SU(2) NJL model: vacuum sector and meson properties

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I. MESONIC PROPERTIES

The action S in this process is

$$S = -N_c \mathrm{Tr}\left[\log D\right] + \int d^4x \, \frac{1}{2G}\left(\sigma^2 + \vec{\pi}^2\right) \tag{1}$$

$$= -\frac{1}{2} N_c \mathrm{Tr}\left[\log D^\dagger D\right] + \int d^4x \, \frac{1}{2G}\left(\sigma^2 + \vec{\pi}^2\right) \tag{2}$$

where $D^\dagger D$ is

$$D^\dagger D = -\partial^2 + \sigma^2 + \vec{\pi}^2 - \left\{\not{\partial}(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\} = V^{-1} + A \tag{3}$$

and $V^{-1} = -\partial^2 + M^2,\ A = \sigma^2 + \vec{\pi}^2 - M^2 - \left\{\not{\partial}(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}$. By the expansion for log,

$$\begin{aligned}
&\mathrm{Tr}\left[\log\left(V^{-1} + A\right)\right] \approx \mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[VAVA\right] + 8I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right)\\
&\Longrightarrow S = -\frac{1}{2}N_c\left[\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[VAVA\right] + 8I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right)\right] + \int d^4x\,\frac{1}{2G}\left((\sigma - \hat{m})^2 + \vec{\pi}^2\right)
\end{aligned} \tag{4}$$

We have to calculate $\mathrm{Tr}\left[VAVA\right]$ hand by hand.

A. Calculation of $\mathrm{Tr}\left[VAVA\right]$

$$\begin{aligned}
\mathrm{Tr}\left[VAVA\right] &= \mathrm{Tr}\left[V\left(\sigma^2 + \vec{\pi}^2 - M^2 - \left\{\not{\partial}(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(\sigma^2 + \vec{\pi}^2 - M^2 - \left\{\not{\partial}(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}\right)\right]\\
&= \mathrm{Tr}\left[V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)\right]\\
&= \mathrm{Tr}\left[V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)\right]\\
&= \mathrm{Tr}\Big[4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} + 2i\sigma_0 V\tilde{\sigma}V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2i\sigma_0 V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\tilde{\sigma} - V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\\
&\quad \times V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2\sigma_0 V\tilde{\sigma}V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) + 2\sigma_0 V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\tilde{\sigma} + iV\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\\
&\quad + iV\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) + V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big]
\end{aligned} \tag{5}$$

We can erase terms using relations $\mathrm{tr}[\gamma^\mu] = 0$ and $\mathrm{tr}[\gamma^\mu\gamma_5] = 0$ in the spin space.

$$\begin{aligned}
\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} - V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\left\{\not{\partial}(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2\sigma_0\left\{V\tilde{\sigma}, V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\right\}\\
&+ V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big].
\end{aligned} \tag{6}$$

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To treat second term, following relation is needed.

$$\begin{aligned}\gamma_5\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau})\gamma_5\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau}) &= -\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau})\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau}) = -\partial_\mu\gamma^\mu\tilde{\pi}_i\tau_i\partial_\nu\gamma^\nu\tilde{\pi}_j\tau_j = -\partial_\mu\gamma^\mu\tilde{\pi}_i\partial_\nu\gamma^\nu\tilde{\pi}_j(\delta_{ij}+i\epsilon_{ijk}\tau_k)\\ &= -\partial_\mu\gamma^\mu\tilde{\pi}_i\partial_\nu\gamma^\nu\tilde{\pi}_i - \epsilon_{ijk}\partial_\mu\gamma^\mu\tilde{\pi}_i\partial_\nu\gamma^\nu\tilde{\pi}_j\tau_k\end{aligned} \tag{7}$$

Because this is in the trace and tr$[\tau_i]$=0, last term is a zero. Therefore

$$\gamma_5\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau})\gamma_5\partial\!\!\!/(\tilde{\vec{\pi}}\cdot\vec{\tau}) = -\partial\!\!\!/\tilde{\vec{\pi}}\cdot\partial\!\!\!/\tilde{\vec{\pi}} \tag{8}$$

$$\begin{aligned}\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2V\tilde{\sigma}V\tilde{\sigma} - V\partial\!\!\!/\tilde{\sigma}V\partial\!\!\!/\tilde{\sigma} - V\partial\!\!\!/\tilde{\vec{\pi}}\cdot V\partial\!\!\!/\tilde{\vec{\pi}} - 2i\gamma_5V\partial\!\!\!/\tilde{\sigma}\cdot\partial\!\!\!/\left(\tilde{\vec{\pi}}\cdot\vec{\tau}\right) + 2\sigma_0\left\{V\tilde{\sigma},V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)\right\}\\ &+V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)\Big].\end{aligned} \tag{9}$$

Since $\mathrm{tr}[\gamma_5]=0$,

$$\begin{aligned}\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2V\tilde{\sigma}V\tilde{\sigma} - V\partial\!\!\!/\tilde{\sigma}V\partial\!\!\!/\tilde{\sigma} - V\partial\!\!\!/\tilde{\vec{\pi}}\cdot V\partial\!\!\!/\tilde{\vec{\pi}} + 2\sigma_0\left\{V\tilde{\sigma},V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)\right\}\\ &+V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2+\tilde{\vec{\pi}}^2\right)\Big].\end{aligned} \tag{10}$$

From $\phi=(\sigma,\vec{\pi})$,

$$\begin{aligned}&\phi = \phi_0+\tilde{\phi} = (\sigma_c = M, 0) + (\tilde{\sigma},\tilde{\vec{\pi}})\\ &\Longrightarrow \phi_0\tilde{\phi} = \sigma_0\tilde{\sigma},\ \ \tilde{\phi}\tilde{\phi} = \tilde{\sigma}^2+\tilde{\vec{\pi}}^2\end{aligned} \tag{11}$$

substituting Eq (10),

$$\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\left[4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi},V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right]. \tag{12}$$

Therefore the action Eq (4) is

$$\begin{aligned}S = -\frac{1}{2}N_c\Big[&\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi},V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right]\\ &+8I_1(M)\int d^4x\left(\sigma^2+\vec{\pi}^2-M^2\right)\Big] + \int d^4x\,\frac{1}{2G}\left((\sigma-\hat{m})^2+\vec{\pi}^2\right)\end{aligned} \tag{13}$$

Since we use saddle-point approximation, the action can be expressed the expansion for ϕ with a zero first derivative:

$$\begin{aligned}S[\phi] &\approx S[\phi_0] + \frac{\delta}{\delta\phi}S[\phi_0]\tilde{\phi} + \frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2\\ &= S[\phi_0] + \frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2.\end{aligned} \tag{14}$$

Comparing with Eq (13), we can find the second derivative term

$$\begin{aligned}\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 &= -\frac{1}{2}N_c\left[-\frac{1}{2}\mathrm{Tr}\left[4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right]\right] = \frac{N_c}{4}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left\langle p\left|4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle\\ &= 2N_c\int\frac{d^4p}{(2\pi)^4}\left\langle p\left|4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle.\end{aligned} \tag{15}$$

Now we will use the completeness relation:

$$\mathbb{I} = \frac{d^4p'}{(2\pi)^4}|p'\rangle\langle p'| \tag{16}$$

$$\begin{aligned}\Longrightarrow\int\frac{d^4p}{(2\pi)^4}\left\langle p\left|4\phi_0^2V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\Big(&\left\langle p\left|4\phi_0^2V\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|V\tilde{\phi}\right|p\right\rangle\\ &-\left\langle p\left|V\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle\Big)\end{aligned} \tag{17}$$

In the momentum space representation, $V = 1/(p^2+M^2)$. So

$$\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 = 2N_c\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\frac{1}{(p^2+M^2)(p'^2+M^2)}\left(\left\langle p\left|4\phi_0^2\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|\tilde{\phi}\right|p\right\rangle - \left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|\partial\!\!\!/\tilde{\phi}\right|p\right\rangle\right) \tag{18}$$

Let us calculate term by term.

$$\begin{aligned}
\left\langle p\left|4\phi_0^2\tilde{\phi}\right|p'\right\rangle &= 4M^2\int d^4x\left\langle p\left|\tilde{\phi}\right|x\right\rangle\langle x\,|\,p'\rangle = 4M^2\int d^4x\,\tilde{\phi}(x)\,\langle p\,|\,x\rangle\,\langle x\,|\,p'\rangle = 4M^2\int d^4x\,\tilde{\phi}(x)e^{ix(p-p')}\\
&= 4M^2\tilde{\phi}(p-p'),\\
\left\langle p'\left|\tilde{\phi}\right|p\right\rangle &= \tilde{\phi}(p'-p),\\
\left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle &= \int d^4x\left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|x\right\rangle\langle x\,|\,p'\rangle = \int d^4x\,\partial\!\!\!/\tilde{\phi}(x)\,\langle p\,|\,x\rangle\,\langle x\,|\,p'\rangle = \int d^4x\,\partial\!\!\!/\tilde{\phi}(x)e^{ix(p-p')}\\
&= \int d^4x\,\partial\!\!\!/\left(\tilde{\phi}(x)e^{ix(p-p')}\right) - \int d^4x\,\tilde{\phi}(x)\partial\!\!\!/e^{ix(p-p')} = -\int d^4x\,\tilde{\phi}(x)i(p-p')e^{ix(p-p')}\\
&= -i(p-p')\tilde{\phi}(p-p'),\\
\left\langle p'\left|\partial\!\!\!/\tilde{\phi}\right|p\right\rangle &= -i(p'-p)\tilde{\phi}(p'-p).
\end{aligned} \tag{19}$$

Then Eq (18) is

$$\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 = 2N_c\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\frac{4M^2\tilde{\phi}(p-p')\tilde{\phi}(p'-p)+(p-p')(p'-p)\tilde{\phi}(p-p')\tilde{\phi}(p'-p)}{(p^2+M^2)(p'^2+M^2)}. \tag{20}$$

For convenience, let us introduce changing variable:

$$p'+p = 2k,\;\; p'-p = q \tag{21}$$

and

$$p' = k+\frac{q}{2},\;\; p = k-\frac{q}{2}. \tag{22}$$

Since we change two variables (p', p) to (k, q), 4-integral variables becomes d^4kd^4q. Hence

$$\begin{aligned}
\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 &= 2N_c\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\frac{4M^2\tilde{\phi}(-q)\tilde{\phi}(q)-q^2\tilde{\phi}(-q)\tilde{\phi}(q)}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}\\
&= 2N_c\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\frac{4M^2-q^2}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}\tilde{\phi}(-q)\tilde{\phi}(q)\\
&= 2N_c\int\frac{d^4q}{(2\pi)^4}(4M^2-q^2)f(q)\tilde{\phi}(-q)\tilde{\phi}(q),\;\; f(q) = \int\frac{d^4k}{(2\pi)^4}\frac{1}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}.
\end{aligned} \tag{23}$$