Jump Detection

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Here we consider time series and ways to detect discontinuities or sudden jumps in such time series. We begin by reviewing a wavelet decomposition of time series data. For details, see Ref. [1].

1 Wavelet

1.1 Continuous Wavelet

What is a wavelet? A function $\psi(x)$ is called a wavelet if it satisfies

$$\int_{-\infty}^{\infty} dx\, \psi(x) = 0 \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|\psi(\omega)|^2}{\omega} = C_\psi < 0 \tag{2}$$

Here C_ψ is called admissible constant. A wavelet family can be generated from the above mother wavelet by translation and dilation

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi \left(\frac{x-b}{a} \right) \tag{3}$$

The parameter a controls the dilation, while b controls the shift of the wavelet.

The *continuous wavelet transformation* (CWT) of a function $f(x)$ is defined by

$$Wf(a,b) = \int_{-\infty}^{\infty} dx\, f(x) \psi^*_{a,b}(x) \equiv \langle f, \psi_{a,b} \rangle \tag{4}$$

The angle bracket denotes L^2-inner product. The inverse transformation is defined as

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db\, Wf(a,b)\, \psi_{a,b}(x). \tag{5}$$

This continuous wavelet decomposition can be used to detect jump. Consider a Heaviside step function $H(x)$. The continuous wavelet transformation of $H(x)$ is

$$Wf(a,b) = \int_{-\infty}^{\infty} dx\, H(x) \psi^*_{a,b}(x) = \int_0^{\infty} \frac{dx}{\sqrt{a}} \psi^* \Big(\frac{x-b}{a} \Big). \tag{6}$$

From the above expression, we see that $Wf(a,0)/\sqrt{a}$ takes a constant value regardless of the value a. This indicates a discontinuity at $b = 0$. In general, if $Wf(a,b)/\sqrt{a}$ takes a constant value for all a at a particular b, this signals a discontinuity of the original function $f(x)$ at $x = b$.

1.2 Discrete Wavelet

In the discrete case, a family of wavelets is constructed from the mother wavelet via

$$\psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j} x - k) \tag{7}$$

for $j, k \in \mathbb{Z}$. The discrete wavelet transformation is defined as

$$Wf_{j,k} = \langle f,\, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} dx\, \psi^{*}_{j,k}(x) f(x) \tag{8}$$

The inverse transformation is

$$f(x) = \sum_{j,k} \psi_{j,k}(x) W f_{j,k} \tag{9}$$

A

The jump time series is defined as

$$x_t = \frac{r_t}{f_t \sigma_t} \tag{10}$$

where $r_t = \ln(p_t/p_{t-1})$ is 1-minute return, f_t is an estimator of intraday periodicity, and σ_t is an estimator for the local volatility. The σ_t is defined as

$$\sigma_t^2 = \frac{\pi}{2K} \sum_{i=1}^{390} |r_{t-i}||r_{t-i+1}| \tag{11}$$

References

[1] I. Daubechies, ed., *Ten lectures on wavelets.* Jan., 1992.