1 Set Theory

1.1 Set Axioms

1.1.1 Undefined notions

Set: $A, B, C, \ldots$

1.1.2 Axioms

1. *Extension:* $\forall A \forall B[\forall C(C \in A \Leftrightarrow C \in B) \Rightarrow A = B]$
2. *Regularity:* $\forall A[\exists C(C \in A) \Rightarrow \exists B(B \in A \wedge \neg\exists D(D \in B \wedge D \in A))]$
   (Every nonempty set contains a set that is disjoint from it. Also know as "Axiom of Foundation.")
3. *Schema of Specification:* $\forall B \forall X_1 \forall X_2 \ldots \forall X_n \exists A \forall C[C \in A \Leftrightarrow (C \in B \wedge \phi)]$
4. *Pairing:* $\forall X_1 \forall X_2 \exists A(X_1 \in A \wedge X_2 \in A)$
5. *Union:* $\forall \mathcal{F}_A \exists U \forall A \forall X[(X \in A \wedge A \in \mathcal{F}_A) \Rightarrow X \in U]$
6. *Schema of Replacement:* $\forall A \forall X_1 \forall X_2 \ldots \forall X_n[\forall B(B \in A \Rightarrow \exists! D\phi) \Rightarrow \exists B \forall C(C \in A \Rightarrow \exists D(D \in B \wedge \phi))]$
7. *Infinity:* $\exists \omega_0[\emptyset \in \omega_0 \wedge \forall X(X \in \omega_0 \Rightarrow X \cup X\} \in \omega_0)]$
8. *Power Set:* $\forall X \exists \mathcal{P}(X) \forall S[S \subseteq X \Rightarrow S \in \mathcal{P}(X)]$
9. *Empty Set:* $\exists A \forall X(X \notin A)$
10. *Choice:* $\forall X[\emptyset \notin X \Rightarrow \exists(f : X \to \bigcup X)\forall A \in X(f(A) \in A)]$

Proposition 1.1. *The empty set axiom is implied by the other nine axioms.*

Proof. Just choose any formula that is always false such as $\phi(X) = X \in B \wedge X \notin B$ and apply the axiom schema of specification. This will give the empty set. The axiom of extension proves uniqueness vacuously. □

1.1.3 Universe

A set U is defined with the following properties...

1. $x \in u \in U \Rightarrow x \in U$
2. $u \in U \wedge v \in U \Rightarrow \{u, v\}, \langle u, v \rangle, u \times v \in U$
3. $X \in U \Rightarrow \mathcal{P}(X) \in U \wedge \bigcup X \in U$
4. $\omega_0 \in U$ is the set of finite ordinals
5. if $f : A \to B$ is a surjective function with $A \in U \wedge B \subset U$, then $B \in U$

(See: Set Constructions.)

In category theory, *small sets* are members of U.

1.2 Set Constructions

1.2.1 Union

- $A \cup B := \{x | x \in A \vee x \in B\}$
- $\bigcup \mathcal{F} := \{x | x \in X \text{ for some } X \in \mathcal{F}\}$

Proposition 1.2. *For sets* $A, B, C \ldots$

- Property Name?: $A \cup \emptyset = A$
- Idempotence: $A \cup A = A$
- Property Name?: $A \subseteq B \Leftrightarrow A \cup B = B$
- Commutative: $A \cup B = B \cup A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$

1.2.2 Intersection

- $A \cap B := \{x \in A | x \in B\} = \{x \in B | x \in A\}$
- $\bigcap \mathcal{F} := \{x | x \in X \text{ for all } X \in \mathcal{F}\}$

1.2.3 Complement

- *Relative Complement:* $A \setminus B := \{x \in A | x \notin B\}$
- *Absolute Complement:* For some universe U and $A \subseteq U$, $A^c := U \setminus A$

1.2.4 Symmetric Difference

$A \triangle B := (A \setminus B) \cup (B \setminus A))$

1.2.5 Power Set

$\mathcal{P}(X) := \{S | S \subseteq X\}$

1.2.6 n-Tuple

- *Ordered pair:* $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$
- $\langle a_1, a_2, a_3, \ldots a_n \rangle := \langle\langle\langle\langle a_1, a_2 \rangle, a_3 \rangle \ldots \rangle, a_n \rangle$

1.2.7 Cartesian Product

- $A \times B := \{\langle a, b \rangle | \text{ for some } a \in A \text{ and for some } b \in B\}$
- $\bigtimes \mathcal{F} := \{\langle a_1, a_2, \ldots a_n \rangle | \text{ for } a_1 \in A_1,\ a_2 \in A_2, \ldots\ a_n \in A_n \text{ where } A_1, A_2, \ldots, A_n \in \mathcal{F}\}$

1.2.8 Quotient by Equivalence Relation

$X/\sim := \{[a]_\sim | a \in X\}$ (See: equivalence relations)

1.2.9 Family

Given a set X and an index set I, a family is a function $\mathcal{F} : I \to X$. A cleaner way of denoting the concept is...

$\mathcal{F}(i) := S_i,\ \{S_i\}_{i \in I}$

1.3 Relations

$\mathcal{R} :\subseteq A \times B$ for some $A \times B$

1.3.1 Equivalence Relations

Relations $\sim \subseteq A \times A$ such that $\forall a, b, c \in A$...

- $a \sim a$
- $a \sim b \Rightarrow b \sim a$
- $a \sim b \wedge b \sim c \Rightarrow a \sim c$

Equivalence Class: $[a]_\sim := \{b \in S | b \sim a\}$

Set Partition: A set $P :\subseteq \mathcal{P}(X)$ such that...

- $\bigcup P = X$
- $\forall S_1, S_2 \in P(S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 = S_2)$

1.3.2 Functions

A relation $f : A \to B$ satisfying $\forall a \in A\ \exists! b \in B$ such that afb, denoted $f(a) = b$.

Injection: A function $f : A \hookrightarrow B$ such that $\forall\, x, y \in A$ if $x \neq y,$ then $f(x) \neq f(y)$. (See: monomorphism. Injections have right inverses.)

Surjection: A function $f : A \twoheadrightarrow B$ such that $\forall\, b \in B\ \exists\, a \in A$ such that $f(a) = b$. (See: epimorphism. Surjections have left inverses, called *sections*.)

Bijection: A function $f : A \xrightarrow{\sim} B$ which is an injection and a surjection. (See: isomorphism)

Restriction: For $C \subseteq A$ and $f : A \to B$, $f{\restriction_C} : C \to B$ where $\forall c \in C\ f{\restriction_C}(c) := f(c)$

Image: $f(A) := \{f(a) | a \in A\}$

Preimage: $f^{-1}(A) := \{a \in A | f(a) \in B\}$

Inclusion Map: For $A \subseteq X$, $\iota : A \hookrightarrow X$ where $\forall\, a \in A\ \iota(a) := a \in X$

Function Composition: $f : X \to Y$ and $g : Y \to Z \Rightarrow g \circ f : X \to Z$ where...
$\forall\, x \in X$, $g \circ f(x) := g(f(x))$

Characteristic Function of a subset: For $A \subseteq X$, $\chi_A : X \to 2$ where...

$$\chi_A(x) := \begin{cases} 0 & x \in X \setminus A \\ 1 & x \in A \end{cases}$$

2 Category Theory

2.1 Metacategories

2.1.1 Undefined notions

- *Objects:* $a, b, c \ldots$
- *Arrows:* $f, g, h \ldots$

2.1.2 Operations

Given $f : a \to b$...

- *Domain:* **dom**: arrows $\to$ objects, $f \mapsto a$
- *Codomain:* **cod**: arrows $\to$ objects, $f \mapsto b$
- *Identity:* **id**: objects $\to$ arrows, $a \mapsto \mathrm{id}_a = 1_a$
- *Composition:* **comp**: arrows $\times :$ arrows $\to$ arrows, $\langle g, f\rangle \mapsto g \circ f$,
  $g \circ f : \mathrm{dom} f \to \mathrm{cod} g$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & c \end{tikzcd}$$

2.1.3 Axioms

- *Associativity:* $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$, $k \circ (g \circ f) = (k \circ g) \circ f$
- *Unit Law:* $1_a \circ f = f$ and $g \circ 1_b = g$

$$
\begin{tikzcd}
a \arrow[rr, "k\circ(g\circ f)=(k\circ g)\circ f"] \arrow[d, "f"] \arrow[drr, "g\circ f"] & & d \\
b \arrow[rr, "k\circ g"] \arrow[urr, "k\circ g"] & & c \arrow[u, "k"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "f"] & b \arrow[d, "1_b"] \arrow[dr, "g"] & \\
& b \arrow[r, "g"] & c
\end{tikzcd}
$$

2.2 Categories

2.2.1 Directed Graph

- A - a set of arrows
- O - a set of objects
- $\mathbf{dom} : A \to O$, $\mathbf{cod} : A \to O$

Set of composable pairs of arrows:

$$A \times_O A = \{\langle g, f\rangle | g, f \in A \text{ and } \mathbf{dom}(g) = \mathbf{cod}(f)\}$$

2.2.2 Categories

Add the following structure to a directed graph...

- $O \xrightarrow{id} A$, $c \mapsto id_C$
- $A \times_O A \xrightarrow{\circ} A$, $\langle g, f\rangle \mapsto g \circ f$

which satisfy $\forall a \in O$ and $\forall \langle g, f\rangle \in A \times_O A$...

- $\mathbf{dom}(\mathbf{id}(a)) = a = \mathbf{cod}(\mathbf{id}(a))$
- $\mathbf{dom}(g \circ f) = \mathbf{dom}(f)$
- $\mathbf{cod}(g \circ f) = \mathbf{cod}(g)$
- metacategorical axioms

Small categories use small sets for their objects.

2.2.3 Hom Sets

$hom(b, c) = \{f | f \in C, \mathbf{dom}(f) = b, \mathbf{cod}(f) = c\}$

2.2.4 Groupoids

A category in which every arrow is an isomorphism.

2.3 Morphisms

Arrows in categories.

2.3.1 Isomorphisms

A morphism $f \in hom(b, c)$ that has a two-sided inverse $g \in hom(c, b)$ under composition such that

$$gf = 1_b, \ fg = 1_c.$$

Proposition 2.1. *The inverse of an isomorphism is unique.*

Proof. For inverses g_1, g_2 of f observe...

$$g_1 = g_1 1_c = g_1(fg_2) = (g_1 f)g_2 = 1_b g_2 = g_2$$

□

Proposition 2.2. *Supposing f^{-1} is the inverse of f...*

- *Each identity 1_c is an isomorphism and is its own inverse.*
- *If f is an isomorphism, then f^{-1} is an isomorphism and further $(f^{-1})^{-1} = f$.*
- *If $f \in hom(a, b)$, $g \in hom(b, c)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.*

2.3.2 Automorphisms

An isomorphism of an object to itself. Denoted:

$$hom(c, c) = aut(c)$$

Observe $aut(c)$ is a group.

2.3.3 Monomorphisms

A morphism $f \in hom(b, c)$ such that $\forall z \in C$ and $\forall \alpha', \alpha'' \in hom(z, b)$:

$$f \circ \alpha' = f \circ \alpha'' \Rightarrow \alpha' = \alpha''$$

2.3.4 Epimorphisms

A morphism $f \in hom(b, c)$ such that $\forall z \in C$ and $\forall \beta', \beta'' \in hom(b, z)$:

$$\beta' \circ f = \beta'' \circ f \Rightarrow \beta' = \beta''$$

2.4 Functors

Morphisms $T : C \to B$ with domain and codomain both categories. It consists of two suitably related functions

- object function T, $c \mapsto Tc$
- arrow function T, $f : c \to c' \mapsto Tf : Tc \to Tc'$

which satisfy...

- $T(1_c) = 1_c$
- $T(g \circ f) = T_g \circ T_f$

2.4.1 Full

$\forall c, c' \in C$ and $g : Tc \to Tc' \in B, \exists f : c \to c' \in C$ s.t. $g \in Tf$

2.4.2 Faithful

$\forall c, c' \in C$ and $f_1, f_2 : c \to c', Tf_1 = Tf_2 \Rightarrow f_1 = f_2$

3 Group Theory