=\frac{Q_k^n(t/\sqrt{n})}{\sqrt{n}},\quad \bar{O}_k^n(t)=\frac{O_k^n(t/\sqrt{n})}{\sqrt{n}},\\
&\bar{\lambda}_k^n(t)=\frac{1}{\sqrt{n}}\int_0^{t/\sqrt{n}}\lambda_k^n(s)ds=\lambda_k^*t+\int_0^{t/\sqrt{n}}\zeta_k^n(s)ds=\lambda_k^*t+\frac{1}{\sqrt{n}}\int_0^t\zeta_k^n(u/\sqrt{n})du.
\end{aligned}\tag{54}$$

Then we have

$$\bar{Q}_k^n(t) = \bar{Q}_k^n(0) + \bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t). \tag{55}$$

Finally, define

$$\bar{\mathfrak{X}}^n(t) = (\bar{Q}^n(t), \bar{C}^n(t)), \quad t \geq 0,$$

where $\bar{Q}^n(t) := (\bar{Q}_1^n(t), \ldots, \bar{Q}_K^n(t))$, $\bar{C}^n(t) = C^n(t/\sqrt{n})$ and recall that $C^n(t)$ denotes the customer class in service at time t in the nth system. It is clear that $\widetilde{\mathfrak{X}}_\diamond^n(\infty)$ is also a stationary distribution of $\bar{\mathfrak{X}}^n(\cdot)$. In the following, we use $\bar{\mathfrak{X}}^n(\infty)$ to denote $\widetilde{\mathfrak{X}}_\diamond^n(\infty)$, to emphasize that we are analyzing hydrodynamic-scaled processes. It is then enough to prove the tightness of $\{\bar{\mathfrak{X}}^n(\infty)\}$.

We use the approach in (Gamarnik and Zeevi 2006, Section 3), where the main idea is to construct appropriate Lyapunov functions to obtain tail bounds on $\bar{\mathfrak{X}}^n(\infty)$. Following Gamarnik and Zeevi (2006), we define for $\theta > 0$ and a function Φ (with slight abuse of notations):

$$L_1^n(\theta, t) := \sup_{(q,\xi)} \mathbb{E}_{(q,\xi)} \left[\exp\left(\theta |\Phi(\bar{\mathfrak{X}}^n(t)) - \Phi(q,\xi)|\right)\right], \tag{56}$$

$$L_2^n(\theta, t) := \sup_{(q,\xi)} \mathbb{E}_{(q,\xi)} \left[(\Phi(\bar{\mathfrak{X}}^n) - \Phi(q,\xi))^2 \cdot \exp\left(\theta(\Phi(\bar{\mathfrak{X}}^n(t)) - \Phi(q,\xi))^+\right)\right], \tag{57}$$

for $t \geq 0$, where $\mathbb{E}_{(q,\xi)}$ stands for the conditional expectation operator $\mathbb{E}[\cdot|\bar{\mathfrak{X}}^n(0) = (q,\xi)]$ with $\bar{Q}^n(0) = q \in \mathbb{R}^K$ and $\bar{C}^n(0) = \xi \in \mathcal{K} \cup \{0\}$.

We divide the proof of Proposition 2 into two parts. In Section C.1, we prove that the sequence of random variables $\left\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\right\}$ is tight. In Section C.2, we show that the sequence of random variables $\left\{\left(\sum_{k\in\mathcal{K}} \frac{\bar{Q}_k^n(\infty)}{\mu_k}\right)^-\right\}$ is tight. Combining these two parts together, and using $\sum_k |x_k| \leq 2\sum_k x_k^+ + (\sum_k x_k)^-$, we obtain the tightness of $\left\{\sum_{k\in\mathcal{K}} \frac{|\bar{Q}_k^n(\infty)|}{\mu_k}\right\}$. Since $\bar{C}^n(\infty)$ takes value in a compact set $\mathcal{K} \cup \{0\}$, the tightness of $\{\bar{\mathfrak{X}}^n(\infty)\}$ then readily follows.

In the following, we denote by $\bar{\pi}^n$ the distribution of $\bar{\mathfrak{X}}^n(\infty)$, and $\mathbb{P}_{\bar{\pi}^n}$ the probability conditional on that $\bar{\mathfrak{X}}^n(0)$ follows $\bar{\pi}^n$.

C.1. Tightness of $\left\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\right\}$. We have the following result.

LEMMA 7. *There exist constants $C_1, C_2 > 0$ which are independent of n such that for all sufficiently large n,*

$$\mathbb{P}_{\bar{\pi}^n}\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(0)-1)^+}{\mu_k} > s\right) \leq C_1 e^{-C_2\cdot s}, \quad \textit{for all } s > 0.$$

As a consequence, the sequence of random variables $\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\}$ is tight.

The key step in proving Lemma 7 is the following result.

LEMMA 8. *There exist constants $t_0, c_0, \gamma > 0$ which are independent of n, such that for all sufficiently large n,*

$$\sup_{(q,\xi): \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} > c_0} \left\{\mathbb{E}_{(q,\xi)}\left[\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}\right] - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}\right\} \leq -\gamma. \tag{58}$$

Using the terminology from Gamarnik and Zeevi (2006), Lemma 8 says that the function

$$\Phi(q,\xi) = \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}, \tag{59}$$

is a Lyapunov function with drift size parameter γ, drift parameter $t_0 > 0$, and exception parameter c_0. With Lemma 8, we now proceed to prove Lemma 7.

Proof of Lemma 7. We use Lemma 8 and apply Theorem 6 in Gamarnik and Zeevi (2006). In the following, $L_1^n(\theta,t)$ and $L_2^n(\theta,t)$ are defined in (56) and (57), with Φ given in (59), respectively. Recall that

$$\bar{Q}_k^n(t) = \bar{Q}_k^n(0) + \bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t). \tag{60}$$

Since the function $\Phi(q,\xi)$ in (59) does not depend on the value of ξ, with slight abuse of notations, we write $\Phi(q) = \Phi(q,\xi) = \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}$. Then for each $t \ge 0$,

$$\begin{aligned}
|\Phi(\bar{Q}^n(t)) - \Phi(\bar{Q}^n(0))| &\le \sum_{k\in\mathcal{K}} \left|\bar{Q}_k^n(t) - \bar{Q}_k^n(0)\right| / \mu_k = \sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \left|\bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t)\right| \\
&\le \left|\sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \bar{S}_k^n(\bar{T}_k^n(t))\right| + \left|\sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \bar{N}_k^n(\bar{\lambda}_k^n(t))\right| \\
&\le \sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \bar{S}_k^n(t) + \sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \bar{N}_k^n(\bar{\lambda}_k^n(t)),
\end{aligned} \tag{61}$$

where we have used the fact that $0 \le \bar{O}_k^n(t) \le \bar{N}_k^n(\bar{\lambda}_k^n(t))$. It follows that for each $t \ge 0$,

$$\begin{aligned}
L_1^n(\theta,t) &\le \mathbb{E}\left[\exp\left(\sum_{k\in\mathcal{K}} \frac{\theta}{\mu_k} \bar{S}_k^n(t) + \sum_{k\in\mathcal{K}} \frac{\theta}{\mu_k} \bar{N}_k^n(\bar{\lambda}_k^n(t))\right)\right] \\
&= \Pi_{k=1}^K \mathbb{E}[\exp(\frac{\theta}{\mu_k} \bar{S}_k^n(t))] \cdot \mathbb{E}[\exp(\frac{\theta}{\mu_k} \bar{N}_k^n(\bar{\lambda}_k^n(t)))] \\
&\le \Pi_{k=1}^K \exp\left(\sqrt{n} t \mu_k (e^{\frac{\theta}{\sqrt{n}\mu_k}} - 1)\right) \cdot \exp\left([\sqrt{n}(\lambda_k^* + a_n)t](e^{\frac{\theta}{\sqrt{n}\mu_k}} - 1)\right),
\end{aligned} \tag{62}$$

where in the second inequality we have used the moment generating functions of Poisson distributions and $\bar{\lambda}_k^n(t) = \lambda_k^* t + \frac{1}{\sqrt{n}} \int_0^t \zeta_k^n(u/\sqrt{n}) du \le (\lambda_k^* + a_n)t$. Hence for any $\theta > 0$,

$$\limsup_{n\to\infty} L_1^n(\theta,t_0) \le \Pi_{k=1}^K \exp\left(\frac{\theta t_0(\mu_k + \lambda_k^*)}{\mu_k}\right) := G(\theta,t_0) < \infty. \tag{63}$$

Next we verify that there exists $\theta > 0$ such that $\theta L_2^n(\theta,t_0) \le \gamma$ for all sufficiently large n. Using the fact that $x^2 \le 2e^x$ for all $x \ge 0$, we can obtain that $\mathbb{E}[Y^2 e^{\theta|Y|}] \le 2\mathbb{E}[e^{(\theta+1)|Y|}]$ for a random variable Y. Hence, we infer from the definitions of L_1^n, L_2^n and (63) that

$$\limsup_{n\to\infty} L_2^n(\theta,t_0) \le 2G((1+\theta),t_0).$$

It is clear that we can choose sufficiently small $\theta > 0$ so that $\theta \cdot G((1+\theta),t_0) \le \gamma$. Hence, we obtain that $\theta L_2^n(\theta,t_0) \le \gamma$ for all n large enough.

Using Lemma 8 and Theorem 6 in Gamarnik and Zeevi (2006), we infer that

$$\mathbb{P}_{\bar{\pi}^n}\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(0)-1)^+}{\mu_k} > s\right) \le (1-\gamma\theta/2)^{-1} L_1^n(\theta,t_0) \exp(-\theta(s-c_0)),$$

where t_0, c_0 are from Lemma 8. The result in Lemma 7 then follows from Equation (63). □

Proof of Lemma 8. To show (58), it is enough to prove that

$$\sup_{(q,\xi):\sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} > c_0} \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k} - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} + 2\gamma\right)^+\right] \le \gamma. \tag{64}$$

Fix $t_0 > 0$. Let $c_0 > u_\diamond$. Define an event $A := \{\sum_{k\in\mathcal{K}} \frac{1}{\mu_k} \bar{N}_k^n(\bar{\lambda}_k^n(t_0)) < c_0 - u_\diamond\}$. First, from (61),

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k} - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} + 2\gamma\right)^+ \cdot 1_{A^c}\right]$$

$$\leq \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0))+2\gamma\right)\cdot 1_{A^c}\right]. \tag{65}$$

From (62), the collection of random variables $\{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0)) : n\geq 1\}$ is uniformly integrable. Given $\epsilon>0$ and $t_0>0$, we can choose $c_0>0$ large so that $\mathbb{P}(A^c)<\epsilon$, and

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0))+2\gamma\right)\cdot 1_{A^c}\right]\leq (2\gamma+1)\epsilon. \tag{66}$$

We next control the expectation when event A holds. Note that for $\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}>c_0$ and on the event A, we can obtain from (60) that for $t\in[0,t_0]$

$$\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t)-1)^+}{\mu_k}\geq\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))>c_0-\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))>u_\diamond.$$

Then under the policy $\psi_\diamond^n$, the system will not work on any new product from a class k with $\bar{Q}_k^n(t)\geq\frac{\alpha_k}{\sqrt{n}}$. As one product may be initially in production, for each class, we have $(\bar{Q}_k^n(t)-1)^+\leq (q_k-1)^+ + \frac{1}{\sqrt{n}}$ for $t\in[0,t_0]$. Also note that there must exist a class k_0 such that $\bar{Q}_{k_0}^n(t)>1$ for $s\in[0,t_0]$. Then $(\bar{Q}_{k_0}^n(t_0)-1)^+-(q_{k_0}-1)^+=\bar{Q}_{k_0}^n(t_0)-q_{k_0}\leq -\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}$. From (60),

$$\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}+2\gamma\leq -\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}+2\gamma.$$

As a result,

$$\begin{aligned}&\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}+2\gamma\right)^+\cdot 1_A\right]\\&\leq \mathbb{E}_{(q,\xi)}\left[\left(-\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}+2\gamma\right)^+\right]\\&\leq\left(2\gamma+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\right)\cdot\mathbb{P}\left(\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))<2\gamma+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\right).\end{aligned} \tag{67}$$

Note that when n is sufficiently large, we can choose an appropriately small $\gamma<\frac{\lambda_{k_0}^* t_0}{4\mu_{k_0}}$, so that the term in (67) is upper bounded by $\frac{\gamma}{2}\leq\gamma-(2\gamma+1)\epsilon$ where $0<\epsilon<\gamma/2(2\gamma+1)$. On combining with (65) and (66), we obtain (64). The proof is therefore complete. ☐

C.2. Tightness of $\left\{\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}_k^n(\infty)}{\mu_k}\right)^-\right\}$. With slight abuse of notations, we use the Lyapunov function

$$\Phi(q,\xi)=\Phi(q)=-\min\left(\sum_{k\neq i^\diamond}\frac{q_k}{\mu_k}+\frac{1}{\mu_{i^\diamond}}\min(q_{i^\diamond},\frac{\alpha_{i^\diamond}}{\sqrt{n}}),0\right)=\left(\sum_{k\in\mathcal{K}}\frac{q_k}{\mu_k}-\frac{1}{\mu_{i^\diamond}}\left(q_{i^\diamond}-\frac{\alpha_{i^\diamond}}{\sqrt{n}}\right)^+\right)^-. \tag{68}$$

We have the following result.

Lemma 9. *There exist constants $t_0,c_0,\gamma>0$ which are independent of n, such that for $\Phi(\cdot)$ in (68),*

$$\sup_{(q,\xi):\Phi(q)>c_0}\left\{\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t_0))\right]-\Phi(q)\right\}\leq-\gamma,\quad \textit{for all sufficiently large } n. \tag{69}$$

Using Lemma 9, we can then obtain the sought tightness result in the following lemma, whose proof is similar to the one of Lemma 7 and is thus omitted.

LEMMA 10. *There exist constants $C_1, C_2 > 0$ which are independent of n such that for all sufficiently large n,*

$$\mathbb{P}_{\bar{\pi}^n}\left(\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}_k^n(0)}{\mu_k}-\frac{1}{\mu_{i^\diamond}}\left(\bar{Q}_{i^\diamond}^n(0)-\frac{\alpha_{i^\diamond}}{\sqrt{n}}\right)^+\right)^- > s\right)\le C_1e^{-C_2\cdot s},\quad \textit{for all } s>0.$$

As a consequence, the sequence of random variables $\left\{\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}_k^n(\infty)}{\mu_k}\right)^-\right\}$ *is tight.*

Proof of Lemma 9. Pick $c_0 > -l_\diamond$ and let $B = \{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t_0)) \le c_0 + l_\diamond\}$. From (62), the collection $\{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0)) : n\ge 1\}$ is uniformly integrable. Given $\epsilon>0$ and $t_0>0$, similar to (65), one can choose c_0 large enough so that $\mathbb{P}(B^c)<\epsilon$, and

$$\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t_0))-\Phi(q))\cdot 1_{B^c}\right]\le\epsilon. \tag{70}$$

For $\Phi(q)>c_0$, we can infer from (60) that on the event B, for $t\in[0,t_0]$

$$\begin{aligned}\sum_{k\neq i^\diamond}\frac{\bar{Q}_k^n(t)}{\mu_k}+\frac{1}{\mu_{i^\diamond}}\min(\bar{Q}_{i^\diamond}^n(t),\alpha_{i^\diamond}/\sqrt{n}) &\le \sum_{k\neq i^\diamond}\frac{\bar{Q}_k^n(0)}{\mu_k}+\frac{1}{\mu_{i^\diamond}}\min(\bar{Q}_{i^\diamond}^n(0),\alpha_{i^\diamond}/\sqrt{n})+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))\\ &< -c_0+c_0+l_\diamond=l_\diamond.\end{aligned}$$

We next consider the following two cases to bound $\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t))-\Phi(x)\right)\cdot 1_B\right]$.

Case 1: $Q_{i^\diamond}^n(0)<\alpha_{i^\diamond}$. Note that the system will not work on class $i^\diamond$ products if $Q_{i^\diamond}^n$ reaches $\alpha_{i^\diamond}$ (because there are customers waiting in other classes), hence $Q_{i^\diamond}^n(t)\le\alpha_{i^\diamond}$ for $t\in[0,t_0]$, and class $i^\diamond$ orders are always outsourced during $[0,t_0]$. This implies that

$$\begin{aligned}&\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t))-\Phi(q))\cdot 1_B\right]\\ &=\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))-\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))\right)\cdot 1_B\right]\\ &=\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda_k^*}{\mu_k}t-t\right)\cdot 1_B\right]-\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda_k^*}{\mu_k}t-\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))-t\right)\cdot 1_B\right]\\ &=\frac{-\lambda_{i^\diamond}^*t}{\mu_{i^\diamond}}\cdot\mathbb{P}(B)-\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda_k^*}{\mu_k}t-\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))-t\right)\cdot 1_B\right].\end{aligned}$$

Because the system is always busy, we have $\sum_{k\in\mathcal{K}}\bar{T}_k^n(t)=t$, and then for n sufficiently large,

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda_k^*}{\mu_k}t-\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))\right)\cdot 1_B\right]\le\epsilon \text{ and } \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))-t\right)\cdot 1_B\right]<\epsilon.$$

Using the fact that $\mathbb{P}(B)>1-\epsilon$, we infer that

$$\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t))-\Phi(q)\right)\cdot 1_B\right]\le-\frac{\lambda_{i^\diamond}^*t}{\mu_{i^\diamond}}\cdot(1-\epsilon)+2\epsilon. \tag{71}$$

Case 2: $Q_{i^\diamond}^n(0)\ge\alpha_{i^\diamond}$. We can compute that

$$\begin{aligned}\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t))-\Phi(q)\right)\cdot 1_B\right] &= \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))-\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))\right)\cdot 1_B\right]\\ &\quad-\frac{1}{\mu_{i^\diamond}}\mathbb{E}_{(q,\xi)}\left[\left(\min(\bar{Q}_{i^\diamond}^n(t),\frac{\alpha_{i^\diamond}}{\sqrt{n}})-\min(\bar{Q}_{i^\diamond}^n(0),\frac{\alpha_{i^\diamond}}{\sqrt{n}})\right)\cdot 1_B\right],\end{aligned}$$

One can verify that $\min(\bar{Q}^n_{i^\diamond}(t),\frac{\alpha_{i^\diamond}}{\sqrt{n}})-\min(\bar{Q}^n_{i^\diamond}(0),\frac{\alpha_{i^\diamond}}{\sqrt{n}})=0$ and $0\le \bar{S}^n_{i^\diamond}(\bar{T}^n_{i^\diamond}(t))\le\frac{1}{\sqrt{n}}$: $\min(\bar{Q}^n_{i^\diamond}(t),\frac{\alpha_{i^\diamond}}{\sqrt{n}})-\min(\bar{Q}^n_{i^\diamond}(0),\frac{\alpha_{i^\diamond}}{\sqrt{n}})=0$ holds because whenever $Q^n_{i^\diamond}$ reaches $\alpha_{i^\diamond}$, new orders would be outsourced, hence $Q^n_{i^\diamond}$ would not be smaller than $\alpha_{i^\diamond}$; $0\le \bar{S}^n_{i^\diamond}(\bar{T}^n_{i^\diamond}(t))\le\frac{1}{\sqrt{n}}$ holds because at most one class $i^\diamond$ product can be produced (the one initially in production), and after that, the system will allocate no capacity to class $i^\diamond$. As a result,

$$\begin{aligned}\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t))-\Phi(q))\cdot 1_B\right]&\le \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t))-\sum_{k=1}^{K}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t))\right)\cdot 1_B\right]+\frac{1}{\sqrt{n}\mu_{i^\diamond}}\\&\le -\frac{\lambda^*_{i^\diamond}t}{\mu_{i^\diamond}}\cdot(1-\epsilon)+2\epsilon+\frac{1}{\sqrt{n}\mu_{i^\diamond}},\end{aligned}$$

where the last inequality follows from a similar argument for (71).

Combing the above two cases, we have $\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t))-\Phi(x)\right]\cdot 1_B\le -\frac{\lambda^*_{i^\diamond}t}{\mu_{i^\diamond}}\cdot(1-\epsilon)+3\epsilon$ for n sufficiently large and $t\in[0,t_0]$. Together with (70), we have for all sufficiently large n,

$$\sup_{(q,\xi):\Phi(q)>c_0}\left\{\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t_0))\right]-\Phi(q)\right\}\le -\frac{\lambda^*_{i^\diamond}t_0}{\mu_{i^\diamond}}\cdot(1-\epsilon)+4\epsilon.$$

Hence we obtain (69) by choosing $\gamma>0$ small so that $-\frac{\lambda^*_{i^\diamond}t_0}{\mu_{i^\diamond}}\cdot(1-\epsilon)+4\epsilon<-\gamma$. □

D. Proof of Proposition 3 We first describe the hydrodynamic limit and its uniform attraction property in Section D.1, and then use it to prove Proposition 3 in Section D.2.

D.1. Hydrodynamic Limit and Uniform Attraction For the simplicity of the presentation, we omit the subscript $*$ notation in the processes under the policy ψ^n_*. We consider the hydrodynamic-scaled processes $\{(\bar{Q}^n,\bar{W}^n,\bar{O}^n,\bar{T}^n,\bar{I}^n)\}$ with $\bar{Q}^n,\bar{O}^n,\bar{T}^n$ defined in (54) and

$$\bar{W}^n(t):=\frac{1}{\sqrt{n}}W^n\left(\frac{t}{\sqrt{n}}\right),\quad \bar{I}^n(t):=\sqrt{n}I^n\left(\frac{t}{\sqrt{n}}\right).$$

In Lemma 11, we establish the convergence of the hydrodynamic-scaled processes under the proposed policies. Its proof is standard, hence is omitted.

Lemma 11. *Fix a constant $M>0$ and assume that $\sum_{k\in\mathcal{K}}\frac{|\bar{Q}^n(0)|}{\mu_k}\le M$ for all n. Then for any subsequence of hydrodynamic-scaled processes $\{(\bar{Q}^n,\bar{W}^n,\bar{O}^n,\bar{T}^n,\bar{I}^n)\}$, there is a further subsequence $\mathcal{N}$, such that along this subsequence $\mathcal{N}$,*

$$(\bar{Q}^n,\bar{W}^n,\bar{O}^n,\bar{T}^n,\bar{I}^n)\to(\bar{Q},\bar{W},\bar{O},\bar{T},\bar{I}),\ \text{u.o.c.},$$

for some Lipschitz continuous process $(\bar{Q},\bar{W},\bar{O},\bar{T},\bar{I})$, which is called a hydrodynamic limit.

Due to Liptschitz continuity, the hydrodynamic limit processes are differentiable at almost all $t\ge 0$. Following the convention in literature (e.g., Mandelbaum and Stolyar (2004)), any t such that the limit processes are differentiable is called *regular*. When we write derivatives of the limit processes with respect to t, we assume they are at a regular time.

Lemma 12. *Any hydrodynamic limit satisfies the following properties:*
1. *For $t\ge 0$,*

$$\begin{aligned}&\bar{Q}_k(t)=\bar{Q}_k(0)+\mu_k\bar{T}_k(t)-\lambda^*_k t+\bar{O}_k(t),\\&\bar{I}(t)=t-\sum_{k\in\mathcal{K}}\bar{T}_k(t),\\&\bar{W}(t)=\bar{W}(0)-\bar{I}(t)+\sum_{k\in\mathcal{K}}\frac{\bar{O}_k(t)}{\mu_k},\\&\bar{I},\bar{T},\bar{O}_{i^*}\ \text{are non-decreasing},\\&\bar{O}'_{i^*}(t)\le\lambda^*_{i^*},\ \text{and}\ \bar{O}_k(t)=0,\ \text{for}\ k\neq i^*.\end{aligned}\tag{72}$$

2. *There is a constant $\chi > 0$ such that*
   (a) *If $f_1(t) := \sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k(t))^+}{\mu_k} > u_*$, then $f_1'(t) < -\chi$.*
   (b) *If $f_2(t) := \sum_{k\neq i^*} \frac{\bar{Q}_k(t)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}} < l_*$, then $f_2'(t) > \chi$.*
   (c) *For $k \in \mathcal{K}$, let $q_k^* = \Delta_k(\bar{W}(t))$. If $\bar{O}_k'(t) = 0$ and $\bar{Q}_k(t) > q_k^*$, then $\bar{Q}_k'(t) = -\lambda_k^* < 0$.*
   (d) *If $\bar{W}(t) < u_*$, then $\bar{I}'(t) = 0$; if $\bar{W}(t) > l_*$, then $\bar{O}_{i^*}'(t) = 0$.*

Proof. 1. The equation of $\bar{Q}$ follows from (55), the law of large numbers, random-time change, and that $\bar{\lambda}_k^n(\cdot)$ converges to $\lambda_k^* e(\cdot)$. The equation of $\bar{I}$ is from $\bar{I}^n(t) = \sqrt{n} I^n(t/\sqrt{n}) = t - \sum_{k\in\mathcal{K}} \bar{T}_k^n(t)$. The equation of $\bar{W}$ is from $\bar{W}^n(t) = \sum_{k\in\mathcal{K}} \frac{\bar{Q}_k^n(t)}{\mu_k}$ and $\sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k} = 1$. Because $\bar{I}^n, \bar{T}^n, \bar{O}_{i^*}^n$ are non-decreasing, $\bar{I}, \bar{T}, \bar{O}_{i^*}$ are non-decreasing. For any $t, s \geq 0$, $\bar{O}_{i^*}^n(t+s) - \bar{O}_{i^*}^n(t) \leq \bar{E}_{i^*}^n(t+s) - \bar{E}_{i^*}^n(t)$, and $\bar{O}_k^n(t) = 0$ for $k \neq i^*$, hence $\bar{O}_{i^*}'(t) \leq \lambda_{i^*}^*$ and $\bar{O}_k(t) = 0$, for $k \neq i^*$.

2. We prove the results one by one.

(a) If $f_1(t) > u_*$, because f_1 is continuous, there exists $\epsilon > 0$ such that $f_1(s) > u_*$ for $s \in [t, t+\epsilon]$. Then for n large enough, $\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(s))^+}{\mu_k} > u_*$ for $s \in [t, t+\epsilon]$. As a result, under the proposed policy, the system will not work on any new product from a class k with $\bar{Q}_k^n(s) \geq \frac{\alpha_k}{\sqrt{n}}$, hence for each $k \in \mathcal{K}$, $\frac{(\bar{Q}_k^n(u))^+}{\mu_k}$ cannot be larger than $\frac{(\bar{Q}_k^n(s))^+}{\mu_k} + \frac{\alpha_k+1}{\mu_k\sqrt{n}}$ for $s \leq u \leq t+\epsilon$. Taking $n \to \infty$, one has $(\bar{Q}_k(u))^+ \leq (\bar{Q}_k(s))^+$ for any $t \leq s \leq u \leq t+\epsilon$. That is, for $k \in \mathcal{K}$, $(\bar{Q}_k)^+$ is non-increasing in $[t, t+\epsilon]$. Note that $f_1(s) > u_*$ for $s \in [t, t+\epsilon]$, there exists a class k_0 such that $\bar{Q}_{k_0}(s) > 0$ for $s \in [t, t+\epsilon]$. For n large enough, $\bar{Q}_{k_0}^n(s) > \frac{\alpha_{k_0}}{\sqrt{n}}$ for $s \in [t, t+\epsilon]$, hence the system will not work a new product from class k_0. Then $0 \leq \bar{S}_{k_0}^n(\bar{T}_{k_0}^n(s)) - \bar{S}_{k_0}^n(\bar{T}_{k_0}^n(t)) \leq \frac{1}{\sqrt{n}}$ for $s \in [t, t+\epsilon]$. Taking $n \to \infty$, one has $\bar{T}_{k_0}(s) = \bar{T}_{k_0}(t)$ for $s \in [t, t+\epsilon]$, which gives $\bar{T}_{k_0}'(t) = 0$. If $k_0 = i^*$, then because $\bar{Q}_{k_0}^n(s) > \frac{\alpha_{k_0}}{\sqrt{n}}$ for $s \in [t, t+\epsilon]$ and n large enough, no class k_0 order is outsourced. If $k_0 \neq i^*$, then no class k_0 order is outsourced. In both cases, $\bar{O}_{k_0}^n(s) = \bar{O}_{k_0}^n(t)$ for $s \in [t, t+\epsilon]$. Taking $n \to \infty$, one has $\bar{O}_{k_0}(s) = \bar{O}_{k_0}(t)$ for $s \in [t, t+\epsilon]$ and hence $\bar{O}_{k_0}'(t) = 0$. Because the derivative of $(\bar{Q}_k)^+$ cannot be larger than 0 for $k \in \mathcal{K}$ and $\bar{Q}_{k_0}(s) > 0$ for $s \in [t, t+\epsilon]$,

$$f_1'(t) \leq \frac{\bar{Q}_{k_0}'(t)}{\mu_{k_0}} = -\frac{\lambda_{k_0}^*}{\mu_{k_0}} \leq -\chi := -\min_k \left\{ \frac{\lambda_k^*}{\mu_k} \right\}.$$

(b) If $f_2(t) < l_*$, then there are three cases:

i) If $\bar{Q}_{i^*}(t) < 0$, then there exists $\epsilon > 0$ such that $\bar{Q}_{i^*}(s) < 0$ and $f_2(s) < l_*$ for $s \in [t, t+\epsilon]$. For n large enough, one has $\sum_{k\neq i^*} \frac{\bar{Q}_k^n(s)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}^n(s),0)}{\mu_{i^*}} < l_*$ and $\bar{Q}_{i^*}^n(s) < 0$ for $s \in [t, t+\epsilon]$. During this period, class i^* orders will be outsourced, that is, $\bar{O}_{i^*}^n(s) - \bar{O}_{i^*}^n(t) = \bar{E}_{i^*}^n(s) - \bar{E}_{i^*}^n(t)$ for $s \in [t, t+\epsilon]$. Taking $n \to \infty$, $\bar{O}_{i^*}(s) - \bar{O}_{i^*}(t) = \lambda_{i^*}^*(s-t)$ for $s \in [t, t+\epsilon]$, hence $\bar{O}_{i^*}'(t) = \lambda_{i^*}^*$. The system is always busy during this period, then $\bar{I}^n(s) = \bar{I}^n(t)$ for $s \in [t, t+\epsilon]$. Taking $n \to \infty$, $\bar{I}(s) = \bar{I}(t)$ for $s \in [t, t+\epsilon]$ and hence $\bar{I}'(t) = 0$. Note that $\bar{Q}_{i^*}(s) < 0$ for $s \in [t, t+\epsilon]$, then

$$f_2(s) = \bar{W}(s) = \bar{W}(0) - \bar{I}(s) + \sum_{k\in\mathcal{K}} \frac{\bar{O}_k(s)}{\mu_k}, \quad \text{for} \quad s \in [t, t+\epsilon].$$

Because $\bar{I}'(t) = 0$ and $\bar{O}_k(t) = 0$ for $k \neq i^*$, $f_2'(t) = \frac{\bar{O}_{i^*}'(t)}{\mu_{i^*}} = \frac{\lambda_{i^*}^*}{\mu_{i^*}} > \chi > 0$.

ii) If $\bar{Q}_{i^*}(t) > 0$: then there exists $\epsilon > 0$ such that $f_2(s) < l_*$ and $\bar{Q}_{i^*}(s) > 0$ for $s \in [t, t+\epsilon]$. Hence $f_2'(t) = \sum_{k\neq i^*} \frac{\bar{Q}_k'(t)}{\mu_k}$. For n large enough, $\bar{Q}_{i^*}^n(s) \geq \frac{\alpha_{i^*}+1}{\sqrt{n}}$ and $\sum_{k\neq i^*} \frac{\bar{Q}_k^n(s)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}^n(s),0)}{\mu_{i^*}} < l_*$ for $s \in [t, t+\epsilon]$, hence the system will not produce new class i^* products. Then $0 \leq \bar{S}_{i^*}^n(\bar{T}_{i^*}^n(s)) - \bar{S}_{i^*}^n(\bar{T}_{i^*}^n(t)) \leq \frac{1}{\sqrt{n}}$ for $s \in [t, t+\epsilon]$. Taking $n \to \infty$, one has $\bar{T}_{i^*}(s) = \bar{T}_{i^*}(t)$ for $s \in [t, t+\epsilon]$, which gives $\bar{T}_{i^*}'(t) = 0$. Similarly, one can prove that $\sum_{k\in\mathcal{K}} \bar{T}_k'(t) = 1$, hence $\sum_{k\neq i^*} \bar{T}_k'(t) = 1$. Then

$$f_2'(t) = \sum_{k\neq i^*} \frac{\bar{Q}_k'(t)}{\mu_k} = \sum_{k\neq i^*} \bar{T}_k'(t) - \sum_{k\neq i^*} \frac{\lambda_k^*}{\mu_k} = \left(1 - \sum_{k\neq i^*} \frac{\lambda_k^*}{\mu_k}\right) = \frac{\lambda_{i^*}^*}{\mu_{i^*}} \geq \chi > 0.$$

iii) If $\bar{Q}_{i^*}(t)=0$: note that there exists $\epsilon>0$ such that $\sum_{k\neq i^*}\frac{\bar{Q}_k(s)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(s),0)}{\mu_{i^*}}<l_*$ for $s\in[t,t+\epsilon]$. We next prove $\bar{Q}_{i^*}(s)=0$ for $s\in[t,t+\epsilon]$. We prove it by contradiction: if $\bar{Q}_{i^*}(s)<0$ for some $s\in(t,t+\epsilon]$, then due to the continuity of $\bar{Q}_{i^*}$, there is $s_0\in(t,s)$ such that $\bar{Q}_{i^*}(s_0)=\bar{Q}_{i^*}(s)/2<0$ and $\bar{Q}_{i^*}(u)\leq\bar{Q}_{i^*}(s)/2$ for $u\in[s_0,s]$; however, from the discussion in i) above, $\bar{O}'_{i^*}(u)=\lambda^*_{i^*}$ for $u\in[s_0,s]$, hence $\bar{Q}_{i^*}$ cannot decrease in $[s_0,s]$ and a contradiction; if $\bar{Q}_{i^*}(s)>0$ for some $s\in(t,t+\epsilon]$, then there is $s_0\in(t,s)$ such that $\bar{Q}_{i^*}(s_0)=\bar{Q}_{i^*}(s)/2>0$ and $\bar{Q}_{i^*}(u)\geq\bar{Q}_{i^*}(s)/2>0$ for $u\in[s_0,s]$; however, from the discussion in ii) above, $\bar{T}'_{i^*}(u)=0$ for $u\in[s_0,s]$, hence $\bar{Q}_{i^*}$ cannot increase in $[s_0,s]$ and then a contradiction.

As a result, $f_2'(t)=\sum_{k\neq i^*}\frac{\bar{Q}'_k(t)}{\mu_k}$. Next we prove $\sum_{k\neq i^*}\bar{T}'_k(t)=1$, which is equivalent to $\bar{T}'_{i^*}(t)=0$. Because $\lim_{n\to\infty}\bar{Q}^n_{i^*}(t)=\bar{Q}_{i^*}(t)=0$, consider two subsequences: $\{n:\bar{Q}^n_{i^*}(t)\geq\frac{\alpha_k}{\sqrt{n}}\}$ and $\{n:\bar{Q}^n_{i^*}(t)<\frac{\alpha_k}{\sqrt{n}}\}$. For any system in the first subsequence, $\bar{Q}^n_{i^*}(s)\geq\frac{\alpha_{i^*}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ because once $\bar{Q}^n_{i^*}$ reaches $\frac{\alpha_{i^*}}{\sqrt{n}}$, new arrivals are outsourced, hence $\bar{Q}^n_{i^*}$ cannot decrease anymore. Then the system will not choose to produce new class i^* items, hence $\bar{S}^n_{i^*}(\bar{T}^n_{i^*}(s))-\bar{S}^n_{i^*}(\bar{T}^n_{i^*}(t))\leq\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{T}_{i^*}(s)-\bar{T}_{i^*}(t)=0$ for $s\in[t,t+\epsilon]$ and hence $\bar{T}'_{i^*}(t)=0$. For systems in the second subsequence, $\bar{Q}^n_{i^*}(s)\leq\frac{\alpha_{i^*}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ because if $\bar{Q}^n_{i^*}$ reaches $\frac{\alpha_{i^*}}{\sqrt{n}}$, then the system allocates no capacity to class i^* and $\bar{Q}^n_{i^*}$ cannot increase to $\frac{\alpha_k+1}{\sqrt{n}}$. As a result, class i^* orders will be outsourced. Following the argument in i) above, one has $\bar{O}'_{i^*}(t)=\lambda^*_{i^*}$, and hence $\bar{T}'_{i^*}(t)=\bar{Q}'_{i^*}(t)$, which equals 0 because $\bar{Q}_{i^*}(s)=0$ for $s\in[t,t+\epsilon]$. Because the limits of both subsequences have the same derivative, one has $\bar{T}'_{i^*}(t)=0$ for any hydrodynamic limits.

Then
$$f_2'(t)=\sum_{k\neq i^*}\frac{\bar{Q}'_k(t)}{\mu_k}=\sum_{k\neq i^*}\bar{T}'_k(t)-\sum_{k\neq i^*}\frac{\lambda^*_k}{\mu_k}=\left(1-\sum_{k\neq i^*}\frac{\lambda^*_k}{\mu_k}\right)=\frac{\lambda^*_{i^*}}{\mu_{i^*}}>\chi>0.$$

(c) If $\bar{Q}_k(t)>q^*_k$, then there must be a class l such that $\bar{Q}_l(t)<q^*_l:=\Delta_l(\bar{W}(t))$. There is $\epsilon>0$ such that $\bar{Q}_k(s)>\Delta_k(\bar{W}(s))$ and $\bar{Q}_l(s)<\Delta_l(\bar{W}(s))$ for $s\in[t,t+\epsilon]$. Then $\bar{Q}^n_k(s)>\Delta_k(\bar{W}^n(s))$ and $\bar{Q}^n_l(s)<\Delta_l(\bar{W}^n(s))$ for $s\in[t,t+\epsilon]$ and n large enough. We first argue that, *under the proposed policies, the system will not allocate capacity to new class k products during the interval* $[t,t+\epsilon]$. For the setting with strictly convex holding/waiting cost functions, using the KKT condition, one can verify that $g'_k(\bar{Q}^n_k(s))\mu_k>g'_k(\bar{Q}^n_l(s))\mu_l$ for $s\in[t,t+\epsilon]$, hence the system will not allocate capacity to new class k products during that interval. For the setting with linear holding/waiting cost functions, if $\bar{W}^n(s)>u^n$, then $k\notin\mathcal{C}^n(s/\sqrt{n})$ and the system will not choose to produce new class k products; if $\bar{W}^n(s)\leq u^n$, then $\mathcal{C}^n(s/\sqrt{n})=\mathcal{K}$. We consider the case $\bar{W}^n(s)>0$ and the case $\bar{W}^n(s)\leq 0$ can be argued similarly. Note that $k\notin\mathcal{N}^n(s/\sqrt{n})$. If $\mathcal{N}^n(s/\sqrt{n})\neq\emptyset$, then the system will choose to produce a product from a class in $\mathcal{N}^n(s/\sqrt{n})$ hence not k; if $\mathcal{N}^n(s/\sqrt{n})=\emptyset$, then $\mathcal{P}^n(t)=\mathcal{K}$ and $k\neq\arg\min_{k\in\mathcal{P}^n(t)}h_k\mu_k$ (because if $k=\arg\min_{k\in\mathcal{P}^n(t)}h_k\mu_k$, then $l\in\mathcal{N}^n(s/\sqrt{n})$).

Because the system will not allocate capacity to new class k products during $[t,t+\epsilon]$, $\bar{S}^n_k(\bar{T}^n_k(s))-\bar{S}^n_k(\bar{T}^n_k(t))\leq\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, one has $\bar{T}_k(s)=\bar{T}_k(t)$ for $s\in[t,t+\epsilon]$. Hence $\bar{T}'_k(t)=0$. Together with the assumption that $\bar{O}'_k(t)=0$, one has $\bar{Q}'_k(t)=-\lambda^*_k<0$.

(d) If $\bar{W}(t)<u_*$, there exists $\epsilon>0$ such that $\bar{W}(s)<u_*$ for $s\in[t,t+\epsilon]$. Then $\bar{W}^n(s)<u_*$ for $s\in[t,t+\epsilon]$ and n large enough. As a result, $\bar{Q}^n_k(s)\geq\frac{\alpha_k}{\sqrt{n}}$ for all $k\in\mathcal{K}$ and $\sum_{k\in\mathcal{K}}\frac{(\bar{Q}^n_k(s))^+}{\mu_k}\geq u_*$ cannot hold simultaneously for $s\in[t,t+\epsilon]$. The system is then always busy, that is $\bar{I}^n(s)=\bar{I}^n(t)$ for $s\in[t,t+\epsilon]$. Taking limit, one has $\bar{I}(s)=\bar{I}(t)$ for $s\in[t,t+\epsilon]$, hence $\bar{I}'(t)=0$.

If $\bar{W}(t)>l_*$, there exists $\epsilon>0$ such that no class i^* order is outsourced during $[t,t+\epsilon]$. This is because: i) if $f_2(t)\leq l_*$, one gets $\bar{Q}_{i^*}(t)>0$, and then there is $\epsilon>0$ such that $\bar{Q}_{i^*}(s)>0$ for $s\in[t,t+\epsilon]$, hence $\bar{Q}^n_{i^*}(s)>\frac{\alpha_k}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ and n large enough; ii) if $f_2(t)>l_*$, then similarly $\sum_{k\neq i^*}\frac{\bar{Q}^n_k(s)}{\mu_k}+\frac{\min(\bar{Q}^n_{i^*}(s),0)}{\mu_{i^*}}>l_*$ for $s\in[t,t+\epsilon]$ and n large enough. Hence $\bar{O}^n_{i^*}(s)=\bar{O}^n_{i^*}(t)$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{O}_{i^*}(s)=\bar{O}_{i^*}(t)$ for $s\in[t,t+\epsilon]$ and hence $\bar{O}'_{i^*}(t)=0$. $\square$

LEMMA 13 **(Uniform attraction)**. *Consider any hydrodynamic limit derived in Lemma 11 with $\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_k(0)|}{\mu_k}\leq M$ for a constant $M>0$. There exist $T_1,T_2>0$ such that*

1. *If* $\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k(0))^+}{\mu_k} > u_*$*, then* $\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k(t))^+}{\mu_k} \le u_*$ *for all* $t \ge T_1$*.*
2. *If* $\sum_{k\neq i^*} \frac{\bar{Q}_k(0)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}(0),0)}{\mu_{i^*}} < l_*$*, then* $\sum_{k\neq i^*} \frac{\bar{Q}_k(t)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}} \ge l_*$ *for all* $t \ge T_2$*.*
3. *For all* $t \ge T := \max\{T_1, T_2\}$*, we have* $\bar{W}(t) = \bar{W}(T) \in [l_*, u_*]$*.*
4. *There exists a time* $T_M > T$*, such that* $\bar{Q}(t) = \Delta(\bar{W}(t))$ *for all* $t \ge T_M$*.*

Proof. 1. If $f_1(t) := \sum_{k\in\mathcal{K}} \frac{(\bar{Q}(t))^+}{\mu_k} > u_{\gamma_*}$, then from Lemma 12, item 2a), $f_1'(t) < -\chi < 0$. Also note that $f_1(0) \le \sum_{k\in\mathcal{K}} \frac{|\bar{Q}_k(0)|}{\mu_k} \le M$, hence within a finite time $T_1 > 0$, f_1 will return to u_*, and cannot be larger than u_* again.

2. If $f_2(t) := \sum_{k\neq i^*} \frac{\bar{Q}_k(t)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}} < l_*$, then from Lemma 12, item 2b), $f_2'(t) > \chi > 0$. Also note that $f_2(0) \ge -\sum_{k\in\mathcal{K}} \frac{|\bar{Q}_k(0)|}{\mu_k} \ge -M$, hence within a finite time $T_2 > 0$, f_2 will return to l_* and cannot be smaller than l_* again.

3. Then for $t \ge T = \max\{T_1, T_2\}$, $f_1(t) := \sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k(t))^+}{\mu_k} \le u_*$ and $f_2(t) := \sum_{k\neq i^*} \frac{\bar{Q}_k(t)}{\mu_k} + \frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}} \ge l_*$. One then gets $\bar{W}(t) \in [l_*, u_*]$. Next we prove $\bar{W}'(t) = 0$ and $\bar{I}'(t) = 0$.

(a) If $\bar{W}(t) \in (l_*, u_*)$, then from Lemma 12 item 2d), we have $\bar{I}'(t) = 0$, $\bar{O}_{i^*}'(t) = 0$. Together with $\bar{O}_k(t)=0$ for $k \neq i^*$, and the equation $\bar{W}(t) = \bar{W}(0) - \bar{I}(t) + \sum_{k\in\mathcal{K}} \frac{\bar{O}_k(t)}{\mu_k}$, we have $\bar{W}'(t) = 0$.

(b) If $\bar{W}(t) = l_*$, then $\bar{W}(\cdot)$ cannot decrease. This is because $f_2 \le \bar{W}$ and if $\bar{W}(\cdot)$ decreases, then $f_2(\cdot)$ will become strictly smaller than l_*. This is a contradiction to the choice of T. On the other hand, $\bar{W}(\cdot)$ will not increase. This is because if it increases to $l_* + \delta$ with $\delta > 0$, it will first reach $l_* + \frac{\delta}{2} > l_*$, but from the discussion in (a), $\bar{W}(\cdot)$ will stay at $l_* + \frac{\delta}{2}$ and cannot increase to $l_* + \delta$. This is a contradiction. As a result, $\bar{W}'(t) = 0$. From Lemma 12 item 2d), $\bar{I}'(t) = 0$. Together with (72), one has $\bar{O}_k'(t) = 0$ for all $k \in \mathcal{K}$.

(c) If $\bar{W}(t) = u_*$, then using an argument similar to the one for (b) above, we have $\bar{W}'(t) = 0$. From Lemma 12 item 2d), $\bar{O}_k'(t) = 0$ for all $k \in \mathcal{K}$. Together with (72), we then have $\bar{I}'(t) = 0$.

4. Then for $t \ge T$, $\bar{W}(t) \in [l_*, u_*]$. Also from the proof of item 3, $\bar{O}_k'(t) = 0$ for all $k \in \mathcal{K}$ and $\bar{I}'(t) = 0$, for $t \ge T$. If $\bar{Q}_k(t) > q_k^* := \Delta_k(\bar{W}(t))$, then from Lemma 12, item 2c), $\bar{Q}_k'(t) = -\lambda_k^* < 0$. As a result, after a finite time $T_M > T$, $\bar{Q}_k(t) \le q_k^*$ for all $k \in \mathcal{K}$. By the definition of q^*, one then has $\bar{Q}_k(t) = q_k^*$ for all $k \in \mathcal{K}$ and $t \ge T_M$. $\square$

D.2. Proof of Proposition 3.

We first show that as $n \to \infty$,

$$\left| \widetilde{Q}_*^n(\infty) - \Delta(l_* \vee (\widetilde{W}_*^n(\infty) \wedge u_*)) \right| \Rightarrow 0. \tag{73}$$

Assume $\bar{\mathfrak{X}}^n(0)$, the initial state of the Markov process $\bar{\mathfrak{X}}^n(\cdot)$, follows a stationary distribution $\bar{\pi}^n$. From Proposition 2, $\{\bar{\mathfrak{X}}^n(0)\}$ is tight, hence for each subsequence, there is a further subsequence (still indexed by n) such that $\bar{\mathfrak{X}}^n(0) \Rightarrow \bar{\mathfrak{X}}(0)$. We assume that this is almost sure convergence by Skorohod representation theorem. Then for $\epsilon > 0$, we can choose M such that $\mathbb{P}(\sum_{k\in\mathcal{K}} \frac{|\bar{Q}_{k*}(0)|}{\mu_k} < M) \ge 1 - \epsilon$. For the hydrodynamic limits with $\sum_{k\in\mathcal{K}} \frac{|\bar{Q}_{k*}(0)|}{\mu_k} < M$, from Lemma 13 item 4, there is $T_M > 0$ such that the corresponding hydrodynamic limits should satisfy $\bar{Q}_*(t) = \Delta(l_* \vee (\bar{W}_*(t) \wedge u_*))$ for $t \ge T_M$. Here we use the fact that $\bar{W}_*(t) \in [l_*, u_*]$ for $t \ge T_M$. We fix such a $t \ge T_M$. Then from Lemma 11 and the continuity of the lifting function Δ, for n large enough in the further subsequence, we can obtain that $\mathbb{P}(|\bar{Q}_*^n(t) - \Delta(l_* \vee (\bar{W}_*^n(t) \wedge u_*))| \ge 2\epsilon) \le 2\epsilon$. Because $\bar{\mathfrak{X}}^n(0)$ follows a stationary distribution, $\bar{\mathfrak{X}}^n(t)$ also follows the stationary distribution. As a result, for n large enough in the further subsequence, $\mathbb{P}(|\bar{Q}_*^n(\infty) - \Delta(l_* \vee (\bar{W}_*^n(\infty) \wedge u_*))| \ge 2\epsilon) \le 2\epsilon$. Because every subsequence has this property, we can conclude that as $n \to \infty$, $|\bar{Q}_*^n(\infty) - \Delta(l_* \vee (\bar{W}_*^n(\infty) \wedge u_*))| \Rightarrow 0$. As $\bar{\mathfrak{X}}^n(\infty)$ and $\widetilde{\mathfrak{X}}^n(\infty)$ have the same distribution, we then obtain (73).

Next note that because $\widetilde{\mathfrak{X}}^n(\infty)$ is tight, for every subsequence, there is a further subsequence such that $\widetilde{Q}_*^n(\infty) \Rightarrow \widetilde{Q}_*(\infty)$ for some random vector $\widetilde{Q}_*(\infty)$. Then from (73) and the convergence-together theorem (Whitt 2002, Theorem 11.4.7), one has $(\widetilde{Q}_*^n(\infty), \Delta(l_* \vee (\widetilde{W}_*^n(\infty) \wedge u_*))) \Rightarrow (\widetilde{Q}_*(\infty), \widetilde{Q}_*(\infty))$. From Lemma 1, $h(l_* \vee (\widetilde{W}^n(\infty) \wedge u_*)) = \sum_{k\in\mathcal{K}} g_k(\Delta_k(l_* \vee (\widetilde{W}^n(\infty) \wedge u_*)))$. For $(x, y) \in \mathbb{R}^{2K}$, introduce a continuous mapping $f(x, y) = \sum_{k\in\mathcal{K}} g_k(x_k) - \sum_{k\in\mathcal{K}} g_k(y_k)$, then one can verify that $g(\cdot,\cdot)$ is a continuous function. Applying the continuous mapping theorem (Whitt 2002, Theorem 3.4.3) with f to the further subsequence, one then has

$$\left|\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) - h(l_* \vee (\widetilde{W}^n(\infty)\wedge u_*))\right| \Rightarrow 0. \tag{74}$$

Because all of these subsequences have the same limit, (74) holds for the whole sequence.

In view of (74), to prove Proposition 3, it is enough to prove the uniform integrability of the sequence of random variables $\left\{\left|\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) - h(l_*\vee(\widetilde{W}^n(\infty)\wedge u_*))\right| : n\geq 1\right\}$. Note that $l_*\vee(\widetilde{W}^n(\infty)\wedge u_*)$ is uniformly bounded for all n, hence it suffices to show the uniform integrability of $\{\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) : n\geq 1\}$. Due to the sub-polynomial assumption in Assumption 1, it is enough to prove $\sup_{n\geq 1}\mathbb{E}\left[|\widetilde{Q}^n_*(\infty)|^{m+1}\right] < \infty$ (Ethier and Kurtz 1986, Theorem A.2.2, page 494). This can be directly proved by using the tail probability bound established in Lemmas 7 and 10. We omit the details. Hence the proof is complete. □

E. Proof of Lemma 2 We first introduce a lemma.

LEMMA 14. *Assume $\pi(\cdot)$ is Lipschitz continuous on $\mathbb{R}$, and $h(\cdot)$ is continuous on $\mathbb{R}$, strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$, $h(0)=0$ and $\lim_{|x|\to\infty} h(x)=\infty$. For each $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, there is a unique continuously differentiable solution $w(x;w_0,\gamma)$, which is jointly continuous in w_0 and γ, to the following ODE:*

$$\begin{aligned} \frac{1}{2}\sigma^2 w'(x) + \pi(w(x)) + h(x) = \gamma, \quad &\textit{for} \quad x\in\mathbb{R}, \\ \textit{subject to} \quad &w(0)=w_0. \end{aligned} \tag{75}$$

Furthermore, $w'(x;w_0,\gamma)$ is continuous in x, w_0 and γ, and the following hold:

1. *For $\omega_0\in\mathbb{R}$, the solution $w(x;w_0,\gamma)$ is strictly increasing in γ for fixed $x>0$ and strictly decreasing in γ for fixed $x<0$. The solution $w(x;w_0,\gamma)$ is strictly increasing in w_0 for fixed $x,\gamma\in\mathbb{R}$. For fixed $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, $w(x;w_0,\gamma)$ cannot have a local minimizer in $(0,\infty)$, cannot have a local maximizer in $(-\infty,0)$, and cannot be a constant in any interval on $\mathbb{R}$.*
2. *Assume $\gamma_1(w)$ is continuous and strictly increasing in w, then $\max_{x\geq 0} w(x;w_0,\gamma_1(w_0))$ is continuous and strictly increasing in w_0; assume $\gamma_2(w)$ is continuous and strictly decreasing in w, then $\min_{x\leq 0} w(x;w_0,\gamma_2(w_0))$ is continuous and strictly increasing in w_0.*
3. *For $k>w_0$, there exists a unique number $\gamma_+(w_0)$ such that $\max_{x\geq 0} w(x;w_0,\gamma_+(w_0))=k$, and the maximizer is unique. The functions $\gamma_+(w_0)$ is continuous and strictly decreasing in w_0. For $a<b$, if either i) $\min(-h'(0-),h'(0+))>0$; or ii) $b=0$ and $\pi(x)=-\mu x^2$ for $x\in[a,b]$ with $\mu>0$, then there exists $w_*\in(a,b)$ such that $\max_{x\geq 0} w(x;w_*,\gamma_+(w_*))=b$ (with a unique maximizer u_*) and $\min_{x\leq 0} w(x;w_*,\gamma_+(w_*))=a$ (with a unique minimizer l_*). The function $w(x;w_*,\gamma_+(w_*))$ is strictly increasing on $[l_*,u_*]$ and $w(x;w_*,\gamma_+(w_*))\in[a,b]$ for $x\in[l_*,u_*]$.*

We now prove Lemma 2 by using Lemma 14. Let $a=-\kappa$ (recall κ in (14)) and $b=0$. Fix $M\geq\kappa^2$ and let $\mu=\frac{m'H^{-1}m}{4}$, $\sigma^2=\sum_{k\in\mathcal{K}}\frac{2\lambda_k^*}{\mu_k^2}$, and $\pi(u)=-\mu\min(u^2,M)$. Then π is Lipschitz continuous. One can verify that h in (13) satisfies the requirement in Lemma 14. With $w(x;w_*,\gamma_+(w_*)), w_*, l_*, u_*$ in Lemma 14 item 3, introduce $v(x)=w(x;w_*,\gamma_+(w_*))$ on $[l_*,u_*]$. Then $|v(x)|^2\leq\kappa^2\leq M$ and hence $\pi(v(x))=-\mu v^2(x)$ for $x\in[l_*,u_*]$. Thus the constants $l_*<0<u_*, \gamma^*\in\mathbb{R}$ and a function $v\in C^1[l_*,u_*]$ satisfying

$$\frac{1}{2}\sigma^2 v'(x) - \mu v^2(x) + h(x) = \gamma^*, \quad \text{for} \quad x\in[l_*,u_*], \tag{76}$$

subject to $v(x)\in[-\kappa,0]$ for all $x\in[l_*,u_*]$ and the boundary and smooth pasting conditions

$$v(l_*)=-\kappa, \quad v(u_*)=0, \quad v'(l_*+)=0, \quad \text{and} \quad v'(u_*-)=0.$$

Now we prove the uniqueness of l_*, u_*, γ^* and v satisfying (76) and the conditions. Assume that there exist $\tilde{l}_* < 0 < \tilde{u}_*, \tilde{\gamma}^*$ and $\tilde{v}$ satisfying (76) with the conditions. We first prove that $\tilde{\gamma}^* \neq \gamma^*$ is impossible. We consider the case $\tilde{\gamma}^* > \gamma^*$ (the case $\tilde{\gamma}^* < \gamma^*$ is similar). From the boundary and smooth pasting conditions one has $h(\tilde{u}_*) > h(u_*)$ and $h(\tilde{l}_*) > h(l_*)$, which ensure $\tilde{u}_* > u_* > 0$ and $\tilde{l}_* < l_* < 0$. Assume $\tilde{v}(0) \geq v(0)$ (the case $\tilde{v}(0) < v(0)$ can be argued similarly). Then from Lemma 14 item 1, one has $\tilde{v}(x) > v(x)$ for $x \in (0, u_*]$, especially $\tilde{v}(u_*) > v(u_*) = b = \tilde{v}(\tilde{u}_*)$. This is a contradiction to $\tilde{v}(x) \in [a,b]$ for $x \in [\tilde{l}_*, \tilde{u}_*]$. As a result, $\tilde{\gamma}^* = \gamma^*$. Then from the boundary conditions, one has $h(\tilde{u}_*) = h(u_*)$ and $h(\tilde{l}_*) = h(l_*)$, which ensure $\tilde{u}_* = u_*$ and $\tilde{l}_* = l_*$. From these, one can also verify that $\tilde{v} = v$. This proves the uniqueness.

Let

$$\bar{v}(x) = -\kappa \times 1_{\{x \in (-\infty, l^*)\}} + v(x) \times 1_{\{[l^*, u^*]\}} + 0 \times 1_{\{(u^*, \infty)\}},$$

and define $\Phi(x) = \int_{l^*}^x \bar{v}(y)dy$. It is easy to verify that $\Phi \in C^2(\mathbb{R})$ and l_*, u_*, γ^* satisfy the differential equation with the corresponding boundary and smooth pasting conditions. The uniqueness of Φ follows from that of v. The constant γ^* is positive by checking the ODE (16) at the u_*.

Next we verify Φ satisfies (17)–(18). Note that $\Phi \in C^2(\mathbb{R})$ and $\Phi'' = \bar{v}'$ is locally L^1, hence it has the third-order derivative almost everywhere. It is easy to verify (18) and $|\Phi'''(x)| \leq C$ for $C > 0$ whenever it exists, and we will focus on the verification of (17).

From the expression of $c(\cdot)$ in (15), one has $-u\Phi'(x) + c(u) = -u\Phi'(x) + \frac{1}{m'H^{-1}m}u^2 \geq -\frac{m'H^{-1}m}{4}(\Phi'(x))^2$. Hence it is enough to verify that $\frac{\sigma^2}{2}\Phi''(x) - \frac{m'H^{-1}m}{4}(\Phi'(x))^2 + h(x) \geq \gamma^*$. It suffices to consider the cases $x > u^*$ and $x < l^*$. Note that for $x > u^*$, we have $\Phi'(x) = \bar{v}(x) = 0 = v(u^*) = \bar{v}(u^*) = \Phi'(u^*)$ and $\Phi''(x) = \bar{v}'(x) = 0 = \bar{v}'(u^*) = \Phi''(u^*)$. It follows that for $x > u^*$,

$$\begin{aligned}\frac{\sigma^2}{2}\Phi''(x) - \frac{m'H^{-1}m}{4}(\Phi'(x))^2 + h(x) &= \frac{\sigma^2}{2}\Phi''(u^*) - \frac{m'H^{-1}m}{4}(\Phi'(u^*))^2 + h(u^*) + h(x) - h(u^*) \\ &= \gamma^* + h(x) - h(u^*) \geq \gamma^*,\end{aligned}$$

where we use the fact that h is increasing on $[0,\infty)$. This verifies the inequality (17) for $x > u^*$. A similar argument (with h decreasing on $(-\infty, 0]$) can yield the inequality for $x < l^*$. □

Proof of Lemma 14. We first state two auxiliary lemmas. Lemma 15 summarizes several results from Cao and Yao (2018) (see Lemmas 5, 6, and 9 there).

LEMMA 15. *Assume $\pi(\cdot)$ is a Lipschitz continuous function on $\mathbb{R}$, and h is continuous and strictly increasing on $[0,\infty)$, $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. For each $w_0 \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, there is a unique continuously differentiable solution $w(x; w_0, \gamma)$ on $[0, \infty)$, which is jointly continuous in w_0 and γ to the following ODE:*

$$\begin{aligned}\frac{1}{2}\sigma^2 w'(x) + \pi(w(x)) + h(x) = \gamma \quad &\textit{for} \quad x \geq 0, \\ \textit{subject to} \quad &w(0) = w_0.\end{aligned} \tag{77}$$

Furthermore, $w'(x; w_0, \gamma)$ is continuous in $x \geq 0$, $w_0 \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, and the following hold:

1. *The solution $w(x; w_0, \gamma)$ is strictly increasing in $\gamma \in \mathbb{R}$ for fixed $x > 0$ and $w_0 \in \mathbb{R}$, and is strictly increasing in $w_0 \in \mathbb{R}$ for fixed $x \geq 0$ and $\gamma \in \mathbb{R}$. For fixed $w_0 \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $w(x; w_0, \gamma)$ cannot have a local minimizer in $x \in (0, \infty)$, and cannot be a constant in any interval on $[0, \infty)$.*
2. *For $k > w_0$, there exists a unique number $\gamma_+(w_0)$ such that $\max_{x\geq 0} w(x; w_0, \gamma_+(w_0)) = k$ with the unique maximizer denoted by $x^*(w_0, \gamma_+(w_0))$. The functions $\gamma_+(w_0)$ and $x^*(w_0, \gamma_+(w_0))$ are both continuous and strictly decreasing in w_0.*

LEMMA 16. *Consider the ODE in (77). Assume $\gamma_1(w)$ is continuous and strictly increasing in w, then $\max_{x\geq 0} w(x; w_0, \gamma_1(w_0))$ is continuous and strictly increasing in w_0.*

We now prove Lemma 14. For $w_0, \gamma \in \mathbb{R}$, from Lemma 15, there is a unique solution $w(x; w_0, \gamma)$ to (75) for $x \geq 0$. Let $\pi_1(x) = \pi(-x)$ and $h_1(x) = h(-x)$ for $x \geq 0$ and from Lemma 15, there is a unique $\bar{w}(x; -w_0, \gamma)$ solving (75) for $x \geq 0$ but with π, h and w_0 replaced by π_1, h_1 and $-w_0$. Let $w(x; w_0, \gamma) = -\bar{w}(-x; -w_0, \gamma)$ for $x \leq 0$. This proves the existence of $w(x; w_0, \gamma)$ on $\mathbb{R}$. Items 1 and 2 can be easily verified using Lemmas 15 and 16. Next we focus on proving item 3.

The existence of the function $\gamma_+(\cdot)$ follows from Lemma 15 item 2. Similarly, there exists a unique number $\gamma_-(w_0)$ such that $\min_{x\leq 0} w(x;w_0,\gamma_-(w_0)) = a$ for all $\omega_0 > a$, and the function $\gamma_-(w_0)$ is continuous and strictly increasing in w_0. If $\gamma_-(\frac{a+b}{2}) = \gamma_+(\frac{a+b}{2})$, then the conclusion holds with $w^* = \frac{a+b}{2}$. In the following, we consider $\gamma_-(\frac{a+b}{2}) \neq \gamma_+(\frac{a+b}{2})$.

If $\gamma_-(\frac{a+b}{2}) < \gamma_+(\frac{a+b}{2})$, let $\gamma(w_0) = \gamma_-(w_0)$ for $w_0 \in (a,b)$; otherwise, let $\gamma(w_0) = \gamma_+(w_0)$ for $w_0 \in (a,b)$. We will consider the case $\gamma(w_0) = \gamma_-(w_0)$ because the other one is similar.

Because $\gamma(\frac{a+b}{2}) = \gamma_-(\frac{a+b}{2}) < \gamma_+(\frac{a+b}{2})$, from item 1,

$$\max_{x\geq 0} w\left(x;\frac{a+b}{2},\gamma\left(\frac{a+b}{2}\right)\right) < \max_{x\geq 0} w\left(x;\frac{a+b}{2},\gamma_+\left(\frac{a+b}{2}\right)\right) = b. \tag{78}$$

Because $\gamma(w_0) = \gamma_-(w_0)$ is continuous and strictly increasing in w_0, hence from item 2, $\max_{x\geq 0} w(x;w_0,\gamma(w_0))$ is continuous in w_0. Note that because $\min_{x\leq 0} w(x;b,\gamma(b)) = a < b = w(0;b,\gamma(b))$ and $w(x;b,\gamma(b))$ cannot have a local maximizer on $(-\infty,0)$, $w(x;b,\gamma(b))$ is strictly increasing for $x\leq 0$ around 0, and hence $w'(0;b,\gamma(b)) \geq 0$. Next we prove $w'(0;b,\gamma(b)) \neq 0$.

For case i), if $w'(0;b,\gamma(b)) = 0$, then for $x<0$ around 0, we have $w''(x;b,\gamma(b)) = -\frac{2}{\sigma^2}\pi'(w(x;b,\gamma(b)))w'(x;b,\gamma(b)) - \frac{2}{\sigma^2}h'(x)$. Since we assume $h'(0-) < 0$, it follows that $w'(x;b,\gamma(b)) < 0$ for $x<0$ around 0. hence $w(x;b,\gamma(b))$ decreases for such x and a contradiction. Hence $w'(0;b,\gamma(b)) \neq 0$.

For case ii), if $w'(0;b,\gamma(b)) = 0$, because $h(0) = 0$ and $\pi(b) = 0$ (because $b = 0$) then $\gamma(b) = 0$. Then $w'(x;b,\gamma(b)) = \frac{2\mu}{\sigma^2}w(x;b,\gamma(b))^2 - \frac{2h(x)}{\sigma^2}$ for $x\leq 0$ around 0, where $w(0;b,\gamma(b)) = 0$ This is a Riccati equation, so we obtain $w(x;b,\gamma(b)) = -\frac{\sigma^2 u'(x)}{2\mu u(x)}$ with u solving $u''(x) - \frac{4\mu}{\sigma^4}h(x)u(x) = 0$ and $u'(0) = 0, u(0) = 1$. Because $h(x) > 0$ for $x<0$, then $u(x) > 0$ implies $u''(x) > 0$. Hence $u'(x)$ is increasing around 0, which gives $u'(x) < 0$ for $x<0$. As a result, $w(x;b,\gamma(b)) > 0$ for $x<0$ around 0. This is a contradiction to the fact that $w(x;b,\gamma(b)) > 0$ strictly increases to 0 for $x<0$. Hence $w'(0;b,\gamma(b)) \neq 0$.

As a result, we must have $w'(0;b,\gamma(b)) > 0$, and hence

$$\lim_{w_0\uparrow b}\max_{x\geq 0} w(x;w_0,\gamma(w_0)) = \max_{x\geq 0} w(x;b,\gamma(b)) > b. \tag{79}$$

Combining (78), (79) and item 2, there is $w_* \in (\frac{a+b}{2}, b)$ such that $\max_{x\geq 0} w(x;w_*,\gamma(w_*)) = b$. Then $\gamma(w_*) = \gamma_-(w_*) = \gamma_+(w_*)$, and the proof is complete. □

Proof of Lemma 16. The monotonicity of $\max_{x\geq 0} w(x;w_0,\gamma_1(w_0))$ follows from that if $w_{01} < w_{02}$, then

$$w(x;w_{01},\gamma_1(w_{01})) < w(x;w_{02},\gamma_1(w_{01})) < w(x;w_{02},\gamma_1(w_{02})), \quad \text{for each } x\geq 0, \tag{80}$$

where the first inequality follows from Lemma 15 (1) and the second inequality follows from the assumption that $\gamma_1(w_0)$ is strictly increasing in w_0 and Lemma 15 (1).

Next we prove the continuity, i.e., $G(w_0) := \max_{x\geq 0} w(x;w_0,\gamma_1(w_0))$ is a continuous function of w_0. If not, then there exists $\tilde{w}_0$ such that at least one of the following occurs: (1) $\lim_{w_0\uparrow\tilde{w}_0} G(w_0) < G(\tilde{w}_0)$; (2) $\lim_{w_0\downarrow\tilde{w}_0} G(w_0) > G(\tilde{w}_0)$. The limits are well defined due to the monotonicity of G. We will focus on Case (2) as Case (1) can be argued similarly.

Suppose $\lim_{w_0\downarrow\tilde{w}_0} G(w_0) > G(\tilde{w}_0)$. Then $G(\tilde{w}_0) < \infty$. Denote by $\bar{x}_0 = \text{argmax}_{x\geq 0} w(x;\tilde{w}_0,\gamma_1(\tilde{w}_0))$, which is finite due to $\lim_{x\to\infty} h(x) = \infty$ and $G(\tilde{w}_0) < \infty$. Then there exists $x_1 > \bar{x}_0$ such that $w(x_1;\tilde{w}_0,\gamma_1(\tilde{w}_0)) < w(\bar{x}_0;\tilde{w}_0,\gamma_1(\tilde{w}_0))$. Denote by $\epsilon = w(\bar{x}_0;\tilde{w}_0,\gamma_1(\tilde{w}_0)) - w(x_1;\tilde{w}_0,\gamma_1(\tilde{w}_0)) > 0$. Because $w(x_1;w_0,\gamma_1(w_0))$ is continuous in w_0, when $w_0 > \tilde{w}_0$ is sufficiently close to $\tilde{w}_0$,

$$w(x_1;w_0,\gamma_1(w_0)) < w(x_1;\tilde{w}_0,\gamma_1(\tilde{w}_0)) + \epsilon = w(\bar{x}_0;\tilde{w}_0,\gamma_1(\tilde{w}_0)) \leq w(\bar{x}_0;w_0,\gamma_1(w_0)),$$

where the last inequality follows from (80). Hence, for a fixed $w_0 > \tilde{w}_0$ sufficiently close to $\tilde{w}_0$, we can deduce that $w(x;w_0,\gamma_1(w_0))$ achieves its maximum in $[0,x_1]$ by Lemma 15 (1). Now for each $x\in[0,x_1]$, we have $w(x;w_0,\gamma_1(w_0)) \downarrow w(x;\tilde{w}_0,\gamma_1(\tilde{w}_0))$ as $w_0 \downarrow \tilde{w}_0$. Using Dini's Theorem, the convergence of $w(\cdot;w_0,\gamma_1(w_0))$ to $w(\cdot;\tilde{w}_0,\gamma_1(\tilde{w}_0))$ on $[0,x_1]$ as $w_0\downarrow\tilde{w}_0$ is uniform. Hence $G(w_0)\downarrow G(\tilde{w}_0)$ as $w_0\downarrow\tilde{w}_0$, which leads to a contradiction. Therefore, we must have $\lim_{w_0\uparrow\tilde{w}_0} G(w_0) = G(\tilde{w}_0) = \lim_{w_0\downarrow\tilde{w}_0} G(w_0)$. This proves the continuity of G. The proof is therefore complete. □

Acknowledgments. We would like to thank Ton Dieker, Itai Gurvich, Hanqin Zhang and Sean Zhou for their helpful comments on an earlier version of this paper.

References

Arapostathis, A., Biswas, A., and Pang, G. (2015). Ergodic control of multi-class M/M/N+M queues in the Halfin-Whitt regime. *Annals of Applied Probability*, 25(6):3511–3570.

Ata, B. and Barjesteh, N. (2019). An approximate analysis of dynamic pricing, outsourcing, and scheduling for a multiclass make-to-stock queue in the heavy traffic regime. *Working Paper, URL: https://ssrn.com/abstract=3464763*.

Bramson, M. (1998). State space collapse with application to heavy traffic limits for multiclass queueing networks. *Queueing Systems*, 30(1-2):89–140.

Budhiraja, A., Ghosh, A., and Lee, C. (2011). Ergodic rate control problem for single class queueing networks. *SIAM Journal on Control and Optimization*, 49(4):1570–1606.

Cao, P. and Yao, D. (2018). Optimal drift rate control and impulse control for a stochastic inventory/production system. *SIAM Journal on Control and Optimization*, 56(3):1856–1883.

Dai, J. G. and Dai, W. (1999). A heavy traffic limit theorem for a class of open queueing networks with finite buffers. *Queueing Systems*, 32:5–40.

Dai, J. G. and Yao, D. (2013). Brownian inventory models with convex holding cost, part 1: Average-optimal controls. *Stochastic Systems*, 3(2):442–499.

Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley.

Gamarnik, D. and Zeevi, A. (2006). Validity of heavy traffic steady-state approximations in generalized Jackson networks. *Annals of Applied Probability*, 16(1):56–90.

Goldberg, D. A., Reiman, M. I., and Wang, Q. (2021). A survey of recent progress in the asymptotic analysis of inventory systems. *Production and Operations Management*, Forthcoming.

Harrison, J. M. (1988). Brownian models of queueing networks with heterogeneous customer populations. In *Stochastic Differential Systems, Stochastic Control Theory and Applications*, pages 147–186. Springer.

Harrison, J. M. and Taksar, M. I. (1983). Instantaneous control of Brownian motion. *Mathematics of Operations Research*, 8(3):439–453.

Huang, J. and Gurvich, I. (2018). Beyond heavy-traffic regimes: Universal bounds and controls for the single-server queue. *Operations Research*, 66(4):1168–1188.

Jacod, J. and Shiryaev, A. (2013). *Limit theorems for stochastic processes*. Springer.

Mandelbaum, A. and Stolyar, A. L. (2004). Scheduling flexible servers with convex delay costs: Heavy-traffic optimality of the generalized $c\mu$-rule. *Operations Research*, 52(6):836–855.

Meyn, S. and Tweedie, R. (1993). Stability of markovian processes III: Foster-lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25(3):518–548.

Plambeck, E., Kumar, S., and Harrison, J. M. (2001). A multiclass queue in heavy traffic with throughput time constraints: Asymptotically optimal dynamic controls. *Queueing Systems*, 39(1):23–54.

Plambeck, E. L. and Ward, A. R. (2006). Optimal control of a high-volume assemble-to-order system. *Mathematics of Operations Research*, 31(3):453–477.

Prato, G. D. (2006). *An Introduction to Infinite-Dimensional Analysis*. Springer.

Reiman, M. I. and Wang, Q. (2015). Asymptotically optimal inventory control for assemble-to-order systems with identical lead times. *Operations Research*, 63(3):716–732.

van Mieghem, J. A. (1995). Dynamic scheduling with convex delay costs: The generalized $c\mu$ rule. *Annals of Applied Probability*, 5(3):809–833.

Ward, A. R. and Kumar, S. (2008). Asymptotically optimal admission control of a queue with impatient customers. *Mathematics of Operations Research*, 33(1):167–202.

Wein, L. M. (1992). Dynamic scheduling of a multiclass make-to-stock queue. *Operations Research*, 40(4):724–735.

Whitt, W. (2002). *Stochastic-Process Limits*. Springer-Verlag, New York.

Xu, Y. and Chao, X. (2009). Dynamic pricing and inventory control for a production system with average profit criterion. *Probability in the Engineering and Informational Sciences*, 23(3):489–513.

Zheng, Y. and Zipkin, P. H. (1990). A queueing model to analyze the value of centralized inventory information. *Operations Research*, 38(2):296–307.