# Chapter 4 : Time Response

## ELG 3155 : Introduction to Control Systems

❑ Objective:

- state-space representation,
- Model electrical and mechanical systems in state space.
- Convert a transfer function to state space.
- Convert a state-space representation to a transfer function.
- Linearize a state-space representation.

Hitham Jleed

uOttawa

# Design process of control systems

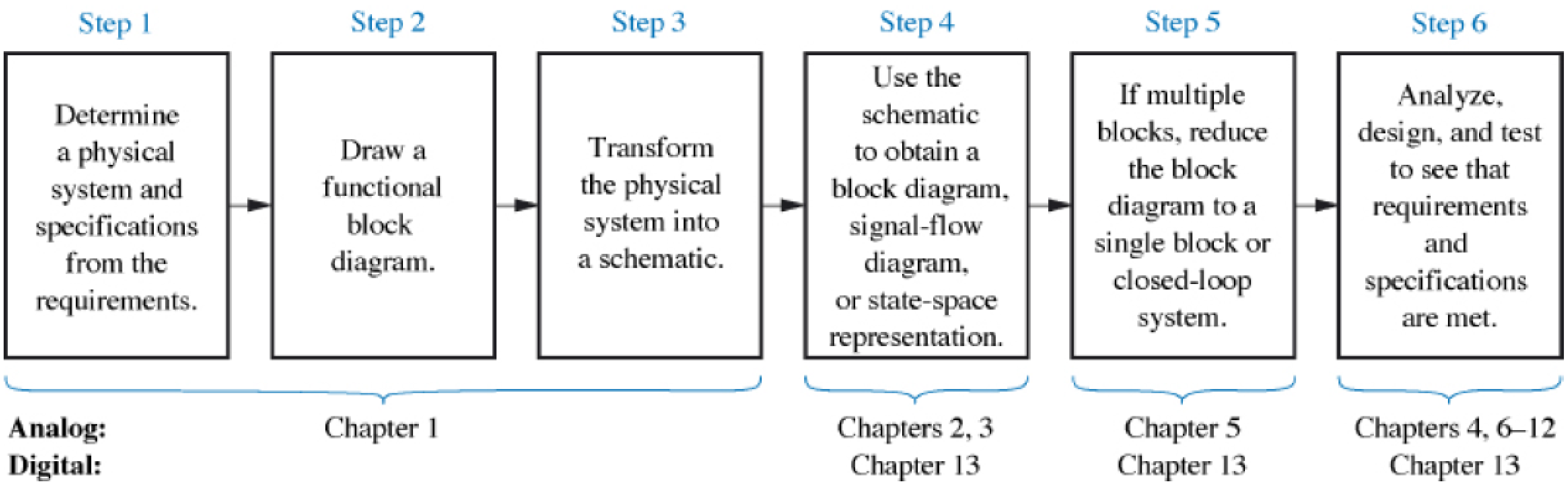



Figure 4 (ch.1): Design process of control systems.

# Outline

1. Introduction
2. General Form of an LTI System Response
3. Time Characteristics of a System's Response
4. First-Order Systems
5. Second Order Systems: Introduction
    1. Overdamped response
    2. Underdamped response
    3. Undamped response
    4. Critically damped response
6. Second Order System: General Analysis
7. Underdamped Second-Order System ($0 < \xi < 1$)
8. System Response with Additional Poles

# Objectives

- Find the time response of a system from its transfer function.
- Use poles and zeros to determine the system’s response.
- Describe quantitatively the transient response of first and second-order systems.

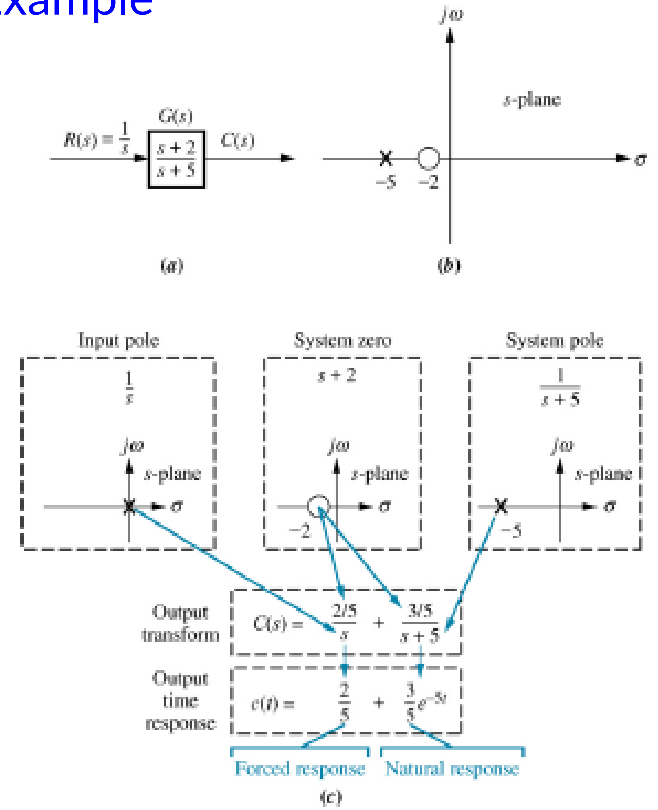
# 1. Introduction

- A system's transient response is very important as it allows to verify if the system satisfies certain desired (design) criteria.

- This chapter is dedicated to analyze the transient responses of several types of control systems.

# Definitions, [1, p. 159]

## Poles and Zeros of a First-Order System: An Example




### *Poles of a TF*

- Values of the Laplace transform variable, $s$, that cause the TF to become infinite
- Any roots of the denominator of the TF that are common to the roots of the numerator

Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

# Definitions, [1, p. 159]

*Zeros of a TF*

- Values of the Laplace transform variable, $s$, that cause the TF to become zero
- Any roots of the numerator of the TF that are common to the roots of the denominator

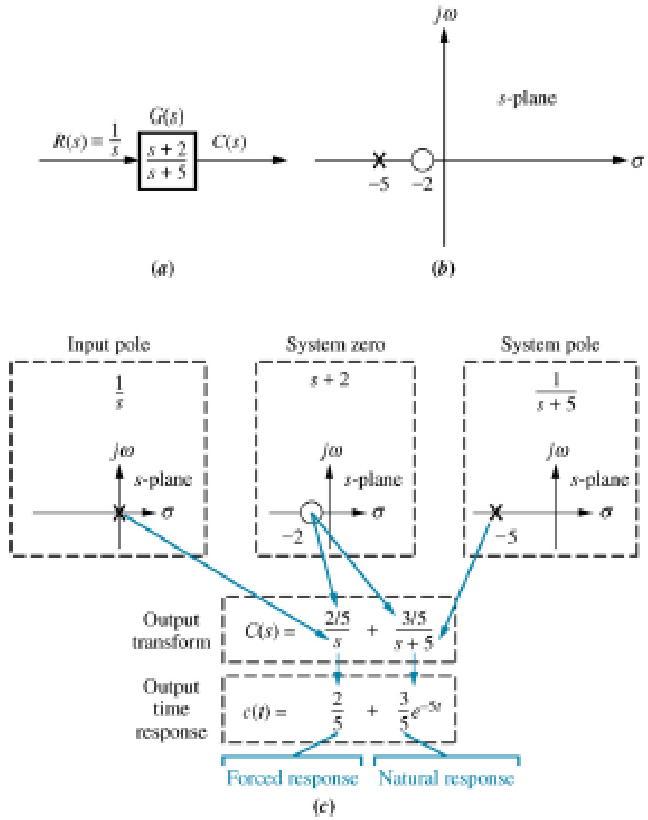



Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

# 2. General Form of an LTI System Response

- This section is a generalization of the textbook's section 4.2.

Consider an LTI system with

- $p$ real non-zero poles $\alpha_1, \alpha_2, \ldots, \alpha_p$ with multiplicities $m_1, m_2, \ldots, m_p$ respectively; and
- $q$ pairs of complex conjugate poles $(\beta_1 \mp j\omega_1), \ldots, (\beta_q \mp j\omega_q)$ with multiplicities $n_1, n_2, \ldots, n_q$, respectively, where $\omega_i > 0$, $i = 1, \ldots, q$ (see Fig. 1)

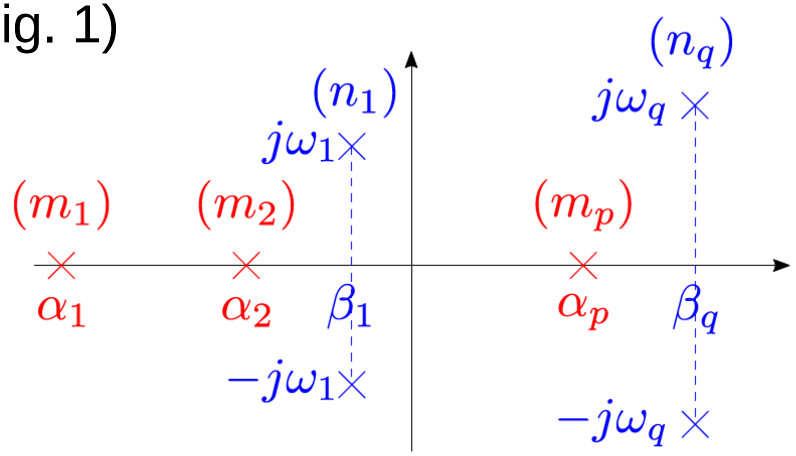



Figure 1: Pole distribution on the s-plane

Then, the system's transfer function is in the form of:

$$G(s) = \frac{N(s)}{(s-\alpha_1)^{m_1}\cdots(s-\alpha_p)^{m_p}\,[(s-\beta_1-j\omega_1)(s-\beta_1+j\omega_1)]^{n_1}\cdots[(s-\beta_q-j\omega_q)(s-\beta_q+j\omega_q)]^{n_q}},$$

where $N(s)$ is a polynomial in $s$.

Assume that the order of $N(s)$ is less than the order of the system (i.e., order of $N(s) < m_1 + m_2 + \ldots + m_p + 2n_1 + 2n_2 + \ldots + 2n_q$).

For an input $R(s)$, the output $C(s) = R(s)G(s)$.

If $R(s) = 1/s$ , and assuming that the system has no pole at $s = 0$, the output can be expressed as (after partial fraction expansion)

$$C(s) = \underbrace{\text{Gain}\,\frac{1}{s}}_{C_f(s)} + \underbrace{\frac{N'(s)}{(s-\alpha_1)^{m_1}\cdots(s-\alpha_p)^{m_p}\,[(s-\beta_1-j\omega_1)(s-\beta_1+j\omega_1)]^{n_1}\cdots[(s-\beta_q-j\omega_q)(s-\beta_q+j\omega_q)]^{n_q}}}_{C_n(s)},$$

where $N'(s)$ is a polynomial in $s$.

$C_f(s) \equiv$ forced response, $C_n(s) \equiv$ natural response

- The forced response represents the contribution of the input to the system's response.
- The natural response represents the contribution of the transfer function $G(s)$ to the system's response.

- Hence, the general form of the system's response to a step input is

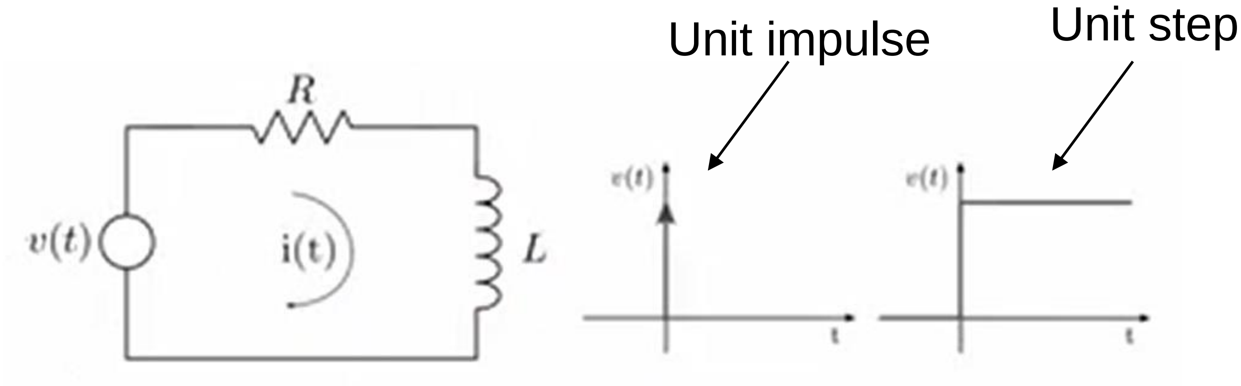
$$c(t) \equiv c_f(t) + c_n(t)$$

where

$$c_f(t) \equiv A_r\, u(t)$$

for some real gain $A_r$ ; and

- Consider RL circuit,




$$V(s) = (R + Ls)I(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1/R}{s(L/R) + 1}$$

- Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R}$$

- Put the denominator in the form $\tau S + 1$

$i(t) = \mathcal{L}^{-1}(I(s))$

Impulse response: $v(t) = \delta(t) \Rightarrow V(s) = 1$

$\tau = \frac{L}{R}$

$$\boxed{I(s) = \frac{1}{R}\frac{1}{\tau s + 1}} = \frac{1}{\tau R}\left(\frac{1}{s + \frac{1}{\tau}}\right)$$

$s = -\frac{1}{\tau}$

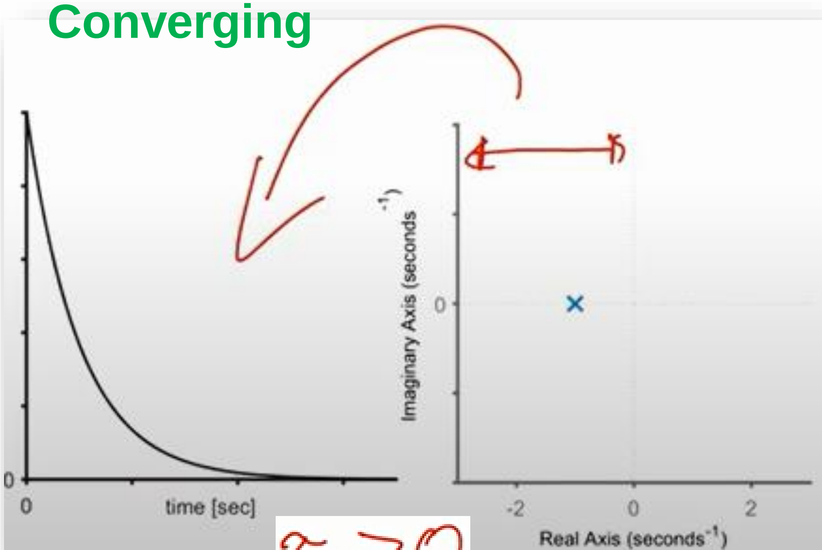
The pole is $s = -1/\tau$. The time response is:

$$i(t) = \frac{1}{\tau R} e^{-\frac{t}{\tau}}$$

$\tau > 0$

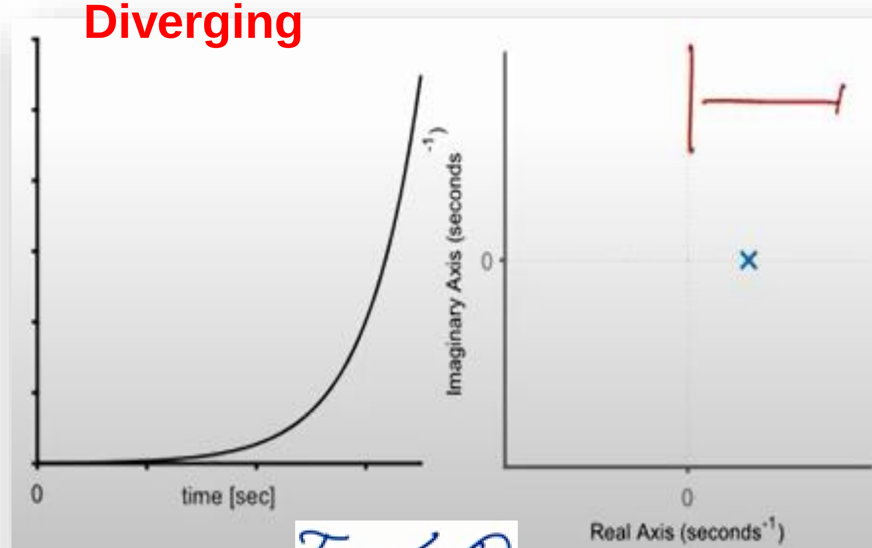
$\tau < 0$

**Converging**




$\tau > 0$

**Diverging**




$\tau < 0$

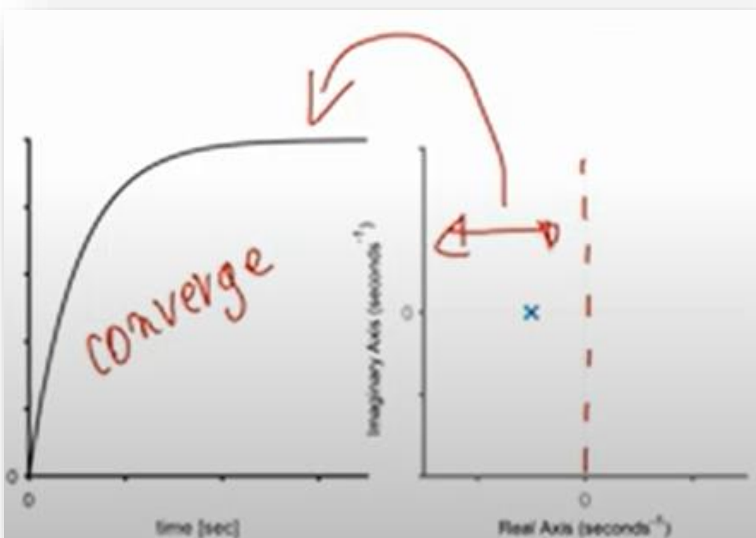
$s = -\frac{1}{\tau}$

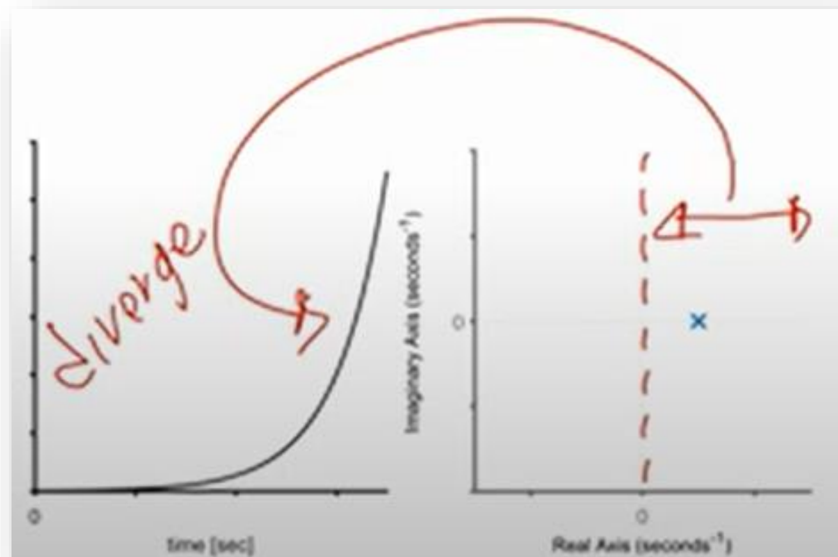
Step response: $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$I(s) = \frac{1}{\tau R}\left(\frac{1}{s}\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{1}{s}\right)\left(\frac{1}{s+\frac{1}{\tau}}\right) = \frac{1}{\tau R}\left(\frac{k_1}{s} + \frac{k_2}{s+\frac{1}{\tau}}\right)$$

Solving for the partial fraction coefficients: $k_1 = \tau$, $k_2 = -\tau$, thus:

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$



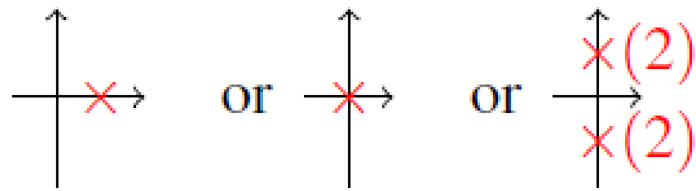



# Conclusions

- The real parts of the poles are responsible for the convergence or divergence of the response.
- The imaginary parts are responsible for the oscillations.

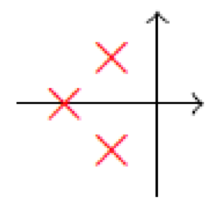
**Diverging amplitude:**

- At least one pole on the RHS of the s-plane or at the origin: or
- At least one pair of complex conjugate poles on the j$\omega$-axis with multiplicity 2 or more.



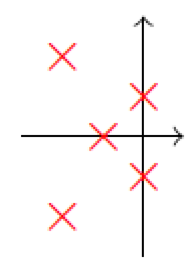

**Converging response:**

- All the poles are on the LHS of the s-plane.



**Oscillatory response with constant amplitude:**

- At least one pair of complex conjugate poles on the j$\omega$-axis; and
- No poles on the RHS of the s-plane; and
- All the poles which are on the j$\omega$-axis have a multiplicity of 1.



# 3. Time Characteristics of a System's Response

## Definition (Time constant)

The time constant is the time for a step response to reach 63% of its final value.

$$\boxed{\text{time constant} \approx \frac{1}{|\text{Re(pole)}|}}$$

depends on pole only!

## Definition (Settling time)

The settling time $T_s$ is the time for the response to reach and stay within $\pm$2% of its final value. (Certain references consider 5%)

## Definition (Rise time)

The rising time $T_r$ is the time taken by the response to go from 10% to 90% of its final value.

- All the parameters are illustrated in Fig. 2.

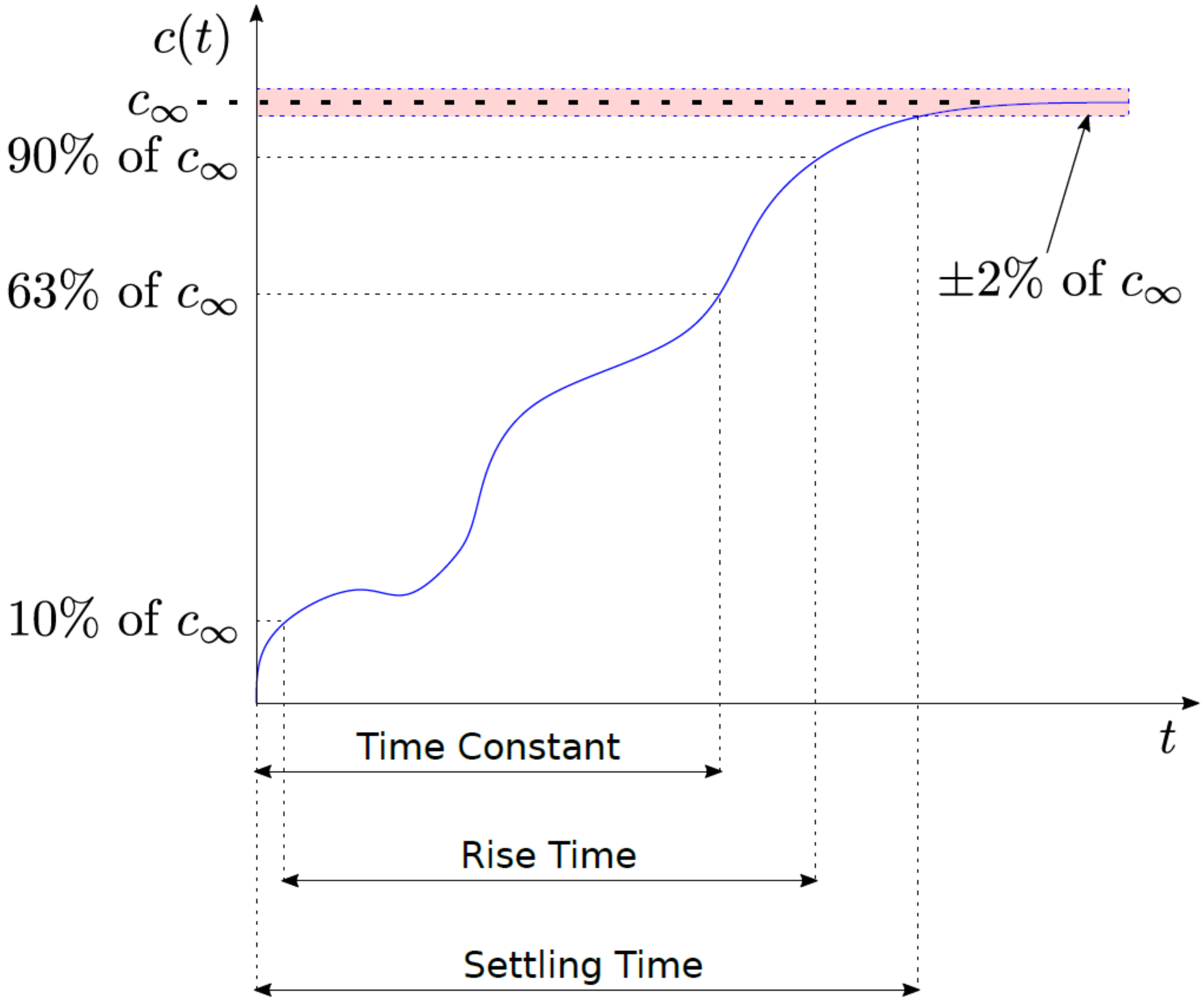
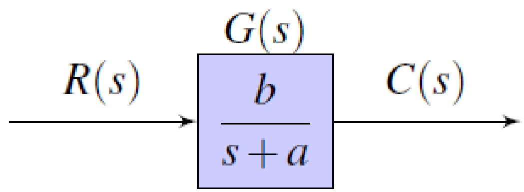


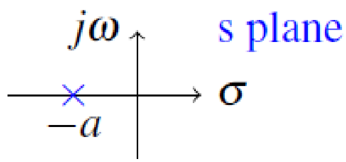

Figure 2: Time characteristics of a system's response

# 4.3 First-Order Systems

Consider the following first-order system where $a > 0$,

$$G(s) = \frac{b}{s+a}$$




Let $R(s) = 1/s$ (unit step)

$$C(s) = \frac{b}{s(s+a)} = \frac{b}{a}\left[\frac{1}{s} + \frac{-1}{s+a}\right] \quad \Rightarrow \boxed{c(t) = c_f(t) + c_n(t) = \frac{b}{a}(1 - e^{-at})}$$

⇒ A non-oscillatory and convergent output.

## Rise time

For a first-order system, the rising time $T_r$ may be approximated as

$$\boxed{T_r \approx 2.2 \times \text{time constant}}$$

depends on pole only!

## Settling time

For a first- or second-order system, the settling time $T_s$ may be approximated as

$$\boxed{T_s \approx 4 \times \text{time constant}}$$

depends on pole only!

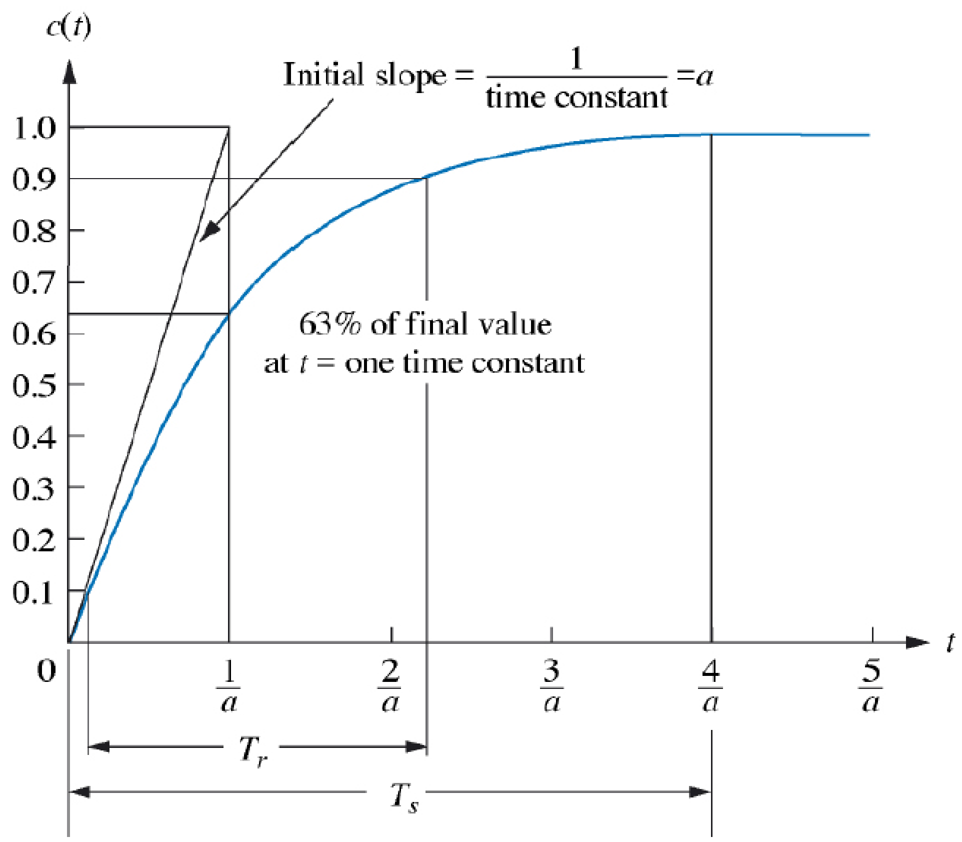



Figure 3: First-order system response to a unit step (Fig. 4.5 of [Nise, 2015])

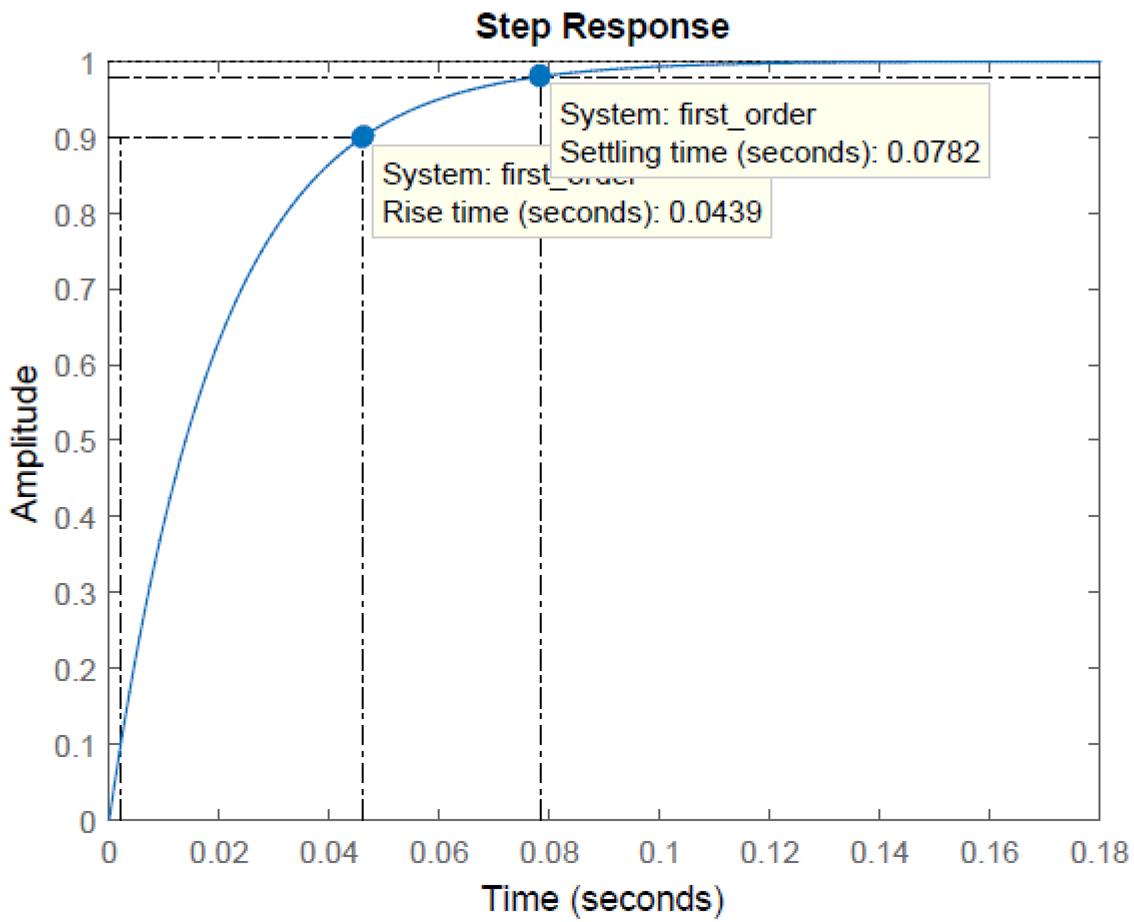
# Exercise 1

A system has a transfer function $G(s) = 50 / (s + 50)$. Find the time constant, settling time, and rise time.

# Verification

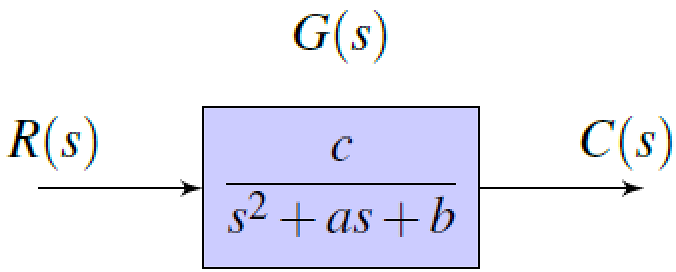
Listing 4: Matlab Code

```
s=tf([1 0],1);
first_order=50/(s+50);
step(first_order);
```



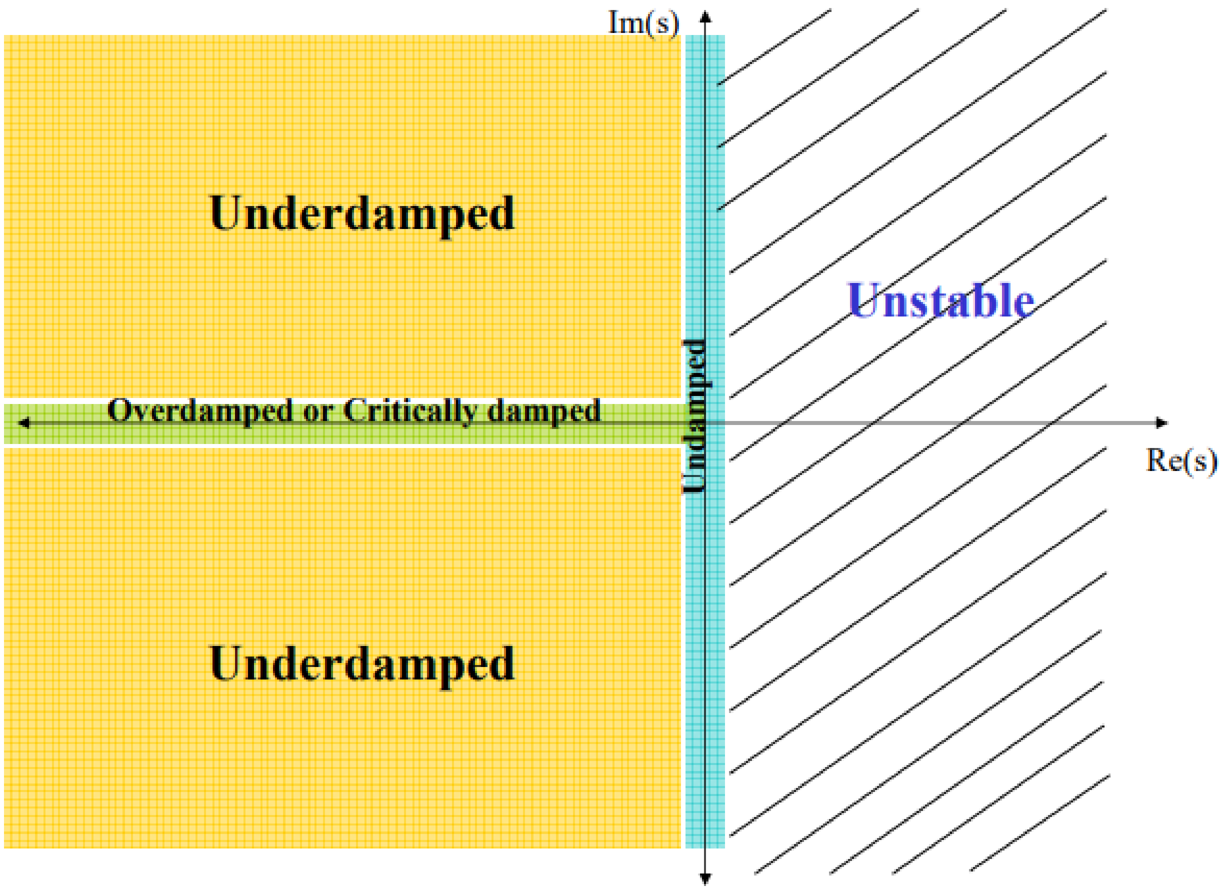
# 4.4 Second-Order Systems: Introduction

In general, a second order system can be represented in the form:




$\Rightarrow$ 4 interesting possibilities.

# Second order system response.



# 5.1. Overdamped response

2 negative real poles, $-\sigma_1 < 0$ and $-\sigma_2 < 0$.

$\Rightarrow$ 2 time constants:

$$\tau_1 = \frac{1}{|\text{pole}_1|} = \frac{1}{|-\sigma_1|} \qquad \tau_2 = \frac{1}{|\text{pole}_2|} = \frac{1}{|-\sigma_2|}$$

$$c(t) = c_f(t) + c_n(t),$$

$c_n(t)$ is made of two exponential signals with the two time constants.

$$\Rightarrow c_n(t) = k_1 e^{-\sigma_1 t} + k_2 e^{-\sigma_2 t}$$

See Fig. 4(b).

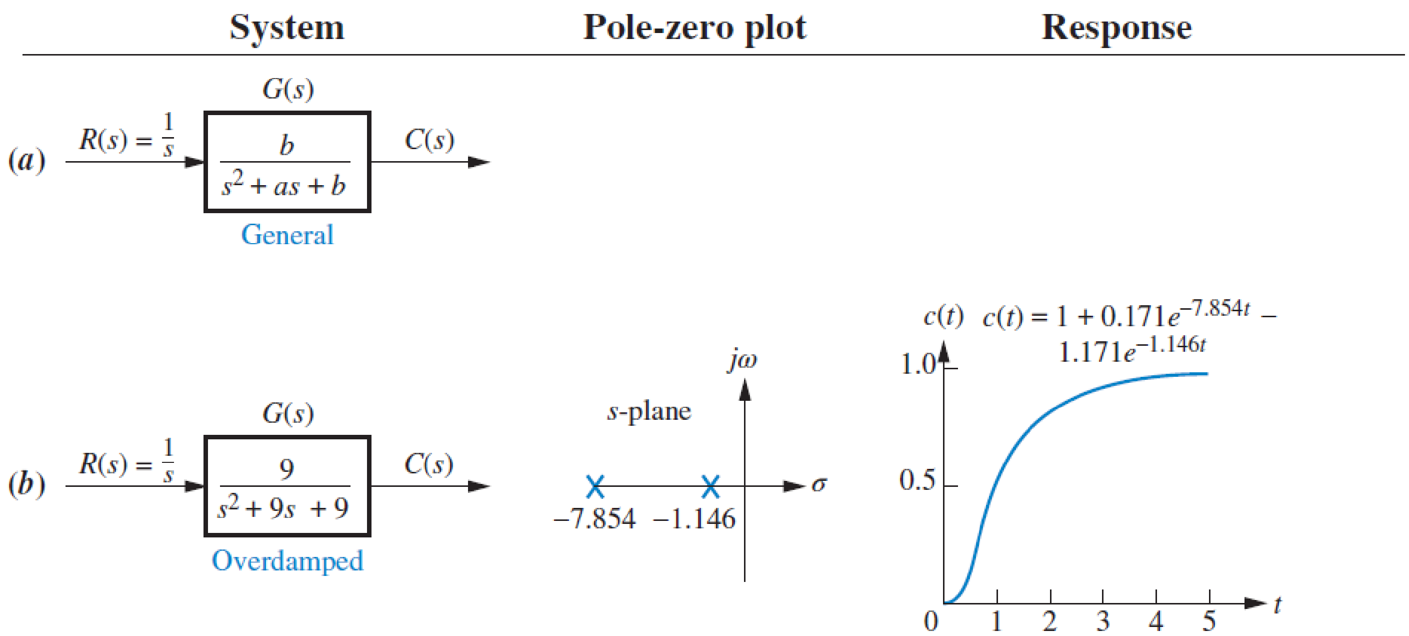



Figure 4: Responses of a second order system to a unit step input ( Fig. 4.7 of [Nise, 2015].)

# 5.2. Underdamped response

2 conjugate complex poles, $-\sigma_d \pm j\omega_d$.

⇒ 1 time constant:

$$\tau = \frac{1}{|\text{poles}|} = \frac{1}{|-\sigma_d|}$$

In this case, $c_n(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$

⇒ A natural response in the form of a damped sinusoidal signal with an exponential term.

See Fig. 4(c).

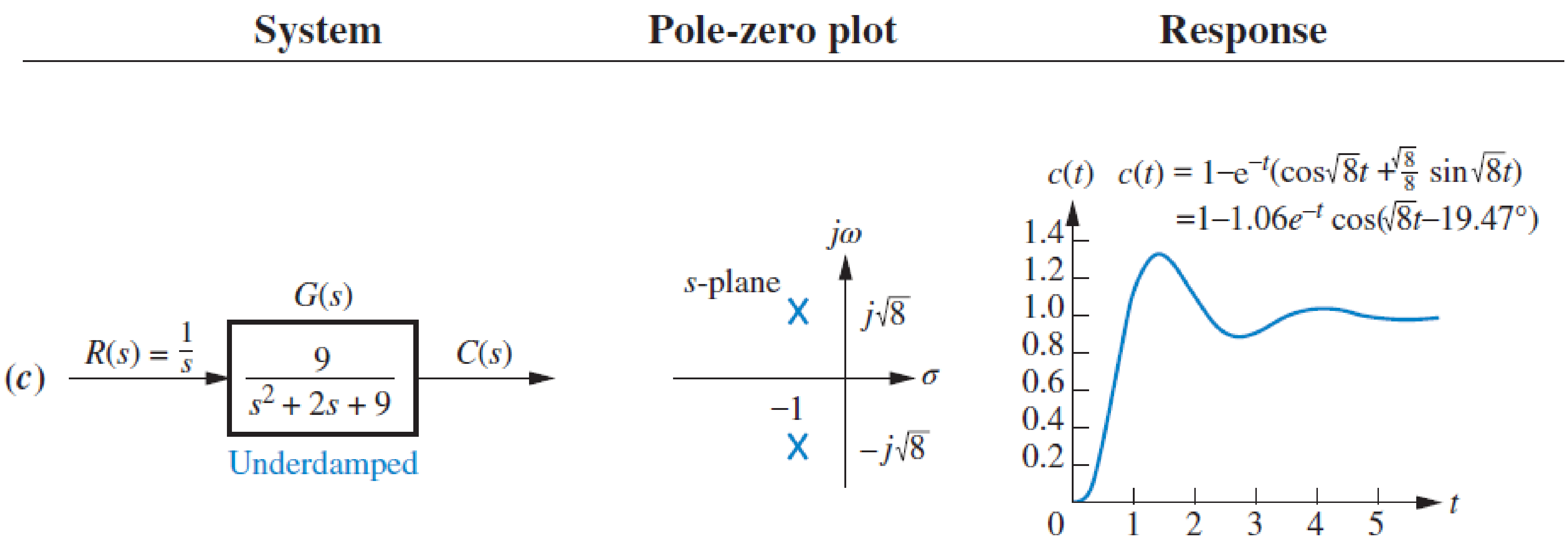


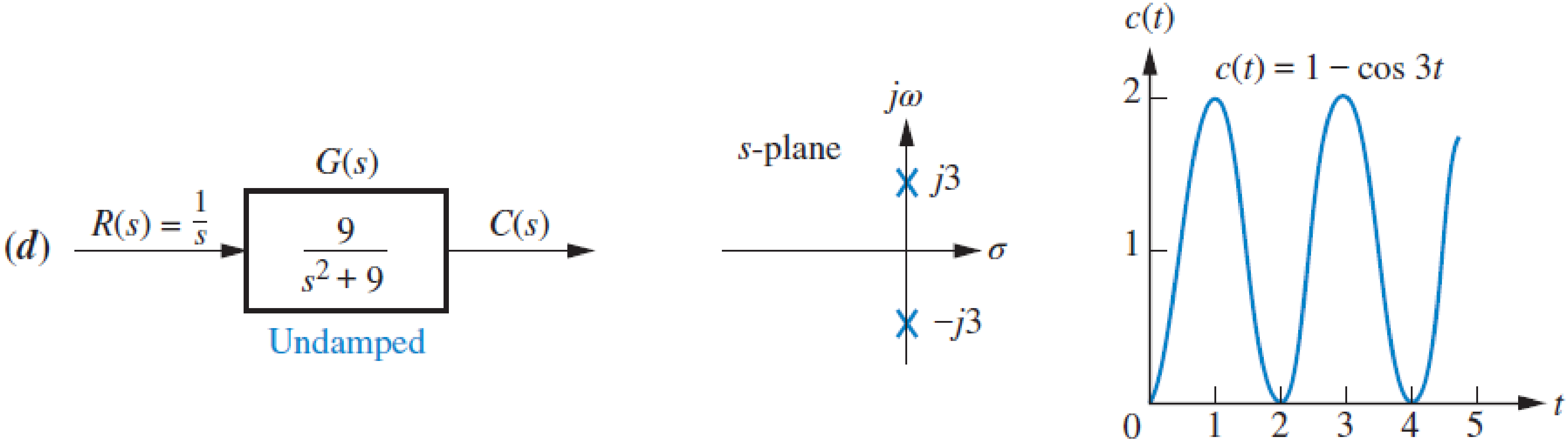

Figure 4 (c): Underdamped Response

# 5.3. Undamped response

2 imaginary poles at $\pm j\omega_d$.

In this case, $c_n(t) = A\cos(\omega_1 t - \phi)$

$\Rightarrow$ An undamped sinusoidal natural response with a frequency $\omega_1$.



# 5.4. Critically damped response

2 real negative poles, $-\sigma_1$.

⇒ 1 time constant:

$$\tau = \frac{1}{|\text{poles}|} = \frac{1}{|-\sigma_1|}$$

In this case, $$c_n(t) = k_1 t e^{-\sigma_1 t} + k_2 e^{-\sigma_1 t}$$

⇒ A critically damped response is the fastest response to reach steady state without oscillations.

See Fig. 4(e).

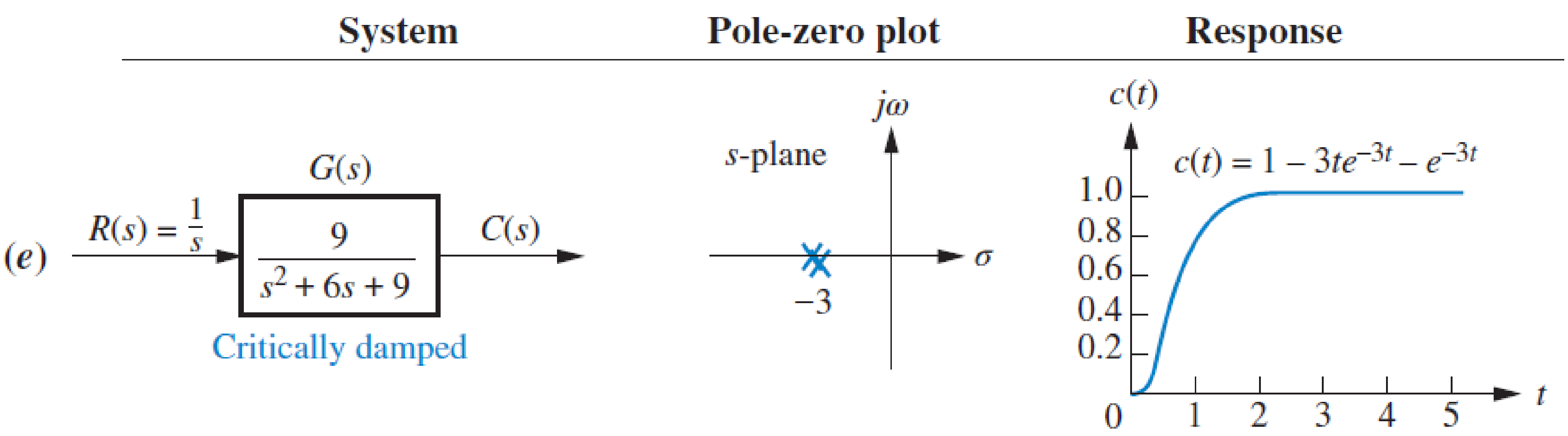



Figure 4 (e): Critically Damped Response

A graphical illustration of the 4 types of these responses is shown in Fig. 5.

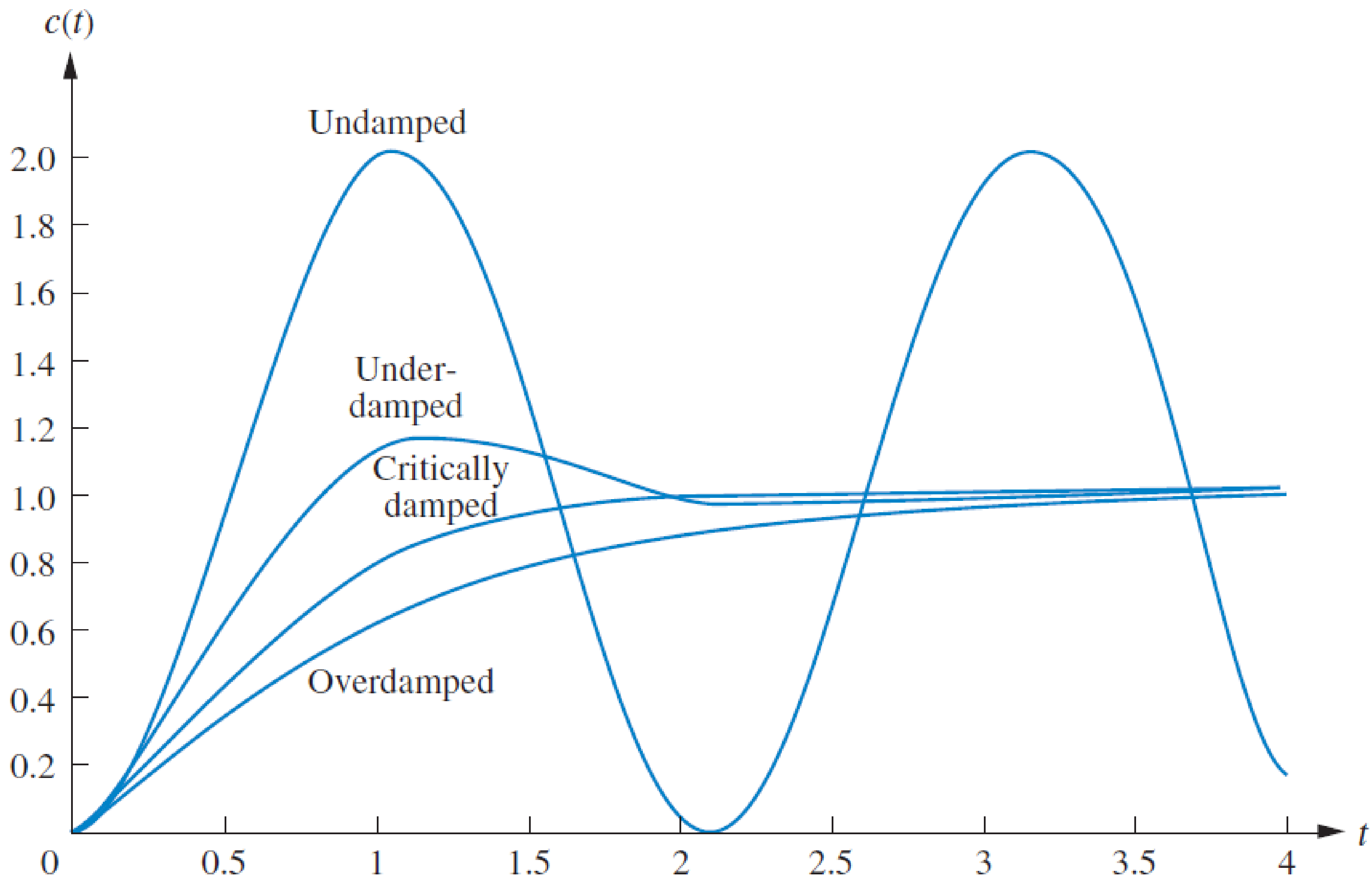



Figure 5: Responses of a second-order system (Fig. 4.10 of [Nise, 2015])

## 4.5 The General Second-Order System

The general form of a second-order system transfer function is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

$\omega_n$ ≡ system's natural frequency

(response frequency in the case of no damping)

ξ ≡ damping ratio

The system's poles are $\boxed{p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$

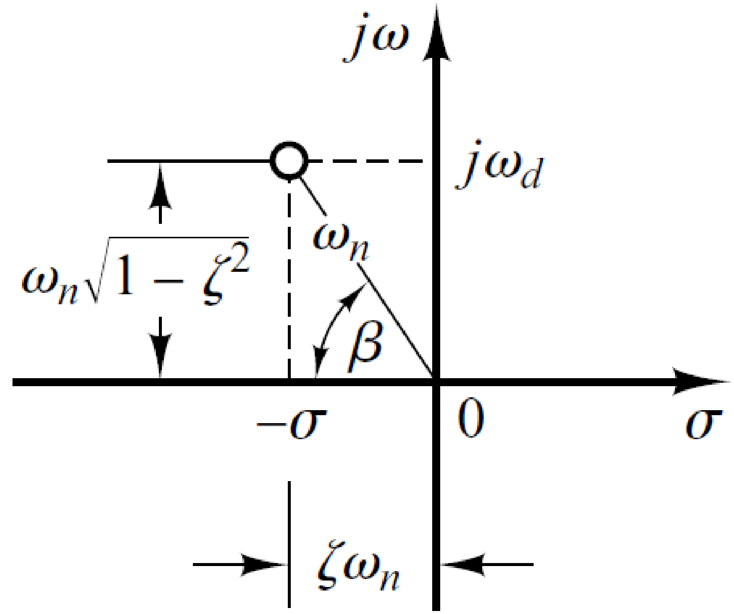
## Definition

The **Natural Frequency** of a pole at $p = \sigma + \imath\omega_d$ is $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\omega_n = \sqrt{b}$.
- Radius of the pole in complex plane.

Resonant Frequency.

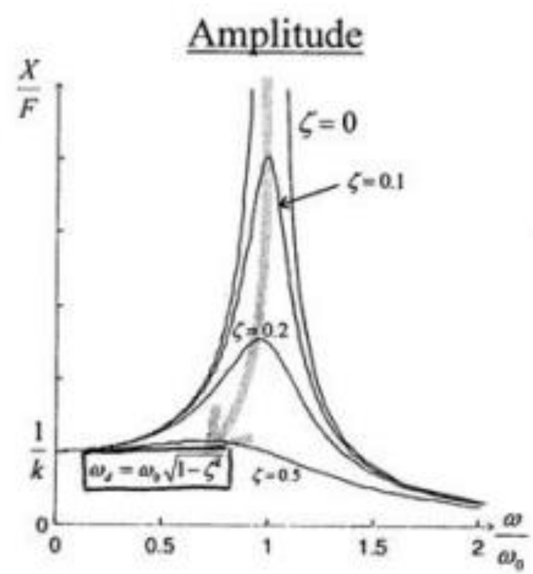
- Also known as resonant frequency




## Definition

The **Damping Ratio** of a pole at $p = \sigma + \imath\omega$ is $\zeta = \frac{|\sigma|}{\omega_n}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\zeta = \frac{a}{2\sqrt{b}}$.
- Gives the ratio by which the amplitude decreases per oscillation (almost...).



$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

## Effect of ξ

$\xi = 0 \Leftrightarrow$ undamped system

$0 < \xi < 1 \Leftrightarrow$ underdamped system

$\xi = 1 \Leftrightarrow$ critically damped system

$\xi > 1 \Leftrightarrow$ overdamped system

Fig. 6 shows a second-order system step response as a function of the damping ratio.

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

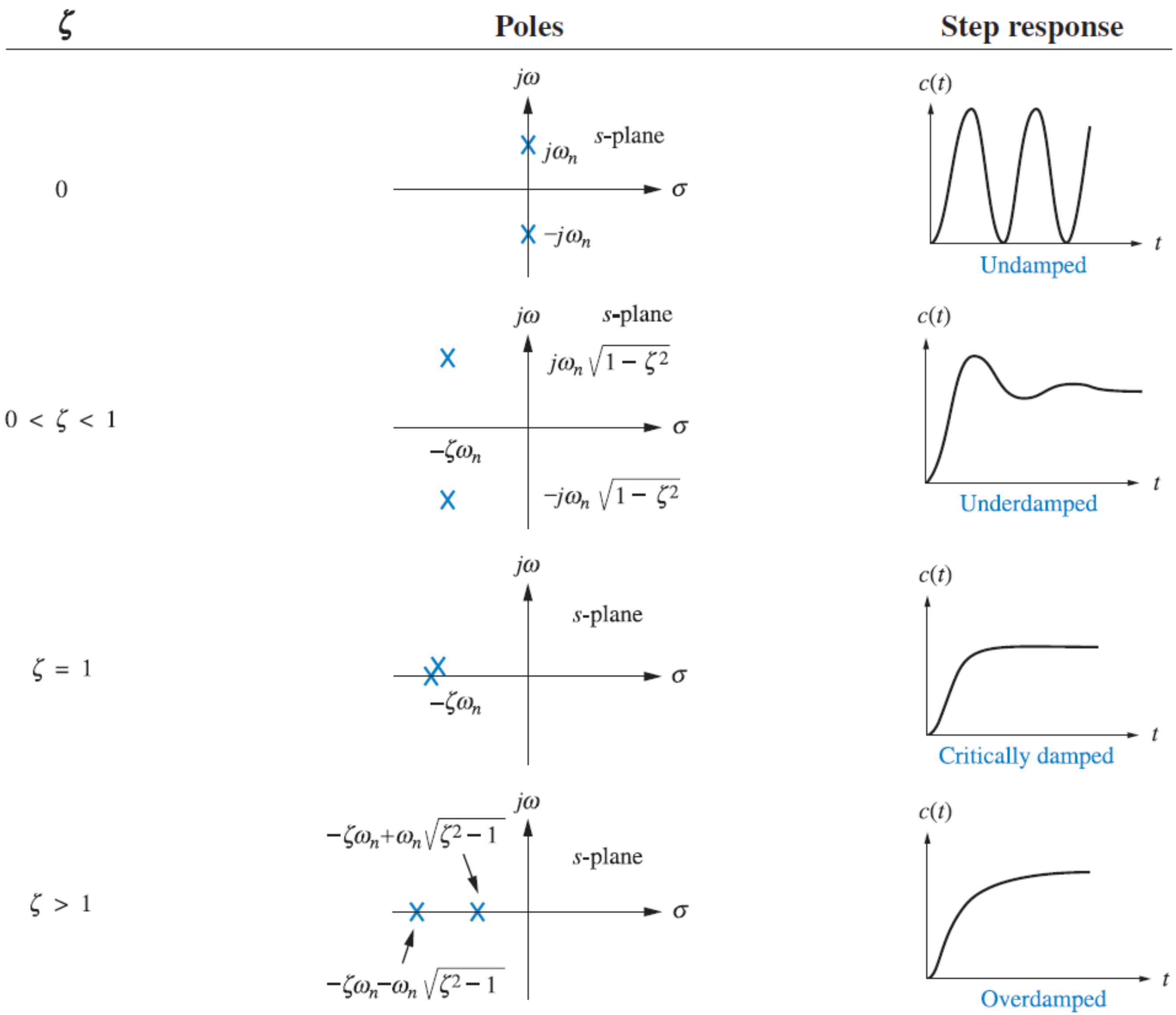
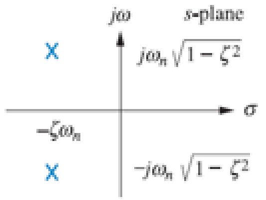
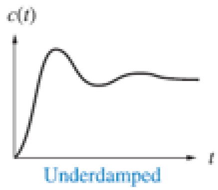



Figure 6: Second-order step response as a function of damping ratio (Fig. 4.11 of [Nise, 2015])

- When $0 < \xi < 1$




  - $\omega_n = \sqrt{\sigma_d^2 + \omega_d^2}$
  - The 2 poles: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
  - In all cases, except in the case of overdamped system
    $\zeta = \frac{\sigma_d}{\omega_n} = \cos\theta \ , \ 0 \le \zeta \le 1.$

$\omega_d$ ≡ damped frequency of oscillations

$\sigma_d$ ≡ exponential decay frequency

The practical meanings of $\omega_d$ and $\sigma_d$ are illustrated graphically in Fig. 7.

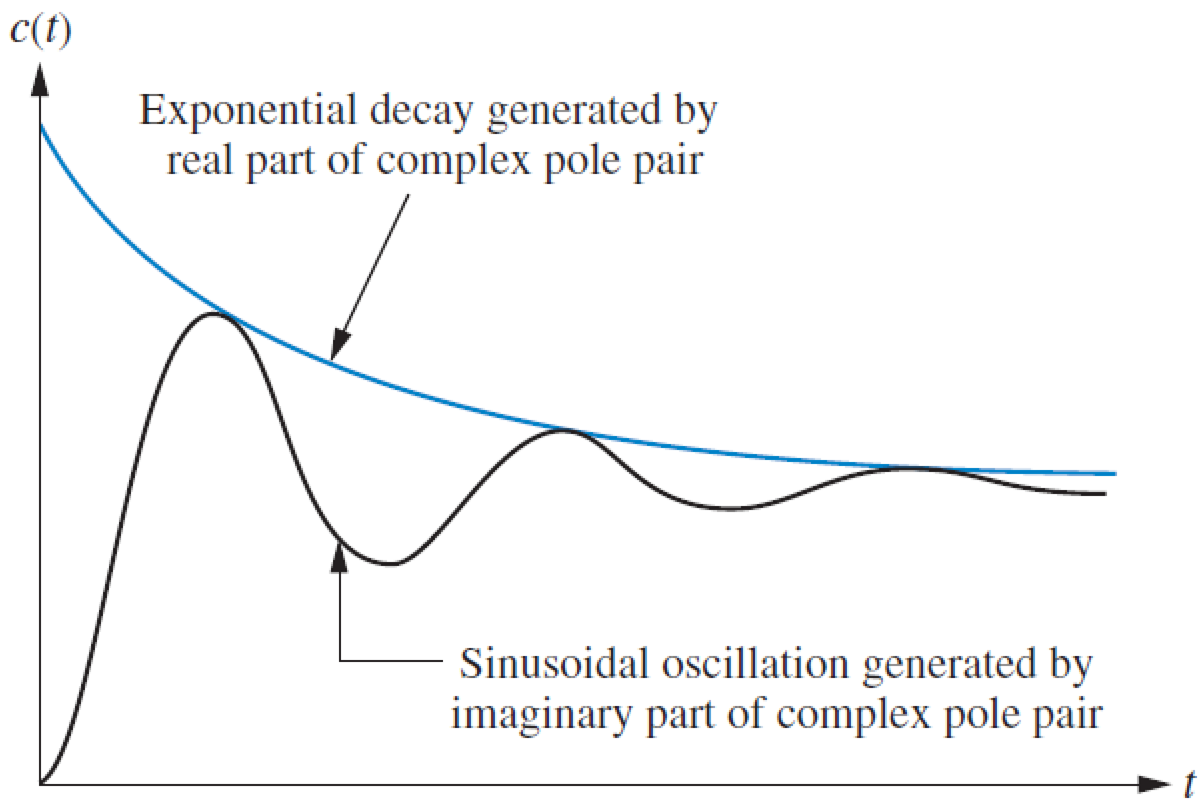
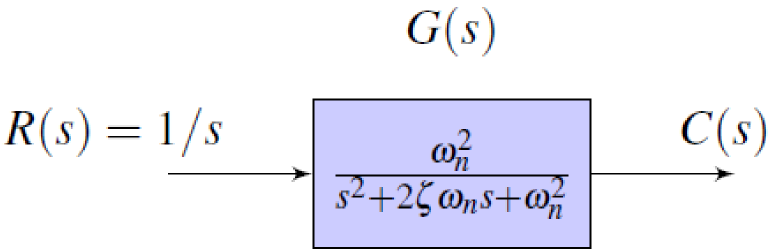
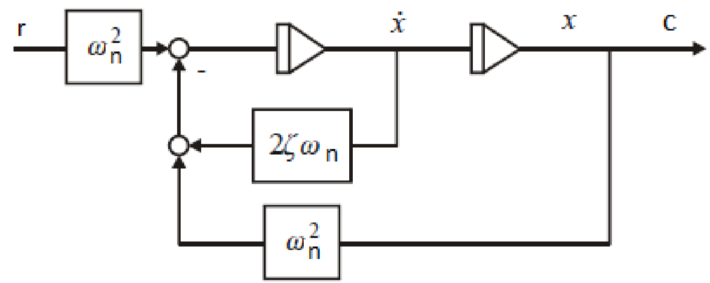



Figure 7: Second-order step response components generated by complex poles (Fig. 4.8 of [Nise, 2015]).

# 7. Underdamped Second-Order System ($0 < \xi < 1$)

- In this part, we study the **step response** of an underdamped second-order system.




$$\Rightarrow C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

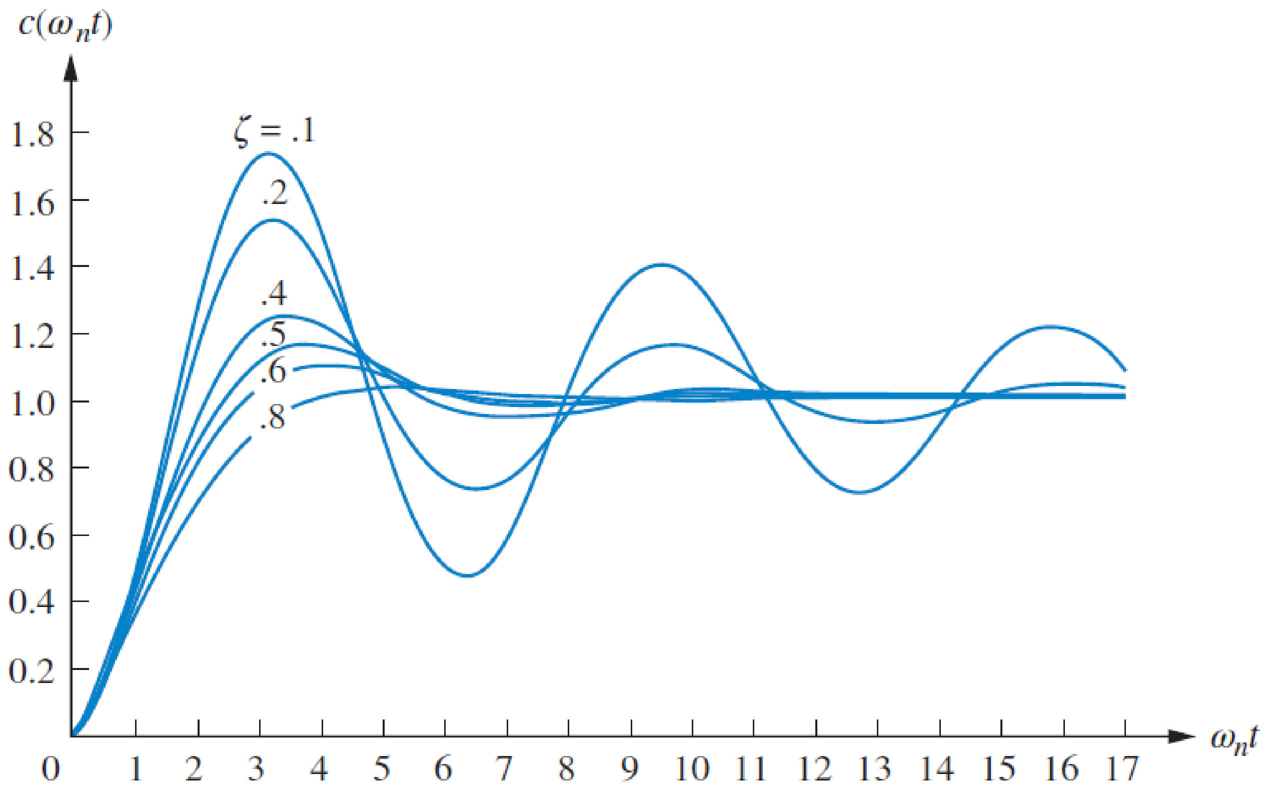
$$\Rightarrow c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t-\phi\right),$$

$$\text{where } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

- Note that the time response c(t) depends on two parameters only: $\omega_n$ and ξ (i.e., on the pair of conjugate poles).

The effect of the damping ratio on the step response of an underdamped system is illustrated in Fig. 8.

Figure 8: Second-order underdamped step-responses for various damping ratio values. (Fig. 4.13 of [Nise, 2015])



- The oscillation magnitude is inversely proportional to the value of $\xi$ ($0 < \xi < 1$).

- In addition to the time constant, rise time ($T_r$), and settling time ($T_s$), which have the same definitions as in first-order systems, the step-response of an underdamped second-order system is also characterized by two other characteristics: the percent overshoot ($\%OS$) and the peak time ($T_p$). (See Fig. 9)

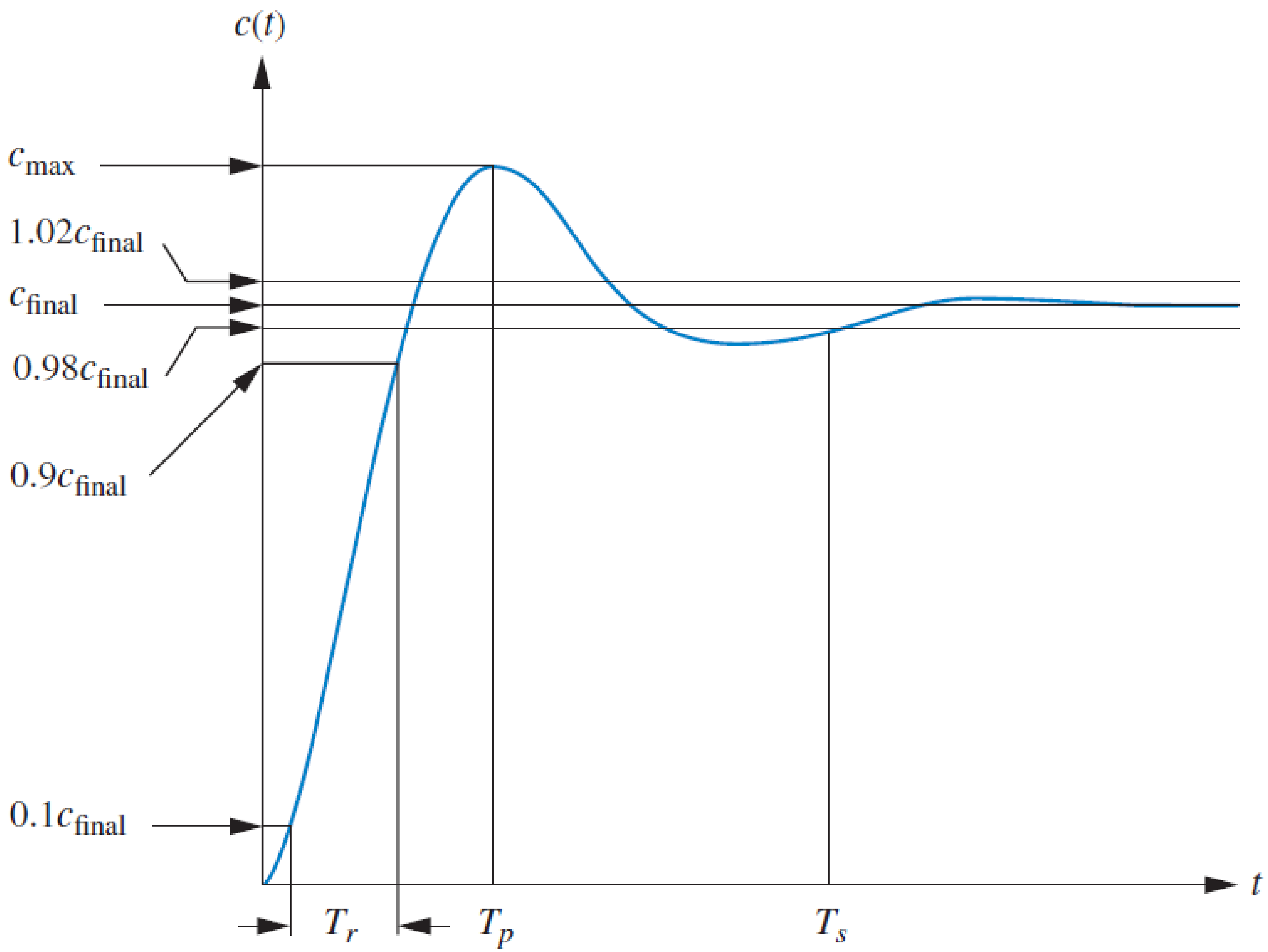



Figure 9: Second-order underdamped step-response specifications. (Fig. 4.14 of [Nise, 2015])

## Definition (Overshoot)

The percent overshoot $\%OS$ is the ratio of the difference between the maximum value and the steady-sate value of the output to the steady-state value (expressed in %).

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100\% \qquad \text{(by definition)}$$

$$\Rightarrow \quad \%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100\% = e^{-\pi/\tan\theta} \times 100\% \qquad \text{(exact value)}$$

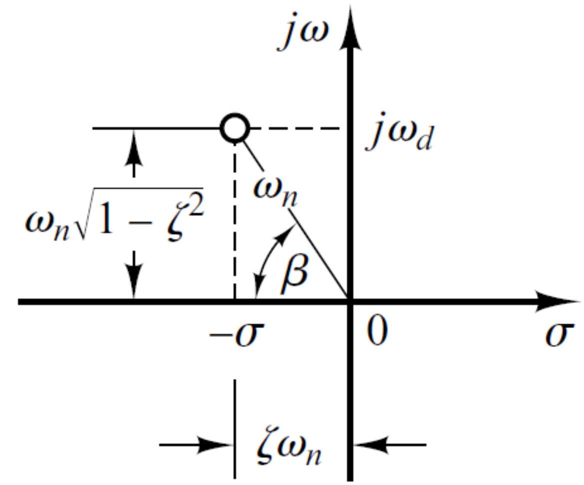
## Definition (Peak time)

The peak time $T_p$ is the time required to reach the first (maximum) peak.

$$\boxed{T_p = \frac{\pi}{|\mathrm{Im(poles)}|} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \qquad \text{(exact value)}$$

## Settling time $T_s$

$$\boxed{T_s \approx 4 \times \text{time constant} = \frac{4}{|\mathrm{Re(poles)}|} = \frac{4}{\zeta\omega_n}},$$




## Rising time $T_r$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

where angle $\beta$ is defined in Figure above. Clearly, for a small value of $t_r$, $v_d$ must be large.

## Note

All these parameters: $\%OS, T_p, T_s, T_r$ , and the time constants depend only on the poles of the system.

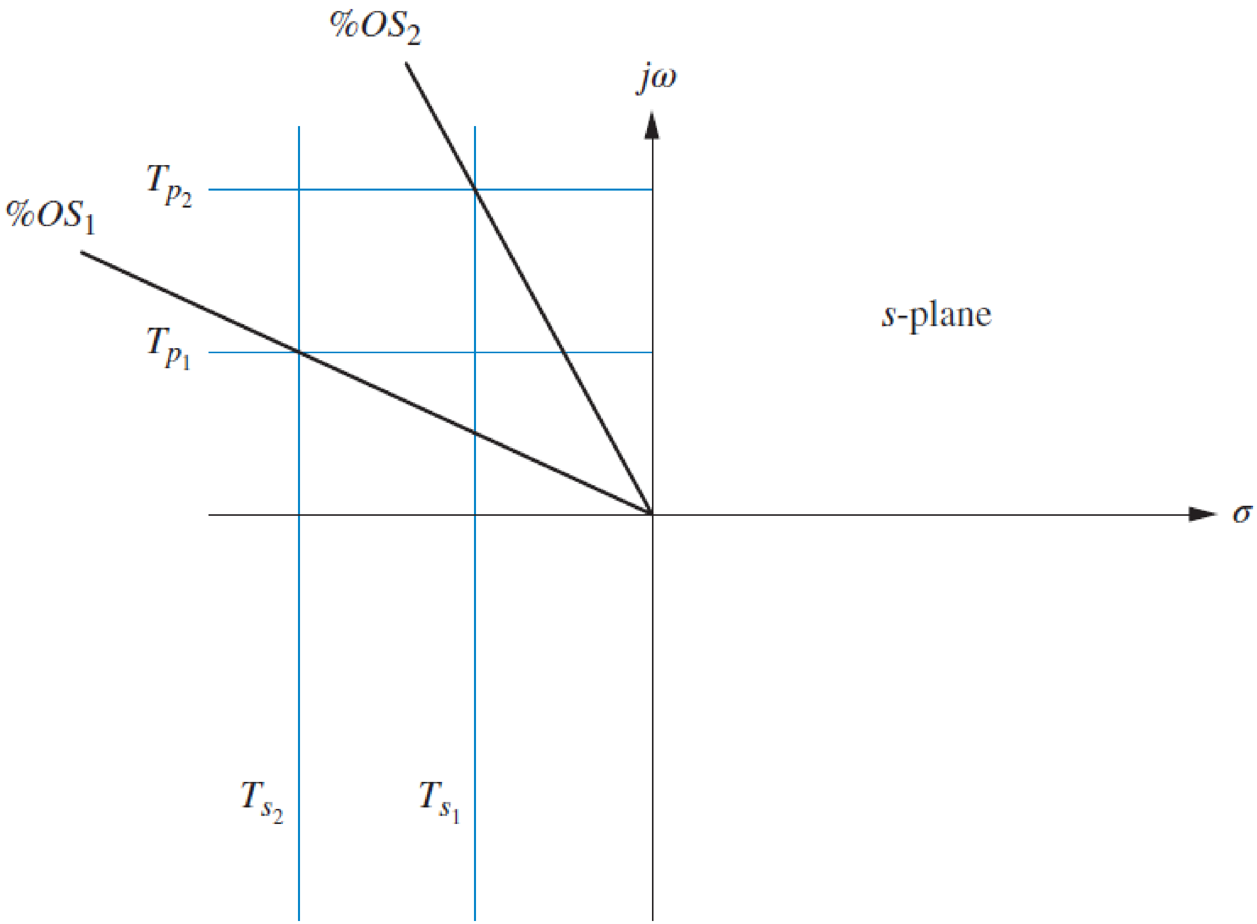



Figure 10: Lines of constant $T_p$, $T_s$ , and $\%OS$, Note: $T_{s2} < T_{s1}$ ; $T_{p2} < T_{p1}$ ; $\%OS_1 < \%OS_2$ (Fig. 4.18 of [Nise, 2015])

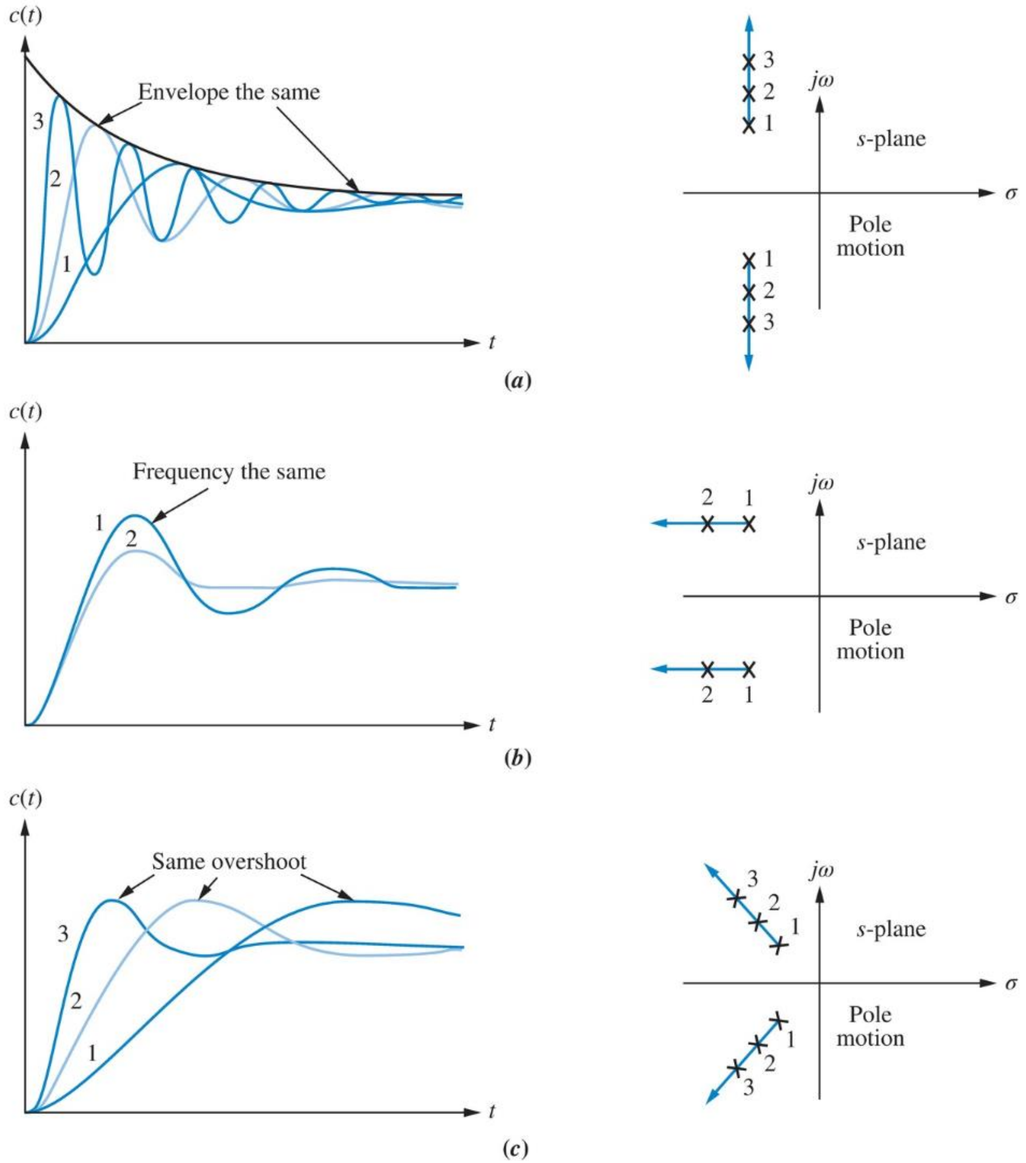
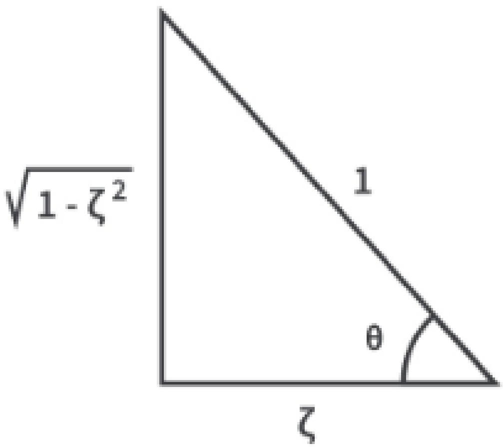



Figure 11: Step-responses of an underdamped second-order system with varying poles locations. (Fig. 4.19 of [Nise, 2015])

# Hint

Take a look at this triangle if you're confused.




$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + \theta)$$

&

$$\theta = cos^{-1}\zeta = sin^{-1}\sqrt{1-\zeta^2}$$

# Matlab

The Matlab code ch4p1 (in Appendix B and SCS) shows how to generate a polynomial of a second order from 2 conjugate complex poles and how to extract the polynomial coefficients to calculate $T_p$, $T_s$, and *%OS*.

```
'(ch4p1) Example 4.6'                      % Display label.
p1=[1 3+7*i];                              % Define polynomial containing
                                           % first pole.
p2=[1 3-7*i];                              % Define polynomial containing
                                           % second pole.
deng=conv(p1,p2);                          % Multiply the two polynomials to
                                           % find the 2nd order polynomial,
                                           % as^2+bs+c.
omegan=sqrt(deng(3)/deng(1))               % Calculate the natural frequency,
                                           % sqrt(c/a).
zeta=(deng(2)/deng(1))/(2*omegan)
                                           % Calculate damping ratio,
                                           % ((b/a)/2*wn).
Ts=4/(zeta*omegan)                         % Calculate settling time,
                                           % (4/z*wn).
Tp=pi/(omegan*sqrt(1-zeta^2))              % Calculate peak time,
                                           % pi/wn*sqrt(1-z^2).
pos=100*exp(-zeta*pi/sqrt(1-zeta^2))
                                           % Calculate percent overshoot
                                           % (100*e^(-z*pi/sqrt(1-z^2)).
pause
```

## Example 2

A system has a transfer function $G(s) = 36/(s^2 + 4.2s + 36)$. Find the natural frequency and damping ratio.

## Solution

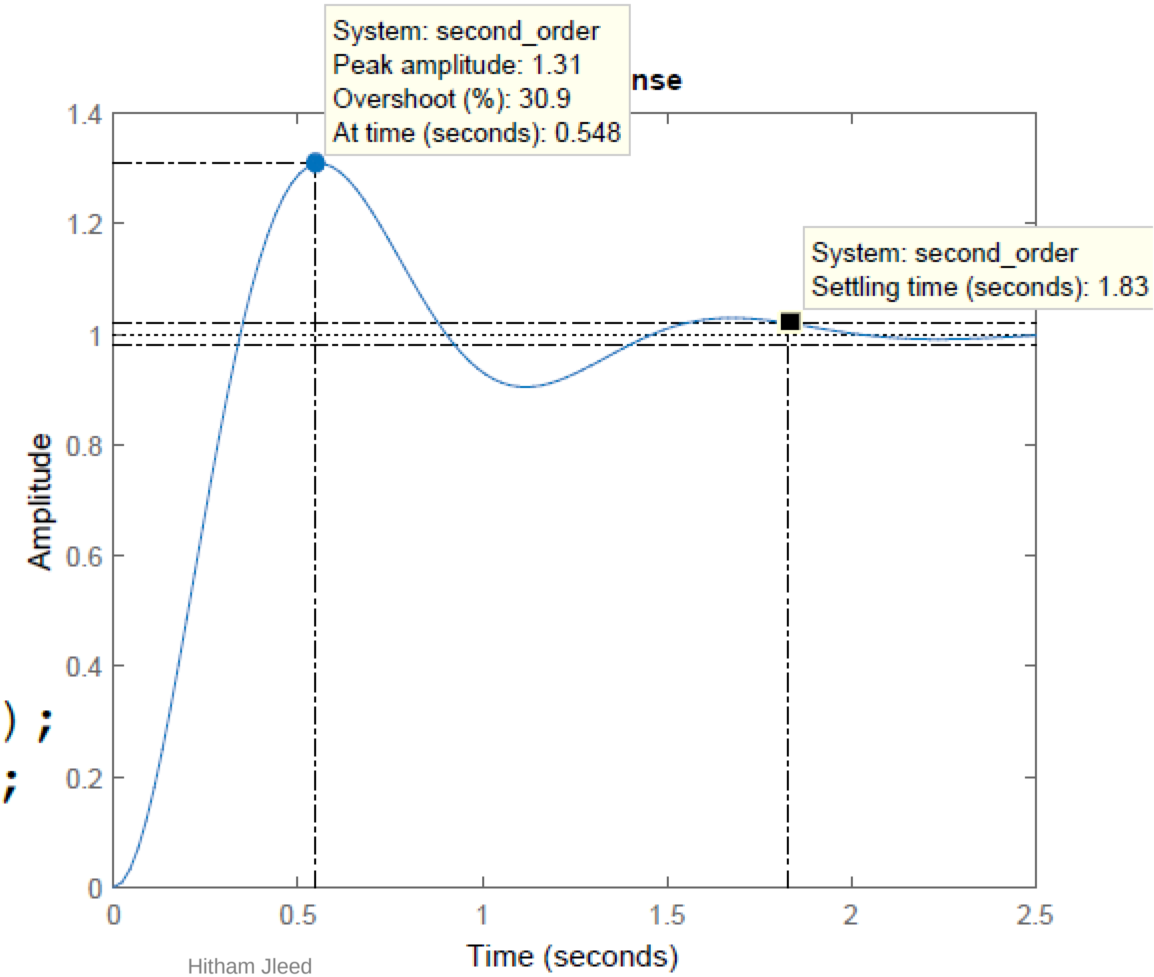
$$\omega_n^2 = 36 \;\Rightarrow\; \omega_n = 6$$

$$2\zeta\omega_n = 4.2 \;\Rightarrow\; \zeta = 0.35$$

## Verification

Listing 6: Matlab Code

```
s=tf([1 0],1);
second_order=36/
 (s^2 + 4.2*s + 36);
step(second_order);
```



## Example 3

A system has a transfer function $G(s) = 100/(s^2 + 15s + 100)$
Find the peak time, settling time, and the overshoot.

## Solution

$\omega_n = 10$ and $\xi = 0.75$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \qquad \Rightarrow T_p = 0.475$$

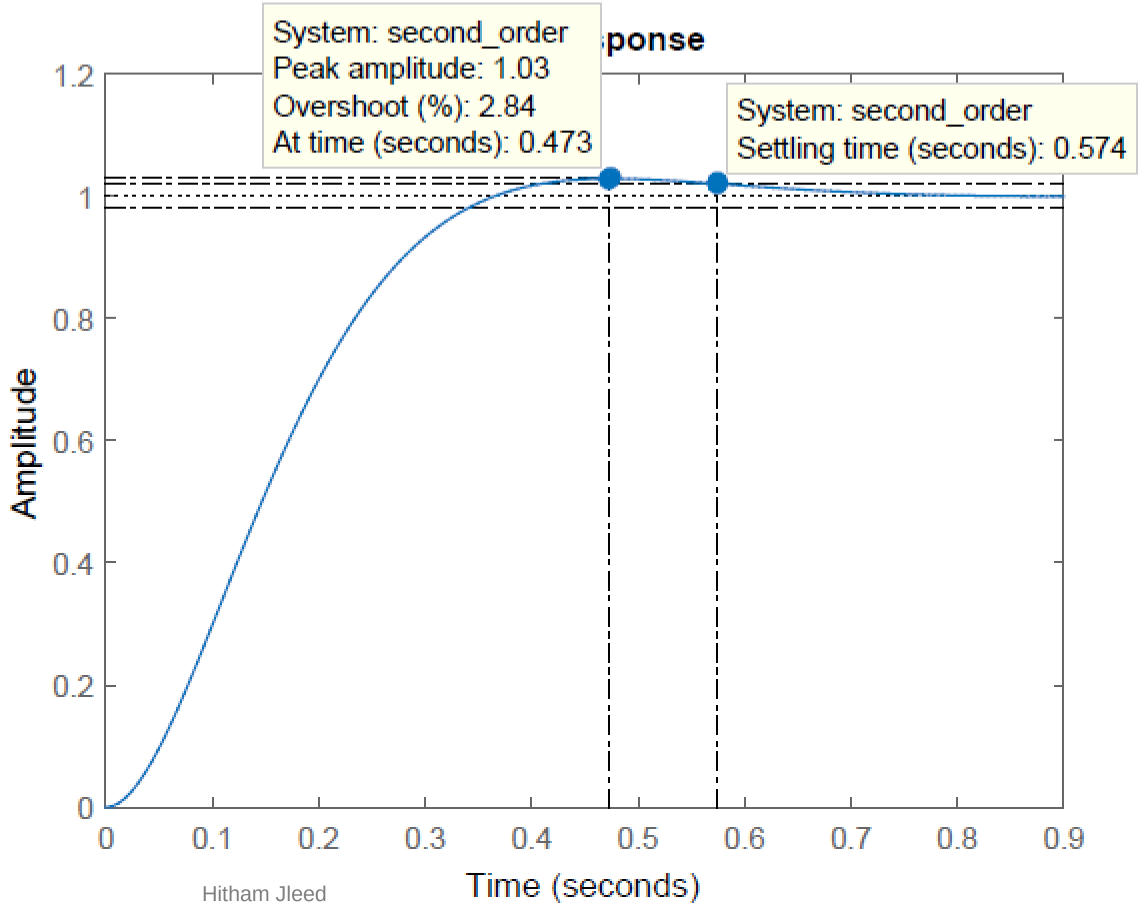
$$\boxed{\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100} \qquad \Rightarrow \%OS = 2.833$$

$$\boxed{T_s = \frac{4}{\zeta\omega_n}} \qquad \Rightarrow T_s = 0.533$$

# Verification

Listing 8: Matlab Code

```
s=tf([1 0],1);
second_order=100/(s^2 + 15*s + 100);
step(second_order);
```



## Example 4

The poles of a second-order system are $3 \pm j7$. Find the peak time, settling time, the overshoot, the natural frequency, and the damping ratio.

## Solution

$$\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$$

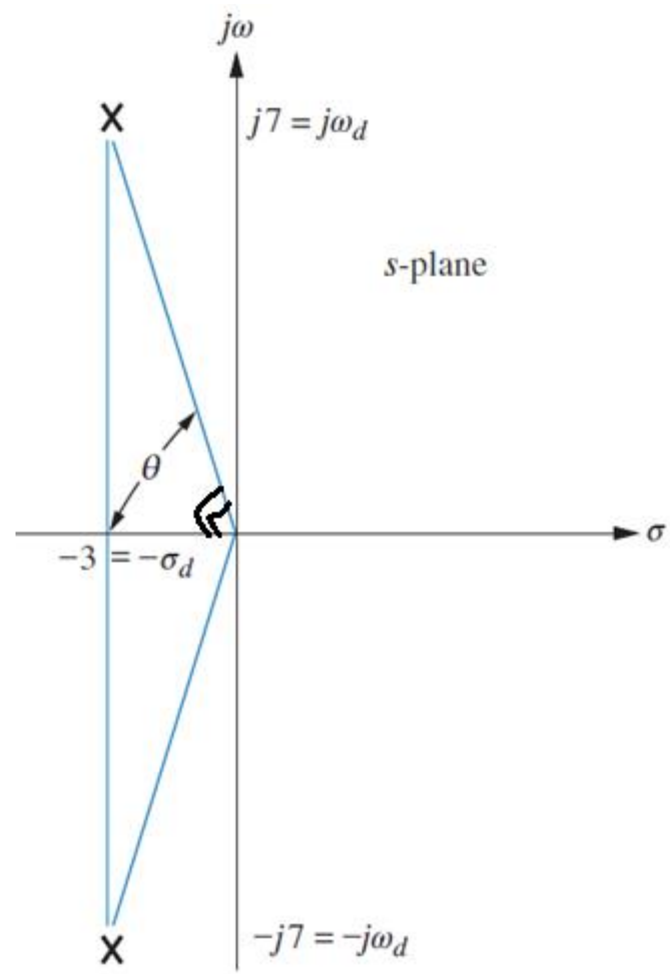
$$\omega_n = \sqrt{7^2 + 3^2} = 7.616$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\%$$

The approximate settling time is

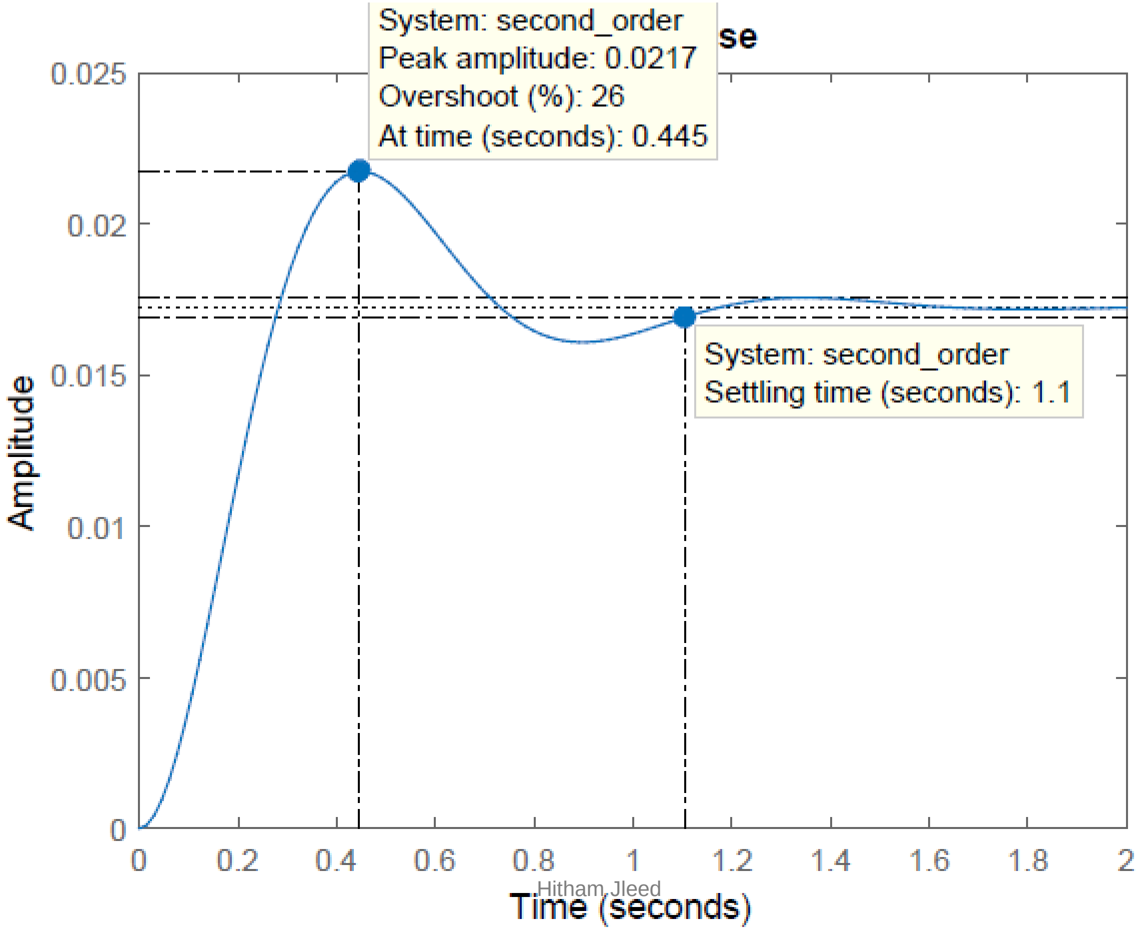
$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds}$$



# Verification

Listing 10: Matlab Code

```
s=tf([1 0],1);
second_order=1/((s+3-7i)*(s+3+7i));
step(second_order);
```



## Example 5

A control system is composed of a plant $G(s) = K / (s(s + 1))$ and a feedback sensor of transfer function $H(s) = 0.5$. Find $K$ so that the system is:

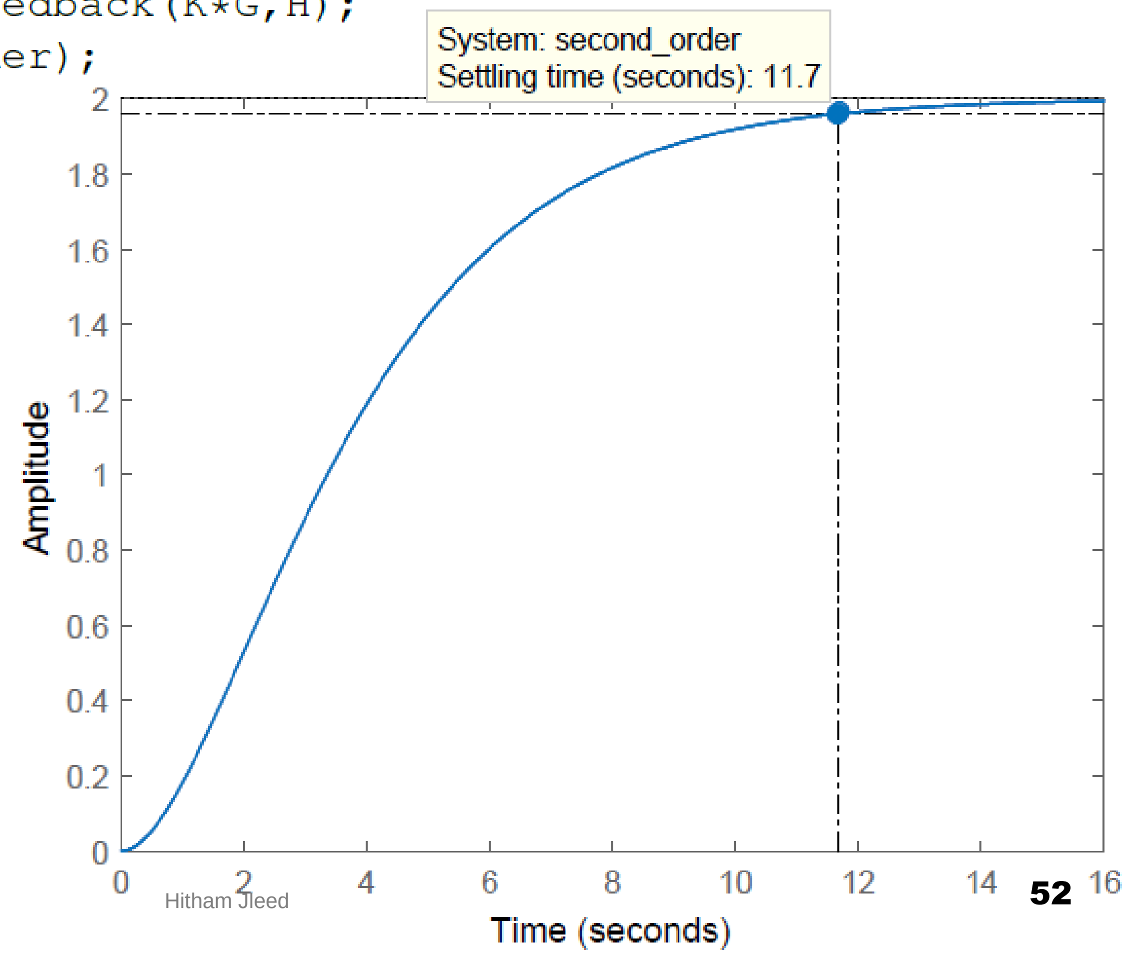
1. critically damped
2. Undamped

## Solution

# Verification

Listing 12: Matlab Code

```
s=tf([1 0],1);
K=0.5;
G=1/(s*(s+1));
H=0.5;
second_order=feedback(K*G,H);
step(second_order);
```



## Exercise 2

A control system is composed of a plant $G(s) = K / (s(s + a))$ and a feedback sensor of transfer function $H(s) = 5$. Find the positive gains $K$ and $a$ so that the following conditions are simultaneously satisfied:

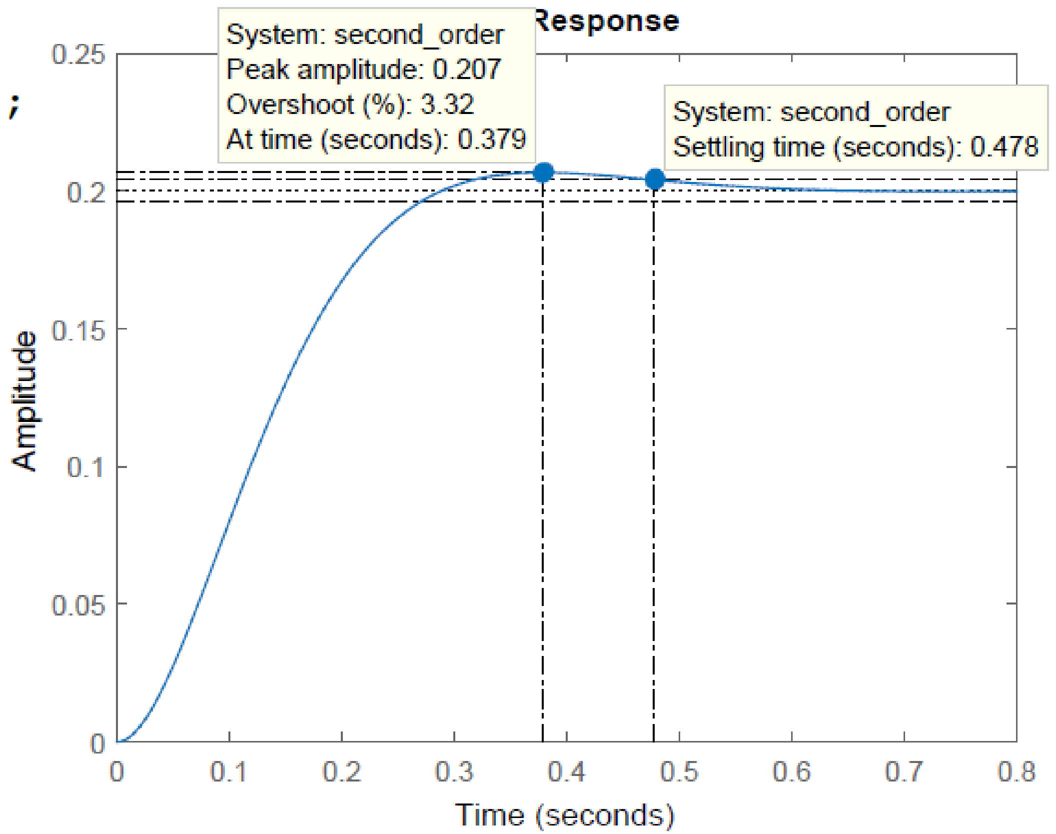
1. an overshoot less than 5%
2. a settling time less than 0.5 s

On the s-plane, indicate the regions of the poles corresponding to these specifications.

# Verification

Listing 14: Matlab Code

```
s=tf([1 0],1);
a=18;
K=30;
G=K/(s*(s+a));
H=5;
second_order=feedback(G,H);
step(second_order);
```



## ■ Example

Determine the differential equations in the state variables $x_1(t)$ and $x_2(t)$ for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

Find the undamped natural frequency $\omega_n$ and damping ratio $\zeta$ for this system.

**Solution**

For this system

$$[sI - A] = \begin{bmatrix} s+1 & 2 \\ -2 & s+3 \end{bmatrix}$$

and

$$\det[sI - A] = s^2 + 4s + 7$$

and therefore for state variable $x_1(t)$:

$$\frac{d^2x_1}{dt^2} + 4\frac{dx_1}{dt} + 7x_1 = \frac{du}{dt} + 3u.$$

and for $x_2(t)$:

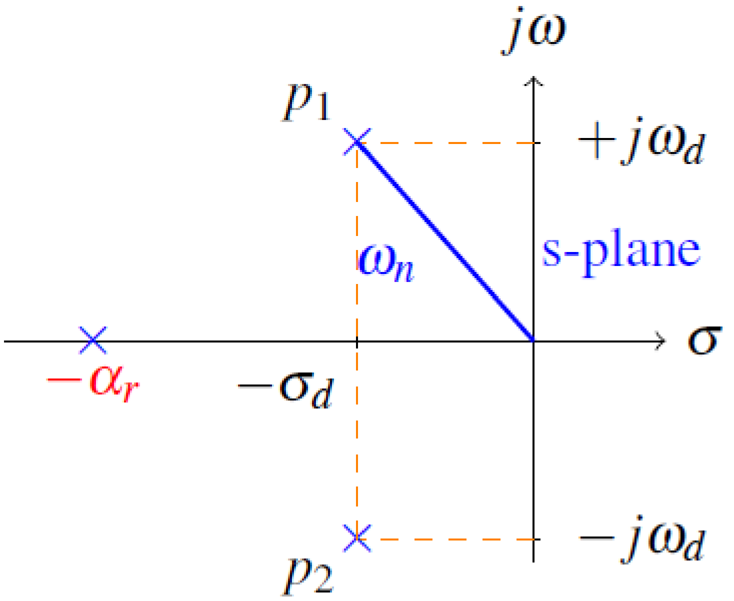
$$\frac{d^2x_2}{dt^2} + 4\frac{dx_2}{dt} + 7x_2 = 2u.$$

By inspection, $\omega_n^2 = 7$, and $2\zeta\omega_n = 4$, giving $\omega_n = \sqrt{7}$ rad/s, and $\zeta = 2/\sqrt{7} = 0.755$.

# 8. System Response with Additional Poles

## the question

What is the effect of adding one (or more) pole(s) on the step-response of a second-order system?




Consider the following 2nd order system with poles

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\Rightarrow G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Suppose that a third pole $p_3 = -\alpha_r$ is added to the system.

⇒ the system becomes of order 3.

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \alpha_r)}$$

After some mathematical manipulations and applying partial-fraction expansion, the system's step response can be expressed as:

$$C(s) = \underbrace{\frac{A}{s}}_{\substack{\uparrow \\ \text{forced response}}} + \underbrace{\frac{B(s+\zeta\omega_n)+C\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} + \frac{D}{s+\alpha_r}}_{\substack{\uparrow \\ \text{natural response}}}$$

$$\Rightarrow c(t) = \overbrace{\underbrace{A\,u(t)}_{\substack{\uparrow \\ c_f(t)}} + e^{-\zeta\omega_n t}\left(B\cos(\omega_d t) + C\sin(\omega_d t)\right)}^{\text{original output of the 2nd order system}} + \overbrace{D\,e^{-\alpha_r t}}^{\text{effect of additional pole}}$$

where the natural part $c_n(t) = e^{-\zeta\omega_n t}\left(B\cos(\omega_d t) + C\sin(\omega_d t)\right) + D\,e^{-\alpha_r t}$.

The effect of the term $De^{-\alpha_r t}$ on the original 2nd order system's response is illustrated in Fig. 12.

Practically, when $\alpha_r \geq 5\sigma_d$ , the effect of pole $p_3 = -\alpha_r$ can be considered to be negligible.

⇒ In this case, the pole $p_3$ can be simply ignored so that the third-order system is approximated by the original 2nd order system. ($p_1$ and $p_2$ are called the dominant poles)

$$G(s) = \frac{N(s)}{(s+\alpha_r)\,(s^2+2\zeta\,\omega_n s+\omega_n^2)} \overset{\text{if } \alpha_r > 5\sigma_d}{\Longrightarrow} G(s) \approx \frac{N'(s)}{(s^2+2\zeta\,\omega_n s+\omega_n^2)}$$

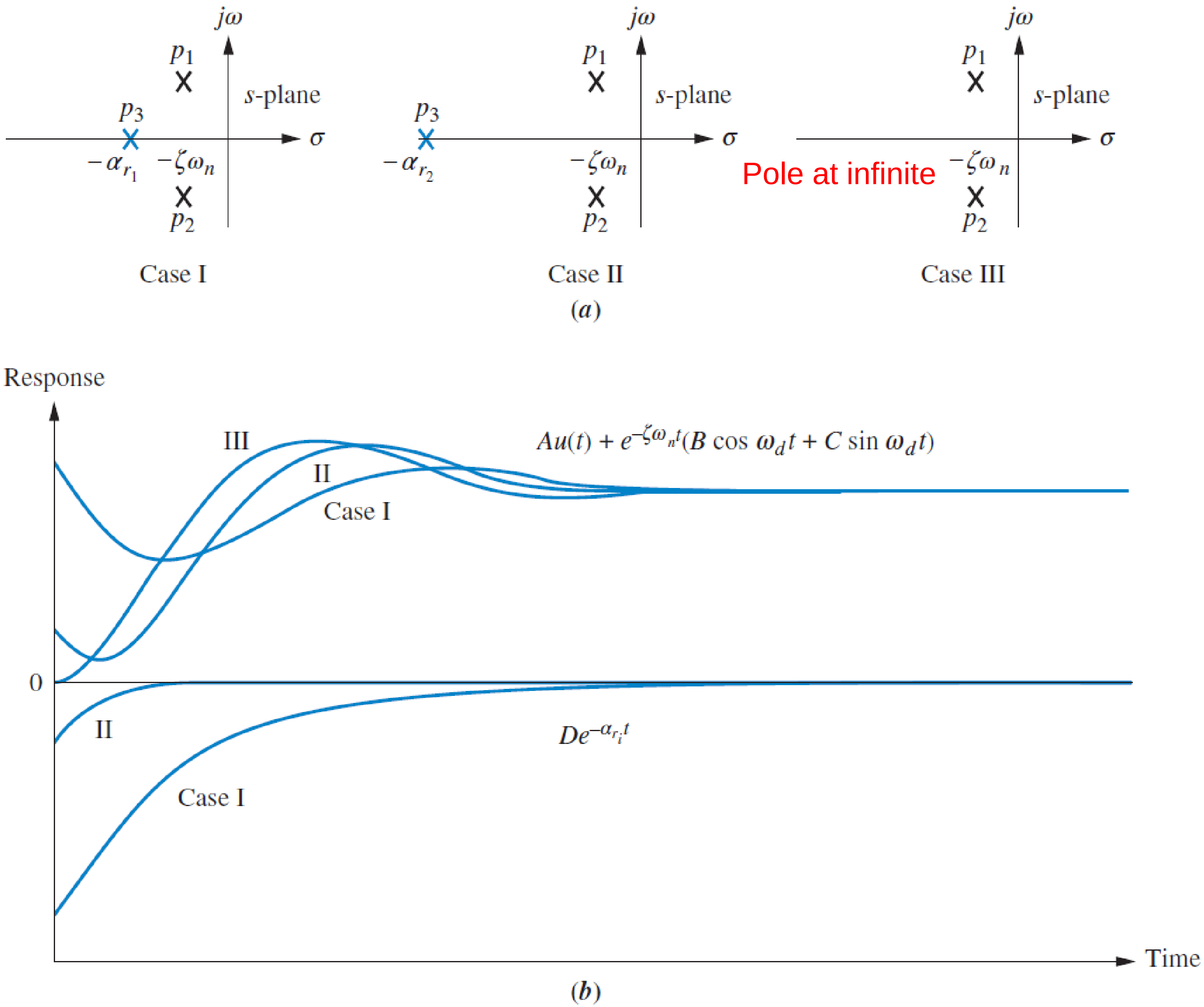



Figure 12: Component responses of a three-pole system. (Fig. 4.23 of [Nise, 2015])

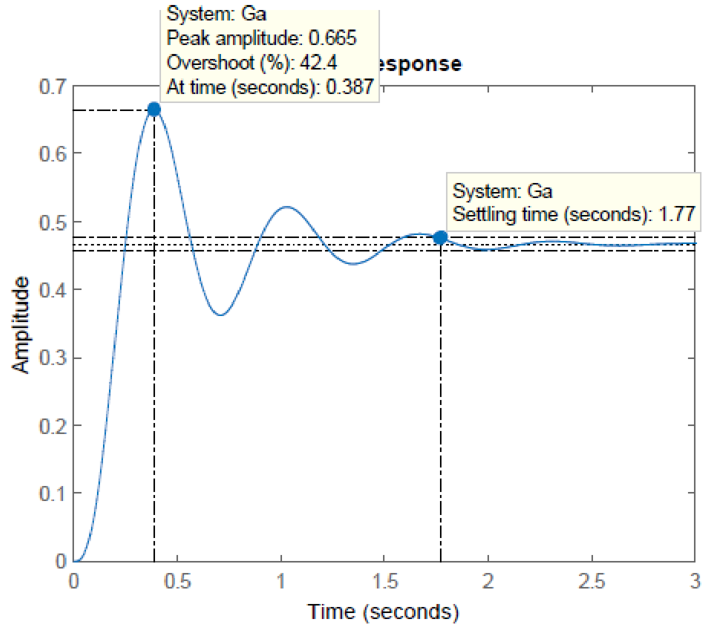
## Exercise 3

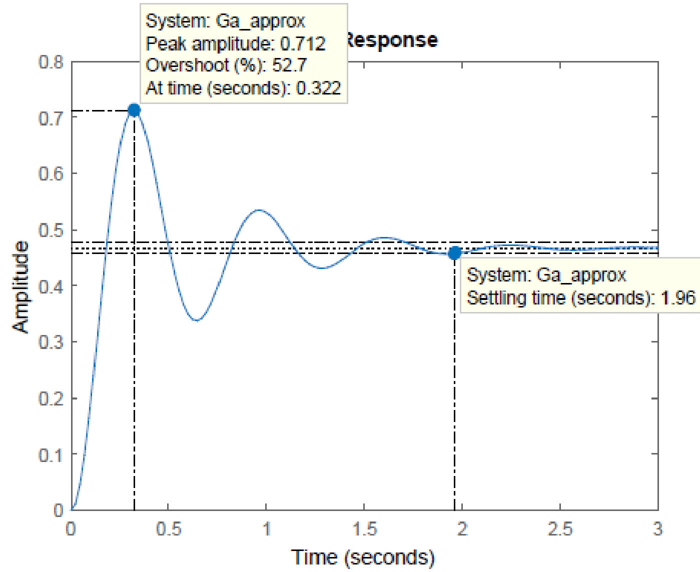
Which of the following transfer functions can be approximated by a second-order model?

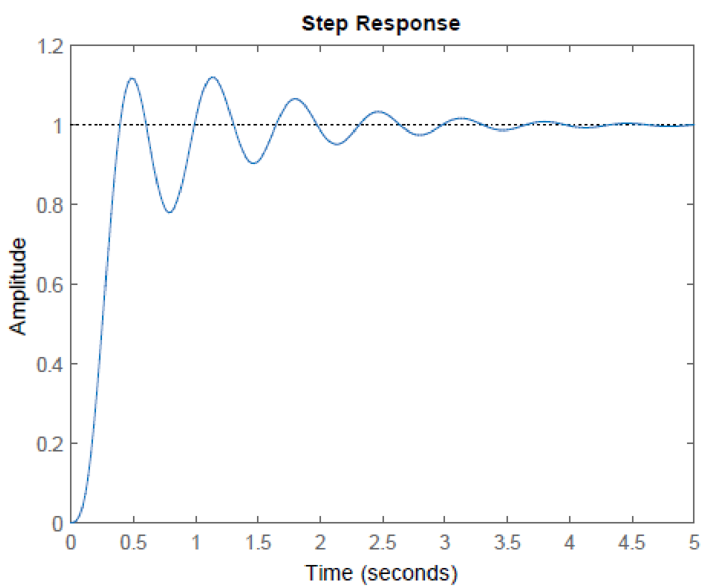
$$G(s) = \frac{700}{(s+15)(s^2+4s+100)}$$

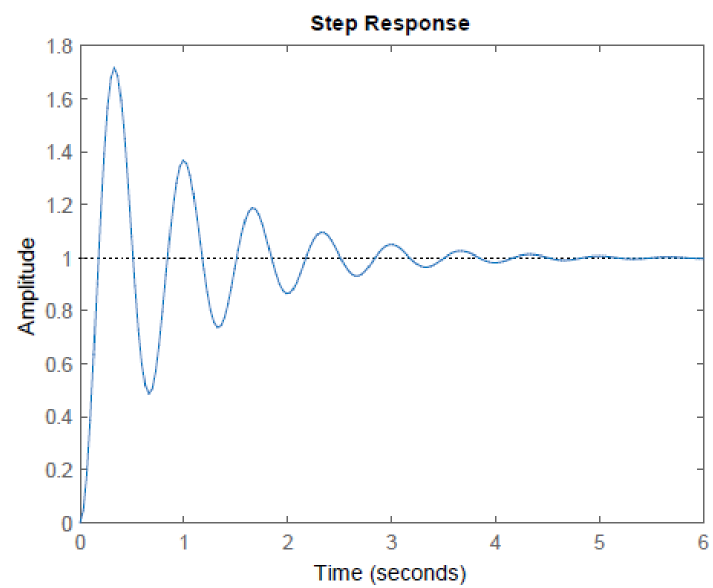
$$G(s) = \frac{360}{(s+4)(s^2+2s+90)}$$

# Verification



# Solved problems:

1. A single degree of freedom spring-mass-damper system has the following data: spring stiffness 20 kN/m; mass 0.05 kg; damping coefficient 20 N-s/m. Determine

   (a) undamped natural frequency in rad/s and Hz
   (b) damping factor
   (c) damped natural frequency n rad/s and Hz.

If the above system is given an initial displacement of 0.1 m, trace the phasor of the system for three cycles of free vibration.

**Solution:**

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20\times10^3}{0.05}} = 632.46 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{632.46}{2\pi} = 100.66\,\text{Hz}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{20\times10^3\times0.05}} = 0.32$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 632.46\sqrt{1-0.32^2} = 600\,\text{rad/s}$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{600}{2\pi} = 95.37\text{ Hz}$$

$$y(t) = Ae^{-\zeta\omega_n t} = 0.1e^{-0.32\times632.46t}$$

# Summary

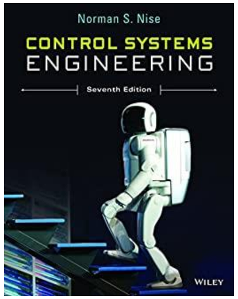
- **Definitions of Transient-Response Specifications:** Frequently, the performance characteristics of a control system are specified in terms of the transient response to a **unit-step input**, since it is easy to generate and is sufficiently drastic. it is common to specify the following:

1. time constant, $t_d$
2. Rise time, $t_r$
3. Settling time, $t_s$
4. Peak time, $t_p$
5. percent overshoot $\%OS$

- **2ND-ORDER SYSTEM DYNAMICS**

https://controlsystemsacademy.com/0024/0024.html#zeta1

# References




Nise, N. S. (2015).
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