SU(2) NJL model: vacuum sector and meson properties

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I. MESONIC PROPERTIES

The action S in this process is

$$S = -N_c \mathrm{Tr}\left[\log D\right] + \int d^4x \, \frac{1}{2G}\left(\sigma^2 + \vec{\pi}^2\right) \tag{1}$$

$$= -\frac{1}{2} N_c \mathrm{Tr}\left[\log D^\dagger D\right] + \int d^4x \, \frac{1}{2G}\left(\sigma^2 + \vec{\pi}^2\right) \tag{2}$$

where $D^\dagger D$ is

$$D^\dagger D = -\partial^2 + \sigma^2 + \vec{\pi}^2 - \left\{\partial\!\!\!/(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\} = V^{-1} + A \tag{3}$$

and $V^{-1} = -\partial^2 + M^2,\ A = \sigma^2 + \vec{\pi}^2 - M^2 - \left\{\partial\!\!\!/(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}$. By the expansion for log,

$$\begin{aligned}
&\mathrm{Tr}\left[\log\left(V^{-1} + A\right)\right] \approx \mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[VAVA\right] + 8I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right) \\
&\Longrightarrow S = -\frac{1}{2}N_c\left[\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[VAVA\right] + 8I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right)\right] + \int d^4x\,\frac{1}{2G}\left((\sigma - \hat{m})^2 + \vec{\pi}^2\right)
\end{aligned} \tag{4}$$

We have to calculate $\mathrm{Tr}\left[VAVA\right]$ hand by hand.

A. Calculation of $\mathrm{Tr}\left[VAVA\right]$

$$\begin{aligned}
\mathrm{Tr}\left[VAVA\right] &= \mathrm{Tr}\left[V\left(\sigma^2 + \vec{\pi}^2 - M^2 - \left\{\partial\!\!\!/(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(\sigma^2 + \vec{\pi}^2 - M^2 - \left\{\partial\!\!\!/(\sigma + i\vec{\pi}\cdot\vec{\tau}\gamma_5)\right\}\right)\right] \\
&= \mathrm{Tr}\left[V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)\right] \\
&= \mathrm{Tr}\left[V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)V\left(2\sigma_0\tilde{\sigma} + 2\vec{\pi}_0\tilde{\vec{\pi}} + \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2 - \left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}\right)\right] \\
&= \mathrm{Tr}\Big[4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} + 2i\sigma_0 V\tilde{\sigma}V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2i\sigma_0 V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\tilde{\sigma} - V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} \\
&\quad \times V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2\sigma_0 V\tilde{\sigma}V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) + 2\sigma_0 V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\tilde{\sigma} + iV\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} \\
&\quad + iV\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) + V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big]
\end{aligned} \tag{5}$$

We can erase terms using relations $\mathrm{tr}[\gamma^\mu] = 0$ and $\mathrm{tr}[\gamma^\mu\gamma_5] = 0$ in the spin space.

$$\begin{aligned}
\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} - V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\}V\left\{\partial\!\!\!/(\tilde{\sigma} + i\tilde{\vec{\pi}}\cdot\vec{\tau}\gamma_5)\right\} + 2\sigma_0\left\{V\tilde{\sigma}, V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\right\} \\
&+ V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big].
\end{aligned} \tag{6}$$

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To treat second term, following relation is needed.

$$\begin{aligned}\gamma_5 \partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) \gamma_5 \partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) &= -\partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) \partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) = -\partial_\mu \gamma^\mu \tilde{\pi}_i \tau_i \partial_\nu \gamma^\nu \tilde{\pi}_j \tau_j = -\partial_\mu \gamma^\mu \tilde{\pi}_i \partial_\nu \gamma^\nu \tilde{\pi}_j (\delta_{ij} + i\epsilon_{ijk}\tau_k) \\ &= -\partial_\mu \gamma^\mu \tilde{\pi}_i \partial_\nu \gamma^\nu \tilde{\pi}_i - \epsilon_{ijk} \partial_\mu \gamma^\mu \tilde{\pi}_i \partial_\nu \gamma^\nu \tilde{\pi}_j \tau_k\end{aligned} \tag{7}$$

Because this is in the trace and tr$[\tau_i]$=0, last term is a zero. Therefore

$$\gamma_5 \partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) \gamma_5 \partial\!\!\!/ (\tilde{\vec{\pi}} \cdot \vec{\tau}) = -\partial\!\!\!/ \tilde{\vec{\pi}} \cdot \partial\!\!\!/ \tilde{\vec{\pi}} \tag{8}$$

$$\begin{aligned}\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} - V\partial\!\!\!/\tilde{\sigma}V\partial\!\!\!/\tilde{\sigma} - V\partial\!\!\!/\tilde{\vec{\pi}} \cdot V\partial\!\!\!/\tilde{\vec{\pi}} - 2i\gamma_5 V\partial\!\!\!/\tilde{\sigma} \cdot \partial\!\!\!/ \left(\tilde{\vec{\pi}} \cdot \vec{\tau}\right) + 2\sigma_0 \left\{V\tilde{\sigma}, V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\right\} \\ &+ V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big].\end{aligned} \tag{9}$$

Since $\mathrm{tr}[\gamma_5] = 0$,

$$\begin{aligned}\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\Big[&4\sigma_0^2 V\tilde{\sigma}V\tilde{\sigma} - V\partial\!\!\!/\tilde{\sigma}V\partial\!\!\!/\tilde{\sigma} - V\partial\!\!\!/\tilde{\vec{\pi}} \cdot V\partial\!\!\!/\tilde{\vec{\pi}} + 2\sigma_0 \left\{V\tilde{\sigma}, V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\right\} \\ &+ V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right) V\left(\tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\right)\Big].\end{aligned} \tag{10}$$

From $\phi = (\sigma, \vec{\pi})$,

$$\begin{aligned}&\phi = \phi_0 + \tilde{\phi} = (\sigma_c = M, 0) + (\tilde{\sigma}, \tilde{\vec{\pi}}) \\ &\Longrightarrow \phi_0\tilde{\phi} = \sigma_0\tilde{\sigma}, \;\; \tilde{\phi}\tilde{\phi} = \tilde{\sigma}^2 + \tilde{\vec{\pi}}^2\end{aligned} \tag{11}$$

substituting Eq (10),

$$\mathrm{Tr}\left[VAVA\right] = \mathrm{Tr}\left[4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi}, V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right]. \tag{12}$$

B. Action and Propagator

Therefore the action Eq (4) is

$$\begin{aligned}S = -\frac{1}{2}N_c\Big[&\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi}, V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right] \\ &+ 8I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right)\Big] + \int d^4x\,\frac{1}{2G}\left((\sigma - \hat{m})^2 + \vec{\pi}^2\right)\end{aligned} \tag{13}$$

We already know from the previous calculation in the paper that

$$8\sigma_c G N_c I_1(\sigma_c) = \sigma_c - \hat{m}. \tag{14}$$

Then last two terms are

$$\begin{aligned}&-4N_c I_1(M)\int d^4x\left(\sigma^2 + \vec{\pi}^2 - M^2\right) + \int d^4x\,\frac{1}{2G}\left((\sigma - \hat{m})^2 + \vec{\pi}^2\right) \\ &= -\frac{M - \hat{m}}{2MG}\int d^4x\left(\phi^2 - M^2\right) + \int d^4x\,\frac{1}{2G}\left(\phi^2 - 2\sigma\hat{m} + \hat{m}^2\right) \\ &= \left(\frac{\hat{m}}{2MG} - \frac{1}{2G}\right)\int d^4x\left(\phi^2 + 2\phi\tilde{\phi} + \tilde{\phi}^2 - M^2\right) + \frac{1}{2G}\int d^4x\,\left(\phi^2 - 2\sigma\hat{m} + \hat{m}^2\right) \\ &= \left(\frac{\hat{m}}{2MG} - \frac{1}{2G}\right)\int d^4x\left(2M\tilde{\sigma} + \tilde{\phi}^2\right) + \frac{1}{2G}\int d^4x\,\left(M^2 + 2M\tilde{\sigma} + \tilde{\phi}^2 - 2\sigma\hat{m} + \hat{m}^2\right) \\ &= \int d^4x\left(\frac{\hat{m}\tilde{\sigma}}{G} + \frac{\hat{m}}{2MG}\tilde{\phi}^2\right) - \int d^4x\left(\frac{M\tilde{\sigma}}{G} + \frac{1}{2G}\tilde{\phi}^2\right) + \int d^4x\,\left(\frac{M^2}{2G} + \frac{M\tilde{\sigma}}{G} + \frac{1}{2G}\tilde{\phi}^2 - \frac{M\hat{m}}{G} + \frac{\tilde{\sigma}\hat{m}}{G} + \frac{\hat{m}^2}{2G}\right) \\ &= \int d^4x\left(\frac{2\hat{m}\tilde{\sigma} - M\hat{m}}{G} + \frac{M^2 + \hat{m}^2}{2G}\right) + \int d^4x\frac{\hat{m}}{2MG}\tilde{\phi}^2.\end{aligned} \tag{15}$$

We have to keep in mind the last term which will be included the second derivative of the action. Let us write the action one more time:

$$
\begin{aligned}
S = -\frac{1}{2}N_c \left[\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi}, V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right]\right] \\
+ \int d^4x \left(\frac{2\hat{m}\tilde{\sigma} - M\hat{m}}{G} + \frac{M^2 + \hat{m}^2}{2G}\right) + \int d^4x \frac{\hat{m}}{2MG}\tilde{\phi}^2,
\end{aligned}
\tag{16}
$$

$$
S = -\frac{1}{2}N_c \left[\mathrm{Tr}\left[\log\left(V^{-1}\right)\right] - \frac{1}{2}\mathrm{Tr}\left[4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi} + 2\left\{V\phi_0\tilde{\phi}, V\tilde{\phi}\tilde{\phi}\right\} + V\tilde{\phi}\tilde{\phi}V\tilde{\phi}\tilde{\phi}\right]\right] + R_1 + R_2\tilde{\phi}^2. \tag{17}
$$

Since we use saddle-point approximation, the action can be expressed the expansion for ϕ with a zero first derivative:

$$
\begin{aligned}
S[\phi] &\approx S[\phi_0] + \frac{\delta}{\delta\phi}S[\phi_0]\tilde{\phi} + \frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 \\
&= S[\phi_0] + \frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2.
\end{aligned}
\tag{18}
$$

Comparing with Eq (17), we can find the second derivative term

$$
\begin{aligned}
\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 &= -\frac{1}{2}N_c\left[-\frac{1}{2}\mathrm{Tr}\left[4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right]\right] + R_2\tilde{\phi}^2 \\
&= \frac{N_c}{4}\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left\langle p\left|4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle + R_2\tilde{\phi}^2 \\
&= 2N_c\int\frac{d^4p}{(2\pi)^4}\left\langle p\left|4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle + R_2\tilde{\phi}^2.
\end{aligned}
\tag{19}
$$

Now we will use the completeness relation:

$$
\mathbb{I} = \frac{d^4p'}{(2\pi)^4}|p'\rangle\langle p'| \tag{20}
$$

$$
\begin{aligned}
\Longrightarrow \int\frac{d^4p}{(2\pi)^4}\left\langle p\left|4\phi_0^2 V\tilde{\phi}V\tilde{\phi} - V\partial\!\!\!/\tilde{\phi}V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\Big(\left\langle p\left|4\phi_0^2 V\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|V\tilde{\phi}\right|p\right\rangle \\
- \left\langle p\left|V\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|V\partial\!\!\!/\tilde{\phi}\right|p\right\rangle\Big)
\end{aligned}
\tag{21}
$$

In the momentum space representation, $V = 1/(p^2 + M^2)$. So

$$
\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 = 2N_c\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\frac{1}{(p^2+M^2)(p'^2+M^2)}\Big(\left\langle p\left|4\phi_0^2\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|\tilde{\phi}\right|p\right\rangle - \left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle\left\langle p'\left|\partial\!\!\!/\tilde{\phi}\right|p\right\rangle\Big) + R_2\tilde{\phi}^2 \tag{22}
$$

Let us calculate term by term.

$$
\begin{aligned}
\left\langle p\left|4\phi_0^2\tilde{\phi}\right|p'\right\rangle &= 4M^2\int d^4x\left\langle p\left|\tilde{\phi}\right|x\right\rangle\langle x\,|\,p'\rangle = 4M^2\int d^4x\,\tilde{\phi}(x)\,\langle p\,|\,x\rangle\,\langle x\,|\,p'\rangle = 4M^2\int d^4x\,\tilde{\phi}(x)e^{ix(p-p')} \\
&= 4M^2\tilde{\phi}(p-p'), \\
\left\langle p'\left|\tilde{\phi}\right|p\right\rangle &= \tilde{\phi}(p'-p), \\
\left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|p'\right\rangle &= \int d^4x\left\langle p\left|\partial\!\!\!/\tilde{\phi}\right|x\right\rangle\langle x\,|\,p'\rangle = \int d^4x\,\partial\!\!\!/\tilde{\phi}(x)\,\langle p\,|\,x\rangle\,\langle x\,|\,p'\rangle = \int d^4x\,\partial\!\!\!/\tilde{\phi}(x)e^{ix(p-p')} \\
&= \int d^4x\,\partial\!\!\!/\left(\tilde{\phi}(x)e^{ix(p-p')}\right) - \int d^4x\,\tilde{\phi}(x)\partial\!\!\!/e^{ix(p-p')} = -\int d^4x\,\tilde{\phi}(x)i(p-p')e^{ix(p-p')} \\
&= -i(p-p')\tilde{\phi}(p-p'), \\
\left\langle p'\left|\partial\!\!\!/\tilde{\phi}\right|p\right\rangle &= -i(p'-p)\tilde{\phi}(p'-p).
\end{aligned}
\tag{23}
$$

Then Eq (22) is

$$\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 = 2N_c\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\frac{4M^2\tilde{\phi}(p-p')\tilde{\phi}(p'-p)-(p-p')(p'-p)\tilde{\phi}(p-p')\tilde{\phi}(p'-p)}{(p^2+M^2)(p'^2+M^2)}+R_2\tilde{\phi}^2. \quad (24)$$

For convenience, let us introduce changing variable:

$$p'+p=2k,\ \ p'-p=q \quad (25)$$

and

$$p'=k+\frac{q}{2},\ \ p=k-\frac{q}{2}. \quad (26)$$

Since we change two variables (p',p) to (k,q), 4-integral variables becomes d^4kd^4q. Hence

$$\begin{aligned}\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 &= 2N_c\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\frac{4M^2\tilde{\phi}(-q)\tilde{\phi}(q)+q^2\tilde{\phi}(-q)\tilde{\phi}(q)}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}+R_2\tilde{\phi}^2\\ &= 2N_c\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\frac{4M^2+q^2}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}\tilde{\phi}(-q)\tilde{\phi}(q)+R_2\tilde{\phi}^2\\ &= 2N_c\int\frac{d^4q}{(2\pi)^4}(4M^2+q^2)f(q)\tilde{\phi}(-q)\tilde{\phi}(q)+R_2\tilde{\phi}^2,\ \ f(q)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left\{\left(k-\frac{q}{2}\right)^2+M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2+M^2\right\}}.\end{aligned} \quad (27)$$

$\tilde{\phi}(q)$ can be expressed into the Fourier transformation

$$\begin{aligned}&\tilde{\phi}(-q)\tilde{\phi}(q)=\int d^4x\int d^4y\,\tilde{\phi}(y)e^{iq(x-y)}\tilde{\phi}(x)\\ &\Longrightarrow \frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2 = 2N_c\int d^4x\int d^4y\int\frac{d^4q}{(2\pi)^4}(4M^2+q^2)f(q)\tilde{\phi}(y)e^{iq(x-y)}\tilde{\phi}(x)\\ &\qquad\qquad+\int d^4x\int d^4y\frac{\hat{m}}{2MG}\delta^{(4)}(x-y)\tilde{\phi}(y)\tilde{\phi}(x).\end{aligned} \quad (28)$$

Actually, the exact expression of the second derivative is

$$\frac{1}{2}\frac{\delta^2}{\delta\phi^2}S[\phi_0]\tilde{\phi}^2=\frac{1}{2}\int d^4x\int d^4y\ \left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\phi^0}\tilde{\phi}(x)\tilde{\phi}(y) \quad (29)$$

and this form is seem to be comparable with the LHS of Eq (28). Therefore

$$\begin{aligned}\frac{1}{2}\left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\phi^0} &= 2N_c\int\frac{d^4q}{(2\pi)^4}(4M^2+q^2)f(q)e^{iq(x-y)}+\frac{\hat{m}}{2MG}\delta^{(4)}(x-y)\\ &= \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left[4N_c(4M^2+q^2)f(q)+\frac{\hat{m}}{MG}\right]e^{iq(x-y)}.\end{aligned} \quad (30)$$

Finally the meson propagator $K_\phi(q^2)$ is

$$\begin{aligned}&\left.\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right|_{\phi^0}=\int\frac{d^4q}{(2\pi)^4}K^{-1}(q^2)e^{iq(x-y)}=\int\frac{d^4q}{(2\pi)^4}\left[4N_c(4M^2+q^2)f(q)+\frac{\hat{m}}{MG}\right]e^{iq(x-y)}\\ &=K(q^2)=\frac{1}{4N_c(4M^2+q^2)f(q)+\frac{\hat{m}}{MG}}=\frac{1}{Z_q(q^2)}\frac{1}{4M^2+q^2+\frac{\hat{m}}{MGZ_q(q^2)}},\ \ Z_q(q^2)=4N_cf(q).\end{aligned} \quad (31)$$

C. $Z_q(q^2)$ Calculation

Now only one process is remain: calculating $Z_q(q^2)$. From Eq (27),

$$\begin{aligned} Z_q(q^2) &= 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left\{\left(k-\frac{q}{2}\right)^2 + M^2\right\}\left\{\left(k+\frac{q}{2}\right)^2 + M^2\right\}} \\ &= 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 + M^2 + \frac{q^2}{4} - qk\right)\left(k^2 + M^2 + \frac{q^2}{4} + qk\right)}. \end{aligned} \tag{32}$$

There are useful integral rules:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2}, \quad \frac{1}{Q^2} = \int_0^\infty du \, u e^{-uQ}. \tag{33}$$

Let us $A = \left(k^2 + M^2 + \frac{q^2}{4} - qk\right)$ and $B = \left(k^2 + M^2 + \frac{q^2}{4} + qk\right)$. substituting to the first rule,

$$\begin{aligned} Z_q(q^2) &= 4N_c \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left[x\left(k^2 + M^2 + \frac{q^2}{4} - qk\right) + (1-x)\left(k^2 + M^2 + \frac{q^2}{4} + qk\right)\right]^{-2} \\ &= 4N_c \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left[-2qkx + \left(k^2 + M^2 + \frac{q^2}{4} + qk\right)\right]^{-2} \\ &= 4N_c \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{(Kx+B)^2} = 4N_c \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{(Kx+B)^2} \end{aligned} \tag{34}$$

where $K = -2qk$ and $B = k^2 + M^2 + \frac{q^2}{4} + qk$. By the second rule,

$$\begin{aligned} Z_q(q^2) &= 4N_c \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \int_0^\infty du \, u \exp\left[-u\left(-2qkx + k^2 + M^2 + \frac{q^2}{4} + qk\right)\right] \\ &= 4N_c \int_0^1 dx \int_0^\infty du \, u \exp\left[-\frac{u}{4}\left\{1 - (1-2x)^2\right\} q^2 - uM^2\right] \int \frac{d^4k}{(2\pi)^4} \exp\left[-u\left(k + \frac{1}{2}q(1-2x)\right)^2\right]. \end{aligned} \tag{35}$$

For convenience, let $1 - 2x = y$, and $-2dx = dy$

$$\begin{aligned} Z_q(q^2) &= 4N_c \int_0^1 dx \int_0^\infty du \, u \exp\left[-\frac{u}{4}\left\{1 - (1-2x)^2\right\} q^2 - uM^2\right] \int \frac{d^4k}{(2\pi)^4} \exp\left[-u\left(k + \frac{1}{2}q(1-2x)\right)^2\right] \\ &= -2N_c \int_1^{-1} dy \int_0^\infty du \, u \exp\left[-\frac{u}{4}\left(1 - y^2\right) q^2 - uM^2\right] \int \frac{d^4k}{(2\pi)^4} \exp\left[-u\left(k + \frac{1}{2}qy\right)^2\right] \\ &= 2N_c \int_{-1}^1 dy \int_0^\infty du \, u \exp\left[-\frac{u}{4}\left(1 - y^2\right) q^2 - uM^2\right] \int \frac{d^4k}{(2\pi)^4} \exp\left[-u\left(k + \frac{1}{2}qy\right)^2\right]. \end{aligned} \tag{36}$$

The last integral is the 4-dimensional spherical integral.

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \exp\left[-u\left(k + \frac{1}{2}qy\right)^2\right] &= \int_0^\infty \frac{dk}{(4\pi)^2} \, k^3 \exp\left[-u\left(k + \frac{1}{2}qy\right)^2\right] \\ &= \int_0^\infty \frac{dk}{(4\pi)^2} \left(k - \frac{1}{2}qy\right)^3 e^{-uk^2} \end{aligned} \tag{37}$$

Expanding quadratic, there are only gaussian integrals. But the terms which have the odd degree of y are erase since of the integral for y is symmetric.

$$\begin{aligned}
\int_0^\infty \frac{dk}{(4\pi)^2}\left(k-\frac{1}{2}qy\right)^3 e^{-uk^2} &= \int_0^\infty \frac{dk}{(4\pi)^2}\left(k^3-\frac{3}{2}qyk^2+\frac{3}{4}q^2y^2k-\frac{1}{8}q^3y^3\right)e^{-uk^2} \\
&= \int_0^\infty \frac{dk}{(4\pi)^2}\left(k^3+\frac{3}{4}q^2y^2k\right)e^{-uk^2} \\
&= \frac{1}{(4\pi)^2}\left(\frac{1}{2u^2}+\frac{3q^2y^2}{8u}\right).
\end{aligned} \tag{38}$$

Hence

$$\begin{aligned}
Z_q(q^2) &= \frac{2N_c}{(4\pi)^2}\int_{-1}^1 dy \int_0^\infty du\, u \exp\left[-\frac{u}{4}\left(1-y^2\right)q^2-uM^2\right]\left(\frac{1}{2u^2}+\frac{3q^2y^2}{8u}\right) \\
&= \frac{2N_c}{(4\pi)^2}\int_{-1}^1 dy \int_0^\infty du\,\left(\frac{1}{2u}e^{-u\left(\frac{q^2}{4}(1-y^2)+M^2\right)}+\frac{3q^2y^2}{8}e^{-u\left(\frac{q^2}{4}(1-y^2)+M^2\right)}\right).
\end{aligned} \tag{39}$$

The last term diverges at $y^2=\frac{M^2}{q^2}+1$. So we remain only first term and cut the integral interval for u(regularization). Finally

$$\begin{aligned}
Z_q(q^2) &= \frac{2N_c}{(4\pi)^2}\int_{-1}^1 dy \int_0^\infty du\, u \exp\left[-\frac{u}{4}\left(1-y^2\right)q^2-uM^2\right]\left(\frac{1}{2u^2}+\frac{3q^2y^2}{8u}\right) \\
&= \frac{2N_c}{(4\pi)^2}\int_{-1}^0 dy \int_{\Lambda^{-2}}^\infty du\,\frac{1}{u}e^{-u\left(M^2+\frac{q^2}{4}(1-y^2)\right)}.
\end{aligned} \tag{40}$$

This is the Pauli-Vilars regularized expression.