Chapter 2: Linear Regression

Distribution of the RSS Values and Hypothesis Testing for 수정하기

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Outline

1 Distribution of the RSS Values

2 Hypothesis Testing for $\hat{\beta}_j \neq 0$

Hat matrix

- Hat matrix defined by $\hat{y} = Hy$

$\hat{y} = X\hat{\beta} = X(X^TX)^{-1}X^Ty = Hy$

$$H \triangleq X(X^TX)^{-1}X^T$$

- Some properties

$$\begin{aligned} H^2 &= X(X^TX)^{-1}X^T \cdot X(X^TX)^{-1}X^T = X(X^TX)^{-1}X^T = H \\ (I-H)^2 &= I - 2H + H^2 = I - H \\ HX &= X(X^TX)^{-1}X^T \cdot X = X \end{aligned}$$

Residual sum of square

- RSS defined

$$\text{RSS} \triangleq ||y - \hat{y}||^2$$

- Using hat matrix

$$\begin{aligned} y - \hat{y} &= y - Hy = (I - H)y = (I - H)(X\beta + \varepsilon) \\ &= (X - HX)\beta + (I - H)\varepsilon = (I - H)\varepsilon \end{aligned}$$

$$\text{RSS} \triangleq ||y - \hat{y}||^2 = \{(I - H)\varepsilon\}^T (I - H)\varepsilon = \varepsilon^T (I - H)^2 \varepsilon = \varepsilon^T (I - H)\varepsilon$$

Eigenvalues H and $I - H$

- They are only zeros and ones

- Dimensions of the eigenspaces of H and $I - H$ are both $p + 1$

  Proof using $\text{rank}(X) = p + 1$

$$\text{rank}(H) \leq \min\{\text{rank}(X(X^T X)^{-1}), \text{rank}(X)\} \leq \text{rank}(X) = p + 1$$
$$\text{rank}(H) \geq \text{rank}(HX) = \text{rank}(X) = p + 1$$

Hat matrix

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$$RSS \triangleq ||y - \hat{y}||^2 = \{(I - H)\varepsilon\}^T (I - H)\varepsilon = \varepsilon^T (I - H)^2 \varepsilon = \varepsilon^T (I - H)\varepsilon$$

Eigenvalues of H and Null space of $(I-H)$

- Proof by contrapositive

$$Hx = x \Rightarrow (I-H)x = 0$$
$$(I-H)x = 0 \Rightarrow Hx = x$$

- Dimensions of the eigenspaces of H is $p+1$

Proof using $\text{rank}(X) = p+1$

$$\text{rank}(H) \leq \min\{\text{rank}(X(X^TX)^{-1}), \text{rank}(X)\} \leq \text{rank}(X) = p+1$$
$$\text{rank}(H) \geq \text{rank}(HX) = \text{rank}(X) = p+1$$

- Dimensions of the null space of $I-H$ is $N-(p+1)$

$$P(I-H)P^T = \text{diag}(\underbrace{1,\dots,1}_{N-p-1},\underbrace{0,\dots,0}_{p+1})$$

- We define $v = P\varepsilon \in \mathbb{R}^N$, then from $\varepsilon = P^T v$

$$\mathrm{RSS} = \varepsilon^T (I - H)\varepsilon = (P^T v)^T (I - H) P^T v = v^T P(I - H) P^T v$$

$$= [v_1, \cdots, v_{N-p-1}, v_{N-p}, \cdots, v_n] \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{N-p-1} \\ v_{N-p} \\ \vdots \\ v_N \end{bmatrix} = \sum_{i=1}^{N-p-1} v_i^2$$

dd

- Let $w \in \mathbb{R}^{N-p-1}$

- Average

  $E[v] = E[P\varepsilon] = 0$

  $E[w] = 0$

- Covariance

  $E[vv^t] = E[P\varepsilon(P\varepsilon)^T] = PE[\varepsilon\varepsilon^t]P = P\sigma^2 IP^T = \sigma^2 I$

  $E[ww^T] = \sigma^2 I$

- We have RSS

$$\frac{RSS}{\sigma^2} \sim \chi^2_{N-p-1}$$

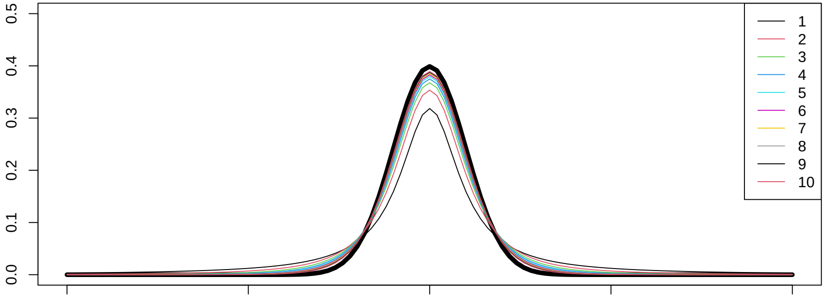
Outline

Test statistic T

- A t distribution with $N-P-1$ degrees of freedom
- We decide that hypothesis $\beta_j = 0$ should be rejected.
- $U \sim N(0,1)$, $V \sim \chi^2_m$,

$$T \triangleq U/\sqrt{V/m}$$

```r
curve(dnorm(x), -10, 10, ann = FALSE, ylim = c(0, 0.5), lwd = 5)
for(i in 1:10)curve(dt(x, df= i), -10, 10, col = i, add = TRUE, ann = FALSE)
legend("topright", legend = 1:10, lty = 1, col = 1:10)
```



Significance level

- $\alpha = 0.01,\ 0.05$
- Reject the null hypothesis

Q & A

Thank you :)