Chapter 2: Linear Regression

Distribution of the RSS Values and Hypothesis Testing for $\hat{\beta}_j \neq 0$

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Outline

Hat Matrix

- Hat matrix defined by

  $\hat{y} = X\hat{\beta} = X(X^T X)^{-1} X^T y = Hy$

$$H := X(X^T X)^{-1} X^T$$

- Some properties

  1. $H^2 = H$
  2. $(I - H)^2 = I - H$
  3. $HX = X$
  4. $H^T = H$

Residual Sum of Squares

- RSS defined

$$\text{RSS} := ||y - \hat{y}||^2$$

- Using hat matrix

$$\begin{aligned} y - \hat{y} &= y - Hy = (I - H)y = (I - H)(X\beta + \varepsilon) \\ &= (X - HX)\beta + (I - H)\varepsilon = (I - H)\varepsilon \end{aligned}$$

$$||y - \hat{y}||^2 = \{(I - H)\varepsilon\}^T (I - H)\varepsilon = \varepsilon^T (I - H)^2 \varepsilon = \varepsilon^T (I - H)\varepsilon$$

Eigenvalues H and $I - H$

- They are only zeros and ones
- Dimensions of the eigenspaces of H and $I - H$ are both $p + 1$

  Proof using $\text{rank}(X) = p + 1$

$$\text{rank}(H) \leq \min\{\text{rank}(X(X^T X)^{-1}), \text{rank}(X)\} \leq \text{rank}(X) = p + 1$$

$$\text{rank}(H) \geq \text{rank}(HX) = \text{rank}(X) = p + 1$$

Eigenvalues of H and Null space of $(I - H)$

- Proof by contrapositive

$$Hx = x \Rightarrow (I - H)x = 0$$

$$(I - H)x = 0 \Rightarrow Hx = x$$

- Dimensions of the eigenspaces of H is $p + 1$

  Proof using $\text{rank}(X) = p + 1$

$$\text{rank}(H) \leq \min\{\text{rank}(X(X^T X)^{-1}), \text{rank}(X)\} \leq \text{rank}(X) = p + 1$$

$$\text{rank}(H) \geq \text{rank}(HX) = \text{rank}(X) = p + 1$$

- Dimensions of the null space of $I - H$ is $N - (p + 1)$

$$P(I - H)P^T = \text{diag}(\underbrace{1, \dots, 1}_{N-p-1}, \underbrace{0, \dots, 0}_{p+1})$$

RSS

- We define $v = P\varepsilon \in \mathbb{R}^N$, then from $\varepsilon = P^T v$

$$\text{RSS} = \varepsilon^T(I-H)\varepsilon = (P^T v)^T (I-H) P^T v = v^T P(I-H)P^T v$$

$$= [v_1, \cdots, v_{N-p-1}, v_{N-p}, \cdots, v_n] \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{N-p-1} \\ v_{N-p} \\ \vdots \\ v_N \end{bmatrix} = \sum_{i=1}^{N-p-1} v_i^2$$

Distribution of RSS

- Let $w \in \mathbb{R}^{N-p-1}$
- Average

  $E[v] = E[P\varepsilon] = 0$

  $E[w] = 0$
- Covariance

  $E[vv^t] = E[P\varepsilon(P\varepsilon)^T] = PE[\varepsilon\varepsilon^t]P = P\sigma^2 IP^T = \sigma^2 I$

  $E[ww^T] = \sigma^2 I$
- We have RSS

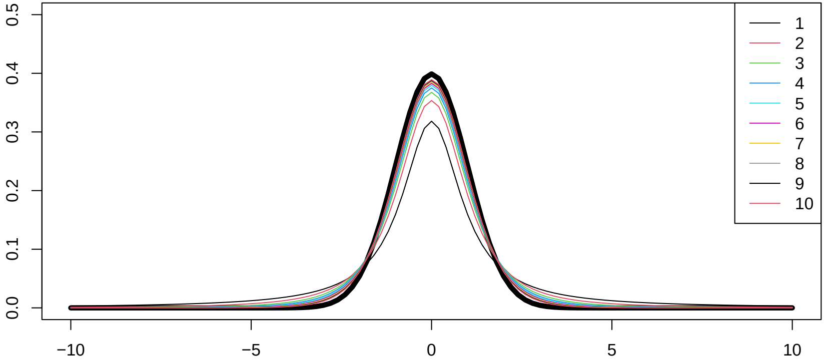
$$\frac{RSS}{\sigma^2} \sim \chi^2_{N-p-1}$$

Outline

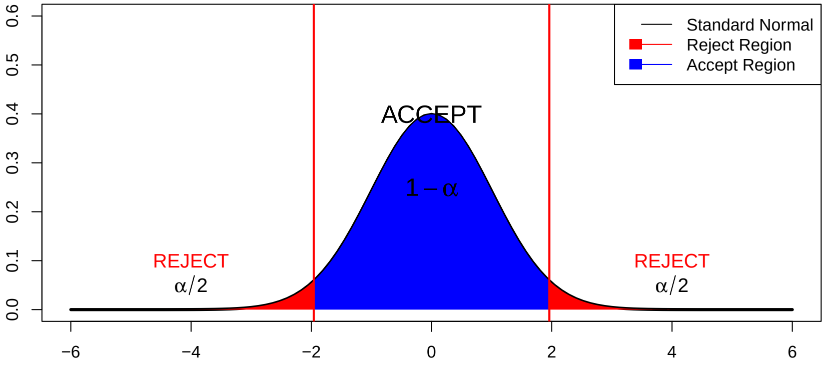
Test Statistic T

- A t distribution with $N-P-1$ degrees of freedom
- We decide that hypothesis $\beta_j = 0$ should be rejected.
- $U \sim N(0,1),\ V \sim \chi^2_m,$

$$T \triangleq U/\sqrt{V/m}$$



- $\alpha = 0.01,\ 0.05$
- Null hypothesis $\beta_j = 0$



Example 23

- For $p = 1$, since

$$X^T X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = N \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & \frac{1}{N}\sum_{i=1}^{N} x_i^2 \end{bmatrix}$$

- The inverse is

$$(X^T X)^{-1} = \frac{1}{N} \begin{bmatrix} 1 & \bar{x} \\ \bar{x} & \frac{1}{N}\sum_{i=1}^{N} x_i^2 \end{bmatrix}^{-1} = \frac{1}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix}$$

- Which means that

$$B_0 = \frac{\frac{1}{N}\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \quad \text{and} \quad B_1 = \frac{1}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

- For $B = (X^T X)^{-1}$, $B\sigma^2$ is covariance matrix of $\hat{\beta}$
- $B_j \sigma^2$ is the variance of $\hat{\beta}_j$
- Because $\bar{x}$ is positive, the correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$ is negative

$$t = \frac{\hat{\beta}_j - \beta_j}{\mathrm{SE}(\hat{\beta}_j)} \sim t_{N-p-1}$$

Statistical Independence in Regression

- It remains to be shown that U and V are independent

$$U \triangleq \frac{\hat{\beta}_j - \beta_j}{\sqrt{B_j}\sigma} \sim N(0,1) \quad \text{and} \quad V \triangleq \chi^2_{N-p-1}$$

- Sufficient to show tha $y - \hat{y}$ and $\hat{\beta} - \beta$ are independent

$$(\hat{\beta} - \beta)(y - \hat{y})^T = (X^T X)^{-1} X^T \varepsilon\varepsilon^T (I - H)$$

- From $E\varepsilon\varepsilon^T = \sigma^2 I$ and $HX = X$,

$$E(\hat{\beta} - \beta)(y - \hat{y})^T = 0$$

Q & A

Thank you :)