## 1 Set Theory

### 1.1 Set Axioms

#### 1.1.1 Undefined notions

Set:  $A, B, C, \ldots$ 

#### 1.1.2 Axioms

- 1. Extension:  $\forall A \forall B [\forall C (C \in A \Leftrightarrow C \in B) \Rightarrow A = B]$
- 2. Regularity:  $\forall A[\exists C(C \in A) \Rightarrow \exists B(B \in A \land \neg \exists D(D \in B \land D \in A))]$  (Every nonempty set contains a set that is disjoint from it. Also know as "Axiom of Foundation.")
- 3. Schema of Specification:  $\forall B \forall X_1 \forall X_2 \dots \forall X_n \exists A \forall C [C \in A \Leftrightarrow (C \in B \land \phi)]$
- 4. Pairing:  $\forall X_1 \forall X_2 \exists A(X_1 \in A \land X_2 \in A)$
- 5. Union:  $\forall \mathcal{F}_A \exists U \forall A \forall X [(X \in A \land A \in \mathcal{F}_A) \Rightarrow X \in U]$
- 6. Schema of Replacement:  $\forall A \forall X_1 \forall X_2 \dots \forall X_n [\forall B (B \in A \Rightarrow \exists ! D\phi) \Rightarrow \exists B \forall C (C \in A \Rightarrow \exists D (D \in B \land \phi))]$
- 7. Infinity:  $\exists \omega_0 [\emptyset \in \omega_0 \land \forall X (X \in \omega_0 \Rightarrow X \cup X) \in \omega_0)]$
- 8. Power Set:  $\forall X \exists \mathcal{P}(X) \forall S[S \subseteq X \Rightarrow S \in \mathcal{P}(X)]$
- 9. Empty Set:  $\exists A \forall X (X \notin A)$
- 10. Choice:  $\forall X [\emptyset \notin X \Rightarrow \exists (f : X \to \bigcup X) \forall A \in X (f(A) \in A)]$

**Proposition 1.1.** The empty set axiom is implied by the other nine axioms.

*Proof.* Just choose any formula that is always false such as  $\phi(X) = X \in B \land X \notin B$  and apply the axiom schema of specification. This will give the empty set. The axiom of extension proves uniqueness vacuously.

#### 1.1.3 Universe

A set U is defined with the following properties...

- 1.  $x \in u \in U \Rightarrow x \in U$
- 2.  $u \in U \land v \in U \Rightarrow \{u, v\}, \langle u, v \rangle, u \times v \in U$
- 3.  $X \in U \Rightarrow \mathcal{P}(X) \in U \land \bigcup X \in U$
- 4.  $\omega_0 \in U$  is the set of finite ordinals
- 5. if  $f: A \to B$  is a surjective function with  $A \in U \land B \subset U$ , then  $B \in U$  (See: Set Constructions.)

In category theory,  $small\ sets$  are members of U.

## 1.2 Set Constructions

#### 1.2.1 Union

- $\bullet \ A \cup B := \{x | x \in A \lor x \in B\}$
- $\bigcup \mathcal{F} := \{x | x \in X \text{ for some } X \in \mathcal{F}\}$

**Proposition 1.2.** For sets A, B, C...

- Property Name?:  $A \cup \emptyset = A$
- Idempotence:  $A \cup A = A$
- Property Name?:  $A \subseteq B \Leftrightarrow A \cup B = B$
- Commutative:  $A \cup B = B \cup A$
- Associative:  $A \cup (B \cup C) = (A \cup B) \cup C$

#### 1.2.2 Intersection

- $A \cap B := \{x \in A | x \in B\} = \{x \in B | x \in A\}$
- $\bigcap \mathcal{F} := \{x | x \in X \text{ for all } X \in \mathcal{F}\}$

#### 1.2.3 Complement

- Relative Complement:  $A \setminus B := \{x \in A | x \notin B\}$
- Absolute Complement: For some universe U and  $A \subseteq U$ ,  $A^c := U \setminus A$

#### 1.2.4 Symmetric Difference

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

#### 1.2.5 Power Set

$$\mathcal{P}(X) := \{ S | S \subseteq X \}$$

### **1.2.6** *n***-Tuple**

- Ordered pair:  $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$
- $\langle a_1, a_2, a_3, \dots a_n \rangle := \langle \langle \langle \langle a_1, a_2 \rangle, a_3 \rangle \dots \rangle, a_n \rangle$

#### 1.2.7 Cartesian Product

- $A \times B := \{ \langle a, b \rangle | \text{ for some } a \in A \text{ and for some } b \in B \}$
- $\times \mathcal{F} := \{ \langle a_1, a_2, \dots a_n \rangle | \text{ for } a_1 \in A_1, a_2 \in A_2, \dots a_n \in A_n \text{ where } A_1, A_2, \dots, A_n \in \mathcal{F} \}$

## 1.2.8 Quotient by Equivalence Relation

 $X/\sim:=\{[a]_{\sim}|a\in X\}$  (See: equivalence relations)

## 1.2.9 Family

Given a set X and an index set I, a family is a function  $\mathcal{F}: I \to X$ . A cleaner way of denoting the concept is...

$$\mathcal{F}(i) := S_i, \ \{S_i\}_{i \in I}$$

## 1.3 Relations

 $\mathcal{R} :\subseteq A \times B$  for some  $A \times B$ 

### 1.3.1 Equivalence Relations

Relations  $\sim \subseteq A \times A$  such that  $\forall a, b, c \in A$ ...

- a ~ a
- $a \sim b \Rightarrow b \sim a$
- $a \sim b \wedge b \sim c \Rightarrow a \sim c$

Equivalence Class:  $[a]_{\sim} := \{b \in S | b \sim a\}$ 

Set Partition: A set  $P :\subseteq \mathcal{P}(X)$  such that...

- $\bullet \mid \exists P = X$
- $\forall S_1, S_2 \in P(S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 = S_2)$

#### 1.3.2 Functions

A relation  $f: A \to B$  satisfying  $\forall a \in A \exists ! b \in B$  such that afb, denoted f(a) = b.

*Injection:* A function  $f: A \hookrightarrow B$  such that  $\forall x, y \in A$  if  $x \neq y$ , then  $f(x) \neq f(y)$ . (See: monomorphism. Injections have right inverses.)

Surjection: A function f:A woheadrightarrow B such that  $\forall b \in B \ \exists \ a \in A \ \text{such that} \ f(a) = b$ . (See: epimorphism. Surjections have left inverses, called sections.)

*Bijection:* A function  $f: A \xrightarrow{\sim} B$  which is an injection and a surjection. (See: isomorphism)

Restriction: For  $C \subseteq A$  and  $f: A \to B$ ,  $f \upharpoonright_C : C \to B$  where  $\forall c \in C \ f \upharpoonright_C (c) := f(c)$ 

Image:  $f(A) := \{f(a) | a \in A\}$ 

 $Preimage: \ f^{-1}(A):=\{a\in A|f(a)\in B\}$ 

Inclusion Map: For  $A \subseteq X$ ,  $\iota : A \hookrightarrow X$  where  $\forall a \in A \ \iota(a) := a \in X$ 

Function Composition:  $f: X \to Y$  and  $g: Y \to Z \Rightarrow g \circ f: X \to Z$  where ...  $\forall x \in X, g \circ f(x) := g(f(x))$ 

Characteristic Function of a subset: For  $A \subseteq X$ ,  $\chi_A : X \to 2$  where...

$$\chi_A(x) := \begin{cases} 0 & x \in X \setminus A \\ 1 & x \in A \end{cases}$$

# 2 Category Theory

## 2.1 Metacategories

## 2.1.1 Undefined notions

• Objects:  $a, b, c \dots$ 

• Arrows:  $f, g, h \dots$ 

## 2.1.2 Operations

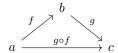
Given  $f: a \to b \dots$ 

• Domain: dom: arrows  $\rightarrow$  objects,  $f \mapsto a$ 

• Codomain: cod: arrows  $\rightarrow$  objects,  $f \mapsto b$ 

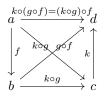
• *Identity:* **id**: objects  $\rightarrow$  arrows,  $a \mapsto id_a = 1_a$ 

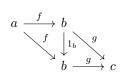
• Composition: comp: arrows  $\times$ : arrows  $\rightarrow$  arrows,  $\langle g, f \rangle \mapsto g \circ f$ ,  $g \circ f$ : dom  $f \rightarrow \operatorname{cod} g$ 



#### 2.1.3 Axioms

- $\bullet \ \textit{Associativity:} \ a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d, \ k \circ (g \circ f) = (k \circ g) \circ f$
- Unit Law:  $1_a \circ f = f$  and  $g \circ 1_b = g$





## 2.2 Categories

## 2.2.1 Directed Graph

- $\bullet$  A a set of arrows
- $\bullet$  O a set of objects
- $\bullet \ \mathbf{dom}: A \to O, \ \mathbf{cod}: A \to O$

Set of composable pairs of arrows:

$$A \times_O A = \{\langle g, f \rangle | g, f \in A \text{ and } \mathbf{dom}(g) = \mathbf{cod}(f)\}$$

### 2.2.2 Categories

Add the following structure to a directed graph...

- $O \xrightarrow{id} A, c \mapsto id_C$
- $A \times_O A \xrightarrow{\circ} A$ ,  $\langle g, f \rangle \mapsto g \circ f$

which satisfy  $\forall a \in O$  and  $\forall \langle g,f \rangle \in A \times_O A.$  . .

- $\operatorname{dom}(\operatorname{id}(a)) = a = \operatorname{cod}(\operatorname{id}(a))$
- $\bullet \ \mathbf{dom}(g \circ f) = \mathbf{dom}(f)$
- $\mathbf{cod}(g \circ f) = \mathbf{cod}(g)$
- metacategorical axioms

Small categories use small sets for their objects.

### 2.2.3 Hom Sets

$$hom(b,c) = \{f|f \in C, \mathbf{dom}(f) = b, \mathbf{cod}(f) = c\}$$

## 2.2.4 Groupoids

A category in which every arrow is an isomorphism.

## 2.3 Morphisms

Arrows in categories.

#### 2.3.1 Isomorphisms

A morphism  $f \in hom(b,c)$  that has a two-sided inverse  $g \in hom(c,b)$  under composition such that

$$gf = 1_b, fg = 1_c.$$

**Proposition 2.1.** The inverse of an isomorphism is unique.

*Proof.* For inverses  $g_1, g_2$  of f observe...

$$g_1 = g_1 1_c = g_1(fg_2) = (g_1 f)g_2 = 1_b g_2 = g_2$$

**Proposition 2.2.** Supposing  $f^{-1}$  is the inverse of f...

- $\bullet$  Each identity  $1_c$  is an isomorphism and is its own inverse.
- If f is an isomorphism, then  $f^{-1}$  is an isomorphism and further  $(f^{-1})^{-1} = f$ .
- If  $f \in hom(a,b)$ ,  $g \in hom(b,c)$  are isomorphisms, then the composition gf is an isomorphism and  $(gf)^{-1} = f^{-1}g^{-1}$ .

#### 2.3.2 Automorphisms

An isomorphism of an object to itself. Denoted:

$$hom(c, c) = aut(c)$$

Observe aut(c) is a group.

#### 2.3.3 Monomorphisms

A morphism  $f \in hom(b,c)$  such that  $\forall z \in C$  and  $\forall \alpha', \alpha'' \in hom(z,b)$ :

$$f \circ \alpha' = f \circ \alpha'' \Rightarrow \alpha' = \alpha''$$

#### 2.3.4 Epimorphisms

A morphism  $f \in hom(b, c)$  such that  $\forall z \in C$  and  $\forall \beta', \beta'' \in hom(b, z)$ :

$$\beta' \circ f = \beta'' \circ f \Rightarrow \beta' = \beta''$$

## 2.4 Functors

Morphisms  $T:C\to B$  with domain and codomain both categories. It consists of two suitably related functions

- object function  $T, c \mapsto Tc$
- arrow function  $T,\,f:c\to c'\mapsto Tf:Tc\to Tc'$

which satisfy...

- $T(1_c) = 1_c$
- $T(g \circ f) = T_g \circ T_f$

## 2.4.1 Full

$$\forall c, c' \in C \text{ and } g: Tc \to Tc' \in B, \exists f: c \to c' \in C \text{ s.t. } g \in Tf$$

## 2.4.2 Faithful

$$\forall c, c' \in C \text{ and } f_1, f_2 : c \to c', Tf_1 = Tf_2 \Rightarrow f_1 = f_2$$

# 3 Group Theory