http://www.ssc.wisc.edu/~kfukushi/econ702

user name: econ702

password: rainey

Grading:

Problem sets: 20%

Mid term: 40% : Mar, 18th

Final Exam 40%

Tom Sargent's website

1. Introduction.

Recall

GDP

real GDP = total production (using labor, capital, ...)

= total expenditures $(= C + I + G + NX)$

= total income $(= R + W)$

"Rule of 70"

If x grows at rate $g\%$ per year, then x doubles approximately every $\frac{70}{g}\%$ years

$(\because)$ suppose x grows at $g\%$ per year and doubles in T years

$$\Rightarrow X_T = 2x_0$$

$$\Vert$$

$$(1+g\%)^T x_0 = 2x_0$$

$$\Rightarrow T \cdot \underbrace{\log\left(1+\frac{g}{100}\right)}_{\approx \frac{g}{100}} = \underbrace{\log(2)}_{\approx 0.7}$$

$$\Rightarrow T \doteqdot \frac{70}{g}$$

Fact 2. Roughly constant average hours worked ($\approx$ 40 hours/wk)

Fact 3: Growing capital stock roughly constant $K/Y \simeq 3$

Fact 4: Roughly constant expenditure share

Fact 5: Roughly constant income share

Fact 6: Fluctuating per capita real GDP growth, sometimes < 0

Fact 7: Positive comovement: Volatility $I \gg Y > C$

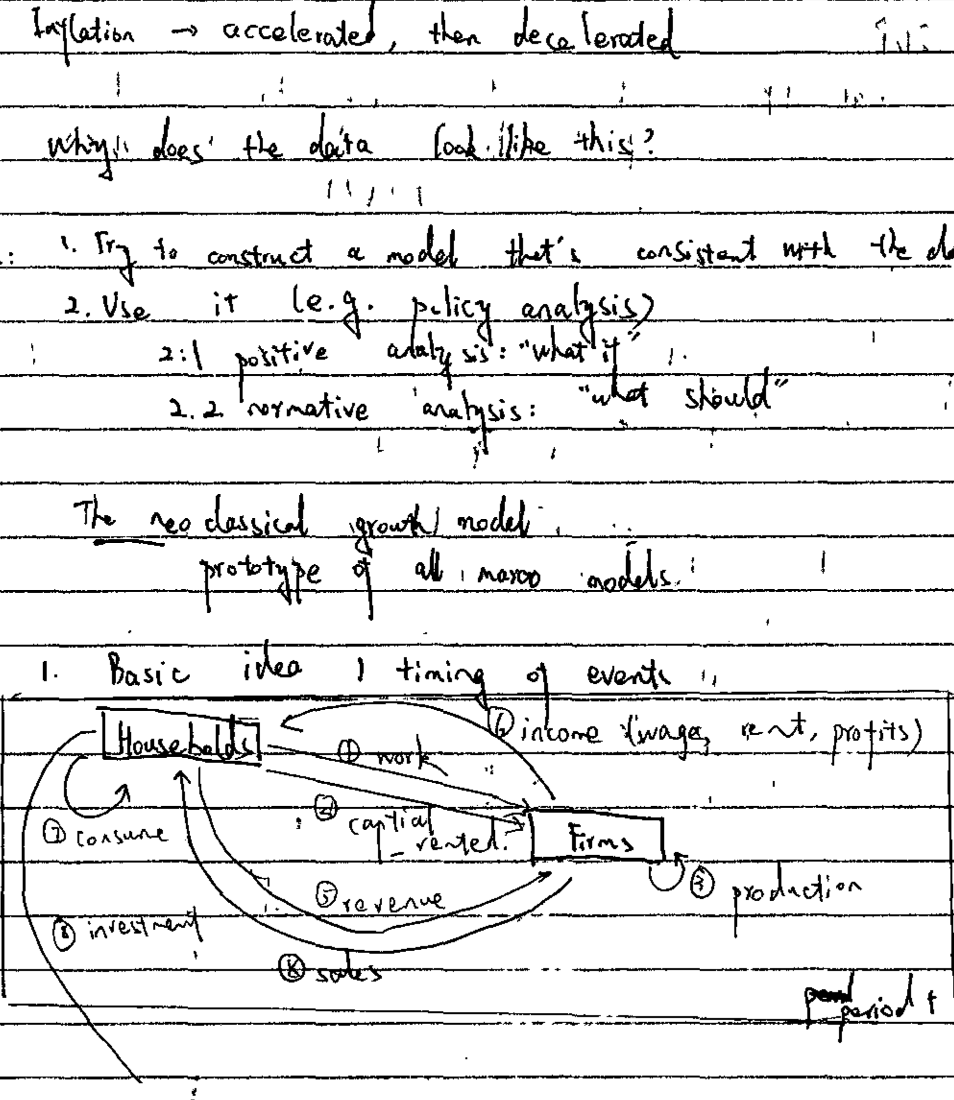
Fact 8: Inflation $\rightarrow$ accelerated, then decelerated

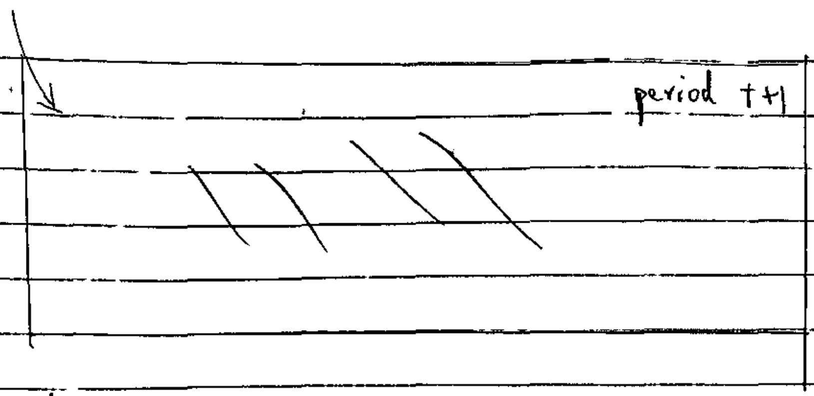
Question: Why does the data look like this?

Agenda: 1. Try to construct a model that's consistant with the data.
2. Use it (e.g. policy analysis)
2.1 positive analysis: "what if"
2.2 normative analysis: "what should"

II. The neoclassical growth model
prototype of all macro models.

1. Basic idea / timing of events





2. Model

Time : $t = 0, 1, 2, 3 \ldots$

Firms:

- hire workers, rent capital, produce / sell goods
- production function:

$$\underline{Y_t} = F(\underline{K_t}, L_t) \quad \leftarrow \text{shape?}$$

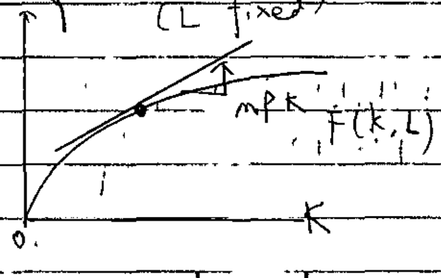
output, capital, labor

Shape of F ?

- F (K, L) is increasing
  (∵) more puts ⇒ more output

- F(K, L) is concave (L fixed)

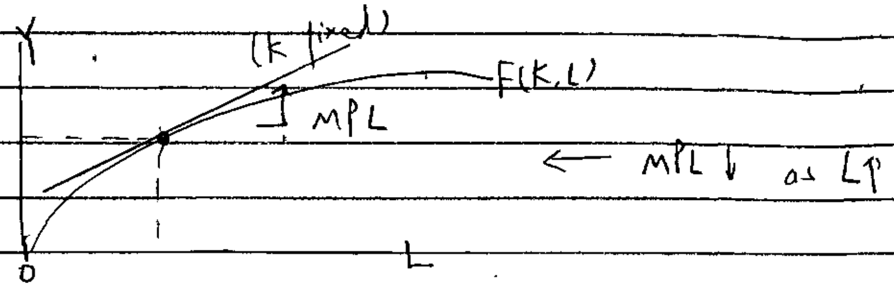
① ⟹



← MPK↓ as K↑ ("diminishing MPK")

② MPK (marginal product of capital) $= F_K(K, L)$

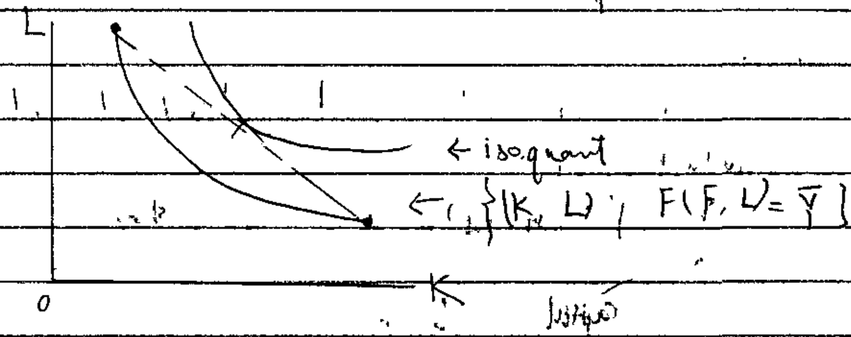
(∵) "anti - congestion"



$MPL = F_L(K, L)$

(∵) congestion.

③ →



(∵) extreme values of $\frac{K}{L}$ are inefficient

• $F(\lambda K, \lambda L) = \lambda F(K, L)$

($\lambda \geq 0$)

(∵) standard replication argument

$$\left.\begin{array}{l} Y = F(K, L) \\ Y = F(K, L) \end{array}\right\} \; F(2K, 2L) = 2Y$$

1.26.

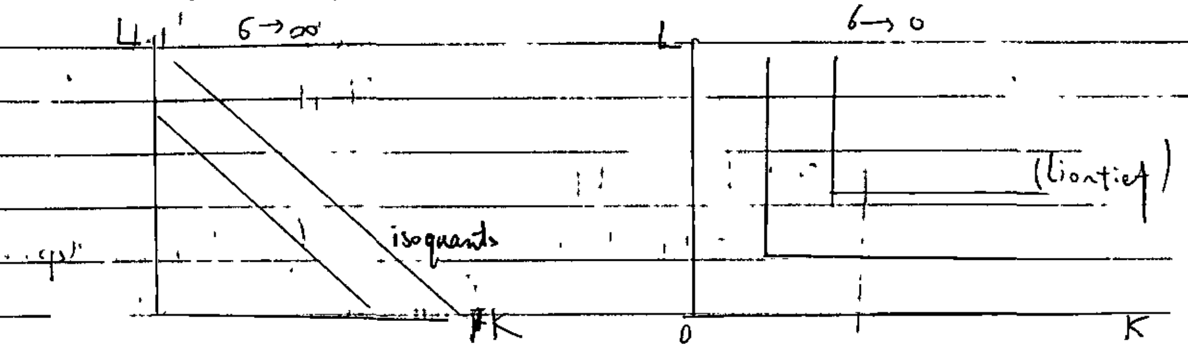
Ex1. (Cobb-Douglas)

special case: $F(K, L) = a K^{\alpha} L^{1-\alpha}$ $(a>0, \alpha \in (0,1))$

Ex2. (CES) - constant elasticity of substitution

$F(K, L) = a\left(\phi K^{\frac{\sigma-1}{\sigma}} + (1-\phi) L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$ $(a>0,\ \phi \in [0,1),\ \sigma>0)$

$\sigma \to \infty$



∘ profit maximization

$$\underset{\text{profit}}{\pi_t} = \begin{cases} \max\limits_{Y_t, K_t, L_t} \underbrace{Y_t}_{\text{output = revenue}} - \underbrace{r_t}_{\text{rental rate of capital}} K_t - \underbrace{w_t}_{\text{wage rate}} \cdot L_t \\ \text{s.t. } Y_t = F(K_t, L_t) \quad K_t, L_t \geq 0 \end{cases}$$

FOCs (necessary & sufficient for optimality):

$$\begin{cases} F_K(K_t, L_t) = r_t \\ F_L(K_t, L_t) = w_t \end{cases} \text{ FOCs}$$

$$\Rightarrow \pi_t = F(K_t, L_t) - \underset{F_K(K_t, L_t)}{r_t} K_t - \underset{F_L(K_t, L_t)}{w_t} L_t$$

$$= F(K_t, L_t) - F_K(K_t, L_t) K_t - F_L(K_t, L_t) \cdot L_t$$

$= 0$

Euler's theorem: $F(K, L) = F_K(K, L)K + F_L(K, L)L$

- Households (HH) — identical ("representative household"):
  - Work, rent capital, earn income, and use income to consume / invest.
  - budget constraints:

$$\underset{\substack{\uparrow \\ \text{consumption}}}{C_t} + \underset{\substack{\uparrow \\ \text{investment}}}{X_t} \le \overbrace{W_t \underset{\substack{\uparrow \\ \text{average time of working} \\ \in[0,1]}}{L_t}}^{\text{Labor income}} + \overbrace{r_t \underset{\substack{\uparrow \\ \text{capital rented}}}{K_t}}^{\text{capital income}} + \underset{\substack{\uparrow \\ \text{profits/dividends}}}{\pi_t}$$

capital accumulation

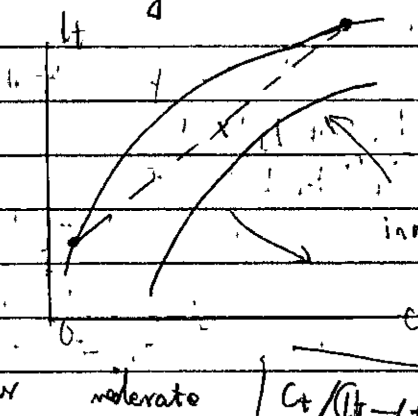
$$K_{t+1} = K_t + X_t - \underbrace{\delta K_t}_{\text{depreciation}} \quad (\delta \in (0,1) \text{ .... depreciation rate})$$

- initial capital: $K_0 = \bar{K}_0$
- goal: maximize "happiness" (= utility)

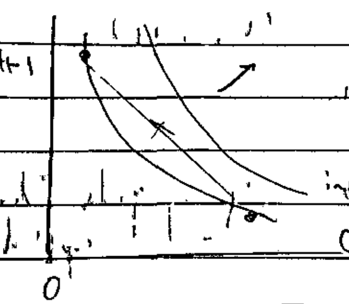
$$U(C_0, C_1, C_2, \ldots L_0, L_1, L_2, \ldots)$$

shape of U?

- U is increasing in C_t ($\forall t$), decreasing in L_t ($\forall t$) keeping other C_t constant.
- U is concave $\Longrightarrow$



desire for moderate $\boxed{C_t / L_t}$



desire for smooth consumption over time.

- Examples: (time separable utility)

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t, l_t)$$

$\beta \in (0,1)$ discount factor, measure of "patience"

β is lower, care more about now

u: period utility

- Utility maximization;

$$\max_{(C_t, x_t, l_t, K_t)_{t=0}^{\infty}} U(C_0, C_1, \dots, l_0, l_1 \dots)$$

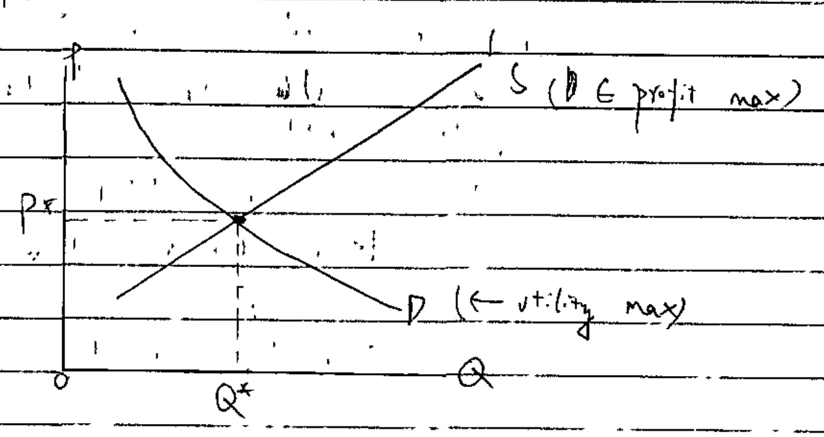
s.t. $C_t + x_t \le w_t l_t + r_t K_t + \pi_t \quad (\forall t)$

$K_{t+1} = (1-\delta) K_t + X_t \quad (\forall t)$

$K_0 = \bar{K}$,

$C_t, l_t, \underbrace{1-l_t}_{\text{leisure}}, K_t \geqq 0 \quad (\forall t)$

Equilibrium:



• Idea: Given prices $(w_t, r_t)_{t=0}^{\infty}$

firms & HHs

optimize: $\longrightarrow$

demand & supply of goods, labor, capital

prices adjust so that these equal

(e.g. excess of labor demand in t)

$\Rightarrow w_t \uparrow$

Market clearing:

	S		D
goods:	Y_t	$=$	$C_t + X_t$
Labor:	l_t	$=$	L_t
capital:	k_t	$=$	K_t
			$(\forall t)$

• Definition:

An equilibrium is a sequence

$(C_t^*, l_t^*, x_t^*, k_t^*,$

$K_t^*, L_t^*, \pi_t^*, w_t^*, Y_t^*)_{t=0}^{\infty}$

such that (Q^*, P^*) : equ.

(i) (H.H optimization) $(C_t^*, l_t^*, x_t^*, k_t^*)_{t=1}^{\infty}$ solves

$$\max_{(c_t, l_t, x_t, k_t)_{t=0}^{\infty}} U(c_0, c_1, \dots, l_0, l_1, \dots)$$

s.t. $c_t + x_t \leq w_t^* l_t + r_t^* k_t + \pi_t^*$ $(\forall t)$

$k_{t+1} = (1-\delta) k_t + x_t$ $(\forall t)$

$k_0 = \bar{k}_0$

$c_t, l_t, 1-l_t, k_t \geq 0$

(ii) (Firm Optimization): (K_t^*, L_t^*) solves $\forall t$:

$$\max_{(K_t, L_t)} F(K_t, L_t) - r_t^* K_t - w_t^* L_t$$

s.t. $K_t, L_t \geq 0$

and π_t^* is the maximum value of the objective function.

(iii) (Market clear):

$\forall t$

$$C_t^* + X_t^* = F(K_t^*, L_t^*)$$

$$L_t^* = l_t^*$$

$$K_t^* = k_t^*$$

Note: Def of equ : declaration of analyst's prediction.

4. Mapping b/w model & data variables

"Product Accounts". $Y = C + I + G + NX$

"Income Accounts" $Y = W + R$

Note: data (GDP) = (Cons) + (Inv.) + (Gov purchase) + (Net imports)

model: $Y_t = C_t + X_t + 0$

(?) gross inv

Yes, by market clearing for goods.

data : (GDP) = (labor income) + (capital inc)

model: $Y_t = W_t L_t + r_t K_t + \pi_t$

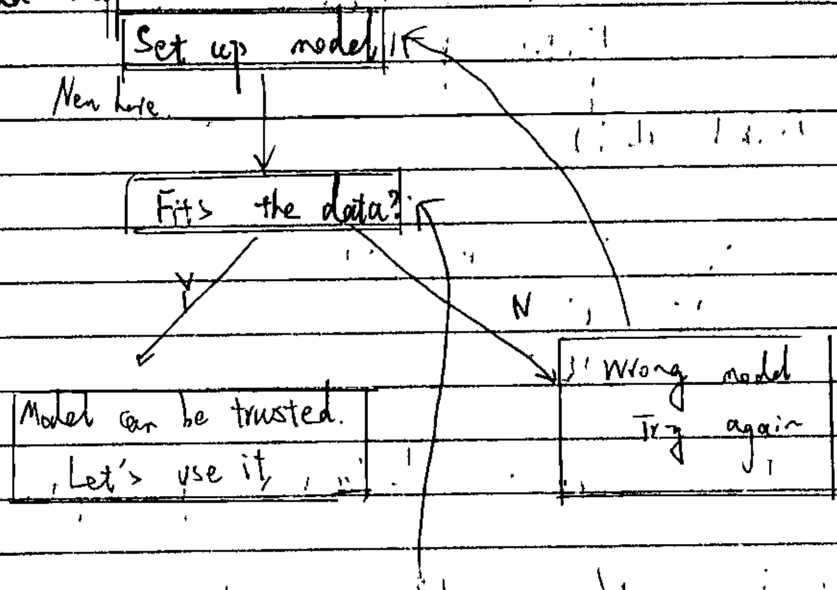
(?)

Yes,

By the definition of π_t

$$\pi_t = \underbrace{F(K_t^*, L_t^*)}_{= Y_t^*} - r_t K_t - w_t L_t$$

5. Road map



① Work out what the model predicts

② check if predictions match the data well.

III. The Solow model

1. Solow (1956, 1957)

Idea: ~~short cut~~

equ cond ~~(i) - HH~~ optimization

replace this with something easier;

(i) $L_t = \bar{L}$ $(\forall t)$

(ii) $w + L_t^* + r_t^* K_t^* + \pi_t^*$

fraction s ↓ X_t^* ; $1-s$ ↘ C_t^*

Recall Facts 2 & 4 $\Rightarrow$ empirical motivation for above assumptions

How does the model work?

$$\underbrace{X_t^*}_{} = s\,\underbrace{\left(W_t^* L_t^* + r_t^* K_t^* + \pi_t^*\right)}_{\overset{\shortparallel}{Y_t^* = F(K_t^*, L_t^*)}}$$

L_t^* K_t^*

$$\underbrace{K_{t+1}}_{\overset{\shortparallel}{K_{t+1}^*}} = (1-\delta)\underbrace{K_t}_{\overset{\shortparallel}{K_t^*}}$$

$$Y_t^* = F(K_t^*, \underbrace{L_t^*}_{})$$

$L_t^* = \bar{L}$

$$\Rightarrow \boxed{K_{t+1}^* = (1-\delta)K_t^* + sF(K_t^*, \bar{L}) \quad (\forall t)}$$

↑ 1st order difference e.g

How this works?

$(t=0)$: $K_1^* = (1-\delta)K_0^* + sF(K_0^*, \bar{L})$

$(t=1)$: $K_2^* = (1-\delta)K_1^* + sF(K_1^*, \bar{L})$

$(t=2)$: ....

$\Rightarrow$ gives us $(K_t^*)_{t=0}^{\infty}$

once we know $(K_t^*)_{t=0}^{\infty}$, we can recover the other variables as:

$Y_t^* = F(K_t^*, \bar{L})$

$X_t^* = sY_t^* = sF(K_t^*, \bar{L})$

$C_t^* = (1-s)Y_t^* = (1-s)\,F(K_t^*, \bar{L})$

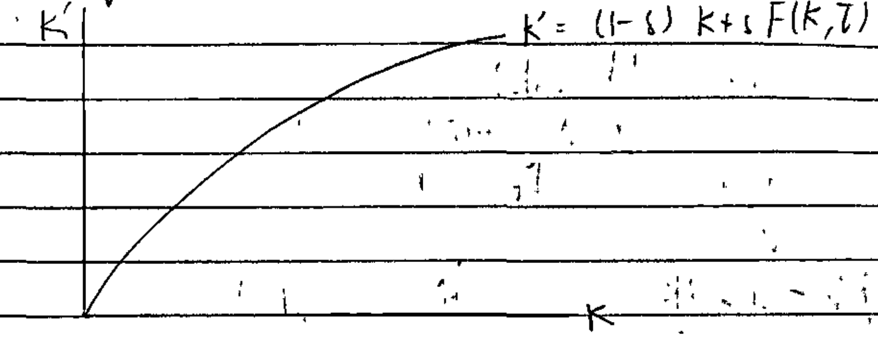
$l_t^* = L_t^* = \bar{L}$

$W_t^* = \ldots F_L(K_t^*, L_t^*)$

$r_t^* = \ldots F_K(K_t^*, L_t^*)$

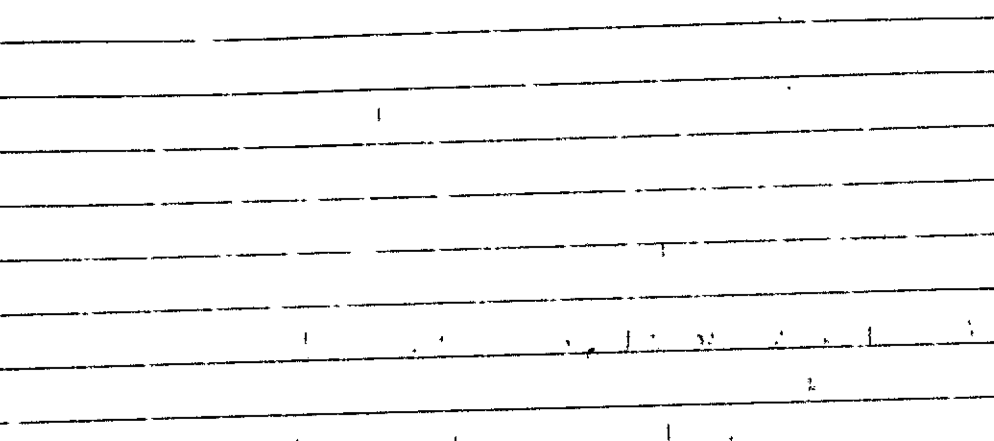
$\pi_t^* = 0$

Empirical device:



case 3)

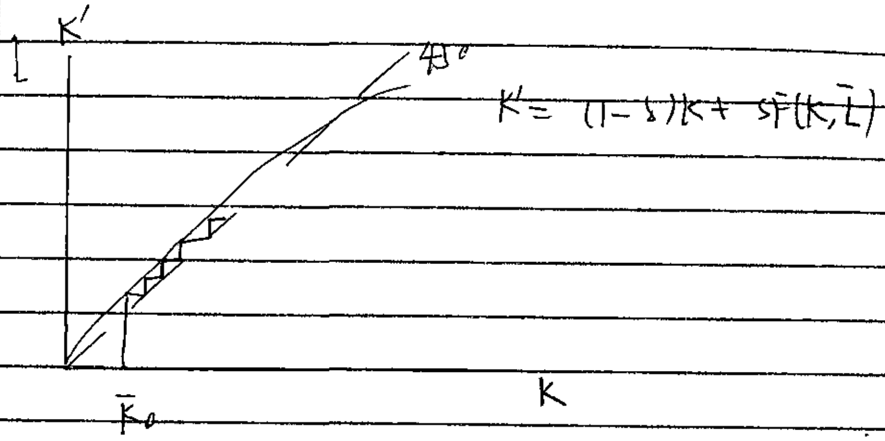
$K' = (1-\delta)K + sF(K, l)$



Q : Is the model consistent with the data?

Q1. Fact 1 (sustainable growth in Y_t)?

(can the model generate this?) ✓



Yes, in case 1 & 3

Q2 Fact 2 (constant, hours per week

↑ Model consistent with this?

OK: hours $= \bar{L} =$ constant)

Q3 Facts 3−5?

↑ Model has trouble replicating these

(Trouble reconciling Facts 1, 2, 3, 5)

(∵) In model

$Y_t^* = F(K_t^*, \bar{L}) \quad (\forall t) \qquad \boxed{\Delta x_t = x_t - x_{t-1}}$

$\Longrightarrow \Delta Y_{t+1}^* \doteq F_K(K_t^*, \bar{L})\, \Delta K_{t+1}$

(taylor approx) $\dfrac{\Delta Y_{t+1}^*}{\Delta Y_t} \doteq \dfrac{F_K(K_t^*, \bar{L})\, K_t^*}{Y_t^*} \cdot \dfrac{\Delta K_{t+1}^*}{K_t^*} \quad (\forall t)$

? $\Longrightarrow \dfrac{\Delta Y_{t+1}^*}{\Delta Y_t} = \left(\dfrac{r_t^* K_t^* + \pi_t^*}{Y_t^*} \right) \cdot \dfrac{\Delta K_{t+1}^*}{K_t^*} \quad (\forall t)$

Data: $g > \alpha$ ... $g < \dfrac{K_t}{Y_t} = 3$

$\alpha \in (0, 1)$

3% >> 1/3 × 3%

- Problem:
  - In model, all Y growth comes from K growth
  - In the data, rate of K growth isn't high enough to ~~ex~~ fully explain output growth.

$\Rightarrow$ Idea: Try adding something to the model — besides capital accumulation — that can make Y growth.

2.2

Notes: How to think about HH's in the model.

- HH's indexed by $j \in [0, 1]$ (they are identical)

HH j chooses $(c_t^j, l_t^j, x_t^j, k_t^j)_{t=0}^{\infty}$

Market clearing:

$$\begin{cases} L_t = \int_0^1 l_t^j \, dj \quad l_t^j = l_t \\ K_t = \int_0^1 \underbrace{k_t^j}_{= k_t} dj = k_t \\ \underbrace{\int_0^1 c_t^j \, dj}_{= C_t} + \underbrace{\int_0^1 x_t^j \, dj}_{= X_t} = F(K_t, L_t) \end{cases}$$

C_t $\quad$ X_t

Idea #1: $Y_t = F(K_t, \underset{\text{allow employment growth}}{L_t})$

- This helps, but only partially
- Model implies:

$$\frac{\Delta Y_{t+1}}{Y_t} \doteq \left(\frac{r_t K_t}{Y_t}\right) \frac{\Delta K_{t+1}}{K_t} + \left(\frac{W_t L_t}{Y_t}\right) \frac{\Delta L_{t+1}}{L_t} \quad (\forall t)$$

check?

Data: $3\% \quad \underbrace{1/3 \times 3\%}_{1\%} + \underbrace{2/3 \times 2\%}_{1.3\%}$

$\Rightarrow$ Gap persists

Idea #2:

$Y_t = F_t(K_t, L_t)$

[production function shifts upward over time due to "techology growth"

$\Rightarrow$ Works quite well ...

2. Solow model w/ TFP growth

- Replace "$Y_t = F(K_t, L_t)$" with:

$Y_t = F(K_t, A_t L_t) \qquad A_t = (1+g)^t, \quad g > 0$

A_t: Total Factor Productivity (TFP)

= an index of technology

... works like an improvement in the quality of labor inputs.

Note: $Y_t = F(K_t, A_t L_t)$ .. motivation?

(Labor augmenting technology progress) (Harrod neutral)

possible alternatives:

$Y_t = A_t F(K_t, L_t) \leftarrow$ (Hicks neutral tech progress.)

$Y_t = F(A_t K_t, L_t) \leftarrow$ (capital augmenting tech progress a.k.a. Solow neutral)

- Why Harrod neutral?

- Start w/ (a) completely general specification

$$Y_t = F_t(K_t, L_t)$$

- In the data:

$$\circledast \begin{cases} \dfrac{\Delta Y^*_{t+1}}{Y^*_t} = g = \dfrac{\Delta K^*_{t+1}}{K_t} \\ L^*_t = \bar{L} \quad (\forall t) \end{cases}$$

- If we have these, we must have:

$$Y^*_0 = F_0(K^*_0, L^*_0)$$

$$\frac{Y^*_t}{(1+g)^t} \qquad \frac{K^*_t}{(1+g)^t}$$

$$\Rightarrow Y^*_t = (1+g)^t F_0\left(\frac{K^*_t}{(1+g)^t}, L^*_0\right)$$

$$= F_0\left(K^*_t, (1+g)^t L^*_0\right) \quad (\text{CRS})$$

$$\bar{L} = L^*_t$$

$$= F_0\left(K^*_t, (1+g)^t L^*_t\right) \quad (\forall t)$$

$$F_0 = F$$

Proposition: Let F_t be the true (CRS) production function in period t. If the equ $(Y^*_t, K^*_t, L^*_t)_{t=0}^{\infty}$ satisfies $\circledast$ then $\exists F$ (CRS) such that

$$\star \rightarrow Y^*_t = F(K^*_t, (1+g)^t L^*_t) \quad (\forall t)$$

I.e.: If $\circledast$ holds, then the data behaves as if there's Harrod neutral tech. change in the back growth

N.B. $\circledast$ only holds on the equ path.
It doesn't say that $\exists F$.

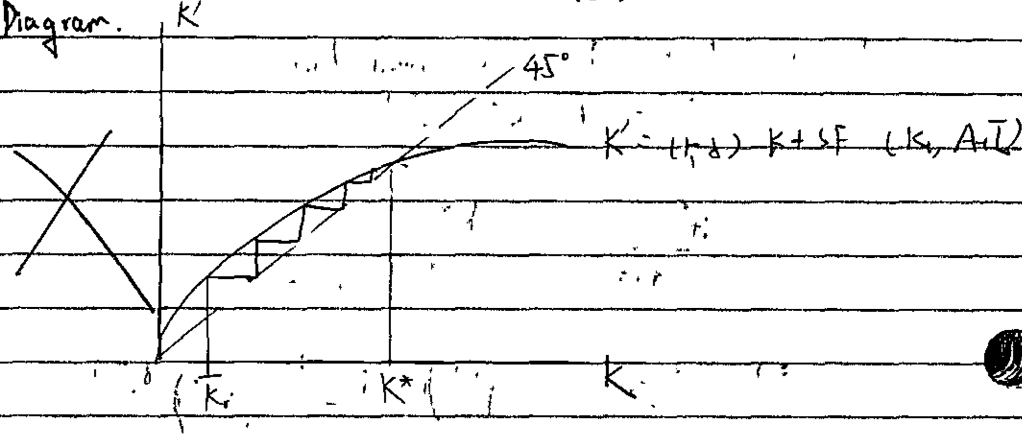
$F_t(K_t, L_t) = F(K_t, (1+g)^t L_t)$ $(* K_t, L_t)$

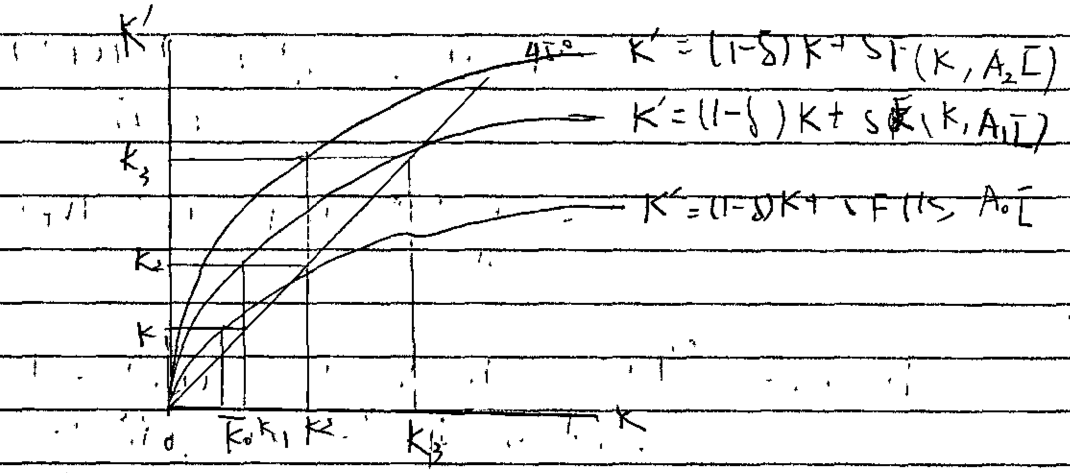
- How does the model work?
  - Repeat what we did before....

$$K_{t+1} = (1-\delta) K_t + sF(K_t, A_t \bar{L}) \quad (\forall t)$$

new

Diagram.





A better way: de-trending

$$\tilde{K}_t = K_t / A_t$$
$$\tilde{Y}_t = Y_t / A_t$$
$$\tilde{X}_t = X_t / A_t$$

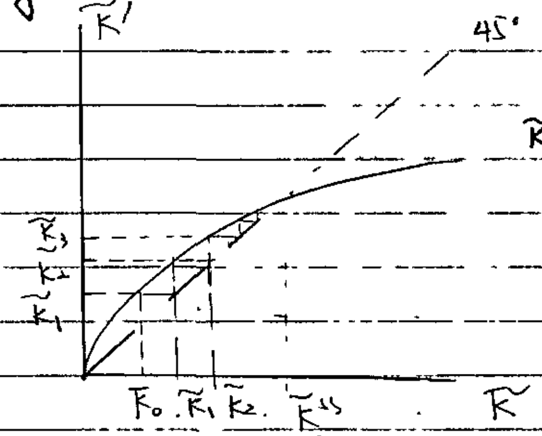
Recall:

$$K_{t+1} = (1-\delta)K_t + sF(K_t, A_t\bar{L}) \quad (\forall t)$$

$$\Rightarrow \underbrace{\frac{K_{t+1}}{A_t}}_{\frac{K_{t+1}}{A_{t+1}} \cdot \frac{A_{t+1}}{A_t} = \tilde{k}_{t+1}(1+g)} = (1-\delta)\underbrace{\frac{K_t}{A_t}}_{\tilde{k}_t} + sF\Big(\underbrace{\frac{K_t}{A_t}}_{\tilde{k}_t}, \bar{L}\Big) \quad (\forall t)$$

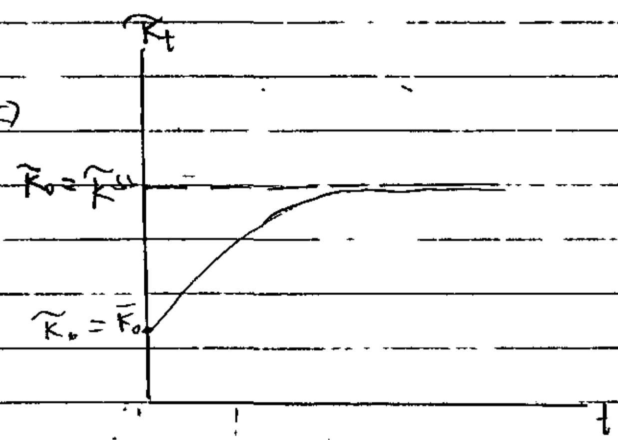
$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{1+g}\left\{(1-\delta)\tilde{k}_t + sF(\tilde{k}_t, \bar{L})\right\} \quad (\forall t)$$

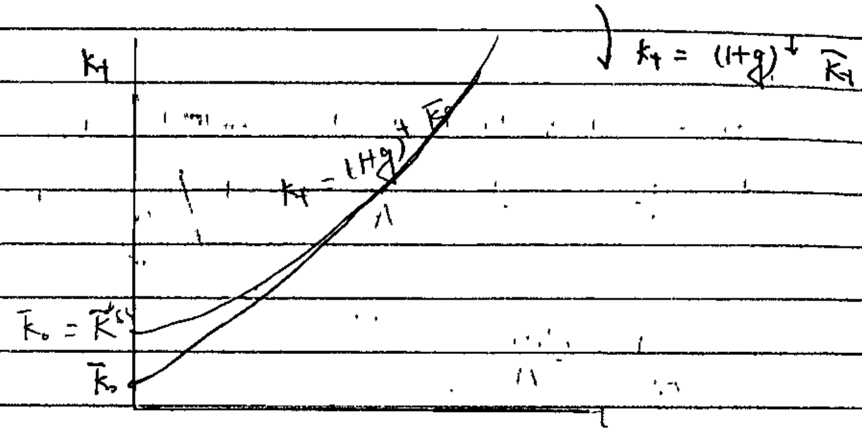
Visually:



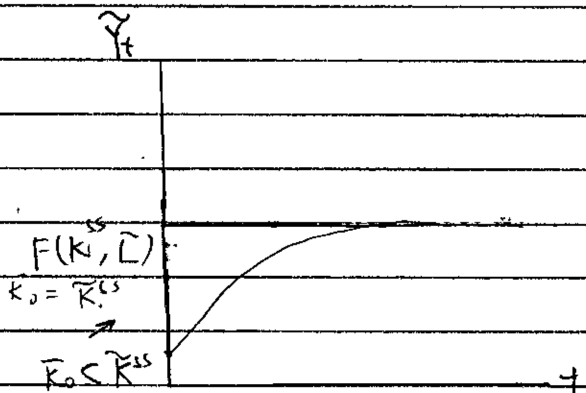
↑ steady state detrended capital stock

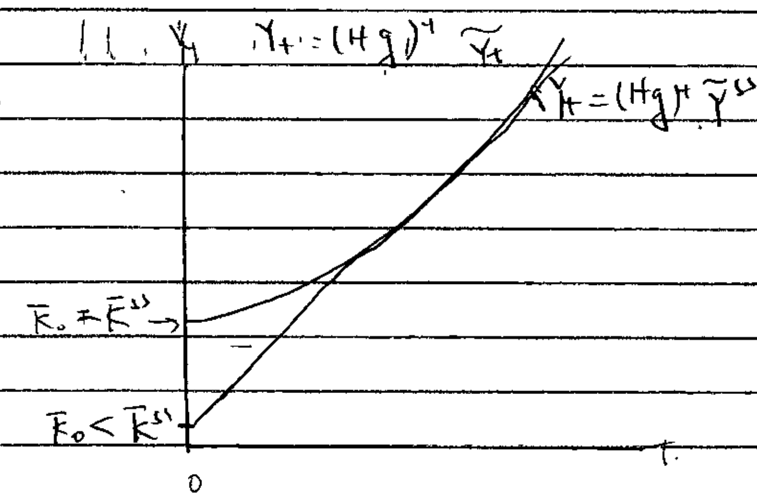
$\Rightarrow$





Also: $\tilde{Y}_t = Y_t / A_t = F\left(\underbrace{\frac{K_t}{A_t}}_{= \tilde{K}_t}, \bar{L}\right)$





Def. An equ exhibits balanced growth if each variable grows at a constant rate

(Different variables may have different growth rates)

Here: - If $\bar{K}_0 = \tilde{K}^{ss}$ then the equm exhibits balanced growth (← Verified for K, Y; check other variables)

- If $\bar{K}_0 \neq \tilde{K}^{ss}$ then the equm converges to an equm that exhibits balanced growth. (= balanced growth path)

Q: Does this model "fit the data facts?

Assume first that:

$$\bar{K}_0 = \tilde{K}^{ss}$$

$$\Rightarrow \tilde{K}_t = \tilde{K}^{ss} \ (\forall t) \Rightarrow$$

$$K_t = A_t \tilde{K}_t = (1+g)^t \tilde{K}^{ss} \ (\forall t)$$

$$\tilde{Y}_t = F(\tilde{K}^{ss}, \bar{L}) \ (\forall t) \Rightarrow Y_t = A_t \tilde{Y}_t = (1+g)^t F(\tilde{K}^{ss}, \bar{L}) \ (\forall t)$$

$$\vdots$$

Fact 1: Roughly constant growth of real GDP

Model: $\frac{Y_{t+1} - Y_t}{Y_t} = g > 0 \Rightarrow OK$

Fact 2: Roughly constant hours per worker.

Model: $l_t = \bar{l} \rightarrow OK$

Fact 3: Roughly constant K_t / Y_t

Model: $\frac{K_t}{Y_t} = \frac{\cancel{(1+g)^t} \tilde{K}^{ss}}{\cancel{(1+g)^t} F(\tilde{K}^{ss}, \bar{L})}$ ← const $\Rightarrow$ OK

Fact 4. Roughly constant expenditure shares

Model: $\frac{C_t}{Y_t} = 1 - s$

$\frac{X_t}{Y_t} = s$ $\Rightarrow$ OK

Fact 5: Roughly constant income shares.

Model: $\frac{r_t K_t + \pi_t}{Y_t} = \frac{F_K(K_t, A_t \bar{L}) K_t}{Y_t}$ "$A_t \bar{K}^{ss}$", $= F_K(\bar{K}^{ss}, \bar{L})$, ↑ constant (above)

$\frac{W_t L_t}{Y_t} = 1 - \frac{r_t K_t + \pi_t}{Y_t}$ (constant)

$\Rightarrow$ OK

F: CRS

$\Rightarrow_{\text{math}}$ $F_K(\lambda K, \lambda L) = F_K(K, L)$

$F_L(\lambda K, \lambda L) = F_L(K, L)$

Facts 6, 7, ...: NG – model generates no business cycles

" has nothing to say about inflation

Assessment: Not too bad as a starting point.

3. Policy analysis.

Q: Given the model's fit with the data, can we use it for policy analysis?

E.g. Use model to predict impact of a tax reform.

A: Probably not

Reason:

We expect: " " "

change taxes

$\Downarrow$

(change incentives to work/invest)

$\Downarrow$

change in $\bar{l}$, s

But model has no way of predicting how $\bar{l}$, s, etc will change. (b/c they are exogenously fixed parameters)

Version of Lucas Critique (Lucas 1976)

Consider a policy change & a model parameter

The parameter is invariant with respect to the policy change if it does not change in response to it.

Ex policy change: change in capital income taxes

δ: (likely) invariant to above

s: (likely) not invariant to above

Lucas Critique If you do policy analysis using a model that assumes a parameter is policy invariant when in fact, it's not, you will (likely) get a wrong answer.

Response: Go back to model with explicit HH optimization

III Back to $\underbrace{\text{neo classical growth}}_{\text{w/ HH optimization}}$

1. A special case.

$t = 0, 1$ (two periods)

no labor

$Y_t = F(K_t) = AK_t \quad (A > 0)$

utility: $V(c_0, c_1)$

$\bar{\delta} = 1$

$\Rightarrow$ Eqm characterization:

- Suppose $(c_0^*,\ c_1^*,\ x_0^*,\ k_0^*,\ k_1^*,\ r_0^*,\ r_1^*,\ \pi_0^*,\ \pi_1^*)$ is a eqm.

By firm optimization:

$F_K(K_t^*) = r_t^* \quad (\forall t)$

$\underset{\overset{\|}{A}}{F_K(K_t^*)} = r_t^* \quad (\forall t)$; $\pi_t^* = 0$

o HH optimization:

$$\max_{c_0, c_1, x_0} \ U(c_0, c_1) \qquad A$$

$$c_0 + x_0 \le r_0^* \bar{K}_0 + \pi_0^*$$

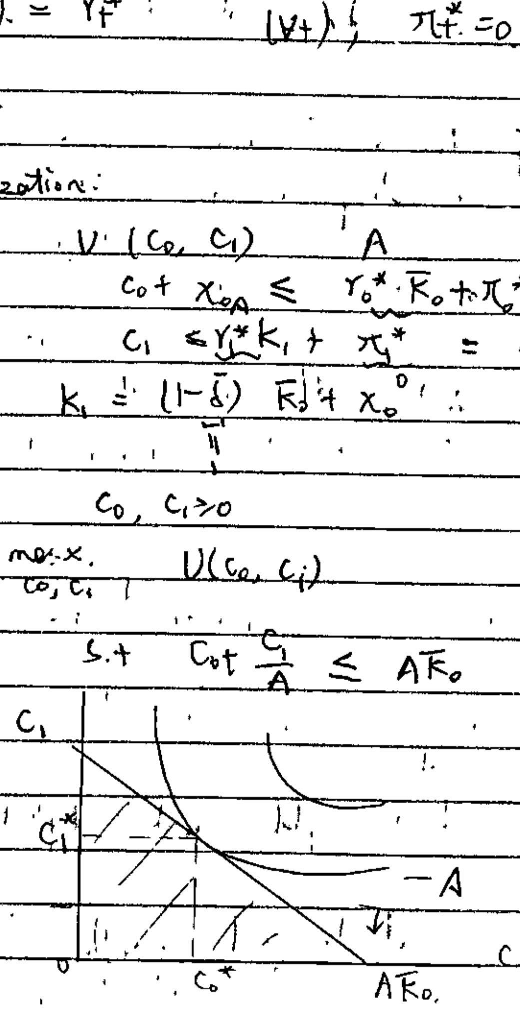
$$c_1 \le r_1^* k_1 + \overset{0}{\pi_1^*} = A x_0$$

$$k_1 = \underset{\overset{\|}{0}}{(1-\delta)} \bar{K}_0 + x_0$$

$$c_0,\ c_1 \ge 0$$

substitute out x_0 $\Longrightarrow$ $\max_{c_0, c_1} \ U(c_0, c_1)$

$$\text{s.t.} \quad c_0 + \frac{c_1}{A} \le A\bar{K}_0$$



$x_0^* = A\bar{K}_0 - c_0^*$

$k_0^* = \bar{K}_0$ <Done!>

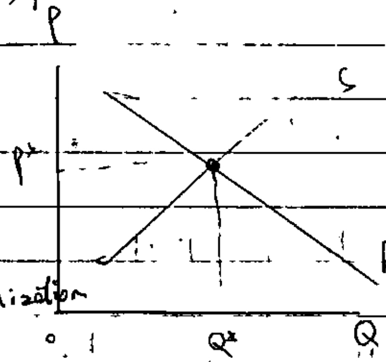
$c_1^* = A x_0^* = A k_1^* \qquad k_1^* = x_0^*$

2. More generally...

- Graphical construction like the above doesn't work ($\because$ too many variables)

- 2 approaches:

Approach #1 (Direct)



Step 1: Derive demand / supply curves from HH / firm optimization

$\mathbb{P} = (w_0, w_1, \ldots, r_0, r_1, \ldots)$

$\underset{(d)}{\check{C}_t(\mathbb{P})}, \ \underset{S}{\hat{L}_t(\mathbb{P})}, \ \check{x}_t(\mathbb{P}), \ \hat{K}_t(\mathbb{P}), \ \check{K}_t(\mathbb{P}), \ \check{L}_t(\mathbb{P})$

Step 2: Find $\mathbb{P}^*$ such that markets clear, i.e.,

$$\begin{cases} \check{C}_t(\mathbb{P}^*) + \check{x}_t(\mathbb{P}^*) = F(\check{K}_t(\mathbb{P}^*), \check{L}(\mathbb{P}^*)) & (\forall t) \\ \hat{L}_t(\mathbb{P}^*) = \check{L}_t(\mathbb{P}^*) & (\forall t) \\ \underset{D}{\check{K}_t(\mathbb{P}^*)} = \underset{S}{\hat{K}_t(\mathbb{P}^*)} & (\forall t) \end{cases}$$

$\leftarrow$ Conceptually straightforward but hard to use.
(Step 2 is an "$\infty \times \infty$" system of non-linear eq's)

Approach #2 (Negishi)
$\leftarrow$ Less straightforward but easier to use.

2.9

5. Negishi approach to eq^n characterization

Imagine an alternative setup:

- Economy run by a benevolent dictator (social planner)
- No markets; planner directly chooses quantities
- Planner's objective $= U(c_0, c_1, \dots\ l_0, l_1 \dots)$
- planner has to obey the laws of nature:

$$\circledast \begin{cases} c_t + x_t \le F(k_t, l_t) \\ k_{t+1} \le (1-\delta) k_t + x_t \\ k_0 = \bar{k}_0 \\ c_t, l_t, 1-l_t, x_t, k_t \ge 0 \end{cases} \quad (\forall t)$$

- Social planner's problem: SPP

$$(SPP) \begin{cases} \max\limits_{(c_t, l_t, x_t, k_t)_{t=0}^{\infty}} U(c_0, c_1, \dots, l_0, l_1 \dots) \\ \text{s.t. } \circledast \end{cases}$$

- Proposition: If $(c_t^*, l_t^*, x_t^*, k_t^*, K_t^*, L_t^*, \pi_t^*, w_t^*, r_t^*)_{t=0}^{\infty}$ is an eq^m then $(c_t^*, l_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$ solves (SPP).

Version of the first Welfare Theorem.

- Social significance: In terms of quantities, economy with free markets behaves as if it were run by a benevolent directator.
(I.e: "capitalism works")

- Analytically useful for equ characterization

(∵) Suppose we had an equ $(C_t^*, l_t^*, x_t^*, k_t^*, K_t^*, L_t^*, \pi_t^*, w_t^*, r_t^*)_{t=0}^{\infty}$

$\Rightarrow$ By the Proposition, $(C_t^*, l_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$ solves (SPP)

- Back out remaining variables as:

$$(A)\quad \begin{cases} K_t^* = k_t^* \\ L_t^* = l_t^* \\ \pi_t^* = 0 & \forall t \\ w_t^* = F_L(K_t^*, L_t^*) \\ r_t^* = F_K(K_t^*, L_t^*) \end{cases}$$

- Suggests:

Algorithm (Negishi):

1. solve (SPP) to get $(C_t^*, l_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$
2. Back out other variables using (A)
3. (Verify)

$\rightarrow$ [No need to solve an "$\infty \times \infty$" system of eq's

Proof: of Proposition.

Let $(C_t^*, l_t^*, x_t^*, k_t^*, K_t^*, L_t^*, \pi_t^*, w_t^*, r_t^*)_{t=0}^{\infty}$ be an equ.

Want to show: $(C_t^*, l_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$ solves (SPP)

Math x^* solves $\left\{ \begin{array}{l} \max_x f(x) \\ \text{s.t } g(x) \geq 0 \end{array} \right.$

$\updownarrow$

$\left\{ \begin{array}{l} \circ \text{ if } \tilde{x} \text{ satisfies } g(\tilde{x}) \geq 0 \text{ then } f(x^*) \geq f(\tilde{x}) \\ \circ \ g(x^*) \geq 0 \end{array} \right.$

(a) $(C_t^*, l_t^*, x_t^*, k_t^*)_{t=0}^{\infty}$ satisfies the constraints of (SPP). i.e:

$$\begin{cases} C_t^* + x_t^* \leq F(K_t^*, l_t^*) & \text{— ①} \\ K_{t+1}^* = (1-\delta) K_t^* + x_t^* \quad \forall t & \text{— ②} \\ K_0^* = \bar{K}_0 & \text{— ③} \\ C_t^*, l_t^*, 1-l_t^*, K_t^* \geq 0 & \text{— ④} \end{cases}$$

① ← OK: $C_t^* + x_t^* = F(\underset{\substack{\| \\ k_t^*}}{K_t^*}, \underset{\substack{\| \\ l_t^*}}{L_t^*})$ $(\forall t)$

market clearing

② ← OK: By constraints in HH's optimization condition

③ ← OK: $K_t^* = \bar{K}_0$

④ ← OK.

AND:

(b) If $(\tilde{c}_t, \tilde{l}_t, \tilde{x}_t, \tilde{k}_t)_{t=0}^{\infty}$ satisfies

$$\begin{aligned} \tilde{c}_t + \tilde{x}_t &\leq F(\tilde{k}_t, \tilde{l}_t) \quad & -⑤ \\ \tilde{k}_{t+1} &= (1-\delta)\tilde{k}_t + \tilde{x}_t \quad \forall t & -⑥ \\ \tilde{k}_0 &= \bar{k}_0 & -⑦ \\ \tilde{c}_t, \tilde{l}_t, 1-\tilde{l}_t&, \tilde{k}_t \geq 0 & -⑧ \end{aligned}$$

then $U(\tilde{c}_0, \tilde{c}_1, \dots, \tilde{l}_0, \tilde{l}_1, \dots) \leq U(C_0^*, C_1^*, \dots, l_0^*, l_1^*, \dots)$

To show this

(def of π_t^*) $\pi_t^* = F(K_t^*, L_t^*) - r_t^* K_t^* - w_t^* L_t^*$

(profit max) $\geq F(\tilde{k}_t, \tilde{l}_t) - r_t^* \tilde{k}_t - w_t^* \tilde{l}_t$

$\Rightarrow w_t^* \tilde{l}_t + r_t^* \tilde{k}_t + \pi_t^* \geq F(\tilde{k}_t, \tilde{l}_t)$

$\overset{⑤}{\geq} \tilde{c}_t + \tilde{x}_t$ — ⓐ

so $(\tilde{c}_t, \tilde{l}_t, \tilde{x}_t, \tilde{k}_t)_{t=0}^{\infty}$ satisfies

(9) : $\tilde{c}_t + \tilde{x}_t \le w_t^* \tilde{l}_t + r_t^* \tilde{k}_t + \pi_t^* \quad (\forall t)$

(1) : $\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{x}_t$

(7) : $\tilde{k}_0 = \bar{k}_0$

(8) : $\tilde{c}_t, \tilde{l}_t, 1-\tilde{l}_t, \tilde{k}_t \ge 0$

which are the constraints in the HH's optimization problem given eqm price.

So by HH's optimization condition (i)

$$U(\tilde{c}_0, \tilde{c}_1, \dots, \tilde{l}_0, \tilde{l}_1, \dots) \le U(c_0^*, c_1^*, \dots, l_0^*, l_1^* \dots)$$

Next task – solve (SPP)

Two approaches

#1 Variational approach

← FOCs Kuhn-Tucker conditions.

#2. Recursive approach

← Dynamic programming

You should know both.

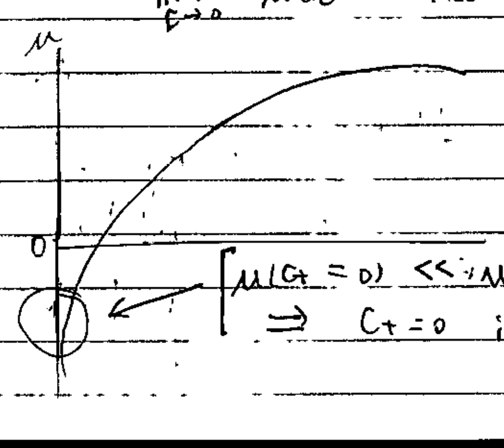
Assume for now;

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- time-seperable
- no disutility from labor $\Rightarrow l_t = 1$

$u' > 0$, $u'' < 0$, u is bounded.

$\lim_{c \to 0} u'(c) = +\infty$ "Inada condition"



$u(c_t = 0) \ll u(c_t = \hat{\varepsilon})$

$\Rightarrow c_t = 0$ is never optimal

∘ $F(0, L) = 0$

∘ So (SPP) becomes:

$$\begin{cases} \max_{(c_t, k_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } c_t + k_{t+1} \leq F(k_t, 1) + (1-\delta)k_t \quad (\forall t) \\ k_0 = \bar{k}_0 \\ c_t, k_t \geq 0 \quad (\forall t) \end{cases}$$

∘ Prop: SPP has a unique solution

(∵) ∘ Existence of solution

$\max_{x \in X} f(x)$ [Weirstrass's Theorem]

X: compact, f: continuous

⇓ solution

∘ Uniqueness of solution [← constraint set is convex & objective is strictly concave]

4. Variational approach

(a) The math

(1) Two-period case

$$\max_{(c,k)} u(c_0) + \beta u(c_1)$$

s.t. $c_0 + k_1 \leq F(k_0, 1) + (1-\delta)k_0$ $[\lambda_0]$

$c_1 + k_2 \leq F(k_1, 1) + (1-\delta)k_1$ ~ ⊛ $[\lambda_1]$

$k_0 = \bar{k}_0$

$c_0 \geq 0$

$c_1 \geq 0$

"=" at solution ($\because u' > 0$)

">" at solution ($\because \lim_{c \to 0} u'(c) = +\infty$)

Pattern of binding constraints?

$k_1 \geq 0$ at solution, ($\because k_1 = 0$) $\Rightarrow c_1 = 0 \Rightarrow$ contradicts this $c_0 > 0$

$k_2 \geq 0$ $[\mu]$

Lagrangean:

$$\mathcal{L} = u(c_0) + \beta u(c_1)$$
$$+ \lambda_0 [F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0 - c_0 - k_1]$$
$$+ \lambda_1 [F(k_1, 1) + (1-\delta) k_1 - c_1 - k_2]$$
$$+ \mu k_2$$

FOCs: (c_0): $u'(c_0^*) = \lambda_0^*$ ①

(c_1): $\beta u'(c_1^*) = \lambda_1^*$ ②

(k_1): $\lambda_0^* = \lambda_1^* [F_k(k_1^*, 1) + 1 - \delta]$ ③

(k_2): $\lambda_1^* = \mu^*$ ④

Complementary slackness: $\mu k_2 = 0$ ⑤

Constraint: $(\ldots)$

1 K-T Theorem: these are necessary & sufficient for optimality

simplify:

①, ②, ③ $\Rightarrow u'(c_0^*) = \beta u'(c_1^*) (F_k(k_1^*, 1) + 1 - \delta)$

⑤: $\mu^* k_2^* = 0$

$$\mu^* = \lambda_1^* \overset{②}{=} \beta u'(c_1^*) > 0 \quad (\beta > 0,\ u' > 0)$$

$\Rightarrow k_2^* = 0$

Prop: $(c_0^*, c_1^*, k_0^*, k_1^*, k_2^*)$ solves (SPP) iff:

$u'(c_0^*) = \beta u'(c_1^*) (F_k(k_1^*, 1) + 1 - \delta)$ Euler equation (EE)

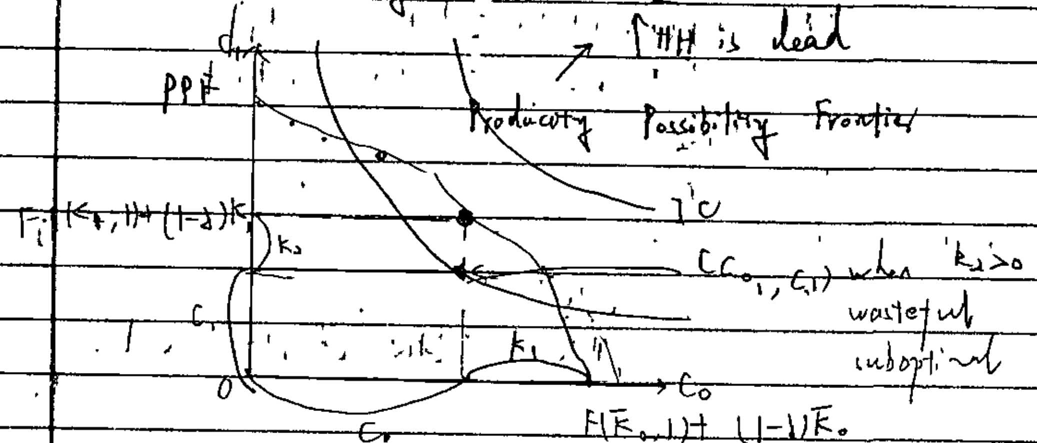
$k_2^* = 0$ Transversality condition (TVC)

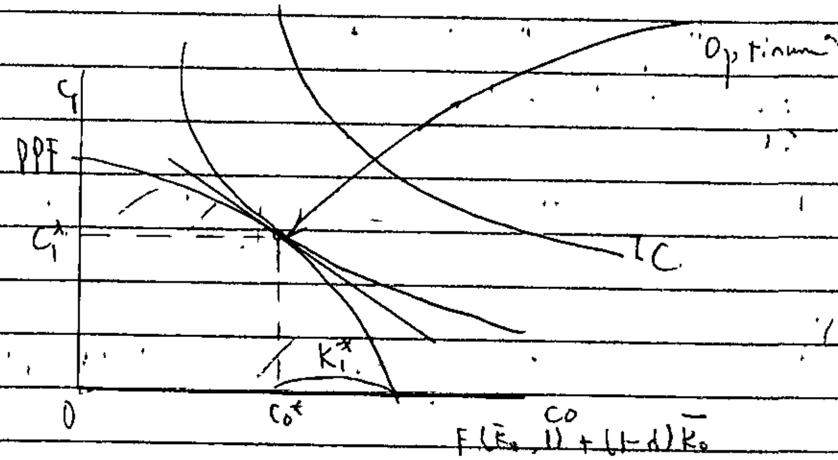
$c_0^* + k_1^* = F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0$

$c_1^* + k_2^* = F(k_1^*, 1) + (1-\delta) k_1^*$

$\begin{cases} c_0^*, c_1^*, k_1^* \geq 0 \\ k_1^* = \bar{k}_0 \end{cases}$

- Interpretation of TVC:
  "Don't bring K into t=2"



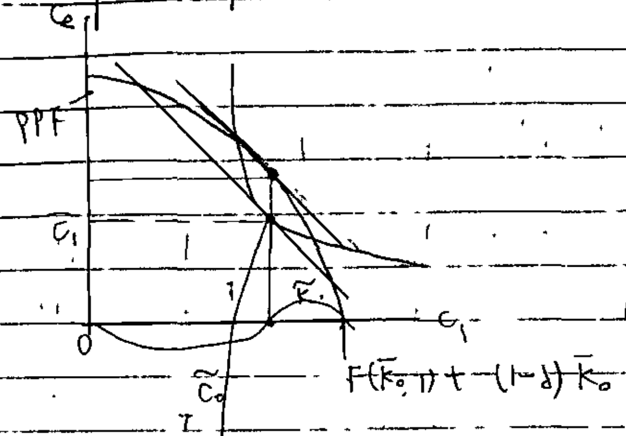


$$MRS = MRT$$

$$\| \quad F_K(K_1^*,1)+1-\delta$$

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)}$$

Note: — importance of TVC



sub optimal - obviously

∴ EE is satisfied (and, so are the constraints)

(ii) $T \, (<\infty)$ period case

$$(*) \begin{cases} \max\limits_{(c,k)} \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{s.t. } c_t + k_{t+1} \le F(k_t, 1) + (1-\delta)k_t \quad (\forall t = 0, 1, \dots, T) \quad [\lambda_t] \\ k_0 = \bar{k}_0 \\ c_t \ge 0 \quad (\forall t = 0, 1, \dots, T) \\ k_t \ge 0 \quad (\forall t = 0, 1, \dots, T) \\ k_{T+1} \ge 0 \quad [\mu] \end{cases}$$

$\le$: = at solution ($\because u' > 0$)

$c_t \ge 0$: >0 at solution ($\because u'(0) = \infty$)

$k_t \ge 0$: >0 at solution ($\because k_t = 0 \Rightarrow c_t = 0 \Rightarrow$ contradicts this)

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t [F(k_t, 1) + (1-\delta)k_t - c_t - k_{t+1}] + \mu k_{T+1}$$

$$\begin{aligned} \| \\ &\lambda_0 [F(k_0, 1) + (1-\delta)k_0 - c_0 - k_1] \\ &+ \lambda_1 [F(k_1, 1) + (1-\delta)k_1 - c_1 - k_2] \\ &+ \dots \\ &+ \lambda_t [F(k_t, 1) + (1-\delta)k_t - c_t - k_{t+1}] \\ &+ \lambda_{t+1} [F(k_{t+1}, 1) + (1-\delta)k_{t+1} - c_{t+1} - k_{t+2}] \\ &+ \dots \\ &+ \lambda_T [F(k_T, 1) + (1-\delta)k_T - c_T - k_{T+1}] \end{aligned}$$

FOCs: (C_t), $0 = \frac{\partial \mathcal{L}}{\partial C_t}$

$= \beta^t u'(C_t^*) - \lambda_t^*$

$\Rightarrow \beta^t u'(C_t^*) = \lambda_t^*$ — ①

(K_{t+1}): $0 = \frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t^* + \lambda_{t+1}^* [F_K(K_{t+1}^*, 1) + 1 - \delta]$

$\forall t = 0, \dots, T-1$ $\Rightarrow \lambda_t^* = \lambda_{t+1}^* [F_K(K_{t+1}^*, 1) + 1 - \delta]$ — ②

K_{T+1}: $0 = \frac{\partial \mathcal{L}}{\partial K_{T+1}} = \mu^* - \lambda_T^*$

$\Rightarrow \mu^* = \lambda_T^*$ — ③

Complementary slackness: $\mu^* K_{T+1}^* = 0$

Constraints

Simplicity: ①, ② $\Rightarrow$

$u'(C_t^*) = \beta u'(C_{t+1}^*) [F_K(K_{t+1}^*, 1) + 1 - \delta]$ $(\forall t = 0, \dots, T-1)$

$\mu^* \overset{③}{=} \lambda_T^* \overset{①}{=} \beta^T u'(C_T^*) > 0$

$\mu^* K_{T+1}^* = 0$ ④

$\Downarrow$

$K_{T+1}^* = 0$

Prop. $(C_0^*, \dots C_T^*, K_0^*, K_1^*, \dots K_{T+1}^*)$ solves ⊛ iff:

$u'(C_t^*) = \beta u'(C_{t+1}^*) [F_K(K_{t+1}^*, 1) + 1 - \delta]$ $(\forall t = 0, \dots, T-1)$ ← Euler eq (E.E)

$K_{T+1}^* = 0$ ← Transversality condition (TVC)

$C_t^* + K_{t+1}^* = F(K_t^*, 1) + (1-\delta) K_t^*$ $(\forall t = 0, \dots, T)$

$K_0^* = \bar{K}_0$

$C_t^*, K_t^* > 0$ $(\forall t = 0, 1, \dots, T)$

Proof: Kuhn-Tucker theorem + above ✓

(iii) infinite horizon

$$(*)\begin{cases} \max_{(C_t, K_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \\ \text{s.t.} \quad C_t + K_{t+1} \le F(K_t, 1) + (1-\delta)K_t \quad (\forall t \ge 0) \\ \quad\quad K_0 = \bar{K}_0 \\ \quad\quad C_t, K_t \ge 0 \quad (\forall t \ge 0) \end{cases}$$

A guess ... generalize / extrapolate case (ii) above:

Prop $(C_t^*, K_t^*)_{t=0}^{\infty}$ solves $(*)$ iff: $(\forall t \ge 0)$

$u'(C_t^*) = \beta u'(C_{t+1}^*)(F_K(K_{t+1}^*, 1) + (1-\delta))$ Euler eq (EE)

$\cancel{\lim_{T\to\infty} K_{T+1}^* = 0} \Leftarrow \lim_{T\to\infty} \beta^T u'(C_T^*) K_{T+1}^* = 0$

$C_t^* + K_{t+1}^* = F(K_t^*, 1) + (1-\delta)K_t^*$ Euler eq (EE)

$K_0^* = \bar{K}_0^*$

$C_t^*, K_t^* > 0$

math intuition : recall - for $T < \infty$

$\mu^* K_{T+1}^* = 0$, $\mu^* = \lambda_T^* = \beta^T u'(C_T^*)$ $\Rightarrow \underbrace{\beta^T}_{>0} \underbrace{u'(C_T^*)}_{>0} K_{T+1}^* = 0$ $\xrightarrow{T\to\infty}$

Note : $\underbrace{\lim_{T\to\infty} K_{T+1}^* = 0}_{\text{wrong TVC}} \not\Leftrightarrow \underbrace{\lim_{T\to\infty} \beta^T u'(C_T^*) K_{T+1}^* = 0}_{\text{right TVC}}$

E.g. $(C_t^*, K_t^*) \xrightarrow{t\to\infty} (\bar{C}, \bar{K})$, $\bar{C} > 0$, $\bar{K} > 0$

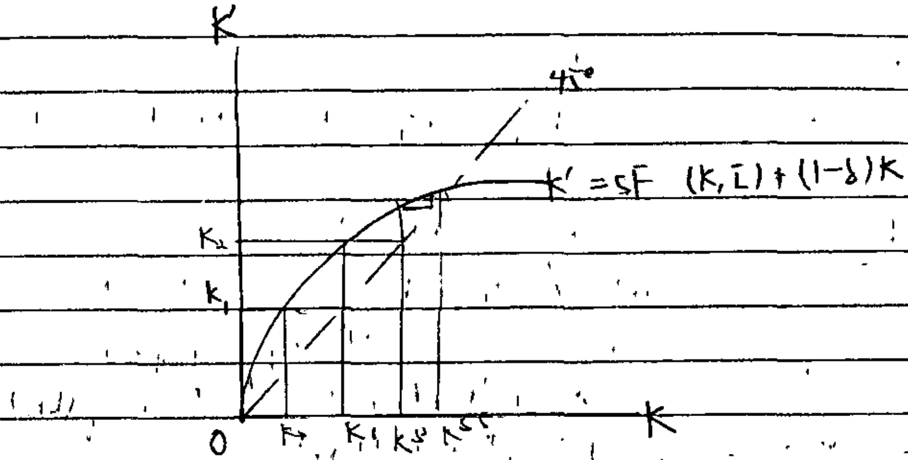
$\lim_{T\to\infty} K_{T+1}^* = \bar{K} > 0$ ... Wrong TVC doesn't hold.

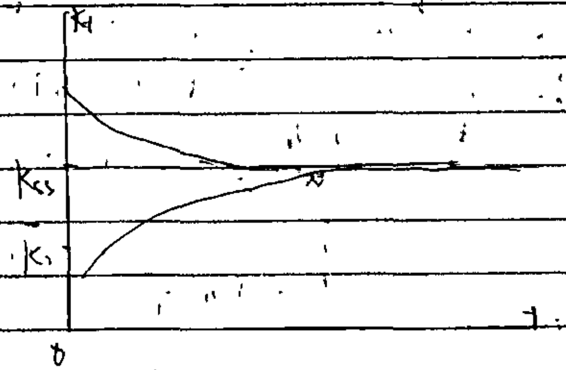
$\underbrace{\beta^T}_{0} \underbrace{u'(C_T^*)}_{u'(\bar{C})} \underbrace{K_{T+1}^*}_{\bar{K}} \to 0$

Proof: Notes: //

(b) Application: SPP solution characterization.

In Solow :





$\Rightarrow$ Under certain conditions: $\exists K^{ss} > 0$ | such that

(i) if $\bar{K}_0 = K^{ss}$ then $K_t^* = K^{ss}$ $(\forall t)$ in eqn

(ii) if $\bar{K}_0 \neq K^{ss}$ then $K_t^* \to K^{ss}$ (as $t \to \infty$) in eqn

The (non-Solow) model have works similarly.

Example:

$$\text{SPP:} \begin{cases} \max_{(C_t, K_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \overbrace{\log(C_t)}^{u(C_t)} \quad \swarrow L=1 \\ \text{s.t.} \quad C_t + K_{t+1} \leq \underbrace{A K_t^{\alpha}}_{F(K_t,1)} \quad (\forall t \geq 0) \\ K_0 = \bar{K}_0 \\ C_t, K_t \geq 0 \quad (\forall t \geq 0) \end{cases}$$

$\cdot$ $(C_t^*, K_t^*)_{t=0}^{\infty}$ solves this SPP iff:

$$\text{(EE)} \quad \underbrace{u'(C_t^*)}_{= \frac{1}{C_t}} = \beta \underbrace{u'(C_{t+1}^*)}_{= \frac{1}{C_{t+1}^*}} [1 - \underbrace{\delta}_{=0} + \underbrace{F_K(K_{t+1}^*, 1)}_{\alpha A K_{t+1}^{\alpha-1}}]$$

Here: $u(c) = \log(c)$
$F(K,1) = AK^{\alpha}$

$$\text{(TVC)}: \lim_{T \to \infty} \beta^T u'(C_T^*) K_{T+1}^* = 0$$

$$\text{FOC}\begin{cases} \text{(EE)}: & \dfrac{1}{C_t^*} = \beta \dfrac{1}{C_{t+1}^*} \alpha A K_{t+1}^{*\,\alpha-1} \quad (\forall t \geq 0) \\ \text{(TVC)}: & \lim_{T\to\infty} \beta^T \dfrac{1}{C_T^*} K_{T+1}^* = 0 \\ \text{(RC)}: & C_t^* + K_{t+1}^* = A K_t^{*\,\alpha} \quad (\forall t \geq 0) \end{cases}$$

$$K_0^* = \bar{K}_0$$

$$C_t^*, K_t^* > 0 \quad (\forall t \geq 0)$$

• How to solve (FOCs)?

(a) Usually need a computer.

(b) For this example, I can give you the solution and you can use the FOCs to verify it.

Proposed solution: let $(C_t^*, K_t^*)_{t=0}^{\infty}$ satisfy

$$\text{(*)} \begin{cases} K_{t+1}^* = \alpha\beta A K_t^{*\,\alpha} \quad (\forall t \geq 0), \quad K_0^* = \bar{K}_0 \\ C_t^* = (1-\alpha\beta) A K_t^{*\,\alpha} \quad (\forall t \geq 0) \end{cases}$$

Then $(C_t^*, K_t^*)_{t=0}^{\infty}$ solves (SPP).

Proof: let $(C_t^*, K_t^*)_{t=0}^{\infty}$ satisfy (*)

To show that this solves (SPP), we only need to verify that it satisfies (FOCs)

(EE): $\text{LHS} = \dfrac{1}{C_t^*} = \dfrac{1}{(1-\alpha\beta) A K_t^{*\,\alpha}}$ → OK

$$\text{RHS} = \beta \cdot \frac{1}{C_{t+1}^*} \alpha A K_{t+1}^{*\,\alpha-1} = \beta \frac{1}{(1-\alpha\beta) \cancel{A K_{t+1}^{*\,\alpha}}} \times A K_{t+1}^{*\,\alpha-1}$$

$$= \alpha\beta \frac{1}{(1-\alpha\beta) K_{t+1}^*}$$

$$= \cancel{\alpha\beta} \frac{1}{(1-\alpha\beta) \cancel{\alpha\beta} A K_t^{*\,\alpha}}$$

(TVC): $\lim_{T\to\infty} \underbrace{\beta^T \frac{1}{C_T^*} K_{T+1}^*}$

$$\beta^T \frac{1}{(1-\alpha\beta)A\cancel{K_t^{*\alpha}}} \alpha\beta A\cancel{K_t^{*\alpha}}$$

$$\beta^T \cdot \frac{\alpha\beta}{1-\alpha\beta} \xrightarrow[T\to\infty]{} 0$$

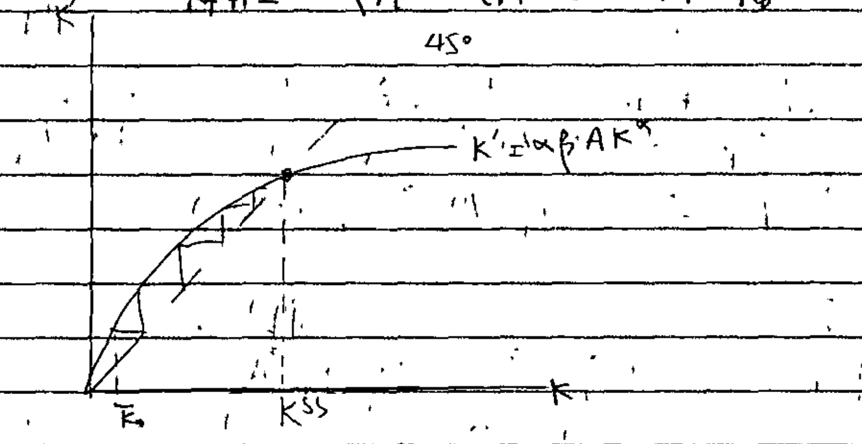
~~(RC)~~ $C_t^* + K_{t+1}^* = (1-\alpha\beta) AK_t^{*\alpha} + \alpha\beta \; AK_t^{*\alpha}$

$= AK_t^{*\alpha}$ ← OK

$\left.\begin{array}{l} K_0^* = \bar{K}_0 \\ C_t^*,\ K_t^* \geq 0 \quad (\forall t \geq 0 \end{array}\right\} \Rightarrow$ OK //

So: solution to (SPP) here satisfies

$$K_{t+1}^* = \alpha\beta K_t^{*\alpha} \quad (\forall t \geq 0) \quad : K_t^* = \bar{K}_0$$

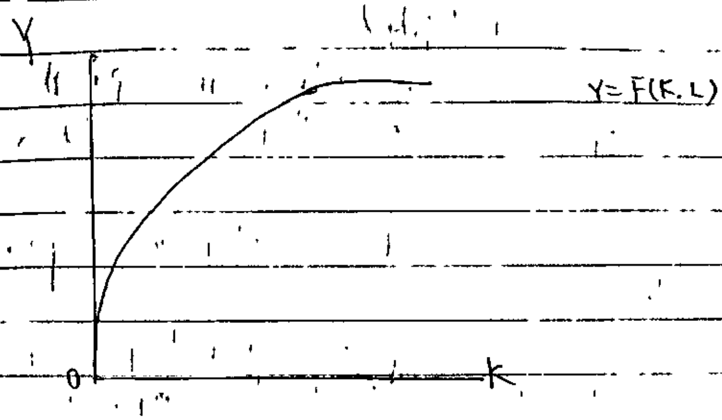


$\begin{cases} \text{if} \quad \bar{K}_0 = K^{ss} \text{ then } K_t^* = K^{ss} \quad (\forall t \geq 0) \\ \text{if} \quad \bar{K}_0 \neq K^{ss} \text{ then } K_t \to K^{ss} \quad (\forall t \geq 0) \end{cases}$

- Q: Does this hold more generally?
- A: Yes, (under Inada conditions for F)

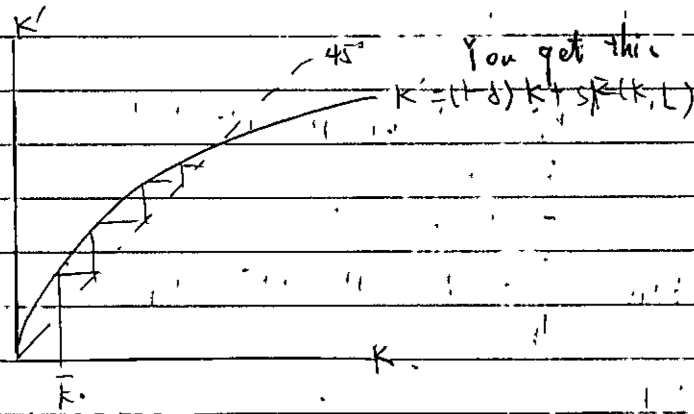
$$\lim_{K\to 0} F_K(K, L) = +\infty$$

$$\lim_{K\to\infty} F_K(K, L) = 0$$



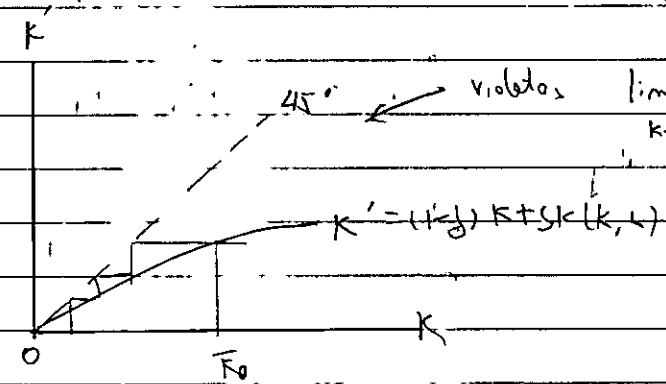
Solow.

<case1>



You get this w/

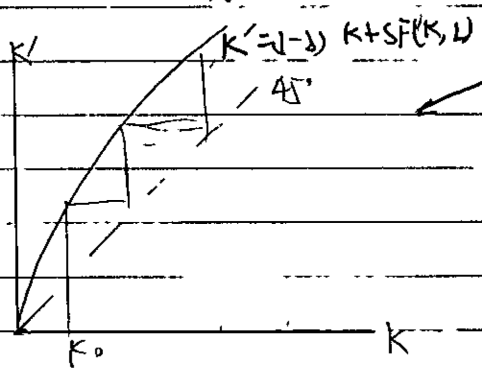
<case2>



violates $\lim_{K\to 0} F_K(K,L) = +\infty$

<case3>

~~$K^*_t \to \infty$~~



violates $\lim_{K\to\infty} F_K(K,L) = 0$

Proposition (Turnpike)

If $(c_t^*, k_t^*)_{t=0}^{\infty}$ solves the SPP then $\exists c^{ss} > 0$ $\exists k^{ss} > 0$ such that $(c_t^*, k_t^*) \to (c^{ss}, k^{ss})$

Explanation ($\Leftarrow$ not exactly a proof): using diagram:

FOC: ($\leftarrow$ necessary & sufficient for optimality):

(EE): $u'(c_t^*) = \beta u'(c_{t+1}^*)[1-\delta + F_K(k_{t+1}^*, 1)]$ $(\forall t)$

(TVC): $\lim_{T\to\infty} \beta^T u'(c_T^*) k_{T+1}^* = 0.$

(RC): $c_t^* + k_{t+1}^* = F(k_t^*, 1) + (1-\delta) k_t^*$ $(\forall t)$

$k_0^* = \bar{k}_0$

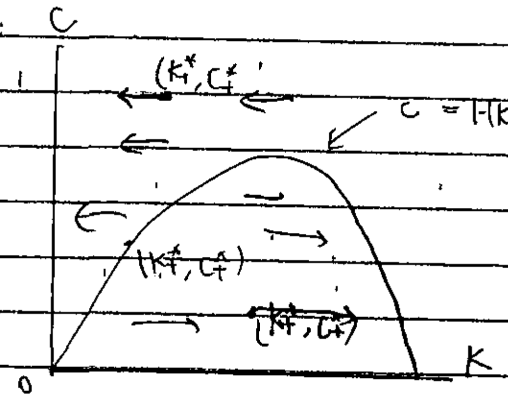
$c_t^*, k_t^* > 0$ $(\forall t)$

$k_{t+1}^* - k_t^* = F(k_t^*, 1) - \delta k_t^* - c_t^*$

$\Downarrow$

$F(k_t^*, 1) - \delta k_t^* - c_t^* \begin{cases} > 0 & \Rightarrow k_{t+1}^* > k_t^* \\ = 0 & = \\ < 0 & \Rightarrow k_{t+1}^* < k_t^* \end{cases}$

graphically:



• (EE)

$\Updownarrow$

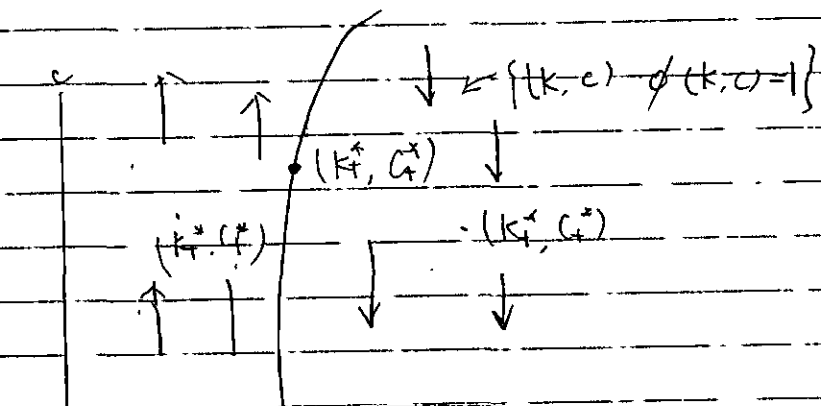
$$\frac{u'(c_t^*)}{u'(c_{t+1}^*)} = \beta[1-\delta + F_K(\underbrace{k_{t+1}^*}_{\overset{(RC)}{=}\ F(k_t^*, 1) + (1-\delta)k_t^* - c_t^*}, 1)]$$

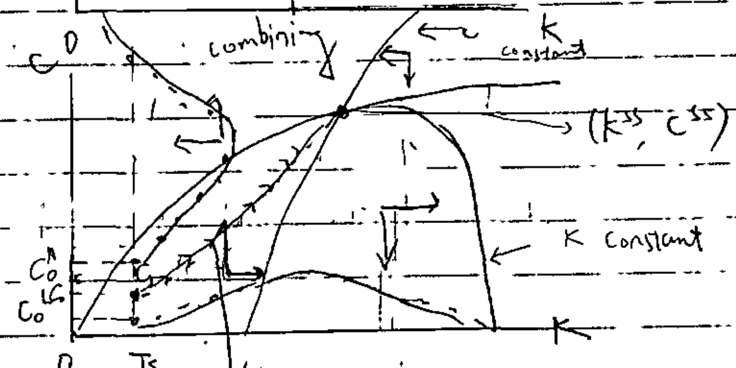
$$= \beta \underbrace{\left[1-\delta + F_K\left(F(K_t^*,1) + (1-\delta)K_t^* - C_t^*, 1\right)\right]}_{\phi(K_t^*,\, C_t^*)}$$

$\ominus \quad \oplus$

$$\phi(K_t^*, C_t^*) \begin{cases} >1 & \Rightarrow u'(C_t^*) > u'(C_{t+1}^*) \Rightarrow C_t^* < C_{t+1}^* \\ =1 & \qquad = \qquad\qquad\qquad\quad = \\ <1 & \Rightarrow u'(C_t^*) < u'(C_{t+1}^*) \Rightarrow \qquad > \end{cases}$$

graphically:



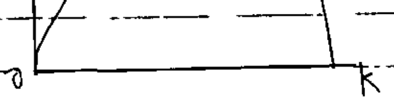


saddle path

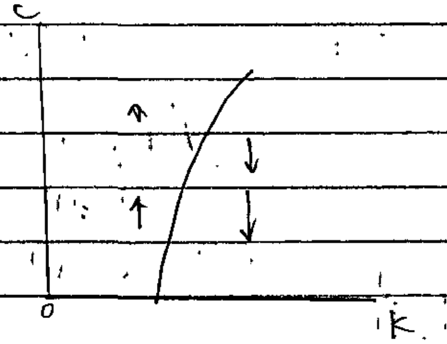
(← SPP solution follows this path)

Recap:

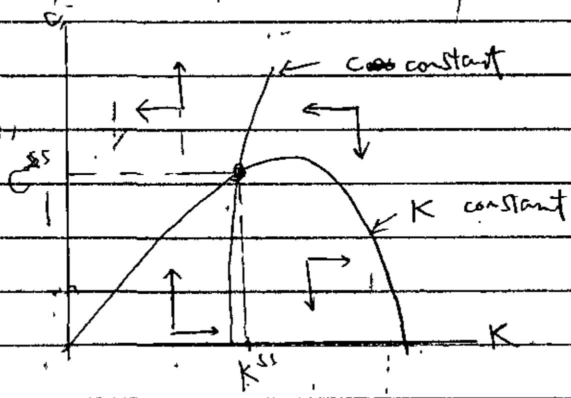
(RC): ⇒



(EE) $\Longrightarrow$



Combine



$(K^{ss}, C^{ss}) = ?$

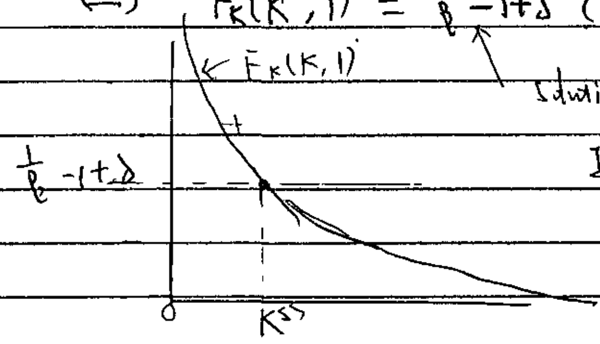
- K - constant $\quad F(K^{ss},1) - \delta K^{ss} - C^{ss} = 0$ — ①
- C - constant $\quad \phi(K^{ss}, C^{ss}) = 1$

$$\phi(K^{ss}, C^{ss}) = \beta[1-\delta + F_K(\underbrace{F(K^{ss},1) + (1-\delta)K^{ss} - C^{ss}}_{=K^{ss}}, 1)]$$

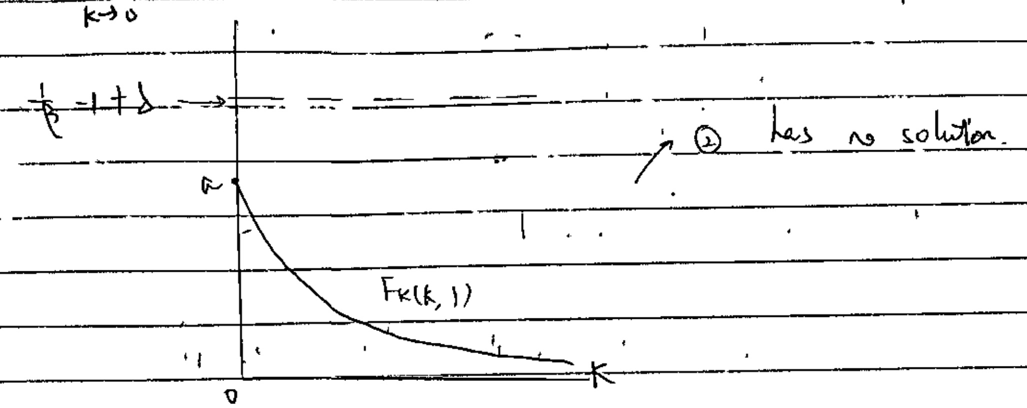
$$= \beta[1-\delta + F_K(K^{ss},1)]$$

$$\Longleftrightarrow \quad F_K(K^{ss},1) = \frac{1}{\beta} - 1 + \delta \quad (>0) \quad \text{— ②}$$

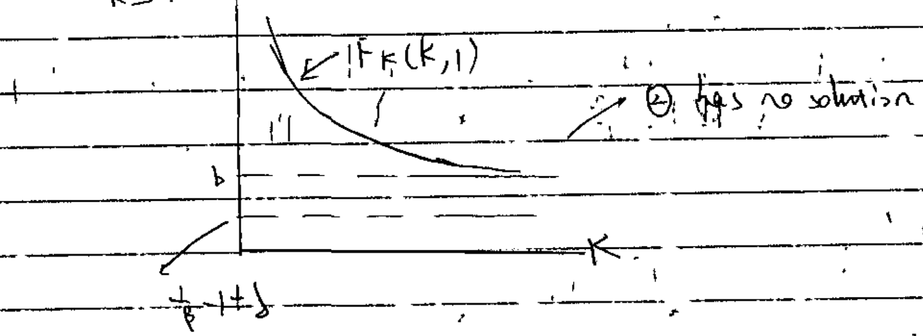
solution exists by Inada conditions



$\lim_{K \to 0} K_0(K,1) \to a < \infty$.



$\lim_{K \to 0} F_K(K,1) = b > 0$



$\therefore$ (K^{ss}, C^{ss}) are given by ①, ②

Eg: $F(K,L) = AK^{\alpha}L^{1-\alpha}$

$\Rightarrow K^{ss} = ? \longleftarrow F_K(K^{ss},1) = \frac{1}{\beta} - 1 + \delta$

$\qquad\qquad\qquad\qquad\qquad \alpha A K^{\alpha-1}$

$\Rightarrow K^{ss} = \left[\left(\frac{1}{\beta} - 1 + \delta\right) \frac{1}{\alpha A} \right]^{\frac{1}{\alpha-1}}$

$C^{ss} = F(K^{ss}, 1) - \delta K^{ss}$

This suggests that $\exists \{C_t^*, K_t^*\}_{t=1}^{\infty}$ such that.

(i) $K_0^* = \bar{K}_0$

(ii) $C_t^*, K_t^* > 0$ $(\forall t)$

(iii) on a path dictated by the arrows in the phase digram

$\Rightarrow$ (RC) (EE) are satisfied $(\forall t)$.

(iv) $(C_t^*, K_t^*) \to (C^{ss}, K^{ss})$ as $t \to \infty$

$\Rightarrow \underbrace{\beta^T}_{0} \underbrace{u'(C_T^*)}_{u'(C^{ss})} \underbrace{K_{T+1}^*}_{K^{ss}} \to 0$ as $T \to \infty$

$\Rightarrow$ (TVC) is satisfied.

$\Rightarrow$ Conclusion: This is the unique solution to SPP.

(c) Computations:

- Shooting algorithm
- Goal: Calculate (a numerical approximation) of $(C_t^*, K_t^*)_{t=0}^{\infty}$ that solves SPP

- Algorithm:

Step 1: Calculate (K^{ss}, C^{ss})

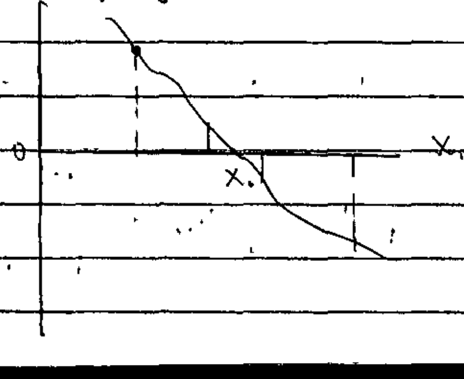
Step 2: Guess C_0^*

Step 3: Starting with $(\bar{K}_0, C_0^*)$ use (RC), (EE) to calculate $(C_t^*, K_t^*)_{t=0}^{H}$ (where H is large, e.g. 1000 years)

Step 4: check if $(C_H^*, K_H^*) \approx (C^{ss}, K^{ss})$ numerically

Yes → Done!

Re-guess C_0^* and try again;



5. Recursive approach

(a) The math

- Key idea: backward induction (BI)

- 2 periods: $t=0$, $t=1$ ← choose $a_1 \in \{T, B\}$

  choose $a_0 \in \{T, B\}$

- Final payoff (utility): $u(a_0, a_1)$

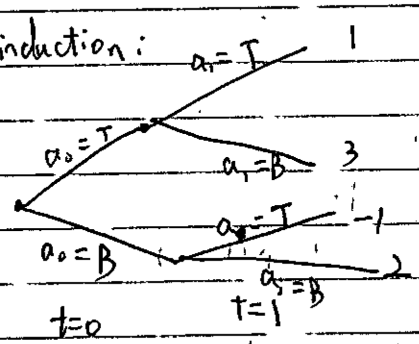
$u(T, T) = 1$

$u(T, B) = 3$ ← optimal choice

$u(B, T) = -1$

$u(B, B) = 2$

- Backward induction:



In $t=1$: $\forall a_0$ solve — not a choice variable

$$\max_{a_1 \in \{T, B\}} u(a_0, a_1)$$

$\Rightarrow$ solution $a_1^*(a_0)$

In $t=0$. solves:

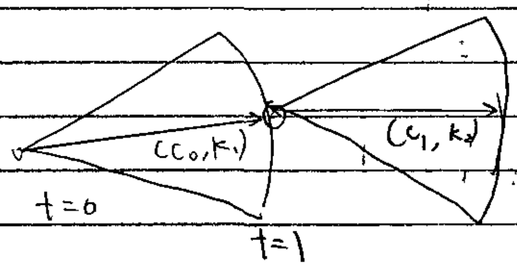
$$\max_{a_0 \in \{T, B\}} u(a_0, a_1^*(a_0))$$

(i) 2 period case

$$\max_{(c,k)} \; u(c_0) + \beta\, u(c_1)$$

s.t.
$c_0 + k_1 \leq F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0$
$c_1 + k_2 \leq F(k_1, 1) + (1-\delta)k_1$
$c_0 \geq 0$
$c_1 \geq 0$
$k_1 \geq 0$
$k_2 \geq 0$

Apply BI:



$t-1$: $\forall (c_0, k_1)$, solves:

$$\max_{(c_1, k_2)} \; u(c_0) + \beta\, u(c_1)$$

s.t.
$c_0 + k_1 \leq F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0$
$c_1 + k_2 \leq F(k_1, 1) + (1-\delta)k_1$
$\cancel{c_0 \geq 0}$
$c_1 \geq 0$
$\cancel{k_1 \geq 0}$
$k_2 \geq 0$

$\Leftrightarrow \max_{(c_1, k_2)} \; u(c_1)$ s.t. $c_1 + k_2 \leq F(k_1, 1) + (1-\delta)k_1$
$c_1 \geq 0$
$k_2 \geq 0$

$\Rightarrow$ solution : $\left(g^c_1(k_1),\; g^k_1(k_1)\right)$

• $t=0$: Solve

$$\begin{cases} \max_{(c_0,k_1)} & u(c_0) + \beta \underbrace{u(g_1^c(k_1))}_{V_1(k_1)} \\ \text{s.t.} & c_0 + k_1 \le F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0 \\ & \cancel{g_1^c(k_1) + g_1^k(k_1) \le F(k_1,1) + (1-\delta)k_1} \\ & c_0 \ge 0 \\ & \cancel{g_1^c(k_1) \ge 0} \\ & k_1 \ge 0 \\ & \cancel{g_1^k(k_1) \ge 0} \end{cases}$$

(*) holds, $\forall k$

$$\Longleftrightarrow \underbrace{\begin{cases} \max_{(c_0,k_1)} & u(c_0) + \beta V_1(k_1) \\ \text{s.t.} & c_0 + k_1 \le F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0 \\ & c_0 \ge 0 \\ & k_1 \ge 0 \end{cases}}_{V_0(\bar{k}_0)}$$

$\Rightarrow$ solution: $(g_0^c(\bar{k}_0), g_0^k(\bar{k}_0))$

• Result: $(c_0^*, k_1^*, c_1^*, k_2^*) = (g_0^c(\bar{k}_0), g_0^k(\bar{k}_0), g_1^c(g_0^k(\bar{k}_0)), g_1^k(g_0^k(\bar{k}_0)))$

solves the SPP.

Terminology: V_t .... value functions.

g_t ... policy functions.

(ii) 3 period case:

$$\max_{(c,k)} \; u(c_0) + \beta u(c_1) + \beta^2 u(c_2)$$

(SPP) s.t.

$$c_0 + k_1 \le F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0$$
$$c_1 + k_2 \le F(k_1, 1) + (1-\delta)k_1$$
$$c_2 + k_3 \le F(k_2, 1) + (1-\delta)k_2$$
$$c_0 \ge 0 \qquad k_1 \ge 0$$
$$c_1 \ge 0 \qquad k_2 \ge 0$$
$$c_2 \ge 0 \qquad k_3 \ge 0$$

choice variables : $(\underbrace{c_0, k_1}_{t=0}, \underbrace{c_1, k_2}_{t=1}, \underbrace{c_2, k_3}_{t=2})$

$\longleftarrow$ BI

$t=2$, $\forall (c_0, k_1, c_1, k_2)$ solves:

$$\max_{(c_1, k_2)} u(c_0) + \beta u(c_1) + \beta^2 u(c_2)$$

$$\text{s.t.} \begin{cases} c_0 + k_1 \leq F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0 \\ c_1 + k_2 \leq F(k_1, 1) + (1-\delta) k_1 \\ c_2 + k_3 \leq F(k_2, 1) + (1-\delta) k_2 \\ c_0 \geq 0 \quad k_1 \geq 0 \\ c_1 \geq 0 \quad k_2 \geq 0 \\ c_2 \geq 0 \quad k_3 \geq 0 \end{cases}$$

$$\Longleftrightarrow \max_{(c_2, k_3)} u(c_2)$$

$$\text{s.t. } c_2 + k_3 \leq F(k_2, 1) + (1-\delta) k_2$$

$$c_2, k_3 \geq 0$$

solution: $g_2^c(k_2)$

$t=1$ $\forall (c_0, k_1)$ solve:

$$\max_{(c_1, k_2)} \cancel{u(c_0)} + \cancel{\beta} u(c_1) + \underbrace{\cancel{\beta^2} u(g_2^c(k_2))}_{V_2(\,)}$$

$$\text{s.t. } \cancel{c_0 + k_1 \leq F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0}$$

$$c_1 + k_2 \leq F(k_1, 1) + (1-\delta) k_1$$

$$g_2^c(k_2) + g_2^k(k_2) \leq F(k_2, 1) + (1-\delta) k_2$$

$c_0 \geq 0$ $\quad c_1 \geq 0$ $\quad g_2^c(k_2) \geq 0$ $\quad k_1 \geq 0$ $\quad k_2 \geq 0$ Holds $\forall k_2$

$g_2^k(k_2) \geq 0$

$\Longleftrightarrow$

$$\begin{cases} \max_{(c_1, k_2)} u(c_1) + \beta V_2(k_2) \\ \text{s.t. } c_1 + k_2 \leq F(k_1, 1) + (1-\delta) k_1 \\ c_1, k_2 \geq 0 \end{cases}$$

solution: $(g_1^c(k_1), g_1^k(k_1))$

c_1 $\quad c_2$

$t=0$. Solves:

$$\max_{(c_0,k_1)} u(c_0) + \beta \underbrace{\left\{ u(g_1^c(k_1)) + \beta^2 \cdot u(g_2^c(\underbrace{g_1^k(k_1)}_{}))\right\}}_{V_1(k_1)}$$

$V_2(g_1^k(k_1))$

s.t. $c_0 + k_1 \le F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0$

$\cancel{g_1^c(k_1) + g_1^k(k_1) \le F(k_1,1) + (1-\delta)k_1}$

$g_2^c(g_1^k(k_1)) + g_2^k(g_1^k(k_1)) \le F(g_1^k(k_1), 1) + (1-\delta) g_1^k(k_1)$

Holds $\forall k_1$

$c_0 \geq 0$ $\qquad$ $k_1 \geq 0$

$\cancel{g_1^c(k_1) \geq 0}$ $\qquad$ $\cancel{g_1^k(k_1) \geq 0}$

$g_2^c(g_1^k(k_1)) \geq 0$ $\qquad$ $g_2^k(g_1^k(k_1)) \geq 0$

$$\Leftrightarrow V_0(\bar{k}_0) = \left\{ \begin{array}{ll} \max_{(c_0,k_1)} & u(c_0) + \beta V_1(k_1) \\ \text{s.t} & c_0 + k_1 \le F(\bar{k}_0, 1) + (1-\delta)\bar{k}_0 \\ & c_0, k_1 \geq 0 \end{array} \right.$$

solution: $(\underset{c_0}{g_0^c(\bar{k}_0)}, \underset{k_1}{g_0^k(\bar{k}_0)})$

Result: $c_0^* = g_0^c(\bar{k}_0)$

$k_1^* = g_0^k(\bar{k}_0)$

$c_1^* = g_1^c(k_1^*)$

$k_2^* = g_1^k(k_1^*)$

$c_2^* = g_2^c(k_2^*)$

$k_3^* = g_2^k(k_2^*)$

solves the SPP

(iii) $T (< \infty)$ period case

$$(\text{SPP}) \left\{ \begin{array}{ll} \max_{(c_t, k_{t+1})_{t=0}^T} & \sum_{t=0}^T \beta^t u(c_t) \\ \text{s.t} & c_t + k_{t+1} \le F(k_t, 1) + (1-\delta) k_t \quad (\forall t) \\ & k_0 = \bar{k}_0 \\ & c_0 \end{array} \right.$$

observation : In the

solves

V

Proposition : let $V_{T+1}(k_{T+1}) \equiv 0$, and let for each $t = T, T-1, \dots, 0$, $(g_t^c(k_t), g_t^k(k_t))$ solves:

$$V_t(k_t) = \begin{cases} \max\limits_{(c_t, k_{t+1})} & u(c_t) + \beta V_{t+1}(k_{t+1}) \\ \text{s.t} & c_t + k_{t+1} \leq F(k_t, 1) + (1-\delta) k_t \\ & c_t, k_{t+1} \geq 0 \end{cases}$$

Bellman equation

Then $(c_t^*, k_{t+1}^*)_{t=0}^T$ given by

$$\begin{cases} k_{t+1}^* = g_t^{*k}(k_t^*) & (\forall t) \\ c_t^* = g_t^{*c}(k_t^*) & (\forall t) \\ k_0^* = \bar{k}_0 & \end{cases}$$

solves (SPP)

V_t ... value function

g_t ... policy function

(iv) infinite horizon case

. No final period! $\Rightarrow$ Not clear where to start BI

. We'll take the following route to generalizing the results above to the infite horizon case.

. observe that (w/ $T < \infty$);

$$V_t(k_t) = \max_{(c_s, k_{s+1})_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s)$$

$$\text{s.t. } C_s + K_{s+1} \le F(K_s, 1) + (1-\delta) K_s \quad (\forall s \ge t)$$
$$C_s, K_{s+1} \ge 0 \quad (\forall s \ge t)$$

(apply the proposition above but stop at period t)

For $T = \infty$ we could define

$$V_t(K_t) = \max_{(C_s, K_{s+1})_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$
$$\text{s.t. } C_s + K_{s+1} \le F(K_s, 1) + (1-\delta) K_s \quad (\forall s \ge t)$$
$$C_s, K_{s+1} \ge 0 \quad (\forall s \ge t)$$

$$V_0(K) = V_1(K) = V_2(K_2) = V_3(K) = \dots$$
$V(K)$ value function

$$V(K_t) = \begin{cases} \max_{C_t, K_{t+1}} u(C_t) + \beta V(K_{t+1}) \\ \text{s.t. } C_t + K_{t+1} \le F(K_t, 1) + (1-\delta) K_t \\ C_t, K_{t+1} \ge 0 \end{cases}$$

$$\Updownarrow$$

$$V(K) = \max_{(C, K')} u(C) + \beta V(K')$$
$$\text{s.t. } C + K' \le F(K, 1) + (1-\delta) K$$
$$C, K' \ge 0$$

$$V(K_t) = \max\ u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}) + \dots$$
$$\text{s.t. } C_t + K_{t+1} \le F(K_t, 1) + (1-\delta) K_t$$
$$C_{t+1} + K_{t+2} \le F(K_{t+1}, 1) + (1-\delta) K_{t+1}$$

$$V(K) = \begin{cases} \max_{(C, K')} u(C) + \beta V(K') \\ \text{s.t. } C + K' \le F(K, 1) + (1-\delta) K \\ C, K' \ge 0 \end{cases}$$

Bellman equation.

Mathematically, a functional equation

Formally

Proposition: Let V satisfy the Bellman equation

$$V(K) = \begin{cases} \max\limits_{(C,K')} & u(C) + \beta V(K') \\ \text{s.t.} & C + K' \leq F(K,1) + (1-\delta)K \\ & C, K' \geq 0 \end{cases}$$

and let $(g^c(K), g^k(K))$ be the maximizers of the RHS

Then $(C_t^*, K_t^*)_{t=0}^{\infty}$ given by

$$\begin{cases} K_{t+1}^* = g^k(K_t^*) \quad (\forall t) \\ C_t^* = g^c(K_t^*) \quad (\forall t) \\ K_0^* = \bar{K}_0 \end{cases}$$

solves (SPP)

Result 1: The Bellman eq. has a unique solution.

Result 2: For any (bounded) function $V^{(0)}$, if we let

$$V^{(i+1)}(K) = \max_{(C,K')} u(C) + \beta V^{(i)}(K')$$

$$\text{s.t. } C + K' \leq F(K,1) + (1-\delta)K$$

$$C, K' \geq 0$$

for $i = 0, 1, 2, \ldots$, then $V^{(i)} \to V$

Result 3: The value function V is concave increasing, differentiable, and

$$V'(K) = u'(g^c(K))\,[F_K(K,1) + 1 - \delta].$$

[Envelope condition / Benveniste/Scheinkman formula

Informal proof:

First, look at the RHS of the Bellman eq:

$$\max_{(C,K')} \begin{cases} u(C) + \beta V(K') \\ \text{s.t. } C + K' \leq F(K,1) + (1-\delta)K \quad [\lambda] \\ C, K' \geq 0 \end{cases}$$

FOCs: $u'(c) = \lambda$, $\beta v'(k') = \lambda$ $\Big\} \Rightarrow u'(c) = \beta v'(k')$

$(\because)$ $u'(g^c(k)) = \beta v'(g^k(k))$ $(\forall k)$

Note: $v(k) = u(\underbrace{g^c(k)}_{= F(k,1) + (1-\delta)k - g^k(k)}) + \beta v(g^k(k))$ $(\ast k)$

$\Rightarrow \left(\frac{d}{dk}\right)$ $v'(k) = u'(g^c(k)) \left[F_k(k,1) + 1 - \delta) - \frac{dg^k(k)}{dk}\right] + \beta v'(g^k(k)) \frac{dg^k(k)}{dk}$

$= u'(g^c(k)) [F_k(k,1) + 1 - \delta] + \underbrace{[\beta v'(g^k(k)) - u'(g^c(k))]}_{= 0 \ \leftarrow \text{FOC above}} \frac{dg^k(k)}{dk}$

$v'(k) = u'(g^c(k)) [F_k(k,1) + 1 - \delta]$

math issue:
(g^k may be non-differentiable)

Result 4: We can derive EE from above:

- Recall FOC: $u'(g^c(k)) = \beta v'(g^k(k))$ $(\forall k)$
- Env condition: $v'(k) = u'(g^c(k)) [F_k(k,1) + 1 - \delta]$ $(\forall k)$

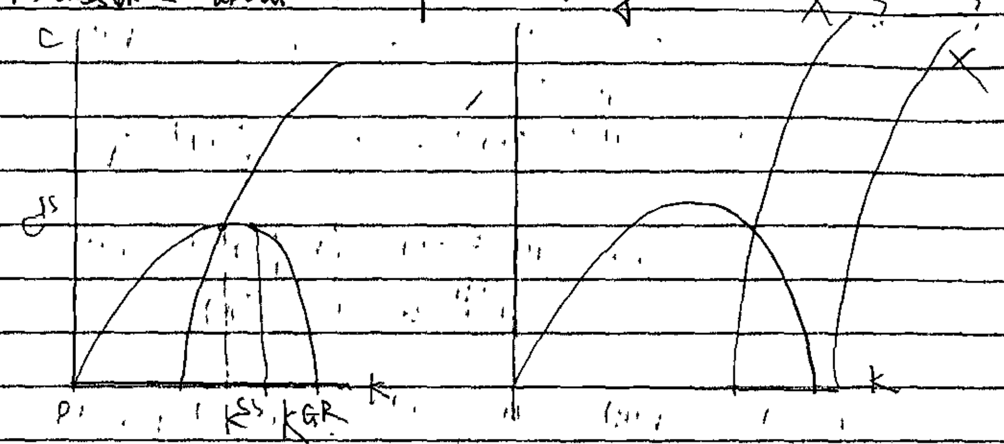
$\Rightarrow u'(g^c(k)) = \beta u'(g^c(g^k(k))) [F_k(g_k(k), 1) + 1 - \delta]$ $(\forall k)$

so if $(c_t^*, k_t^*)_{t=0}^{\infty}$ solves SPP, then $\forall t$:

$u'(\underbrace{g^c(k_t^*)}_{c_t^*}) = \beta u'(\underbrace{g^c(\underbrace{g^k(k_t^*)}_{k_{t+1}^*})}_{= c_{t+1}^*}) [F_k(\underbrace{g_k(k_t^*)}_{k_{t+1}^*}, 1) + 1 - \delta]$

$\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*) [F_k(k_{t+1}^*, 1) + 1 - \delta]$

Discussion - about the phase diagram.



$K^{GR} = \max_k F(K,1) - \delta K$ ← [Consumption sustainable given a constant K

$C_t = F(\underbrace{K_t}_{K},1) - (1-\delta)\underbrace{K_t}_{K} - \underbrace{K_{t+1}}_{K}$

$C_t = F(K,1) - \delta K$

Golden rule capital stock

I.e. K that maximizes the consumption that's sustainable given constant K.

FOC: $F_K(K^{GR},1) = \delta$

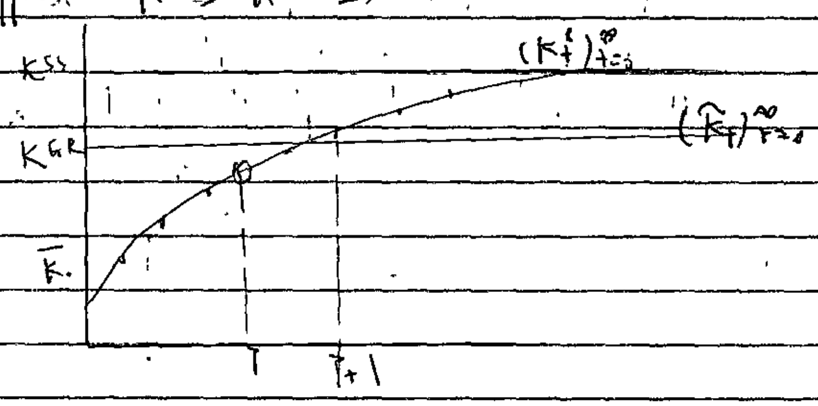
Prop: $K^{ss} < K^{GR}$

Proof: $F_K(K^{ss},1) = \frac{1}{\beta} - 1 + \delta > \delta = F_K(K^{GR},1)$

$\Rightarrow K^{ss} < K^{GR}$

Economic mechanism behind this:

Suppose: $K^{ss} > K^{GR} \Rightarrow$



$(\tilde{K}_t)_{t=0}^{\infty}$

$\Downarrow \quad \tilde{c}_t = F(\tilde{K}_t, 1) + (1-\delta)\tilde{K} - \tilde{K}_{t+1}$

Comparing $(c_t^*)_{t=0}^{\infty}$ vs: $(\tilde{c}_t)_{t=0}^{\infty}$;

$\forall t = 0, \dots, T-1$: $\tilde{c}_t = c_t^*$

$$\tilde{c}_T = F(\underset{\underset{K_T^*}{\shortparallel}}{\tilde{K}_T}, 1) + (1-\delta)\underset{\underset{K_T^*}{\shortparallel}}{\tilde{K}_T} - \underset{\underset{K^{GR} < K_{T+1}^*}{\shortparallel}}{\tilde{K}_{T+1}}$$

$$> F(K_T^*, 1) + (1-\delta) K_T^* - K_{T+1}^* = c_T^*$$

$\forall t \geq T+1$:

$$\tilde{c}_t = F(\underset{K^{GR}}{\tilde{K}_t}, 1) + \dots (1-\delta)\underset{K^{GR}}{\tilde{K}_t} - \underset{K^{GR}}{\tilde{K}_{t+1}}$$

$\#$

Proposed solution: The value function is:

$$V(K) = \frac{\alpha}{1-\alpha\beta}\log(K) + \underbrace{\frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right]}_{\text{Const}}$$

and the policy functions are:

$$\begin{cases} g^k(K) = \alpha\beta K^{\alpha} \\ g^c(K) = (1-\alpha\beta) K^{\alpha} \end{cases} \qquad K_{t+1}^* = g^k(K_t^*) = \alpha\beta(K_t^*)^{\alpha}$$

3.4th 2015

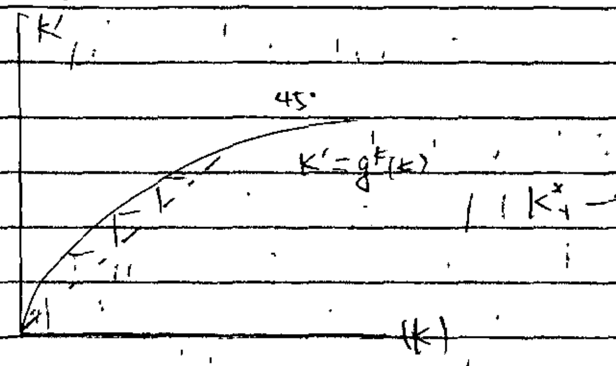
More generally: SP solution satisfies

$$K_{t+1}^* = g^k(K_t^*) \quad (\forall t \geq 0) \quad K_0^* = \bar{K}_0$$

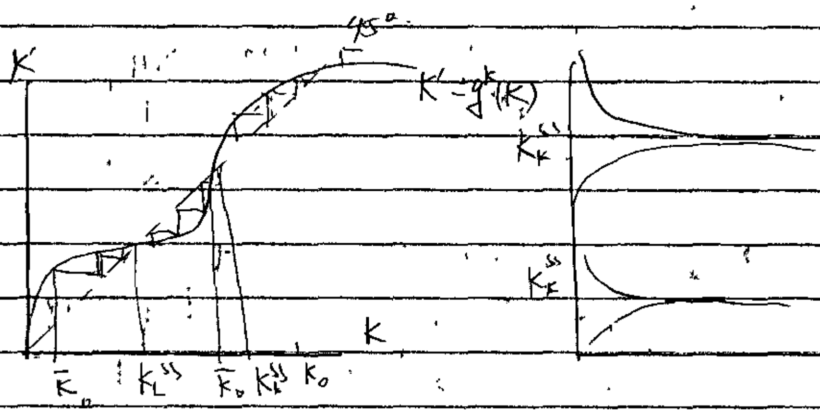
$\Rightarrow$ shape of g^k determines what happens to K_t^* over time.

g^k?

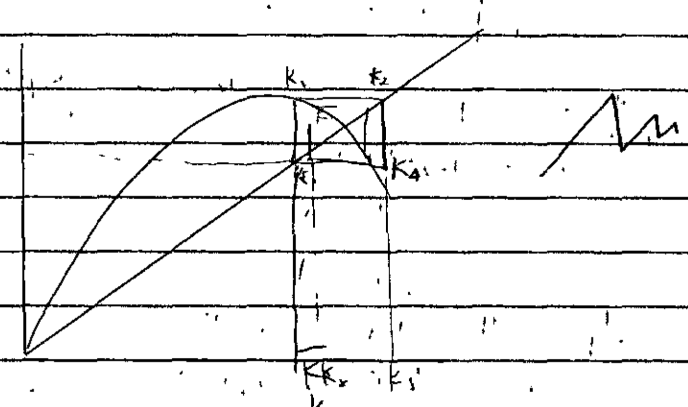
Possibilities.



(5)



(6)



Result: Possibilities (2) - (7) can't happen in this model

(c) Computations

Value function iteration w/ discretization.

1 2 3 4 5

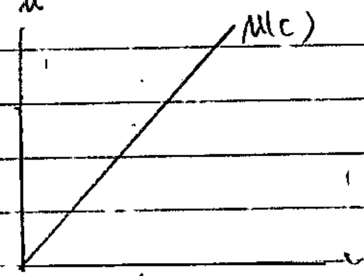
6. set $K_0^* = \bar{K}_0$

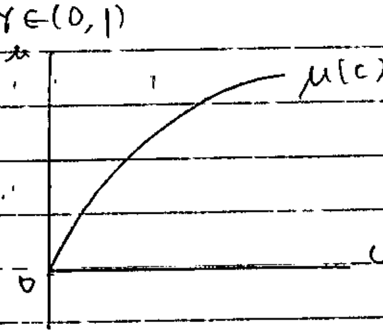
$K_{t+1}^* = g^k(K_t^*)$

$C_t^* = g^c(K_t^*)$

$$u(C_t) = \begin{cases} \log(C_t) \\ \dfrac{C_t^{1-\gamma}}{1-\gamma} \end{cases}$$

$\gamma = 0$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $\gamma \in (0,1)$





$\gamma > 1$

3.16th Utility

Set up model

Need to solve the model

Negishi model

variational model

Recursive methods

Does the model fit the data?

OK: Facts 1-5 Y

model can be trusted

Let's using it, e.g.

policy analysis

N

Wrong model

Try again

added

TFP growth

balanced growth

~~preference~~

preference

V: Fiscal and Monetary Policies

Two pillars of macroeconomics policies

Fiscal policies: govt. spending, taxation, govt debt.

Monetary policies: money supply, ...

Condition #2: Govt's no-Ponzi-game condition (NPG)

$\limsup_{i \to \infty}$

$T=0$: $P_0 G_0 + (1+i_0) B_0 \leq T_0 + B_1 + (M_1 - M_0)$

Note that (NPG) rules out the above scheme.

$$\frac{B_{t+1}}{(1+i_1) \cdots (1+i_t)} = \varepsilon > 0$$

$\uparrow$

limsup is > 0 $\rightarrow$ NG violated.

Key ingredient: cash-in-advance (CIA)

idea: HHs need cash (m_t) to buy things (C_t, X_t)

Fiscal Model

Each HH consists of "worker" and "shopper".

Timing: period $t+1$

AM PM

time

Household's participants in an asset market

constraint:

$b_{t+1} + m_{t+1} \leq W_t l_t + R_t K_t + \pi_t$
$+ (1+i_t) b_t + m_t$
$- p_t c_t - p_t x_t - \tau_t$

choose

$p_t \theta^c \underbrace{c_{t+1}}_{\text{Units}} + p_{t+1} \theta^x \underbrace{x_{t+1}}_{\text{Units}} \leq m_{t+1}$

$[0,1]$ parameters b/w 0 & 1
- fraction of goods that require cash to purchase

Price

cash-in-advance constraint

(Trade-off: m_{t+1} yields no interest but is required to buy goods

b_{t+1} yields interest but does not allow you to purchase some goods.

Wealth at the beginning of $t=0$ $\equiv \overline{M}_0$
$\Rightarrow$ asset market constraint for $t=0$: $b_0 + m_0 \leq \overline{M}_0$

goods $C_t^* + x_t^* + G_t = F(K_t^*, L_t^*)$
capital K_t^* k_t^*

Equilibrium gives a mapping
$GP = (B_t, \tau_t, P_t, M_t)_{t=0}^{\infty} \to (C_t^*, l_t^*, \dots)_{t=0}^{\infty}$
(eqm)

If you change this $\to$ This changes

2. Govt spending
HH op
Firm op
Market clearing
Govt's budget constraint $p^* G = T$
Equ characterization:
Firm op $F_l = \frac{W^*}{p^*}$

Given this

Govn $\max_{c,l} u(c,l) + \gamma(G)$

$$\max \; u(c,l)$$
$$c \leq Al - G$$
$$c, l, 1-l \geq 0$$

$G\uparrow \quad c\downarrow \quad Y\uparrow$

Economic mechanism

$G\uparrow \underset{(GBC)}{\Rightarrow} T\uparrow \Rightarrow$ people feel poorer → $l\uparrow$ work harder to (make up for loss)

$C^*\downarrow$ (cut back on spending)

Govt spending multiplier $= \frac{\Delta Y}{\Delta G} = ?$

Here: Suppose $G\uparrow$ by 100
Then:

$$\underbrace{y^*}_{\uparrow ?} = \underbrace{C^*}_{\downarrow ?} + \underbrace{G}_{\uparrow 100}$$

$$0 < \frac{\Delta Y}{\Delta G} < 1$$

optimal govt spending:

$$G = A \Rightarrow \begin{array}{l} C = 0 \\ l = 1 \end{array}$$

If govt wants to maximize output, set $G = A$
$(\Rightarrow Y^* = A)$

A more reasonable objective:

$$\max_G \underbrace{u(\underset{\ominus}{C^*(G)}, \underset{\oplus}{l^*(G)})}_{\ominus \text{ cost of } G} + \underbrace{\overline{\gamma(G)}}_{\oplus \text{ is } G \text{ benefits of } G}$$

3. Labor income taxation (vs. lump sum taxation)

HH optimization

Firm optimization.

Equ characterization:

Firm optimization $\Rightarrow \frac{W^*}{P^*} = A$

$\pi^* = 0$

Difference # 2: Laffer curve

$\downarrow$ log

Typical estimate of $\varepsilon \approx 0.5$

Economic impact of govt debt?
Some first thoughts:

high debt $\Rightarrow$ high bond supply $\Rightarrow$ high int. rates

$\Downarrow$

high taxes (to repay) $\Longrightarrow$ HHs worse off (in future)

which effect dominates

Offsetting effects

high debt [low taxes in the past, high taxes in the future] high HHs saving $\Rightarrow$ low int rates

$\Downarrow$

HHs own a lot of govt bonds
$\Rightarrow$ HHs are wealthy $\Rightarrow$ HHs are better off.

<u>Ans</u>: The two effects exactly cancel out (under certain conditions), and debt has essentially no effect on quantities (except quantity of debt), prices, or welfare.

Simple model to illustrate this result.

- 2 periods ($t=0,1$)
- No labor, capital
- output: Y_0, Y_1 ($\leftarrow$ requires no inputs)
- $GP = (G_0, G_1, \ldots, \tau_0, \tau_1, B_1)$
- No initial debt: $B_0 = 0$
- No M.

An eqm given $GP = (G_0, G_1, \tau_0, \tau_1, B_1)$ is $(C_0^*, C_1^*, b_p^*, \pi_0^*, \pi_1^*, P_0^*, P_1^*, i_1^*)$ such that

(i) (HH optimization) (C_0^*, C_1^*, b_1^*) solves:

$$\begin{cases} \max_{(C_0, C_1, b_1)} & u(C_0) + \beta u(C_1) + \underbrace{v(G_0, G_1)}_{\text{can ignore}} \\ \text{s.t.} & P_0^* C_0 + b_1 \leq \pi_0^* - \tau_0 \\ & P_1^* C_1 \leq \pi_1^* + (1+i^*) b_1 - \tau_1 \\ & C_0, C_1 \geq 0 \end{cases}$$

(ii) (Firm optimization) $\forall t$

$$\pi_t^* = \cancel{\max} \quad P_t^* Y_t$$

(iii) (Market clearing)

$$\begin{cases} C_0^* + G_0 = Y_0 \\ C_1^* + G_1 = Y_1 \\ b_1^* = B_1 \end{cases}$$

GBC:

$$P_0^* G_0 = \tau_0 + B_1$$

$$P_1^* G_1 + (1+i_1^*) B_1 = \tau_1$$

o Eqn. characterization.

HH solve: $\max_{(C_0, C_1, b_1)} u(C_0) + \beta u(C_1)$

$$\text{s.t.} \quad P_0^* C_0 + b_1 \leq \pi_0^* - \tau_0$$

$$P_1^* C_1^* \leq \pi_1^* + (1+i^*) b_1 - \tau_1$$

$$C_0, C_1 \geq 0$$

$$\Downarrow$$

$$C_0 + \underbrace{\frac{b_1}{P_0^*}}_{= \tilde{b}_1 \text{ (real saving)}} \leq \frac{\pi_0^*}{P_0^*} - \frac{\tau_0}{P_0^*}$$

$$C_1 \leq \frac{\pi_1^*}{P_1^*} + \underbrace{\frac{(1+i_1^*) b_1}{P_1^*}}_{= (1+i_1^*) \frac{P_0^*}{P_1^*} \tilde{b}_1} - \frac{\tau_1}{P_1^*}$$

$$(1+i_1^*) \frac{P_0^*}{P_1^*} = 1 + r_1^r \quad \text{real interest rate}$$

$$\Longleftrightarrow \begin{cases} \max\limits_{(C_0, C_1)} & u(C_0) + \beta u(C_1) \\ \text{s.t.} & \underbrace{C_0 + \frac{C_1}{1+r_1^*}}_{(\text{PV of } C_0)} \le \left(\frac{\pi_0}{P_0^*}\right) + \frac{\frac{\pi_1^*}{P_1^*}}{1+r_1^*} \underbrace{- \left(\frac{\tau_0}{P_0^*}\right) - \frac{\left(\frac{\tau_1}{P_1^*}\right)}{1+r_1^*}} \\ & C_0, C_1 \ge 0 \end{cases}$$

$Y_0 + \frac{Y_1}{1+r_1^*}$

$= G_0 + \frac{G_1}{1+r_1^*}$

PV of after tax (non-interest) income.

intertemporal budget constraint

From firm opt: $\pi_t^* = P_t^* Y_t \Rightarrow \frac{\pi_t^*}{P_t^*} = Y_t$

From GBC

$$\begin{cases} G_0 = \frac{\tau_0}{P_0^*} + \frac{B_1}{P_0^*} = \tilde{B}_1 \\ G_1 + \frac{(1+i_1^*)B_1}{P_1^*} = \frac{\tau_1}{P_1^*} \end{cases}$$

$\frac{(1+i_1^*)B_1}{P_1^*} = (1+r_1^*)\tilde{B}_1$

subst.

$$\Rightarrow \underbrace{G_0 + \frac{G_1}{1+r_1^*} = \left(\frac{\tau_0}{P_0^*}\right) + \frac{\left(\frac{\tau_1}{P_1^*}\right)}{1+r_1^*}}$$

so HHs solve:

$$\max_{(C_0, C_1)} \begin{cases} u(C_0) + \beta u(C_1) \\ \text{s.t.} \quad C_0 + \frac{C_1}{1+r_1^*} \le (Y_0 - G_0) + \frac{Y_1 - G_1}{1+r_1^*} \\ C_0, C_1 \ge 0 \end{cases}$$

graphically:

Q: Is this an equi?

A: No $(\because)$ $\begin{cases} C_0^* + G_0 < Y_0 \\ C_1^* + G_1 > Y_1 \end{cases}$ ← markets do not clear

Another try...

Q: Is this an eqn?

A: No! (:) $C_0^* + G_0 > Y_0$ ← Markets do not clear

$C_1^* + G_1 < Y_1$

In eqm r_1^* has to be such that:

graphic

↑ Here, We are in eqm.

$\Longrightarrow$ Equ consumption : real interest rate, r_1^*

HH Utility $u(C_0^*) + \beta u(C_1^*)$ depend on Government Policy only through G_0, G_1, but not on B_0, B_1

Role of debt

Suppose $B_1 \uparrow \Rightarrow \tilde{B}_1 \uparrow$ by ε

GBC : $G_0 = \frac{\tau_0}{P_0^*} + \tilde{B}_1 + \varepsilon$

$G_1 + (1+r_1^*)\tilde{B}_1 = \frac{\tau_1}{P_1^*} + (1+r_1^*)\varepsilon$

$\Longrightarrow$ No change in budget line.

$\Longrightarrow$ No change in C_0^*, C_1^*, r_1^*

No change in HH utility

(called Ricardian Equivalent)

Economic mechanism behind Ricardian equivalence

5. Govt debt w/ labor income taxes

We know: With lump sum taxes, we got

Q. What if we have labor income taxes instead?

~~Then~~ To see this, we'll use the following model:

- 2 periods: $t = 0, 1$
- output $Y_t = L_t$
- utility $u(C_0, l_0) + \beta u(C_1, l_1) + \gamma(G_0, G_1)$

$$= \left(C_0 - \frac{1}{1+\frac{1}{\varepsilon}} l_0^{1+\frac{1}{\varepsilon}}\right) + \beta\left(C_1 - \frac{1}{1+\frac{1}{\varepsilon}} l_1^{1+\frac{1}{\varepsilon}}\right) + \gamma(G_0, G_1)$$

An eqm given $GP = (G_0, G_1, \tau_0^l, \tau_1^l, B_1)$ is $(C_0^*, C_1^*, l_0^*, l_1^*, b_1^*, L_0^*, L_1^* \ldots$

(i) (HH) optimization

$$p_0^* C_0 + b_1 \le w_0^* l_0 + \pi_0^* - \tau_0^l w_0^* l_0$$
$$p_1^* C_1 \le w_1^* l_1 + \pi_1^* - \tau_1^l w_1^* l_1 + (1+i_1^*) b_1$$

$$p_t^* L_t - w_t^* L_t$$

GBC

$$p_0^* G_0 = \tau_0^l w_0^* l_0^* + B_1$$
$$p_1^* G_1 + (1+i_1^*) B_1 = \tau_1^l w_1^* l_1^*$$

Economic mechanism ($t=1$)

high debt $\Rightarrow$ high taxes on earnings
(to repay debt)
$\Rightarrow$ lower incentives to work
$\Rightarrow$ lower output
$\Rightarrow$ lower consumption

with lump sum taxes, we didn't have this going on

Q: We know that B_1 affects HH utility
what's the optimal B_1 given this?
i.e. maximize HH utility

set $\tau_0^l = \tau_1^l$ ($\leftarrow$ labor income tax smoothing)
and use B_1 as a buffer to make this possible

Application:
Suppose there is a war in $t=0$, and peace follows in $t=1$
$\Rightarrow$ Want: $G_0 >> G_1$

How should the govt finance this?
A: (Given the above) set $\tau_0^l = \tau_1^l$
tax rev is $t=0$
(tax rev in $t=1$)

• Intuition:

Proof of labor income tax smoothly result.

We know in an given GP:
equin

$$\begin{cases} l_0^* = (1-\tau_0^l)^\varepsilon \\ l_1^* = (1-\tau_1^l)^\varepsilon \\ r_1^* = \frac{1}{\beta} - 1 \end{cases}$$

$\xrightarrow[\text{in } U]{\text{substitute}}$ $U = \left(\frac{1}{1+\varepsilon}\right) \{(1-\tau_0^l)^{1+\varepsilon} + \beta(1-\tau_1^l)^{1+\varepsilon}\}$

So govt's optimization problem:

6. Capital income taxes

Q: Impact of capital income taxes on economy?

Model:

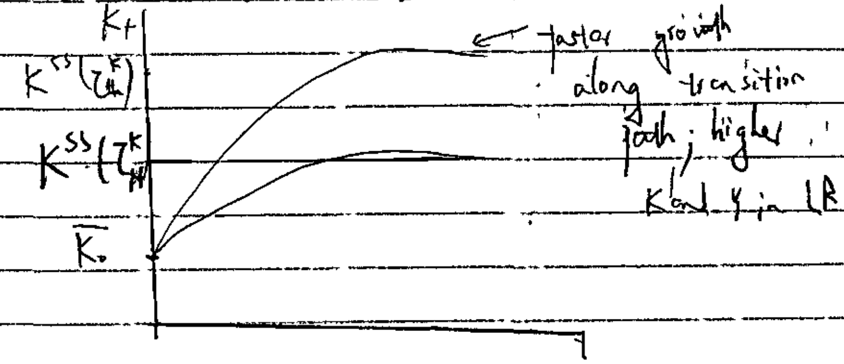
$t = 0, 1, 2, \ldots$ (∞ horizon)

$$\begin{cases} \text{Util} = \sum_{t=0}^{\infty} \beta^t u(C_t) \quad (\Rightarrow \text{no dist from labor}) \\ GP = (G_t = G, T_t, \underbrace{\tau_t^k = \tau^k}) \end{cases}$$

↑ tax rate on capital income $R_t^x k_t$

No money

So if there are two identical countries, one with high τ_H^k and one w/ low τ_L^k.



Mechanism: $\tau^k \downarrow \Rightarrow$ HHs have stronger incentive to invest,

$\Rightarrow X_t \uparrow, K_t \uparrow, Y_t \uparrow$

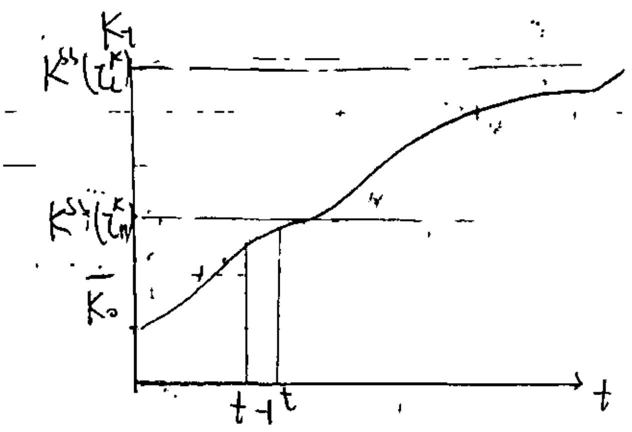
Above was a comparison of two ~~cou~~ countries.
What if govt of a given country decides to reduce τ^k at a certain point?

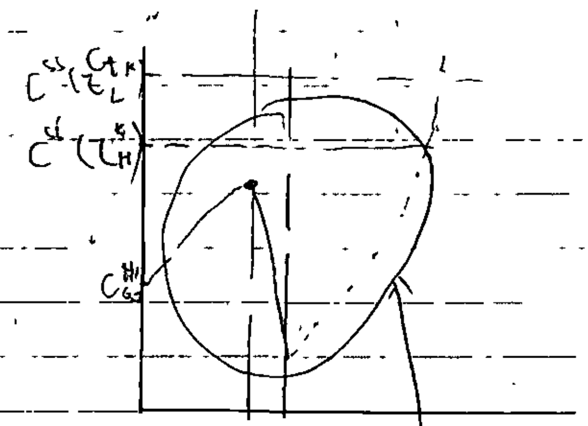
Scenario:

- In $t=0$ govt commits to τ_H^k ($\forall t \geq 0$)
  $\Rightarrow$ private sector believes this.
  $\Rightarrow$

In $t=T$ govt suddenly (w/o advance notice) reduces tax rate to τ_L^k ($\forall t \geq T$)
$\Rightarrow$ private sector believes this

Initially economy is in an eq'm given τ_H^k
$(C_t$





suprise $\downarrow$

Intuition $\tau^K \downarrow \Rightarrow x_t \uparrow$ incentive

$\Rightarrow C_t \downarrow$ is the only way to do this

Matters for welfare (among other things)

$\tau^K \downarrow$ ~~long-run~~ $C^{ss} \uparrow \Longrightarrow V \uparrow$

short-run $C_t \downarrow$ (in short run) $\Rightarrow V \downarrow$

Important not to miss this if the goal is to evaluate the desirability of $\tau^K \downarrow$

7. Money and Prices

Q: Macroeconomics impact of money supply change?

E.g $M_t \uparrow$ (b/c govt tries to finance spending by printing money)

Model:

- $t = 0, 1, 2, 3 \cdots$ (∞ horizon)
- output $= Y$ ($\leftarrow$ no inputs required)

$$\text{utility} = \sum_{t=0}^{\infty} \beta^t u(C_t) + \gamma(G_0, G_1, \ldots)$$

$$GP = (G_t = G, \tau_t, \underbrace{M_t, B_t}_{M_1, B_1 > 0\ (\theta = 1)})_{t=0}^{\infty}$$

$\theta^0 = 1$

Equ given $GP = (G_t = G, \tau_t, M_t, B_t)_{t=0}^{\infty}$ is a sequence $(C_t^*, b_t^*, M_t^*, \pi_t^*, P_t^*, i_t^*)_{t=0}^{\infty}$ such that

(i) (HH optimization) $(C_t^*, b_t^*, m_t^*)_{t=0}^{\infty}$ solves:

$$\max_{(C_t, b_t, m_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) + \gamma(G_0, G_1, \ldots)$$

$$\text{s.t. } P_t^* c_t + b_{t+1} + m_{t+1} \overset{(=)}{\leq} \pi_t^* + (1 + i_t^*) b_t + m_t - \tau_t \ (\forall t) \quad (\lambda_t)$$

$$P_t^* C_t \leq m_t \ (\forall t) \quad [\mu_t]$$

$$m_0 + b_0 = M_0 + B_0 \quad [\lambda_{-1}]$$

$$C_t, b_t, m_t \geq 0 \quad (\forall t)$$

(ii) (Firm optimization) $\forall t$

$$\pi_t^* = P_t^* Y$$

(iii) (Market clearing) $\forall t$:

$$c_t^* + G = Y$$
$$b_t^* = B_t,$$
$$m_t^* = M_t$$

GBC: $P_t^* G + (1+i_t^*) B_t = \tau_t + \underbrace{(M_{t+1} - M_t)}_{\text{seigniorage}} + B_{t+1} \quad (\forall t)$

Eqm characterization

- Market clearing $\Rightarrow \begin{cases} c_t^* = Y - G \\ b_t^* = B_t \\ m_t^* = M_t \end{cases} \quad (\forall t)$

Remaining variables: $\underbrace{P_t^* = ? \quad i_t^* = ?}$

Focus below.

Key: HH's portfolio choice

→ In each t, HHs choose how to allocate their savings between cash m_t and bond b_t

(If $m_t \uparrow \$1 \Rightarrow b_t \downarrow \$1$, if we keep savings constant).

m_t: ⊖ no interest

⊕ helps with CIA

Trade-off

$\left(\Leftarrow \text{e.g. } M_t = 0 \underset{\text{CIA}}{\Rightarrow} P^*_t G \le M_t = 0 \right.$

$\left. c_t^* = 0 \Rightarrow \text{bad} \right)$

b_t: ⊕ you get interest

⊖ bonds don't help w/ CIA

Formally: FOCs for HHs

$$\mathcal{L} = \sum_{t=0}^{\infty} \Big\{ \beta^t u(c_t) + \lambda_t \big[\pi_t^* + (1+i_{t-1}^*) b_t + m_t - \tau_t - P_t^* c_t - b_{t+1} - m_{t+1} \big] + \mu_t \big[m_t - P_t^* c_t \big] + \lambda_{-1} \big[M_0 + B_0 - m_0 - b_0 \big] \Big\}$$

$$(c_t): \quad \beta^t u'(c_t^*) = P_t^* (\lambda_t + \mu_t) \quad (\forall t) \quad ①$$

$$(m_t): \quad \lambda_t + \mu_t = \lambda_{t-1} \quad (\forall t) \quad ②$$

$$(b_t): \quad \lambda_t (1 + i_t^*) = \lambda_{t-1} \quad (\forall t) \quad ③$$

$$(TVC): \quad \lim_{t \to \infty} \lambda_t m_{t+1}^* = 0 \quad ④$$

$$\lim_{t \to \infty} \lambda_t b_{t+1}^* = 0 \quad ⑤$$

$$(CS) \quad \mu_t \big[m_t^* - P_t^* c_t^* \big] = 0 \quad (\forall t) \quad ⑥$$

$$\mu_t = \lambda_t i_t^*$$

Result 1: $i_t^* \geq 0 \ (\forall t)$

$(\because) \ \mu_t = \lambda_t i_t^* \ = \ i_t^* = \frac{\mu_t}{\lambda_t} \geq 0$ //

Called the zero lower bound on nominal interest rates.

Economic mechanism

$i_t^* < 0 \implies m_t$ strictly dominates bonds b_t

$\because m_t^*$ gives strictly higher return AND can be used in transactions via CIA.

$\implies b_t^* = 0 < B_t$

$\implies$ Bond market doesn't clear

$\implies$ no equ.

Result 2: If $i_t^* > 0$, then the CIA constraint binds, i.e., $P_t^* C_t^* = m_t^*$ and therefore

$$P_t^* = \frac{m_t^*}{C_t^*} = \frac{M_t}{Y-G}$$

$(\because)$ $i_t^* > 0 \Rightarrow 0 < i_t^* = \frac{\mu_t}{\lambda_t} \Rightarrow \mu_t > 0$

⑥

$$\underbrace{P_t^* C_t^*}_{\substack{\| \\ Y-G}} = \underbrace{m_t^*}_{\substack{\| \\ M_t^*}}$$

$$\Rightarrow P_t^* = \frac{M_t}{Y-G}$$

Mechanism:

$i_t^* > 0 \Rightarrow$ holding M_t has an opportunity cost $\overset{\|}{i_t^*} > 0$

$\Rightarrow$ HHs don't want to hold more ~~and~~ cash than they need to satisfy CIA.

$\Rightarrow P_t^* C_t^* = M_t^*$

Q: Given the above, what's the economic impact of monetary policy?

1. Output $= Y$, Consumption $C_t^* = Y - G$ regardless of $\{M_t\}_{t=0}^{\infty}$

$\Rightarrow \{M_t\}_{t=0}^{\infty}$ has no impact on output, Consumption, HH utility.

Money is neutral, (i.e., level of money supply doesn't affect real variables), and superneutral (i.e. growth rate of money supply doesn't affect real variables.)

2. (If $i_t^* > 0$):

$$P_t^* = \frac{M_t}{Y-G} \quad \left(\Leftrightarrow \underbrace{P_t^* Y = M_t \left(\frac{Y}{Y-G}\right)}_{\text{quantity equation}}\right)$$

$\frac{Y}{Y-G}$: velocity of money

$\Rightarrow$ If we double M_t ($\forall t$), then P_t^* doubles ($\forall t$)

If $\frac{M_{t+1} - M_t}{M_t} \uparrow \Rightarrow$ infl rate $\frac{P_{t+1} - P_t}{P_t} \uparrow$

empirically well-supported

Note: In this model:
real variables (output, consumption ...) are determined by $\boxed{1}$ independently of money supply.

$(M_t)_{t=0}^{\infty}$ only affects nominal variables in $\boxed{+}$
$\longleftarrow$ called the classical dichotomy.

Q: Here, govt can finance its spending by (lump sum) taxes, debt, seigniorage.
Difference?

A: No difference for real variables. —
output, cons, HH utility
Seigniorage typically causes inflation, while taxes do not.

key mechanism:
- taxes: HHs pay govt money
- seigniorage: HHs "pay" govt in (real terms) by accepting higher prices.
  (a called an inflation tax)

$\Rightarrow$ In this model, the two are equivalent in real terms.

Q: Do we have $i_t^* > 0$ $(\forall t)$ or $i_t^* = 0$ $(\exists t)$?

A. $i_t^* > 0$ $(\forall t)$ in any equ. at least if $M_{t+1} \geq M_t (\forall t)$

(∵) Suppose: $M_{t+1} \geq M_t$ $(\forall t)$

Suppose toward a contradiction that $i_T^* = 0$ $(\exists t)$

$\Longrightarrow_{②,③} \mu_T = 0$

So: $\dfrac{\beta^T u'(C_T^*)}{P_T^*} \underset{①}{=} \lambda_T + \mu_T = \lambda_T \underset{②}{=} \lambda_{T+1} + \mu_{T+1} \underset{①}{=} \dfrac{\beta^{T+1} u'(C_{T+1}^*)}{P_{T+1}^*}$

$\Rightarrow P_{T+1}^* = \beta P_T^* \underset{CIA}{\leq} \dfrac{M_T^*}{C_T^*}$ $= Y - G = C_{t+1}^*$

$\Rightarrow P_{T+1}^* C_{T+1}^* \leq \beta M_T < M_T \leq M_{T+1}$

$\Rightarrow P_{T+1}^* C_{T+1}^* < M_{T+1}$

$\underset{④}{\Rightarrow} \mu_{T+1} = 0 \underset{②③}{\Longrightarrow} i_{T+1}^* = 0.$

By induction: $i_T^* = i_{T+1}^* = i_{T+2}^* = \cdots = 0$

$\underset{②,③}{\Longrightarrow} \begin{cases} \mu_T = \mu_{T+1} = \mu_{T+2} = \cdots = 0 \\ \lambda_T = \lambda_{T+1} = \lambda_{T+2} = \cdots = (>0) \end{cases}$

this implies:

$$\lim_{T\to\infty} \lambda_T^* \underset{M_{T+1}}{m_{T+1}^*} = \underbrace{\left(\lim_{t\to\infty} \lambda_t\right)}_{\lambda_t > 0} \left(\lim_{t\to\infty} M_{t+1}\right)$$

8 The inflation tax — distortionary effects

In the previous section, we studied inflation tax in economy w/ exogeneous production.

no inputs required.

$\searrow$ [inflation tax and lump sum taxes are equivalent in real terms]

Q: what if production requires inputs e.g. labor?
A: Inflation tax is distortionary.
$\Rightarrow$ Inflation has negative impact on output, employment, consumption, HH utility.

Modify previous model so that

$$Y_t = L_t$$

and utility is:

$$\sum_{t=0}^{\infty} \beta^t u(C_t, L_t)$$
$$\oplus \quad \ominus$$

(1) (HH optimization) $(C_t^*, L_t^*, b_t^*, m_t^*)_{t=0}^{\infty}$

Result: The following is an equi:

$C_t^* = C^*$
$L_t^* = L^*$
$b_t^* = B_t$
⋮

$P_t^* =$

(assuming that e.g. $\left(\frac{B_t}{P_t^* Y_t^*}\right)_{t=0}^{\infty}$ debt-to-GDP ratio is bounded

Proof: Firm optimization
(check eqm conditions hold)
[← OK (by $P_t^* = W_t^*$)]
market clearing
[← PF]
HH optimization (← just need to check FOCs)
sol $\begin{cases} \lambda_t^* = \dfrac{\beta^{t+1} \mu_t (C_t^*, l_t^*)}{P_{t+1}^*} & (\forall t) \\ \mu_t = \lambda_t i^* & (\forall t) \end{cases}$

Effect of monetary policy in above eqm?
Result 1: Money is neutral
(∵) If we multiply M_t by $\phi > 0$
$\Rightarrow$ In eqm: P_t^* W_t^* are multiplied by ϕ
But no change in real variables
C_t^*, l_t^* ... HH utility

Result 2: Money is not supernatural
(∵) $g^m \uparrow \longrightarrow i^* \uparrow$

High M profit

VI The Real Business Cycle (RBC) Model

1. Motivation

Recall. Fact G - fluctuation in real GDP growth rates
[← called business cycles (booms / recessions)

Neo classical growth model: no fluctuations

⟹ Want to extend model

2. RBC model

Idea: Extend neo

Interpretations:

weather shocks: { good weather ⟹ high agricultural productivity $\rightarrow A_t \uparrow$

$A_t \downarrow$

oil price shocks

Mathematically: $(A_0, A_1, \ldots A_t, A_{t+1}, \ldots)$ are random variables

Notation $\begin{cases} Pr(A_0, A_1, .. A_T) = pdf \\ Pr(A_{t+1} \mid A_t, A_{t-1}, \ldots A_0) = \text{conditional pdf.} \end{cases}$

E ... expectation

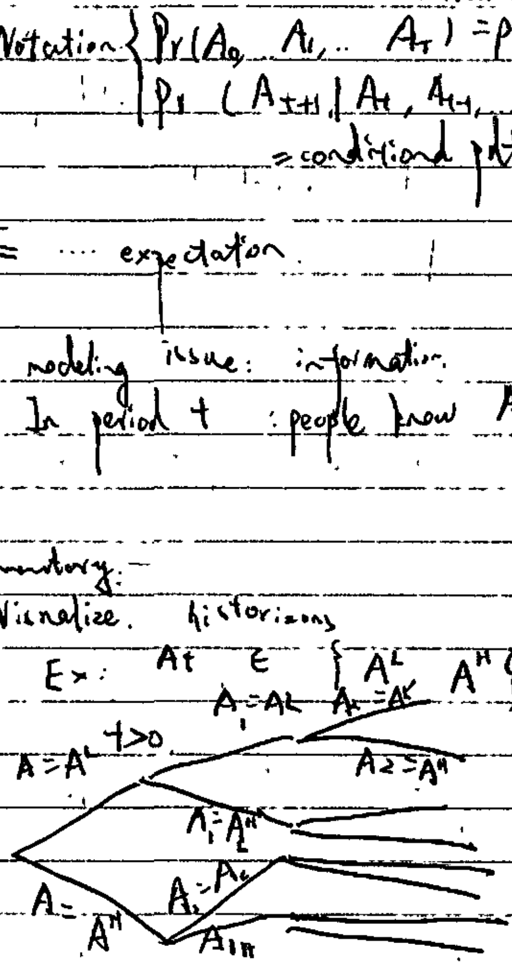
Key modeling issue: information

In period t: people know A_t

Commentary:

Visualize histories

Ex: $A_t \in \{A^L, A^H\}$



Alternative information structures:

Ex 1 : C_t is chosen in period $t-1$

$\Rightarrow C_t(h_{t-1})$

Ex 2. C_t is chosen ~~in period t+1~~ (after A_{t+1} is known)

$\Rightarrow C_t(h_{t+1})$

$\sim$ These are all different models.

· HH utility

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t(h_t), l_t(h_t))\right]$$

$(A_1, \dots A_t)$

expectation over $A_1, A_2, \dots A_t$ conditional on A_0, which is known in $t=0$

3. Negishi algorithm.

Result : The First Welfare Theorem holds

(Proof: Essentially the same as the growth model)

(4) Recursive approach

Need $(A_t)_{t=0}^{\infty}$ to be Markov process for this to work well.

[i.e. $\Pr(A_{t+1} \mid A_t, A_{t-1}, \dots, A_0) = \Pr(A_{t+1} \mid A_t)$]

doesn't depend on $A_t, A_{t-1}, \dots, A_0$

If $(A_t)_{t=0}^{\infty}$ is Markov, the Bellman eq is:

$$v(K, A) = \max_{c, l, k'} \{ u(c, l) + \beta E[v(K', A') \mid A] \}$$

s.t. $c + k' \le F(k, A l) + (1-\delta)k$

$c, l, 1-l, k' \ge 0.$

Proof: Let v satisfies the Bellman eq (*).

and $(g^c)(K, A)$ $g^l(K, A)$, $g^k(K, A)$ be maximizes of the RHS of (*)

then $(c_t^*(h_t), l_t^*(h_t), K_t^*(h_{t-1}))$ given by

$$\begin{cases} K_{t+1}^*(h_t) = g^k(K_t^*(h_{t-1}), A_t) & (\forall t, h_t) \\ k_0^*(h_{-1}) = \bar{K}_0 & \\ c_t^*(h_t) = g^c(K_t^*(h_{t-1}), A_t) & (\forall t, h_t) \\ l_t^*(h_t) = g^l(K_t^*(h_{t-1}), A_t) & (\forall t, h_t) \end{cases}$$

solves the (SPP)

To see why the Bellman eq % like the above

consider 2-period ex:

$$\max_{(c,k)} u(c_0(A_0)) + \beta u(c_1(A_0, A^L)) P_r(A^L | A_0) + \beta u(c_1(A_0, A^H)) P_r(A^H | A_0)$$

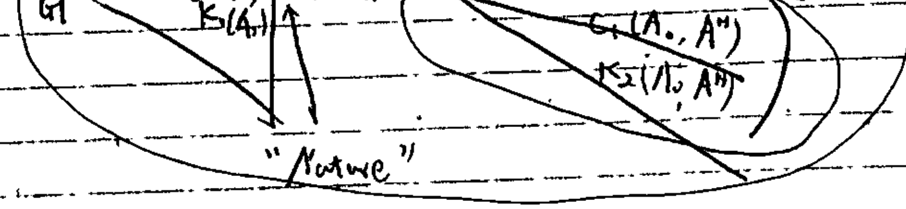
s.t. $c_0(A_0) + k_1(A_0) \le F(\bar{K}_0, 1) + (1-\delta)\bar{K}_0$

$c_1(A_0, A^L) + K_2(A_0, A^L) \le F(K_1(A_0), A^L) + (1-\delta)K_1(A_0)$

$c_1(A_0, A^H) + K_2(A_0, A^H) \le F(K_1(A_0, A^H) + (1-\delta)K_1(A_0^H)$

$c, k \ge 0.$

BI...



(BI)

Problem ①: $t=1$, given $A_1 = A^L$

$$\max_{(c_1(A_0,A^L),\ K_2(A_0 A^L))} U(c_0(A_0)) + \beta u(c_1(A_0, A^L)) \Pr(A^L | A_0) + \beta u(c_1(A_0, A^H)) \Pr(A^H | A_0)$$

s.t. $\cancel{c_0(A_0) + K_1(A_0) \le F(K_0, A_0) + (1-\delta) K_0}$

$c_1(A_0, A^L) + K_2(A_0, A^L) \le F(K_1(A_0), A^L) + (1-\delta) K_1(A_0)$

$\cancel{c_1(A_0, A^H) + K_2(A_0, A^H) \le F(K_1(A_0), A^H) + (1-\delta) K_1(A_0)}$

$c, K \ge 0.$

$$\Longleftrightarrow \begin{cases} \max_{(c_1(A_0,A^L),\ K_2(A_0,A^H))} u(c_1(A_0, A^L)) \\ c_1(A_0 A^L) + K_2(A_0, A^L) \le F(K_1(A_0), A^L) + (1-\delta) K_1(A_0) \\ c, K \ge 0. \end{cases}$$

$\|$

$V_1(K_1(A_0), A^L)$

$\uparrow$ Solution: $\left(g_1^c(K_1(A_0), A^L),\ g_1^K(K_1(A_0), A^L)\right)$

Problem ②: $t=1$, given $A_1 = A^H$

...

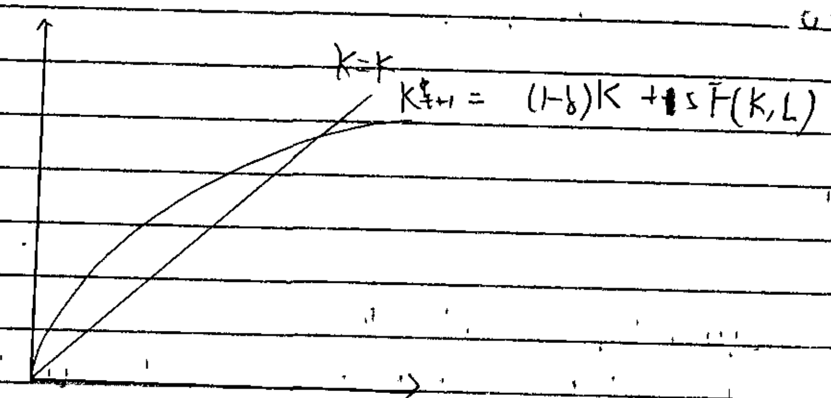
Problem ③ $t=0$.

$$\max_{(c_0(A_0),\ K_1(A_0))} u(c_0(A_0)) + \beta u(g_1^c(K_1(A_0), A^L)) \Pr(A^L | A_0) + \beta u(g_1^c(K_1(A_0), A^H)) \Pr(A^H | A_0)$$

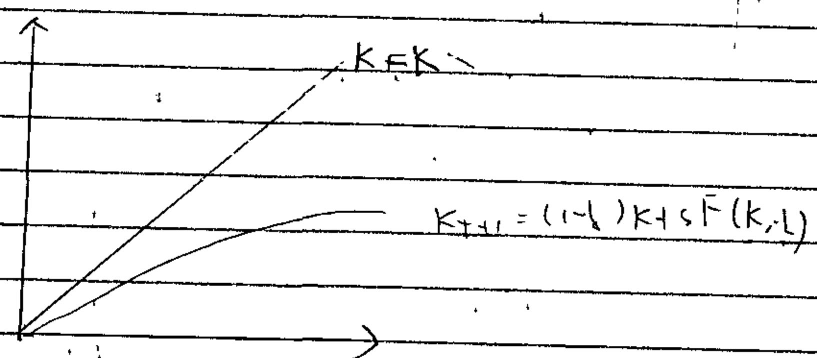
s.t. $c_0(A_0) + K_1(A_0) \le F(\bar{K}_0, A_0) + (1-\delta) \bar{K}_0$

$c_1(A_0, A^L) + \dots$

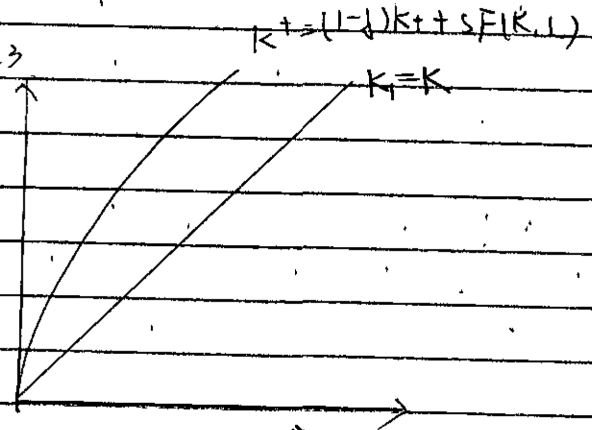
case1



case2



case3



(1) $K_{t+1} = (1-\delta)\ K_t + s(K_t + L_t)$

$= 1.1K_t + 2$

PiS 1

CRS

$\Rightarrow F(\lambda K, \lambda L) = \lambda F(K, L)$

Differentiate ↑ w.r.t. λ

$\Rightarrow F_\lambda(\lambda K, \lambda L) \cdot K + F_L(\lambda K, \lambda L) \cdot L = F(K, L)$ $\quad \lambda > 0$

$\lambda = 1$

$\Rightarrow F_K(K, L)K + F_L(K, L)L = F(K, L)$

Firms maximize profit

$MR = MC$

$MRK = F_K(K_t, L_t) = r_t$

$MRL = F_L(K_t, L_t) = w_t$

$\Rightarrow F(K_t, L_t) = r_t K_t + w_t L_t$

$\rightarrow \pi_t = 0$

Q2 $\frac{r_t K_t}{Y_t}$ = const in data

$F(K,L) = AK^{\alpha}L^{1-\alpha}$ & Profit Maximization

$\Rightarrow \frac{r_t K_t}{Y_t}$ = constant $= \alpha$

Profit Max $\Rightarrow r_t = MPK = F_K(K_t, L_t)$

$= A\alpha K_t^{\alpha-1} L_t^{1-\alpha}$

$r_t K_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} \cdot K_t^{1}$

$= \alpha A K_t^{\alpha} L_t^{1-\alpha} = \alpha Y_t$

$F(K_t, L_t) = A[\phi K^{\frac{6-1}{6}} + (1-\phi)L^{\frac{6-1}{6}}]^{\frac{6}{6-1}}$ $6 \neq 1$

$\frac{r_t K_t}{Y_t}$ = constant ?

$r_t = MPK = A[\phi K^{\frac{6-1}{6}} + (1-\phi)L^{\frac{6-1}{6}}]^{\frac{6}{6-1}-1} \cdot \frac{6-1}{6} \phi \cdot K_t^{\frac{6-1}{6}-1}$

$= A[\phi K_t^{\frac{6-1}{6}} + (1-\phi) L_t^{\frac{6-1}{6}}]^{\frac{1}{6-1}} \cdot \phi K^{\frac{6-1}{6}-1}$

$= A[\phi K^{\frac{6-1}{6}} + (1-\phi) L_t^{\frac{6-1}{6}}] \cdot \phi K_t^{\frac{6-1}{6}}$

Q3 $F(K_t, L_t) = K_t + L_t$

$L_t = 10$ $\delta = 0.2$ $X_t = s \cdot Y_t$

$d = 0.1$ $K_{t+1} = (1-\delta)K_t + X_t$

	K_t	Y_t	X_t
2000	100	110	22
2001	102	122	24.4
2002	128.2	135.2	28.04

Q4.

Dis 6

(2) $V^{(0)} = 0$ $\quad V^{(1)} \quad V^{(2)} \quad V^{(3)}$

(3) $V^{(i)}$

(4) $V = \lim_{i \to \infty} V^{(i)}$

$$V^{(1)}(k) = \max_{(c,k')} \{ \log(c) + \beta V^{(0)}(k') \} = \max \{\log(c)\}$$
$$c + k' \leq k^{\alpha} \qquad c, k' \geq 0 \qquad = \log(k^{\alpha}) = \alpha \log k$$

$$\begin{cases} k' = g_0^k(k) = 0 \\ c = g_0^c(k) = k^{\alpha} \end{cases}$$

$$V^{(2)}(k) = \max_{(c,k')} \{ \log(k^{\alpha} - k') + \beta\alpha \log(k') \}$$
$$k^{\alpha} \geq k' \geq 0$$

FO $=$' with k'

$$\frac{1}{k^{\alpha} - k'} = \frac{\alpha\beta}{k'}$$

$$\Rightarrow k' = g^k(k) = \frac{\alpha\beta}{1+\alpha\beta} k^{\alpha}$$
$$g^c(k) = \frac{1}{1+\alpha\beta} k^{\alpha}$$

$$\alpha(1+\alpha\beta)\log(k) + \alpha\beta(1+\alpha\beta) - (1+\alpha\beta)\log(1+\alpha\beta))$$
$$V^{(3)}(k) = \max \{ \log(k^{\alpha} - k') + \beta V^{(2)}(k) \}$$
$$= \alpha(1 + \alpha\beta + (\alpha\beta)^2)\log(k) + \text{const}$$

$$V^{(0)}(k) = 0$$
$$V^{(1)}(k) = \alpha \log(k)$$
$$V^{(2)}(k) = \alpha(1+\alpha\beta)\log(k) + \cdots$$
$$V^{(3)}(k) = \alpha(1 + \alpha\beta + (\alpha\beta)^2)\log(k) + \cdots$$

3. $$V^{(i)}k = \alpha\left(\sum_{j=0}^{i-1} (\alpha\beta)^{j}\right)\Big]\log(k) + \text{const}.$$

$$\text{const}_{i+1} = \beta\,\text{const}_i - \log\left[\textstyle\sum_{j=0} (\alpha\beta)^{j}\right] + \left[\sum_{j=1}^{i} (\alpha\beta)^{j}\right]$$
$$\times \log\left[\alpha\beta\right.$$

4. $V(k) = \lim_{i\to\infty} V^{(i)}(k)$

$= \lim_{i\to\infty} \alpha \left[\sum_{j=0}^{i-1} (\alpha\beta)^j \right] \log(k) \cdot \lim_{i\to\infty} \text{const}_i$

$= \frac{\alpha}{1-\alpha\beta} \log(k)$

5. $V^{(0)}(k) = -k$

$V^{(1)}(k) = \max_{(c,k')} \{ \log(c) - \beta k' \}$

s.t $c + k' \leq k^\alpha$

$k' = 0$

$c = k^\alpha$

$V^{(0)}(k) = 1$

$V^{(1)}(k) = \alpha \log(k) + \beta$

$V^{(i)}(k) = \alpha \left(\sum_{j=0}^{i-1} (\alpha\beta)^j \right) \log(k) + \beta^i + \text{const}_i$

7. Envelop.

$V'(k) = u'(g(k)) \, (F_k(k,1) + 1 - \delta)$

$V'(k) = \frac{\alpha}{1-\alpha\beta} \frac{1}{k}$ $\quad \frac{1}{g(k)} \alpha k^{\alpha-1}$

$\cancel{(1-\alpha\beta) k^\alpha}$

$V(k) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$

$c + k' = F(k, 1) + (1-\delta)k$

$k \uparrow \to V(k) \uparrow$

$\hookrightarrow F_k(k,1) + 1 - \delta) = c \uparrow$

$\to u'(c) \times (F_k(k,1) + 1 - \delta)$

Q1 $\frac{rK}{K} = \frac{\frac{rK}{Y}}{\frac{K}{Y}}$

PP8

Q1 $(G, \tau) \rightarrow$ Eqm $(c^*, l^*, L^*, p^*, W^*, \pi^*)$

a. (c^*, l^*) solves

$$\max_{c,l} \log(c) + \log(1-l) + \gamma G$$

$$p^* c \leq W^* l + \pi^* - \tau$$

$$c, l, 1-l \geq 0$$

b. $\pi^* = \max_{L \geq 0} p^* AL - W^* L = (p^* A - W^*) L$

c. $c^* + G = AL^*$

$L^* = l^* \in (0, 1)$

d. $p^* G \leq p^* \tau$

1. $(G, \tau) \rightarrow$ solves eqm

$p^* A - W^* > 0 \quad (\Leftrightarrow A > \frac{W^*}{p^*})$

$\Rightarrow L^* = \infty$ or I

$p^* A - W^* < 0 \Rightarrow L^* = 0$

$PA^* - W^* \Leftrightarrow PA = \frac{W^*}{p^*}$

$L^* \in (0, \infty)$

$c = Al - \tau$

$= Al - G$

$\max \log (Al - G) + \log (1 - l) + \gamma G$

$\Rightarrow$ FOC $\frac{A}{Al^* - G} = \frac{1}{1 - l^*} \Rightarrow$

$l^* = \frac{A + G}{2A} = L^*$

$$\begin{cases} C^* = \dfrac{A-G}{2} \\ l^* = \dfrac{A+G}{2} = L^* \\ \dfrac{W^*}{P^*} = A \qquad \Rightarrow P^* \in (0,\infty) \\ \qquad\qquad\qquad\qquad W^* = AP^* \\ \pi^* = 0 \end{cases}$$

$$Y^* = Al^* = \frac{A+G}{2}$$

$$\frac{\partial Y^*}{\partial G} = \frac{1}{2} < 1$$

2. $G > A$

$$C^* + G = Al^* \leq A$$

3. Optimal G? If G^* maximize Y^*

$$\max_{G \leq A} \left\{ \log(C^*(G)) + \log(1 - l^*(G)) + \gamma G \right\} \xrightarrow{G = A} \log\left(\frac{A-G}{2A}\right)$$

$$\max_{G \leq A} \left\{ 2\log(A-G) - 2\log(2) - \log(A) + \gamma G \right\}$$

FOC $\dfrac{2}{A-G^*} = \gamma \Rightarrow G^* = A - \dfrac{2}{\gamma}$

m_t ⊕ can be used for shopping

⊖ no return (nominal)

b_t ⊕ nominal return : $i_t \doteq 0$

⊖ can't be used for shopping , m^*

t

m_t, b_t $\rightarrow$ $\rightarrow$ b_{t+1}, m_{t+1}

$P_t^* C_t^* \le m_t^*$

P R

Q1. Production $F(K, A_t L_t) = (K_t)^\alpha (A_t L_t)^{1-\alpha}$

A_t is uncertain.

1. $C_1(A^H, A^H)$

$C_1(A^H, A^L)$

(SPP)

$$\max_{(C_t(h_t),\ K_t(h_{t-1}),\ L_t(h_t))} E_0\left[\sum_{t=0}^{\infty} \beta^t \log(C_t)\right]$$

$$C_t(h_t) + K_t(h_t) \le (K_t(h_{t-1}))^\alpha \times (A_t L_t(h_t))^{\alpha} \quad \forall t\ \forall h_t$$

$$K_0(h_{-1}) = \bar{K}_0 > 0.$$

$$C_t(h_t),\ K_t(h_{t-1}),\ L_t(h_t),\ 1-L(h_t) \ge 0$$

2. FOC: Variational Approach.

$\Rightarrow$ FOC

(RC) $C_t^*(h_t) + K_{t+1}^*(h_t) = k_t^*(h_{t-1})^\alpha A_t^{1-\alpha}$

$L_t^*(h_t) = 1$

(NC) $K_0^*(h_{-1}) = \bar{K}_0$

(EE) $\frac{1}{C_t^*(h_t)} = \beta E\left[\frac{1}{C_{t+1}^*(h_{t+1})}\left[\alpha k_{t+1}^*(h_t)\right]^{\alpha-1}(A_{t+1})^{1-\alpha}\right]$

(TVC) $\lim_{T\to\infty} E_0\left[\beta^t \frac{K_{t+1}(h_t)}{C_t^*(h_t)}\right] = 0.$

3. $C_t^*(h_t) = (1-\alpha\beta)(K_t^*(h_{t-1}))^\alpha (A_t)^{1-\alpha}$ $\quad E_t[\ ] = E[\ \ |h_t]$

$K_{t+1}^*(h_t) = \alpha\beta(K_t^*(h_{t-1}))^\alpha (A_t)^{1-\alpha}$ $\quad E[f(x)\ |x]$

$L_t^*(h_t) = 1.$ $\quad E[f(x)|x] = f(x)$

$K_0^*(h_{-1}) = \bar{K}_0 > 0.$

$\frac{\alpha\beta}{1-\alpha\beta}\frac{1}{K_{t+1}^*(h_t)}$

check (EE)

(RHS) $= E_t\left[\beta \frac{\alpha \cancel{(K_{t+1}^*(h_t))^{\alpha-1}} A_{t+1}^{1-\alpha}}{(1-\alpha\beta)\cancel{(K_{t+1}^*(h_t))^\alpha} A_{t+1}^{1-\alpha}}\right]$ $\quad \underbrace{\sum_{A_{t+1}} P(A_{t+1}|h_t)}_{=1}$

$= E_0\left[\frac{\alpha\beta}{1-\alpha\beta}\frac{1}{K_{t+1}^*(h_t)}\right]$ $\quad \sum_{A_{t+1}} \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{K_{t+1}^*(h_t)} P(A_{t+1}|h_t)$

$= \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{K_{t+1}^*(h_t)}$

$$= \frac{\alpha\beta}{1-\alpha\beta} \underbrace{\frac{1}{\alpha\beta (k_t^*(h_{t-1}))^\alpha A_t^{1-\alpha}}}$$

$$= \frac{1}{C_t^*(h_t)} = (\text{LHS})$$

$\boxed{\text{TVC}}$ $\lim_{T\to\infty} \mathbb{E}_0 \left[\beta^t \frac{k_{t+1}^*(h_t)}{C_t^*(h_t)}\right]$

$$= \lim_{t\to\infty} \mathbb{E} \left[\beta^t \frac{\alpha\beta}{1-\alpha\beta}\right] = 0.$$

#3 Recursive Approach

$\Rightarrow$ Bellman Eq.

$$P(A_t | A_{t-1} \dots A_0) = P(A_t | A_{t-1})$$

$$v(k, A) = \max_{(c,k',l)} \left\{ \log(c) + \beta \mathbb{E} \left[V(k', A') | A \right] \right\}$$

$$\text{s.t.} \quad c + k' \leq k^\alpha (Al)^{1-\alpha}$$

$$c, k', 1-l, l \geq 0.$$