1 Set Theory

1.1 Set Axioms

1.1.1 Undefined notions

Set: $A, B, C, \ldots$

1.1.2 Axioms

1. *Extension:* $\forall A \forall B[\forall C(C \in A \Leftrightarrow C \in B) \Rightarrow A = B]$
2. *Regularity:* $\forall A[\exists C(C \in A) \Rightarrow \exists B(B \in A \wedge \neg \exists D(D \in B \wedge D \in A))]$
(Every nonempty set contains a set that is disjoint from it. Also know as "Axiom of Foundation.")
3. *Schema of Specification:* $\forall B \forall X_1 \forall X_2 \ldots \forall X_n \exists A \forall C[C \in A \Leftrightarrow (C \in B \wedge \phi)]$
4. *Pairing:* $\forall X_1 \forall X_2 \exists A(X_1 \in A \wedge X_2 \in A)$
5. *Union:* $\forall \mathcal{F}_A \exists U \forall A \forall X[(X \in A \wedge A \in \mathcal{F}_A) \Rightarrow X \in U]$
6. *Schema of Replacement:* $\forall A \forall X_1 \forall X_2 \ldots \forall X_n[\forall B(B \in A \Rightarrow \exists! D\phi) \Rightarrow \exists B \forall C(C \in A \Rightarrow \exists D(D \in B \wedge \phi))]$
7. *Infinity:* $\exists \omega[\emptyset \in \omega \wedge \forall X(X \in \omega \Rightarrow X \cup X\} \in \omega)]$
8. *Power Set:* $\forall X \exists \mathcal{P}(X) \forall S[S \subseteq X \Rightarrow S \in \mathcal{P}(X)]$
9. *Empty Set:* $\exists A \forall X(X \notin A)$
10. *Choice:* $\forall X[\emptyset \notin X \Rightarrow \exists(f : X \to \bigcup X) \forall A \in X(f(A) \in A)]$

Proposition 1.1.1. *The empty set axiom is implied by the other nine axioms.*

Proof. Just choose any formula that is always false such as $\phi(X) = X \in B \wedge X \notin B$ and apply the axiom schema of specification. This will give the empty set. The axiom of extension proves uniqueness vacuously. $\square$

1.1.3 Universe

A set U is defined with the following properties...

1. $x \in u \in U \Rightarrow x \in U$
2. $u \in U \wedge v \in U \Rightarrow \{u, v\}, \langle u, v \rangle, u \times v \in U$
3. $X \in U \Rightarrow \mathcal{P}(X) \in U \wedge \bigcup X \in U$
4. $\omega \in U$ is the set of finite ordinals
5. if $f : A \to B$ is a surjective function with $A \in U \wedge B \subset U$, then $B \in U$

(See: Set Constructions.)

In category theory, *small sets* are members of U.

1.2 Set Constructions

1.2.1 Union

- $A \cup B := \{x | x \in A \vee x \in B\}$
- $\bigcup \mathcal{F} := \{x | x \in X \text{ for some } X \in \mathcal{F}\}$

Proposition 1.2.1. *For sets A, B, C, the following hold...*

- Identity: $A \cup \emptyset = A$
- Idempotence: $A \cup A = A$
- Absorption: $A \subseteq B \Leftrightarrow A \cup B = B$
- Commutative: $A \cup B = B \cup A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$

1.2.2 Intersection

- $A \cap B := \{x \in A | x \in B\} = \{x \in B | x \in A\}$
- $\bigcap \mathcal{F} := \{x | x \in X \text{ for all } X \in \mathcal{F}\}$

Proposition 1.2.2. *For sets A, B, C, the following hold...*

- Zero: $A \cap \emptyset = \emptyset$
- Idempotence: $A \cap A = A$
- Absorption: $A \subseteq B \Leftrightarrow A \cap B = A$
- Commutative: $A \cap B = B \cap A$
- Associative: $A \cap (B \cap C) = (A \cap B) \cap C$

1.2.3 Complement

- *Relative Complement:* $A \setminus B := \{x \in A | x \notin B\}$
- *Absolute Complement:* For some universe U and $A \subseteq U$, $A^c := U \setminus A$

Proposition 1.2.3. *For a universe U and sets $A, B \subseteq U$...*

- $(A^c)^c = A$
- $\emptyset^c = U$
- $U^c = \emptyset$
- $A \cap A^c = \emptyset$

- $A \cup A^c = U$
- $A \subseteq B \Leftrightarrow B^c \subseteq A^c$

Proposition 1.2.4 (DeMorgan's Laws)**.** *For a universe U and sets $A, B \subseteq U \ldots$*

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Proposition 1.2.5. *For sets $A, B \ldots$*

- $A \setminus B = A \cap B^c$
- $A \subseteq B \Leftrightarrow A \setminus B = \emptyset$
- $A \setminus (A \setminus B) = A \cap B$
- $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
- $A \cap B \subseteq (A \cap C) \cup (B \cap C^c)$
- $(A \cup C) \cap (B \cup C^c) \subseteq A \cup B$

Proposition 1.2.6. *For a family $\mathcal{F} \ldots$*

- $\forall X \in \mathcal{F}, \bigcup_{k \in K} X_k = \bigcup_{j \in J}(\bigcup_{i \in I_j} X_i)$
- $\forall X \in \mathcal{F}, \bigcap_{k \in K} X_k = \bigcap_{j \in J}(\bigcap_{i \in I_j} X_i)$
- $\forall X \in \mathcal{F}, \bigcup_{i \in I} X_i = \bigcup_{j \in J} X_j$
- $\forall X \in \mathcal{F}, \bigcap_{i \in I} X_i = \bigcap_{j \in J} X_j$
- $(\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{i,j}(A_i \cap B_j)$
- $(\bigcap_{i \in I} A_i) \cup (\bigcap_{j \in J} B_j) = \bigcap_{i,j}(A_i \cup B_j)$

Proposition 1.2.7 (Generalized DeMorgan's Laws)**.** *For a universe U and a family $\mathcal{F} \ldots$*

- $(\bigcup_{X \in \mathcal{F}} X)^c = \bigcap_{X \in \mathcal{F}} X^c$
- $(\bigcap_{X \in \mathcal{F}} X)^c = \bigcup_{X \in \mathcal{F}} X^c$

1.2.4 Symmetric Difference

$A \triangle B := (A \setminus B) \cup (B \setminus A))$

1.2.5 Power Set

$\mathcal{P}(X) := \{S | S \subseteq X\}$

Proposition 1.2.8. *For sets A, B and a family $\mathcal{F}$...*

- $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$
- $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$
- $\bigcap_{X \in \mathcal{F}} \mathcal{P}(X) = \mathcal{P}(\bigcap_{X \in \mathcal{F}} X)$
- $\bigcup_{X \in \mathcal{F}} \mathcal{P}(X) \subseteq \mathcal{P}(\bigcup_{X \in \mathcal{F}} X)$

1.2.5.1 Characteristic Function of a subset

For $A \subseteq X$, $\chi_A : X \to 2$ where...

$$\chi_A(x) := \begin{cases} 0 & x \in X \setminus A \\ 1 & x \in A \end{cases}$$

1.2.6 n-Tuple

- *Ordered pair:* $\langle a, b \rangle := \{\{a\}, \{a, b\}\}$
- $\langle a_1, a_2, a_3, \ldots a_n \rangle := \langle \langle \langle \langle a_1, a_2 \rangle, a_3 \rangle \ldots \rangle, a_n \rangle$

1.2.7 Cartesian Product

- $A \times B := \{\langle a, b \rangle |$ for some $a \in A$ and for some $b \in B\}$
- $\bigtimes \mathcal{F} := \{\langle a_1, a_2, \ldots a_n \rangle |$ for $a_1 \in A_1$, $a_2 \in A_2, \ldots$ $a_n \in A_n$ where $A_1, A_2, \ldots, A_n \in \mathcal{F}\}$

Proposition 1.2.9. *For sets A, B...*

- $(A \cup B) \times X = (A \times X) \cup (B \times X)$
- $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times X)$
- $(A \setminus B) \times X = (A \times X) \setminus (B \times X)$

Proposition 1.2.10. *For families $\{A_i\}_{i \in I}, \{B_j\}_{j \in J}, \{X_i\}_{i \in I}$...*

- $(\bigcup_{i \in I} A_i) \times (\bigcup_{j \in J} B_j) = \bigcup_{i,j} (A_i \times B_j)$
- $(\bigcap_{i \in I} A_i) \times (\bigcap_{j \in J} B_j) = \bigcap_{i,j} (A_i \times B_j)$
- $\bigcap_i X_i \subseteq X_j \subseteq \bigcup_i X_i$

1.2.8 Quotient by Equivalence Relation

$X/ \sim := \{[a]_\sim | a \in X\}$ (See: equivalence relations)

1.2.9 Family

Given a set X and an index set I, a family is a function $\mathcal{F} : I \to X$. A cleaner way of denoting the concept is...

$$\mathcal{F}(i) := S_i,\ \{S_i\}_{i \in I}$$

1.3 Relations

$\mathcal{R} :\subseteq A \times B$ for some $A \times B$

1.3.1 Equivalence Relations

Relations $\sim \subseteq A \times A$ such that $\forall a, b, c \in A$...

- *Reflexive:* $a \sim a$
- *Symmetric:* $a \sim b \Rightarrow b \sim a$
- *Transitive:* $a \sim b \wedge b \sim c \Rightarrow a \sim c$

1.3.1.1 Equivalence Class

$[a]_\sim := \{b \in S | b \sim a\}$

1.3.1.2 Set Partition

A set $P :\subseteq \mathcal{P}(X)$ such that...

- $\bigcup P = X$
- $\forall S_1, S_2 \in P(S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 = S_2)$

1.3.2 Functions

A relation $f : A \to B$ satisfying $\forall a \in A\ \exists! b \in B$ such that afb, denoted $f(a) = b$.

1.3.2.1 Injection

A function $f : A \hookrightarrow B$ such that $\forall\, x, y \in A$ if $x \neq y,$ then $f(x) \neq f(y)$. (See: monomorphism. Injections have right inverses.)

1.3.2.2 Surjection

A function $f : A \twoheadrightarrow B$ such that $\forall\, b \in B\ \exists\, a \in A$ such that $f(a) = b$. (See: epimorphism, Stirling numbers of the second kind. Surjections have left inverses, called *sections*.)

1.3.2.3 Bijection

A function $f : A \xrightarrow{\sim} B$ which is an injection and a surjection. (See: isomorphism)

1.3.2.4 Restriction

For $C \subseteq A$ and $f : A \to B$, $f\restriction_C : C \to B$ where $\forall c \in C\; f\restriction_C(c) := f(c)$

1.3.2.5 Image

$f(A) := \{f(a) | a \in A\}$

Proposition 1.3.1. *For a function $f : A \to B$ and a family $\{X_i\}_{i \in I}$ where $\forall i \in I\; X_i \subseteq A \ldots$*

- $f(\bigcup_i X_i) = \bigcup_i f(X_i)$
- *In general,* $f(\bigcap_i X_i) \neq \bigcap_i f(X_i)$
- *In general,* $f(X)^c \neq f(X^c)$

1.3.2.6 Preimage

$f^{-1}(A) := \{a \in A | f(a) \in B\}$

Proposition 1.3.2. *Given a function $f : X \to Y$, f is surjective if and only if $\forall A \subseteq Y$, where $A \neq \emptyset$, $f^{-1}(A) \neq \emptyset$.*

Proposition 1.3.3. *Given a function $f : X \to Y$, f is injective if and only if $\forall A \subseteq ran\ f$, where A is a singleton, $f^{-1}(A)$ is a singleton.*

Proposition 1.3.4. *Given a function $f : X \to Y \ldots$*

- *If $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$.*
- *If f is surjective, then $f(f^{-1}(B)) = B$.*
- *If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$.*
- *If f is injective, then $A = f(f^{-1}(A))$.*
- *If $\{B_i\}$ is a family of subset of Y, then $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$ and $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.*

1.3.2.7 Function Composition

$f : X \to Y$ and $g : Y \to Z \Rightarrow g \circ f : X \to Z$ where $\forall\, x \in X$, $g \circ f(x) := g(f(x))$

1.4 Natural Numbers

1.4.1 Successor

For a set n, its *successor* n^+ is defined by...

$$n^+ = n \cup \{n\}$$

1.4.2 Inductive

A set N is *inductive* if and only if $\emptyset \in N$ and $(\forall n \in N)\, n^+ \in N$.

The Axiom of Infinity may be restated in terms of "inductiveness," i.e....

There exists an inductive set ω.

1.4.3 Natural Number

A *natural number* is a set that belongs to every inductive set, i.e. the intersection of them all.

The following theorem is a consequence of the definition...

Theorem 1.4.1 (Induction on ω)**.** *Any inductive subset of ω coincides with ω.*

Proposition 1.4.1. *Every natural number except 0 is the successor of some natural number.*

Proof. Let $T = \{n \in \omega | n = 0 \vee (\exists p \in \omega) n = p^+\}$ and use induction. □

1.4.4 Peano's Postulates

1.4.4.1 Peano System

An ordered triple $\langle N, S, e\rangle$ consiting of a set N, a function $S : N \to N$, and a member $e \in N$ such that the following three conditions are met:

1. $e \notin \text{ran} S$.
2. S is injective.
3. Any subset $A \subseteq N$ that contains e and is closed under S equals N itself.

Proposition 1.4.2. *Let $\sigma = \{\langle n, n^+\rangle | n \in \omega\}$. Then $\langle \omega, \sigma, 0\rangle$ is a Peano system.*

1.4.4.2 Transitive Set

A set A is said to be a *transitive set* if and only if $x \in a \in A \Rightarrow x \in A$.

Proposition 1.4.3. *For a transitive set a,*

$$\bigcup (a^+) = a.$$

Proposition 1.4.4. *Every natural number is a transitive set and ω is a transitive set.*

Proof. Use induction. □

1.4.5 Recursion

Theorem 1.4.2 (Recursion Theorem on ω)**.** *Let A be a set, $a \in A$, and $F : A \to A$. Then there exists an unique function $h : \omega \to A$ such that...*

$$h(0) = a,$$

and for every $n \in \omega$,

$$h(n^+) = F(h(n)).$$

Proof. The idea is to lef h be the union of many approximating functions. For the purposes of this proof, call a function v *acceptable* if and only if dom $v \subseteq \omega$, ran $v \subseteq A$, and the following conditions hold:

1. If $0 \in$ dom v, then $v(0) = a$.
2. If $n^+ \in$ dom v (where $n \in \omega$), then also $n \in$ dom v and $v(n^+) = F(v(n))$.

Let $\mathcal{H}$ be the collection of all acceptable functions, and let $h = \bigcup \mathcal{H}$. Thus...

$$\begin{aligned} (\star) \quad \langle n, y \rangle \in h &\Leftrightarrow \langle n, y \rangle \text{ is a member of some acceptable } v \\ &\Leftrightarrow v(n) = y \text{ for some acceptabe } v. \end{aligned}$$

We claim that this h meets the demands of the theorem. This claim can be broken down into four parts. The four parts involve showing that (I) h is a function, (II) h is acceptable, (III) dom h is all of ω, and (IV) h is unique.

I. We first claim that h is a function. Let...

$$S = \{n \in \omega | \text{ for at most one } y, \langle n, y \rangle \in h\}.$$

We must check that S is inductive. If $\langle 0, y_1 \rangle \in h$ and $\langle 0, y_2 \rangle \in h$, then by $(\star)$ there exist acceptable v_1 and v_2 such that $v_1(0) = y_1$ and $v_2(0) = y_2$. But by (1) it follows that $y_1 = a = y_2$. Thus $0 \in S$.

Next suppose that $k \in S$. Consider $\langle k^+, y_1 \rangle \in h$ and $\langle k^+, y_2 \rangle \in h$. As before there must exist acceptabel v_1 and v_2 such that $v_1(k^+) = y_1$ and $v_2(k+) = y_2$. By condition (2) it follows that...

$$y_1 = v_1(k^+) = F(v_1(k)) \quad \text{and} \quad y_2 = v_2(k^+) = F(v_2(k)).$$

But since $k \in S$, we have $v_1(k) = v_2(k)$. Therefore...

$$y_1 = F(v_1(k)) = F(v_2(k)) = y_2.$$

So $k^+ \in S$, proving S is inductive and conincides with ω. Consequently h is a function.

II. Next we claime that h itself is acceptable. We have just seen that h is a function, and it is clear from $(\star)$ that dom $h \subseteq \omega$ and ran $h \subseteq A$.

First examine (1). If $0 \in$ dom h, then there must be some acceptable v with $v(0) = h(0)$. Since $v(0) = a$, we have $h(0) = a$.

Next examine (2). Assume $n^+ \in$ dom h. Again there must be some acceptable v with $v(n^+) = h(n^+)$. Since v is acceptable we have $n \in$ dom v (and $v(n) = h(n)$) and

$$h(n^+) = v(n^+) = F(v(n)) = F(h(n)).$$

Thus h satisfies (2) and so is acceptable.

III. We now claim that dom $h = \omega$ (the function is nonempty). It suffices to show that dom h is inductive. The function $\{\langle 0, a\rangle\}$ is acceptable and hence $0 \in$ dom h. Suppose the $k \in$ dome h. If $k^+ \notin$ dom h, then let...

$$v = h \cup \{\langle k^+, F(h(k))\rangle\}.$$

Then v is a function, dom $v \subseteq \omega$, and ran $v \subseteq A$. We will show that v is acceptable.

Condition (1) holds since $v(0) = h(0) = a$. For condition (2) there are two cases. If $n^+ \in$ dom v where $n^+ \neq k^+$, then $n^+ \in$ dom h and $v(n^+) = h(n^+) = F(h(n)) = F(v(n))$. The other case occurs if $n^+ = k^+$. Since the successor operation is injective, $n = k$. By assumption $k \in$ dom h. Thus...

$$v(k^+) = F(h(k)) = F(v(k))$$

and (2) holds. Hence v is acceptable. But then $v \subseteq h$, so that $k^+ \in$ dom h after all. So dom h is inductive and therefore coincides with ω.

IV. Finally we claim that h is unique. For let h_1 and h_2 both satisfy the conclusion fo the theorem. Let...

$$S = \{n \in \omega | h_1(n) = h_2(n)\}.$$

S is inductive, showing $h_1 = h_2$. Thus h is unique. $\square$

Example 1.4.2.1. *There is no function $h : \mathbb{Z} \to \mathbb{Z}$ such that for every $a \in \mathbb{Z}$,*

$$h(a + 1) = h(a)^2 + 1.$$

Proof. Note $h(a) > h(a-1) > h(a-2) > \cdots > 0$. Recursion on ω reliex on there being a starting point 0. $\mathbb{Z}$ has no analogous starting point. $\square$

Theorem 1.4.3. *Let $\langle N, S, e\rangle$ be a Peano system. Then $\langle \omega, \sigma, 0\rangle$ is isomorphic to $\langle N, S, e\rangle$, i.e. there is a funciton h mapping ω bijectively to N in a way that preserves the successor operation*

$$h(\sigma(n)) = S(h(n))$$

and the zero element

$$h(0) = e.$$

1.4.6 Arithmetic

1.4.6.1 Addition

Addition $(+)$ is the binary operation on ω such that for any m and $n \in \omega$,

$$m + n = A_m(n),$$

where $A_m : \omega \to \omega$ is the unique function given by the recursion theorem for which...

- $A_m(0) = m$
- $A_m(n^+) = A_m(n)^+ \; \forall n \in \omega$.

Proposition 1.4.5. *For natural numbers m and n,*

- $m + 0 = m$,
- $m + n^+ = (m + n)^+$

1.4.6.2 Multiplication

Multiplication $(\cdot)$ is the binary operation on ω such that for any m and $n \in \omega$,

$$m \cdot n = M_m(n),$$

where $M_m : \omega \to \omega$ is the unique function given by the recursion theorem for which...

- $M_m(0) = 0$
- $M_m(n^+) = M_m(n) + m$.

Proposition 1.4.6. *For natural numbers m and n,*

- $m \cdot 0 = 0$,
- $m \cdot n^+ = m \cdot n + m$

1.4.6.3 Exponentiation

Exponentiation is the binary operation on ω such that for any m and $n \in \omega$,

$$m^n = E_m(n),$$

where $E_m : \omega \to \omega$ is the unique function given by the recursion theorem for which...

- $E_m(0) = 1$
- $M_m(n^+) = E_m(n) \cdot m$.

Proposition 1.4.7. *For natural numbers m and n,*

- $m^0 = 1$,
- $m^{(n^+)} = m^n \cdot m$.

1.4.7 Ordering on the natural numbers

Define $m < n$ if and only if $m \in n$.

Lemma 1.4.4. *For any natural numbers m and n...*

- $m \in n \Leftrightarrow m^+ \in n^+$.
- $n \notin n$

Theorem 1.4.5 (Trichotomy Law for ω)**.** *For any natural numbers m and n, exactly one of the three conditions...*

- $m \in n$
- $m = n$
- $n \in m$

holds.

Corollary 1.4.5.1. *For any natural numbers m and n,*

- $m \in n \Leftrightarrow m \subset n$
- $(m \in n) \vee (m = n) \Leftrightarrow m \subseteq n$

Proposition 1.4.8. *For any natural numbers m, n and p,...*

- $m \in n \Leftrightarrow m + p \in n + p$.
- *If, in addition, $p \neq 0$, then $m \in n \Leftrightarrow m \cdot p \in n \cdot p$.*

Corollary 1.4.5.2. *The following canncellation laws hold for $m, n, p \in \omega$...*

- $m + p \in n + p \Rightarrow m = n$

- *If, in addition, $p \neq 0$, then $m \cdot p \in n \cdot p \Rightarrow m = n$*

Theorem 1.4.6 (Well Ordering of ω)**.** *Let A be a nonempty set of ω. Then there is some $m \in A$ such that $(m \in n) \vee (m = n)$ for all $n \in A$.*

Proof. Assume that A is a subset of ω without a least element; we will show that $A = \emptyset$. We could attempt to do this by showing that the complement $\omega \setminus A$ is inductive. But in order to show that $k^+ \in \omega - A$, it is not enough to know merely that $k \in \omega \setminus A$, we must know that all numbers smaller than k are in $\omega \setminus A$ as well. Given this additional information, we can argue that $k^+ \in \omega \setminus A$ lest it be a least element of A.

To write down what is approximately this argument, let...

$$B = \{m \in \omega | \text{ no number less than } m \text{ belongs to } A\}.$$

We claim that B is inductive. $0 \in B$ vacuously. Suppose that $k \in B$. Then if n is less that k^+, either n is less than k (in which case $n \notin A$ since $k \in B$) or $n = k$ (in which case $n \notin A$ lest, by trichotomy, it be least in A). In either case, n is outside of A. Hence $k^+ \in B$ and B is inductive. It clearly follows that $A = \emptyset$. □

Corollary 1.4.6.1. *There is no function $f : \omega \to \omega$ such that $f(n^+) \in f(n)$ for every natural number n.*

Theorem 1.4.7 (Strong Induction Principle for ω)**.** *Let A be a subset of ω, and assume the for every $n \in \omega$, if every number less than n is in A, then $n \in A$. Then $A = \omega$.*

1.5 Constructing Number Systems

For the purposes of this subsection let $\mathbb{N} := \omega$.

1.5.1 The Integers

Let $\sim_{\mathbb{Z}}$ be the equivalence relation on $\mathbb{N} \times \mathbb{N}$ for which...

$$\langle m, n \rangle \Leftrightarrow m + q = p + n.$$

Then the set of *Integers*, denoted $\mathbb{Z}$, is the set $\mathbb{N} \times \mathbb{N} / \sim_{\mathbb{Z}}$.

1.5.1.1 Addition

Addition of integers $a = \langle m, n \rangle$ and $b = \langle p, q \rangle$ is defined as...

$$a +_{\mathbb{Z}} b = [\langle m + p, n + q \rangle]$$

Lemma 1.5.1. *Addition of integers ($+_{\mathbb{Z}}$) is well defined, i.e. if $\langle m, n \rangle \sim_{\mathbb{Z}} \langle m', n' \rangle$ and $\langle p, q \rangle \sim_{\mathbb{Z}} \langle p', q' \rangle$, then...*

$$\langle m + p, n + q \rangle \sim_{\mathbb{Z}} \langle m' + p', n' + q' \rangle$$

The integers under addition form an abelian group.

1.5.1.2 Multiplication

Multiplication of integers $a = \langle m, n \rangle$ and $b = \langle p, q \rangle$ is defined as...

$$a \cdot_{\mathbb{Z}} b = [\langle mp + nq, mq + np \rangle]$$

Lemma 1.5.2. *Multiplication of integers ($\cdot_{\mathbb{Z}}$) is well defined, i.e. if $\langle m, n \rangle \sim_{\mathbb{Z}} \langle m', n' \rangle$ and $\langle p, q \rangle \sim_{\mathbb{Z}} \langle p', q' \rangle$, then...*

$$\langle mp + nq, mq + np \rangle \sim_{\mathbb{Z}} \langle m'p' + n'q', m'q' + n'p' \rangle$$

The integers under multiplication form an abelian group.

1.5.1.3 Order

Order of integers $a = \langle m, n \rangle$ and $b = \langle p, q \rangle$ is defined as...

$$a <_{\mathbb{Z}} b \Leftrightarrow m + q \in p + n$$

Lemma 1.5.3. *Order of integers ($<_{\mathbb{Z}}$) is well defined, i.e. if $\langle m, n \rangle \sim_{\mathbb{Z}} \langle m', n' \rangle$ and $\langle p, q \rangle \sim_{\mathbb{Z}} \langle p', q' \rangle$, then...*

$$m + q \in p + n \Leftrightarrow m' + q' \in p' + n'$$

The order relation so defined linearly orders the integers.

1.5.2 The Rational Numbers

Let $\sim_{\mathbb{Q}}$ be the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\})$ for which...

$$\langle a, b \rangle \sim \langle c, d \rangle \Leftrightarrow a \cdot_{\mathbb{Z}} d = c \cdot_{\mathbb{Z}} b.$$

Then the set of *Rational Numbers*, denoted $\mathbb{Q}$, is the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}/ \sim_{\mathbb{Q}}$.

1.5.2.1 Addition

Addition of rational numbers $p = \langle a, b \rangle$ and $q = \langle c, d \rangle$ is defined as...

$$p +_{\mathbb{Q}} q = [\langle ad + cb, bd \rangle]$$

Lemma 1.5.4. *Addition of rational numbers is well defined.*

The rational numbers under addition form an abelian group.

1.5.2.2 Multiplication

Multiplication of rational numbers $p = \langle a, b \rangle$ and $q = \langle c, d \rangle$ is defined as...

$$p \cdot_{\mathbb{Q}} q = [\langle ac, bd \rangle]$$

Lemma 1.5.5. *Multiplication of rational numbers is well defined.*

The rational numbers under addition and multiplication form a field.

1.5.2.3 Order

Order of rational numbers $p = \langle a, b\rangle$ and $q = \langle c, d\rangle$ is defined as...

$$p <_{\mathbb{Q}} q \Leftrightarrow ad < cb.$$

Lemma 1.5.6. *The order of rational numbers is well-defined.*

The order relation so defined linearly orders the rational numbers.

1.5.3 The Real Numbers with Cauchy Sequences

Define a *Cauchy sequence* to be a function $s : \omega \to \mathbb{Q}$ such that...

$$(\forall \varepsilon > 0)(\exists k \in \omega)(\forall m > k)(\forall n > k)|s_m - s_n| < \varepsilon.$$

Let C be the set of all Cauchy sequences. For $r, s \in C$, define $r \sim_{\mathbb{R}} s$ if and only if $|r_n - s_n|$ is arbitrarily small for large n.

With more work we can define $\mathbb{R} := C/\sim$.

1.5.4 The Real Numbers with Dedekind Cuts

A *Dedekind cut* is a subset x of $\mathbb{Q}$ such that:

1. $\emptyset \neq x \neq \mathbb{Q}$
2. x is "closed downward," i.e.,

$$q \in x \wedge r < q \Rightarrow r \in x.$$

3. x has no largest member

$\mathbb{R}$ is the set of Dedekind cuts.

1.5.4.1 Order

Define an ordering on $\mathbb{R}$ as...

$$x <_{\mathbb{R}} y \Leftrightarrow x \subset y$$

Proposition 1.5.1. $<_{\mathbb{R}}$ *is a linear ordering.*

Proof. $<_{\mathbb{R}}$ is clearly transitive; so it suffices to show that $<_{\mathbb{R}}$ satisfies trichotomy on $\mathbb{R}$. So consider $x, y \in \mathbb{R}$. Obviously *at most* one of the alternatives,

$$x \subset y, \;\; x = y, \;\; y \subset x,$$

can hold, but we must prove that at least one holds. Without loss of generality, suppose that the first two fail, i.e., that $x \not\subseteq y$.

Since $x \not\subseteq y$ there is some rational r in the relative complement $x \backslash y$. Consider any $q \in y$. If $r \subseteq q$, then since y is closed downward, we would have $r \in y$. But $r \notin y$, so we must have $q < r$. Since x is closed downward, it follows that $q \in x$. Since q was arbitrary (and $x \neq y$), we have $y \subset x$. $\square$

Theorem 1.5.7 (Least Upper Bound Property)**.** *Any bounded nonempty subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.*

Proof. Let A be a set of real numbers. The least upper bound is just $\bigcup A$. $\square$

1.5.4.2 Addition

Addition of real number x, y is defined as...

$$x +_{\mathbb{R}} y = q + r | q \in x \wedge r \in y$$

1.5.4.3 Multiplication

The *absolute value* of a real number x is defined as...

$$|x| = x \cup -x$$

Multiplication of real number x, y is defined as follows...

- If x and y are nonnegative real numbers, then...

$$x \cdot_{\mathbb{R}} y = 0_{\mathbb{R}} \cup \{rs | 0 \leq r \in x \wedge 0 \leq s \in y\}.$$

- It x and y are both negative real numbers, then...

$$x \cdot_{\mathbb{R}} y = |x| \cdot_{\mathbb{R}} |y|.$$

- If one of the real numbers x and y is negative and one is nonnegative, then...

$$x \cdot_{\mathbb{R}} y = -(|x| \cdot_{\mathbb{R}} |y|).$$

Real numbers under addition, multiplication, and their order relation form an ordered field.

1.6 Cardinality

1.6.1 Equinumerosity

Two sets A and B are *equinumerous*, denoted $A \approx B$, if and only if there is a bijection $f : A \to B$.

Proposition 1.6.1. *Equinumerosity is an equivalence relation. (See: isomorphism)*

Theorem 1.6.1 (Diagonalization)**.** *The set ω is not equinumerous to the set $\mathbb{R}$ of real numbers.*

Proof. Suppose for the sake of contradition that there is a bijection $f : \omega \to \mathbb{R}$. Thus we can imagine a list of successive values...

$$f(0) = 236.001\ldots$$

$$f(1) = -7.777\ldots$$

$$f(2) = 3.1415\ldots$$

$$\vdots$$

Then consider the real number $0.a_1 a_2 a_3 \ldots$ where:

$$a_n = \begin{cases} 7 & \text{if the nth decimal of } f(n) \neq 7 \\ 6 & \text{otherwise.} \end{cases}$$

This number cannot be in the range of f, so it is not a bijection. ↯

Theorem 1.6.2 (Diagonalization)**.** *No set is equinumerous to its power set.*

Proof. Let $g : A \to \mathcal{P}(A)$. Consider...

$$B = \{x \in A | x \notin g(x)\}.$$

Then $B \subseteq A$, but for each $x \in A$,

$$x \in B \Leftrightarrow x \notin g(x).$$

Hence $B \notin \text{ran } g$ and g is not a bijection. □

1.6.2 Finite/Infinite

A set is *finite* if and only if it is equinumerous to some natural number. Otherwise it is *infinite*.

Theorem 1.6.3 (Pigeonhole Principle)**.** *No natural number is equinumerous to a proper subset of itself.*

Proof. Suppose $f : N \to N$ is a bijection from a finite set to itself. We will show that ran f is all of the set n. This suffices to prove the theorem.

We use the induction on n. Define:

$$T = \{n \in \omega | \text{ every injection from } n \text{ into } n \text{ has range } n\}$$

We have that $0 \in T$; the only function from the set 0 into the set 0 is the empty function, which has range 0. Now suppose that $k \in T$ and that f is an injection from k^+ into $k+$. Note that the restriction $f\restriction_k$ maps k injectively into k^+. There are two cases...

Case I: The set k is closed under f. Then $f\restriction_k$ maps the set k into the set k. Then because $k \in T$ we may conclude that ran $(f\restriction_k) = k$. Since f is injective, the only possible value for $f(k)$ is the number k. Hence ran f is $k \cup \{k\}$, which is the set k^+.

Case II: Otherwise $f(p) = k$ for some number p less than k. In this case we interchange two values of teh functino. Define $\hat{f}$ by...

$$\hat{f}(p) = f(k),$$

$$\hat{f}(k) = f(p) = k,$$

$$\hat{f}(x) = f(x) \text{ for other } x \in k^+.$$

The $\hat{f}$ maps the set k^+ injectively into the set k^+, and the set k is closed under $\hat{f}$. So we can apply Case I.

Thus ran $f = k^+$. □

Corollary 1.6.3.1. *No finite set is equinumerous to a proper subset of itself.*

Corollary 1.6.3.2. *Any set equinumerous to a proper subset of itself is infinite.*

Corollary 1.6.3.3. *The set ω is infinite.*

Corollary 1.6.3.4. *Any finite set is equinumerous to a unique natural number.*

Lemma 1.6.4. *If C is a proper subset of a natural number n, the $C \approx m$ for some m less than n.*

Corollary 1.6.4.1. *Any subset of a finite set if finite.*

1.6.3 Cardinal Numbers

For any set A, the cardinal number of A, denoted card A, is a set...

1. For any sets A, B...

$$\text{card } A = \text{card } B \Leftrightarrow A \approx B.$$

2. For a finite set A, card A is the natural number n for which $A \approx n$.

(See: cardinal number definition using ordinals)

1.6.3.1 Cardinal Arithmetic

Let κ and λ be any cardinal numbers.

- $\kappa + \lambda = \text{card}(K \cup L)$, where K and L are any disjoint sets of cardinality κ and λ, respectively.
- $\kappa \cdot \lambda = \text{card}(K \times L)$, where K and L are any sets of cardinality κ and λ, respectively.
- $\kappa^\lambda = \text{card}^L K$, where K and L are any sets of cardinality κ and λ, respectively.

Proposition 1.6.2. *Assume that* $K_1 \approx K_2$ *and* $L_1 \approx L_2$*.*

1. *If* $K_1 \cap L_1 = K_2 \cap L_2 = \emptyset$*, then* $K_1 \cup L_1 \approx K_2 \cup L_2$*.*
2. $K_1 \times L_1 \approx K_2 \times L_2$*.*
3. ${}^{L_1}K_1 \approx^{L_2} K_2$*.*

Proposition 1.6.3. *For any cardinal numbers* κ, λ*, and* μ*...*

- $\kappa + \lambda = \lambda + \kappa$ *and* $\kappa \cdot \lambda = \lambda \cdot \kappa$*.*
- $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ *and* $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$*.*
- $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$*.*
- $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$*.*
- $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$*.*
- $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$

Proposition 1.6.4. *Let* m *and* n *be finite cardinals. Then...*

- $m + n = m +_{\omega} n$
- $m \cdot n = m \cdot_{\omega} n$
- $m^n = m^n$

(See: natural number arithmetic.)

Corollary 1.6.4.2. *If* A *and* B *are finite, then* $A \cup B$*,* $A \times B$*, and* ${}^{B}A$ *are also finite.*

1.6.3.2 Ordering Cardinal Numbers

A set A is *dominated* by a set B (written $A \preceq B$) if and only if there is an injective function from A into B.

Theorem 1.6.5 (Schröder-Bernstein Theorem)**.** *If* $A \preceq B$ *and* $B \preceq A$*, then* $A \approx B$*.*

Proof. The proof is accomplished with mirrors. Given injections $f : A \to B$ and $g : B \to A$. Define C_n by recursion, using the formulas

$$C_0 = A \setminus \operatorname{ran} g \quad \text{and} \quad C_{n^+} = g[f[C_n]].$$

Thus C_0 is the troublesome part that keeps g from being a bijection. We bounce it back and forth, obtaining $C_1, C_2, \ldots$. This function showing that $A \approx B$ is the function $h : A \to B$ defined by...

$$h(x) = \begin{cases} f(x) & \text{if } x \in C_n \text{ for some } n, \\ g^{-1}(x) & \text{otherwise.} \end{cases}$$

Note that in the second case ($x \in A$ but $x \notin C_n$ for any n) it follows that $x \notin C_0$ and hence $x \in \text{ran } g$. So $g^{-1}(x)$ makes sense in this case. We verify that h is indeed a bijection. Define $D_n = f[C_n]$, so that $C_{n^+} = g[D_n]$. Consider distinct $x, y \in A$. Since both f abd g^{-1} are injective, the only possible problem arises when, say, $x \in C_m$ and $y \in \bigcup_{n\in\omega} C_n$. In this case,

$$h(x) = f(x) \in D_m,$$

whereas,

$$h(y) = g^{-1}(y) \notin D_m,$$

lest $y \in C_{m^+}$. So $h(x) \neq h(x')$, showing h is injective.

Finally, we show h is surjective. Certainly each $D_n \subseteq \text{ran } h$, because $D_n = h[C_n]$. Consider then a point y in $B \setminus \bigcup_{n\in\omega} D_n$. Where is $g(y)$? Certainly $g(y) \notin C_0$. Also $g(y) \notin C_{n^+}$, because $C_{n^+} = g[D_n]$, $y \notin D_n$, and g is injective. So $g(y) \notin C_n$ for any n. Therefore $h(g(y)) = g^{-1}(g(y)) = y$. This shows that $y \in \text{ran } h$, thereby proving part (a). □

Theorem 1.6.6 (Restated Schröder-Bernstein Theorem)**.** *For cardinal numbers κ and λ, if $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.*

Proposition 1.6.5. *Let κ, λ and μ be cardinal numbers.*

- $\kappa \leq \lambda \Rightarrow \kappa + \mu \leq \lambda + \mu$
- $\kappa \leq \lambda \Rightarrow \kappa \cdot \mu \leq \lambda \cdot \mu$
- $\kappa \leq \lambda \Rightarrow \kappa^\mu \leq \lambda^\mu$
- $\kappa \leq \lambda \Rightarrow \mu^\kappa \leq \mu^\lambda$*; if not both κ and μ equal zero.*

1.6.3.3 Infinite Cardinal Arithmetic

Lemma 1.6.7. *For any infinite cardinal κ, we have $\kappa \cdot \kappa = \kappa$.*

Theorem 1.6.8 (Absorption Law of Cardinal Arithmetic)**.** *Let κ and λ be cardinal numbers, the larger of which is infinite and the smaller of which is nonzero. Then...*

$$\kappa + \lambda = \kappa \cdot \lambda = max(\kappa, \lambda).$$

1.7 Countable Sets

A set A is *countable* if and only if $A \preceq \omega$, i.e. if and only if card $A \leq \aleph_0$.

Theorem 1.7.1. *A countable union of countable sets is countable.*

Proof. We may suppose that $\notin \mathcal{A}$, for otherwise we could simply remove it without affecting $\bigcup \mathcal{A}$. We may further suppose that $\mathcal{A} \neq \emptyset$, since $\bigcup \emptyset$ is certainly countable. Thus $\mathcal{A}$ is a countable (but nonempty) function from $\omega \times \omega$ onto $\bigcup \mathcal{A}$. It is easy to find a function from ω onto $\omega \times \omega$, and the composition will map ω onto $\bigcup \mathcal{A}$, thereby showing that $\bigcup \mathcal{A}$ is countable. Since $\mathcal{A}$ is countable but nonempty, there is a function G from ω onto $\mathcal{A}$. We are given that each set $G(m)$ is countable and nonempty. Hence for each m there is a function from ω onto $G(m)$. We must then use the axiom of choice to select such a function for each m. Let $H : \omega \to^{\omega} (\bigcup \mathcal{A})$ be defined by...

$$H(m) = \{g | g \text{ is a function from } \omega \text{ onto } G(m)\}.$$

We know that $H(m)$ is nonempty for each m. Hence there is function F with domain ω such that for each m, $F(m)$ is a function from ω ontop $G(m)$. To conclude the proof we have only to let $f(m, n) = F(m)(n)$. Then f is a function from $\omega \times \omega$ onto $\bigcup \mathcal{A}$. □

1.8 Axiom of Choice

(See: set axioms)

Theorem 1.8.1 (Axiom of Choice)**.** *The following statements are equivalent.*

1. *For any relation R, there is a function $F \subseteq R$ with dom $F =$ dom R.*
2. *The Cartesian product of nonempty sets is always nonempty. That is, if H is a function with domain I and if $(\forall i \in I) H(i) \neq \emptyset$, then there is a function f with domain I such that $(\forall i \in I) f(i) \in H(i)$.*
3. *For any set A there is a function F (a "choice function" for A) such that $F(B) \in B$ for every nonempty $B \subseteq A$.*
4. *Let $\mathcal{A}$ be a set such that (a) each member of $\mathcal{A}$ is a nonempty set, and (b) any two distinct members of $\mathcal{A}$ are disjoint. Then there exists a set C containing exactly one element from each member of $\mathcal{A}$ (i.e., for each $B \in \mathcal{A}$ the set $C \cap B$ is a singleton $\{x\}$ for some x).*

There are other theorems that are equivalent to the axiom of choice.

Theorem 1.8.2 (Cardinal Comparability)**.** *For any sets C and D, either $C \preceq D$ or $D \preceq C$. For any two cardinal numbers κ and λ, either $\kappa \leq \lambda$ or $\lambda \leq \kappa$.*

Theorem 1.8.3 (Zorn's Lemma)**.** *Let $\mathcal{A}$ be a set such that for every chain $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$. ($\mathcal{B}$ is called a* chain *if and only if for any C and D in $\mathcal{B}$, either $C \subseteq D$ or $D \subseteq C$.) Then $\mathcal{A}$ contains an element M (a "maximal" element) such that M is not a subset of any other set in $\mathcal{A}$.*

1.9 Continuum Hypothesis

Proposition 1.9.1. *For any infinite set A, we have $\omega \preceq A$.*

Proposition 1.9.2. *$\aleph_0 \leq \kappa$ for any infinite cardinal κ.*

Corollary 1.9.0.1. *A set is infinite if and only if it is equinumerous to a proper subset of itself.*

The *continuum hypothesis* is:

$$\text{There is no set } \mathcal{S} \text{ such that } \aleph_0 \prec \text{card } \mathcal{S} \prec 2^{\aleph_0}.$$

1.10 Ordinal Numbers

1.10.1 Partial Orderings

A *partial ordering* is a relation R such that...

1. R is transitive
2. R is irreflexive, that is for all x we have $x \not R x$

Proposition 1.10.1. *Assume that $<$ is a partial ordering. Then for x, y, and z:*

1. At most *one of the alternatives,*
$$x < y, \quad x = y, \quad y < x,$$
can hold.
2. $x \leq y \leq x \Rightarrow x = y$.

1.10.2 Linear Orderings

A *linear ordering* is a partial ordering R that satisfies trichotomy.

1.10.3 Well Orderings

A *well ordering* is a linear ordering R on A such that every nonempty subset of A has a least element.

Theorem 1.10.1. *Let $<$ be a linear ordering on A. Then if is a well ordering if and only if there does not exist any function $f : \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$.*

Theorem 1.10.2 (Transifinite Induction Principle)**.** *Assume that $<$ is a well ordering on A. Assume that B is a subset of A with the special property that for every $t \in A$,*
$$seg\ t \subseteq B \Rightarrow t \in B.$$
Then B coincides with A.

Proof. If $B \subset A$, then $A \setminus B$ has a least element m. Bu the leastness, $y \in B$ for any $y < m$. But this is to say that seg $m \subseteq B$, so by assumption $m \in B$ after all. $\square$

Proposition 1.10.2. *Assume that $<$ is a linear ordering on A. Further assume that the only subset of A such that $\forall\, t \in A,\ seg\ t \subseteq B \Rightarrow t \in B$ is A itself. Then $<$ is a well ordering on A.*

1.10.4 Transfinite Recursion

Theorem 1.10.3 (Transfinite Recursion Theorem Schema)**.** *For any formula $\gamma(x, y)$ the following is a theorem:*

Assume that $<$ is a well ordering on a set A. Assume that for any f there is a unique y such that $\gamma(f, y)$. Then there exists a unique function F with domain A such that...

$$\gamma(F \restriction \ seg\ t,\ F(t))$$

for all $t \in A$.

The following axiom is used to prove the transfinite recursion theorem schema.

For any formula $\varphi(x, y)$ not containing the letter B, the following is an axiom:

$$\forall[(\forall x \in A)\forall y_1 \forall y_2(\varphi(x, y_1) \wedge \varphi(x, y_2) \Rightarrow y_1 = y_2)$$
$$\Rightarrow \exists B \forall y(y \in B \Leftrightarrow (\exists x \in A)\varphi(x, y))].$$

1.10.5 Epsilon Images

Let $<$ be a well ordering on A and let $\gamma(x, y)$ be the formulat $y = \text{ ran } x$. Then the transfinite recursion theorem gives an unique function E with domain A such that $\forall t \in A$:

$$\begin{aligned} E(t) &= \text{ ran } (E \restriction \text{ seg } t) \\ &= E[\text{seg } t] \\ &= \{E(x) | x < t\}. \end{aligned}$$

The ϵ-image of $\langle A, < \rangle$ is the range of E.

Proposition 1.10.3. *Let $<$ be a well ordering on A and let E be as above and α its epsilon image.*

1. *$E(t) \notin E(t)$ for any $t \in A$.*
2. *E maps A bijectively to α.*
3. *For any s and t in A,*

   $$s < t \textit{ if and only if } E(s) \in E(t)$$

4. *α is a transitive set.*

1.10.6 Ordinal Numbers

Proposition 1.10.4. *Two well-ordered structures are isomorphic if and only if they have the same ϵ-image. That is, if $<_1$ and $<_2$ are well orderings on A_1 and A_2, respectively, then $\langle A_1, <_1 \rangle \cong \langle A_2, <_2 \rangle$ if and only if the ϵ-image of $\langle A_1, <_1 \rangle$ is the same as the ϵ-image of $\langle A_2, <_2 \rangle$.*

The *ordinal number* of $\langle A, < \rangle$ is its ϵ-image. An *ordinal number* is a set that is the ordinal number of some well-ordered structure.

1.10.7 Cardinal Numbers

Theorem 1.10.4 (Numeration Theorem)**.** *Any set is equinumerous to some ordinal number.*

For any set A, define the cardinal number of A (card A) to be the least ordinal equinumerous to A.

2 Combinatorics

2.1 Basic Methods

Use Cardinality to derive the most basic results.

2.1.1 Addition

Theorem 2.1.1 (Addition principle)**.** *If A and B are two disjoint finite sets, then...*

$$|A \cup B| = |A| + |B|.$$

Theorem 2.1.2 (Generalized addition principle)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets that are pairwise disjoint. Then...*

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$$

2.1.2 Subtraction

Theorem 2.1.3 (Subtraction principle)**.** *Let A be a finite set, and let $B \subseteq A$. Then $|A \setminus B| = |A| - |B|$.*

Proof. Observe $|A \setminus B| + |B| = |A|$ by the addition principle. □

2.1.3 Multiplication

Theorem 2.1.4 (Product principle)**.** *Let X and Y be two finite sets. Then $|X \times Y| = |X| \times |Y|$.*

Theorem 2.1.5 (Generalized product principle)**.** *Let $X_1, X_2, \ldots, X_n$ be finite sets. Then $|\bigtimes_{i \in I}^{n} X_i| = \prod_{i \in I}^{n} |X_i|$.*

2.1.4 Division

Theorem 2.1.6. *Let S and T be finite sets so that a d-to-one function $f : T \to S$ exists. Then*

$$|S| = \frac{|T|}{d}.$$

2.1.5 Binomial Coefficients

See permutations.

Theorem 2.1.7. *Let n be a positive integer, and let $k \leq n$ be a nonnegative integer. Then the number of all k-element subsets of $[n]$ is*

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Note: $\binom{n}{k} = \binom{n}{n-k}$ exhibits duality.

Theorem 2.1.8 (Binomial theorem)**.** *If n is a positive integer, then...*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. The left-hand side of the equation contains the factor $(x+y)$ n times. To compute the product we choose an x or y term from each factor and multiply those n terms together, then do this in all 2^n possible ways, adding all the resulting products. It suffices to show that there are exactly $\binom{n}{k}$ products of the form $x^k y^{n-k}$, which is immediately obvious from the way we compute the product. □

Theorem 2.1.9. *Let n and k be nonnegative integers so that $k < n$. Then...*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Theorem 2.1.10. *For all positive integers n,*

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

2.1.6 Pigeonhole Principle

Theorem 2.1.11 (Pigeonhole Principle)**.** *Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint. Let us assume that*

$$|A_1 \cup A_2 \cup \cdots \cup A_k| > kr.$$

Then there exists at least one index i so that $|A_i| > r$. (See: Pigeonhole Priciple in Set Theory)

Example 2.1.11.1. *Consider the sequence $1, 3, 7, 15, 31, \ldots,$ in other words, the sequence whose ith element is $a_i = 2^i - 1$. Let q be any odd integer. Then our sequence contains an element that is divisible by q.*

Proof. Consider the first q elements of our sequence. If one of them is divisible by q, then we are done. If not, then consider their remainders modulo q. That is, let us write...

$$a_i = d_i q + r_i$$

where $0 < r_i < q$, and $d_i = \lfloor a_i/q \rfloor$. As the integers $r_1, r_2, \ldots, r_q$ all come from the open interval $(0, q)$, there are $q-1$ possibilities for their values. On the other hand, their number is q, so, by the pigeonhole principle, there have to be two of them that are equal. Say these are r_n and r_m, with $n > m$. Then $a_n = d_n q + r_n$ and $a_m = d_m q + r_n$, so...

$$a_n - a_m = (d_n - d_m)q$$

or, after rearranging,

$$\begin{aligned}(d_n - d_m)q &= a_n - a_m \\ &= (2^n - 1) - (2^m - 1) \\ &= 2^m(2^{n-m} - 1) \\ &= 2^m a_{n-m}\end{aligned}$$

As the first expression of our chain of equations is divisible by q, so too must be the last expression. Note that 2^{n-m} is relatively prime to any odd number q, that is, the largest common divisor of 2^{n-m} and q is 1. Therefore, the equality $(d_n - d_m)q = 2^{n-m}a_{n-m}$ implies that a_{n-m} is divisible by q. □

2.2 Applications of Basic Methods

2.2.1 Inclusion-Exclusion

Theorem 2.2.1 (Inclusin-exclusion principle). *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then...*

$$|A_1 \cup A_2 \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j},$$

where $(i_1, i_2, \ldots, i_j)$ ranges all j-element subsets of $[n]$.

Proof. We prove the two follwoing claims:

1. If x is contained in the set represented on the left side of the equation, then the right side conts it exactly once.

2. If x is not contained in any A_i, then the right-hand side counts x zero times.

(1) Assume that x is contained in exactly k of the n A_i-sets, with $k > 0$. Certainly, x is not in any j-fold intersection where $j > k$. On the otherhand

$j \leq k$, then x is contained in exactly $\binom{k}{j}$ different j-fold intersections. If we take the signs into accoount, this means that the right side counts x exactly...

$$m = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j}$$

times. Now we show that $m = 1$ necessarily. Observe...

$$1 - m = \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} = (1-1)^k = 0,$$

since k is positive.

(2) We repeat the above argument with $k = 0$. Then the binomial theorem technique we use above gives us $(1-1)^0 = 1$, implying $m = 0$.

Thus the left-hand side and the right-hand side count the same objects. □

2.2.2 Multisets

Given a set A, a *multiset* is defined via a function $m : A \to \mathbb{N} \cup \{0\}$. It is a set containing $a \in A$ $m(a)$ many times.

2.2.2.1 Multinomial Coefficients

Theorem 2.2.2. *Given a multiset A of n elements over a k element sets. The number of ways to linearly order the elements of A is...*

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1! a_2! \ldots a_k!}.$$

2.2.3 Weak Compositions

Let $a_1, a_2, \ldots, a_k$ be nonnegative integers satisfying

$$\sum_{i=1}^{k} a_i = n.$$

Then the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ is called a *weak composition* of n into k parts.

Theorem 2.2.3. *The number of weak compositions of n into k parts is...*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Corollary 2.2.3.1. *The number of n-element multisets over a k-elemnt set is...*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

2.2.4 Compositions

Let $a_1, a_2, \ldots, a_k$ be positive integers satisfying

$$\sum_{i=1}^{k} a_i = n.$$

Then the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ is called a *composition* of n into k parts.

Corollary 2.2.3.2. *The number of compositions of n into k parts is...*

$$\binom{n-1}{k-1}.$$

2.2.5 Stirling numbers of the second kind

Given a finite set A, $|A| = n$, the number of set partitions of A into $0 < k \leq n$ classes is denoted $S(n, k)$, the *Stirling number of the second kind.*

Theorem 2.2.4. *For all positive integers n and k satisfying $n \leq k$, the equality...*

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Theorem 2.2.5. *For all positive integers n and k satisfying $n \geq k$.*

$$S(n+1, k) = \sum_{i=0}^{n} \binom{n}{i} S(n-i, k-1)$$

Theorem 2.2.6. *The number of surjections from $[n]$ to $[k]$ is equal to*

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

Corollary 2.2.6.1. *For all positive integers $k \leq n$,*

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

2.2.5.1 Bell numbers

The number of all partitions of a finite set A, where $|A| = n$, is denoted $B(n)$ and is called a *Bell number.*

Theorem 2.2.7. *Set $B(0) = 1$. Then, for all positive integers n,*

$$B(n+1) = \sum_{k=0}^{n} B(k) \binom{n}{k}.$$

2.2.6 Partitions of integers

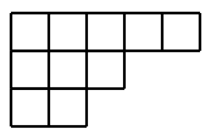
A *partition of an integer n* is a finite sequence $(a_1, a_2, \ldots, a_k)$ of positive integers satisfying $a_1 \geq a_2 \geq \cdots \geq a_k$ and $a_1 + a_2 + \cdot + a_k = n$.

Theorem 2.2.8. *As $n \to \infty$, the function $p(n)$ satisfies...*

$$p(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

2.2.7 Ferrers shapes

The *Ferrers shape* of the partition $(a_1, a_2, \ldots, a_k)$ is a row diagram of squares, with non-increasing amounts of squares in lower rows. For example the Ferrers shape fo $(5, 3, 2)$ is...



Proposition 2.2.1. *For all positive integers $k \leq n$, the number of partitions of n that have at least k parts is equal to the number of partitions of n in which the largest part is at least k.*

Proposition 2.2.2. *For every positive integer n, the number of partitions of n in which the first two parts are equal is equal to the number of partitions of n in which each part is at least 2.*

Lemma 2.2.9. *Let $m > k \geq 1$. Let S be the set of partitions of n into m parts, the smallest of which is equal to k, and let T be the set of partitions of n into $m - 1$ parts, in which the kth part is larger than the $(k+1)$st part and the smallest part is at least k. Then $|S| = |T|$.*

2.2.8 Euler's totient function

For any positive integer n, let $\phi(n)$ denote the number of positive integers $k \leq n$ that are relatively prime to n.

Proposition 2.2.3. *Let $n = pq$, where p and q are distinct prinmes. Then $\phi(n) = (p-1)(q-1)$.*

Proof. Use the inclusion-exclusion principle on $[pq]$, followed by the subtraction principle. □

Proof. Let $n = p_1 p_2 \ldots p_t$, where the p_i are pairwise distinct primes. Then...

$$\phi(n) = \prod_{i=1}^{t} (p_i - 1).$$

□

Lemma 2.2.10. *Let a and b be two positive integers whose greates common divisor is 1, and let $n = ab$. Then $\phi(n) = \phi(a)\phi(b)$.*

Proposition 2.2.4. *For any prime p, and any positive integer d,*

$$\phi(p^d) = (p-1)p^{d-1}.$$

Proposition 2.2.5. *Let $n = p_1^{d_1} p_2^{d_2} \dots p_t^{d_t}$, where the p_i are distinct primes. Then…*

$$\phi(n) = \prod_{i=1}^{t} p_i^{d_i - 1}(p_i - 1)$$

2.3 Permutations

Given a set A, a *permutation* of A is a bijection $f : A \to A$.

Proposition 2.3.1. *Given a finite set A, if $n = |A|$ the number of permutations of A is $n!$.*

Intuitively permutations represent the reordering of an ordered list. Looking at the idea of "sub-orderings" of lists we come up with the following proposition…

Proposition 2.3.2 (k-lists)**.** *Let n and k be positive integers so that $n \geq k$. Then the number of injections $f : [k] \to [n]$ is…*

$$(n)_k := n(n-1)(n-2)\cdots(n-k+1).$$

2.4 Twelvefold Way

There are 12 fundamental counting problems. Sometimes they are formulated in terms of putting *balls* into *baskets*.

Let N and K be finite sets and n and k be their cardinality respectively…

2.4.1 Functions from K to N

Count with sequences of k elements in N, $|{}^K N|$.

2.4.2 Injections from K to N

Count with k-lists, $(n)_k$.

2.4.3 Surjections from K to N

Count with the number of surjections from $[k]$ to $[n]$, $\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k$.

2.4.4 Injections from K to N, up to a permutation of K

Count subsets, k-lists without order, $\binom{n}{k}$.

2.4.5 Functions from K to N, up to a permutation of K

Count multisets with k elements from N, $\binom{n+k-1}{k}$.

2.4.6 Surjections from K to N, up to a permutation of K

Count compositions of k into n parts, $\binom{k-1}{n-1}$.

2.4.7 Injections from K to N, up to a permutation of N

Provided $k \leq n$, there is only 1 of these.

2.4.8 Surjections from K to N, up to a permutation of N

Count partitions of K into n non-empty subsets, $S(k,n)$.

2.4.9 Functions from K to N, up to a permutation of N

Count all the partitions of K up to n classes, $\sum_{i=0}^{n}\binom{k}{i}$. If $k \leq n$, $B(k)$.

2.4.10 Functions from K to N, up to a permutation of K and N

Count partitions of k into $\leq n$ non-empty subsets, $\sum_{i=0}^{n} p_i(k)$.

2.4.11 Injections from K to N, up to a permutation of K and N

Provided $k \leq n$, there is only 1 of these.

2.4.12 Surjections from K to N, up to a permutation of K and N

Count partitions of k into n non-empty subsets, $p_n(k)$.

2.5 Graphs

3 Category Theory

3.1 Metacategories

3.1.1 Undefined notions

- *Objects:* $a, b, c \ldots$
- *Arrows:* $f, g, h \ldots$

3.1.2 Operations

Given $f : a \to b \ldots$

- *Domain:* **dom**: arrows $\to$ objects, $f \mapsto a$
- *Codomain:* **cod**: arrows $\to$ objects, $f \mapsto b$
- *Identity:* **id**: objects $\to$ arrows, $a \mapsto \mathrm{id}_a = 1_a$
- *Composition:* **comp**: arrows $\times :$ arrows $\to$ arrows, $\langle g, f\rangle \mapsto g \circ f$, $g \circ f : \mathrm{dom} f \to \mathrm{cod} g$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & c \end{tikzcd}$$

3.1.3 Axioms

- *Associativity:* $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$, $k \circ (g \circ f) = (k \circ g) \circ f$

$$\begin{tikzcd} a \arrow[r, "k\circ(g\circ f)=(k\circ g)\circ f"] \arrow[d, "f"] \arrow[dr, "g\circ f"] & d \\ b \arrow[r, "k\circ g"] \arrow[ur, "k\circ g"] & c \arrow[u, "k"] \end{tikzcd}$$

- *Unit Law:* $1_a \circ f = f$ and $g \circ 1_b = g$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"] & b \arrow[d, "1_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"] & c \end{tikzcd}$$

3.2 Categories

3.2.1 Directed Graph

- A - a set of arrows
- O - a set of objects
- **dom** : $A \to O$, **cod** : $A \to O$

Set of composable pairs of arrows:

$$A \times_O A = \{\langle g, f\rangle | g, f \in A \text{ and } \mathbf{dom}(g) = \mathbf{cod}(f)\}$$

3.2.2 Categories

Add the following structure to a directed graph...

- $O \xrightarrow{id} A$, $c \mapsto id_C$
- $A \times_O A \xrightarrow{\circ} A$, $\langle g, f\rangle \mapsto g \circ f$

which satisfy $\forall a \in O$ and $\forall \langle g, f\rangle \in A \times_O A$...

- $\mathbf{dom}(\mathbf{id}(a)) = a = \mathbf{cod}(\mathbf{id}(a))$
- $\mathbf{dom}(g \circ f) = \mathbf{dom}(f)$
- $\mathbf{cod}(g \circ f) = \mathbf{cod}(g)$
- metacategorical axioms

Small categories use small sets for their objects.

3.2.3 Hom Sets

$hom(b, c) = \{f | f \in C, \mathbf{dom}(f) = b, \mathbf{cod}(f) = c\}$

3.2.4 Groupoids

A category in which every arrow is an isomorphism.

3.3 Morphisms

Arrows in categories.

3.3.1 Isomorphisms

A morphism $f \in hom(b, c)$ that has a two-sided inverse $g \in hom(c, b)$ under composition such that

$$gf = 1_b, \;\; fg = 1_c.$$

Proposition 3.3.1. *The inverse of an isomorphism is unique.*

Proof. For inverses g_1, g_2 of f observe...

$$g_1 = g_1 1_c = g_1(fg_2) = (g_1 f)g_2 = 1_b g_2 = g_2$$

□

Proposition 3.3.2. *Supposing f^{-1} is the inverse of f...*

- *Each identity 1_c is an isomorphism and is its own inverse.*
- *If f is an isomorphism, then f^{-1} is an isomorphism and further $(f^{-1})^{-1} = f$.*
- *If $f \in hom(a, b)$, $g \in hom(b, c)$ are isomorphisms, then the composition gf is an isomorphism and $(gf)^{-1} = f^{-1}g^{-1}$.*

3.3.2 Automorphisms

An isomorphism of an object to itself. Denoted:

$$hom(c, c) = aut(c)$$

Observe $aut(c)$ is a group.

3.3.3 Monomorphisms

A morphism $f \in hom(b, c)$ such that $\forall z \in C$ and $\forall \alpha', \alpha'' \in hom(z, b)$:

$$f \circ \alpha' = f \circ \alpha'' \Rightarrow \alpha' = \alpha''$$

3.3.4 Epimorphisms

A morphism $f \in hom(b, c)$ such that $\forall z \in C$ and $\forall \beta', \beta'' \in hom(b, z)$:

$$\beta' \circ f = \beta'' \circ f \Rightarrow \beta' = \beta''$$

3.4 Functors

Morphisms $T : C \to B$ with domain and codomain both categories. It consists of two suitably related functions

- object function T, $c \mapsto Tc$
- arrow function T, $f : c \to c' \mapsto Tf : Tc \to Tc'$

which satisfy...

- $T(1_c) = 1_c$
- $T(g \circ f) = T_g \circ T_f$

3.4.1 Full

$\forall c, c' \in C$ and $g : Tc \to Tc' \in B, \exists f : c \to c' \in C$ s.t. $g \in Tf$

3.4.2 Faithful

$\forall c, c' \in C$ and $f_1, f_2 : c \to c', Tf_1 = Tf_2 \Rightarrow f_1 = f_2$

3.5 Duality

4 Group Theory

5 Ring Theory

6 Modules

7 Homology

8 Topology

9 Homotopy