Module 1 | Lesson 2

Recursive Least Squares

Recursive Least Squares

By the end of this video, you will be able to…

- Extend the (batch) least squares formulation to a recursive one
- Use this method to compute a ‘running estimate’ of the least squares solution as measurements stream in

Batch Least Squares

In our previous formulation, we assumed we had all of our measurements available when we computed our estimate:

	Resistance Measurements (Ohms)	
#	Multimeter A ($\sigma = 20$ Ohms)	Multimeter B ($\sigma = 2$ Ohms)
1	1068	
2	988	
3		1002
4		996

'Batch Solution' $$\hat{x}_{\text{WLS}} = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}$$

Recursive Estimation

- What happens if we have a *stream* of data? Do we need to re-solve for our solution every time? Can we do something smarter?

$$\hat{x}_1 = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}_1$$

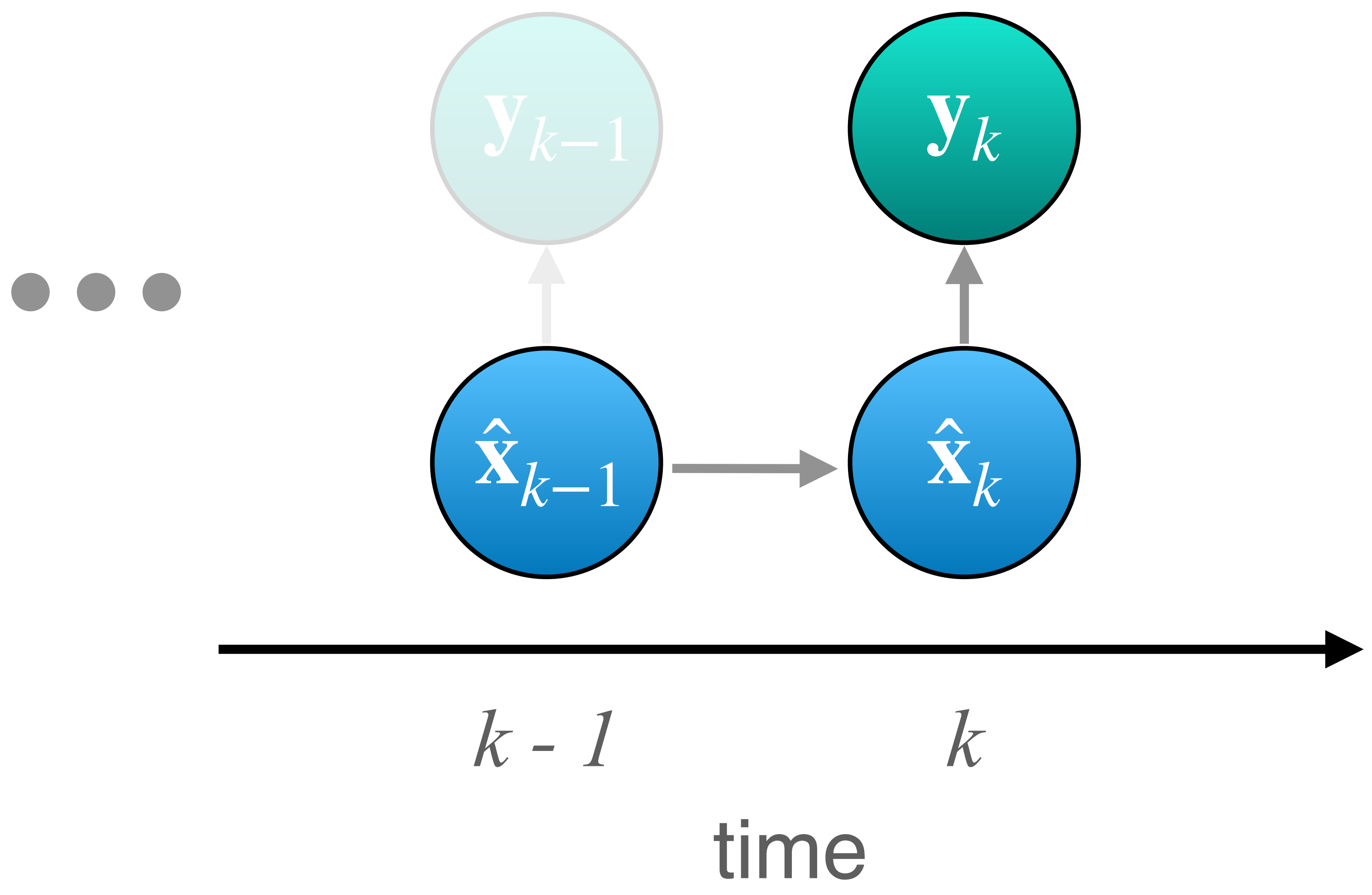
$$\hat{x}_2 = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y}_{1:2}$$

$$\vdots$$

	Resistance (Ohms)	
Time	Multimeter A	Multimeter B
t = 1 sec	1068	
t = 2 sec	988	
t = 3 sec		1002
t = 4 sec		996

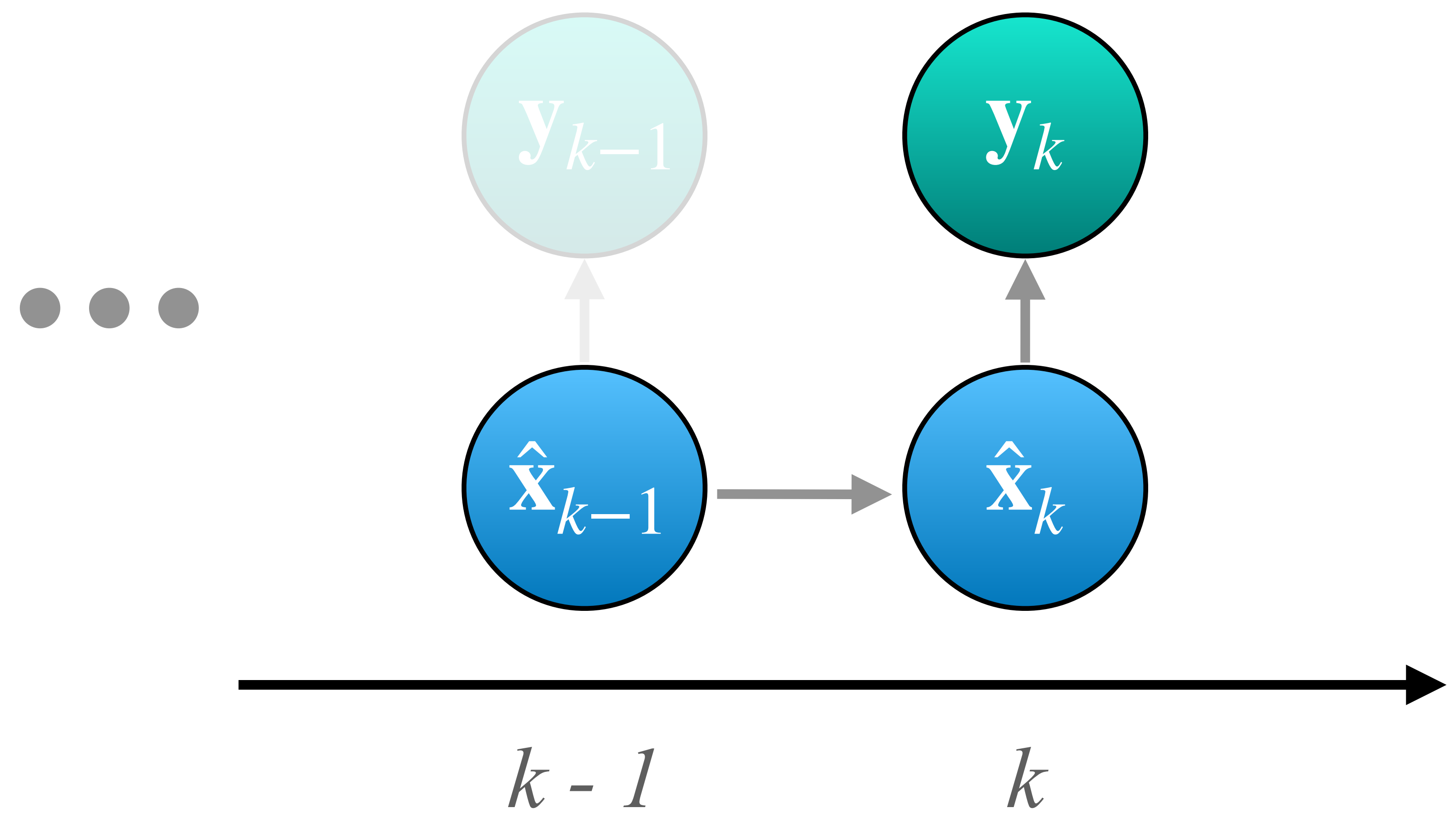
Linear Recursive Estimator

- We can use a *linear recursive estimator*
- Suppose we have an optimal estimate, $\hat{\mathbf{x}}_{k-1}$, of our unknown parameters at time interval $k-1$
- Then we obtain a new measurement at time k: $\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{v}_k$



Goal: compute $\hat{\mathbf{x}}_k$ as a function of $\mathbf{y}_k$ and $\hat{\mathbf{x}}_{k-1}$!

Linear Recursive Estimator



- We can use a *linear recursive update:*

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k-1} \right)$$

- We update our new state as a linear combination of the previous best guess and the current measurement *residual (or error)*, weighted by a gain matrix $\mathbf{K}_k$

Recursive Least Squares

- But what is the gain matrix $\mathbf{K}_k$?
- We can compute it by minimizing a similar least squares criterion, but this time we'll use a probabilistic formulation.
- We wish to minimize the **expected value of the sum of squared errors** of our current estimate at time step k:

$$
\begin{aligned}
\mathscr{L}_{\mathrm{RLS}} &= \mathbb{E}[(x_k - \hat{x}_k)^2] \\
&= \sigma_k^2
\end{aligned}
$$

- If we have n unknown parameters at time step k, we generalize this to

$$
\begin{aligned}
\mathscr{L}_{\mathrm{RLS}} &= \mathbb{E}[(x_{1k} - \hat{x}_{1k})^2 + \ldots + (x_{nk} - \hat{x}_{nk})^2] \\
&= \mathrm{Trace}(\mathbf{P}_k)
\end{aligned}
$$

Estimator ***covariance***

Recursive Least Squares

- Using our linear recursive formulation, we can express covariance as a function of $\mathbf{K}_k$

$$\mathbf{P}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}(\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T$$

- We can show (through matrix calculus) that this is minimized when

$$\mathbf{K}_k = \mathbf{P}_{k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

- With this expression, we can also simplify our expression for $\mathbf{P}_k$:

$$\begin{aligned}\mathbf{P}_k &= \mathbf{P}_{k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k-1} \\ &= (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}\end{aligned}$$

Our covariance ‘shrinks’ with each measurement

Recursive Least Squares | Algorithm

1. Initialize the estimator

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}]$$

$$\mathbf{P}_0 = \mathbb{E}[(\mathbf{x} - \hat{\mathbf{x}}_0)(\mathbf{x} - \hat{\mathbf{x}}_0)^T]$$

2. Set up the measurement model, defining the Jacobian and the measurement covariance matrix:

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{v}_k$$

3. Update the estimate of $\hat{\mathbf{x}}_k$ and the covariance $\mathbf{P}_k$ using:

$$\mathbf{K}_k = \mathbf{P}_{k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k-1} + \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k-1}\right)$$

$$\mathbf{P}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k-1}$$

Important! Our parameter covariance ‘shrinks’ with each measurement

Summary | Recursive Least Squares

- RLS produces a 'running estimate' of parameter(s) for *a stream of measurements*

- RLS is a linear recursive estimator that minimizes the (co)variance of the parameter(s) at the current time