

Module 3 | Lesson 1

3D Geometry and Reference Frames

Module 3 | GPS & INS Sensing for Pose Estimation

In this module...

- 3D kinematics, important reference frames
- Rotation representations
- Inertial Measurement Unit (IMU)
- Global Navigation Satellite Systems (GNSS)

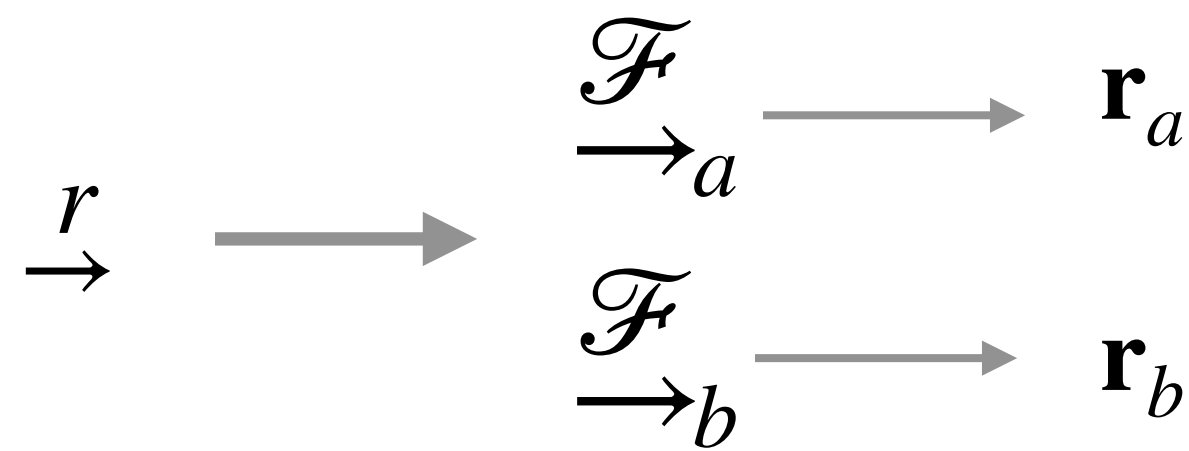
3D Geometry and Reference Frames

By the end of this video, you will be able to...

- Understand how reference frames affect vector coordinates
- Compare and contrast different rotation representations
- Understand the importance of the ECEF, ECIF and Navigation reference frames

Coordinate Rotations

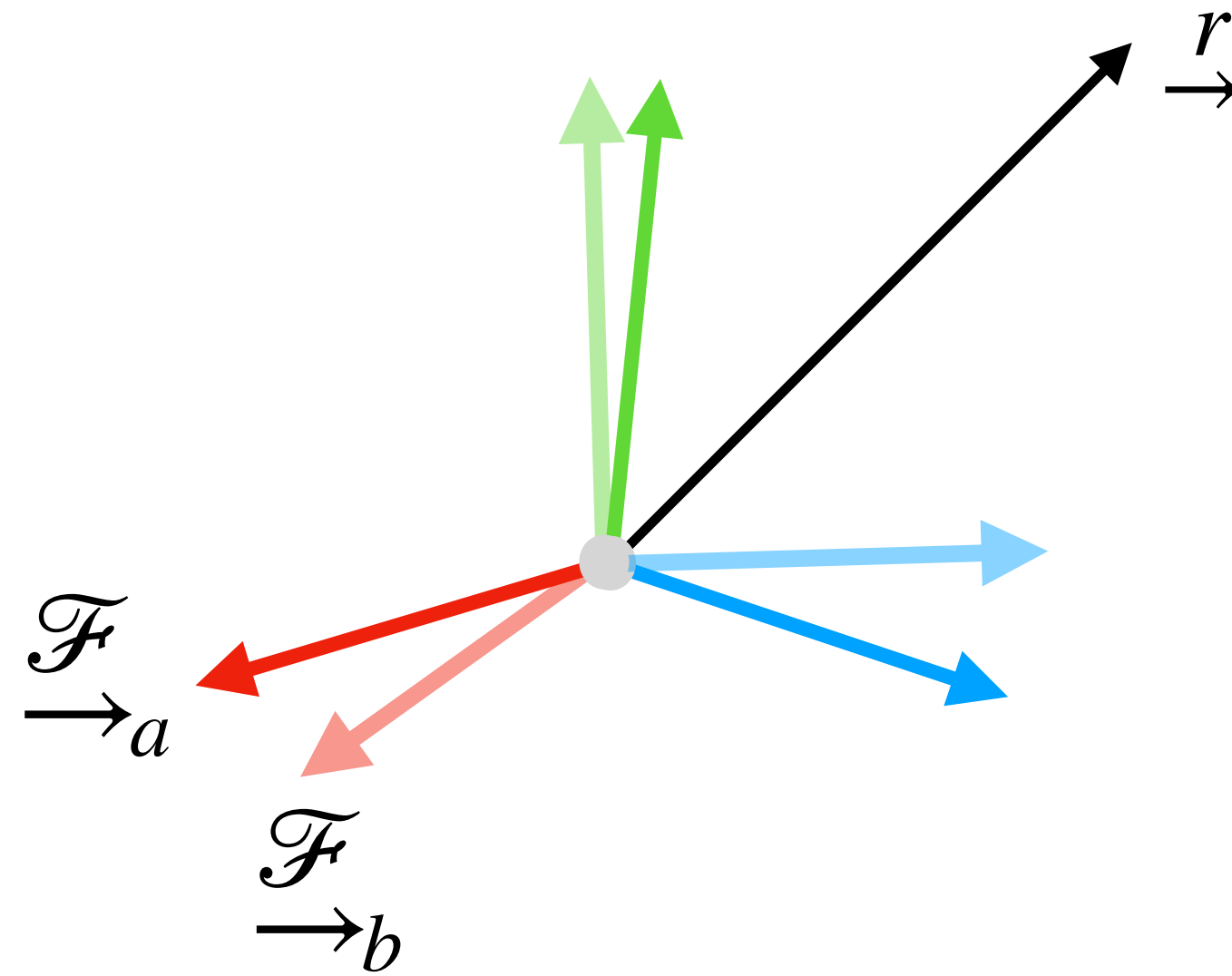
Vectors can be expressed in different coordinate frames:



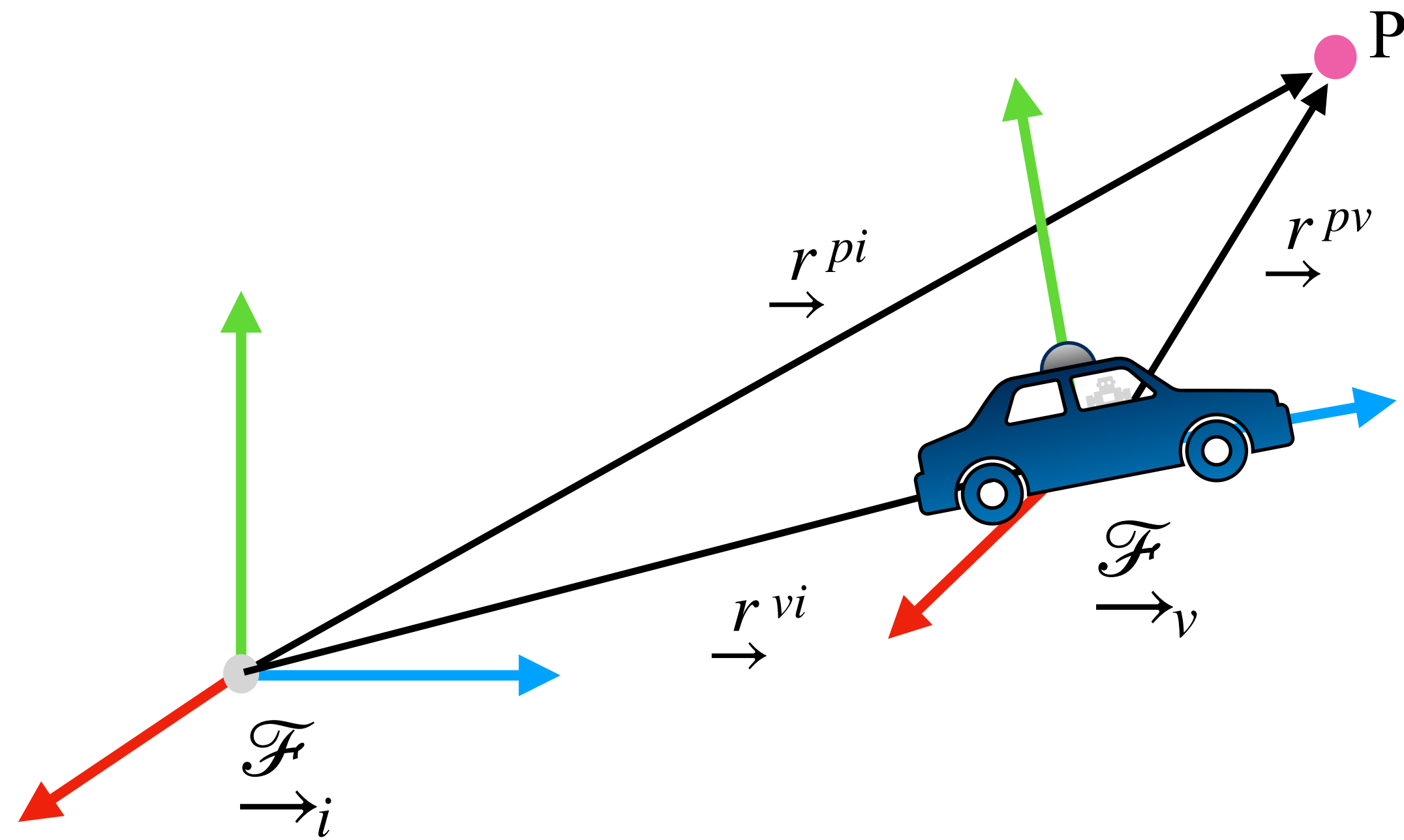
The coordinates of the vector are related through a *rotation matrix*:

$$\mathbf{r}_b = \mathbf{C}_{ba} \mathbf{r}_a$$

← takes coordinates in frame a and rotates them into frame b



Transformations



$$\vec{r}^{pi} = \vec{r}^{vi} + \vec{r}^{pv}$$



$$\mathbf{r}_i^{pi} = \mathbf{r}_i^{vi} + \mathbf{C}_{iv} \mathbf{r}_v^{pv}$$

How Can We Represent a Rotation?

1

Rotation matrix

$$\mathbf{C}_{ba} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \vec{b}_1 \cdot \vec{a}_3 \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \vec{b}_2 \cdot \vec{a}_3 \\ \vec{b}_3 \cdot \vec{a}_1 & \vec{b}_3 \cdot \vec{a}_2 & \vec{b}_3 \cdot \vec{a}_3 \end{bmatrix}$$

“direction
cosine matrix”
(DCM)

$$\mathbf{C}_{ba} \in \mathcal{R}^{3 \times 3}$$

$$\mathbf{r}_b = \mathbf{C}_{ba} \mathbf{r}_a$$

$$\mathbf{C}_{ba} \mathbf{C}_{ba}^T = \mathbf{C}_{ba} \mathbf{C}_{ab} = \mathbf{1}$$

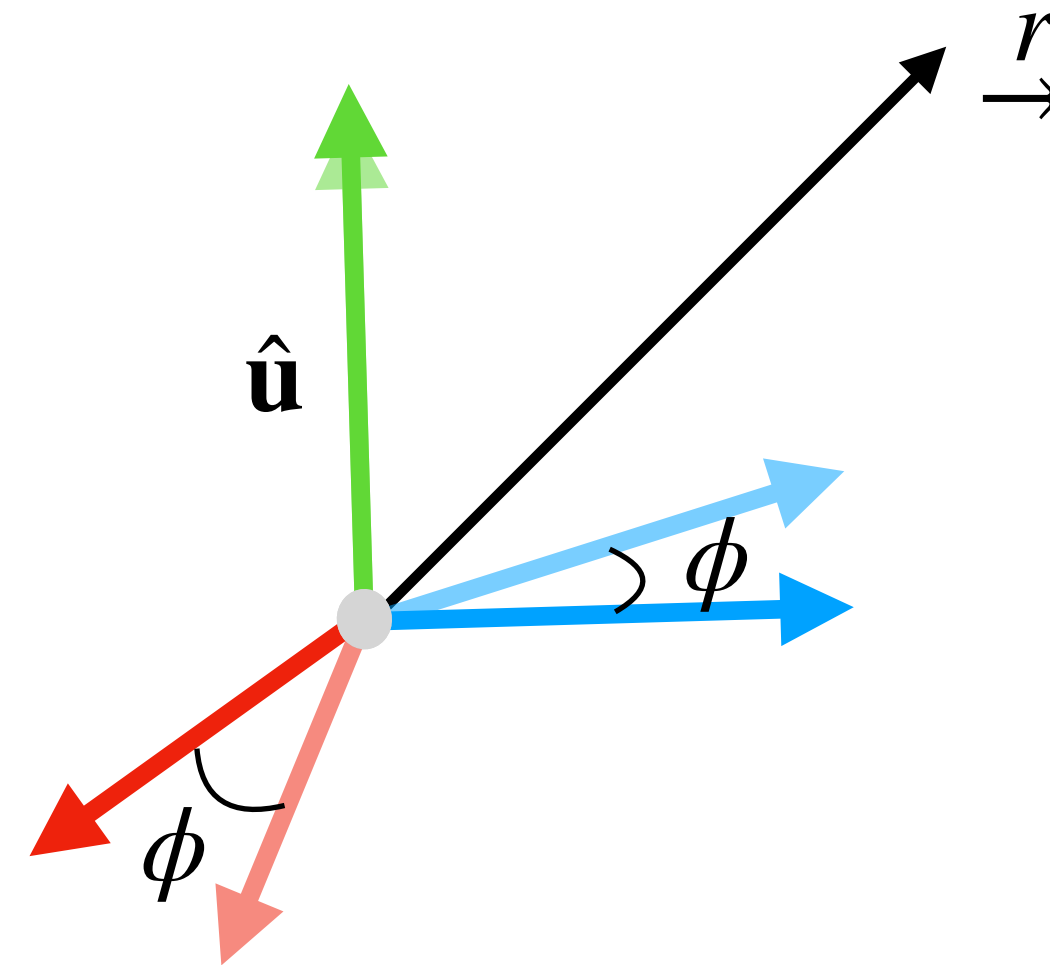
How can we represent a rotation?

2

Unit quaternions

$$\mathbf{q} = \begin{bmatrix} q_w \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} \cos \frac{\phi}{2} \\ \hat{\mathbf{u}} \sin \frac{\phi}{2} \end{bmatrix}$$

$$\|\mathbf{q}\| = 1$$



$$\mathbf{r}_b = \mathbf{C}(\mathbf{q}_{ba})\mathbf{r}_a$$

$$\mathbf{C}(\mathbf{q}) = (q_w^2 - \mathbf{q}_v^T \mathbf{q}_v)\mathbf{1} + 2\mathbf{q}_v \mathbf{q}_v^T + 2q_w[\mathbf{q}_v]_{\times}$$

where

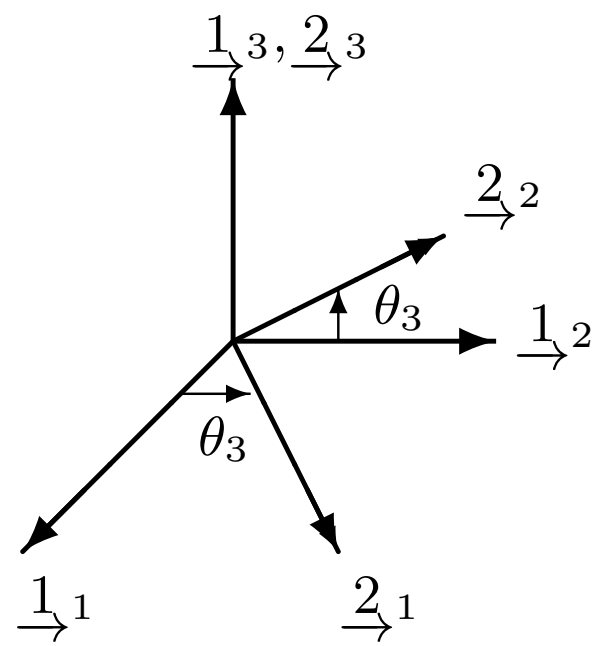
$$[\mathbf{a}]_{\times} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

How can we represent a rotation?

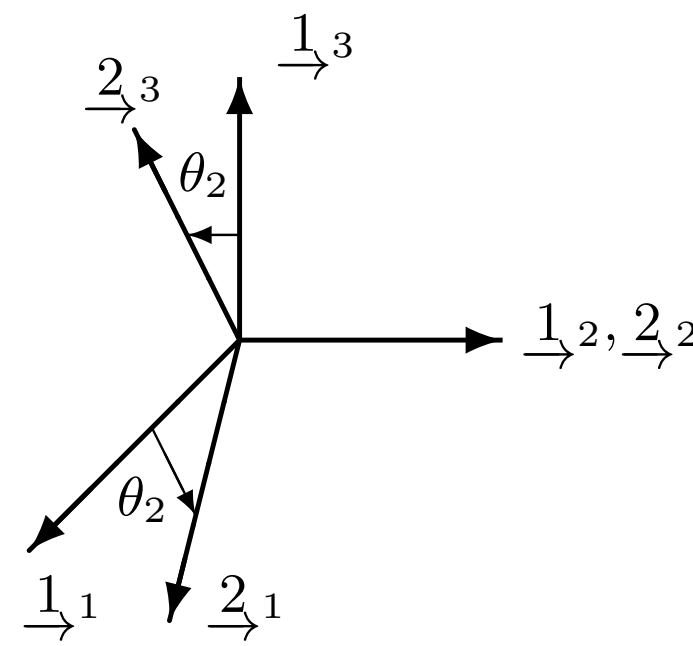
3

Euler angles

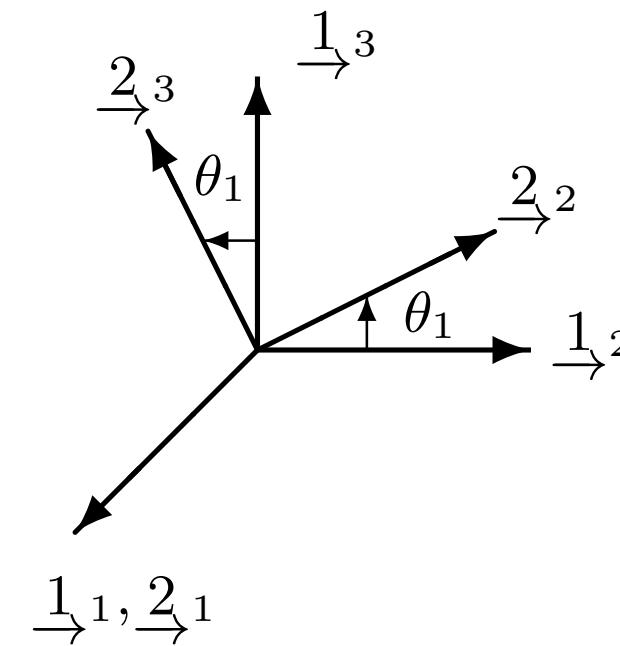
$$\mathbf{C}(\theta_3, \theta_2, \theta_1) = \mathbf{C}_3(\theta_3) \mathbf{C}_2(\theta_2) \mathbf{C}_1(\theta_1)$$



about the 3-axis



about the 2-axis



about the 1-axis

$$\mathbf{C}_3 = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

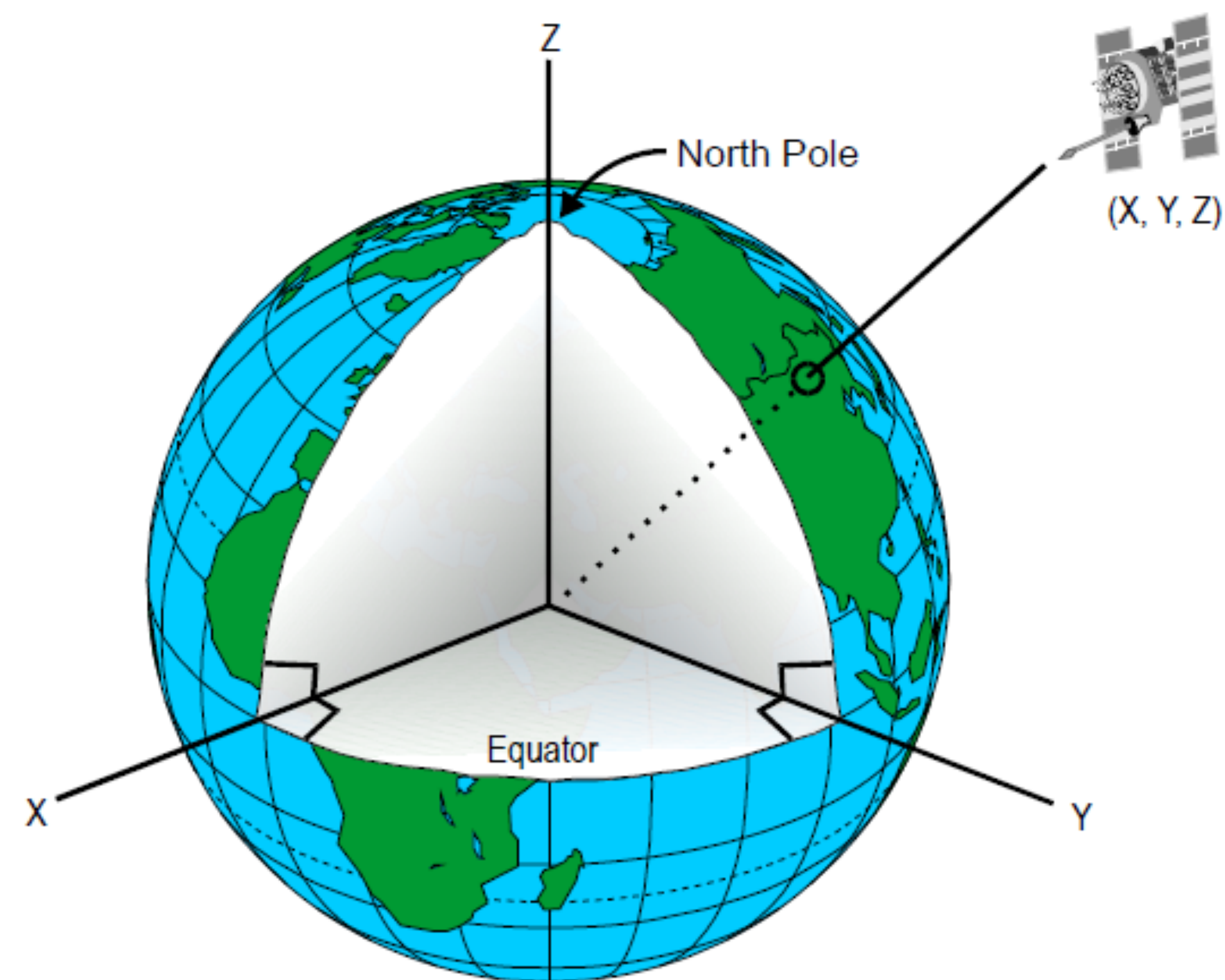
$$\mathbf{C}_2 = \begin{bmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix}$$

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix}$$

Which Rotation Representation Should I Use?

	Rotation Matrix	Unit quaternion	Euler angles
<i>Expression</i>	\mathbf{C}	$\mathbf{q} = \begin{bmatrix} \cos \frac{\phi}{2} \\ \hat{\mathbf{u}} \sin \frac{\phi}{2} \end{bmatrix}$	$\{\theta_3, \theta_2, \theta_1\}$
<i>Parameters</i>	9	4	3
<i>Constraints</i>	$\mathbf{C}\mathbf{C}^T = \mathbf{1}$	$ \mathbf{q} = 1$	<i>None*</i>
<i>Singularities?</i>	No	No	<i>Yes!</i>

Reference Frames | ECIF



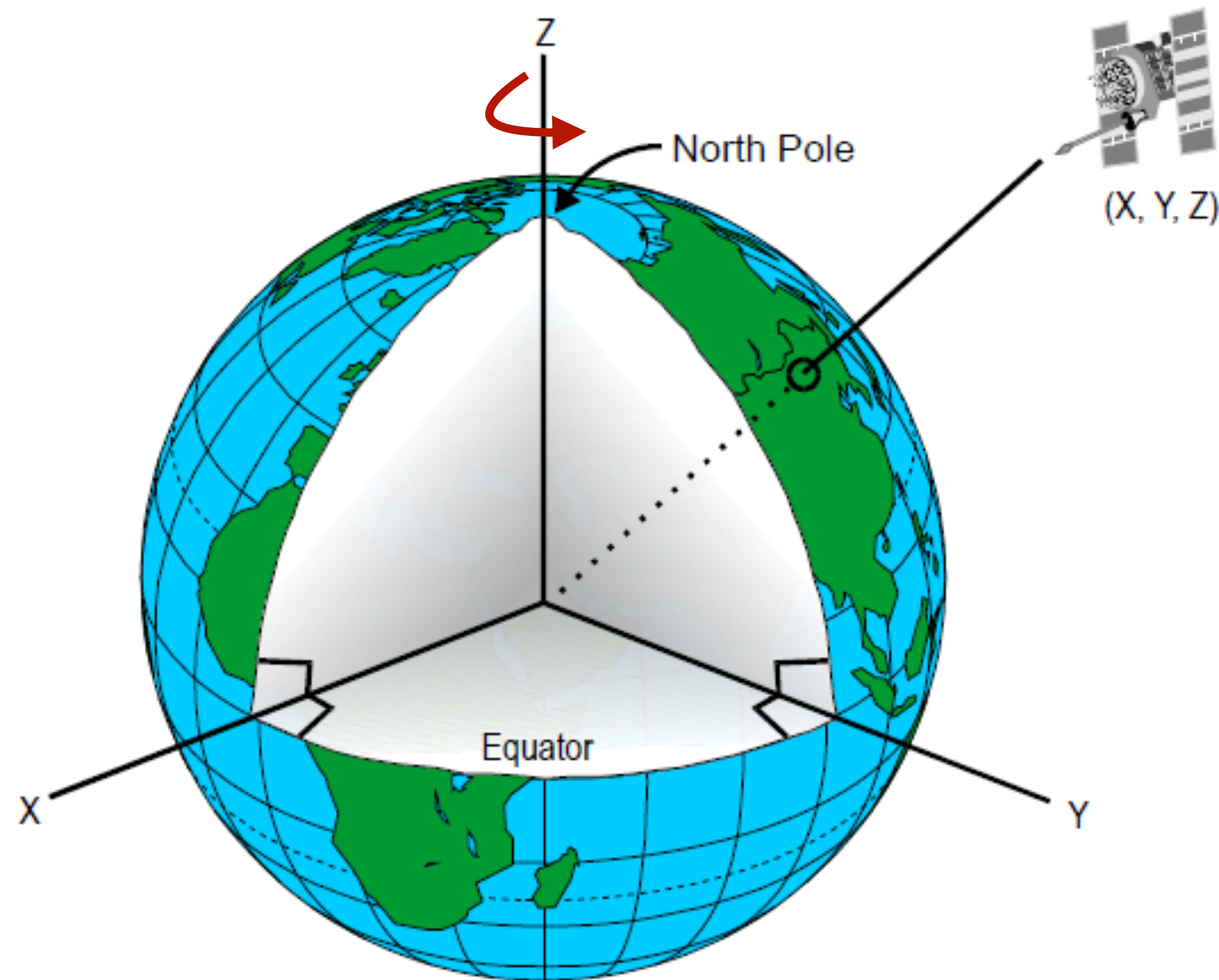
ECIF coordinate frame is fixed,
Earth rotates about the z axis.

$$\mathcal{F} \rightarrow \text{ECIF}$$

x	<i>fixed w.r.t. stars</i>
y	<i>fixed w.r.t. stars</i>
z	true north

Earth-Centred Inertial Frame

Reference Frames | ECEF



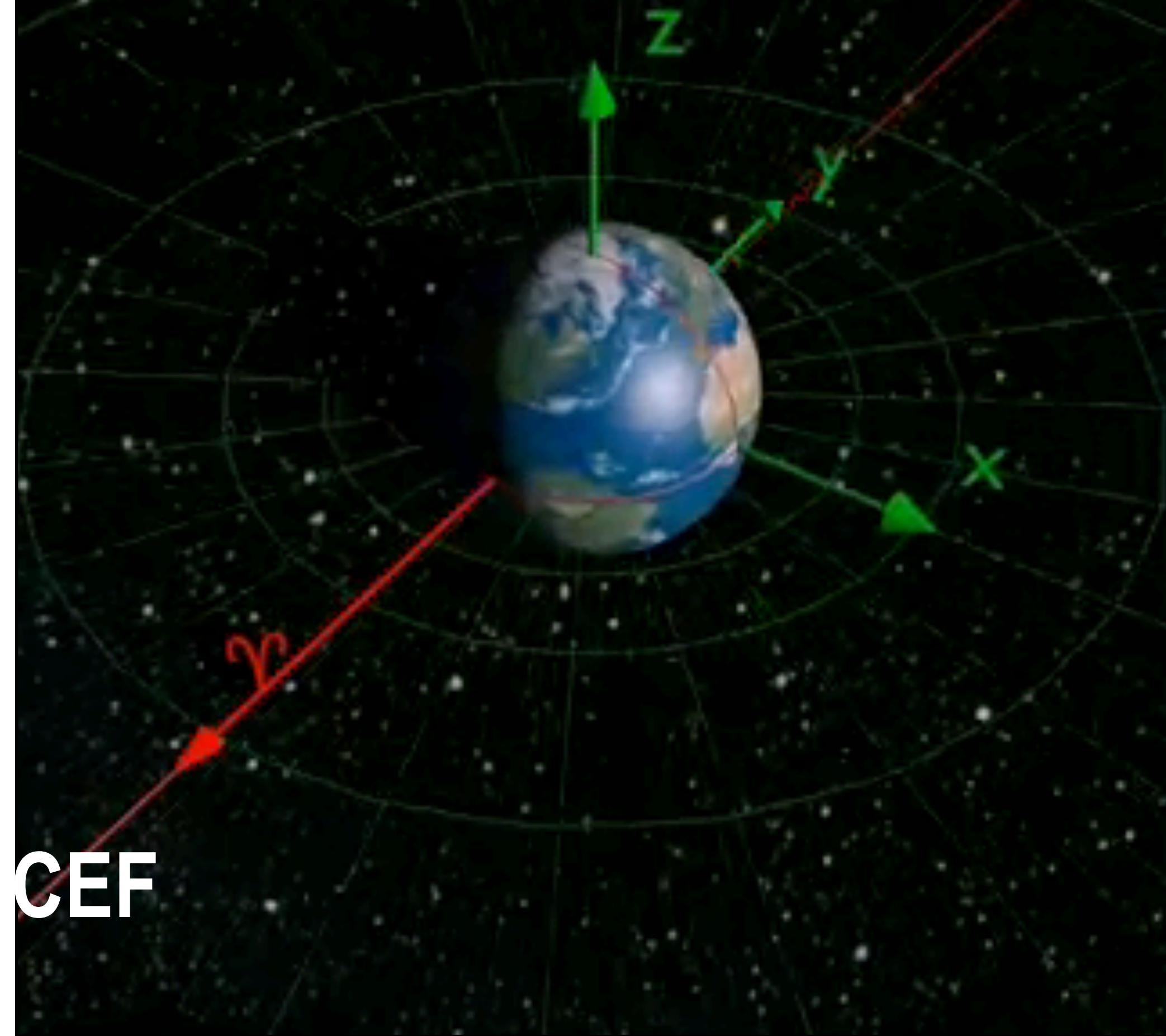
Earth-Centred Earth-Fixed Frame

ECEF coordinate frame rotates with the Earth.

$$\mathcal{F} \rightarrow \text{ECEF}$$

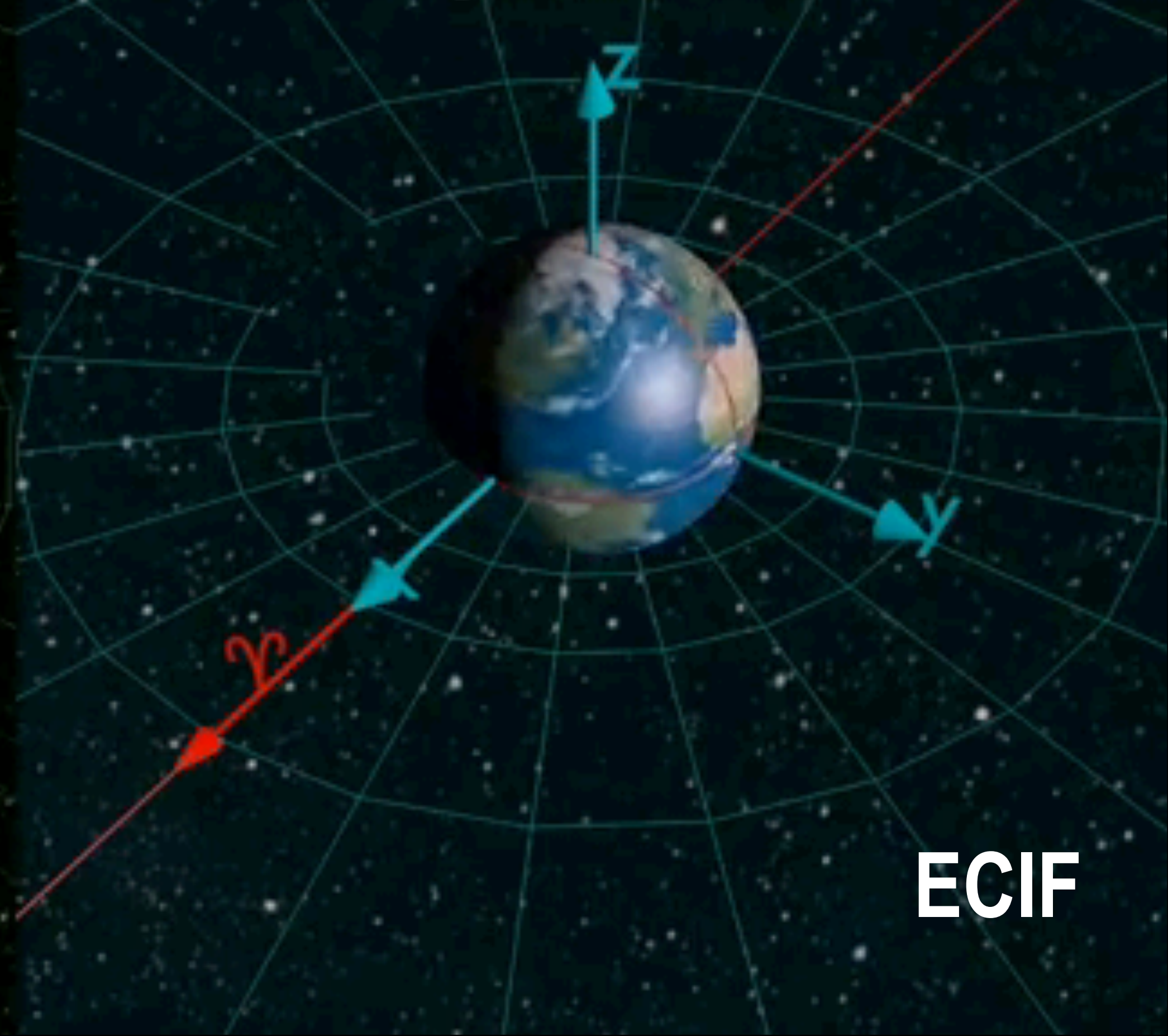
x	prime meridian (on equator)
y	RHR
z	true north

Earth Centered Earth Fixed



CEF

Earth Centered Inertial Equatorial



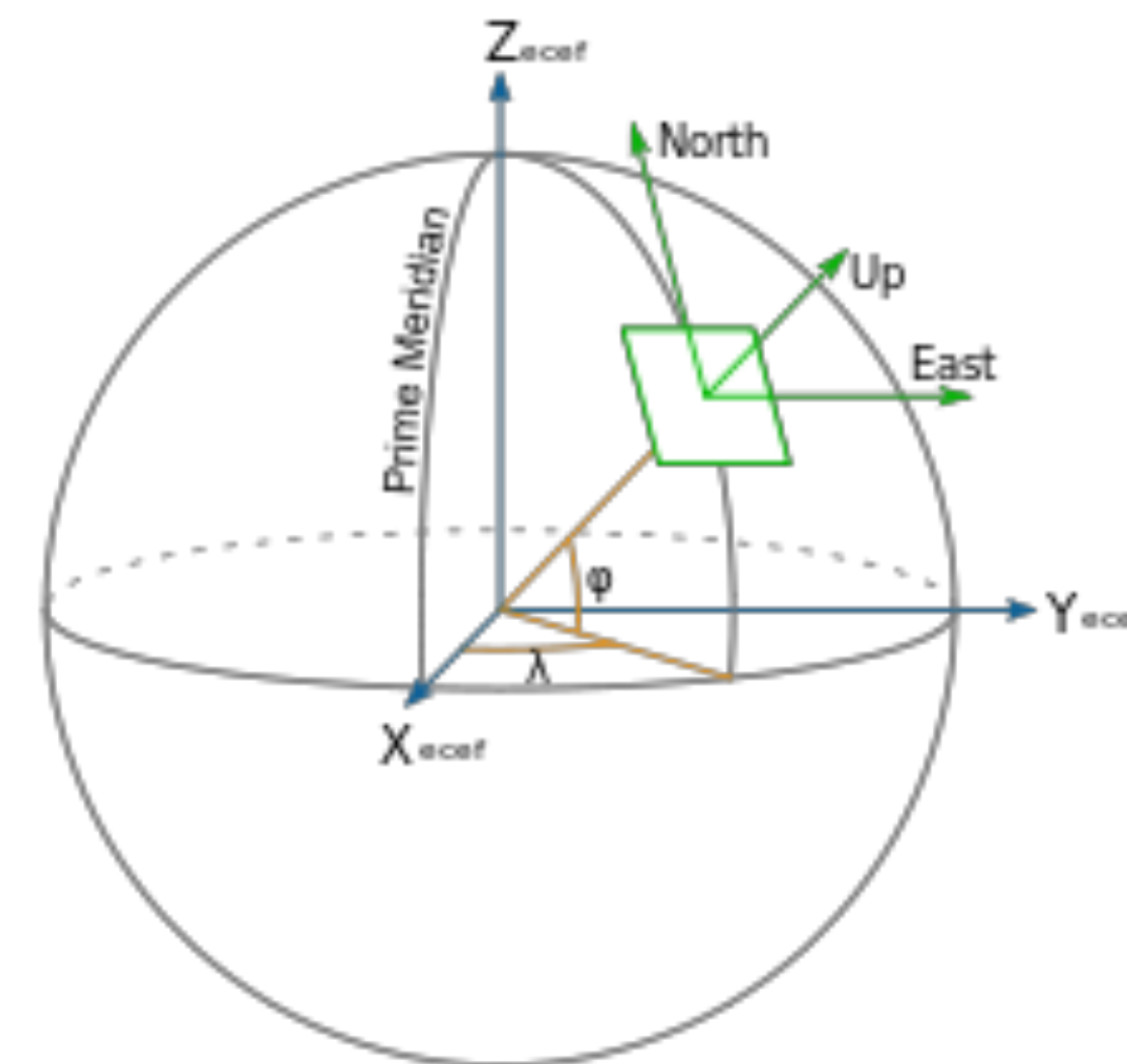
ECIF

Reference Frames | Navigation

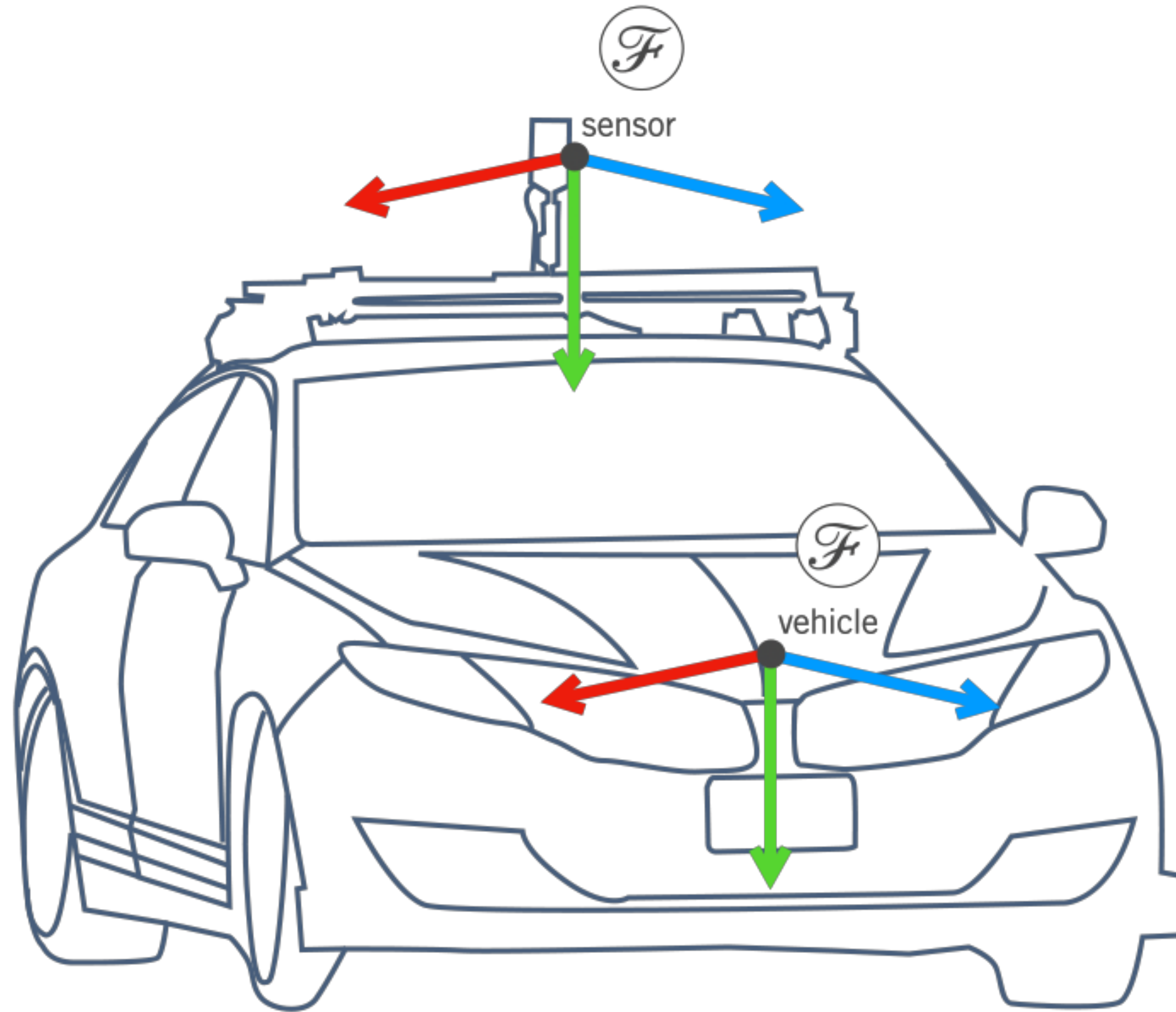


NED Frame

x	True North
y	True East
z	Down (along gravity)



Reference Frames | Sensor & Vehicle



Summary | 3D Geometry and Reference Frames

- Vector quantities can be expressed in different reference frames
- Rotations can be parametrized by rotation matrices, quaternions or Euler angles
- ECEF, ECIF and Navigation frames are important in localization