Asymptotically Optimal Control of Make-to-Stock Systems

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We consider multi-class make-to-stock production/inventory systems in which the manager makes three decisions including pricing, outsourcing and scheduling to maximize the long-run average profit. For a sequence of systems in the heavy-traffic regime, with linear or strictly convex holding/waiting cost functions, we propose a sequence of policies and establish its asymptotic optimality. Our proof combines the lower bound approach and a thorough steady-state analysis of the systems. We also establish general results on the existence and tightness of the stationary distributions of the state processes under a more general family of policies.

Key words: Make-to-stock, heavy traffic, steady-state analysis, asymptotically optimal control

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1. Introduction We consider the dynamic control of a make-to-stock production/inventory system, in which multiple types of products are produced and stored in inventory to satisfy customer demand. Customers are price-sensitive in the sense that the demand arrival processes depend on the prices of the products. A customer's demand is satisfied if the desired product is available in the inventory; otherwise, the customer waits in a queue or the manager may decide to outsource the new order if too many orders are waiting. The cost of outsourcing a customer order is fixed. Products held in the inventory incur inventory holding costs, and customers waiting for products incur waiting costs. The system manager's objective is to maximize the long-run average profit by dynamically making three types of decisions: pricing, outsourcing (whether to outsource a new order or not) and scheduling (which product to prioritize in the production process).

The above system, with *linear* holding/waiting cost functions, was recently proposed and analyzed in Ata and Barjesteh (2019), which generalized the classical model in Wein (1992) by incorporating pricing and outsourcing decisions. In general, the problem of finding the optimal control policy for the system is analytically intractable. Hence, Ata and Barjesteh (2019) considered the heavy-traffic regime (see e.g. Harrison (1988)) where both system capacity and demand are large, with server utilization close to one, and conducted an approximate analysis. They formulated and analyzed the corresponding Brownian control problem (BCP); i.e., the limiting control problem associated with Brownian motions. Then they interpreted the solutions of the BCP in the context of the original control problem for the make-to-stock system. Although Ata and Barjesteh (2019) illustrated the effectiveness of their policy via simulation studies, they did not analyze their proposed policy directly for the original system.

In this paper, we consider the above system with both *linear* and *strictly convex* holding/waiting cost functions. We propose a dynamic control policy and *theoretically* prove its asymptotic optimality in the heavy-traffic regime. We also establish general results on the existence and tightness of stationary distributions for multi-class make-to-stock systems under a more general family of policies. These steady-state results do not depend on the holding/waiting cost functions, hence they are of independent interest and may be used to analyze multi-class make-to-stock systems under other scheduling policies. Despite being closely related to Ata and Barjesteh (2019), our paper is significantly different in several aspects.

First, our focus is to propose a policy and prove its asymptotic optimality for multi-class make-to-stock systems modeled by multi-dimensional continuous-time Markov chains with discrete state spaces. In contrast, their focus is to solve the Bellman equation for the BCP, and then use the continuous state solution to inform a policy. They do not prove the asymptotic optimality of their policy. As such, we must employ new methods that are totally different from theirs. Specifically, we develop a new method by combining the lower bound approach with steady-state State Space Collapse (SSC) to prove the asymptotic optimality of our proposed policy for the multi-class make-to-stock system. To the best of our knowledge, our paper is the first to use such a method to analyze a production/inventory system. The lower bound approach (also known as the verification theorem) is a standard approach to analyze BCPs with continuous states; for example, see Harrison and Taksar (1983) and Dai and Yao (2013). We adapt the method to analyze our make-to-stock systems with discrete states. Depending on whether products are held in inventory or customer orders are waiting, the system states can be either positive or negative, which results in several difficulties for the steady-state analysis of the underlying multi-dimensional Markov chains. To prove the existence of stationary distributions, we use the Foster-Lyapunov criteria (for example, see Meyn and Tweedie (1993)), and to establish the tightness of stationary distributions, we adopt the approach in Gamarnik and Zeevi (2006) to obtain tail bounds for stationary distributions. Despite using standard frameworks, because the system states can be either positive or negative, difficulties come in the form of carefully designing appropriate Lyapunov functions, which require us to explore the details of the proposed policies. In order to prove the steady-state SSC, we apply the framework in Bramson (1998) to establish uniform attractions of the hydrodynamic limits, with the difficulty mainly lying in a detailed analysis of different scenarios for the systems under the proposed policy.

Second, our proposed policy has some important differences compared to Ata and Barjesteh (2019). When the holding/waiting cost functions are linear, our proposed policy is almost identical to their policy. However, there is an important difference in the outsourcing decision, where we will not outsource an order if the inventory level of the only class to be outsourced exceeds the safety stock. Our outsourcing decision is intuitive but cannot be derived from the corresponding BCP, and this difference appears to be essential to the construction of Lyapunov functions in the steady-state analysis. When the holding/waiting cost functions are strictly convex, the pricing and outsourcing policies share similar structures as in the linear cost setting. However, the scheduling policy is different and extends the generalized $c\mu$ rule (van Mieghem (1995)) to make-to-stock systems with both positive and negative states.

Third, we consider both linear and strictly convex holding/waiting cost functions, while Ata and Barjesteh (2019) only consider linear holding/waiting cost functions. There are many applications that motivate convex holding/waiting costs; see e.g. Dai and Yao (2013), Mandelbaum and Stolyar (2004), and van Mieghem (1995). Mathematically, generalization from linear to strictly convex cost functions for our problem is non-trivial, as additional technical difficulties arise from the analysis of certain free boundary problems. In Ata and Barjesteh (2019), the authors derive the Bellman equation for the BCP, which consists of an ODE with several free boundary conditions. They solve this free-boundary ODE explicitly by solving a Riccati equation, and relying on the linear cost structure. In contrast, we need to consider more general free-boundary ODEs to incorporate both linear and strictly convex holding/waiting cost functions. The solution to the new ODEs plays an important role in determining the parameters of our proposed policy and establishing its asymptotic optimality. We contribute to the literature by proving the existence of a unique smooth solution to this new class of free-boundary ODEs.

Our contributions. To summarize, although the contribution of modelling is credited to Ata and Barjesteh (2019) and our proposed policy is inspired by theirs, we still make the following significant contributions:

1. We propose a policy for a multi-class make-to-stock system with linear or strictly convex holding/waiting cost functions, and establish the asymptotic optimality of the policy.

2. We establish general results on the existence and tightness of the stationary distributions of the inventory processes. These results do not rely on the structures of holding/waiting cost functions, and hence may be applicable to analyze other policies.

3. Our proof illustrates how to combine the lower bound approach with steady-state SSC to analyze discrete-state production/inventory systems with both positive and negative states.

1.1. Literature Review We survey relevant studies and explain the difference between our work and the existing literature. For a comprehensive literature review on make-to-stock systems, see Ata and Barjesteh (2019).

The earlier work on the scheduling of multi-class make-to-stock systems includes Zheng and Zipkin (1990). It is generally difficult to derive exact solutions for the optimal dynamic scheduling policy for multi-class make-to-stock systems. As such, Wein (1992) considers the heavy-traffic regime and solve a related BCP to propose a scheduling policy for the systems. These papers do not consider pricing and outsourcing decisions as in our work. Dynamic pricing has been studied in Xu and Chao (2009) for a make-to-stock production system selling a single-type product. They obtain optimal pricing and production control policy for the system.

In a broader context, our work is related to asymptotic analysis of production/inventory systems. See e.g. Plambeck and Ward (2006), Reiman and Wang (2015) and a recent survey paper Goldberg et al. (2021). Reiman and Wang (2015) study multi-product assemble-to-order (ATO) systems and propose a policy that is asymptotically optimal when the lead time grows. Our paper considers the heavy-traffic regime where demand and system capacity both grow, which is more related to Plambeck and Ward (2006). Plambeck and Ward (2006) study the optimal control of a high-volume ATO system with the objective of maximizing expected infinite horizon discounted profit. They prove asymptotic optimality of their proposed policy in the heavy-traffic regime. Our work differs from theirs in that we consider different systems and focus on long-run average profit as the objective. In turn, this leads to a different method of analysis where steady state analysis plays an important role in our work but not in theirs.

The outsourcing and scheduling decisions are related to admission control and dynamic scheduling in queueing systems. Concerning admission control, Plambeck et al. (2001) show that for a multi-class single-server queueing system, to asymptotically achieve a desired bound on the throughput, it is enough to reject one class of customers. Ward and Kumar (2008) develop asymptotically optimal admission control of a $GI/GI/1$ queue with impatient customers in heavy traffic. Our model is different from theirs. Concerning dynamic scheduling, for a single-server queue with a convex waiting cost function, van Mieghem (1995) established the asymptotic optimality of the generalized-$c\mu$ rule; Mandelbaum and Stolyar (2004) generalized the work of van Mieghem (1995) to the parallel-server setting. In our paper, the scheduling policy also has a form of the generalized-$c\mu$ rule when the state cost functions are convex. However, different from the model in van Mieghem (1995) where the states are nonnegative, the system states in our model can be either positive or negative, which complicates the analyses. The long-run average objective we consider is also related to ergodic control of queueing systems. For instance, Budhiraja et al. (2011) prove that near optimal control policies in an associated diffusion control problem can be used to construct asymptotically optimal rate control policies for the original single class queueing networks, Huang and Gurvich (2018) show that the service rate derived from an intuitive Brownian control problem is universally nearly optimal for a single-server queueing system and Arapostathis et al. (2015) consider the scheduling problem in a many-server queue. Although the lower bound approach has also been applied in the latter two papers, we combine the approach with a thorough steady state analysis and state-space collapse in stationarity to analyze a complex make-to-stock production/inventory system.

Organization. The rest of the paper is organized as follows. We describe the model in Section 2, and introduce the heavy traffic framework in Section 3. We introduce the proposed policy and present the main results in Section 4. We provide detailed guidelines on policy implementation in Section 5. The asymptotic optimality of the proposed policy, and general results about the steady states are established in Section 6. The paper is concluded in Section 7. We leave the proofs of auxiliary results in the appendix.

Notation. All vectors are understood to be column vectors. For $K \in \mathbb{N}$, let $\mathbb{D}^K$ be the space of all $\mathbb{R}^K$-valued functions that are right continuous on $[0,\infty)$ and have left limits on $(0,\infty)$, equipped with the Skorohod J_1-topology. All the stochastic processes are assumed to have sample paths in $\mathbb{D}^K$ for an appropriate K. We use $\Rightarrow$ to denote weak convergence. We use $A \subset B$ to mean that A is a strict subset of B. For a sequence of functions $f^n(\cdot) \in \mathbb{D}^K$, $f^n(t) \to f(t)$ u.o.c. as $n \to \infty$ means that $f^n(t)$ uniformly converges to $f(t)$ on compact sets.

2. A Multi-Class Make-to-Stock System We consider a make-to-stock system as in Ata and Barjesteh (2019), adopting their notation and terminology with slight modifications to accommodate our analysis. The system sells K types of products to customers. Each customer order needs one product and a customer order is class k if a type k product is needed. In the following, we will use customer order and order interchangeably. Denote by $\mathcal{K} := \{1, \ldots, K\}$.

The make-to-stock system is modeled as a multi-class single server queueing system. Let $E(\cdot) = \{(E_k(t)); k \in \mathcal{K}, t \geq 0\}$ be the arrival process of orders, with $E_k(t)$ being the number of class k orders which arrived by time t. We assume the process $E_k(\cdot) = \{E_k(t); t \geq 0\}$ is a non-homogeneous Poisson process with rate to be controlled, that is,

$$E_k(t) = N_k\left(\int_0^t \lambda_k(s)ds\right),$$

where $N_k(\cdot) = \{N_k(t); t \geq 0\}$ is a unit rate Poisson process. The arrival rate vector $\lambda(t) = (\lambda_k(t); k \in \mathcal{K})$ is to be controlled, and can choose values from a set $\mathcal{L} \subset \mathbb{R}_+^K$ (with the usual topology). We assume that interior($\mathcal{L}$) is not empty, so $\mathcal{L}$ is not allowed to be a set of discrete points. The manager can control the arrival rate $\lambda(t)$ by controlling the price vector $p(t) = (p_k(t); k \in \mathcal{K})$, which can choose values from another set $\mathcal{P} \subset \mathbb{R}_+^K$. Here $p_k(t)$ is the unit price of product k at time t. Assume that there is a non-negative demand function $\Lambda : \mathcal{P} \to \mathcal{L}$ so that $\lambda(t) = \Lambda(p(t))$ for $t \geq 0$, and a unique inverse demand function $\Lambda^{-1} : \mathcal{L} \to \mathcal{P}$ such that $p(t) = \Lambda^{-1}(\lambda(t))$, then there is a one-to-one relationship between $p(t) \in \mathcal{P}$ and $\lambda(t) \in \mathcal{L}$. As a result, in the following we consider the control of the arrival rates $\lambda(\cdot)$.

Customer orders may be outsourced to avoid long waiting times. Denote by $O(\cdot) = (O_k(\cdot); k \in \mathcal{K})$ the K-dimensional outsourcing process, with $O_k(t)$ being the number of class k orders outsourced up to time t. Introduce a random variable ξ_{ki} for the ith class k order to indicate whether that order is outsourced ($\xi_{ki} = 1$) or not ($\xi_{ki} = 0$). Then

$$O_k(t) = \sum_{i=1}^{E_k(t)} \xi_{ki}. \tag{1}$$

The actual number of class k orders accepted by time t is then

$$A_k(t) = E_k(t) - O_k(t).$$

The production time of each class k product is assumed to be exponential with rate μ_k. For $k \in \mathcal{K}$, let $S_k(t)$ denote the number of class k products produced until time t if the system were to continuously work on class k products up to time t. Then for $k \in \mathcal{K}$, $S_k(\cdot) = \{S_k(t) : t \geq 0\}$ is a Poisson process with rate μ_k. It is assumed that $N_k(\cdot), k \in \mathcal{K}$, and $S_k(\cdot), k \in \mathcal{K}$, are independent. Let $T_k(t)$ denote the cumulative amount of time the system is devoted to class k products until time t, then the number of class k products produced by time t is

$$F_k(t) = S_k(T_k(t)).$$

The manager makes dynamic scheduling decisions by determining $T(\cdot) = (T_k(\cdot); k \in \mathcal{K})$.

The system manager controls the system by deciding the arrival rate, outsourcing new orders and making scheduling decisions to maximize the long-run average profit. A dynamic control policy is denoted by $\psi = (\lambda, O, T)$. Under a control policy ψ, denote by $Q_k(t, \psi)$ the inventory level of class k products at time t, for $k \in \mathcal{K}$. As the policy will be clear from the context, we will drop the notation ψ for brevity. If $Q_k(t) \geq 0$, it denotes the number of class k products in inventory; if $Q_k(t) < 0$, its absolute value denotes the number of class k orders waiting. Then the dynamics of the inventory process $Q_k(\cdot) = \{Q_k(t) : t \geq 0\}$ is

$$Q_k(t) = Q_k(0) + F_k(t) - E_k(t) + O_k(t).$$

We also introduce $C(t)$ to indicate the product class that is currently in production. That is, $C(t) = k, k \in \mathcal{K}$, means the system is producing a class k product at time t, and $C(t) =$

0 means the system is idle. The system state process is $\mathfrak{X}(\cdot) = \{\mathfrak{X}(t); t \geq 0\}$ with $\mathfrak{X}(t) = (Q_1(t), \ldots, Q_K(t), C(t))$.

Introduce the process $I(\cdot) = \{I(t); t \geq 0\}$, with $I(t) = t - \sum_{k\in\mathcal{K}} T_k(t)$ for $t \geq 0$, which is the cumulative idle time of the system by time t.

Definition 1. A control policy $\psi = (\lambda, O, T)$ is said to be *feasible* if it is non-anticipating with respect to $\mathfrak{X}$, $\lambda(t) \in \mathcal{L}$ for $t \geq 0$, and

1. $I(\cdot), T(\cdot), O(\cdot)$ are non-decreasing with $I(0) = T(0) = O(0) = 0$,
2. $I(\cdot), T(\cdot)$ are continuous.

We will focus on feasible Markov control policies[1]. To describe the Markov control policies, we introduce process $\xi(\cdot) = (\xi_k(\cdot), k \in \mathcal{K})$ with $\xi_k(\cdot) = \{\xi_k(t); t \geq 0\}$, in which $\xi_k(s-)$ indicates the outsourcing decision for a virtual new class k order arriving at time s: if $\xi_k(s-) = 1$ and a class k order arrives at s, then that order is outsourced; if $\xi_k(s-) = 0$, that order is accepted. For convenience, to avoid two (unnecessary) changes in the system state simultaneously (i.e. the inventory first decreases by one due to an order arrival at s and then increases by one if the order is outsourced at s), we use $\xi_k(s-)$ instead of $\xi_k(s)$ to indicate the outsourcing decision is made with the information before time s. Denote by τ_{ki} the arrival epoch of the ith class k order. Then $\xi_{ki} = \xi_k(\tau_{ki}-)$, and from (1) we have

$$O_k(t) = \int_0^t \xi_k(s-) dE_k(s). \tag{2}$$

From (2), controlling $O_k(\cdot)$ is done by controlling the process $\xi_k(\cdot)$.

Definition 2. A policy $\psi = (\lambda, O, T)$ is called a *Markov control policy* if $(\lambda(t), \xi(t))$ can be represented as a measurable function, from $\mathbb{R}^{K+1}$ to $\mathbb{R}^{2K}$, of $\mathfrak{X}(t)$. For notational brevity, in the following, we use $(\lambda(\cdot), \xi(\cdot))$ to denote the corresponding measurable function, hence $(\lambda(t), \xi(t)) = (\lambda(\mathfrak{X}(t)), \xi(\mathfrak{X}(t)))$. Denote by Π the set of all feasible Markov control policies.

Note that although a policy $\psi = (\lambda, O, T)$ has three parts, we only assume $(\lambda(t), \xi(t))$ are functions of $\mathfrak{X}(t)$. This is because $T_k(t) = \int_0^t 1_{\{C(s)=k\}} ds$ for $k \in \mathcal{K}$. From this, controlling $T(\cdot)$ is equivalent to controlling $C(\cdot) = \{C(t); t \geq 0\}$, which is part of the system state process $\mathfrak{X}(\cdot)$.

Under a Markov control policy, one can verify that the process $\mathfrak{X}(\cdot)$ is a continuous-time Markov chain with countable state space $S = \mathbb{Z}^K \times (\mathcal{K} \cup \{0\})$. Denote by ν the distribution of $\mathfrak{X}(0)$, the initial state of $\mathfrak{X}(\cdot)$.

Denote by δ_k the variable cost of producing a class k product, and a vector $\delta = (\delta_k)_{k\in\mathcal{K}}$. Then the total self-production costs for all products until time t is $\sum_{k=1}^K \delta_k(E_k(t) - O_k(t))$. In addition, since outsourcing is generally more expensive than self-production, we denote the unit outsourcing cost for class k products by $\delta_k + \vartheta_k$, where $\vartheta_k \geq 0$ is interpreted as the cost *in excess of the self-production cost*. For simplicity, we call ϑ_k the unit outsourcing cost. Then the total outsourcing cost until time t is $\sum_{k=1}^K (\delta_k + \vartheta_k) O_k(t)$. Furthermore, when a class k order arrives at time s, the price for that order is $p_k(s-)$; hence the total revenue from all products until time t is $\int_0^t p(s-) \cdot dE(s) = \sum_{k=1}^K \int_0^t p_k(s-) dE_k(s)$. By combining these three parts, we obtain that the cumulative revenue minus the production and outsourcing costs until time t is

$$\int_0^t (p(s-) - \delta) \cdot dE(s) - \sum_{k\in\mathcal{K}} \vartheta_k O_k(t) = \int_0^t (\Lambda^{-1}(\lambda(s-)) - \delta) \cdot dE(s) - \sum_{k\in\mathcal{K}} \vartheta_k O_k(t).$$

The state cost function of class k, $q_k : \mathbb{R} \to \mathbb{R}_+$, comprises inventory holding and customer waiting costs. We assume that for $k \in \mathcal{K}$, $q_k(x)$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$, with $q_k(0) = 0$. We consider two types of state cost functions: (1) for $k \in \mathcal{K}$, $q_k(x)$ is strictly convex, and (2) for $k \in \mathcal{K}$, $q_k(x)$ is linear on $(-\infty, 0]$ and linear on $[0, \infty)$.

[1] Because of the Poisson arrivals and exponential production times, the control problem we consider is a continuous-time Markov decision process. By Theorem 11.1.1 in Puterman (2014), we can restrict attention to Markov policies when computing average-cost optimal policies; see also Feinberg et al. (2021).

Then the state cost until time t is $\int_0^t \sum_{k\in\mathcal{K}} q_k(Q_k(s))ds$. Hence, the expected cumulative profit process associated with initial distribution ν and policy $\psi=(\lambda,O,T)$ is

$$V(t,\nu,\psi)=\mathbb{E}_\nu\left[\int_0^t \left(\Lambda^{-1}(\lambda(s-))-\delta\right)\cdot dE(s)-\int_0^t \sum_{k\in\mathcal{K}} q_k(Q_k(s))ds-\sum_{k\in\mathcal{K}}\vartheta_k O_k(t)\right].$$

The system manager seeks to find a feasible Markov control policy to maximize the long-run average profit, that is

$$\max_{\psi\in\Pi}\liminf_{t\to\infty}\frac{1}{t}V(t,\nu,\psi). \tag{3}$$

Define the profit rate function $\mathbf{r}$ as follows:

$$\mathbf{r}(x)=x'(\Lambda^{-1}(x)-\delta),\quad x\in\mathcal{L},$$

where $\delta=(\delta_k)$ and recall that δ_k is the variable cost of producing a class k product. One can verify that $\int_0^t(p(s-)-\delta)\cdot d(E(s)-\int_0^s\lambda(u)du)$ is a martingale. Together with the fact that $\lambda(\cdot)$ (because $\lambda(\cdot)=\lambda(\mathfrak{X}(\cdot))$) has only countable jumps, one has

$$\mathbb{E}_\nu\left[\int_0^t(p(s-)-\delta)\cdot dE(s)\right]=\mathbb{E}_\nu\left[\int_0^t \mathbf{r}(\lambda(s))ds\right].$$

Hence

$$V(t,\nu,\psi)=\mathbb{E}_\nu\left[\int_0^t \mathbf{r}(\lambda(s))ds-\int_0^t \sum_{k\in\mathcal{K}} q_k(Q_k(s))ds-\sum_{k\in\mathcal{K}}\vartheta_k O_k(t)\right].$$

Following Ata and Barjesteh (2019), we also consider the static planning problem

$$\text{maximize}\quad \mathbf{r}(\lambda)\quad\text{ subject to } \lambda\in\mathcal{L}. \tag{4}$$

We assume $\mathbf{r}(\cdot)$ is twice continuously differentiable on $\mathcal{L}$ and the problem (4) has a unique optimal solution $\lambda^*\in\text{interior}(\mathcal{L})$ with $\sum_{k\in\mathcal{K}}\lambda_k^*/\mu_k=1$. That is, ignoring the randomness in the system, the profit maximizing demand rate from problem (4) makes the system critically loaded. Note that since $\mathcal{L}\subset\mathbb{R}_+^K$, we have $\lambda_k^*>0$ for $k\in\mathcal{K}$. We also assume the Hessian matrix of $\mathbf{r}$, $\nabla^2\mathbf{r}(\cdot)$, is continuous at point λ^* and that $\nabla^2\mathbf{r}(\lambda^*)$ is negative definite.

3. Heavy Traffic Framework Problem (3) is difficult to solve. Ata and Barjesteh (2019) solved a related BCP and proposed a policy for the linear holding/waiting cost case. We will adopt the same heavy traffic framework with slight modifications. Different from Ata and Barjesteh (2019), where the effectiveness of their policy is illustrated numerically but not rigorously analyzed for discrete-state make-to-stock systems, our purpose is to establish the asymptotic optimality of a feasible Markov control policy for discrete-state systems with either linear or strictly convex holding/waiting cost functions. See Assumption 1 below for more details.

We consider a sequence of make-to-stock systems as above, under an asymptotic framework known as the *conventional heavy traffic* regime. The systems are indexed by $n\in\mathbb{N}$. The relevant parameters and processes in the nth system will be appended with a superscript n. For example, the control in the nth system is denoted by $\psi^n=(\lambda^n,O^n,T^n)$, in which λ^n is the arrival rate process to the nth system, T_k^n records the time allocated to producing class k products, and O_k^n denotes the number of outsourced class k orders. The outsourcing process of class k orders $O_k^n(t)=\int_0^t\xi_k^n(s-)dE_k^n(s)$, hence the control of O^n is done via ξ^n. We follow Ata and Barjesteh (2019) and consider the arrival rate $\lambda^n(t)$ of the form:

$$\lambda^n(t)=n\lambda^*+\sqrt{n}\zeta^n(t),$$

with $\zeta^n:\mathbb{R}_+\to\mathbb{R}^K$. The control of $\lambda^n(\cdot)$ is done via $\zeta^n(\cdot)$, hence we will write the control as $\psi^n=(\zeta^n,O^n,T^n)$. Slightly different from Ata and Barjesteh (2019), we assume the production to be rate $\mu_k^n=n\mu_k$. This is mainly for notational simplicity, and does not change the results. The system state process in the nth system is denoted by $\mathfrak{X}^n(\cdot) = \{\mathfrak{X}^n(t) := (Q_1^n(t), \ldots, Q_K^n(t), C^n(t)); t \geq 0\}$, whose initial state $\mathfrak{X}^n(0)$ follows distribution ν^n. We will focus on feasible Markov control policies, that is, $(\zeta^n(t), \xi^n(t))$ can be represented as a measurable function of $\mathfrak{X}^n(t)$.

Define the diffusion-scaled processes $\widetilde{Q}^n(\cdot) = \{(\widetilde{Q}_1^n(t), \ldots, \widetilde{Q}_K^n(t)); t \geq 0\}$ and $\widetilde{O}^n(\cdot) = \{(\widetilde{O}_1^n(t), \ldots, \widetilde{O}_K^n(t)); t \geq 0\}$ with

$$\widetilde{Q}^n(t) = \frac{Q^n(t)}{\sqrt{n}}, \quad \text{and} \quad \widetilde{O}^n(t) = \frac{O^n(t)}{\sqrt{n}}.$$

Let $\widetilde{\mathfrak{X}}^n(t) = (\widetilde{Q}^n(t), C^n(t))$, where $C^n(t)$ is unscaled and takes values in $\mathcal{K} \cup \{0\}$. Under a feasible Markov control policy ψ^n, the process $\widetilde{\mathfrak{X}}^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n = \frac{1}{\sqrt{n}} \mathbb{Z}^K \times (\mathcal{K} \cup \{0\})$. Let $\widetilde{\nu}^n$ be the distribution of the initial state $\widetilde{\mathfrak{X}}^n(0)$.

Define $\rho_k = \lambda_k^* / \mu_k$ to be the nominal workload of class k orders for $k \in \mathcal{K}$. Let

$$\widetilde{X}_k^n(t) = \frac{S_k^n(T_k^n(t)) - \mu_k^n T_k^n(t)}{\sqrt{n}} - \frac{N_k(\int_0^t \lambda_k^n(s)ds) - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}}. \tag{5}$$

and

$$\widetilde{Y}_k^n(t) = \sqrt{n}(\rho_k t - T_k^n(t)).$$

Then the dynamics of $\widetilde{Q}^n$ under a control policy ψ^n is

$$\begin{aligned}
\widetilde{Q}_k^n(t) =& \widetilde{Q}_k^n(0) + \frac{S_k^n(T_k^n(t))}{\sqrt{n}} - \frac{E_k^n(t)}{\sqrt{n}} + \tilde{O}_k^n(t) \\
=& \widetilde{Q}_k^n(0) + \frac{S_k^n(T_k^n(t)) - \mu_k^n T_k^n(t)}{\sqrt{n}} - \frac{N_k(\int_0^t \lambda_k^n(s)ds) - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}} + \tilde{O}_k^n(t) \\
&+ \frac{\mu_k^n(T_k^n(t) - \rho_k t) + n\lambda_k^* t - \int_0^t \lambda_k^n(s)ds}{\sqrt{n}} \\
=& \widetilde{Q}_k^n(0) + \widetilde{X}_k^n(t) - \mu_k \widetilde{Y}_k^n(t) - \int_0^t \zeta_k^n(s)ds + \tilde{O}_k^n(t).
\end{aligned} \tag{6}$$

We also define the one-dimensional nominal workload process $\widetilde{W}^n(\cdot) = \{\widetilde{W}^n(t); t \geq 0\}$ with

$$\widetilde{W}^n(t) := \sum_{k \in \mathcal{K}} \frac{\widetilde{Q}_k^n(t)}{\mu_k}, \quad \text{for } t \geq 0. \tag{7}$$

Define $\widetilde{I}^n(t) := \sum_{k \in \mathcal{K}} \widetilde{Y}_k^n(t)$. Using $\sum_{k \in \mathcal{K}} \rho_k = 1$, one has $\widetilde{I}^n(t) = \sqrt{n}(t - \sum_{k \in \mathcal{K}} T_k^n(t)) = \sqrt{n} I^n(t)$, where $I^n(\cdot)$ is the cumulative idle time process. Then from (6) and (7), for $t \geq 0$,

$$\widetilde{W}^n(t) = \widetilde{W}^n(0) + \sum_{k \in \mathcal{K}} \frac{\widetilde{X}_k^n(t)}{\mu_k} - \widetilde{I}^n(t) - \int_0^t \sum_{k \in \mathcal{K}} \frac{\zeta_k^n(s)}{\mu_k} ds + \sum_{k \in \mathcal{K}} \frac{\widetilde{O}_k^n(t)}{\mu_k}. \tag{8}$$

Following Ata and Barjesteh (2019), we assume $\Lambda^n(x) = n\Lambda(x)$, then $(\Lambda^n)^{-1}(nx) = \Lambda^{-1}(x)$. The variable cost of producing $\delta = (\delta_k)$ is assumed to be independent of n and hence will not be scaled. Then the resulted profit rate function $\mathbf{r}^n(nx) = n\mathbf{r}(x)$. The unit outsourcing cost (in excess of the self-production cost) ϑ_k^n is assumed to vary with n: $\vartheta_k^n = \frac{r_k}{\sqrt{n}}$, where r_k is a given constant for each k. The relevant setting is where the extra outsourcing cost is small relative to the self-production cost.

We assume that the state cost function $q_k^n(x)$ in the nth system is given by

$$q_k^n(x) = g_k(x/\sqrt{n}).$$

Here the functions $g_k(\cdot): \mathbb{R} \to \mathbb{R}_+$ are assumed to be one of the following two types:

Assumption 1 **(State cost functions)**. 1. *Strictly convex: for each $k\in\mathcal{K}$, g_k is strictly convex, decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$, with $g_k(0)=0$. Furthermore, $g_k(\cdot)$ is a sub-polynomial function, that is, there exist constants $m\in\mathbb{N}$ and $c>0$ such that*

$$g_k(x)\le c(1+|x|^m),\quad \textit{for } x\in\mathbb{R}.$$

2. *Linear: there exist positive constants $h_k,b_k>0$, $k\in\mathcal{K}$, such that for $k\in\mathcal{K}$ and $x\in\mathbb{R}$,*

$$g_k(x)=\begin{cases} h_k x, & x\ge 0,\\ -b_k x, & x<0.\end{cases}$$

We assume that there is a unique class minimizing $h_k\mu_k$ and a unique class maximizing $b_k\mu_k$.

Then the expected cumulative profit process associated with policy ψ^n is

$$\begin{aligned} V^n(t,\nu^n,\psi^n) =&\mathbb{E}_{\nu^n}\left[\int_0^t \left((\Lambda^n)^{-1}(\lambda^n(s))-\delta\right)\cdot dE^n(s)-\sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds-\sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right]\\ =&\mathbb{E}_{\nu^n}\left[\int_0^t n\mathbf{r}(\lambda^*+\zeta^n(s)/\sqrt{n})ds-\sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds-\sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right].\end{aligned}$$

Here, $\mathbb{E}_{\nu^n}$ denotes the expectation conditioned on that the initial state $\mathfrak{X}^n(0)$ follows ν^n.

From (4), $n\mathbf{r}(\lambda^*)t$ serves as an upper bound on the $V^n(t,\nu^n,\psi^n)$. Maximizing $V^n(t,\nu^n,\psi^n)$ is equivalent to minimizing the deviation of $V^n(t,\nu^n,\psi^n)$ from $n\mathbf{r}(\lambda^*)t$, which is

$$\begin{aligned} \widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n) =&n\mathbf{r}(\lambda^*)t-V^n(t,\nu^n,\psi^n)\\ =&\mathbb{E}_{\nu^n}\left[\int_0^t n\cdot(\mathbf{r}(\lambda^*)-\mathbf{r}(\lambda^*+\zeta^n(s)/\sqrt{n}))ds+\sum_{k\in\mathcal{K}}\int_0^t q_k^n(Q_k^n(s))ds+\sum_{k\in\mathcal{K}}\vartheta_k^n O_k^n(t)\right]\\ =&\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^t n\cdot(\mathbf{r}(\lambda^*)-\mathbf{r}(\lambda^*+\zeta^n(s)/\sqrt{n}))ds+\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds+\sum_{k\in\mathcal{K}} r_k\widetilde{O}_k^n(t)\right].\end{aligned}$$

where $\mathbb{E}_{\widetilde{\nu}^n}$ denotes the expectation conditioned on that $\widetilde{\mathfrak{X}}^n(0)$ follows $\widetilde{\nu}^n$. Define

$$c^n(y):=n(\mathbf{r}(\lambda^*)-\mathbf{r}(\lambda^*+y/\sqrt{n}))\ge 0,\quad y\in\mathbb{R}^K. \tag{9}$$

Then we have

$$\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n) =\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^t c^n(\zeta^n(s))ds+\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds+\sum_{k\in\mathcal{K}} r_k\widetilde{O}_k^n(t)\right]. \tag{10}$$

In the following, we will call $\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n)$ the expected loss (due to randomness and the controls) under an initial state distribution $\widetilde{\nu}^n$ and a control ψ^n. The long-run average expected loss for a given control policy ψ^n is defined by

$$\widetilde{V}^n(\widetilde{\nu}^n,\psi^n):=\limsup_{t\to\infty}\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n). \tag{11}$$

We will focus on feasible Markov control policies. For technical reasons, we also assume that the arrival rate cannot change in the order of n to simplify the analysis.

Definition 3. A sequence of polices $\{\psi^n\}$ is called *asymptotically admissible* if for each n, $\psi^n\in\Pi^n$, that is, ψ^n is a feasible Markov control policy of the nth system, and there exists a sequence of nonnegative numbers $\{a_n\}$ with $\lim_{n\to\infty}a_n=0$, such that $\sup_{t\ge 0}\|\zeta^n(t)/\sqrt{n}\|\le a_n$ almost surely for all $n\ge 1$. This implies $\sup_{t\ge 0}\|\zeta^n(t)\|=o(\sqrt{n})$ almost surely.

Denote by $\bar{\Pi}$ the set of sequences of control policies that are asymptotically admissible.

Definition 4 (Asymptotic optimality). A sequence of polices $\{\psi_*^n\}$ is *asymptotically optimal* if it is asymptotically admissible, and for any other sequence of policies $\{\psi^n\}\in\bar{\Pi}$,

$$\liminf_{n\to\infty}\widetilde{V}^n(\widetilde{\nu}^n,\psi^n)\ge\limsup_{n\to\infty}\widetilde{V}^n(\widetilde{\nu}^n,\psi_*^n) \tag{12}$$

for any sequence of initial distributions $\{\widetilde{\nu}^n\}$, with each $\widetilde{\nu}^n$ having a finite $(2m+1)$-th moment.

The moment assumption on $\{\widetilde{\nu}^n\}$ holds if the system starts from any fixed state.

4. Main Results In this section, we propose a sequence of control policies $\{\psi_*^n\}$ and establish its asymptotic optimality. We first introduce an Ordinary Differential Equation (ODE) in Section 4.1, which will help us identify several parameters of the policy. Then we describe the proposed policy and state our main result (Theorem 1) in Section 4.2.

4.1. An ODE For each $w \in \mathbb{R}$, consider the following minimization problem:

$$\begin{aligned} h(w) := \min_{x \in \mathbb{R}^K} \quad & \sum_{k \in \mathcal{K}} g_k(x_k) \\ \text{s.t.} \quad & \sum_{k \in \mathcal{K}} \frac{x_k}{\mu_k} = w. \end{aligned} \tag{13}$$

Denote by $x^* = (x_1^*, \ldots, x_K^*)$ an optimal solution if it exists. It is clear that x^* depends on w. The following existence and uniqueness results are standard, see van Mieghem (1995).

LEMMA 1. *For state cost functions g_k, $k \in \mathcal{K}$, satisfying Assumption 1 and any fixed $w \in \mathbb{R}$, there exists a unique optimal solution x^* to* (13). *Introduce the lifting function $\Delta(\cdot)$ by $\Delta(w) := x^*$. Then the function $\Delta(w) = (\Delta_1(w), \ldots, \Delta_K(w))$ is well defined, and*

$$h(w) = \sum_{k \in \mathcal{K}} g_k(\Delta_k(w)). \tag{14}$$

Define a column vector $m = (m_k)_{k \in \mathcal{K}}$ where $m_k = 1/\mu_k$ for each k, and let $H = -\nabla^2 \mathbf{r}(\lambda^*)/2$. Recall that $\nabla^2 \mathbf{r}(\lambda^*)$ is the Hessian matrix of $\mathbf{r}$ at λ^* and is assumed to be negative definite, hence H is positive definite. Define the effective outsourcing cost κ as follows:

$$i^* = \operatorname{argmin} \left\{ \frac{r_k}{m_k} : k \in \mathcal{K} \right\} \quad \text{and} \quad \kappa = \frac{r_{i^*}}{m_{i^*}}. \tag{15}$$

If there are multiple i^* minimizing $\frac{r_k}{m_k}$, we specify i^* to be the minimum index.

The following lemma identifies some parameters that we will use in the proposed policies. Its proof can be found in Section F. To facilitate the presentation, we define

$$c(u) := \min\{x'Hx : m'x = u, x \in \mathbb{R}^K\} = \frac{1}{m'H^{-1}m} u^2, \quad \text{for } u \in \mathbb{R}, \tag{16}$$

where the last equality is due to Lemma 1 of Ata and Barjesteh (2019).

LEMMA 2. *Let $\sigma^2 = \sum_{k \in \mathcal{K}} \frac{2\lambda_k^*}{\mu_k^2}$. Then there exist unique constants $l_* < 0 < u_*, \gamma^* > 0$ and a unique (up to an additive constant) real-valued function $\Phi \in C^2(\mathbb{R})$ satisfying the differential equation:*

$$\frac{\sigma^2}{2} \Phi''(x) - \frac{m'H^{-1}m}{4} (\Phi'(x))^2 + h(x) = \gamma^*, \quad x \in (l_*, u_*), \tag{17}$$

and $\Phi'(x) \in [-\kappa, 0]$ for all $x \in \mathbb{R}$, with $\Phi'(x) = -\kappa$ for $x \le l_*$, $\Phi'(x) = 0$ for $x \ge u_*$, and $\Phi''(x) = 0$ for $x \notin (l_*, u_*)$. As a consequence, there exists a positive constant C such that for any $u, x \in \mathbb{R}$:

$$\frac{\sigma^2}{2} \Phi''(x) - u\Phi'(x) + c(u) + h(x) \ge \gamma^*, \tag{18}$$

$$-\kappa \le \Phi'(x) \le 0, \quad |\Phi''(x)| \le C. \tag{19}$$

In addition, $\Phi'''(x)$ exists almost everywhere and $|\Phi'''(x)| \le C$ whenever it exists.

It is clear that (17) is essentially a first-order ODE for the unknown function $\Phi'(\cdot)$. Such an equation falls in the class of Riccati equations which are first-order ODEs that are quadratic in the unknown functions. Note that γ^* and the boundary points l_* and u_* of the ODE (17) are also unknown and need to be determined, hence we call the ODE a free boundary ODE as in Dai and Yao (2013). If h is derived from the linear state cost functions g_k, then Ata and Barjesteh (2019) provided the closed-form expressions of Φ and the constants l_*, u_*, γ^* (see Section 6.3 there). However, if h is derived from the strictly convex state cost functions g_k, to the best of our knowledge, closed-form expressions are unavailable for the function Φ or the constants l_*, u_*, γ^*. We provide a numerical solution to this free boundary ODE in Section 5.1.

4.2. The Proposed Policies We propose a sequence of feasible Markov control policies $\{\psi_*^n\}$ with ψ_*^n applied to the nth system. In the three parts of the policy ψ_*^n, the arrival rate control and the outsourcing decisions have the same structure for both types of state cost functions, while the scheduling decision depends on whether the state cost functions are linear or strictly convex. Following Wein (1992) and Ata and Barjesteh (2019), we adopt the concept of *safety stock* from the inventory management literature, and denote by a non-negative integer α_k the safety stock for class k. These parameters α_k are assumed to be independent of n.

The details of the proposed policy ψ_*^n are as follows:

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1. Arrival rates: Given the nominal workload process $\widetilde{W}^n(t)$ in (7), Φ, l_* and u_* in Lemma 2, the proposed arrival rate vector is $\lambda_*^n(t) = n\lambda^* + \sqrt{n}\cdot\zeta_*^n(t)$ with

$$\zeta_*^n(t) = \frac{H^{-1}m}{2}\Phi'\left(l_* \vee (\widetilde{W}^n(t) \wedge u_*)\right). \tag{20}$$

2. Outsourcing: if $\widetilde{W}^n(t) \le l_* < 0$ and $Q_{i^*}^n(t) \le \alpha_{i^*}$, outsource new class i^* orders at time t; otherwise, do not outsource any new order.

3. Scheduling: First identify $\mathcal{C}^n(t)$, the set of candidate classes at time t: if $\sum_{k\in\mathcal{K}} \frac{(Q_k^n(t))^+}{\mu_k} > u^n = \sqrt{n}u_*$, let $\mathcal{C}^n(t) = \{k \in \mathcal{K} : Q_k^n(t) < \alpha_k\}$; otherwise, let $\mathcal{C}^n(t) = \mathcal{K}$. The scheduling decision depends on the structure of the state cost functions:

(a) Strictly convex: when the system is ready to produce a new product, it will work on the class $\arg\min_{k\in\mathcal{C}^n(t)} g_k'(\widetilde{Q}_k^n(t))\mu_k$.

(b) Linear: let $\mathcal{N}^n(t) = \{k \in \mathcal{C}^n(t) : Q_k^n(t) < \alpha_k\}$. When the system is ready to produce a new product, if $\mathcal{N}^n(t) \neq \emptyset$, it will work on the class $\arg\max_{k\in\mathcal{N}^n(t)} b_k\mu_k$; otherwise, it will work on the class $\arg\min_{k\in\mathcal{C}^n(t)} h_k\mu_k$.

We will append a subscript $*$ to processes under the proposed policy ψ_*^n. For example, $\widetilde{\mathfrak{X}}_*^n(\cdot) = \{\widetilde{\mathfrak{X}}_*^n(t); t \ge 0\}$ is the (scaled) system state process under the control policy ψ_*^n. Note that Φ' is nonpositive, hence for all t, $\sum_{k\in\mathcal{K}} \frac{\lambda_{k*}^n(t)}{\mu_k^n} = \sum_{k\in\mathcal{K}} \frac{\lambda_k^*}{\mu_k} + \frac{m'H^{-1}m}{2\sqrt{n}}\Phi'\left(l_* \vee (\widetilde{W}^n(t) \wedge u_*)\right) \le 1$.

Remark 1. In the case of strictly convex state cost functions, if $\widetilde{Q}_k^n(t) < 0$, then $g_k'(\widetilde{Q}_k^n(t)) < 0$; if $\widetilde{Q}_k^n(t) > 0$, then $g_k'(\widetilde{Q}_k^n(t)) > 0$. As a result, for both types of state cost functions, *if the inventory of class k reaches its safety stock, that is, $Q_k^n(t) \ge \alpha_k$, and there exists another class j such that $Q_j^n(t) < 0$, then the system will not produce new class k products.* This observation will be used frequently in the proofs.

Remark 2 (Safety stocks). Note that $\mathcal{C}^n(t) = \emptyset$ means that there is no candidate class at time t, which implies that the system will become idle if it finishes producing a product at time t. This happens if and only if $\sum_{k\in\mathcal{K}} \frac{(Q_k^n(t))^+}{\mu_k} > u^n = \sqrt{n}u_*$ and $Q_k^n \ge \alpha_k$ for all $k \in \mathcal{K}$. Intuitively, this means that when all classes have enough products on hand (more than the safety stock) and the nominal workload is too large (above a threshold), then there is no risk of being out of stock and hence the system should stop working. Note that the system would keep working if at least one class has an inventory level lower than the safety stock, even if the nominal workload is too large. For the setting with linear state cost functions, if there are not enough products on hand for some classes (i.e., less than the corresponding safety stocks), then the system will give priority to these classes. These observations are consistent with the usual strategy in inventory management to reduce the risk of running out of stock. We also emphasize that the theoretical results in our paper apply for a fixed set of safety stock levels that can be chosen arbitrarily, for instance, one can take $\alpha_k = 0$ for all $k \in \mathcal{K}$. On the other hand, when the policy ψ_*^n is implemented in a make-to-stock system, its performance may depend on the choices of (α_k). The best (α_k) can be computed by using a simulation-based search (as in Wein (1992) and Ata and Barjesteh (2019)), where one varies (α_k) and compares the corresponding long-run average profit in the simulation of the make-to-stock systems under the proposed policy ψ_*^n.

Remark 3 (Difference from Ata and Barjesteh (2019)). For the setting with linear state cost functions, our policy is almost identical to the one in Ata and Barjesteh (2019), except that in the outsourcing part, we have the additional requirement $Q^n_{i^*}(t) \le \alpha_{i^*}$. This requirement is intuitive because if class i^* has many products on hand (higher than the safety stock), it is better to reduce its inventory level and hence not outsource its orders. Mathematically, we need this additional requirement when constructing Lyapunov functions to prove the existence of the stationary distributions of the inventory processes (Proposition 1 in Section 6.1), and when establishing the properties of the hydrodynamic limit (Lemma 13 in Appendix D). In particular, we use the requirement $Q^n_{i^*}(t) \le \alpha_{i^*}$ in Case (2b)-(ii) in the proof of Proposition 1, and in the proof of item 2(a) of Lemma 13. It is unclear how to prove these results without this additional requirement.

Remark 4 (Consistency). If $\alpha_k = 0$ for all $k \in \mathcal{K}$, then the scheduling policies are consistent for both cases, because for the linear state cost functions, b_k and h_k are the corresponding derivatives of the state cost rate functions. Note that the scheduling component of the policy in the setting with strictly convex holding/waiting cost functions is a form of the generalized-$c\mu$ rule (van Mieghem (1995)). If there exists a class k such that $\alpha_k > 0$, then the policies can be different: when there are customers waiting, in the linear cost case the system may work on a class with a positive inventory if this class has the largest $b_k\mu_k$ in $\mathcal{N}^n(t)$ (and the queue length of this class satisfies $0 < Q^n_k(t) < \alpha_k$), while in the strictly convex case, the system will always work on a class with customers waiting, as discussed in Remark 1.

Our main result is the following theorem.

Theorem 1 **(Asymptotic optimality)**. *Suppose Assumption 1 holds. Then the sequence of policies $\{\psi^n_*\}$ is asymptotically optimal.*

In view of (12), the theorem claims that for any initial distribution $\widetilde{\nu}^n$, the long-run average deviation of profit from the upper bound of the nth system is asymptotically minimized under policy ψ^n_*; this is equivalent to that the long-run average profit of the nth system is asymptotically maximized under policy ψ^n_*, for any initial distribution $\widetilde{\nu}^n$.

5. Policy Implementation

In this section we provide a guideline on how to interpret and implement the policies obtained from the heavy traffic analysis to one specific system.

5.1. A Numerical Solution to the Free Boundary ODE (17).

The proposed policy in Section 4.2 requires the inputs Φ', l_* and u_* from the free-boundary ODE (17). In this section, we propose a numerical method to solve this free boundary ODE. Our method works for both linear and convex state cost functions. In the case of linear state cost functions, Ata and Barjesteh (2019) provided an analytical solution to the free boundary ODE.

We need the function $h(\cdot)$ which is defined in (13). When the holding/waiting costs are strictly convex, we can obtain from (13) that h is the objective of a convex optimization problem with equality constraints. Hence it can be efficiently solved using, for example, Newton's method, see (Boyd and Vandenberghe 2004, Chapter 10). When the state cost functions are linear, the function h is piecewise linear and explicit (Ata and Barjesteh 2019).

Denote $v = \Phi'$ where Φ is given in Lemma 2. Then to solve the free boundary ODE (17), we need to numerically find the constants $l_* < 0 < u_*, \gamma^* > 0$ and a function $v \in C^1[l_*, u_*]$ so that

$$\frac{1}{2}\sigma^2 v'(x) - \frac{m'H^{-1}m}{4} v^2(x) + h(x) = \gamma^*, \quad \text{for} \quad x \in [l_*, u_*], \tag{21}$$

subject to $v(x) \in [-\kappa, 0]$ for all $x \in [l_*, u_*]$ and the boundary and smooth pasting conditions

$$v(l_*) = -\kappa, \quad v(u_*) = 0, \quad v'(l_*+) = 0, \quad \text{and} \quad v'(u_*-) = 0. \tag{22}$$

Our method to numerically solve the free boundary ODE (21)–(22) is built on the idea of proving Lemma 2. We create three different modules, where Module 3 calls Module 2, Module 2 calls Module 1, and Module 3 serves as the ultimate algorithm.

1. Module 1: for input parameters (γ, w_0), use an algorithm (e.g., the Runge-Kutta algorithm) to solve the ODE on $[-L, L]$:

$$\frac{1}{2}\sigma^2 w'(x) - \frac{m'H^{-1}m}{4}\min(w^2(x), M) + h(x) = \gamma, \quad \text{for} \quad x \in [-L, L], \qquad (23)$$
$$\text{subject to} \quad w(0) = w_0,$$

where L is a large value so that $[-L, L]$ serves as an approximation of $\mathbb{R}$, and M is a constant with $M \geq \kappa^2$ where κ is defined in (15). The outputs of Module 1 are: (1) the function $w(\cdot)$ (denoted by $w_{\gamma,w_0}(\cdot)$), (2) $w_{\max}(\gamma, w_0)$ and $x_{\max}(\gamma, w_0)$, its maximum value and the maximizer on $(0, L)$, and (3) $w_{\min}(\gamma, w_0)$ and $x_{\min}(\gamma, w_0)$, its minimum and the minimizer on $(-L, 0)$.

2. Module 2: for any input $w_0 < 0$, find the unique constant γ such that the function $w_{\gamma,w_0}(\cdot)$ generated from Module 1 satisfies $w_{\max}(\gamma, w_0) = 0$. This can be done by using the bisection method, see Algorithm 1. The outputs of Module 2 are: (1) the constant γ, and (2) $(w_{\gamma,w_0}(\cdot), x_{\max}(\gamma, w_0), w_{\min}(\gamma, w_0), x_{\min}(\gamma, w_0))$.

Note that using (23) one can verify that $w_{\max}(0, w_0) < 0$ for $w_0 < 0$. In Step 2 of Algorithm 1 the existence of γ_u with $w_{\max}(\gamma_u, w_0) \geq 0$ is guaranteed by item 3 of Lemma 15 in the appendix.

Algorithm 1 Bisection for Module 2

Require: Accuracy parameter $\epsilon > 0$.

1: Initially let $\gamma_l = 0$ and $\gamma_u = 1$;

2: Use Module 1 to get $w_{\max}(\gamma_u, w_0)$.
If $w_{\max}(\gamma_u, w_0) < 0$, update γ_l to be the previous γ_u and γ_u to be 2 times the previous γ_u; and return to the beginning of this step.
This step would yield γ_l and γ_u with $w_{\max}(\gamma_l, w_0) < 0$, and $w_{\max}(\gamma_u, w_0) \geq 0$.

3: Use the bisection method to get the desired γ:
Let $\gamma_0 = \frac{\gamma_l+\gamma_u}{2}$ and use Module 1 to get $w_{\max}(\gamma_0, w_0)$.
If $w_{\max}(\gamma_0, w_0) = 0$, let $\gamma_l = \gamma_u = \gamma_0$, stop;
Else-if $w_{\max}(\gamma_0, w_0) > 0$, let $\gamma_u = \gamma_0$;
Else let $\gamma_l = \gamma_0$;
If $\gamma_u - \gamma_l > \epsilon$, return to the beginning of this step; otherwise, let $\gamma = \frac{\gamma_u+\gamma_l}{2}$, use Module 1 to get $(w_{\gamma,w_0}(\cdot), x_{\max}(\gamma, w_0), w_{\min}(\gamma, w_0), x_{\min}(\gamma, w_0))$, and stop.

3. Module 3: based on Module 2, one can use the bisection method to search over w_0 such that $w_{\min}(\gamma, w_0) = -\kappa$, see Algorithm 2. Note that the desired $w_0 \in [-\kappa, 0]$, see Lemma 2.

Algorithm 2 Bisection for Module 3

Require: Accuracy parameter $\epsilon > 0$.

1: Initially, let $w_l = -\kappa$ and $w_u = 0$;

2: Let $w_0 = \frac{w_l+w_u}{2}$, use Module 2 to get γ and $(w_{\gamma,w_0}(\cdot), x_{\max}(\gamma, w_0), w_{\min}(\gamma, w_0), x_{\min}(\gamma, w_0))$;

3: If $w_{\min}(\gamma, w_0) = -\kappa$, let $\gamma^* = \gamma$, $v = w_{\gamma,w_0}(\cdot)$, $l_* = x_{\min}(\gamma, w_0)$ and $u_* = x_{\max}(\gamma, w_0)$; stop.
Else-if $w_{\min}(\gamma, w_0) > -\kappa$, let $w_l = w_l$ and $w_u = w_0$;
Else let $w_l = w_0$ and $w_u = w_u$;
If $w_u - w_l > \epsilon$, go to step 2; otherwise let $w_0 = \frac{w_u+w_l}{2}$, and use Module 2 to get γ, $w_{\gamma,w_0}(\cdot)$, $x_{\min}(\gamma, w_0)$ and $x_{\max}(\gamma, w_0)$. Let $\gamma^* = \gamma$, $v = w_{\gamma,w_0}(\cdot)$, $l_* = x_{\min}(\gamma, w_0)$ and $u_* = x_{\max}(\gamma, w_0)$, and stop.

5.2. Policy Implementation for One Specific System

5.2.1. Policy Implementation for One Specific System The proposed policy in Section 4.2 is obtained from heavy-traffic analysis where one considers a sequence of make-to-stock systems indexed by n. A natural question is how to interpret and implement this proposed policy for a specific system. For such a system, we propose the following steps for policy implementation. Note that n is not used as an input parameter in the policy implementation below.

1. First, estimate the system parameters. To emphasize that this is one specific system, we attach a subscript 0 to the parameters. Specifically, the parameters include the profit rate function $\mathbf{r}_0(x)$ (which depends on the demand rate function and the self-production cost), the production rate vector μ_0 (or the mean production time vector m_0), the state cost function $q_0(x)$, and the unit outsourcing cost vector ϑ_0. Let $i^* = \text{argmin}\left\{\frac{\vartheta_{0k}}{m_{0k}} : k \in \mathcal{K}\right\}$ and denote by $\kappa_0 = \frac{\vartheta_{0i^*}}{m_{0i^*}}$ the effective outsourcing cost. Choose the safety-stock levels (α_k). We also denote by λ_0^* the unique maximizer of the profit rate function $\mathbf{r}_0(x)$ and the matrix $H_0 = -\mathbf{r}_0''(\lambda_0^*)/2$.

2. Second, compute the function $h_0(\cdot)$, where

$$h_0(y) := \min_{x \in \mathbb{R}^K} \quad \sum_{k \in \mathcal{K}} q_{0k}(x_k) \qquad \text{s.t.} \quad \sum_{k \in \mathcal{K}} \frac{x_k}{\mu_{0k}} = y. \tag{24}$$

3. Third, solve the following free-boundary ODE and obtain the solution $\Phi_0(\cdot)$ and the two boundary points l_{0*} and u_{0*}:

$$\frac{1}{2}\sum_{k \in \mathcal{K}} \frac{2\lambda_{0k}^*}{\mu_{0k}^2} \times \Phi_0''(x) - \frac{m_0' H_0^{-1} m_0}{4} \times (\Phi_0'(x))^2 + h_0(x) = \gamma^*, \quad x \in (l_{0*}, u_{0*}) \tag{25}$$

with $\Phi_0'(x) \in [-\kappa_0, 0]$ for all x, $\Phi_0'(x) = -\kappa_0$ for $x \le l_{0*}$, $\Phi_0'(x) = 0$ for $x \ge u_{0*}$, and $\Phi_0''(x) = 0$ for $x \notin (l_{0*}, u_{0*})$. Note that this can be done by using the method in Section 5.1.

4. Finally, implement the following policy for the given system. Denote by $W(t) = \sum_{k \in \mathcal{K}} \frac{Q_k(t)}{\mu_{0k}}$.

---

1. Arrival rates: the proposed arrival rate is

$$\lambda_0^* + \frac{H_0^{-1} m_0}{2} \times \Phi_0'\left(l_{0*} \vee (W(t) \wedge u_{0*})\right). \tag{26}$$

2. Outsourcing: if $W(t) \le l_{0*}$ and $Q_{i^*}(t) \le \alpha_{i^*}$, outsource new class i^* orders at time t; otherwise, do not outsource any new order.

3. Scheduling: if $\sum_{k \in \mathcal{K}} \frac{(Q_k(t))^+}{\mu_{0k}} > u_{0*}$, let $\mathcal{C}_0(t) = \{k \in \mathcal{K} : Q_k(t) < \alpha_k\}$; otherwise, let $\mathcal{C}_0(t) = \mathcal{K}$. The scheduling decision depends on the structure of the state cost functions:

(a) Strictly convex: when the system is ready to produce a new product, it will work on the class $\arg\min_{k \in \mathcal{C}_0(t)} q_{0k}'(Q_k(t))\mu_{0k}$.

(b) Linear: let $\mathcal{N}_0(t) = \{k \in \mathcal{C}_0(t) : Q_k(t) < \alpha_k\}$. When the system is ready to produce a new product, if $\mathcal{N}_0(t) \ne \emptyset$, it will work on the class $\arg\max_{k \in \mathcal{N}_0(t)} b_{0k}\mu_{0k}$; otherwise, it will work on the class $\arg\min_{k \in \mathcal{C}_0(t)} h_{0k}\mu_{0k}$.

5.2.2. Connection to the Heavy Traffic Analysis Next, we explain how the policy defined in Section 5.2.1 for a specific system can be derived from the proposed policy in Section 4.2 which is obtained from the heavy traffic analysis. We use a two-step procedure.

1. First, we view the specific system as an element (with index n_0) of a sequence of systems, and scale the parameters to obtain parameters of the 'limit system' (see Section 3).

(a) The profit rate function $\mathbf{r}(x) := \frac{\mathbf{r}_0(n_0 x)}{n_0}$. The unique maximizer of $\mathbf{r}$, denoted by λ^*, then satisfies $\lambda^* = \frac{\lambda_0^*}{n_0}$. In addition, one has $H := -\mathbf{r}''(\lambda^*)/2 = -n_0\mathbf{r}_0''(n_0\lambda^*)/2 = -n_0\mathbf{r}_0''(\lambda_0^*)/2 = n_0 H_0$. As a result, we have $H^{-1} = H_0^{-1}/n_0$.

(b) The production rate vector $\mu := \frac{\mu_0}{n_0}$, and the mean production time vector is given by $m = n_0 m_0$.

(c) The state cost rate function $g_k(x) := q_{0k}(\sqrt{n_0}x)$ for $k \in \mathcal{K}$. Then, one can verify that $h(x) = h_0(x/\sqrt{n_0})$.

(d) The unit outsourcing cost vector $r = \sqrt{n_0}\vartheta_0$. Then i^* is also argmin$\left\{\frac{r_k}{m_k} : k \in \mathcal{K}\right\}$ and the effective outsourcing cost $\kappa := r_{i^*}/m_{i^*} = \kappa_0/\sqrt{n_0}$.

With these parameters for the 'limit system', the free-boundary ODE in Lemma 2 becomes

$$\frac{1}{2}\sum_{k\in\mathcal{K}}\frac{2\lambda^*_{0k}/n_0}{(\mu_{0k}/n_0)^2}\Phi''(x)-\frac{(n_0m_0)'H_0^{-1}/n_0(n_0m_0)}{4}(\Phi'(x))^2+h(x)=\gamma^*,\quad x\in(l_*,u_*),\tag{27}$$

and the function Φ and the boundary points l_*,u_* satisfy the conditions in Lemma 2.

2. Next, we apply the policy ψ^n_* obtained from the heavy traffic analysis to the given system with index $n=n_0$. By introducing $\Phi_0(x)=\Phi(\sqrt{n_0}x)$, we have $\Phi_0'(x)=\sqrt{n_0}\Phi'(\sqrt{n_0}x)$ and $\Phi_0''(x)=n_0\Phi''(\sqrt{n_0}x)$. Then solving (27) is equivalent to solving the following ODE:

$$\frac{1}{2}\sum_{k\in\mathcal{K}}\frac{2\lambda^*_{0k}}{\mu^2_{0k}}\times\Phi_0''(x)-\frac{m_0'H_0^{-1}m_0}{4}\times(\Phi_0'(x))^2+h_0(x)=\gamma^*,\quad x\in(l_{0*},u_{0*}),$$

where $l_{0*}=l_*/\sqrt{n_0},u_{0*}=u_*/\sqrt{n_0}$. The boundary conditions are given by: $\Phi_0'(x)\in[-\sqrt{n_0}\kappa,0]=[-\kappa_0,0]$, $\Phi_0'(x)=-\kappa_0$ for $x\le l_{0*}=l_*/\sqrt{n_0}$, $\Phi_0'(x)=0$ for $x\ge u_{0*}=u_*/\sqrt{n_0}$, and $\Phi_0''(x)=0$ for $x\notin(l_{0*},u_{0*})$. Note that this is exactly the ODE in (25).

The proposed arrival rate for the specific n_0th system is $\lambda^{n_0}_*(t)=n_0\lambda^*+\sqrt{n_0}\zeta^{n_0}_*(t)$, where

$$\zeta^{n_0}_*(t)=\frac{H^{-1}m}{2}\Phi'\left(l_*\vee(\widetilde{W}^{n_0}(t)\wedge u_*)\right).$$

See (20). Note that $\lambda^*_0=n_0\lambda^*$. In addition, we have

$$\sqrt{n_0}\zeta^{n_0}_*(t)=\frac{H^{-1}m}{2}\cdot\sqrt{n_0}\Phi'\left(\sqrt{n_0}l_{0*}\vee(\sqrt{n_0}W(t)\wedge\sqrt{n_0}u_{0*})\right)=\frac{H_0^{-1}m_0}{2}\cdot\Phi_0'\left(l_{0*}\vee(W(t)\wedge u_{0*})\right).$$

Therefore, the proposed arrival rate for the specific system is exactly the one in (26). The outsourcing and scheduling decisions can be obtained similarly and we suppress the details.

Remark 5 (Choice of system index n_0). To evaluate the performance of the policy discussed in Section 5.2.1 and to check whether it is nearly optimal in heavy-traffic, one needs to decide the system index n_0 which measures the "largeness" of the given system. Because we assume for system n the extra unit outsourcing cost (ϑ^n_k) is $O(1/\sqrt{n})$ and the self-production cost (δ^n_k) is of $O(1)$, one possible choice is to set $n_0=(\max_{k\in\mathcal{K}}(\delta_{0k})/\min_{k\in\mathcal{K}}(\vartheta_{0k}))^2$.

6. Analyses of the Policies We now analyze systems under the proposed policies and prove Theorem 1. In Section 6.1, we conduct steady state analyses; in Section 6.2, we illustrate how to adapt the lower bound approach to the discrete-state make-to-stock systems.

6.1. Steady-State Analysis In this subsection, we will prove the existence of steady states, and the tightness of the sequence of stationary distributions, for systems under more general policies. These results could be used to analyze multi-class make-to-stock systems under other scheduling policies; hence, we believe they are of independent interest.

To this end, introduce a set $\Psi\subset\bar{\Pi}$, where each element of Ψ is a sequence of control policies that are asymptotically admissible (see Definition 3). We use $\{\psi^n_\diamond\}$ to denote a generic element of Ψ, with the nth policy $\psi^n_\diamond$ for the nth system satisfying:

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1. Arrival rates: the arrival rate vector $\lambda^n(t)=n\lambda^*+\sqrt{n}\cdot\zeta^n(t)$.
2. Outsourcing: fix a class $i^\diamond\in\mathcal{K}$ and a constant $l_\diamond<0$. If $\widetilde{W}^n(t)\le l_\diamond$ and $Q^n_{i^\diamond}(t)\le\alpha_{i^\diamond}$, outsource new class $i^\diamond$ orders at time t; otherwise, do not outsource any new order.
3. Scheduling: there exists $u_\diamond>0$ such that if $\sum_{k\in\mathcal{K}}\frac{(Q^n_k(t))^+}{\mu_k}>\sqrt{n}u_\diamond$, then the system will not work on any class k such that $Q^n_k(t)\ge\alpha_k$; the system will not work on a class if the inventory of that class reaches its safety stock and there exists another class with customers waiting.

It is easy to verify that the sequence of the proposed policies $\{\psi^n_*\}$ is one element of Ψ: the condition for the outsourcing part can be verified with $l_\diamond=l_*$, $i^\diamond=i^*$; and the condition for the scheduling part can be verified with $u_\diamond=u_*$, the definition of $\mathcal{C}^n(t)$ and from Remarks 1.

Fix an element of Ψ, that is, a sequence of policies $\{\psi^n_\diamond\}$. Consider a sequence of systems, with the nth system $\widetilde{\mathfrak{X}}^n_\diamond(\cdot)$ under the policy $\psi^n_\diamond$, we can prove the following two propositions. We defer the proof of Proposition 1 to Appendix B and the proof of Proposition 2 to Appendix C.

PROPOSITION 1 **(Existence of steady states)**. *Under the policy $\psi^n_\diamond$, there exists a stationary distribution for the Markov process $\widetilde{\mathfrak{X}}^n_\diamond(\cdot)$ for all sufficiently large n.*

Denote $\widetilde{\mathfrak{X}}^n_\diamond(\infty)$ the random vector that follows a stationary distribution. The next result establishes the tightness of $\{\widetilde{\mathfrak{X}}^n_\diamond(\infty)\}$.

PROPOSITION 2 **(Tightness of stationary distributions)**. *The sequence of random vectors $\{\widetilde{\mathfrak{X}}^n_\diamond(\infty)\}$ is tight.*

6.2. Proof of Theorem 1 In Section 6.2.1, we show that the constant γ^* is a lower bound for the long-run average expected loss under any asymptotically admissible sequence of policies (Theorem 2). Then, in Section 6.2.2, we verify that the lower bound γ^* is achieved under the sequence of proposed policies $\{\psi^n_*\}$.

We first state a lemma which plays an essential role in both the proofs of Theorems 1 and 2. We use $\triangle\mathbb{Y}(t)=\mathbb{Y}(t)-\mathbb{Y}(t-)$ to denote the jump of a process $\mathbb{Y}$ at time t, and write

$$\widetilde{\mathbb{X}}^n(t) := \sum_{k\in\mathcal{K}} \frac{\widetilde{X}^n_k(t)}{\mu_k} \quad \text{and} \quad \widetilde{\mathbb{O}}^n(t) := \sum_{k\in\mathcal{K}} \frac{\widetilde{O}^n_k(t)}{\mu_k}. \tag{28}$$

We can apply Ito's formula to the semimartingale $\widetilde{W}^n$ in (8), take Taylor expansion of Φ and use the stationarity of $\widetilde{W}^n(\cdot)$ to obtain the following result. The proof is deferred to the end of this section.

LEMMA 3. *Suppose there exists a stationary distribution $\widetilde{\pi}^n$ for the Markov chain $\widetilde{\mathfrak{X}}^n(\cdot)$ under a feasible policy ψ^n and assume $\widetilde{\mathfrak{X}}^n(0)$ follows $\widetilde{\pi}^n$. For Φ in Lemma 2, we have*

$$0=\mathbb{E}[\Phi(\widetilde{W}^n(t))]-\mathbb{E}[\Phi(\widetilde{W}^n(0))]=\Psi_1(t,\widetilde{\pi}^n,\psi^n)+\Psi_2(t,\widetilde{\pi}^n,\psi^n)+\Psi_3(t,\widetilde{\pi}^n,\psi^n), \tag{29}$$

where

$$\Psi_1(t,\widetilde{\pi}^n,\psi^n)=\mathbb{E}\left[\int_0^t\left(-\sum_{k\in\mathcal{K}}\frac{\zeta^n_k(s)}{\mu_k}\Phi'(\widetilde{W}^n(s-))+\sum_{k\in\mathcal{K}}\frac{\lambda^*_k}{\mu^2_k}\Phi''(\widetilde{W}^n(s-))\right)ds\right], \tag{30}$$

$$\Psi_2(t,\widetilde{\pi}^n,\psi^n)=\int_0^t\Phi'(\widetilde{W}^n(s-))d\widetilde{\mathbb{O}}^n(s)-\int_0^t\Phi'(\widetilde{W}^n(s-))d\widetilde{I}^n(s), \tag{31}$$

$$\begin{aligned}\Psi_3(t,\widetilde{\pi}^n,\psi^n)=&\sum_{k\in\mathcal{K}}\frac{1}{\mu^2_k}\times\mathbb{E}\left[\int_0^t\frac{1}{2}\Phi''(\widetilde{W}^n(s-))\left(\frac{\zeta^n_k(s)}{\sqrt{n}}ds-\frac{\mu_k}{\sqrt{n}}d\widetilde{Y}^n_k(s)\right)\right]\\&-\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{\mathbb{O}}^n(s)|>0}\left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right)\right]\\&+\mathbb{E}\left[\sum_{s\le t:|\triangle\widetilde{W}^n(s)|>0}\frac{1}{6}\Phi'''(\Xi^n(s))(\triangle\widetilde{W}^n(s))^3\right],\end{aligned} \tag{32}$$

and in (32)*, $\Xi^n(s)\in(\min\{\widetilde{W}^n(s),\widetilde{W}^n(s-)\},\max\{\widetilde{W}^n(s),\widetilde{W}^n(s-)\})$.*

6.2.1. A Lower Bound We prove a strong version of the lower bound. For this, let

$$\widetilde{V}^n := \inf_{\widetilde{\nu}^n,\psi^n\in\Pi^n}\widetilde{V}^n(\widetilde{\nu}^n,\psi^n). \tag{33}$$

Here $\widetilde{\nu}^n$, the distribution of $\widetilde{\mathfrak{X}}^n(0)$, can be any distribution supported on $\widetilde{S}^n$.

Theorem 2 **(Lower Bound).** *We have*

$$\liminf_{n\to\infty} \widetilde{V}^n \geq \gamma^*.$$

From Theorem 2, for any sequence of initial distributions $\{\widetilde{\nu}^n\}$ and any asymptotically admissible sequence of control policies $\{\psi^n\}$, we have

$$\liminf_{n\to\infty} \widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \geq \liminf_{n\to\infty} \widetilde{V}^n \geq \gamma^*.$$

As a result, γ^* serves as a lower bound for the long-run average expected loss under any asymptotically admissible sequence of control policies $\{\psi^n\}$ and initial distributions $\{\widetilde{\nu}^n\}$.

Proof of Theorem 2. The main idea of the proof is to apply Lemma 3 and carefully bound the terms Ψ_i for $i=1,2,3$, by using properties including those of Φ in (18) and (19).

For notational simplicity, we denote by C generic constants that are independent of n, although the value of C may differ from line to line. Recall $\widetilde{V}^n$ defined in (33). Consider two sets: $A_1 = \{n : \widetilde{V}^n > \gamma^* + 1\}$ and $A_2 = \{n : \widetilde{V}^n \leq \gamma^* + 1\}$. If A_2 is finite, then the conclusion holds. Otherwise, it is enough to consider $\liminf_{n\in A_2 : n\to\infty} \widetilde{V}^n$. In the following, we focus on this $\liminf$ and for notational simplicity, we omit $n \in A_2$ and will always assume $n \in A_2$. From the definition of $\widetilde{V}^n$, for any $\epsilon \in (0,1)$, there exists $\widetilde{\nu}^n$ and ψ^n such that

$$\widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \leq \widetilde{V}^n + \epsilon \leq \gamma^* + 2, \quad \text{for all } n \text{ sufficiently large.} \tag{34}$$

The following lemma ensures us that we can always assume $\widetilde{\nu}^n$ to be a stationary distribution.

Lemma 4. *Suppose* (34) *holds. Then there exists a stationary distribution $\widetilde{\pi}^n$ for the Markov chain $\widetilde{\mathfrak{X}}^n(\cdot)$ under the policy ψ^n for all sufficiently large n and*

$$\widetilde{V}^n(\widetilde{\pi}^n, \psi^n) \leq \widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \leq \widetilde{V}^n + \epsilon \leq \gamma^* + 2. \tag{35}$$

The proof of this result is deferred to Appendix A. In the following, we will assume that $\widetilde{\mathfrak{X}}^n(0)$ follows a stationary distribution $\widetilde{\pi}^n$ so that the systems start from stationarity. For notational simplicity we use $\mathbb{E}$ to represent $\mathbb{E}_{\widetilde{\pi}^n}$ in the analysis below.

We clarify that Lemma 4 and Proposition 1, both of which state the existence of stationary distributions, hold for policies under different assumptions. Lemma 4 applies to a *general* admissible policy ψ^n under condition (34) (where the details of the policy are not specified) and is used to prove the lower bound result in Theorem 2. On the other hand, Proposition 1 applies to the specific policy $\psi^n_\diamond$ defined in Section 5.1 (but condition (34) may not hold) and is needed to analyze the proposed policies in Theorem 1.

We now continue to prove Theorem 2. We will show

$$\liminf_{n\to\infty} \widetilde{V}^n(\widetilde{\pi}^n, \psi^n) \geq \gamma^*, \tag{36}$$

which then yields $\liminf_{n\to\infty} \widetilde{V}^n + \epsilon \geq \gamma^*$ by (35). Then, Theorem 2 holds because ϵ is arbitrary.

Note that because of the existence of the stationary distribution $\widetilde{\pi}^n$, from (29), we have

$$0 = \Psi_1(t, \widetilde{\pi}^n, \psi^n) + \Psi_2(t, \widetilde{\pi}^n, \psi^n) + \Psi_3(t, \widetilde{\pi}^n, \psi^n).$$

We first analyze the term $\Psi_1(t, \widetilde{\pi}^n, \psi^n)$. Since Φ satisfies the condition (18), if we denote by $\bar{c}(\zeta) = \zeta' H \zeta$ for $\zeta \in \mathbb{R}^K$, then from (16), $c(u) \leq \bar{c}(\zeta^n(s))$ for $u = \sum_{k\in\mathcal{K}} \frac{\zeta^n_k(s)}{\mu_k} = m'\zeta^n(s)$. Using (18) with this u, for each x and $s \geq 0$,

$$-\sum_{k\in\mathcal{K}} \frac{\zeta^n_k(s)}{\mu_k} \Phi'(x) + \sum_{k\in\mathcal{K}} \frac{\lambda^*_k}{\mu^2_k} \Phi''(x) + \bar{c}(\zeta^n(s)) + h(x) \geq \gamma^*.$$

Since $\widetilde{W}^n$ has only finite jumps on $[0,t]$, we can then infer that

$$\begin{aligned}\Psi_1(t,\widetilde{\pi}^n,\psi^n) &\geq \gamma^* t - \mathbb{E}\left[\int_0^t \bar{c}(\zeta^n(s))ds + \int_0^t h(\widetilde{W}^n(s))ds\right] \\ &\geq \gamma^* t - \mathbb{E}\left[\int_0^t \bar{c}(\zeta^n(s))ds + \int_0^t \sum_{k\in\mathcal{K}} g_k(\widetilde{Q}_k^n(s))ds\right], \end{aligned} \tag{37}$$

where in the second inequality we have used the facts that $\widetilde{W}^n(t) = \sum_{k\in\mathcal{K}} \frac{\widetilde{Q}_k^n(t)}{\mu_k}$ and that if $\sum_{k\in\mathcal{K}} q_k/\mu_k = x$, then $\sum_{k\in\mathcal{K}} g_k(q_k) \geq h(x)$ from the definition of $h(x)$ in (13).

Next, we analyze the term $\Psi_2(t,\widetilde{\pi}^n,\psi^n)$. Using (19) and the fact that $\widetilde{I}^n$ is nondecreasing, we obtain for each t,

$$\Psi_2(t,\widetilde{\pi}^n,\psi^n) \geq -\kappa\widetilde{\mathbb{O}}^n(t) \geq -\sum_{k\in\mathcal{K}} r_k \widetilde{O}_k^n(t), \tag{38}$$

where the last inequality is from the definition of κ in (15) and the definition of $\widetilde{\mathbb{O}}^n$ in (28).

Finally, we analyze the term $\Psi_3(t,\widetilde{\pi}^n,\psi^n)$. Using the fact that the magnitude of jumps of $\widetilde{X}_k^n(s)$ is $\frac{1}{\sqrt{n}}$, we get

$$\mathbb{E}\left|\sum_{s\leq t: |\triangle\widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle\widetilde{\mathbb{X}}^n(s))^2\right)\right| \leq \mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{C}{2n\mu_k^2} O_k^n(t)\right].$$

Note that $\widetilde{\mathbb{O}}^n(t) := \sum_{k\in\mathcal{K}} \frac{\tilde{O}_k^n(t)}{\mu_k}$. Hence we have

$$\begin{aligned}\mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{C}{2n\mu_k^2} O_k^n(t)\right] &= \mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{C}{2\sqrt{n}\mu_k^2} \tilde{O}_k^n(t)\right] \leq \frac{C}{2\sqrt{n}} \frac{1}{\min_{k\in\mathcal{K}}\{\mu_k\}} \mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{\tilde{O}_k^n(t)}{\mu_k}\right] \\ &= \frac{C}{2\min_{k\in\mathcal{K}}\{\mu_k\}} \times \mathbb{E}\left[\frac{1}{\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right] := \mathbb{E}\left[\frac{C}{\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right],\end{aligned}$$

in which the values of the generic constant C may be different. Since $|\Phi'''(x)| \leq C$, we have

$$\begin{aligned}\mathbb{E}\left|\sum_{s\leq t: |\triangle\widetilde{W}^n(s)|>0} \frac{1}{6}\Phi'''(\Xi^n(s))(\triangle\widetilde{W}^n(s))^3\right| &\leq \mathbb{E}\left[\sum_{s\leq t: |\triangle\widetilde{W}^n(s)|>0} \frac{C}{6}|\triangle\widetilde{W}^n(s)|^3\right] \\ &\leq \frac{C}{6\sqrt{n}} \frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t \lambda_k^n(s)ds + \mu_k^n T_k^n(t))]}{n}.\end{aligned}$$

The last inequality holds because $\widetilde{W}^n$ jumps only when there is a customer arrival or a production completion, and also by the definition of $\widetilde{W}^n$ in (8),

$$\begin{aligned}\mathbb{E}\left[\sum_{s\leq t: |\triangle\widetilde{W}^n(s)|>0} |\triangle\widetilde{W}^n(s)|^3\right] &\leq \frac{1}{\min_{k\in\mathcal{K}} \mu_k^3}\left(\frac{1}{\sqrt{n}}\right)^3 \cdot \sum_{k\in\mathcal{K}} \mathbb{E}\left[N_k\left(\int_0^t \lambda_k^n(u)du\right) + S_k^n(T_k^n(t))\right] \\ &= \frac{C}{\sqrt{n}} \frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t \lambda_k^n(s)ds + \mu_k^n T_k^n(t))]}{n}.\end{aligned}$$

Therefore, using $|\Phi''(x)| \leq C$, we deduce from (32) that

$$|\Psi_3(t,\widetilde{\pi}^n,\psi^n)| \leq \hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n) + \mathbb{E}\left[\frac{C}{\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right], \tag{39}$$

where

$$\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n) = \left|\mathbb{E}\left[\sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\int_0^t \Phi''(\widetilde{W}^n(s-))\frac{1}{2\sqrt{n}} d\widetilde{Y}_k^n(s)\right]\right|$$

$$+\mathbb{E}\left[\int_0^t \sum_{k\in\mathcal{K}} \frac{C}{\mu_k^2}\frac{|\zeta_k^n(s)|}{\sqrt{n}}ds\right] + \frac{C}{6\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t \lambda_k^n(s)ds + \mu_k^n T_k^n(t))]}{n}.$$

Recall from (10) that

$$\widetilde{V}^n(t,\widetilde{\pi}^n,\psi) = \mathbb{E}\left[\int_0^t c^n(\zeta^n(s))ds + \sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds + \sum_{k\in\mathcal{K}} r_k \widetilde{O}_k^n(t)\right], \tag{40}$$

where $c^n(\zeta) = n(\mathbf{r}(\lambda^*) - \mathbf{r}(\lambda^* + \zeta/\sqrt{n}))$. Hence, we obtain

$$\mathbb{E}\left[\frac{C}{\sqrt{n}}\widetilde{\mathbb{O}}^n(t)\right] \le \frac{C}{\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n).$$

It follows that

$$|\Psi_3(t,\widetilde{\pi}^n,\psi^n)| \le \hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n) + \frac{C}{\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n). \tag{41}$$

On combining the estimates (37), (38) and (41), we can deduce from (29) that

$$\left(1+\frac{C}{\sqrt{n}}\right)\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n) - \gamma^* t \ge \mathbb{E}\left[\int_0^t (c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s)))ds\right] - \hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n). \tag{42}$$

Next, we analyze the terms on the right-hand-side of the above inequality. Using Taylor's theorem and the fact that $\nabla\mathbf{r}(\lambda^*) = 0$, we obtain

$$\begin{aligned} c^n(\zeta) &= n(\mathbf{r}(\lambda^*) - \mathbf{r}(\lambda^* + \zeta/\sqrt{n})) = -\zeta'\left(\int_0^1 \nabla^2\mathbf{r}(\lambda^* + \theta\zeta/\sqrt{n})(1-\theta)d\theta\right)\zeta \\ &= \zeta' H\zeta - \zeta'\left(\int_0^1 [\nabla^2\mathbf{r}(\lambda^* + \theta\zeta/\sqrt{n}) - \nabla^2\mathbf{r}(\lambda^*)](1-\theta)d\theta\right)\zeta, \end{aligned}$$

where we recall that $H := -\nabla^2\mathbf{r}(\lambda^*)/2$. Since $\nabla^2\mathbf{r}(\cdot)$ is continuous at λ^* and $|\zeta_k^n(s)/\sqrt{n}| \le a_n$ with $a_n \to 0$ as $n\to\infty$, we have for $\bar{c}(\zeta) = \zeta' H\zeta$ and any $\epsilon > 0$,

$$|c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s))| \le \int_0^1 \|\nabla^2\mathbf{r}(\lambda^* + \theta\zeta^n(s)/\sqrt{n}) - \nabla^2\mathbf{r}(\lambda^*)\|(1-\theta)d\theta\cdot|\zeta^n(s)|^2 \le \epsilon\|\zeta^n(s)\|^2, \tag{43}$$

for all sufficiently large n. Thus, we have

$$\left|\int_0^t (c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s)))ds\right| \le \epsilon\cdot\int_0^t \|\zeta^n(s)\|^2 ds.$$

Denote the smallest eigenvalue of $H = -\nabla^2\mathbf{r}(\lambda^*)/2$ by λ_H. We know $\lambda_H > 0$ since H is positive definite. Then (43) implies that

$$c^n(\zeta^n(s)) \ge (\lambda_H - \epsilon)\|\zeta^n(s)\|^2, \quad \text{for } 0 < \epsilon < \lambda_H. \tag{44}$$

By (40), this leads to

$$\mathbb{E}\left[\left|\int_0^t (c^n(\zeta^n(s)) - \bar{c}(\zeta^n(s)))ds\right|\right] \le \frac{\epsilon}{\lambda_H - \epsilon}\widetilde{V}^n(t,\widetilde{\pi}^n,\psi^n). \tag{45}$$

Next, we study the term $\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n)$. We show that

$$\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\hat{\Psi}_3(t,\widetilde{\pi}^n,\psi^n) = 0. \tag{46}$$

To this end, we can show the following lemma, the proof of which is deferred to Appendix A.

LEMMA 5. *Assume $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$ for all sufficiently large n, and (35) holds. Then*

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\frac{1}{2\sqrt{n}}\int_0^t \Phi''(\widetilde{W}^n(s-))d\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\widetilde{Y}_k^n(s)\right]=0, \tag{47}$$

where we recall that $\widetilde{Y}_k^n(t)=\sqrt{n}(\rho_k t-T_k^n(t))$ for $k\in\mathcal{K}$ and $t\geq 0$.

In addition, by assumption $|\zeta_k^n(s)/\sqrt{n}|\leq a_n\to 0$. Hence, we have

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\int_0^t\sum_{k\in\mathcal{K}}\frac{C}{\mu_k^2}\frac{|\zeta_k^n(s)|}{\sqrt{n}}ds\right]=0.$$

Furthermore, note that $\lambda_k^n(s)=n\lambda^*+\sqrt{n}\zeta_k^n(s)$ with $|\zeta_k^n(s)/\sqrt{n}|\leq a_n\to 0$, $\mu_k^n=n\mu_k$ and $T_k^n(t)\leq t$, we obtain $0\leq \mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t\lambda_k^n(s)ds+\mu_k^nT_k^n(t))]/n\leq \lambda_k^*t+a_nt+\mu_kt$. Hence,

$$\lim_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\frac{C}{6\sqrt{n}}\frac{\mathbb{E}[\sum_{k\in\mathcal{K}}(\int_0^t\lambda_k^n(s)ds+\mu_k^nT_k^n(t))]}{n}=0.$$

It then follows that (46) holds.

Therefore, we can infer from (42), (45) and (46) that

$$\liminf_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\left[\widetilde{V}^n(t,\widetilde{\pi}_n,\psi^n)\left(1+\frac{\epsilon}{\lambda_H-\epsilon}+\frac{C}{\sqrt{n}}\right)-\gamma^*t\right]\geq 0,$$

which implies $\liminf_{n\to\infty}\widetilde{V}^n(\widetilde{\pi}_n,\psi^n)\left(1+\frac{\epsilon}{\lambda_H-\epsilon}\right)\geq\gamma^*$. This yields $\liminf_{n\to\infty}\widetilde{V}^n(\widetilde{\pi}_n,\psi^n)\geq\gamma^*$ by sending $\epsilon\to 0+$. This proves (36). Hence, we have proved the desired result. □

6.2.2. Proof of Theorem 1 In this section, we prove that the lower bound γ^* is achieved asymptotically under the sequence of the proposed policies $\{\psi_*^n\}$. The following proposition will be essential and its proof can be found in Appendix D. Recall that $\widetilde{Q}_{k*}^n(\infty)$ and $\widetilde{W}_*^n(\infty)$ follow the stationary distributions of $\widetilde{Q}_{k*}^n(\cdot)$ and $\widetilde{W}_*^n(\cdot)$ respectively, under the control policy ψ_*^n.

PROPOSITION 3 **(State-Space Collapse)**. *For systems under the proposed policies $\{\psi_*^n\}$,*

$$\lim_{n\to\infty}\mathbb{E}\left[\left|\sum_{k\in\mathcal{K}}g_k(\widetilde{Q}_{k*}^n(\infty))-h\left(l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*)\right)\right|\right]=0.$$

Recall the uniqueness of the lifting function $\Delta(\cdot)$ in Lemma 1. From the above proposition, one can expect that $\widetilde{Q}_{k*}^n(\infty)$ is close to $\Delta_k(l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*))$ in L_1-norm, that is, the K-dimensional inventory levels $\widetilde{Q}_*^n(\infty)$ are close to functions of the one-dimensional nominal workload $\widetilde{W}_*^n(\infty)$. Furthermore, because $l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*)$ is bounded, the scaled inventory levels $\widetilde{Q}_{k*}^n(\infty), k\in\mathcal{K}$, are also expected to be bounded asymptotically, even though we only outsource class i^* orders.

Now we prove Theorem 1. First, we show that for any stationary distribution $\widetilde{\pi}_*^n$ of $\widetilde{\mathfrak{X}}_*^n(\cdot)$,

$$\lim_{n\to\infty}\widetilde{V}^n(\widetilde{\pi}_*^n,\psi_*^n)=\gamma^*. \tag{48}$$

Suppose the initial distribution of $\widetilde{\mathfrak{X}}_*^n(\cdot)$ is $\widetilde{\pi}_*^n$, and in the following we use $\mathbb{E}$ to denote $\mathbb{E}_{\widetilde{\pi}_*^n}$. From Lemma 3, we have

$$0=\Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n)+\Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n)+\Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n).$$

We first compute $\Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n)$ under the policy ψ_*^n. For the function Φ in Lemma 2, one can readily verify, using the expression of ζ_*^n in (20) and $\bar{c}(\zeta)=\zeta'H\zeta$, that

$$-\sum_{k\in\mathcal{K}}\frac{\zeta_{k*}^n(s)}{\mu_k}\Phi'(\widetilde{W}_*^n(s))+\sum_{k\in\mathcal{K}}\frac{\lambda_k^*}{\mu_k^2}\Phi''(\widetilde{W}_*^n(s))+\bar{c}(\zeta_*^n(s))+h\left(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*)\right)=\gamma^*,$$

Then we can deduce from (30) that

$$\Psi_1(t,\widetilde{\pi}_*^n,\psi_*^n)=\gamma^* t-\mathbb{E}\left[\int_0^t \bar{c}(\zeta_*^n(s))ds+\int_0^t h\left(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*)\right)ds\right].$$

Next we show that under the policy ψ_*^n, we have

$$\Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n)=-\sum_{k\in\mathcal{K}} r_k\widetilde{O}_{k*}^n(t), \tag{49}$$

Under the policy ψ_*^n, when $\widetilde{\mathbb{O}}_*^n$ jumps (i.e., an order is outsourced), we have $\widetilde{W}_*^n(t)\le l_*$; On the other hand, when $\widetilde{I}_*^n(s)$ increases (i.e., the system stops production and becomes idle), we have $\sum_{k\in\mathcal{K}}\frac{(Q_{k*}^n(t))^+}{\mu_k}\ge\sqrt{n}u_*$ and $Q_{k*}^n(t)\ge\alpha_k>0$ for all $k\in\mathcal{K}$; hence, $W_*^n(t)=\sum_{k\in\mathcal{K}}\frac{Q_{k*}^n(t)}{\mu_k}\ge\sqrt{n}u_*$. Also note that $\Phi'(x)=-\kappa$ for $x\le l_*$, and $\Phi'(x)=0$ for $x\ge u_*$. Hence,

$$\Psi_2(t,\widetilde{\pi}_*^n,\psi_*^n)=\mathbb{E}\left[\int_0^t\Phi'(\widetilde{W}_*^n(s-))d\widetilde{\mathbb{O}}_*^n(s)-\int_0^t\Phi'(\widetilde{W}_*^n(s-))d\widetilde{I}_*^n(s)\right]=-\kappa\cdot\mathbb{E}[\widetilde{\mathbb{O}}_*^n(t)].$$

In addition, note that under the proposed policy ψ_*^n, only class i^* orders will be outsourced. Using the definition of $\kappa=\frac{r_{i^*}}{m_{i^*}}$ we have

$$\kappa\cdot\widetilde{\mathbb{O}}_*^n(t)=\kappa\cdot\widetilde{O}_{i^**}^n(t)m_{i^*}=r_{i^*}\widetilde{O}_{i^**}^n(t)=\sum_{k\in\mathcal{K}}r_k\widetilde{O}_{k*}^n(t).$$

Hence we obtain (49).

Finally, for $\Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n)$, as in (39) we have

$$|\Psi_3(t,\widetilde{\pi}_*^n,\psi_*^n)|\le\hat{\Psi}_3(t,\widetilde{\pi}_*^n,\psi_*^n)+\frac{C}{\sqrt{n}}\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n).$$

Combining these estimates, we can deduce from (29) that

$$\begin{aligned}\left(1-\frac{C}{\sqrt{n}}\right)\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n)-\gamma^*t\le{}&\mathbb{E}\left[\int_0^t(c^n(\zeta_*^n(s))-\bar{c}(\zeta_*^n(s)))ds\right]+\hat{\Psi}_3(t,\widetilde{\pi}_*^n,\psi_*^n)\\&+\mathbb{E}\left[\left|\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_{k*}^n(s))ds-\int_0^t h(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*))ds\right|\right]\end{aligned}$$

For the last term in the above inequality, we apply Proposition 3 to obtain

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\left|\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_{k*}^n(s))ds-\int_0^t h(l_*\vee(\widetilde{W}_*^n(s)\wedge u_*))ds\right|\right]=0.$$

For the other terms, we can control them similarly as in the proof of Theorem 2 (indeed, ζ_*^n is now bounded, which simplifies matters). Then

$$\limsup_{n\to\infty}\widetilde{V}^n(\widetilde{\pi}_*^n,\psi_*^n)=\limsup_{n\to\infty}\limsup_{t\to\infty}\frac{1}{t}\left[\widetilde{V}^n(t,\widetilde{\pi}_*^n,\psi_*^n)\right]\le\gamma^*.$$

Combining with the lower bound result in Theorem 2, we obtain (48).

To complete the proof of Theorem 1, we need the following lemma. The proof is given in Appendix E.

Lemma 6. *For n large enough, under ψ_*^n and any initial distribution $\widetilde{\nu}^n$, there is a unique stationary distribution $\widetilde{\pi}_*^n$ of $\widetilde{\mathfrak{X}}_*^n(\cdot)$ such that*

$$\lim_{t\to\infty}\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi_*^n)=\widetilde{V}^n(\widetilde{\pi}_*^n,\psi_*^n). \tag{50}$$

Given Lemma 6 and the definition that $\widetilde{V}^n(\widetilde{\nu}^n,\psi_*^n)=\limsup_{t\to\infty}\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi_*^n)$, we can infer from (48) that Theorem 1 holds. □

Proof of Lemma 3. We apply Ito's formula (e.g., see Theorem 32 of Chapter II in Protter (2005)) to the semimartingale $\widetilde{W}^n$ in (8) with the function Φ in Lemma 2 to obtain

$$\begin{aligned}\Phi(\widetilde{W}^n(t)) =&\Phi(\widetilde{W}^n(0)) + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s) + \sum_{s\le t:|\triangle \widetilde{W}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{W}^n(s))^2\right) \\ &+ \sum_{s\le t:|\triangle \widetilde{W}^n(s)|>0} \left(\triangle\Phi(\widetilde{W}^n(s)) - \Phi'(\widetilde{W}^n(s-))\triangle\widetilde{W}^n(s) - \frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{W}^n(s))^2\right).\end{aligned} \tag{51}$$

From the dynamics of $\widetilde{W}^n$ in (8),

$$\begin{aligned}\sum_{s\le t:|\triangle \widetilde{W}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{W}^n(s))^2\right) =& \sum_{s\le t:|\triangle \widetilde{\mathbb{X}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right) \\ &- \sum_{s\le t:|\triangle \widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right).\end{aligned}$$

This equality holds because the epochs at which the process $\widetilde{W}^n$ jumps constitute a subset of those at which $\widetilde{\mathbb{X}}^n$ jumps, and when there is an arrival of a customer order that is outsourced (i.e., $\widetilde{\mathbb{O}}^n$ jumps up), the process $\widetilde{\mathbb{X}}^n$ jumps but $\widetilde{W}^n$ does not.

In addition, using Taylor expansion, we have

$$\triangle\Phi(\widetilde{W}^n(s)) - \Phi'(\widetilde{W}^n(s-))\triangle\widetilde{W}^n(s) - \frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{W}^n(s))^2 = \frac{1}{6}\Phi'''(\Xi^n(s))(\triangle \widetilde{W}^n(s))^3.$$

Hence, from (51)

$$\begin{aligned}\Phi(\widetilde{W}^n(t)) =&\Phi(\widetilde{W}^n(0)) + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s) + \sum_{s\le t:|\triangle \widetilde{\mathbb{X}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right) \\ &- \sum_{s\le t:|\triangle \widetilde{\mathbb{O}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right) + \sum_{s\le t:|\triangle \widetilde{W}^n(s)|>0} \frac{1}{6}\Phi'''(\Xi^n(s))(\triangle \widetilde{W}^n(s))^3.\end{aligned} \tag{52}$$

From the dynamics of $\widetilde{W}^n$ in (8), and using the martingale property of $\widetilde{\mathbb{X}}^n$ and the boundedness of Φ' in (19), we can readily infer that

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{W}^n(s)\right] \\ &= \mathbb{E}\left[-\int_0^t \sum_{k\in\mathcal{K}} \frac{\zeta_k(s)}{\mu_k}\Phi'(\widetilde{W}^n(s-))ds + \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{\mathbb{O}}^n(s) - \int_0^t \Phi'(\widetilde{W}^n(s-))d\widetilde{I}^n(s)\right].\end{aligned}$$

Moreover, since $\widetilde{\mathbb{X}}^n(t)$ is a linear combination of $\widetilde{X}_k^n(t)$, where the jump magnitude of $\widetilde{X}_k^n$ is $1/\sqrt{n}$ whenever there is a jump, and Φ'' is bounded from (19), we then have

$$\begin{aligned}&\mathbb{E}\left[\sum_{s\le t:|\triangle \widetilde{\mathbb{X}}^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{\mathbb{X}}^n(s))^2\right)\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2}\mathbb{E}\left[\sum_{s\le t:|\triangle \widetilde{X}_k^n(s)|>0} \left(\frac{1}{2}\Phi''(\widetilde{W}^n(s-))(\triangle \widetilde{X}_k^n(s))^2\right)\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2} \times \frac{1}{n}\mathbb{E}\left[\int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))dN_k\left(\int_0^s \lambda_k^n(u)du\right) + \int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))dS_k^n(T_k^n(s))\right] \\ &= \sum_{k\in\mathcal{K}} \frac{1}{\mu_k^2} \times \mathbb{E}\left[\int_0^t \frac{1}{2}\Phi''(\widetilde{W}^n(s-))\left(\frac{\lambda_k^n(s)+n\lambda_k^*}{n}ds - \frac{\mu_k}{\sqrt{n}}d\widetilde{Y}_k^n(s)\right)\right].\end{aligned}$$

Recall that the system is assumed to start from stationarity. Hence we can take expectation on both sides of (52), use the stationarity of $\widetilde{W}^n(\cdot)$, and $\frac{\lambda_k^n(s)+n\lambda_k^*}{n} = 2\lambda_k^* + \frac{1}{\sqrt{n}}\zeta_k^n(s)$ to obtain (29). The proof is complete. □

7. Conclusions and Future Research In this paper, we consider the optimal control of a multi-class make-to-stock system where the manager makes pricing, outsourcing, and scheduling decisions to maximize the long-run average profit. We propose a policy and establish the asymptotic optimality of the proposed policy in the heavy-traffic regime.

For future research, there are a few directions that are worth exploring. First, in this paper, we assume the production times are exponentially distributed so that the nominal workload process is a semimartingale and then we can apply Ito's formula. A natural direction is to consider general production time distributions. Second, we consider the demand arrival rates that satisfy Definition 3, which suggests that the fluctuation of the arrival rates is $o(n)$, where n corresponds to the system size. It would be interesting to study the control problem where larger fluctuations of demand arrival rates are allowed. Third, Theorem 1 of this paper shows the asymptotic optimality of the proposed policy in the heavy traffic regime. An intriguing question is whether one can establish the convergence rate. Finally, it would be interesting to extend our current work to a network setting. For that, one may focus on network structures satisfying complete resource pooling conditions, as in Mandelbaum and Stolyar (2004).

Appendix. Proofs

A. Proofs of Auxiliary Lemmas Within the Proof of Theorem 2

A.1. Proof of Lemma 4 To prove Lemma 4, we can infer from (34) and the definition of $\widetilde{V}^n(\widetilde{\nu}^n, \psi^n)$ in (11) that under policy ψ^n, there exists a sequence $\{t_i\}$ such that

$$\lim_{t_i\to\infty} \frac{\widetilde{V}^n(t_i, \widetilde{\nu}^n, \psi^n)}{t_i} = \liminf_{t\to\infty} \frac{\widetilde{V}^n(t, \widetilde{\nu}^n, \psi^n)}{t} \le \gamma^* + 2. \tag{53}$$

For both strictly convex and linear functions $g_k, k\in\mathcal{K}$, there exists a constant $c>0$ such that $g_k(x) > c|x|$, for $|x|>1$. As a result, from (53), there exists a constant C (independent of n) such that $\limsup_{t_i\to\infty} \frac{\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^{t_i}\sum_{k\in\mathcal{K}}|\widetilde{Q}_k^n(s)|ds\right]}{t_i} \le C$. Using Markov inequality, this implies for $\epsilon>0$,

$$\limsup_{t_i\to\infty} \frac{1}{t_i}\int_0^{t_i} \mathbb{P}_{\widetilde{\nu}^n}\left(\sum_{k\in\mathcal{K}}|\widetilde{Q}_k^n(s)| \ge C/\epsilon\right) ds \le \epsilon. \tag{54}$$

Hence if we define $H_{t_i}(\cdot) := \frac{1}{t_i}\int_0^{t_i}\mathbb{P}_{\widetilde{\nu}^n}(\widetilde{\mathfrak{X}}^n(s)\in\cdot)ds$, then from (54) and the definition of $\widetilde{\mathfrak{X}}^n(t) = (\widetilde{Q}_1^n(t),\ldots,\widetilde{Q}_K^n(t), C(t))$, the sequence of probability measures $\{H_{t_i}: t_i>0\}$ is tight. Note that $\widetilde{\mathfrak{X}}^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n = \frac{1}{\sqrt{n}}\mathbb{Z}^n\times(\mathcal{K}\cup\{0\})$, hence we can assume that every function is continuous on $\widetilde{S}^n$ (endowed with discrete topology), and it follows that $\widetilde{\mathfrak{X}}^n(\cdot)$ satisfies the Feller property, i.e., $\mathbb{E}[f(\widetilde{\mathfrak{X}}^n(t))|\widetilde{\mathfrak{X}}^n(0)=\mathfrak{x}]$ is a bounded and continuous function of $\mathfrak{x}$ for all t whenever f is bounded and continuous. We can then follow the proof of the Krylov-Bogoliubov theorem (Prato 2006, Theorem 7.1) and infer that there exists a stationary distribution $\widetilde{\pi}^n$ for $\widetilde{\mathfrak{X}}^n(\cdot)$ and moreover, a subsequence of probability measures $\{H_{t_i}: t_i>0\}$ converges weakly to this invariant measure $\widetilde{\pi}^n$ as $t_i\to\infty$. For notational simplicity, we use $\{t_i\}$ in the following to denote this further subsequence. From (10) and the property of stationary distributions we can infer that

$$\widetilde{V}^n(\widetilde{\pi}^n, \psi^n)$$
$$= \limsup_{t\to\infty}\frac{1}{t}\mathbb{E}_{\widetilde{\pi}^n}\left[\int_0^t c^n(\zeta^n(\widetilde{\mathfrak{X}}^n(s)))ds + \sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_k^n(s))ds + \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}} r_k\int_0^t \xi_k(s-)\lambda_k^n(s)ds\right]$$

$$= \mathbb{E}_{\widetilde{\pi}^n}\left[c^n(\zeta^n(\widetilde{\mathfrak{X}}^n(0))) + \sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_k(0)) + \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}} r_k\xi_k(\widetilde{\mathfrak{X}}^n(0))\lambda^n_k(\widetilde{\mathfrak{X}}^n(0))\right]$$
$$= \mathbb{E}_{\widetilde{\pi}^n}\left[F(\widetilde{\mathfrak{X}}^n(0))\right], \tag{55}$$

where

$$F(\widetilde{\mathfrak{X}}^n(0)) := c^n(\zeta^n(\widetilde{\mathfrak{X}}^n(0))) + \sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_k(0)) + \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}} r_k\xi_k(\widetilde{\mathfrak{X}}^n(0))\lambda^n_k(\widetilde{\mathfrak{X}}^n(0)). \tag{56}$$

In the first equality of (55) we use $\widetilde{O}^n_k(t) = \frac{1}{\sqrt{n}}\int_0^t \xi_k(s-)dE^n_k(s) = \frac{1}{\sqrt{n}}\int_0^t \xi_k(s-)\lambda^n_k(s)ds + \frac{1}{\sqrt{n}}\int_0^t \xi_k(s-)d(E^n_k(s) - \int_0^s \lambda^n_k(u)du)$ and the fact that $\frac{1}{\sqrt{n}}\int_0^t \xi_k(s-)d(E^n_k(s) - \int_0^s \lambda^n_k(u)du)$ is a martingale. The second equality holds because $\widetilde{\pi}^n$ is a stationary distribution of $\mathfrak{X}^n(\cdot)$, which has finite number of jumps on any finite time interval. Note that the function F in (56) is non-negative. It is defined on a discrete space, so it is also continuous. Since $H_{t_i} \Rightarrow \widetilde{\pi}^n$ when $t_i \to \infty$, we can then infer that

$$\begin{aligned}
&\widetilde{V}^n(\widetilde{\pi}^n, \psi^n) = \mathbb{E}_{\widetilde{\pi}^n}[F(\widetilde{\mathfrak{X}}^n(0))] \\
&\le \liminf_{t_i\to\infty} \frac{1}{t_i}\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^{t_i} F(\widetilde{\mathfrak{X}}^n(s))ds\right] \\
&= \liminf_{t_i\to\infty} \frac{1}{t_i}\mathbb{E}_{\widetilde{\nu}^n}\left[\int_0^{t_i} c^n(\zeta^n(\widetilde{\mathfrak{X}}^n(s)))ds + \sum_{k\in\mathcal{K}}\int_0^{t_i} g_k(\widetilde{Q}^n_k(s))ds + \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}} r_k \int_0^t \xi_k(s-)\lambda^n_k(s)ds\right] \\
&\le \widetilde{V}^n(\widetilde{\nu}^n, \psi^n) \le \widetilde{V}^n + \epsilon \le \gamma^* + 2,
\end{aligned} \tag{57}$$

where the first inequality is due to Fatou's lemma for weakly convergent probability measures (e.g., see Theorem 1.1 and Equation (1.5) in Feinberg et al. (2014)) and the last line of (57) is due to (34). Hence, we have proven (35). □

A.2. Proof of Lemma 5 Note that the integral in (47) is a Lebesgue integral and $\widetilde{W}^n$ can only have finite jumps almost surely, it is enough to prove

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\frac{1}{2\sqrt{n}}\int_0^t \Phi''(\widetilde{W}^n(s))d\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\widetilde{Y}^n_k(s)\right] = 0.$$

Now, we first show that for each $k \in \mathcal{K}$,

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \Phi''(\widetilde{W}^n(s))dT^n_k(s)\right] = \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))dT^n_k(s)\right], \tag{58}$$
$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \Phi''(\widetilde{W}^n(s))d(\rho_k s)\right] = \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))d(\rho_k s)\right]. \tag{59}$$

To see this, using the fact that $|\Phi''(x)| \le C$ and $T^n_k(t) - T^n_k(s) \le t - s$ for $0 \le s \le t$, we have

$$\left|\mathbb{E}\left[\frac{1}{t}\int_0^{\lfloor t\rfloor} \Phi''(\widetilde{W}^n(s))dT^n_k(s)\right] - \mathbb{E}\left[\frac{1}{t}\int_0^t \Phi''(\widetilde{W}^n(s))dT^n_k(s)\right]\right| \le \frac{C(t-\lfloor t\rfloor)}{t} \le \frac{C}{t},$$

where $\lfloor t\rfloor$ denotes the floor operator. Because $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, we have $\mathbb{E}[\int_0^{\lfloor t\rfloor} \Phi''(\widetilde{W}^n(s))dT^n_k(s)] = \lfloor t\rfloor \cdot \mathbb{E}[\int_0^1 \Phi''(\widetilde{W}^n(s))dT^n_k(s)]$. Then (58) readily follows. Similarly we can obtain (59). Hence, using the definition of $\widetilde{Y}^n_k$ we have

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\frac{1}{\sqrt{n}}\int_0^t \Phi''(\widetilde{W}^n(s-))d\widetilde{Y}^n_k(s)\right] = \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))d(\rho_k s)\right] - \mathbb{E}\left[\int_0^1 \Phi''(\widetilde{W}^n(s))dT^n_k(s)\right].$$

Second, we show that the process $\{T_k^n(t)-\rho_k t : t\in[0,1]\}$ converges weakly to zero on $C[0,1]$ as $n\to\infty$. Because $\rho_k\le 1$ and $T_k^n(t)-T_k^n(s)\le t-s$ for $0\le s\le t\le 1$, $\{T_k^n(t)-\rho_k t : t\in[0,1]\}$ is tight on $C[0,1]$. Thus it is enough to prove $T_k^n(t)-\rho_k t\Rightarrow 0$ for each $t\in[0,1]$, which is true if

$$\mathbb{E}[|T_k^n(t)-\rho_k t|]\to 0, \quad \text{as } n\to\infty. \tag{60}$$

To prove (60), note that $T_k^n(t)-\rho_k t=-\widetilde{Y}_k^n(t)/\sqrt{n}$. Then from (6) we can compute

$$\mathbb{E}\left[|T_k^n(t)-\rho_k t|\right] \le \frac{1}{\mu}\mathbb{E}\left[\left|\frac{\widetilde{Q}_k^n(t)}{\sqrt{n}}\right|+\left|\frac{\widetilde{Q}_k^n(0)}{\sqrt{n}}\right|+\left|\frac{\widetilde{X}_k^n(t)}{\sqrt{n}}\right|+\left|\int_0^t \frac{\zeta_k^n(s)}{\sqrt{n}}ds\right|+\left|\frac{\widetilde{O}_k^n(t)}{\sqrt{n}}\right|\right].$$

The first, second, and fifth terms on the right-hand side of the above inequality all converge to 0 as $n\to\infty$ because $\widetilde{V}^n(\widetilde{\pi}^n,\psi^n)<\gamma^*$ by our assumption. The fourth term converges to 0 because $|\frac{\zeta^n(s)}{\sqrt{n}}|\le a_n\to 0$. The third term converges to 0 because

$$\mathbb{E}\left[\left|\frac{\widetilde{X}_k^n(t)}{\sqrt{n}}\right|\right]\le \mathbb{E}\left[\sup_{0\le u\le 1}\left|\frac{S_k^n(u)-\mu_k^n u}{n}\right|\right]+\mathbb{E}\left[\sup_{0\le u\le 1}\left|\frac{N_k(\int_0^u \lambda_k^n(s)ds)-\int_0^u \lambda_k^n(s)ds}{n}\right|\right]\to 0.$$

Thus, we obtain (60). To complete the proof, we also need the following lemma.

Lemma 7. *Assume $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$ for all sufficiently large n and* (35) *holds. Then the sequence $\{\widetilde{W}^n(\cdot)=\{\widetilde{W}^n(t):t\in[0,1]\}:n\ge 1\}$ is tight.*

By Lemma 7, for any subsequence, there exists a further subsequence $\{n_i\}$ of $\{n\}$ such that $\widetilde{W}^{n_i}\Rightarrow\widetilde{W}$ on $D[0,1]$ for some $\widetilde{W}$. For this subsequence $\{n_i\}$, we also have $T_k^{n_i}(\cdot)\Rightarrow\rho_k\cdot$ as $n_i\to\infty$. Since $T_k^{n_i}(\cdot)$ is non-decreasing with $T_k^{n_i}(1)\le 1$, and Φ'' is continuous and bounded, we can apply Lemma 8.3 in Dai and Dai (1999) and the dominated convergence theorem to obtain

$$\lim_{n_i\to\infty}\mathbb{E}\int_0^1 \Phi''(\widetilde{W}^{n_i}(s))dT_k^{n_i}(s)=\mathbb{E}\int_0^1 \Phi''(\widetilde{W}(s))d(\rho_k s)=\lim_{n_i\to\infty}\mathbb{E}\int_0^1 \Phi''(\widetilde{W}^{n_i}(s))d(\rho_k s).$$

It follows that (47) holds for the subsequence $\{n_i\}$. Because (47) holds for any subsequence, it also holds for the whole sequence. □

Proof of Lemma 7. Because $\{T_k^n(t)-\rho_k t : t\in[0,1]\}$ converges weakly to zero as $n\to\infty$, one can infer from (5) that $\{\sum_{k\in\mathcal{K}}\frac{\widetilde{X}_k^n(t)}{\mu_k}:t\in[0,1]\}$ converges weakly to a Browian motion with zero drift. From (8) and (Jacod and Shiryaev 2013, Chapter VI, Corollary 3.33), it is then enough to prove the tightness of $\{\widetilde{R}^n(\cdot)=\{\widetilde{R}^n(t):t\in[0,1]\}:n\ge 1\}$, where

$$\widetilde{R}^n(t):=\widetilde{W}^n(t)-\sum_{k\in\mathcal{K}}\frac{\widetilde{X}_k^n(t)}{\mu_k}=\widetilde{W}^n(0)-\widetilde{I}^n(t)-\int_0^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(s)}{\mu_k}ds+\sum_{k\in\mathcal{K}}\frac{\widetilde{O}_k^n(t)}{\mu_k}, \quad t\in[0,1]. \tag{61}$$

We use the tightness criteria in (Jacod and Shiryaev 2013, Chapter VI. Theorem 4.1), and verify the three conditions there; see also (Billingsley 1999, Theorem 13.5 and equation (13.14)). From (35), $\mathbb{P}(|\widetilde{Q}^n(0)|>C/\epsilon)\le\epsilon$ for all n sufficiently large, hence the sequence $\{\widetilde{Q}^n(0)\}$ is tight. Because $\widetilde{R}^n(0)=\widetilde{W}^n(0)=\sum_{k\in\mathcal{K}}\frac{\widetilde{Q}_k^n(0)}{\mu_k}$, then $\{\widetilde{R}^n(0)\}$ is also tight. Thus, Condition (i) holds. To verify Conditions (ii) and (iii), we check the following two conditions: there exists a sequence of bounded numbers $\{C_n\}$ such that for all n sufficiently large and any $0\le s<u<t\le 1$,

$$\mathbb{E}[|\widetilde{R}^n(t)-\widetilde{R}^n(s)|]\le C_n(t-s), \tag{62}$$

$$\mathbb{E}\left[|\widetilde{R}^n(u)-\widetilde{R}^n(s)|\times|\widetilde{R}^n(t)-\widetilde{R}^n(u)|\right]\le C_n(t-s)^2. \tag{63}$$

First, we verify (62). From the dynamics of $\widetilde{R}^n$ in (61), we have

$$\widetilde{R}^n(t)-\widetilde{R}^n(s)=-(\widetilde{I}^n(t)-\widetilde{I}^n(s))-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du+\sum_{k\in\mathcal{K}}\frac{\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)}{\mu_k}.$$

Since $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, it follows that $\mathbb{E}[\widetilde{W}^n(t)-\widetilde{W}^n(s)]=0$. Also note that $\mathbb{E}[\sum_{k\in\mathcal{K}}\frac{\widetilde{X}_k^n(t)-\widetilde{X}_k^n(s)}{\mu_k}]=0$. It then follows from (61) that $\mathbb{E}[\widetilde{R}^n(t)-\widetilde{R}^n(s)]=0$. As a result, we obtain

$$\mathbb{E}[\widetilde{I}^n(t)-\widetilde{I}^n(s)]=\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]+\sum_{k\in\mathcal{K}}\frac{\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]}{\mu_k}.$$

Because the system is stationary when $\widetilde{\mathfrak{X}}^n(0)$ follows the stationary distribution $\widetilde{\pi}^n$, we can then obtain $\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]=a_k^n(t-s)$, where a_k^n are non-negative and bounded (because of (35)). Similarly, we have $\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]=b^n\cdot(t-s)$, where $\{b^n\}$ is a sequence of bounded numbers. To see the boundedness, we can compute that

$$\left|\mathbb{E}\left[-\int_s^t\sum_{k\in\mathcal{K}}\frac{\zeta_k^n(u)}{\mu_k}du\right]\right|\le\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|}{\mu_k}du\right]\le(t-s)\sqrt{\frac{\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|^2}{\mu_k^2}du\right]}{t-s}}\le C(t-s),$$

where C is a generic constant, the second inequality above follows from Cauchy-Schwarz inequality and the last inequality is due to (44) and (35). From these estimates, we then infer that

$$\mathbb{E}[|\widetilde{R}^n(t)-\widetilde{R}^n(s)|]\le 2\mathbb{E}\left[\int_s^t\sum_{k\in\mathcal{K}}\frac{|\zeta_k^n(u)|}{\mu_k}du\right]+2\sum_{k\in\mathcal{K}}\frac{\mathbb{E}[\widetilde{O}_k^n(t)-\widetilde{O}_k^n(s)]}{\mu_k}\le C_n\cdot(t-s),\quad(64)$$

where the sequence of real numbers $\{C_n\}$ is bounded. Hence, we have proven (62).

Next, we prove (63). Let $\mathcal{F}_u^n$ be the natural filtration generated by $\mathfrak{X}^n$ until time u. Then

$$\begin{aligned}\mathbb{E}\left[|\widetilde{R}^n(u)-\widetilde{R}^n(s)|\times|\widetilde{R}^n(t)-\widetilde{R}^n(u)|\right]&=\mathbb{E}\left[|\widetilde{R}^n(u)-\widetilde{R}^n(s)|\times\mathbb{E}\left[|\widetilde{R}^n(t)-\widetilde{R}^n(u)|\Big|\mathcal{F}_u^n\right]\right]\\&\le\mathbb{E}\left[|\widetilde{R}^n(u)-\widetilde{R}^n(s)|\times C_n(t-u)\right]\\&\le C_n^2(t-u)(u-s)\le C_n^2(t-s)^2,\end{aligned}$$

where the first inequality follows from (64). Hence, the tightness of $\{\widetilde{R}^n(\cdot):n\ge1\}$ follows. □

B. Proof of Proposition 1 We use the Foster-Lyapunov criteria; e.g., see (Meyn and Tweedie 1993, Theorem 4.5). Under the policy $\psi_\diamond^n$, the process $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$ is a continuous-time Markov chain with countable state space $\widetilde{S}^n=\frac{1}{\sqrt{n}}\mathbb{Z}^n\times(\mathcal{K}\cup\{0\})$. In addition, it satisfies the Feller property by using a similar argument as in the proof of Lemma 4. Write $G^n=\left(G_{x,x'}^n\right)_{x,x'\in\widetilde{S}^n}$ for the generator matrix of $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$. The Foster-Lyapunov criteria states that if there exist a function $\mathcal{V}:\widetilde{S}^n\to\mathbb{R}_+$ where $\mathcal{V}(x)\to\infty$ as $|x|\to\infty$, and a constant $r>0$ such that

$$G^n\mathcal{V}(x):=\sum_{x'\in\widetilde{S}^n}G_{x,x'}^n(\mathcal{V}(x')-\mathcal{V}(x))\le-1,\quad\text{for } x\in\widetilde{S}^n\text{ with }\sum_{k\in\mathcal{K}}\frac{|x_k|}{\mu_k}>r,\quad(65)$$

then there exists a stationary distribution for the Markov chain $\widetilde{\mathfrak{X}}_\diamond^n(\cdot)$.

Next, we construct the Lyapunov function $\mathcal{V}$ which satisfies the drift inequality in (65). Let Υ be a positive constant such that

$$\Upsilon\cdot\frac{\lambda_i^*}{\mu_i}\ge\sum_{k\neq i}\frac{\lambda_k^*}{\mu_k}+1,\quad\text{for all } i\in\mathcal{K}.\quad(66)$$

This is feasible by picking $\Upsilon\ge\frac{2}{\min_k\left\{\frac{\lambda_k^*}{\mu_k}\right\}}-1$. Now, for $x=(x_1,\ldots,x_K,\xi)\in S$, we define

$$\mathcal{V}_0(x)=1_{\left\{\xi\in\{k\in\mathcal{K}:x_k\ge\alpha_k/\sqrt{n}\}\right\}},\quad(67)$$

$$\mathcal{V}_1(x) = \Upsilon \cdot \sum_{k\in\mathcal{K}} \frac{(x_k - \alpha_k/\sqrt{n})^+}{\mu_k} + \sum_{k\in\mathcal{K}} \frac{(x_k - \alpha_k/\sqrt{n})^-}{\mu_k}, \tag{68}$$

$$\mathcal{V}(x) = \mathcal{V}_0(x) + \mathcal{V}_1(x), \tag{69}$$

where we recall that ξ denotes the index of the customer class that the system is working on. Pick $r > 2\max\{|u_\diamond|, |l_\diamond|\}$. We verify (65) by considering $x \in \widetilde{S}^n$ with $\sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} > r$ and discussing several cases separately. We use e_k to denote a vector in which all elements are zeros except for the kth element, which equals one.

Case (1): $\mathcal{V}_0(x) = 1$ for $x = (x_1, \ldots, x_K, \xi)$; i.e., the inventory level of the product class in production exceeds the safety-stock level. We study the following two subcases.

Case (1a): $\sum_{k\in\mathcal{K}} \frac{x_k^+}{\mu_k} > u_\diamond$. In this case, consider that the Markov chain $\widetilde{\mathfrak{X}}^n_\diamond(\cdot)$ transits to a new state $x' = (x'_1, \ldots, x'_K, \xi')$ from the current state $x = (x_1, \ldots, x_K, \xi)$. If the state transition is due to an order arrival, then $\mathcal{V}_0(x') \le \mathcal{V}_0(x)$ by the definition of $\mathcal{V}_0(\cdot)$. On the other hand, if the state transition is due to a production completion, then x_ξ will increase by one, and as a result the indicator function $\mathcal{V}_0(x') = 0$ and $\xi' \ne \xi$ since the system will not work on a class with $x'_k \ge \alpha_k/\sqrt{n}$ when $\sum_{k\in\mathcal{K}} \frac{(x'_k)^+}{\mu_k} > u_\diamond$ according to the policy $\psi^n_\diamond$. Hence, we have

$$\begin{aligned}
G^n\mathcal{V}(x) &= \sum_{k\in\mathcal{K}} \lambda^n_k(x) \times \left(\mathcal{V}(x - \frac{e_k}{\sqrt{n}}) - \mathcal{V}(x)\right) + \mu^n_\xi \left(\mathcal{V}\left(x + \frac{e_\xi}{\sqrt{n}} + e_{K+1}(\xi' - \xi)\right) - \mathcal{V}(x)\right) \\
&\le \sum_{k\in\mathcal{K}} \lambda^n_k(x) \times \left(\mathcal{V}_1(x - \frac{e_k}{\sqrt{n}}) - \mathcal{V}_1(x)\right) + \mu^n_\xi \left(-1 + \mathcal{V}_1\left(x + \frac{e_\xi}{\sqrt{n}}\right) - \mathcal{V}_1(x)\right) \\
&\le \sum_{k\in\mathcal{K}} \frac{\lambda^n_k(x)}{\mu_k\sqrt{n}} + \Upsilon \frac{\mu^n_\xi}{\sqrt{n}} - \mu^n_\xi = \sum_{k\in\mathcal{K}} \frac{\sqrt{n}\lambda^*_k + \zeta_k(x)}{\mu_k} + \Upsilon\sqrt{n}\mu_\xi - n\mu_\xi \le -1,
\end{aligned}$$

for all sufficiently large n. In the above we use $\lambda^n_k(x) = n\lambda^*_k + \sqrt{n}\zeta^n_k(x)$ where $|\zeta^n_k(x)/\sqrt{n}| \le a_n \to 0$, and $\mu^n_k = n\mu_k$ for each $k \in \mathcal{K}$.

Case (1b): $\sum_{k\in\mathcal{K}} \frac{x_k^+}{\mu_k} \le u_\diamond$. In this case, since $\sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} > r$, then $\sum_{k\in\mathcal{K}} \frac{x_k^-}{\mu_k} = \sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} - \sum_{k\in\mathcal{K}} \frac{x_k^+}{\mu_k} > r - u_\diamond > 0$. For policy $\psi^n_\diamond$, when there is a class with customers waiting, the system would not start to produce a new product from a class with an inventory level higher than the corresponding safety stock. Hence if the state transition is due to a production completion, then for the new state $x' = (x'_1, \ldots, x'_K, \xi')$, we still have $\xi' \ne \xi$, and $\mathcal{V}_0(x') = 0$. Following a similar computation as in Case (1a), we obtain that $G^n\mathcal{V}(x) \le -1$ for all sufficiently large n.

Case (2): $\mathcal{V}_0(x) = 0$ for $x = (x_1, \ldots, x_K, \xi)$. In this case, from the definition of $\mathcal{V}_0$ in (67) we know that there is either no product in production ($\xi = 0$) or the class index ξ of the product in production satisfies $x_\xi < \alpha_\xi/\sqrt{n}$. Suppose the system state x transits to x' at the next event, which can be either an order arrival or a production completion. We argue that $\mathcal{V}_0(x') = 0$ according to the policy $\psi^n_\diamond$. To see this, we first consider the case where the state transition is due to an order arrival. Note that if the system is occupied before an order arrival, the arrival of any class k will not change ξ (the class in production) and will reduce x_k by $1/\sqrt{n}$, so $\mathcal{V}_0$ remains at zero at the new state x'. If the system is idle before the arrival, then $x_k \ge 0$ for each k (i.e., no order is waiting) and we have $\sum_{k\in\mathcal{K}} \frac{x_k^+}{\mu_k} = \sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} > r$. According to the scheduling part of the policy $\psi^n_\diamond$, after the new arrival, the system only chooses from the classes with $x'_k < \alpha_k/\sqrt{n}$, hence we also have $\mathcal{V}_0(x') = 0$ by (67). Next we consider the case where the state transition is due to a production completion. Since $\sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} > r$, we obtain that either $\sum_{k\in\mathcal{K}} \frac{x_k^+}{\mu_k} > r/2 > u_\diamond$ or $\sum_{k\in\mathcal{K}} \frac{x_k^-}{\mu_k} > r/2 > |l_\diamond|$. In the former, we have $\mathcal{V}_0(x') = 0$ from the previous discussion. In the latter, there is at least one class with customers waiting. By the scheduling part of the policy, the system will not work on classes with inventory level higher than the corresponding safety stock level α_k. Hence, we also have $\mathcal{V}_0(x') = 0$. In summary, the indicator function $\mathcal{V}_0$ does not change its value when there is a state transition from x to x' if $\mathcal{V}_0(x) = 0$.

Next, we consider two subcases in order to compute $G^n\mathcal{V}(x)$ when $\sum_{k\in\mathcal{K}} \frac{|x_k|}{\mu_k} > r$.

Case (2a): $x_k \ge \alpha_k/\sqrt{n}$ for all k. Then the system is idle under the proposed policy. Hence only an order arrival can change the state and using the definition of $\mathcal{V}$ in (69) we have

$$\begin{aligned}
G^n\mathcal{V}(x) &= \sum_{k:x_k\ge\alpha_k/\sqrt{n}} \lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)\\
&= \sum_{k:x_k>\alpha_k/\sqrt{n}} \lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)+\sum_{k:x_k=\alpha_k/\sqrt{n}} \lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)\\
&= -\Upsilon\sum_{k:x_k>\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}+\sum_{k:x_k=\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\\
&\le -\sqrt{n}+\sum_{k\in\mathcal{K}}\frac{a_n}{\mu_k}\le -1,\quad \text{provided that } n \text{ is sufficiently large.}
\end{aligned}$$

In the last inequality, we use the fact that $\lambda_k^n(x)=n\lambda_k^*+\sqrt{n}\zeta_k^n(x)$ where $|\zeta_k^n(x)/\sqrt{n}|\le a_n\to 0$, and the inequality $\Upsilon\frac{\lambda_i^*}{\mu_i}\ge\sum_{k\neq i}\frac{\lambda_k^*}{\mu_k}+1$ for all $i\in\mathcal{K}$.

Case (2b): There exists a class k_0 such that $x_{k_0}<\alpha_{k_0}/\sqrt{n}$ and the system is working on this class. We consider the following two situations.

i). If $x_k\le\alpha_k/\sqrt{n}$ for all k, then the arrivals of class $i^\diamond$ orders are outsourced according to the policy. This is because if $\sum_{k\in\mathcal{K}}\frac{|x_k|}{\mu_k}>r$ and $x_k\le\alpha_k/\sqrt{n}$ for all $k\in\mathcal{K}$, we must have $\sum_{k\in\mathcal{K}}\frac{x_k}{\mu_k}<l_\diamond$. Hence, using (69) and the fact that $\sum_{k\in\mathcal{K}}\frac{\lambda_k^*}{\mu_k}=1$, we can compute

$$\begin{aligned}
G^n\mathcal{V}(x) &= \mu_{k_0}^n\Big(\mathcal{V}_1(x+\frac{e_{k_0}}{\sqrt{n}})-\mathcal{V}_1(x)\Big)+\sum_{k\neq i^\diamond}\lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)\\
&= -\Big(\sqrt{n}-\sum_{k\neq i^\diamond}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\Big)\le -\frac{\lambda_{i^\diamond}^*}{\mu_{i^\diamond}}\sqrt{n}+\sum_{k\neq i^\diamond}\frac{a_n}{\mu_k}\le -1,\quad \text{for } n \text{ sufficiently large.}
\end{aligned}$$

ii). Otherwise, there exists a class k_1 such that $x_{k_1}>\alpha_{k_1}/\sqrt{n}$. When the system state is x, if a new class $i^\diamond$ order should not be outsourced, let $\underline{\mathcal{K}}(x)=\mathcal{K}$; otherwise, let $\underline{\mathcal{K}}(x)=\mathcal{K}\backslash\{i^\diamond\}$. Because $x_{k_1}>\alpha_{k_1}/\sqrt{n}$, $k_1\in\underline{\mathcal{K}}(x)$. We can obtain from (69) that for n sufficiently large,

$$\begin{aligned}
G^n\mathcal{V}(x) &= \mu_{k_0}^n\Big(\mathcal{V}_1(x+\frac{e_{k_0}}{\sqrt{n}})-\mathcal{V}_1(x)\Big)+\sum_{k\in\underline{\mathcal{K}}(x)}\lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)\\
&\le \sum_{k\in\underline{\mathcal{K}}(x):x_k>\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)+\sum_{k\in\underline{\mathcal{K}}(x):x_k\le\alpha_k/\sqrt{n}}\lambda_k^n(x)\times\Big(\mathcal{V}_1(x-\frac{e_k}{\sqrt{n}})-\mathcal{V}_1(x)\Big)\\
&\le -\Upsilon\sum_{k\in\underline{\mathcal{K}}(x):x_k>\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}+\sum_{k\in\underline{\mathcal{K}}(x):x_k\le\alpha_k/\sqrt{n}}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\\
&\le -\Upsilon\frac{\lambda_{k_1}^n(x)}{\mu_{k_1}\sqrt{n}}+\sum_{k\neq k_1}\frac{\lambda_k^n(x)}{\mu_k\sqrt{n}}\le -\sqrt{n}+\Upsilon\sum_{k\in\mathcal{K}}\frac{a_n}{\mu_k}\le -1,
\end{aligned}$$

where we have used (66) in the last line. The proof is therefore complete. □

C. Proof of Proposition 2 To prove the tightness of $\{\widetilde{\mathfrak{X}}_\diamond^n(\infty)\}$, we will analyze the hydrodynamic-scaled processes. To this end, we first introduce the hydrodynamic-scaled processes under the proposed policy $\psi_\diamond^n$ as follows. For simplicity of presentation, we omit the subscript $\diamond$ notation in defining these processes. For $k\in\mathcal{K}$ and $t\ge 0$, define

$$\begin{aligned}
&\bar{S}_k^n(t)=\frac{S_k^n(t/\sqrt{n})}{\sqrt{n}},\quad \bar{N}_k^n(t)=\frac{N_k(\sqrt{n}t)}{\sqrt{n}},\quad \bar{T}_k^n(t)=\sqrt{n}T_k^n(t/\sqrt{n}),\\
&\bar{Q}_k^n(t)=\frac{Q_k^n(t/\sqrt{n})}{\sqrt{n}},\quad \bar{O}_k^n(t)=\frac{O_k^n(t/\sqrt{n})}{\sqrt{n}},\\
&\bar{\lambda}_k^n(t)=\frac{1}{\sqrt{n}}\int_0^{t/\sqrt{n}}\lambda_k^n(s)ds=\lambda_k^*t+\int_0^{t/\sqrt{n}}\zeta_k^n(s)ds=\lambda_k^*t+\frac{1}{\sqrt{n}}\int_0^t\zeta_k^n(u/\sqrt{n})du. \qquad (70)
\end{aligned}$$

Then we have

$$\bar{Q}_k^n(t) = \bar{Q}_k^n(0) + \bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t). \tag{71}$$

Finally, define

$$\bar{\mathfrak{X}}^n(t) = (\bar{Q}^n(t), \bar{C}^n(t)), \quad t \geq 0,$$

where $\bar{Q}^n(t) := (\bar{Q}_1^n(t), \dots, \bar{Q}_K^n(t))$, $\bar{C}^n(t) = C^n(t/\sqrt{n})$ and recall that $C^n(t)$ denotes the customer class in production at time t in the nth system. It is clear that $\widetilde{\mathfrak{X}}_\diamond^n(\infty)$ is also a stationary distribution of $\bar{\mathfrak{X}}^n(\cdot)$. In the following, we use $\bar{\mathfrak{X}}^n(\infty)$ to denote $\widetilde{\mathfrak{X}}_\diamond^n(\infty)$, to emphasize that we are analyzing hydrodynamic-scaled processes. It is then enough to prove the tightness of $\{\bar{\mathfrak{X}}^n(\infty)\}$.

We use the approach in (Gamarnik and Zeevi 2006, Section 3), where the main idea is to construct appropriate Lyapunov functions to obtain tail bounds on $\bar{\mathfrak{X}}^n(\infty)$. Following Gamarnik and Zeevi (2006), we define for $\theta > 0$ and a function Φ (with a slight abuse of the notations):

$$L_1^n(\theta, t) := \sup_{(q,\xi)} \mathbb{E}_{(q,\xi)} \left[\exp\left(\theta |\Phi(\bar{\mathfrak{X}}^n(t)) - \Phi(q,\xi)|\right)\right], \tag{72}$$

$$L_2^n(\theta, t) := \sup_{(q,\xi)} \mathbb{E}_{(q,\xi)} \left[(\Phi(\bar{\mathfrak{X}}^n) - \Phi(q,\xi))^2 \cdot \exp\left(\theta(\Phi(\bar{\mathfrak{X}}^n(t)) - \Phi(q,\xi))^+\right)\right], \tag{73}$$

for $t \geq 0$, where $\mathbb{E}_{(q,\xi)}$ stands for the conditional expectation operator $\mathbb{E}[\cdot|\bar{\mathfrak{X}}^n(0) = (q,\xi)]$ with $\bar{Q}^n(0) = q \in \mathbb{R}^K$ and $\bar{C}^n(0) = \xi \in \mathcal{K} \cup \{0\}$.

We divide the proof of Proposition 2 into two parts. In Section C.1, we prove that the sequence of random variables $\left\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\right\}$ is tight. In Section C.2, we show that the sequence of random variables $\left\{\left(\sum_{k\in\mathcal{K}} \frac{\bar{Q}_k^n(\infty)}{\mu_k}\right)^-\right\}$ is tight. Combining these two parts together, and using $\sum_k |x_k| \leq 2\sum_k x_k^+ + (\sum_k x_k)^-$, we obtain the tightness of $\left\{\sum_{k\in\mathcal{K}} \frac{|\bar{Q}_k^n(\infty)|}{\mu_k}\right\}$. Since $\bar{C}^n(\infty)$ takes value in a compact set $\mathcal{K} \cup \{0\}$, the tightness of $\{\bar{\mathfrak{X}}^n(\infty)\}$ then readily follows.

In the following, we denote by $\bar{\pi}^n$ the distribution of $\bar{\mathfrak{X}}^n(\infty)$, and $\mathbb{P}_{\bar{\pi}^n}$ the probability conditional on that $\bar{\mathfrak{X}}^n(0)$ follows $\bar{\pi}^n$.

C.1. Tightness of $\left\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\right\}$. We have the following result.

LEMMA 8. *There exist constants $C_1, C_2 > 0$ which are independent of n such that for all sufficiently large n,*

$$\mathbb{P}_{\bar{\pi}^n}\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(0)-1)^+}{\mu_k} > s\right) \leq C_1 e^{-C_2 \cdot s}, \quad \textit{for all } s > 0.$$

As a consequence, the sequence of random variables $\{\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(\infty)-1)^+}{\mu_k}\}$ is tight.

The key step in proving Lemma 8 is the following result.

LEMMA 9. *There exist constants $t_0, c_0, \gamma > 0$ which are independent of n, such that for all sufficiently large n,*

$$\sup_{(q,\xi): \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} > c_0} \left\{\mathbb{E}_{(q,\xi)}\left[\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}\right] - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}\right\} \leq -\gamma. \tag{74}$$

Using the terminology from Gamarnik and Zeevi (2006), Lemma 9 says that the function

$$\Phi(q,\xi) = \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}, \tag{75}$$

is a Lyapunov function with drift size parameter γ, drift parameter $t_0 > 0$, and exception parameter c_0. With Lemma 9, we now proceed to prove Lemma 8.

Proof of Lemma 8. We use Lemma 9 and apply Theorem 6 in Gamarnik and Zeevi (2006). In the following, $L_1^n(\theta,t)$ and $L_2^n(\theta,t)$ are defined in (72) and (73), with Φ given in (75), respectively. Recall that

$$\bar{Q}_k^n(t) = \bar{Q}_k^n(0) + \bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t). \tag{76}$$

Since the function $\Phi(q,\xi)$ in (75) does not depend on the value of ξ, with a slight abuse of the notations, we write $\Phi(q) = \Phi(q,\xi) = \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k}$. Then for each $t \geq 0$,

$$\begin{aligned}
|\Phi(\bar{Q}^n(t)) - \Phi(\bar{Q}^n(0))| &\leq \sum_{k\in\mathcal{K}} \left|\bar{Q}_k^n(t) - \bar{Q}_k^n(0)\right|/\mu_k = \sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\left|\bar{S}_k^n(\bar{T}_k^n(t)) - \bar{N}_k^n(\bar{\lambda}_k^n(t)) + \bar{O}_k^n(t)\right| \\
&\leq \left|\sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\bar{S}_k^n(\bar{T}_k^n(t))\right| + \left|\sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))\right| \\
&\leq \sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\bar{S}_k^n(t) + \sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t)),
\end{aligned} \tag{77}$$

where we have used the fact that $0 \leq \bar{O}_k^n(t) \leq \bar{N}_k^n(\bar{\lambda}_k^n(t))$. It follows that for each $t \geq 0$,

$$\begin{aligned}
L_1^n(\theta,t) &\leq \mathbb{E}\left[\exp\left(\sum_{k\in\mathcal{K}} \frac{\theta}{\mu_k}\bar{S}_k^n(t) + \sum_{k\in\mathcal{K}} \frac{\theta}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))\right)\right] \\
&= \Pi_{k=1}^K \mathbb{E}[\exp(\frac{\theta}{\mu_k}\bar{S}_k^n(t))] \cdot \mathbb{E}[\exp(\frac{\theta}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t)))] \\
&\leq \Pi_{k=1}^K \exp\left(\sqrt{n}t\mu_k(e^{\frac{\theta}{\sqrt{n}\mu_k}} - 1)\right) \cdot \exp\left([\sqrt{n}(\lambda_k^* + a_n)t](e^{\frac{\theta}{\sqrt{n}\mu_k}} - 1)\right),
\end{aligned} \tag{78}$$

where in the second inequality we have used the moment generating functions of Poisson distributions and $\bar{\lambda}_k^n(t) = \lambda_k^* t + \frac{1}{\sqrt{n}}\int_0^t \zeta_k^n(u/\sqrt{n})du \leq (\lambda_k^* + a_n)t$. Hence for any $\theta > 0$,

$$\limsup_{n\to\infty} L_1^n(\theta,t_0) \leq \Pi_{k=1}^K \exp\left(\frac{\theta t_0(\mu_k + \lambda_k^*)}{\mu_k}\right) := G(\theta,t_0) < \infty. \tag{79}$$

Next we verify that there exists $\theta > 0$ such that $\theta L_2^n(\theta,t_0) \leq \gamma$ for all sufficiently large n. Using the fact that $x^2 \leq 2e^x$ for all $x \geq 0$, we can obtain that $\mathbb{E}[Y^2 e^{\theta|Y|}] \leq 2\mathbb{E}[e^{(\theta+1)|Y|}]$ for a random variable Y. Hence, we infer from the definitions of L_1^n, L_2^n and (79) that

$$\limsup_{n\to\infty} L_2^n(\theta,t_0) \leq 2G((1+\theta),t_0).$$

It is clear that we can choose sufficiently small $\theta > 0$ so that $\theta \cdot G((1+\theta),t_0) \leq \gamma$. Hence, we obtain that $\theta L_2^n(\theta,t_0) \leq \gamma$ for all n large enough.

Using Lemma 9 and Theorem 6 in Gamarnik and Zeevi (2006), we infer that

$$\mathbb{P}_{\bar{\pi}^n}\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(0)-1)^+}{\mu_k} > s\right) \leq (1-\gamma\theta/2)^{-1} L_1^n(\theta,t_0)\exp(-\theta(s-c_0)),$$

where t_0, c_0 are from Lemma 9. The result in Lemma 8 then follows from Equation (79). $\square$

Proof of Lemma 9. To show (74), it is enough to prove that

$$\sup_{(q,\xi):\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k} > c_0} \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k} - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} + 2\gamma\right)^+\right] \leq \gamma. \tag{80}$$

Fix $t_0 > 0$. Let $c_0 > u_\diamond$. Define an event $A := \{\sum_{k\in\mathcal{K}} \frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0)) < c_0 - u_\diamond\}$. First, from (77),

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}} \frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k} - \sum_{k\in\mathcal{K}} \frac{(q_k-1)^+}{\mu_k} + 2\gamma\right)^+ \cdot 1_{A^c}\right]$$

$$\leq \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0))+2\gamma\right)\cdot 1_{A^c}\right]. \tag{81}$$

From (78), the collection of random variables $\{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0)):n\geq 1\}$ is uniformly integrable. Given $\epsilon>0$ and $t_0>0$, we can choose $c_0>0$ large so that $\mathbb{P}(A^c)<\epsilon$, and

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}_k^n(t_0)+\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t_0))+2\gamma\right)\cdot 1_{A^c}\right]\leq (2\gamma+1)\epsilon. \tag{82}$$

Next, we control the expectation when event A holds. Note that for $\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}>c_0$ and on the event A, we can obtain from (76) that for $t\in[0,t_0]$

$$\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t)-1)^+}{\mu_k}\geq\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))>c_0-\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}_k^n(\bar{\lambda}_k^n(t))>u_\diamond.$$

Then, under the policy $\psi_\diamond^n$, the system will not work on any new product from a class k with $\bar{Q}_k^n(t)\geq\frac{\alpha_k}{\sqrt{n}}$. As one product may be initially in production, for each class, we have $(\bar{Q}_k^n(t)-1)^+\leq(q_k-1)^++\frac{1}{\sqrt{n}}$ for $t\in[0,t_0]$. Also note that there must exist a class k_0 such that $\bar{Q}_{k_0}^n(t)>1$ for $s\in[0,t_0]$. Then $(\bar{Q}_{k_0}^n(t_0)-1)^+-(q_{k_0}-1)^+=\bar{Q}_{k_0}^n(t_0)-q_{k_0}\leq-\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}$. From (76),

$$\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}+2\gamma\leq-\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}+2\gamma.$$

As a result,

$$\begin{aligned}&\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(t_0)-1)^+}{\mu_k}-\sum_{k\in\mathcal{K}}\frac{(q_k-1)^+}{\mu_k}+2\gamma\right)^+\cdot 1_A\right]\\&\leq\mathbb{E}_{(q,\xi)}\left[\left(-\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}+2\gamma\right)^+\right]\\&\leq\left(2\gamma+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\right)\cdot\mathbb{P}\left(\frac{1}{\mu_{k_0}}\bar{N}_{k_0}^n(\bar{\lambda}_{k_0}^n(t_0))<2\gamma+\frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\right).\end{aligned} \tag{83}$$

Note that when n is sufficiently large, we can choose an appropriately small $\gamma<\frac{\lambda_{k_0}^*t_0}{4\mu_{k_0}}$, so that the term in (83) is upper bounded by $\frac{\gamma}{2}\leq\gamma-(2\gamma+1)\epsilon$ where $0<\epsilon<\gamma/2(2\gamma+1)$. On combining with (81) and (82), we obtain (80). The proof is therefore complete. ☐

C.2. Tightness of $\left\{\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}_k^n(\infty)}{\mu_k}\right)^-\right\}$. With a slight abuse of the notations, we use the Lyapunov function

$$\Phi(q,\xi)=\Phi(q)=-\min\left(\sum_{k\neq i^\diamond}\frac{q_k}{\mu_k}+\frac{1}{\mu_{i^\diamond}}\min(q_{i^\diamond},\frac{\alpha_{i^\diamond}}{\sqrt{n}}),0\right)=\left(\sum_{k\in\mathcal{K}}\frac{q_k}{\mu_k}-\frac{1}{\mu_{i^\diamond}}\left(q_{i^\diamond}-\frac{\alpha_{i^\diamond}}{\sqrt{n}}\right)^+\right)^-. \tag{84}$$

We have the following result.

Lemma 10. *There exist constants $t_0,c_0,\gamma>0$ which are independent of n, such that for $\Phi(\cdot)$ in (84),*

$$\sup_{(q,\xi):\Phi(q)>c_0}\left\{\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t_0))\right]-\Phi(q)\right\}\leq-\gamma,\quad\textit{for all sufficiently large } n. \tag{85}$$

Using Lemma 10, we can then obtain the sought tightness result in the following lemma, the proof of which is similar to the one of Lemma 8 and is thus omitted.

LEMMA 11. *There exist constants $C_1, C_2 > 0$ which are independent of n such that for all sufficiently large n,*

$$\mathbb{P}_{\bar{\pi}^n}\left(\left(\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}^n_k(0)}{\mu_k} - \frac{1}{\mu_{i^\diamond}}\left(\bar{Q}^n_{i^\diamond}(0) - \frac{\alpha_{i^\diamond}}{\sqrt{n}}\right)^+\right)^-\right) > s\right) \le C_1 e^{-C_2\cdot s}, \quad \textit{for all } s > 0.$$

As a consequence, the sequence of random variables $\left\{\left(\sum_{k\in\mathcal{K}}\frac{\bar{Q}^n_k(\infty)}{\mu_k}\right)^-\right\}$ *is tight.*

Proof of Lemma 10. Pick $c_0 > -l_\diamond$ and let $B = \{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t_0)) \le c_0 + l_\diamond\}$. From (78), the collection $\{\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(t_0) + \sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t_0)) : n \ge 1\}$ is uniformly integrable. Given $\epsilon > 0$ and $t_0 > 0$, similar to (81), one can choose c_0 large enough so that $\mathbb{P}(B^c) < \epsilon$, and

$$\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t_0)) - \Phi(q))\cdot 1_{B^c}\right] \le \epsilon. \tag{86}$$

For $\Phi(q) > c_0$, we can infer from (76) that on the event B, for $t \in [0, t_0]$

$$\begin{aligned}\sum_{k\neq i^\diamond}\frac{\bar{Q}^n_k(t)}{\mu_k} + \frac{1}{\mu_{i^\diamond}}\min(\bar{Q}^n_{i^\diamond}(t), \alpha_{i^\diamond}/\sqrt{n}) &\le \sum_{k\neq i^\diamond}\frac{\bar{Q}^n_k(0)}{\mu_k} + \frac{1}{\mu_{i^\diamond}}\min(\bar{Q}^n_{i^\diamond}(0), \alpha_{i^\diamond}/\sqrt{n}) + \sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t))\\ &< -c_0 + c_0 + l_\diamond = l_\diamond.\end{aligned}$$

Next, we consider the following two cases to bound $\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t)) - \Phi(x)\right)\cdot 1_B\right]$.

Case 1: $Q^n_{i^\diamond}(0) < \alpha_{i^\diamond}$. Note that the system will not work on class $i^\diamond$ products if $Q^n_{i^\diamond}$ reaches $\alpha_{i^\diamond}$ (because there are customers waiting in other classes), hence $Q^n_{i^\diamond}(t) \le \alpha_{i^\diamond}$ for $t \in [0, t_0]$, and class $i^\diamond$ orders are always outsourced during $[0, t_0]$. This implies that

$$\begin{aligned}&\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t)) - \Phi(q))\cdot 1_B\right]\\ &= \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t)) - \sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t))\right)\cdot 1_B\right]\\ &= \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda^*_k}{\mu_k}t - t\right)\cdot 1_B\right] - \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda^*_k}{\mu_k}t - \sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t)) + \sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t)) - t\right)\cdot 1_B\right]\\ &= \frac{-\lambda^*_{i^\diamond}t}{\mu_{i^\diamond}}\cdot\mathbb{P}(B) - \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda^*_k}{\mu_k}t - \sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t)) + \sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t)) - t\right)\cdot 1_B\right].\end{aligned}$$

Because the system is always busy, we have $\sum_{k\in\mathcal{K}}\bar{T}^n_k(t) = t$, and then for n sufficiently large,

$$\mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{\lambda^*_k}{\mu_k}t - \sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t))\right)\cdot 1_B\right] \le \epsilon \text{ and } \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\in\mathcal{K}}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t)) - t\right)\cdot 1_B\right] < \epsilon.$$

Using the fact that $\mathbb{P}(B) > 1 - \epsilon$, we infer that

$$\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t)) - \Phi(q)\right)\cdot 1_B\right] \le -\frac{\lambda^*_{i^\diamond}t}{\mu_{i^\diamond}}\cdot(1-\epsilon) + 2\epsilon. \tag{87}$$

Case 2: $Q^n_{i^\diamond}(0) \ge \alpha_{i^\diamond}$. We can compute that

$$\begin{aligned}\mathbb{E}_{(q,\xi)}\left[\left(\Phi(\bar{Q}^n(t)) - \Phi(q)\right)\cdot 1_B\right] &= \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{N}^n_k(\bar{\lambda}^n_k(t)) - \sum_{k\neq i^\diamond}\frac{1}{\mu_k}\bar{S}^n_k(\bar{T}^n_k(t))\right)\cdot 1_B\right]\\ &\quad - \frac{1}{\mu_{i^\diamond}}\mathbb{E}_{(q,\xi)}\left[\left(\min(\bar{Q}^n_{i^\diamond}(t), \frac{\alpha_{i^\diamond}}{\sqrt{n}}) - \min(\bar{Q}^n_{i^\diamond}(0), \frac{\alpha_{i^\diamond}}{\sqrt{n}})\right)\cdot 1_B\right],\end{aligned}$$

One can verify that $\min(\bar{Q}^n_{i^\diamond}(t), \frac{\alpha_{i^\diamond}}{\sqrt{n}}) - \min(\bar{Q}^n_{i^\diamond}(0), \frac{\alpha_{i^\diamond}}{\sqrt{n}}) = 0$ and $0 \le \bar{S}^n_{i^\diamond}(\bar{T}^n_{i^\diamond}(t)) \le \frac{1}{\sqrt{n}}$: $\min(\bar{Q}^n_{i^\diamond}(t), \frac{\alpha_{i^\diamond}}{\sqrt{n}}) - \min(\bar{Q}^n_{i^\diamond}(0), \frac{\alpha_{i^\diamond}}{\sqrt{n}}) = 0$ holds because whenever $Q^n_{i^\diamond}$ reaches $\alpha_{i^\diamond}$, new orders are outsourced, hence $Q^n_{i^\diamond}$ would not be smaller than $\alpha_{i^\diamond}$; $0 \le \bar{S}^n_{i^\diamond}(\bar{T}^n_{i^\diamond}(t)) \le \frac{1}{\sqrt{n}}$ holds because at most one class $i^\diamond$ product can be produced (the one initially in production), and after that, the system will allocate no capacity to class $i^\diamond$. As a result,

$$\begin{aligned}\mathbb{E}_{(q,\xi)}\left[(\Phi(\bar{Q}^n(t)) - \Phi(q)) \cdot 1_B\right] &\le \mathbb{E}_{(q,\xi)}\left[\left(\sum_{k \ne i^\diamond} \frac{1}{\mu_k} \bar{N}^n_k(\bar{\lambda}^n_k(t)) - \sum_{k=1}^{K} \frac{1}{\mu_k} \bar{S}^n_k(\bar{T}^n_k(t))\right) \cdot 1_B\right] + \frac{1}{\sqrt{n}\mu_{i^\diamond}} \\ &\le -\frac{\lambda^*_{i^\diamond} t}{\mu_{i^\diamond}} \cdot (1-\epsilon) + 2\epsilon + \frac{1}{\sqrt{n}\mu_{i^\diamond}},\end{aligned}$$

where the last inequality follows from a similar argument for (87).

Combing the above two cases, we have $\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t)) - \Phi(x)\right] \cdot 1_B \le -\frac{\lambda^*_{i^\diamond} t}{\mu_{i^\diamond}} \cdot (1-\epsilon) + 3\epsilon$ for n sufficiently large and $t \in [0, t_0]$. Together with (86), we have for all sufficiently large n,

$$\sup_{(q,\xi):\Phi(q) > c_0} \left\{\mathbb{E}_{(q,\xi)}\left[\Phi(\bar{Q}^n(t_0))\right] - \Phi(q)\right\} \le -\frac{\lambda^*_{i^\diamond} t_0}{\mu_{i^\diamond}} \cdot (1-\epsilon) + 4\epsilon.$$

Hence we obtain (85) by choosing $\gamma > 0$ small so that $-\frac{\lambda^*_{i^\diamond} t_0}{\mu_{i^\diamond}} \cdot (1-\epsilon) + 4\epsilon < -\gamma$. □

D. Proof of Proposition 3 We first describe the hydrodynamic limit and its uniform attraction property in Section D.1, and then use it to prove Proposition 3 in Section D.2.

D.1. Hydrodynamic Limit and Uniform Attraction For simplicity of presentation, we omit the subscript $*$ notation in the processes under the policy ψ^n_*. We consider the hydrodynamic-scaled processes $\{(\bar{Q}^n, \bar{W}^n, \bar{O}^n, \bar{T}^n, \bar{I}^n)\}$ with $\bar{Q}^n, \bar{O}^n, \bar{T}^n$ defined in (70) and

$$\bar{W}^n(t) := \frac{1}{\sqrt{n}} W^n\left(\frac{t}{\sqrt{n}}\right), \quad \bar{I}^n(t) := \sqrt{n} I^n\left(\frac{t}{\sqrt{n}}\right).$$

In Lemma 12, we establish the convergence of the hydrodynamic-scaled processes under the proposed policies. Its proof is standard, hence is omitted.

LEMMA 12. *Fix a constant $M > 0$ and assume that $\sum_{k \in \mathcal{K}} \frac{|\bar{Q}^n(0)|}{\mu_k} \le M$ for all n. Then for any subsequence of hydrodynamic-scaled processes $\{(\bar{Q}^n, \bar{W}^n, \bar{O}^n, \bar{T}^n, \bar{I}^n)\}$, there is a further subsequence $\mathcal{N}$, such that along this subsequence $\mathcal{N}$,*

$$(\bar{Q}^n, \bar{W}^n, \bar{O}^n, \bar{T}^n, \bar{I}^n) \to (\bar{Q}, \bar{W}, \bar{O}, \bar{T}, \bar{I}), \ \text{u.o.c.},$$

for some Lipschitz continuous process $(\bar{Q}, \bar{W}, \bar{O}, \bar{T}, \bar{I})$, which is called a hydrodynamic limit.

Due to Liptschitz continuity, the hydrodynamic limit processes are differentiable at almost all $t \ge 0$. Following the convention in the literature (e.g., Mandelbaum and Stolyar (2004)), any t such that the limit processes are differentiable is called *regular*. When we write derivatives of the limit processes with respect to t, we assume they are at a regular time.

LEMMA 13. *Any hydrodynamic limit satisfies the following properties:*
1. *For $t \ge 0$,*

$$\begin{aligned}&\bar{Q}_k(t) = \bar{Q}_k(0) + \mu_k \bar{T}_k(t) - \lambda^*_k t + \bar{O}_k(t), \\ &\bar{I}(t) = t - \sum_{k \in \mathcal{K}} \bar{T}_k(t), \\ &\bar{W}(t) = \bar{W}(0) - \bar{I}(t) + \sum_{k \in \mathcal{K}} \frac{\bar{O}_k(t)}{\mu_k}, \\ &\bar{I}, \bar{T}, \bar{O}_{i^*} \text{ are non-decreasing}, \\ &\bar{O}'_{i^*}(t) \le \lambda^*_{i^*}, \text{ and } \bar{O}_k(t) = 0, \text{ for } k \ne i^*.\end{aligned} \tag{88}$$

2. *There is a constant $\chi>0$ such that*
   (a) *If $f_1(t):=\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k(t))^+}{\mu_k}>u_*$, then $f_1'(t)<-\chi$.*
   (b) *If $f_2(t):=\sum_{k\neq i^*}\frac{\bar{Q}_k(t)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}}<l_*$, then $f_2'(t)>\chi$.*
   (c) *For $k\in\mathcal{K}$, let $q_k^*=\Delta_k(\bar{W}(t))$. If $\bar{O}_k'(t)=0$ and $\bar{Q}_k(t)>q_k^*$, then $\bar{Q}_k'(t)=-\lambda_k^*<0$.*
   (d) *If $\bar{W}(t)<u_*$, then $\bar{I}'(t)=0$; if $\bar{W}(t)>l_*$, then $\bar{O}_{i^*}'(t)=0$.*

Proof. 1. The equation for $\bar{Q}$ follows from (71), the law of large numbers, random-time change, and that $\bar{\lambda}_k^n(\cdot)$ converges to $\lambda_k^*e(\cdot)$. The equation of $\bar{I}$ is from $\bar{I}^n(t)=\sqrt{n}I^n(t/\sqrt{n})=t-\sum_{k\in\mathcal{K}}\bar{T}_k^n(t)$. The equation of $\bar{W}$ is from $\bar{W}^n(t)=\sum_{k\in\mathcal{K}}\frac{\bar{Q}_k^n(t)}{\mu_k}$ and $\sum_{k\in\mathcal{K}}\frac{\lambda_k^*}{\mu_k}=1$. Because $\bar{I}^n,\bar{T}^n,\bar{O}_{i^*}^n$ are non-decreasing, $\bar{I},\bar{T},\bar{O}_{i^*}$ are non-decreasing. For any $t,s\geq 0$, $\bar{O}_{i^*}^n(t+s)-\bar{O}_{i^*}^n(t)\leq\bar{E}_{i^*}^n(t+s)-\bar{E}_{i^*}^n(t)$, and $\bar{O}_k^n(t)=0$ for $k\neq i^*$, hence $\bar{O}_{i^*}'(t)\leq\lambda_{i^*}^*$ and $\bar{O}_k(t)=0$, for $k\neq i^*$.

2. We prove the results one by one.

(a) If $f_1(t)>u_*$, because f_1 is continuous, there exists $\epsilon>0$ such that $f_1(s)>u_*$ for $s\in[t,t+\epsilon]$. Then for n large enough, $\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k^n(s))^+}{\mu_k}>u_*$ for $s\in[t,t+\epsilon]$. As a result, under the proposed policy, the system will not work on any new product from a class k with $\bar{Q}_k^n(s)\geq\frac{\alpha_k}{\sqrt{n}}$, hence for each $k\in\mathcal{K}$, $\frac{(\bar{Q}_k^n(u))^+}{\mu_k}$ cannot be larger than $\frac{(\bar{Q}_k^n(s))^+}{\mu_k}+\frac{\alpha_k+1}{\mu_k\sqrt{n}}$ for $s\leq u\leq t+\epsilon$. Taking $n\to\infty$, one has $(\bar{Q}_k(u))^+\leq(\bar{Q}_k(s))^+$ for any $t\leq s\leq u\leq t+\epsilon$. That is, for $k\in\mathcal{K}$, $(\bar{Q}_k)^+$ is non-increasing in $[t,t+\epsilon]$. Because $f_1(s)>u_*$ for $s\in[t,t+\epsilon]$, there exists a class k_0 such that $\bar{Q}_{k_0}(s)>0$ for $s\in[t,t+\epsilon]$. For n large enough, $\bar{Q}_{k_0}^n(s)>\frac{\alpha_{k_0}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$, hence the system will not work a new product from class k_0. Then $0\leq\bar{S}_{k_0}^n(\bar{T}_{k_0}^n(s))-\bar{S}_{k_0}^n(\bar{T}_{k_0}^n(t))\leq\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, one has $\bar{T}_{k_0}(s)=\bar{T}_{k_0}(t)$ for $s\in[t,t+\epsilon]$, which gives $\bar{T}_{k_0}'(t)=0$. If $k_0=i^*$, then because $\bar{Q}_{k_0}^n(s)>\frac{\alpha_{k_0}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ and n large enough, no class k_0 order is outsourced. If $k_0\neq i^*$, then no class k_0 order is outsourced. In both cases, $\bar{O}_{k_0}^n(s)=\bar{O}_{k_0}^n(t)$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, one has $\bar{O}_{k_0}(s)=\bar{O}_{k_0}(t)$ for $s\in[t,t+\epsilon]$, hence $\bar{O}_{k_0}'(t)=0$. Because the derivative of $(\bar{Q}_k)^+$ cannot be larger than 0 for $k\in\mathcal{K}$ and $\bar{Q}_{k_0}(s)>0$ for $s\in[t,t+\epsilon]$,

$$f_1'(t)\leq\frac{\bar{Q}_{k_0}'(t)}{\mu_{k_0}}=-\frac{\lambda_{k_0}^*}{\mu_{k_0}}\leq-\chi:=-\min_k\left\{\frac{\lambda_k^*}{\mu_k}\right\}. \tag{89}$$

(b) If $f_2(t)<l_*$, then there are three cases:

i) If $\bar{Q}_{i^*}(t)<0$, then there exists $\epsilon>0$ such that $\bar{Q}_{i^*}(s)<0$ and $f_2(s)<l_*$ for $s\in[t,t+\epsilon]$. For n large enough, one has $\sum_{k\neq i^*}\frac{\bar{Q}_k^n(s)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}^n(s),0)}{\mu_{i^*}}<l_*$ and $\bar{Q}_{i^*}^n(s)<0$ for $s\in[t,t+\epsilon]$. During this period, class i^* orders will be outsourced, that is, $\bar{O}_{i^*}^n(s)-\bar{O}_{i^*}^n(t)=\bar{E}_{i^*}^n(s)-\bar{E}_{i^*}^n(t)$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{O}_{i^*}(s)-\bar{O}_{i^*}(t)=\lambda_{i^*}^*(s-t)$ for $s\in[t,t+\epsilon]$, hence $\bar{O}_{i^*}'(t)=\lambda_{i^*}^*$. The system is always busy during this period, then $\bar{I}^n(s)=\bar{I}^n(t)$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{I}(s)=\bar{I}(t)$ for $s\in[t,t+\epsilon]$, hence $\bar{I}'(t)=0$. Note that $\bar{Q}_{i^*}(s)<0$ for $s\in[t,t+\epsilon]$, then

$$f_2(s)=\bar{W}(s)=\bar{W}(0)-\bar{I}(s)+\sum_{k\in\mathcal{K}}\frac{\bar{O}_k(s)}{\mu_k},\quad\text{for}\quad s\in[t,t+\epsilon].$$

Because $\bar{I}'(t)=0$ and $\bar{O}_k(t)=0$ for $k\neq i^*$, $f_2'(t)=\frac{\bar{O}_{i^*}'(t)}{\mu_{i^*}}=\frac{\lambda_{i^*}^*}{\mu_{i^*}}>\chi>0$.

ii) If $\bar{Q}_{i^*}(t)>0$: then there exists $\epsilon>0$ such that $f_2(s)<l_*$ and $\bar{Q}_{i^*}(s)>0$ for $s\in[t,t+\epsilon]$. Hence $f_2'(t)=\sum_{k\neq i^*}\frac{\bar{Q}_k'(t)}{\mu_k}$. For n large enough, $\bar{Q}_{i^*}^n(s)\geq\frac{\alpha_{i^*}+1}{\sqrt{n}}$ and $\sum_{k\neq i^*}\frac{\bar{Q}_k^n(s)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}^n(s),0)}{\mu_{i^*}}<l_*$ for $s\in[t,t+\epsilon]$, hence the system will not produce new class i^* products. Then $0\leq\bar{S}_{i^*}^n(\bar{T}_{i^*}^n(s))-\bar{S}_{i^*}^n(\bar{T}_{i^*}^n(t))\leq\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, one has $\bar{T}_{i^*}(s)=\bar{T}_{i^*}(t)$ for $s\in[t,t+\epsilon]$, which gives $\bar{T}_{i^*}'(t)=0$. Similarly, one can prove that $\sum_{k\in\mathcal{K}}\bar{T}_k'(t)=1$, hence $\sum_{k\neq i^*}\bar{T}_k'(t)=1$. Then

$$f_2'(t)=\sum_{k\neq i^*}\frac{\bar{Q}_k'(t)}{\mu_k}=\sum_{k\neq i^*}\bar{T}_k'(t)-\sum_{k\neq i^*}\frac{\lambda_k^*}{\mu_k}=\left(1-\sum_{k\neq i^*}\frac{\lambda_k^*}{\mu_k}\right)=\frac{\lambda_{i^*}^*}{\mu_{i^*}}\geq\chi>0.$$

iii) If $\bar{Q}_{i^*}(t)=0$: note that there exists $\epsilon>0$ such that $\sum_{k\neq i^*}\frac{\bar{Q}_k(s)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(s),0)}{\mu_{i^*}}<l_*$ for $s\in[t,t+\epsilon]$. Next, we prove $\bar{Q}_{i^*}(s)=0$ for $s\in[t,t+\epsilon]$. We prove it by contradiction: if $\bar{Q}_{i^*}(s)<0$ for some $s\in(t,t+\epsilon]$, then due to the continuity of $\bar{Q}_{i^*}$, there is $s_0\in(t,s)$ such that $\bar{Q}_{i^*}(s_0)=\bar{Q}_{i^*}(s)/2<0$ and $\bar{Q}_{i^*}(u)\le\bar{Q}_{i^*}(s)/2$ for $u\in[s_0,s]$; however, from the discussion in i) above, $\bar{O}'_{i^*}(u)=\lambda^*_{i^*}$ for $u\in[s_0,s]$, hence $\bar{Q}_{i^*}$ cannot decrease in $[s_0,s]$ and a contradiction; if $\bar{Q}_{i^*}(s)>0$ for some $s\in(t,t+\epsilon]$, then there is $s_0\in(t,s)$ such that $\bar{Q}_{i^*}(s_0)=\bar{Q}_{i^*}(s)/2>0$ and $\bar{Q}_{i^*}(u)\ge\bar{Q}_{i^*}(s)/2>0$ for $u\in[s_0,s]$; however, from the discussion in ii) above, $\bar{T}'_{i^*}(u)=0$ for $u\in[s_0,s]$, hence $\bar{Q}_{i^*}$ cannot increase in $[s_0,s]$ and then a contradiction.

As a result, $f'_2(t)=\sum_{k\neq i^*}\frac{\bar{Q}'_k(t)}{\mu_k}$. Next we prove $\sum_{k\neq i^*}\bar{T}'_k(t)=1$, which is equivalent to $\bar{T}'_{i^*}(t)=0$. Because $\lim_{n\to\infty}\bar{Q}^n_{i^*}(t)=\bar{Q}_{i^*}(t)=0$, consider two subsequences: $\{n:\bar{Q}^n_{i^*}(t)\ge\frac{\alpha_k}{\sqrt{n}}\}$ and $\{n:\bar{Q}^n_{i^*}(t)<\frac{\alpha_k}{\sqrt{n}}\}$. For any system in the first subsequence, $\bar{Q}^n_{i^*}(s)\ge\frac{\alpha_{i^*}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ because once $\bar{Q}^n_{i^*}$ reaches $\frac{\alpha_{i^*}}{\sqrt{n}}$, new arrivals are outsourced, hence $\bar{Q}^n_{i^*}$ cannot decrease anymore. Then the system will not choose to produce new class i^* items, hence $\bar{S}^n_{i^*}(\bar{T}^n_{i^*}(s))-\bar{S}^n_{i^*}(\bar{T}^n_{i^*}(t))\le\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{T}_{i^*}(s)-\bar{T}_{i^*}(t)=0$ for $s\in[t,t+\epsilon]$, hence $\bar{T}'_{i^*}(t)=0$. For systems in the second subsequence, $\bar{Q}^n_{i^*}(s)\le\frac{\alpha_{i^*}}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$ because if $\bar{Q}^n_{i^*}$ reaches $\frac{\alpha_{i^*}}{\sqrt{n}}$, then the system allocates no capacity to class i^* and $\bar{Q}^n_{i^*}$ cannot increase to $\frac{\alpha_k+1}{\sqrt{n}}$. As a result, class i^* orders will be outsourced. Following the argument in i) above, one has $\bar{O}'_{i^*}(t)=\lambda^*_{i^*}$, hence $\bar{T}'_{i^*}(t)=\bar{Q}'_{i^*}(t)$, which equals 0 because $\bar{Q}_{i^*}(s)=0$ for $s\in[t,t+\epsilon]$. Because the limits of both subsequences have the same derivative, one has $\bar{T}'_{i^*}(t)=0$ for any hydrodynamic limits.

Then

$$f'_2(t)=\sum_{k\neq i^*}\frac{\bar{Q}'_k(t)}{\mu_k}=\sum_{k\neq i^*}\bar{T}'_k(t)-\sum_{k\neq i^*}\frac{\lambda^*_k}{\mu_k}=\left(1-\sum_{k\neq i^*}\frac{\lambda^*_k}{\mu_k}\right)=\frac{\lambda^*_{i^*}}{\mu_{i^*}}>\chi>0.$$

(c) If $\bar{Q}_k(t)>q^*_k$, then there must be a class l such that $\bar{Q}_l(t)<q^*_l:=\Delta_l(\bar{W}(t))$. There is $\epsilon>0$ such that $\bar{Q}_k(s)>\Delta_k(\bar{W}(s))$ and $\bar{Q}_l(s)<\Delta_l(\bar{W}(s))$ for $s\in[t,t+\epsilon]$. Then $\bar{Q}^n_k(s)>\Delta_k(\bar{W}^n(s))$ and $\bar{Q}^n_l(s)<\Delta_l(\bar{W}^n(s))$ for $s\in[t,t+\epsilon]$ and n large enough. First, we argue that, *under the proposed policies, the system will not allocate capacity to new class k products during the interval* $[t,t+\epsilon]$. For the setting with strictly convex holding/waiting cost functions, using the KKT condition, one can verify that $g'_k(\bar{Q}^n_k(s))\mu_k>g'_k(\bar{Q}^n_l(s))\mu_l$ for $s\in[t,t+\epsilon]$, hence the system will not allocate capacity to new class k products during that interval. For the setting with linear holding/waiting cost functions, if $\bar{W}^n(s)>u^n$, then $k\notin\mathcal{C}^n(s/\sqrt{n})$ and the system will not choose to produce new class k products; if $\bar{W}^n(s)\le u^n$, then $\mathcal{C}^n(s/\sqrt{n})=\mathcal{K}$. We consider the case $\bar{W}^n(s)>0$ and the case $\bar{W}^n(s)\le 0$ can be argued similarly. Note that $k\notin\mathcal{N}^n(s/\sqrt{n})$. If $\mathcal{N}^n(s/\sqrt{n})\neq\emptyset$, then the system will choose to produce a product from a class in $\mathcal{N}^n(s/\sqrt{n})$ hence not k; if $\mathcal{N}^n(s/\sqrt{n})=\emptyset$, then $\mathcal{P}^n(t)=\mathcal{K}$ and $k\neq\arg\min_{k\in\mathcal{P}^n(t)}h_k\mu_k$ (because if $k=\arg\min_{k\in\mathcal{P}^n(t)}h_k\mu_k$, then $l\in\mathcal{N}^n(s/\sqrt{n})$).

Because the system will not allocate capacity to new class k products during $[t,t+\epsilon]$, $\bar{S}^n_k(\bar{T}^n_k(s))-\bar{S}^n_k(\bar{T}^n_k(t))\le\frac{1}{\sqrt{n}}$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, one has $\bar{T}_k(s)=\bar{T}_k(t)$ for $s\in[t,t+\epsilon]$. Hence $\bar{T}'_k(t)=0$. Together with the assumption that $\bar{O}'_k(t)=0$, one has $\bar{Q}'_k(t)=-\lambda^*_k<0$.

(d) If $\bar{W}(t)<u_*$, there exists $\epsilon>0$ such that $\bar{W}(s)<u_*$ for $s\in[t,t+\epsilon]$. Then $\bar{W}^n(s)<u_*$ for $s\in[t,t+\epsilon]$ and n large enough. As a result, $\bar{Q}^n_k(s)\ge\frac{\alpha_k}{\sqrt{n}}$ for all $k\in\mathcal{K}$ and $\sum_{k\in\mathcal{K}}\frac{(\bar{Q}^n_k(s))^+}{\mu_k}\ge u_*$ cannot hold simultaneously for $s\in[t,t+\epsilon]$. The system is then always busy, that is $\bar{I}^n(s)=\bar{I}^n(t)$ for $s\in[t,t+\epsilon]$. Taking limit, one has $\bar{I}(s)=\bar{I}(t)$ for $s\in[t,t+\epsilon]$, hence $\bar{I}'(t)=0$.

If $\bar{W}(t)>l_*$, there exists $\epsilon>0$ such that no class i^* order is outsourced during $[t,t+\epsilon]$. This is because there is $\epsilon>0$ such that $\bar{W}(s)>l_*$ for $s\in[t,t+\epsilon]$, hence $\bar{W}^n_{i^*}(s)>l_*$ for $s\in[t,t+\epsilon]$ and n large enough. Hence $\bar{O}^n_{i^*}(s)=\bar{O}^n_{i^*}(t)$ for $s\in[t,t+\epsilon]$. Taking $n\to\infty$, $\bar{O}_{i^*}(s)=\bar{O}_{i^*}(t)$ for $s\in[t,t+\epsilon]$, hence $\bar{O}'_{i^*}(t)=0$. □

LEMMA 14 **(Uniform attraction)**. *Consider any hydrodynamic limit derived in Lemma 12 with* $\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_k(0)|}{\mu_k}\le M$ *for a constant* $M>0$. *There exist* $T_1,T_2>0$ *depending on* M *such that*

1. *If* $\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k(0))^+}{\mu_k}>u_*$, *then* $\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k(t))^+}{\mu_k}\le u_*$ *for all* $t\ge T_1$.

2. *If* $\sum_{k\neq i^*}\frac{\bar{Q}_k(0)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(0),0)}{\mu_{i^*}}<l_*$, *then* $\sum_{k\neq i^*}\frac{\bar{Q}_k(t)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}}\geq l_*$ *for all* $t\geq T_2$.
3. *For all* $t\geq T:=\max\{T_1,T_2\}$, *we have* $\bar{W}(t)=\bar{W}(T)\in[l_*,u_*]$.
4. *There exists a time* $T_M>T$, *such that* $\bar{Q}(t)=\Delta(\bar{W}(t))$ *for all* $t\geq T_M$.

Proof. 1. If $f_1(t):=\sum_{k\in\mathcal{K}}\frac{(\bar{Q}(t))^+}{\mu_k}>u_{\gamma_*}$, then from Lemma 13, item 2a), $f_1'(t)<-\chi<0$. Also note that $f_1(0)\leq\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_k(0)|}{\mu_k}\leq M$, hence within a finite time $T_1>0$, f_1 will return to u_*, and cannot be larger than u_* again.

2. If $f_2(t):=\sum_{k\neq i^*}\frac{\bar{Q}_k(t)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}}<l_*$, then from Lemma 13, item 2b), $f_2'(t)>\chi>0$. Also note that $f_2(0)\geq-\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_k(0)|}{\mu_k}\geq -M$, hence within a finite time $T_2>0$, f_2 will return to l_* and cannot be smaller than l_* again.

3. Then for $t\geq T=\max\{T_1,T_2\}$, $f_1(t):=\sum_{k\in\mathcal{K}}\frac{(\bar{Q}_k(t))^+}{\mu_k}\leq u_*$ and $f_2(t):=\sum_{k\neq i^*}\frac{\bar{Q}_k(t)}{\mu_k}+\frac{\min(\bar{Q}_{i^*}(t),0)}{\mu_{i^*}}\geq l_*$. One then gets $\bar{W}(t)\in[l_*,u_*]$. Next we prove $\bar{W}'(t)=0$ and $\bar{I}'(t)=0$.

(a) If $\bar{W}(t)\in(l_*,u_*)$, then from Lemma 13 item 2d), we have $\bar{I}'(t)=0$, $\bar{O}_{i^*}'(t)=0$. Together with $\bar{O}_k(t)$=0 for $k\neq i^*$, and the equation $\bar{W}(t)=\bar{W}(0)-\bar{I}(t)+\sum_{k\in\mathcal{K}}\frac{\bar{O}_k(t)}{\mu_k}$, we have $\bar{W}'(t)=0$.

(b) If $\bar{W}(t)=l_*$, then $\bar{W}(\cdot)$ cannot decrease. This is because $f_2\leq\bar{W}$ and if $\bar{W}(\cdot)$ decreases, then $f_2(\cdot)$ will become strictly smaller than l_*. This is a contradiction to the choice of T. On the other hand, $\bar{W}(\cdot)$ will not increase. This is because if it increases to $l_*+\delta$ with $\delta>0$, it will first reach $l_*+\frac{\delta}{2}>l_*$; however, from the discussion in (a), $\bar{W}(\cdot)$ will stay at $l_*+\frac{\delta}{2}$ and cannot increase to $l_*+\delta$. This is a contradiction. As a result, $\bar{W}'(t)=0$. From Lemma 13 item 2d), $\bar{I}'(t)=0$. Together with (88), one has $\bar{O}_k'(t)=0$ for all $k\in\mathcal{K}$.

(c) If $\bar{W}(t)=u_*$, then using an argument similar to the one for (b) above, we have $\bar{W}'(t)=0$. From Lemma 13 item 2d), $\bar{O}_k'(t)=0$ for all $k\in\mathcal{K}$. Together with (88), we then have $\bar{I}'(t)=0$.

4. Then for $t\geq T$, $\bar{W}(t)\in[l_*,u_*]$. Also from the proof of item 3, $\bar{O}_k'(t)=0$ for all $k\in\mathcal{K}$ and $\bar{I}'(t)=0$, for $t\geq T$. If $\bar{Q}_k(t)>q_k^*:=\Delta_k(\bar{W}(t))$, then from Lemma 13, item 2c), $\bar{Q}_k'(t)=-\lambda_k^*<0$. As a result, after a finite time $T_M>T$, $\bar{Q}_k(t)\leq q_k^*$ for all $k\in\mathcal{K}$. By the definition of q^*, one then has $\bar{Q}_k(t)=q_k^*$ for all $k\in\mathcal{K}$ and $t\geq T_M$. □

D.2. Proof of Proposition 3. We first show that as $n\to\infty$,

$$\left|\widetilde{Q}_*^n(\infty)-\Delta(l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*))\right|\Rightarrow 0. \tag{90}$$

Assume $\bar{\mathfrak{X}}^n(0)$, the initial state of the Markov process $\bar{\mathfrak{X}}^n(\cdot)$, follows a stationary distribution $\bar{\pi}^n$. From Proposition 2, $\{\bar{\mathfrak{X}}^n(0)\}$ is tight, hence for each subsequence, there is a further subsequence (still indexed by n) such that $\bar{\mathfrak{X}}^n(0)\Rightarrow\bar{\mathfrak{X}}(0)$. We assume that this is almost sure convergence by Skorohod representation theorem. Then for $\epsilon>0$, we can choose M such that $\mathbb{P}(\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_{k*}(0)|}{\mu_k}<M)\geq 1-\epsilon$. For the hydrodynamic limits with $\sum_{k\in\mathcal{K}}\frac{|\bar{Q}_{k*}(0)|}{\mu_k}<M$, from Lemma 14 item 4, there is $T_M>0$ such that the corresponding hydrodynamic limits should satisfy $\bar{Q}_*(t)=\Delta(l_*\vee(\bar{W}_*(t)\wedge u_*))$ for $t\geq T_M$. Here we use the fact that $\bar{W}_*(t)\in[l_*,u_*]$ for $t\geq T_M$. We fix such a $t\geq T_M$. Then from Lemma 12 and the continuity of the lifting function Δ, for n large enough in the further subsequence, we can obtain that $\mathbb{P}(|\bar{Q}_*^n(t)-\Delta(l_*\vee(\bar{W}_*^n(t)\wedge u_*))|\geq 2\epsilon)\leq 2\epsilon$. Because $\bar{\mathfrak{X}}^n(0)$ follows a stationary distribution, $\bar{\mathfrak{X}}^n(t)$ also follows the stationary distribution. As a result, for n large enough in the further subsequence, $\mathbb{P}(|\bar{Q}_*^n(\infty)-\Delta(l_*\vee(\bar{W}_*^n(\infty)\wedge u_*))|\geq 2\epsilon)\leq 2\epsilon$. Because every subsequence has this property, we can conclude that as $n\to\infty$, $|\bar{Q}_*^n(\infty)-\Delta(l_*\vee(\bar{W}_*^n(\infty)\wedge u_*))|\Rightarrow 0$. As $\bar{\mathfrak{X}}^n(\infty)$ and $\widetilde{\mathfrak{X}}^n(\infty)$ have the same distribution, we then obtain (90).

Next, note that because $\widetilde{\mathfrak{X}}^n(\infty)$ is tight, for every subsequence, there is a further subsequence such that $\widetilde{Q}_*^n(\infty)\Rightarrow\widetilde{Q}_*(\infty)$ for some random vector $\widetilde{Q}_*(\infty)$. Then, from (90) and the convergence-together theorem (Whitt 2002, Theorem 11.4.7), one has $(\widetilde{Q}_*^n(\infty),\Delta(l_*\vee(\widetilde{W}_*^n(\infty)\wedge u_*)))\Rightarrow(\widetilde{Q}_*(\infty),\widetilde{Q}_*(\infty))$. From Lemma 1, $h(l_*\vee(\widetilde{W}^n(\infty)\wedge u_*))=\sum_{k\in\mathcal{K}}g_k(\Delta_k(l_*\vee(\widetilde{W}^n(\infty)\wedge u_*)))$. For $(x,y)\in\mathbb{R}^{2K}$, introduce a continuous mapping $f(x,y)=\sum_{k\in\mathcal{K}}g_k(x_k)-\sum_{k\in\mathcal{K}}g_k(y_k)$, then one can verify that $f(\cdot,\cdot)$ is a continuous function. Applying the continuous mapping theorem (Whitt 2002, Theorem 3.4.3) with f to the further subsequence, one then has

$$\left|\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) - h(l_* \vee (\widetilde{W}^n(\infty)\wedge u_*))\right| \Rightarrow 0. \tag{91}$$

Because all of these subsequences have the same limit, (91) holds for the whole sequence.

In view of (91), to prove Proposition 3, it is enough to prove the uniform integrability of the sequence of random variables $\left\{\left|\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) - h(l_* \vee (\widetilde{W}^n(\infty)\wedge u_*))\right| : n\geq 1\right\}$. Note that $l_* \vee (\widetilde{W}^n(\infty)\wedge u_*)$ is uniformly bounded for all n, hence it suffices to show the uniform integrability of $\{\sum_{k\in\mathcal{K}} g_k(\widetilde{Q}^n_{k*}(\infty)) : n\geq 1\}$. Due to the sub-polynomial assumption in Assumption 1, it is enough to prove $\sup_{n\geq 1}\mathbb{E}\left[|\widetilde{Q}^n_*(\infty)|^{m+1}\right] < \infty$ (Ethier and Kurtz 1986, Proposition A.2.2, page 494). This can be directly proved by using the tail probability bound established in Lemmas 8 and 11. We omit the details. Hence the proof is complete. □

E. Proof of Lemma 6 First, we obtain from (10) that for each $t>0$,

$$\frac{1}{t}\widetilde{V}^n(t,\widetilde{\nu}^n,\psi^n_*) = \mathbb{E}_{\widetilde{\nu}^n}\left[A_t\right],$$

where

$$A_t = \frac{1}{t}\left[\int_0^t c^n(\zeta^n_*(s))ds + \sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}^n_{k*}(s))ds + \frac{1}{\sqrt{n}}\sum_{k\in\mathcal{K}} r_k\int_0^t \xi_{k*}(s-)\lambda^n_{k*}(s)ds\right]. \tag{92}$$

In the above we have used the relation $\widetilde{O}^n_{k*}(t) = \frac{1}{\sqrt{n}}\int_0^t \xi_{k*}(s-)dE^n_{k*}(s) = \frac{1}{\sqrt{n}}\int_0^t \xi_{k*}(s-)\lambda^n_{k*}(s)ds$, which follows from the fact that $\int_0^t \xi_{k*}(s-)d(E^n_{k*}(s) - \int_0^s \lambda^n_{k*}(u)du)$ is a martingale. Note that ζ^n_*, ξ_{k*} and λ^n_{k*} are all functions of the state $\widetilde{\mathfrak{X}}^n_*(\cdot)$ by the definition of Markov controls.

Second, we argue that the state process $\widetilde{\mathfrak{X}}^n_*(\cdot)$ under the policy ψ^n_* is a continuous time Markov chain that has a unique stationary distribution $\widetilde{\pi}^n_*$. By Proposition 1, this chain has at least one stationary distribution. Next we show that the Markov chain $\widetilde{\mathfrak{X}}^n_*(\cdot)$ has at most one closed communicating class, so the stationary distribution of $\widetilde{\mathfrak{X}}^n_*(\cdot)$ is unique. To see this, we use the setting of linear state costs as an illustration. The arguments for the setting of convex state costs are similar. Without loss of generality, we assume $\arg\max_{k\in\mathcal{K}} b_k\mu_k = 1$. We argue that every state of $\widetilde{\mathfrak{X}}^n_*(\cdot)$ can reach the state $(-1/\sqrt{n},0,\cdots,0,1)$, which indicates there is one class 1 customer waiting and class 1 is currently in production. Note that any inventory level $(x_k)_{k\in\mathcal{K}}$ associated with $\widetilde{\mathfrak{X}}^n_*(\cdot)$ can reach $(\max(x_k,\alpha_k/\sqrt{n}))_{k\in\mathcal{K}}$ by productions (and no order arrivals during the period of productions). Starting from $(\max(x_k,\alpha_k/\sqrt{n}))_{k\in\mathcal{K}}$, the inventory level $(\alpha_k/\sqrt{n})_{k\in\mathcal{K}}$ can be reached via order arrivals (with no production completions during this period). Then the system is not idling based on our proposed policy ψ^n_* and we assume class 2 is in production without loss of generality. Then the state $(-1/\sqrt{n},-1/\sqrt{n},0,\cdots,0,2)$ of $\widetilde{\mathfrak{X}}^n_*(\cdot)$ can be reached with further order arrivals before the production of the class 2 product is completed. Finally, when the production of the class 2 product is finished (with no new order arrivals), the machine will produce class 1 products based on the policy ψ^n_*, and the state $(-1/\sqrt{n},0,\cdots,0,1)$ is reached.

Third, by the Lyapunov drift condition (Equation (65)) established in the proof of Proposition 1, we can infer from Theorem 4.4 of Meyn and Tweedie (1993) that $\widetilde{\mathfrak{X}}^n_*(\cdot)$ is positive Harris recurrent. It follows that the strong law of large numbers holds for the Markov chain $\widetilde{\mathfrak{X}}^n_*(\cdot)$, i.e.,

$$\lim_{t\to\infty} A_t = \widetilde{V}^n(\widetilde{\pi}^n_*,\psi^n_*), \quad \text{almost surely for any initial distribution } \widetilde{\nu}^n.$$

See Theorem 17.1.7 of Meyn and Tweedie (2009) and Remark 2 of Dai (1995).

Hence, to prove (50), it remains to show the L_1 convergence, i.e., $\lim_{t\to\infty}\mathbb{E}_{\widetilde{\nu}^n}[A_t] = \widetilde{V}^n(\widetilde{\pi}^n_*,\psi^n_*)$. It suffices to show that $\{A_t : t>0\}$ is uniformly integrable. We proceed to analyze the three terms on the right-hand-side of (92) as follows.

• First, under the policy ψ_*^n, we can obtain from Equation (20) that $|\zeta_*^n(s)| \le \frac{1}{2}|H^{-1}m|\cdot\kappa$ for all $s\ge 0$. By (43), we infer that for n large enough, there exists some constant C such that $|c^n(\zeta_*^n(s))|\le C$ for all $s\ge 0$. Hence we have for all $t>0$,

$$\frac{1}{t}\left|\int_0^t c^n(\zeta_*^n(s))ds\right| \le C.$$

• Second, under the policy ψ_*^n, we have $\lambda_{k*}^n(s) = n\lambda_k^* + \sqrt{n}\cdot\zeta_{k*}^n(s)$. It follows that $|\lambda_{k*}^n(s)|\le n(\lambda_k^*+1)$ for n large enough. Because $|\xi_{k*}(s-)|\le 1$, we obtain that for all $t>0$,

$$\frac{1}{t}\frac{1}{\sqrt{n}}\left|\sum_{k\in\mathcal{K}} r_k\int_0^t \xi_{k*}(s-)\lambda_{k*}^n(s)ds\right| \le \sqrt{n}\sum_{k\in\mathcal{K}} r_k(\lambda_k^*+1).$$

• Finally, we need to show $\{\frac{1}{t}\sum_{k\in\mathcal{K}}\int_0^t g_k(\widetilde{Q}_{k*}^n(s))ds : t>0\}$ is uniformly integrable given an initial distribution $\widetilde{\nu}^n$. Note that $|g_k(x)|\le c(1+|x|^m)$ for some c and $m\in\mathbb{N}$ by Assumption 1. Therefore, it suffices to show $\{\frac{1}{t}\sum_{k\in\mathcal{K}}\int_0^t |\widetilde{Q}_{k*}^n(s)|^m ds : t>0\}$ is uniformly integrable. This follows from

$$\limsup_{t\to\infty}\mathbb{E}_{\widetilde{\nu}^n}\left[\left(\frac{1}{t}\sum_{k\in\mathcal{K}}\int_0^t |\widetilde{Q}_{k*}^n(s)|^m ds\right)^2\right] \le \limsup_{t\to\infty}\frac{K}{t}\mathbb{E}_{\widetilde{\nu}^n}\int_0^t\sum_{k\in\mathcal{K}}|\widetilde{Q}_{k*}^n(s)|^{2m}ds \le C_m,$$

for some constant $C_m>0$. The first inequality above follows from the Cauchy-Schwartz inequality. The second inequality can be proved by a similar argument as in the proof of Theorem 5.5 in Dai and Meyn (1995) and the assumption that $\widetilde{\nu}^n$ having a finite $(2m+1)$-th moment, so we only provide a sketch of the proof. The key step is to show that, for a *fixed* n large enough, there is a constant $\bar{\delta}>0$ such that

$$\lim_{|x|\to\infty}\frac{1}{|x|}\mathbb{E}_x\left[\sum_{k\in\mathcal{K}}\left|Q_{k*}^n\left(|x|\bar{\delta}\right)\right|\right] = 0, \tag{93}$$

where we have slightly abused the notation and used x to denote the initial value of $(Q_{k*}^n)_{k\in\mathcal{K}}$. This can be proved with the following two steps:

1. First, one can show that $\lim_{|x|\to\infty}\frac{1}{|x|}\sum_{k\in\mathcal{K}}\left|Q_{k*}^n\left(|x|\bar{\delta}\right)\right| = 0$ almost surely. This can be proved using the standard framework as in the proof of Theorem 4.1 of Dai (1995), and an argument similar to the proof of item 2 of Lemma 13 in this paper.
2. Second, one can prove the uniform integrability of $\{\frac{1}{|x|}\sum_{k\in\mathcal{K}}\left|Q_{k*}^n\left(|x|\bar{\delta}\right)\right| : |x|>0\}$. Its proof is similar to the proof of Lemma 4.5 of Dai (1995).

By combining these three parts, we infer that $\{A_t : t>0\}$ is uniformly integrable. Hence, the equation (50) holds.

F. Proof of Lemma 2 We first introduce a lemma.

LEMMA 15. *Assume $\mathcal{G}(\cdot)$ is Lipschitz continuous on $\mathbb{R}$, and $h(\cdot)$ is continuous on $\mathbb{R}$, strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$, $h(0)=0$ and $\lim_{|x|\to\infty}h(x)=\infty$. For each $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, there is a unique continuously differentiable solution $w(x;w_0,\gamma)$, which is jointly continuous in w_0 and γ, to the following ODE:*

$$\begin{aligned}\frac{1}{2}\sigma^2 w'(x) + \mathcal{G}(w(x)) + h(x) = \gamma, \quad &\textit{for} \quad x\in\mathbb{R},\\ \textit{subject to} \quad &w(0)=w_0.\end{aligned} \tag{94}$$

Furthermore, $w'(x;w_0,\gamma)$ is continuous in x, w_0 and γ, and the following hold:

1. *For $\omega_0\in\mathbb{R}$, the solution $w(x;w_0,\gamma)$ is strictly increasing in γ for fixed $x>0$ and strictly decreasing in γ for fixed $x<0$. The solution $w(x;w_0,\gamma)$ is strictly increasing in w_0 for fixed $x,\gamma\in\mathbb{R}$. For fixed $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, $w(x;w_0,\gamma)$ cannot have a local minimizer in $(0,\infty)$, cannot have a local maximizer in $(-\infty,0)$, and cannot be a constant in any interval on $\mathbb{R}$.*

2. *Assume* $\gamma_1(w)$ *is continuous and strictly increasing in* w*, then* $\max_{x\geq 0} w(x;w_0,\gamma_1(w_0))$ *is continuous and strictly increasing in* w_0*; assume* $\gamma_2(w)$ *is continuous and strictly decreasing in* w*, then* $\min_{x\leq 0} w(x;w_0,\gamma_2(w_0))$ *is continuous and strictly increasing in* w_0*.*

3. *For* $k > w_0$*, there exists a unique number* $\gamma_+(w_0)$ *such that* $\max_{x\geq 0} w(x;w_0,\gamma_+(w_0)) = k$*, and the maximizer is unique. The functions* $\gamma_+(w_0)$ *is continuous and strictly decreasing in* w_0*. For* $a < b$*, if either i)* $\min(-h'(0-), h'(0+)) > 0$*; or ii)* $b = 0$ *and* $\mathcal{G}(x) = -\mu x^2$ *for* $x \in [a,b]$ *with* $\mu > 0$*, then there exists* $w_* \in (a,b)$ *such that* $\max_{x\geq 0} w(x;w_*,\gamma_+(w_*)) = b$ *(with a unique maximizer* u_**) and* $\min_{x\leq 0} w(x;w_*,\gamma_+(w_*)) = a$ *(with a unique minimizer* l_**). The function* $w(x;w_*,\gamma_+(w_*))$ *is strictly increasing on* $[l_*,u_*]$ *and* $w(x;w_*,\gamma_+(w_*)) \in [a,b]$ *for* $x \in [l_*,u_*]$*.*

We now prove Lemma 2 by using Lemma 15. Let $a = -\kappa$ (recall κ in (15)) and $b = 0$. Fix $M \geq \kappa^2$ and let $\sigma^2 = \sum_{k\in\mathcal{K}} \frac{2\lambda_k^*}{\mu_k^2}$, and $\mathcal{G}(u) = -\frac{m'H^{-1}m}{4}\min(u^2, M)$. Then $\mathcal{G}(\cdot)$ is Lipschitz continuous. One can verify that h in (14) satisfies the requirement in Lemma 15. With $w(x;w_*,\gamma_+(w_*)), w_*, l_*, u_*$ in Lemma 15 item 3, introduce $v(x) = w(x;w_*,\gamma_+(w_*))$ on $[l_*,u_*]$. Then $|v(x)|^2 \leq \kappa^2 \leq M$ and hence $\mathcal{G}(v(x)) = -\frac{m'H^{-1}m}{4}v^2(x)$ for $x \in [l_*,u_*]$. Thus the constants $l_* < 0 < u_*, \gamma^* \in \mathbb{R}$ and the function $v \in C^1[l_*,u_*]$ satisfy

$$\frac{1}{2}\sigma^2 v'(x) - \frac{m'H^{-1}m}{4}v^2(x) + h(x) = \gamma^*, \quad \text{for} \quad x \in [l_*,u_*], \tag{95}$$

subject to $v(x) \in [-\kappa, 0]$ for all $x \in [l_*,u_*]$ and the boundary and smooth pasting conditions

$$v(l_*) = -\kappa, \quad v(u_*) = 0, \quad v'(l_*+) = 0, \quad \text{and} \quad v'(u_*-) = 0. \tag{96}$$

Now we prove the uniqueness of l_*, u_*, γ^* and v satisfying (95) and the conditions (96). Assume that there exist $\tilde{l}_* < 0 < \tilde{u}_*, \tilde{\gamma}^*$ and $\tilde{v}$ satisfying (95) with the conditions (96). First, we prove that $\tilde{\gamma}^* \neq \gamma^*$ is impossible. We consider the case $\tilde{\gamma}^* > \gamma^*$ (the case $\tilde{\gamma}^* < \gamma^*$ is similar). From the boundary and smooth pasting conditions one has $h(\tilde{u}_*) > h(u_*)$ and $h(\tilde{l}_*) > h(l_*)$, which ensure $\tilde{u}_* > u_* > 0$ and $\tilde{l}_* < l_* < 0$. Assume $\tilde{v}(0) \geq v(0)$ (the case $\tilde{v}(0) < v(0)$ can be argued similarly). Then from Lemma 15 item 1, one has $\tilde{v}(x) > v(x)$ for $x \in (0,u_*]$, especially $\tilde{v}(u_*) > v(u_*) = b = \tilde{v}(\tilde{u}_*)$. This is a contradiction to $\tilde{v}(x) \in [a,b]$ for $x \in [\tilde{l}_*, \tilde{u}_*]$. As a result, $\tilde{\gamma}^* = \gamma^*$. Then from the boundary conditions, one has $h(\tilde{u}_*) = h(u_*)$ and $h(\tilde{l}_*) = h(l_*)$, which ensure $\tilde{u}_* = u_*$ and $\tilde{l}_* = l_*$. From these, one can also verify that $\tilde{v} = v$. This proves the uniqueness.

Let

$$\bar{v}(x) = -\kappa \times 1_{\{x\in(-\infty,l^*)\}} + v(x) \times 1_{\{[l^*,u^*]\}} + 0 \times 1_{\{(u^*,\infty)\}},$$

and define $\Phi(x) = \int_{l^*}^x \bar{v}(y)dy$. It is easy to verify that $\Phi \in C^2(\mathbb{R})$ and l_*, u_*, γ^* satisfy the differential equation with the corresponding boundary and smooth pasting conditions. The uniqueness of Φ follows from that of v. The constant γ^* is positive by checking the ODE (17) at the u_*.

Next we verify Φ satisfies (18)–(19). Note that $\Phi \in C^2(\mathbb{R})$ and $\Phi'' = \bar{v}'$ is locally L^1, hence it has the third-order derivative almost everywhere. It is easy to verify (19) and $|\Phi'''(x)| \leq C$ for $C > 0$ whenever it exists, and we will focus on the verification of (18).

From the expression of $c(\cdot)$ in (16), one has $-u\Phi'(x) + c(u) = -u\Phi'(x) + \frac{1}{m'H^{-1}m}u^2 \geq -\frac{m'H^{-1}m}{4}(\Phi'(x))^2$. Hence it is enough to verify that $\frac{\sigma^2}{2}\Phi''(x) - \frac{m'H^{-1}m}{4}(\Phi'(x))^2 + h(x) \geq \gamma^*$. It suffices to consider the cases $x > u^*$ and $x < l^*$. Note that for $x > u^*$, we have $\Phi'(x) = \bar{v}(x) = 0 = v(u^*) = \bar{v}(u^*) = \Phi'(u^*)$ and $\Phi''(x) = \bar{v}'(x) = 0 = \bar{v}'(u^*) = \Phi''(u^*)$. It follows that for $x > u^*$,

$$\begin{aligned}\frac{\sigma^2}{2}\Phi''(x) - \frac{m'H^{-1}m}{4}(\Phi'(x))^2 + h(x) &= \frac{\sigma^2}{2}\Phi''(u^*) - \frac{m'H^{-1}m}{4}(\Phi'(u^*))^2 + h(u^*) + h(x) - h(u^*) \\ &= \gamma^* + h(x) - h(u^*) \geq \gamma^*,\end{aligned}$$

where we use the fact that h is increasing on $[0,\infty)$. This verifies the inequality (18) for $x > u^*$. A similar argument (with h decreasing on $(-\infty,0]$) can yield the inequality for $x < l^*$. □

Proof of Lemma 15. We first state two auxiliary lemmas. Lemma 16 summarizes several results from Cao and Yao (2018) (see Lemmas 5, 6, and 9 there).

LEMMA 16. *Assume $\mathcal{G}(\cdot)$ is a Lipschitz continuous function on $\mathbb{R}$, and h is continuous and strictly increasing on $[0,\infty)$, $h(0)=0$ and $\lim_{x\to\infty} h(x)=\infty$. For each $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, there is a unique continuously differentiable solution $w(x;w_0,\gamma)$ on $[0,\infty)$, which is jointly continuous in w_0 and γ to the following ODE:*

$$\begin{aligned} \frac{1}{2}\sigma^2 w'(x)+\mathcal{G}(w(x))+h(x)=\gamma \quad &\textit{for} \quad x\geq 0, \\ \textit{subject to} \quad &w(0)=w_0. \end{aligned} \tag{97}$$

Furthermore, $w'(x;w_0,\gamma)$ is continuous in $x\geq 0$, $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, and the following hold:

1. *The solution $w(x;w_0,\gamma)$ is strictly increasing in $\gamma\in\mathbb{R}$ for fixed $x>0$ and $w_0\in\mathbb{R}$, and is strictly increasing in $w_0\in\mathbb{R}$ for fixed $x\geq 0$ and $\gamma\in\mathbb{R}$. For fixed $w_0\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, $w(x;w_0,\gamma)$ cannot have a local minimizer in $x\in(0,\infty)$, and cannot be a constant in any interval on $[0,\infty)$.*
2. *For $k>w_0$, there exists a unique number $\gamma_+(w_0)$ such that $\max_{x\geq 0} w(x;w_0,\gamma_+(w_0))=k$ with the unique maximizer denoted by $x^*(w_0,\gamma_+(w_0))$. The functions $\gamma_+(w_0)$ and $x^*(w_0,\gamma_+(w_0))$ are both continuous and strictly decreasing in w_0.*

LEMMA 17. *Consider the ODE in (97). Assume $\gamma_1(w)$ is continuous and strictly increasing in w, then $\max_{x\geq 0} w(x;w_0,\gamma_1(w_0))$ is continuous and strictly increasing in w_0.*

We now prove Lemma 15. For $w_0,\gamma\in\mathbb{R}$, from Lemma 16, there is a unique solution $w(x;w_0,\gamma)$ to (94) for $x\geq 0$. Let $\mathcal{G}_1(x)=\mathcal{G}(-x)$ and $h_1(x)=h(-x)$ for $x\geq 0$ and from Lemma 16, there is a unique $\bar{w}(x;-w_0,\gamma)$ solving (94) for $x\geq 0$ but with $\mathcal{G}$, h and w_0 replaced by $\mathcal{G}_1$, h_1 and $-w_0$. Let $w(x;w_0,\gamma)=-\bar{w}(-x;-w_0,\gamma)$ for $x\leq 0$. This proves the existence of $w(x;w_0,\gamma)$ on $\mathbb{R}$. Items 1 and 2 can be easily verified using Lemmas 16 and 17. Next we focus on proving item 3.

The existence of the function $\gamma_+(\cdot)$ follows from Lemma 16 item 2. Similarly, there exists a unique number $\gamma_-(w_0)$ such that $\min_{x\leq 0} w(x;w_0,\gamma_-(w_0))=a$ for all $\omega_0>a$, and the function $\gamma_-(w_0)$ is continuous and strictly increasing in w_0. If $\gamma_-(\frac{a+b}{2})=\gamma_+(\frac{a+b}{2})$, then the conclusion holds with $w^*=\frac{a+b}{2}$. In the following, we consider $\gamma_-(\frac{a+b}{2})\neq\gamma_+(\frac{a+b}{2})$.

If $\gamma_-(\frac{a+b}{2})<\gamma_+(\frac{a+b}{2})$, let $\gamma(w_0)=\gamma_-(w_0)$ for $w_0\in(a,b)$; otherwise, let $\gamma(w_0)=\gamma_+(w_0)$ for $w_0\in(a,b)$. We consider the case $\gamma(w_0)=\gamma_-(w_0)$ because the other one is similar.

Because $\gamma(\frac{a+b}{2})=\gamma_-(\frac{a+b}{2})<\gamma_+(\frac{a+b}{2})$, from item 1,

$$\max_{x\geq 0} w\left(x;\frac{a+b}{2},\gamma\left(\frac{a+b}{2}\right)\right)<\max_{x\geq 0} w\left(x;\frac{a+b}{2},\gamma_+\left(\frac{a+b}{2}\right)\right)=b. \tag{98}$$

Because $\gamma(w_0)=\gamma_-(w_0)$ is continuous and strictly increasing in w_0, hence from item 2, $\max_{x\geq 0} w(x;w_0,\gamma(w_0))$ is continuous in w_0. Note that because $\min_{x\leq 0} w(x;b,\gamma(b))=a<b=w(0;b,\gamma(b))$ and $w(x;b,\gamma(b))$ cannot have a local maximizer on $(-\infty,0)$, $w(x;b,\gamma(b))$ is strictly increasing for $x\leq 0$ around 0, and hence $w'(0;b,\gamma(b))\geq 0$. Next, we prove $w'(0;b,\gamma(b))\neq 0$.

For case i), if $w'(0;b,\gamma(b))=0$, then for $x<0$ around 0, we have $w''(x;b,\gamma(b))=-\frac{2}{\sigma^2}\pi'(w(x;b,\gamma(b)))w'(x;b,\gamma(b))-\frac{2}{\sigma^2}h'(x)$. Since we assume $h'(0-)<0$, it follows that $w'(x;b,\gamma(b))<0$ for $x<0$ around 0. Hence $w(x;b,\gamma(b))$ decreases for such x and a contradiction. Hence, $w'(0;b,\gamma(b))\neq 0$.

For case ii), if $w'(0;b,\gamma(b))=0$, because $h(0)=0$ and $\mathcal{G}(b)=0$ (because $b=0$) then $\gamma(b)=0$. Then $w'(x;b,\gamma(b))=\frac{2\frac{m'H^{-1}m}{4}}{\sigma^2}w(x;b,\gamma(b))^2-\frac{2h(x)}{\sigma^2}$ for $x\leq 0$ around 0, where $w(0;b,\gamma(b))=0$ This is a Riccati equation, so we obtain $w(x;b,\gamma(b))=-\frac{\sigma^2 u'(x)}{2\frac{m'H^{-1}m}{4}u(x)}$ with u solving $u''(x)-\frac{4\frac{m'H^{-1}m}{4}}{\sigma^4}h(x)u(x)=0$ and $u'(0)=0,u(0)=1$. Because $h(x)>0$ for $x<0$, then $u(x)>0$ implies $u''(x)>0$. Hence $u'(x)$ is increasing around 0, which gives $u'(x)<0$ for $x<0$. As a result, $w(x;b,\gamma(b))>0$ for $x<0$ around 0. This is a contradiction to the fact that $w(x;b,\gamma(b))>0$ strictly increases to 0 for $x<0$. Hence $w'(0;b,\gamma(b))\neq 0$.

As a result, we must have $w'(0;b,\gamma(b))>0$, and hence

$$\lim_{w_0\uparrow b}\max_{x\geq 0} w(x;w_0,\gamma(w_0))=\max_{x\geq 0} w(x;b,\gamma(b))>b. \tag{99}$$

Combining (98), (99), and item 2, there is $w_* \in (\frac{a+b}{2}, b)$ such that $\max_{x\geq 0} w(x; w_*, \gamma(w_*)) = b$. Then $\gamma(w_*) = \gamma_-(w_*) = \gamma_+(w_*)$, and the proof is complete. □

Proof of Lemma 17. The monotonicity of $\max_{x\geq 0} w(x; w_0, \gamma_1(w_0))$ follows from that if $w_{01} < w_{02}$, then

$$w(x; w_{01}, \gamma_1(w_{01})) < w(x; w_{02}, \gamma_1(w_{01})) < w(x; w_{02}, \gamma_1(w_{02})), \quad \text{for each } x \geq 0, \tag{100}$$

where the first inequality follows from Lemma 16 (1) and the second inequality follows from the assumption that $\gamma_1(w_0)$ is strictly increasing in w_0 and Lemma 16 (1).

Next we prove the continuity, i.e., $G(w_0) := \max_{x\geq 0} w(x; w_0, \gamma_1(w_0))$ is a continuous function of w_0. If not, then there exists $\tilde{w}_0$ such that at least one of the following occurs: (1) $\lim_{w_0\uparrow\tilde{w}_0} G(w_0) < G(\tilde{w}_0)$; (2) $\lim_{w_0\downarrow\tilde{w}_0} G(w_0) > G(\tilde{w}_0)$. The limits are well defined due to the monotonicity of G. We will focus on Case (2) as Case (1) can be argued similarly.

Suppose $\lim_{w_0\downarrow\tilde{w}_0} G(w_0) > G(\tilde{w}_0)$. Then $G(\tilde{w}_0) < \infty$. Denote by $\bar{x}_0 = \text{argmax}_{x\geq 0} w(x; \tilde{w}_0, \gamma_1(\tilde{w}_0))$, which is finite due to $\lim_{x\to\infty} h(x) = \infty$ and $G(\tilde{w}_0) < \infty$. Then there exists $x_1 > \bar{x}_0$ such that $w(x_1; \tilde{w}_0, \gamma_1(\tilde{w}_0)) < w(\bar{x}_0; \tilde{w}_0, \gamma_1(\tilde{w}_0))$. Denote by $\epsilon = w(\bar{x}_0; \tilde{w}_0, \gamma_1(\tilde{w}_0)) - w(x_1; \tilde{w}_0, \gamma_1(\tilde{w}_0)) > 0$. Because $w(x_1; w_0, \gamma_1(w_0))$ is continuous in w_0, when $w_0 > \tilde{w}_0$ is sufficiently close to $\tilde{w}_0$,

$$w(x_1; w_0, \gamma_1(w_0)) < w(x_1; \tilde{w}_0, \gamma_1(\tilde{w}_0)) + \epsilon = w(\bar{x}_0; \tilde{w}_0, \gamma_1(\tilde{w}_0)) \leq w(\bar{x}_0; w_0, \gamma_1(w_0)),$$

where the last inequality follows from (100). Hence, for a fixed $w_0 > \tilde{w}_0$ sufficiently close to $\tilde{w}_0$, we can deduce that $w(x; w_0, \gamma_1(w_0))$ achieves its maximum in $[0, x_1]$ by Lemma 16 (1). Now for each $x \in [0, x_1]$, we have $w(x; w_0, \gamma_1(w_0)) \downarrow w(x; \tilde{w}_0, \gamma_1(\tilde{w}_0))$ as $w_0 \downarrow \tilde{w}_0$. Using Dini's Theorem, the convergence of $w(\cdot; w_0, \gamma_1(w_0))$ to $w(\cdot; \tilde{w}_0, \gamma_1(\tilde{w}_0))$ on $[0, x_1]$ as $w_0 \downarrow \tilde{w}_0$ is uniform. Hence $G(w_0) \downarrow G(\tilde{w}_0)$ as $w_0 \downarrow \tilde{w}_0$, which leads to a contradiction. Therefore, we must have $\lim_{w_0\uparrow\tilde{w}_0} G(w_0) = G(\tilde{w}_0) = \lim_{w_0\downarrow\tilde{w}_0} G(w_0)$. This proves the continuity of G. The proof is therefore complete. □

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