Path Planning Optimization

Course 4, Module 7, Lesson 2

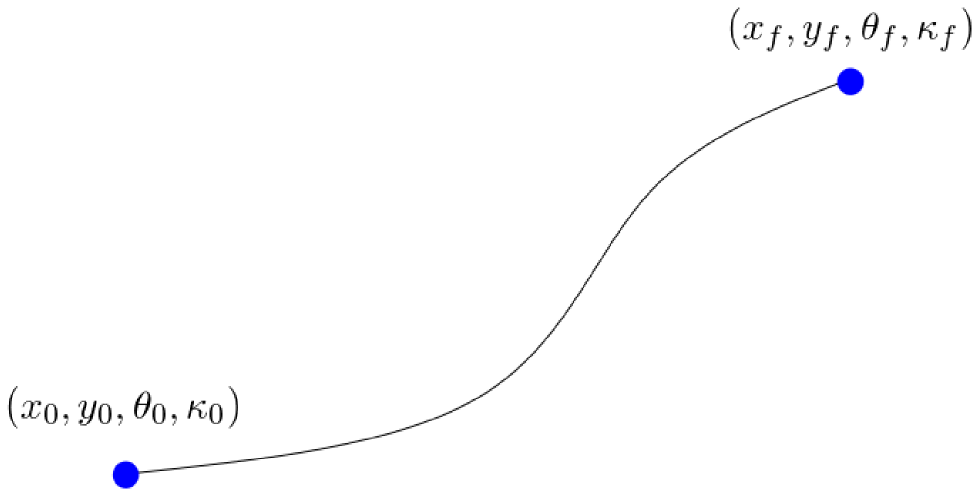
UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Learning Objectives

- Identify required boundary conditions and constraints for spiral path planning
- Know how to approximate the constraints to improve optimization tractability
- Know how to re-map parameters to improve optimization convergence speed

Cubic Spiral and Boundary Conditions

- Boundary conditions specify starting state and required ending state
- Spiral end position lacks closed form solution, requires numerical approximation



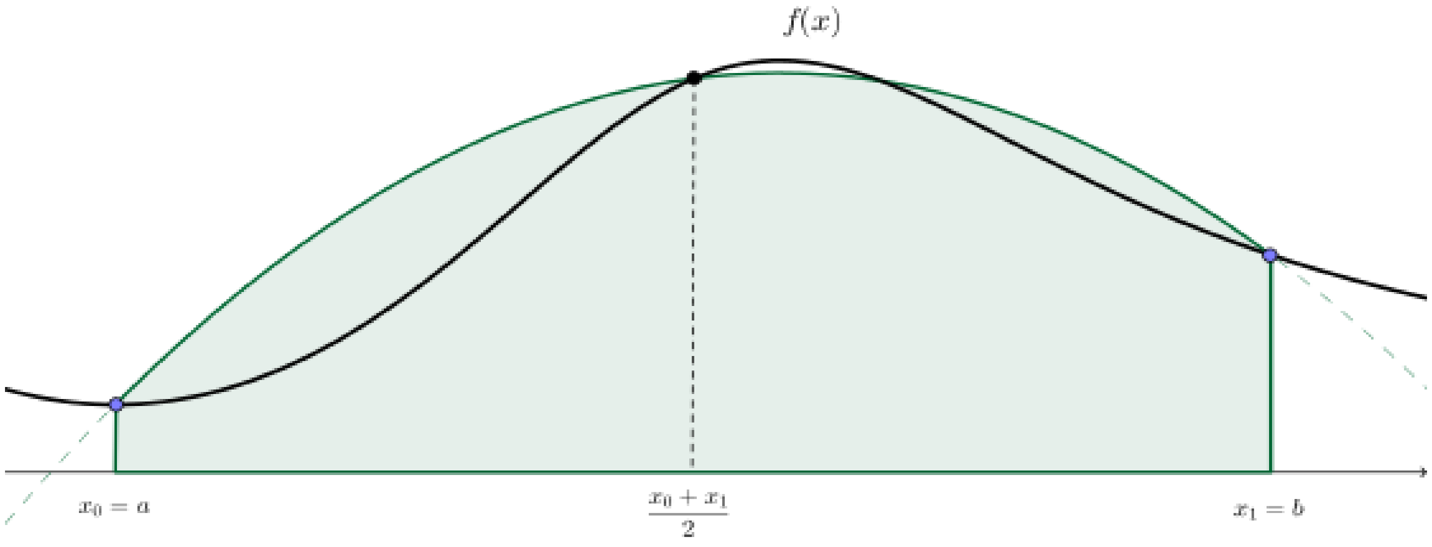
$$\kappa(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$x(s) = x_0 + \int_0^s \cos(\theta(s'))ds'$$

$$y(s) = y_0 + \int_0^s \sin(\theta(s'))ds'$$

Position Integrals and Simpson's Rule

- Simpson's rule has improved accuracy over other methods
- Divides the integration interval into n regions, and evaluates the function at each region boundary



$$\int_0^s f(s')ds' \approx \frac{s}{3n}\left(f(0) + 4f\left(\frac{s}{n}\right) + 2f\left(\frac{2s}{n}\right) + \cdots + f(s)\right)$$

Applying Simpson's Rule

- Applying Simpson's rule with $n = 8$
- $\theta(s)$ has a closed form solution
- Substituting our integrand for $x(s)$ and $y(s)$ into Simpson's rule gives us our approximations $x_S(s)$ and $y_S(s)$

$$\kappa(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$\theta(s) = \theta_0 + \int_0^s a_3 s'^3 + a_2 s'^2 + a_1 s' + a_0 ds'$$

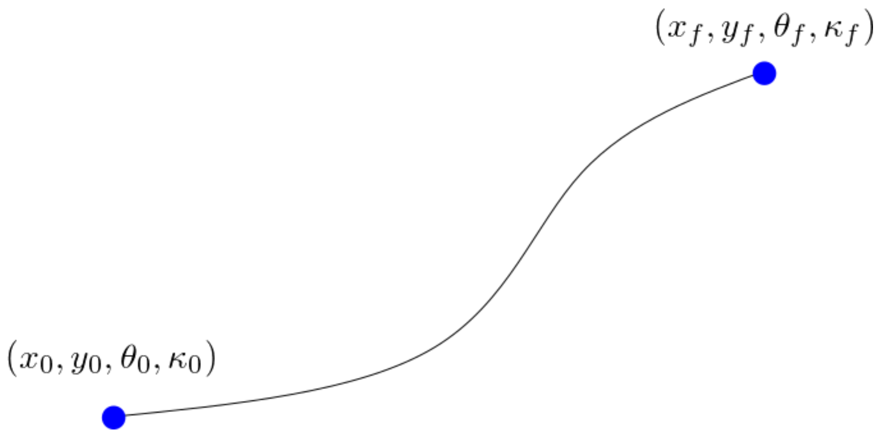
$$= \theta_0 + a_3 \frac{s^4}{4} + a_2 \frac{s^3}{3} + a_1 \frac{s^2}{2} + a_0 s$$

$$x_S(s) = x_0 + \frac{s}{24}\left[\cos(\theta(0)) + 4\cos\left(\theta\left(\frac{s}{8}\right)\right) + 2\cos\left(\theta\left(\frac{2s}{8}\right)\right) + 4\cos\left(\theta\left(\frac{3s}{8}\right)\right) + 2\cos\left(\theta\left(\frac{4s}{8}\right)\right) + 4\cos\left(\theta\left(\frac{5s}{8}\right)\right) + 2\cos\left(\theta\left(\frac{6s}{8}\right)\right) + 4\cos\left(\theta\left(\frac{7s}{8}\right)\right) + \cos(\theta(s))\right]$$

$$y_S(s) = y_0 + \frac{s}{24}\left[\sin(\theta(0)) + 4\sin\left(\theta\left(\frac{s}{8}\right)\right) + 2\sin\left(\theta\left(\frac{2s}{8}\right)\right) + 4\sin\left(\theta\left(\frac{3s}{8}\right)\right) + 2\sin\left(\theta\left(\frac{4s}{8}\right)\right) + 4\sin\left(\theta\left(\frac{5s}{8}\right)\right) + 2\sin\left(\theta\left(\frac{6s}{8}\right)\right) + 4\sin\left(\theta\left(\frac{7s}{8}\right)\right) + \sin(\theta(s))\right]$$

Boundary Conditions via Simpson's Rule

- Using our Simpson's approximations, we can now write out the full boundary conditions in terms of spiral parameters
- Can now generate a spiral that satisfies boundary conditions by optimizing its spiral parameters and its length, s_f



$$x_S(s_f) = x_f$$

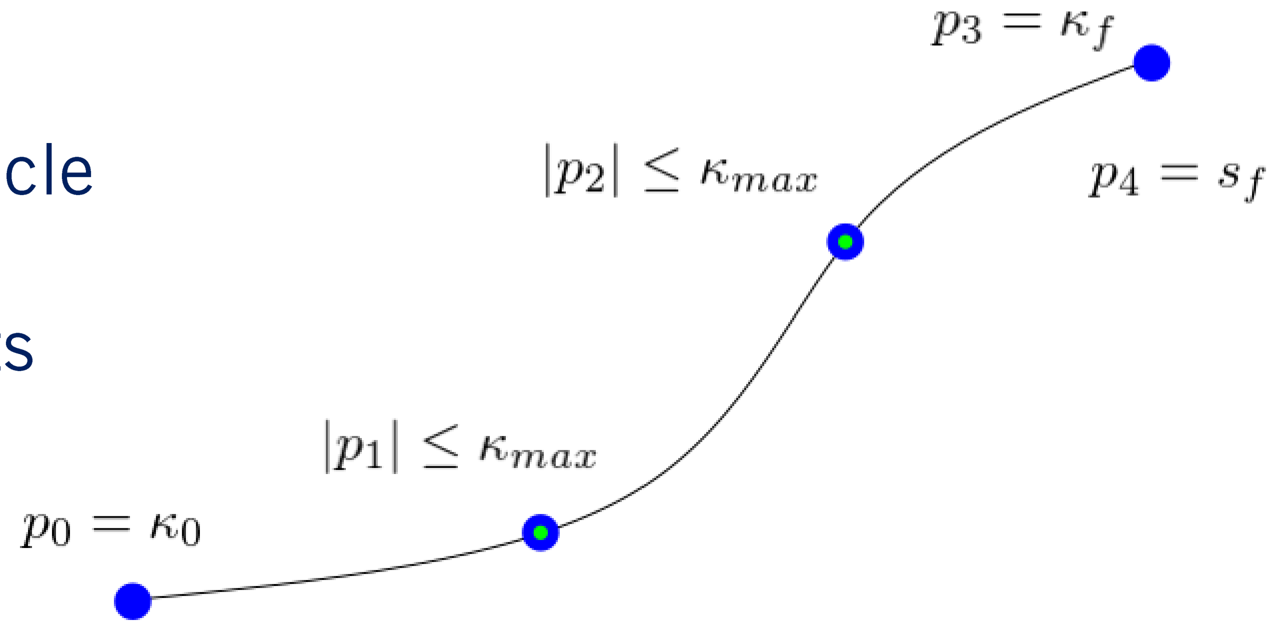
$$y_S(s_f) = y_f$$

$$\theta(s_f) = \theta_f$$

$$\kappa(s_f) = \kappa_f$$

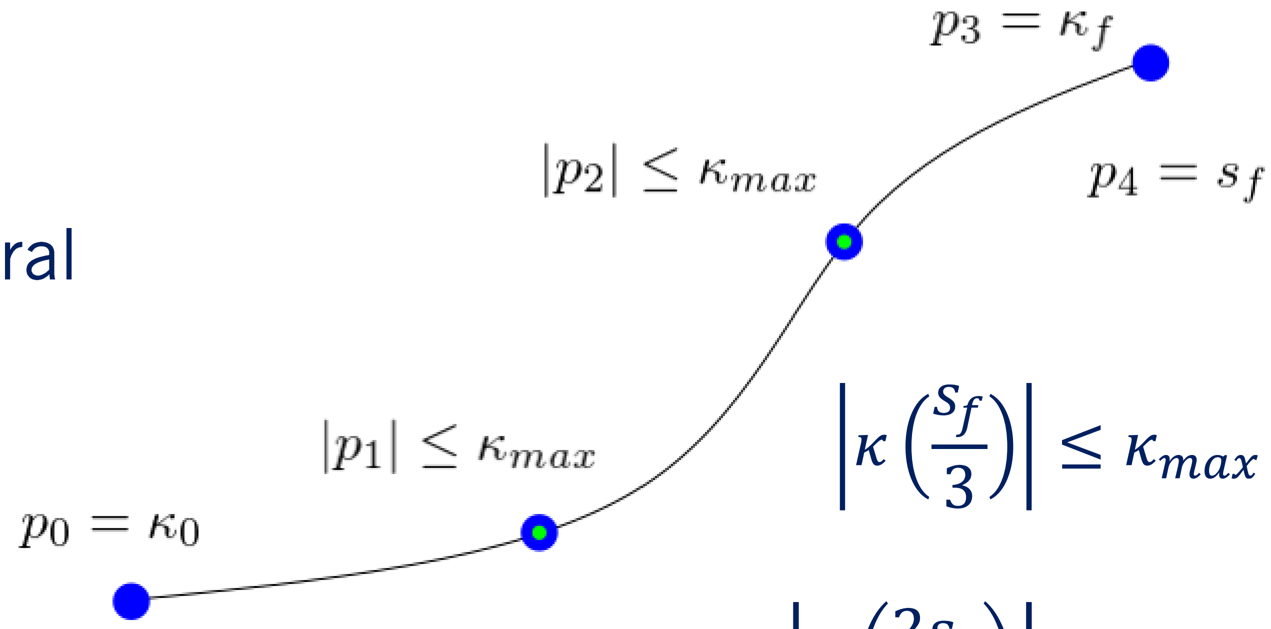
Approximate Curvature Constraints

- Want to apply curvature constraints to path so it is drivable by the vehicle
- Curvature constraints correspond to minimum vehicle turning radius
- Can constrain sampled points along the path due to well-behaved nature of spiral's curvature



Approximate Curvature Constraints

- Can constrain curvature at 1/3rd and 2/3rd's of the way along the path
- Now all constraints and boundary conditions are complete to generate the spiral



$$\left|\kappa\left(\frac{s_f}{3}\right)\right| \leq \kappa_{max}$$

$$\left|\kappa\left(\frac{2s_f}{3}\right)\right| \leq \kappa_{max}$$

Bending Energy Objective

$$f_{be}(a_0, a_1, a_2, a_3, s_f) = \int_0^{s_f} (a_3 s^3 + a_2 s^2 + a_1 s + a_0)^2 \, ds$$

- Bending energy distributes curvature evenly along spiral to promote comfort
  - Equal to integral of square curvature along path, which has closed form for spirals
- Gradient also has a closed form solution
  - Has many terms, so best left to a symbolic solver

Initial Optimization Problem

- Can bring constraints and objective together to form the full optimization problem
  - Can perform optimization in the vehicle's body attached frame to set starting boundary condition to zero

$$\min f_{be}(a_0, a_1, a_2, a_3, s_f) \text{ s.t.} \begin{cases} \left|\kappa\left(\frac{s_f}{3}\right)\right| \leq \kappa_{max}, & \left|\kappa\left(\frac{2s_f}{3}\right)\right| \leq \kappa_{max} \\ x_s(0) = x_0, & x_s(s_f) = x_f \\ y_s(0) = y_0, & y_s(s_f) = y_f \\ \theta(0) = \theta_0, & \theta(s_f) = \theta_f \\ \kappa(0) = \kappa_0, & \kappa(s_f) = \kappa_f \end{cases}$$

Soft Constraints

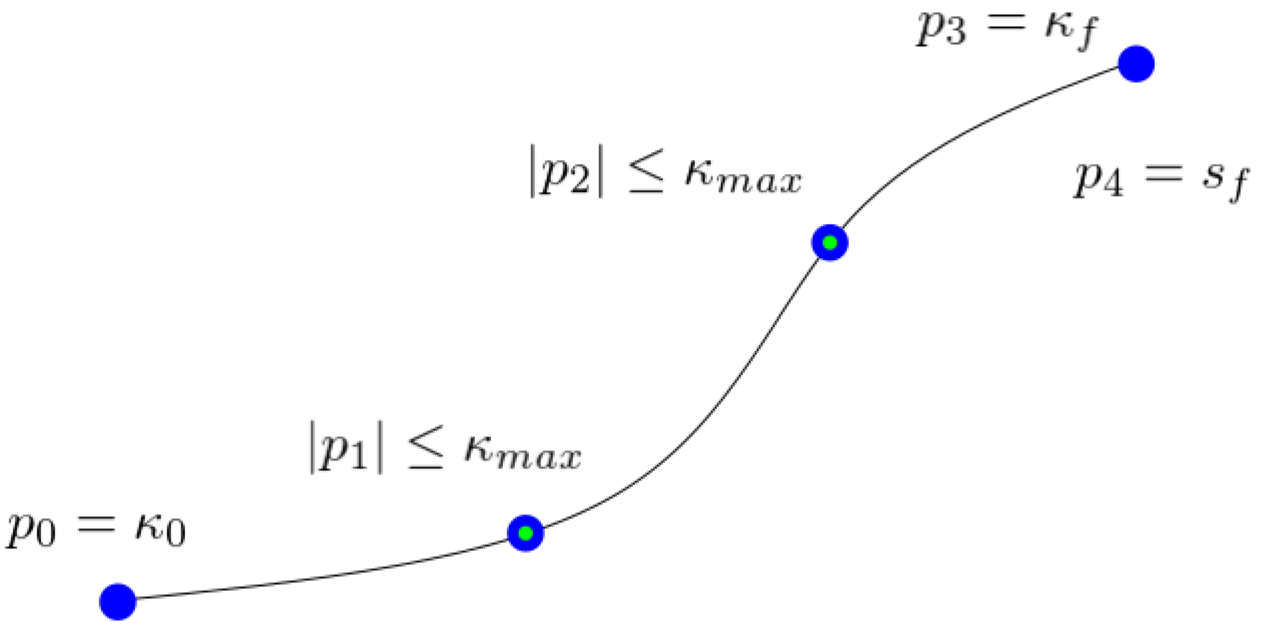
- Challenging for optimizer to satisfy constraints exactly
- Can soften equality constraints by penalizing deviation heavily in the objective function
- We also assume initial curvature is known, which corresponds to a_0

$$\min f_{be}(a_0, a_1, a_2, a_3, s_f) + \alpha\left(x_s(s_f) - x_f\right) + \beta\left(y_s(s_f) - y_f\right) + \gamma\left(\theta_s(s_f) - \theta_f\right)$$

$$\text{s.t.} \begin{cases} \left|\kappa\left(\frac{s_f}{3}\right)\right| \leq \kappa_{max} \\ \left|\kappa\left(\frac{2s_f}{3}\right)\right| \leq \kappa_{max} \\ \kappa(s_f) = \kappa_f \end{cases}$$

Parameter Remapping

- Can remap spiral parameters
- p_0 to p_3 corresponds to curvature at 4 points equally spaced along path
- p_4 corresponds to the arc length of the spiral
- Since initial and final curvature are known, p_0 and p_3 eliminated from optimization, reducing dimensionality



$$a_0 = p_0$$

$$a_1 = -\frac{11p_0/2 - 9p_1 + 9p_2/2 - p_3}{p_4}$$

$$a_2 = \frac{9p_0 - 45p_1/2 + 18p_2 - 9p_3/2}{p_4^2}$$

$$a_3 = -\frac{9p_0/2 - 27p_1/2 + 27p_2/2 - 9p_3/2}{p_4^3}$$

Final Optimization Problem

- Replacing spiral parameters with new parameters leads to new optimization formulation
- Curvature constraints correspond directly to new parameters
- Boundary conditions handled by soft constraints and constant p_0 and p_3

$$\min f_{be}(a_0, a_1, a_2, a_3, s_f) + \alpha\big(x_S(p_4) - x_f\big) + \beta\big(y_S(p_4) - y_f\big) + \gamma\big(\theta_S(p_4) - \theta_f\big)$$

$$\text{s.t.} \begin{cases} |p_1| \leq \kappa_{max} \\ |p_2| \leq \kappa_{max} \end{cases}$$

Summary

- Reviewed boundary conditions on state and curvature constraints
- Introduced Simpson's rule to compute spiral end position
- Devised optimization problem using bending energy
- Developed method to re-map parameters to improve optimization convergence speed

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