Chapter 8: Root Locus Techniques

ELG 3155 : Introduction to Control Systems

❑ Objective:

- Study root locus (RL) techniques.
- How to use RL techniques to find the pole locations of a closed-loop system.
- How to use RL to find the gain values to satisfy certain criteria of the system's transient response.

uOttawa

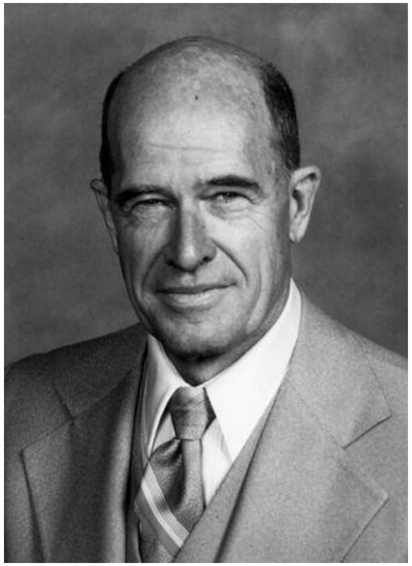
Hitham Jleed

Outline

8.1 Introduction

Root locus, a graphical presentation of the closed-loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response(*Evans, 1948, 1950*).

The root locus can be used to describe qualitatively the performance of a system as various parameters are changed.



Walter Richard Evans

Introduction(cont'd):

- System performance and stability → determined by ***closed-loop*** poles
- Typical closed-loop feedback control system

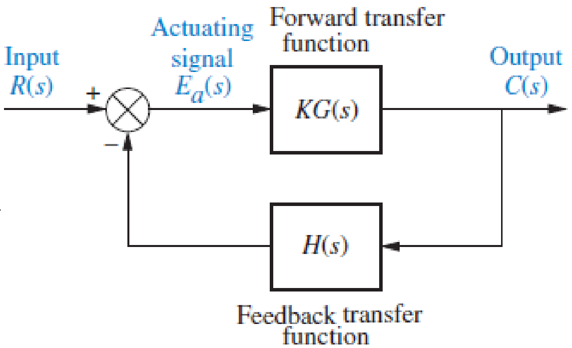
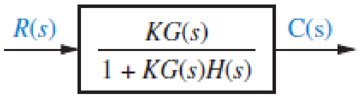


FIGURE 8.1



$$G(s)=\frac{(s+1)}{s(s+2)} \qquad H(s)=\frac{(s+3)}{(s+4)}$$

Open-loop TF

$$KG(s)H(s)=\frac{K(s+1)}{s(s+2)}\frac{(s+3)}{(s+4)}$$

Zeros → -1, -3
Poles → 0, -2, -4

- Location of poles → easily found
- *Variation of gain K → do not affect the location of any poles*

Closed-loop TF

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}$$
$$=\frac{K(s+1)(s+4)}{s^3+(6+K)s^2+(8+4K)s+3K}$$

- Location of poles → need to factor the denominator → difficult
- *variation of gain K → do change the location of poles*

```
for k = 0:10
  p = [1,6+k, 8+4*k,3*k];
  r = roots(p);
  plot(k, r, 'k*');
  hold on
end
hold off
```

- Easy way to know the CL pole locations w/o solving higher-order characteristic equation of CLTF for various gain K?
- Root locus
  - graphical representation of the closed-loop poles as a function of system parameters
  - Can be used to design system parameters of the high order systems to yield a desired system specifications
  - ***Estimating closed-loop poles***' location when gain K is varied ***using open-loop poles***
  - Represent the poles of T(s) as K varies

Open-loop TF

$$KG(s)H(s)=\frac{K(s+1)}{s(s+2)}\frac{(s+3)}{(s+4)}$$

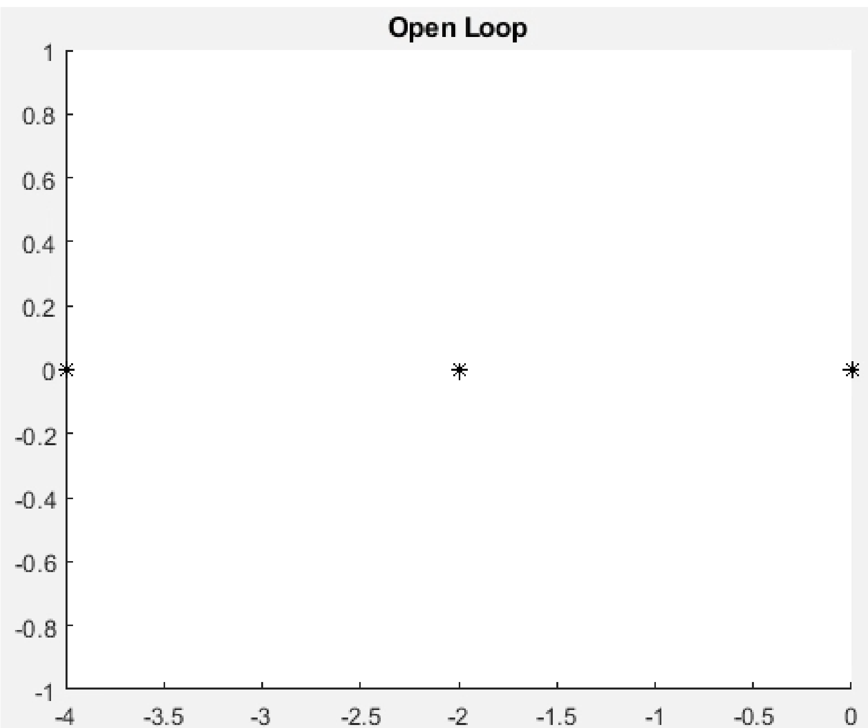
$$= K\frac{N(s)}{D(s)}.$$

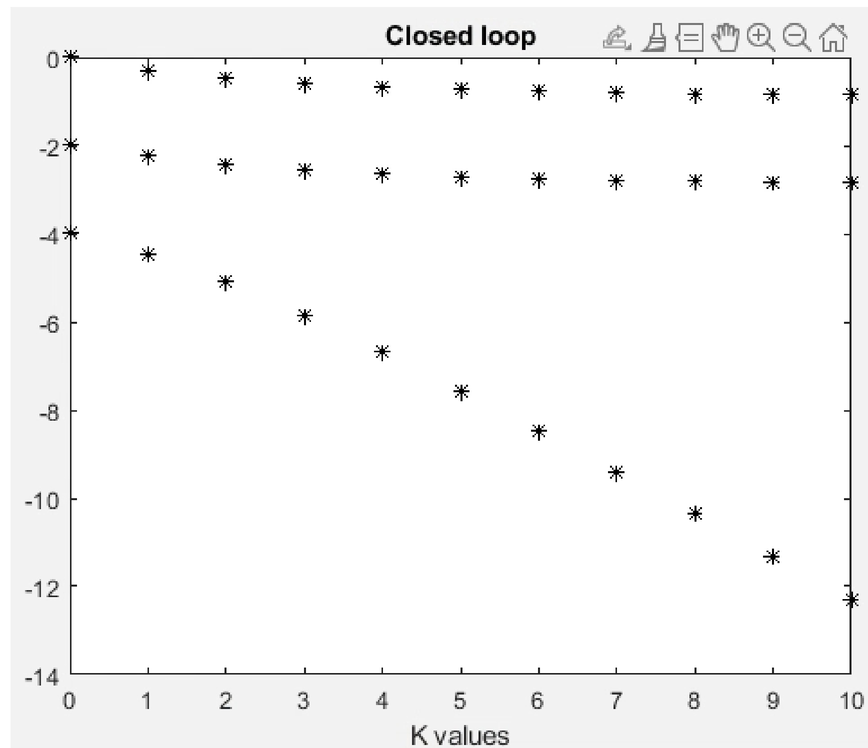
Closed-loop TF

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}=\frac{K(s+1)(s+4)}{s^3+(6+K)s^2+(8+4K)s+3K}$$

```
p = conv(conv([1 0],[1 2]),[1 4])
r = roots(p);
scatter(real(r),imag(r),'k*')
```

```
for k = 0:10
   p = [1,6+k, 8+4*k,3*k];
   r = roots(p);
   plot(k, r, 'k*'); hold on;
end
hold off
```





8.2. Defining the Root Locus

Camera Example

- System below can automatically track subject wearing infrared sensors.
- Solving for the poles using the quadratic equation, we can create the table below for different values of K.

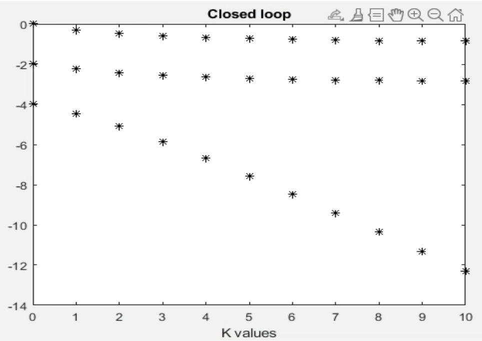
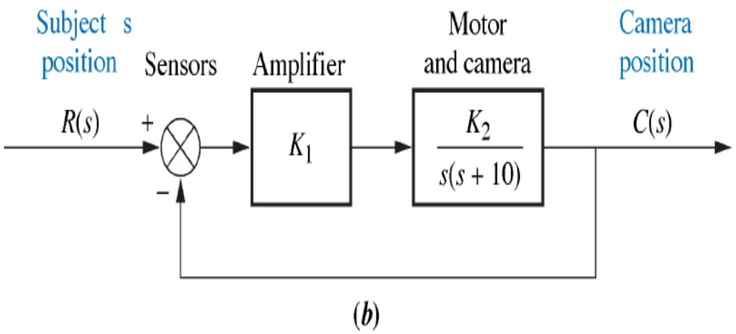
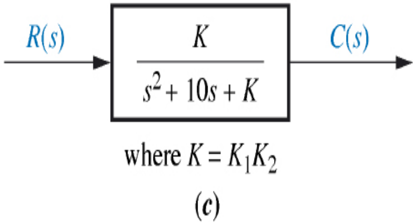
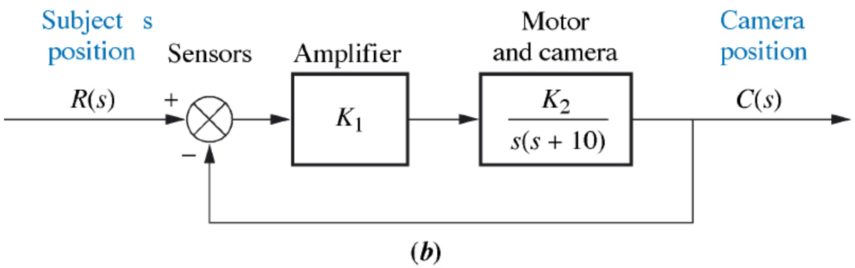


Figure 8.4 (a): Camera with 1 degree of freedom (pitch angle) [Nise, 2015].







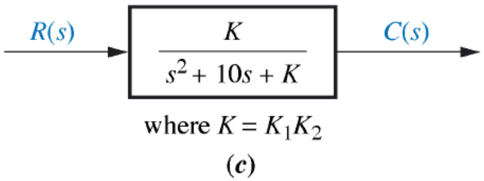


Table 8.1 of [Nise, 2015]. The effect of gain $K = K1K2 \geq 0$ on the placement of the system's poles.

Figure 8.4(b)-(c) : (b) Block diagram of the camera system ; and (c) equivalent diagram. [Nise, 2015].

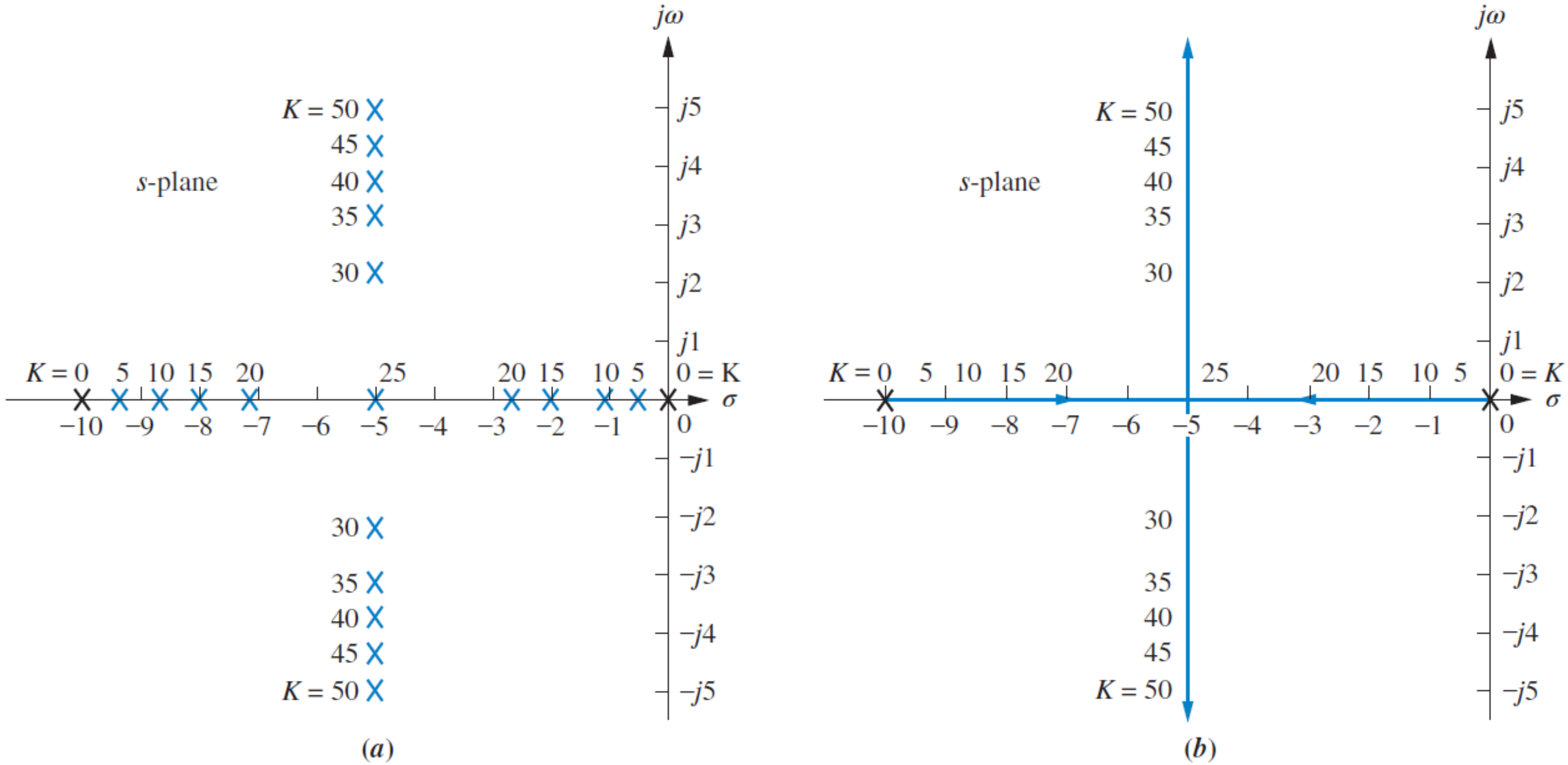
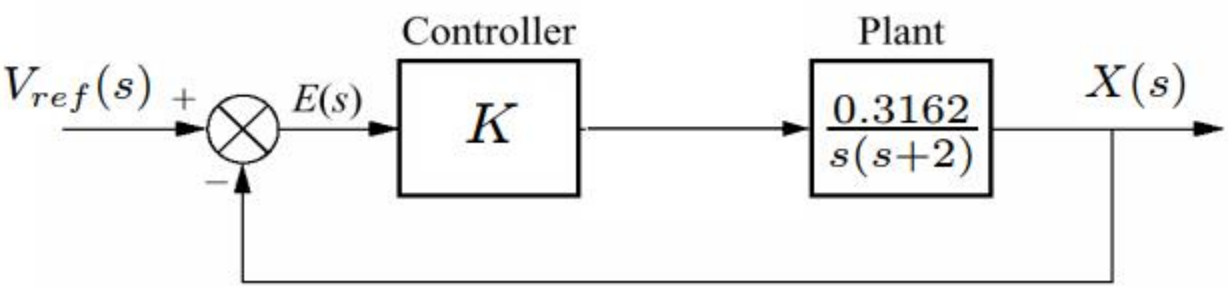


Figure 3: (a) Pole placements for different values of K > 0 ; and (b) root locus. Figure 8.5 of [Nise, 2015].

Type 1 system (no disturbance)



Closed–loop transfer function

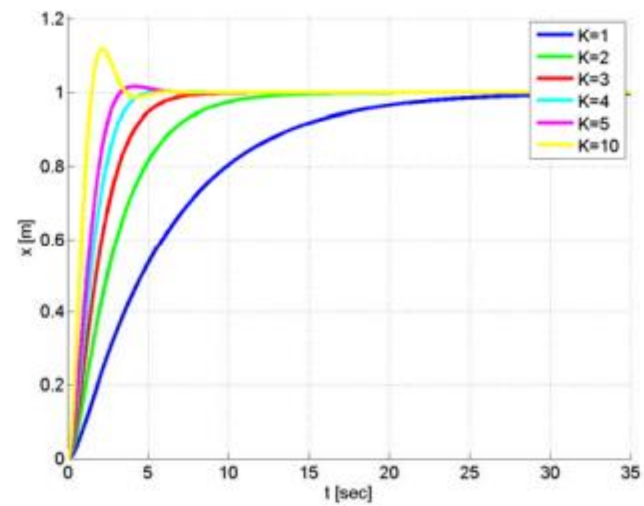
$$\frac{X(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2 + 2s + 0.3162K}$$

Pole locations

$$p_1 = -1 + \sqrt{1 - 0.3162K} \qquad p_2 = -1 - \sqrt{1 - 0.3162K}$$

System becomes **underdamped**$\Rightarrow$
$\Rightarrow$ step response **overshoots** if

$$1 - 0.3162K < 0 \Leftrightarrow K > 3.1626$$

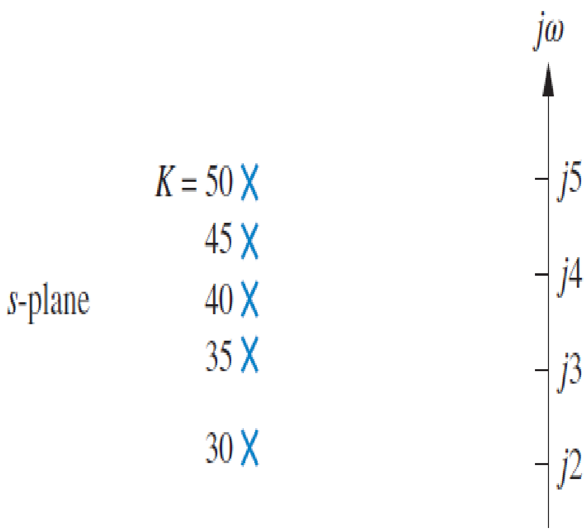


Vector Representation of Complex Number

What is complex number?

$$s = \sigma + j\omega$$

Complex number is a vector



Vector has magnitude and direction

$$s = \sigma + j\omega$$

Therefore, we can also represent complex number s as:

$$M = \sqrt{\sigma^2 + \omega^2}$$

$$\theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right)$$

Vector Representation of Complex Number

But, if s is a variable in a function, how to represent the complex number. For example,

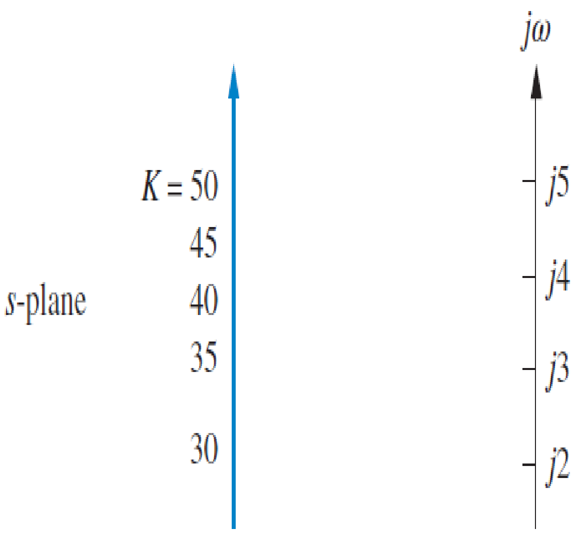
$$s = M\angle\theta$$

Replacing s,

$$F(s) = s + a$$

Another complex number

Graphically,



Vector Representation of Complex Number

$$F(s) = M\angle\theta$$

$$F(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$$M = \frac{\prod \text{zero length}}{\prod \text{pole length}} = \frac{\prod_{i=1}^{m}\left|(s+z_i)\right|}{\prod_{j=1}^{n}\left|(s+p_j)\right|}$$

$$\theta = \sum \text{zero angles} - \sum \text{poles angle}$$

$$\theta = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)$$

Vector Representation of Complex Number

Example

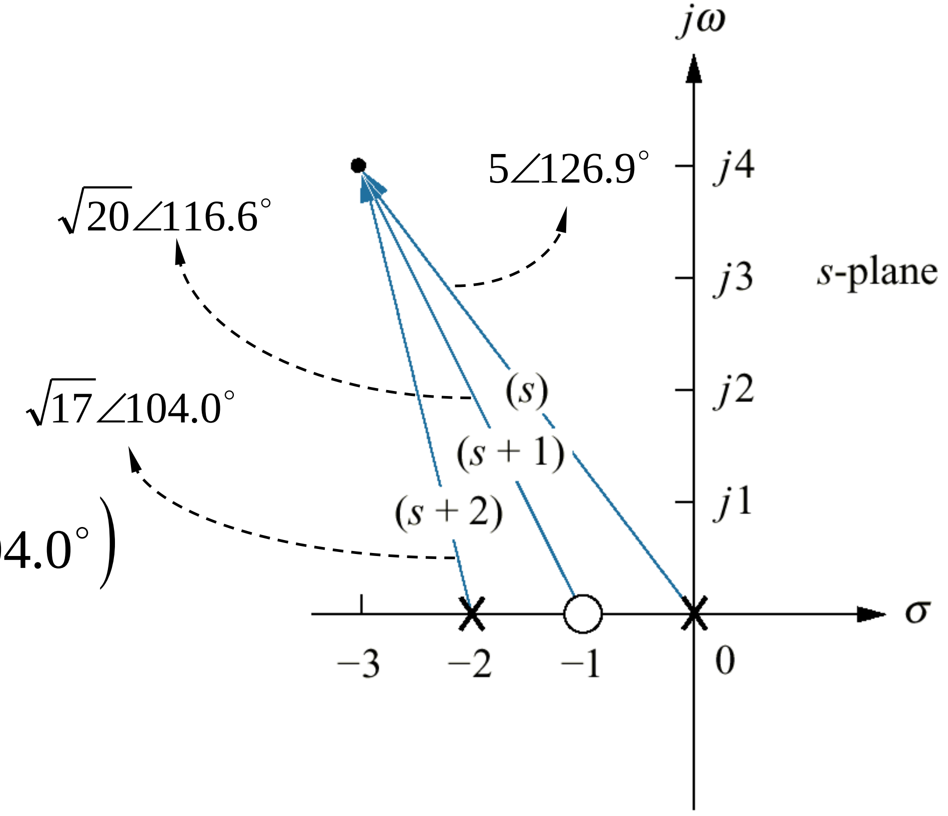
$$F(s)=\frac{s+1}{s(s+2)}$$

$$s=-3+j4$$

What is F(s)?

$$F(s)=\frac{\sqrt{20}}{5\sqrt{17}}\angle\left(116.6^\circ-126.9^\circ-104.0^\circ\right)$$

$$F(s)=0.217\angle-114.3^\circ$$



Root locus terminology

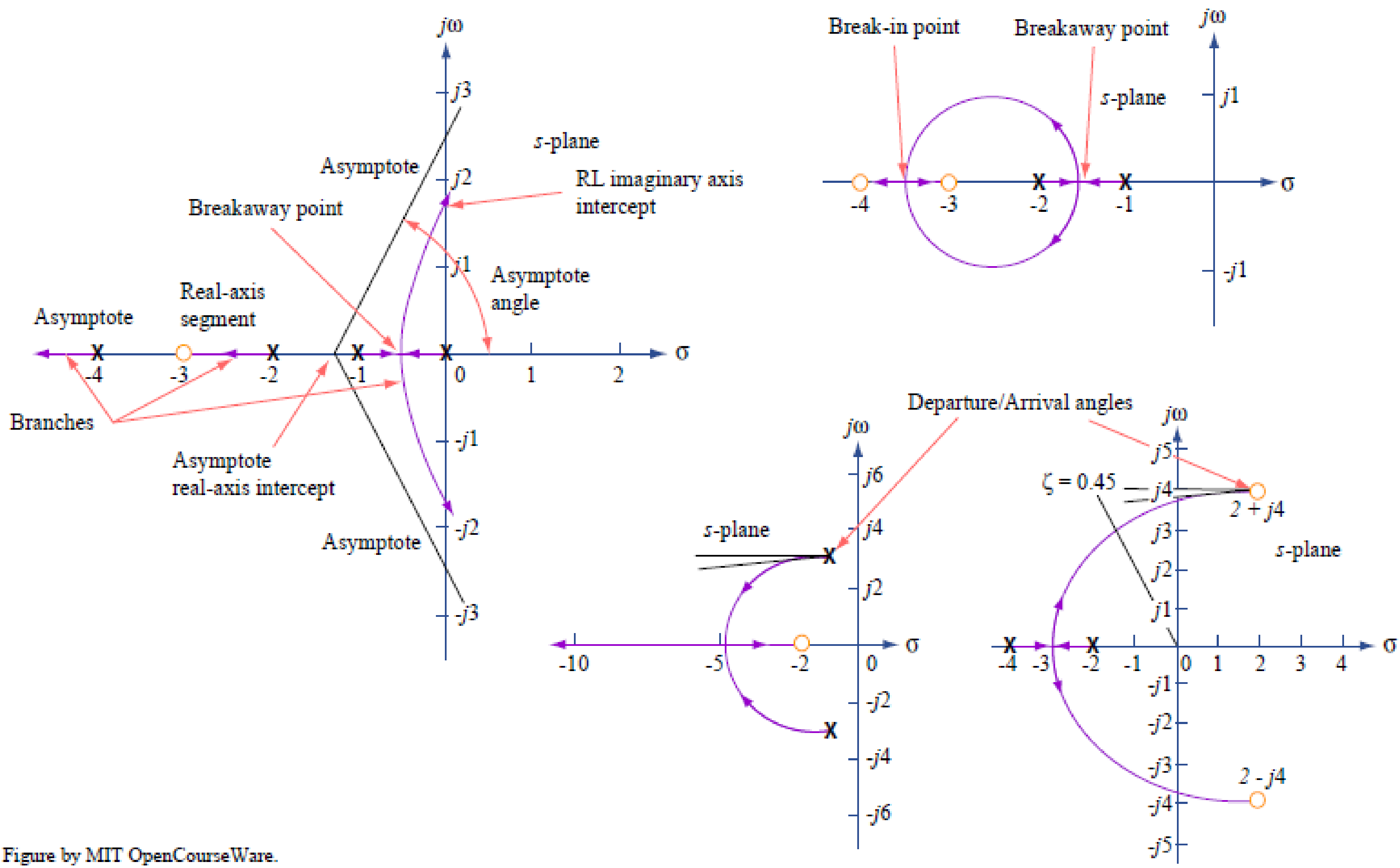


Figure by MIT OpenCourseWare.

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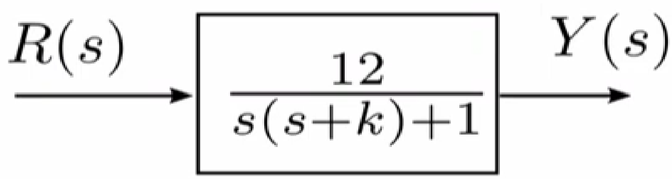
8.3 Properties of the root locus

- FB CL poles, [1, p. 388]

The unknown parameter k affects the location of poles and therefore the response of the system to an input.



What value of k should I choose to meet my system performance requirements?

If the value of k is exactly as predicted, what is the effect of a variation of k on my system?

Brute-force method:

Open-loop transfer function (OLTF)

Consider the closed-loop system in Fig. 4. The OLTF is $KG(s)H(s)$.

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

The angle of the complex number is an *odd* multiple of 180°

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ$$
$$k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

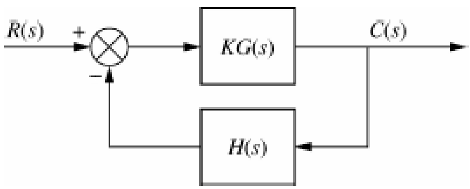


Figure: -FB system

The system gain, K, satisfies *magnitude criterion*

$$|KG(s)H(s)| = 1$$

angle criterion

$$\angle KG(s)H(s) = (2k+1)180^\circ$$

and thus

$$K = \frac{1}{|G(s)||H(s)|}$$

Figure 4: (a) Closed-loop system ; and (b) equivalent transfer function. Figure 8.1 of [Nise, 2015].

- Let the OLTF be of the form OLTF $\equiv K\frac{N(s)}{D(s)}$.
- Then, the system's closed-loop system's characteristic equation is in the form of :

$$1+\mathrm{OLTF}=1+K\frac{N(s)}{D(s)}=1+K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}=0,$$

$$\Rightarrow \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}=-\frac{1}{K}$$

- Any point on the RL must satisfy two conditions (amplitude condition and phase condition) :

i. $$\frac{\prod_{i=1}^{m}|(s-z_i)|}{\prod_{i=1}^{n}|(s-p_i)|}=\left|-\frac{1}{K}\right| \qquad (1)$$

ii. $$\sum_{i=1}^{m}\underline{/(s-z_i)}-\sum_{i=1}^{n}\underline{/(s-p_i)}=\underline{/-\frac{1}{K}} \qquad (2)$$

Root Locus (K > 0)

- When $K > 0$ the amplitude and phase conditions become :

i. $$\frac{\prod_{i=1}^{m} |(s-z_i)|}{\prod_{i=1}^{n} |(s-p_i)|} = \left|-\frac{1}{K}\right| = \frac{1}{K}$$

ii. $$\sum_{i=1}^{m} \angle(s-z_i) - \sum_{i=1}^{n} \angle(s-p_i) = \angle -\frac{1}{K} = (2q+1)180^\circ; \;\; q = 0, \pm 1, \pm 2, \ldots$$

8.4 sketching -FB RL, [1, p. 390]

Basic 5-rules to sketch a root locus **rapidly**.

- 1-Number of branches (Number of Loci): Equals the number of CL poles
- 2-Symmetry: About the real axis
- 3-Real-axis segments: On the real axis, for K >0, the RL exists to the left of an odd number of real-axis, finite OL poles and/or finite OL zeros
- 4-Starting and ending points: The RL begins at the finite & infinite poles of G(s)H(s) and ends at the finite & infinite zeros of G(s)H(s).
- 5-Behavior at ∞: The RL approaches straight lines as asymptotes as the RL approaches ∞. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a, and angle, θ_a, as follows,

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

where k = 0; ±1; ± 2; ± 3 and the angle is given in radians with respect to the positive extension of the real axis.

Example 8.2

Sketching a Root Locus with Asymptotes

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

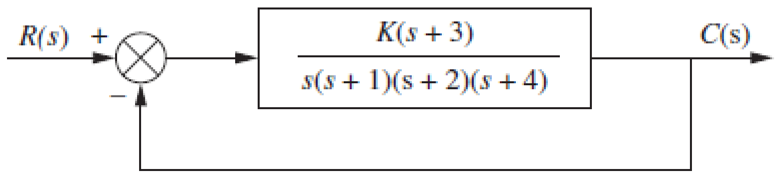


FIGURE 8.11 System for Example 8.2

Basic rules

- 1-Number of branches
- 2-Symmetry
- 3-Real-axis segments
- 4-Starting and ending points
- 5-Behavior at ∞

SOLUTION: Let us begin by calculating the asymptotes.

OLTF $\equiv K\dfrac{N(s)}{D(s)}$. Poles={0,-1,-2,-4} & zeros={-3}

$\forall s \in (0,\infty), = 0 \text{ (even)} \Rightarrow (0,\infty) \notin \text{RL}.$

$\forall s \in (-1,0), = 1 \text{ (odd)} \Rightarrow (-1,0) \in \text{RL}.$

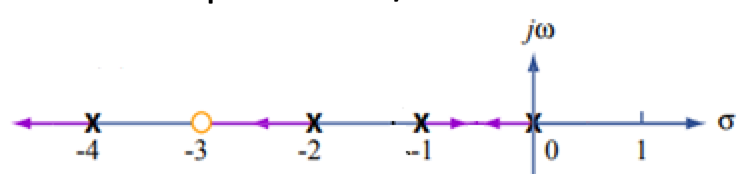
$\forall s \in (-2,-1), = 2 \text{ (even)} \Rightarrow (-2,-1) \notin \text{RL}.$

$\forall s \in (-3,-2), = 3 \text{ (odd)} \Rightarrow (-3,-2) \in \text{RL}.$

$\forall s \in (-4,-3), = 4 \text{ (even)} \Rightarrow (-4,-3) \notin \text{RL}.$

$\forall s \in (-\infty,-4), = 5 \text{ (odd)} \Rightarrow (-\infty,-4) \in \text{RL}.$

The real-axis segments lie to the left of an odd number of poles and/or zeros.



- The locus starts at the open-loop poles and ends at the open-loop zeros.
- There are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.

1-Number of branches

2-Symmetry

3-Real-axis segments

4-Starting and ending points

5-Behavior at ∞

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

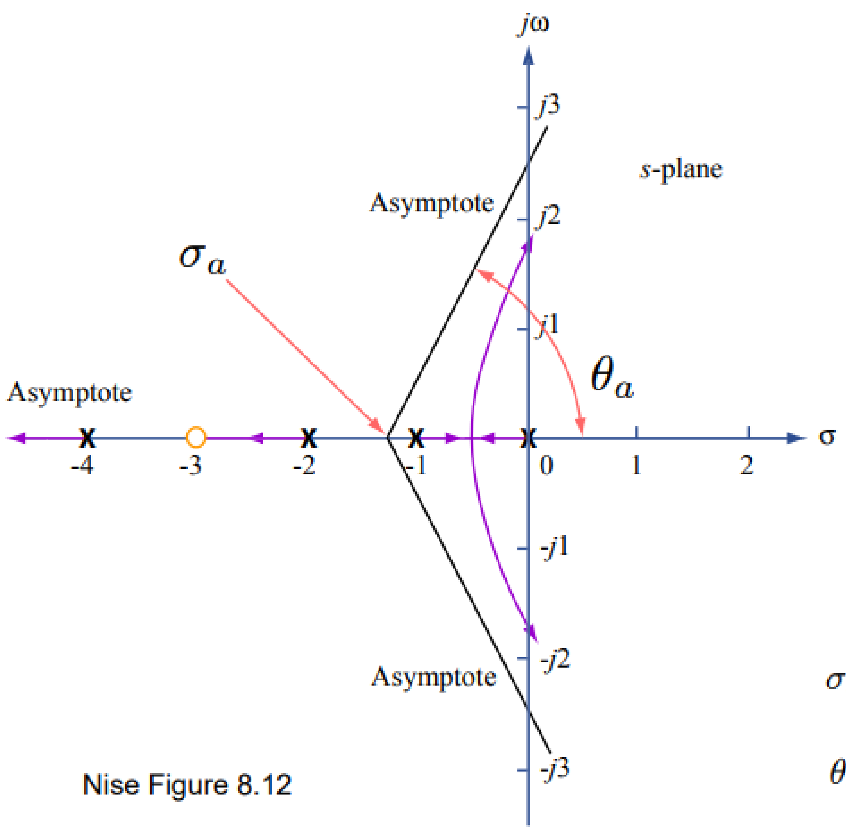
$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$\begin{aligned} &= \pi/3 && \text{for } k = 0 \\ &= \pi && \text{for } k = 1 \\ &= 5\pi/3 && \text{for } k = 2 \end{aligned}$$

If the value for *k* continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

$$\theta_a = \pi/3 \qquad \text{for } k = 3$$



Nise Figure 8.12

In this example, poles = $\{0, -1, -2, -4\}$, zeros = $\{-3\}$ so

$$\sigma_a = \frac{[0 + (-1) + (-2) + (-4)] - [(-3)]}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2m+1)\pi}{4-1} = \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$$

8.5 Refining the Sketch -FB RL, [1, p. 395]

Additional rules for refining a RL sketch

- 6-Real-axis breakaway & break-in points: At the breakaway or break-in point, the branches of the RL form an angle of 180/n with the real axis, where n is the number of CL poles arriving at or departing from the single breakaway or break-in point on the real-axis.
- 7-The $j\omega$-axis crossings: The $j\omega$-crossing is a point on the RL that separates the stable operation of the system from the unstable operation.
- 8-Angles of departure & arrival: The value of ω at the axis crossing yields the frequency of oscillation, while the gain, K, at the $j\omega$ -axis crossing yields the maximum or minimum positive gain for system stability.
- 9-Plotting & calibrating the RL: All points on the RL satisfy the angle criterion, which can be used to solve for the gain, K, at any point on the RL.

Rule6 (a) Breakaway and Break-in Points via Differentiation

6-Break in/out point

7-$j\omega$-axis crossings

8-Angles of dep& arriv

9-9-Plotting & calibrating

- Procedure
- Maximize & minimize the gain, K, using differential calculus: The RL breaks away from the real-axis at a point where the gain is maximum and breaks into the real-axis at a point where the gain is minimum. For all points on the RL

$$K = -\frac{1}{G(s)H(s)}$$

For points along the real-axis segment of the RL where breakaway and break-in points could exist, s = σ. Differentiating with respect to σ and setting the derivative equal to zero, results in points of maximum and minimum gain and hence the breakaway and break-in points.

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

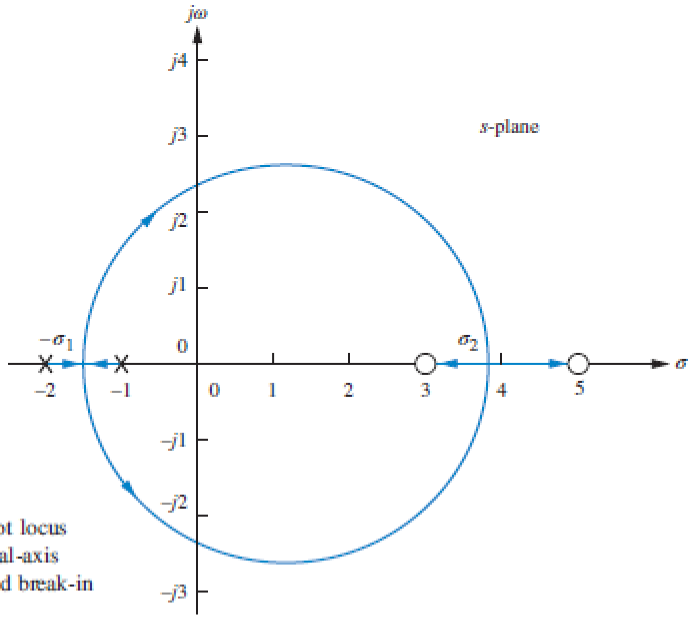


FIGURE 8.13 Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \tag{8.33}$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1 \tag{8.34}$$

Solving for K, we find

$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} \tag{8.35}$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma-61)}{(\sigma^2-8\sigma+15)^2} = 0 \tag{8.36}$$

$$11x^2 - 26x - 61 = 0$$

Solve for x

Solve the equation for x by finding a, b, and c of the quadratic then applying the quadratic formula.

Exact Form:

$$x = \frac{13 + 2\sqrt{210}}{11}, \frac{13 - 2\sqrt{210}}{11}$$

Decimal Form:

$$x = 3.81661395\ldots, -1.45297759\ldots$$

Solving for σ, we find $\sigma = -1.45$ and 3.82, which are the breakaway and break-in points.

Rule6. (b) Breakaway and Break-in Points via Transition procedure

Procedure

- Eliminates the need to differentiate. Breakaway and break-in points satisfy the relationship

$$\sum_{1}^{m} \frac{1}{\sigma + z_i} = \sum_{1}^{n} \frac{1}{\sigma + p_i}$$

where zi and pi are the negative of the zero and pole values, respectively, of G(s)H(s).

Example 8.4

Breakaway and Break-in Points Without Differentiation

PROBLEM: Repeat Example 8.3 without differentiating.

SOLUTION: Using Eq. (8.37),

$$\frac{1}{\sigma-3}+\frac{1}{\sigma-5}=\frac{1}{\sigma+1}+\frac{1}{\sigma+2} \quad (8.38)$$

Simplifying,

$$11\sigma^2-26\sigma-61=0 \quad (8.39)$$

Hence, $\sigma=-1.45$ and 3.82, which agrees with Example 8.3.

$$\frac{1}{x-3}+\frac{1}{x-5}=\frac{1}{x+1}+\frac{1}{x+2}$$

Solve for x

Solve the rational equation by combining expressions and isolating the variable x.

Exact Form:

$$x=\frac{13+2\sqrt{210}}{11},\frac{13-2\sqrt{210}}{11}$$

Decimal Form:

$$x=3.81661395\ldots,-1.45297759\ldots$$

Tap to view steps...

Rule7. The jω-crossings, [1, p. 399]

6-Break in/out point

7-jω-axis crossings

8-Angles of dep& arriv

9-9-Plotting & calibrating

Procedures for finding jω-crossings

- Using the Routh-Hurwitz criterion, forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

- At the jω-crossing, the sum of angles from the finite OL poles & zeros must add to (2k + 1)180

Example 8.5-Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, *K*, for which the root locus crosses the imaginary axis. For what range of *K* is the system stable?

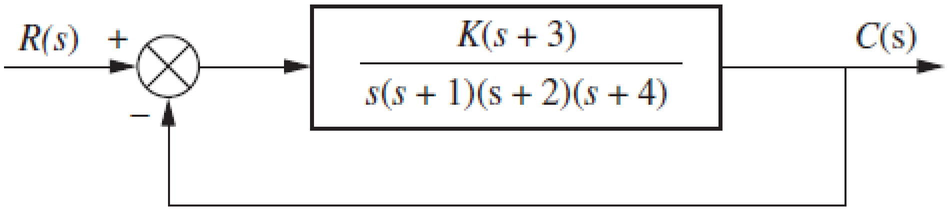


FIGURE 8.11

SOLUTION: The closed-loop transfer function for the system

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the $s1$ row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0$$

K has two numbers, but we need positive as, **$K = 9.65$**

TABLE 8.3 Routh table for Eq (8.40)

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

Forming the even polynomial by using the $s2$ row with K =9.65, we obtain

$$(90-k)s^2 + 21k = 80.35s^2 + 202.7 = 0$$

s is found to be equal to **$\pm j1.59$**.

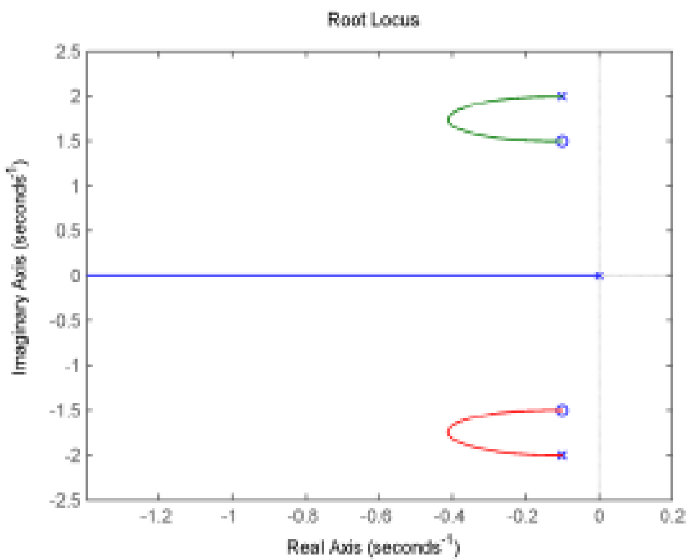
Rule8. Angles of departure & arrival, [1, p. 400]

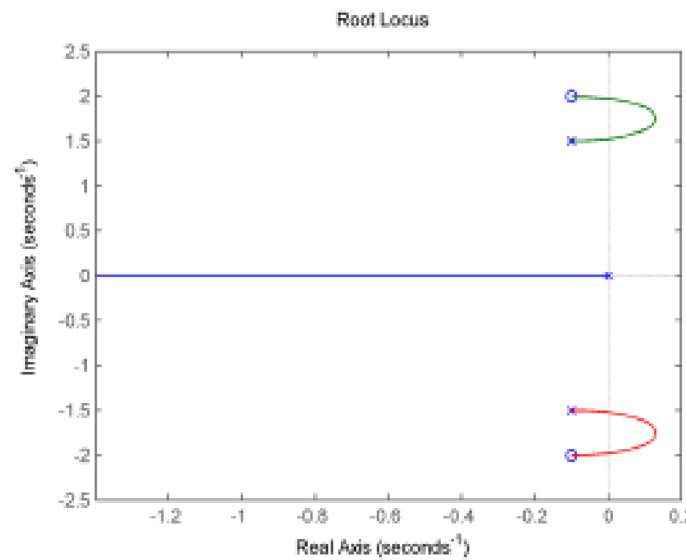
6-Break in/out point
7-$j\omega$-axis crossings
8-Angles of dep& arriv
9-9-Plotting & calibrating

Consider the following two systems

$$G_1(s) = \frac{s^2+0.2s+2.26}{s^2+0.2s+4.01}$$

$$G_2(s) = \frac{s^2+0.2s+4.01}{s^2+0.2s+2.26}$$





Similar systems, with very different stability behavior

⇒ Understanding how to determine ***angles of departure*** from complex poles and ***angles of arrival*** at complex zeros will allow us to predict this

Angle of Departure

- To find the angle of departure from a pole, p_1:
  - Consider a test point, s_0, very close to p_1
  - The angle from p_1 to s_0 is ϕ_1
  - The angle from all other poles/zeros, ϕ_i/ψ_i, to s_0 are approximated as the angle from p_i or z_i to p_1
  - Apply the angle criterion to find ϕ_1

$$\sum_{i=1}^{m}\psi_i-\phi_1-\sum_{i=2}^{n}\phi_i=(2i+1)180^\circ$$

- Solving for the departure angle, ϕ_1:

$$\boxed{\phi_1=\sum_{i=1}^{m}\psi_i-\sum_{i=2}^{n}\phi_i-180^\circ}$$

- In words:

$$\boxed{\phi_{depart}=\Sigma\angle(zeros)-\Sigma\angle(other\ poles)-180^\circ}$$

Angles of departure & arrival, example

The RL departs from complex, OL poles and arrives at complex, OL zeros

Example 8.6

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

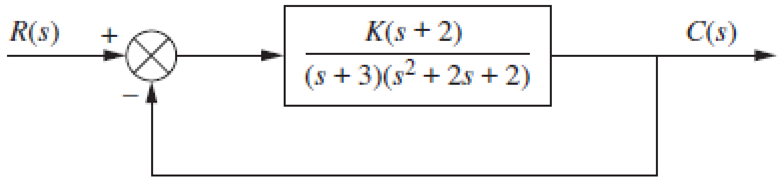


FIGURE 8.16 Unity feedback system with complex poles

Departure: from complex poles. ***Arrival***: to complex zeros.

Angle of departure: Zero: -2 Poles:-3, -1+j, -1- j

we calculate the sum of angles drawn to a point close to the complex pole, -1 + j

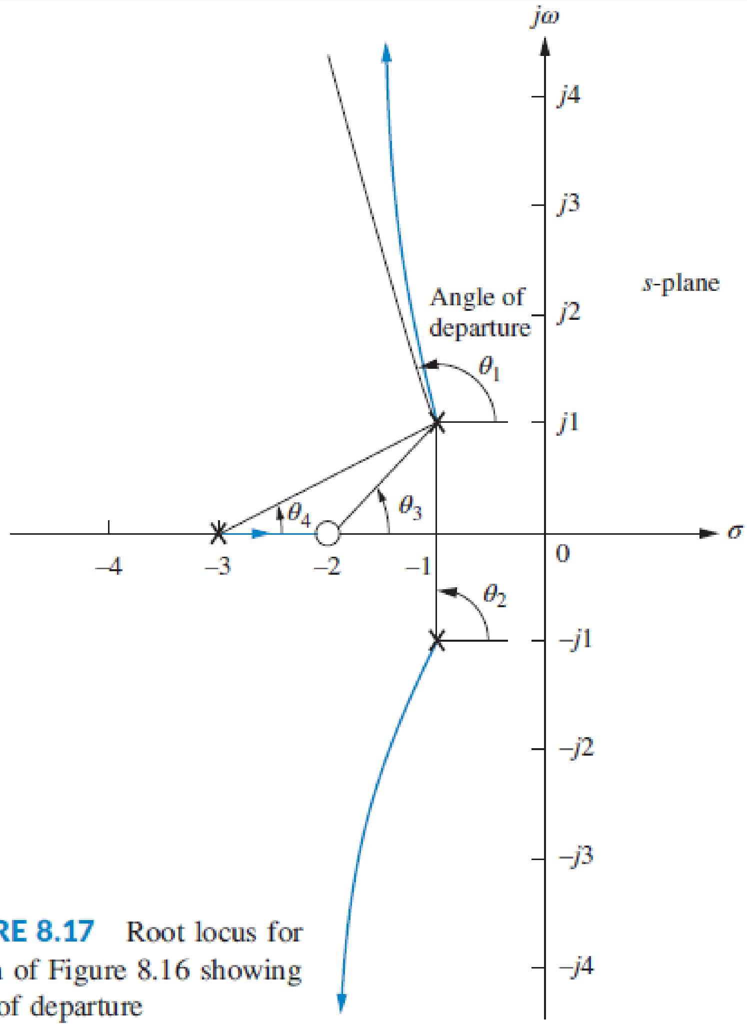


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

Sum (zero angles) – Sum (pole angles) = (2k+1)180°

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4$$

$$= -\theta_1 - 90° + \underbrace{\tan^{-1}\left(\frac{1}{1}\right)}_{\theta_3} - \underbrace{\tan^{-1}\left(\frac{1}{2}\right)}_{\theta_4} = 180°$$

from which $\theta = -251.6° = 108.4°$.

A sketch of the root locus is shown in Figure 8.17.

Notice how the departure angle from the complex poles helps us to refine the shape.

Procedure for Sketching the Root Locus

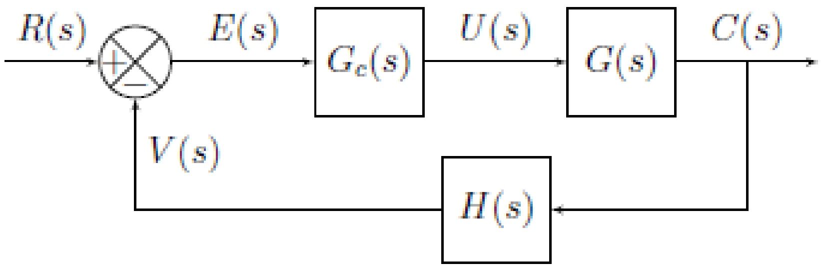
1. Obtain the system's characteristic equation and write it in the form of $1 + KGH(s) = 0$, where K is the parameter of interest.

   Then, determine the poles and zeros of the OLTF $KGH(s)$.
2. Position the poles and zeros of the OLTF on the s-plane and mark them with 'x' and 'o', respectively.
3. Draw the segments of the RL on the real axis.
4. Determine the location of the break-in and break-away points on the real axis, if any.
5. Calculate the points of intersection with the imaginary axis (critical frequency), if any.
6. Compute the angles of departure from complex poles and the angles of arrival to complex zeros, if any.
7. Determine the number of asymptotic directions.
8. Determine the angles of the asymptotes and their intersection point on the real axis.

More

Problem1. Consider the following system, where

$$G_c(s) = \frac{K}{s},\; H(s) = (s+3),\; G(s) = \frac{1}{(s+2)(s^2+2s+2)},$$ and $K > 0$



- Sketch the root locus of this system, as precisely as possible, as K varies in the interval (0, ∞).

1-Number of branches
2-Symmetry
3-Real-axis segments
4-Starting and ending points
5-Behavior at ∞

A locus (branch) is the virtual path of one pole of the closed-loop system on the s plane.

$$1 + k\frac{s+3}{s(s+2)(s^2+2s+2)} = 1 + OLTF \equiv 1 + K\frac{N(s)}{D(s)} = 0 \text{ (right form)}$$

The number of loci is : $\max\{n, m\}$ = 4

one zero, $z_1 = -3 \quad (m = 1)$, and

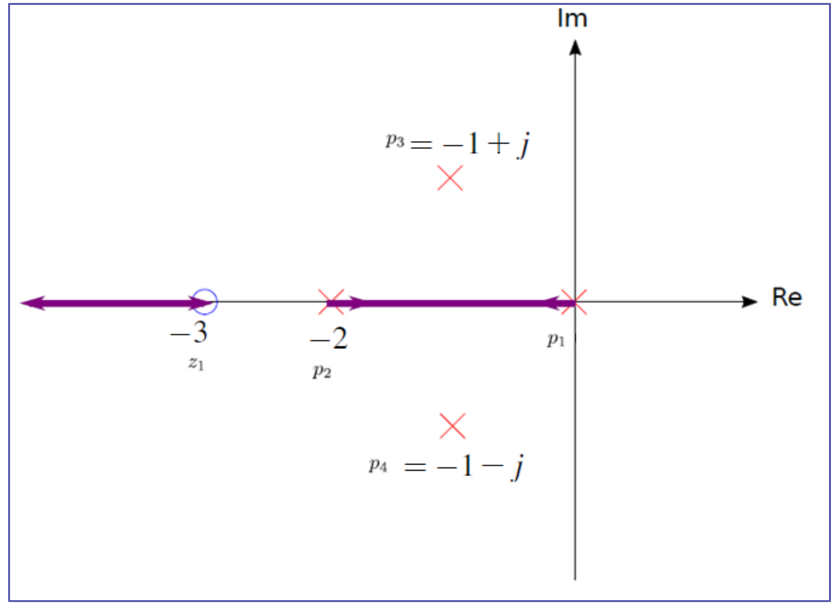
four poles, $p_1 = 0,\ p_2 = -2,\ p_{3,4} = -1 \pm j \quad (n = 4)$

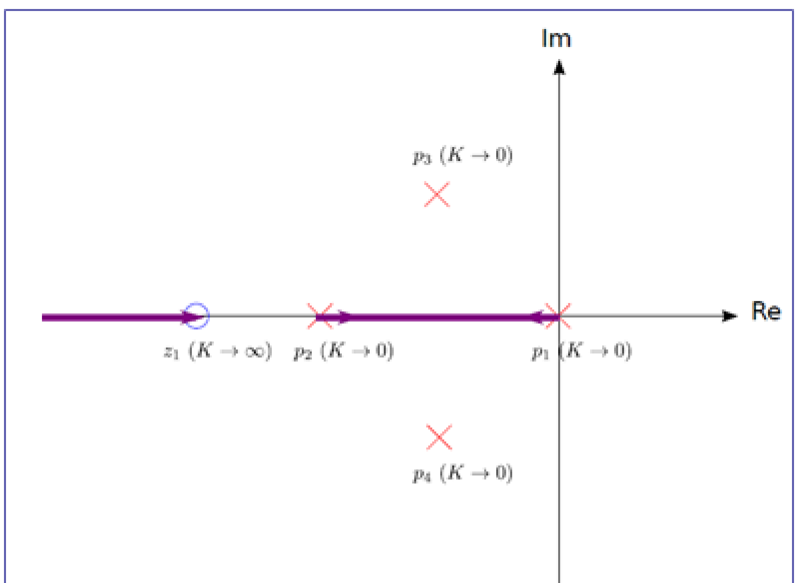
1-Number of branches
2-Symmetry
3-Real-axis segments
4-Starting and ending points
5-Behavior at ∞

The complex roots of the characteristic equation are in the form of conjugate pairs.

Non-Real-Axis Root-Locus Segments

1-Number of branches
2-Symmetry
3-Real-axis segments
4-Starting and ending points
5-Behavior at ∞





In general: The branches start at the open-loop poles and end at the open-loop zeros.

1-Number of branches
2-Symmetry
3-Real-axis segments
4-Starting and ending points
5-Behavior at ∞

1-Number of branches

2-Symmetry

3-Real-axis segments

4-Starting and ending points

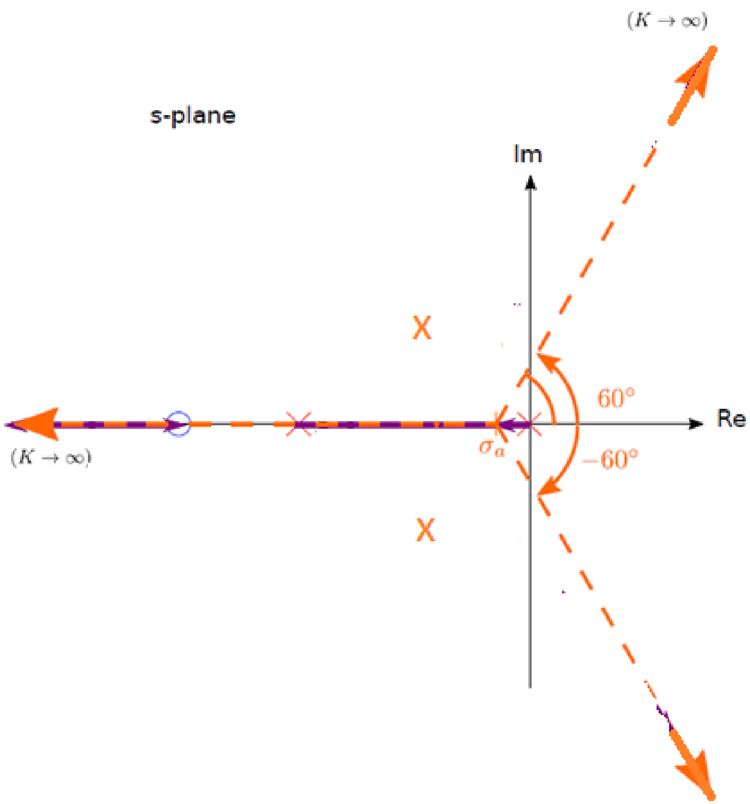
5-Behavior at ∞

If $n > m$, $(n-m)$ branches of the root locus take asymptotic directions as $K \rightarrow \infty$.

- Number of asymptotic branches : $n-m$
- All asymptotes intersect at a single point σ_a on the real axis :

$$\sigma_a = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

$$\sigma_d = \frac{-2-1-j-1+j+3}{4-1} = -\frac{1}{3}$$



- Angles of asymptotes :

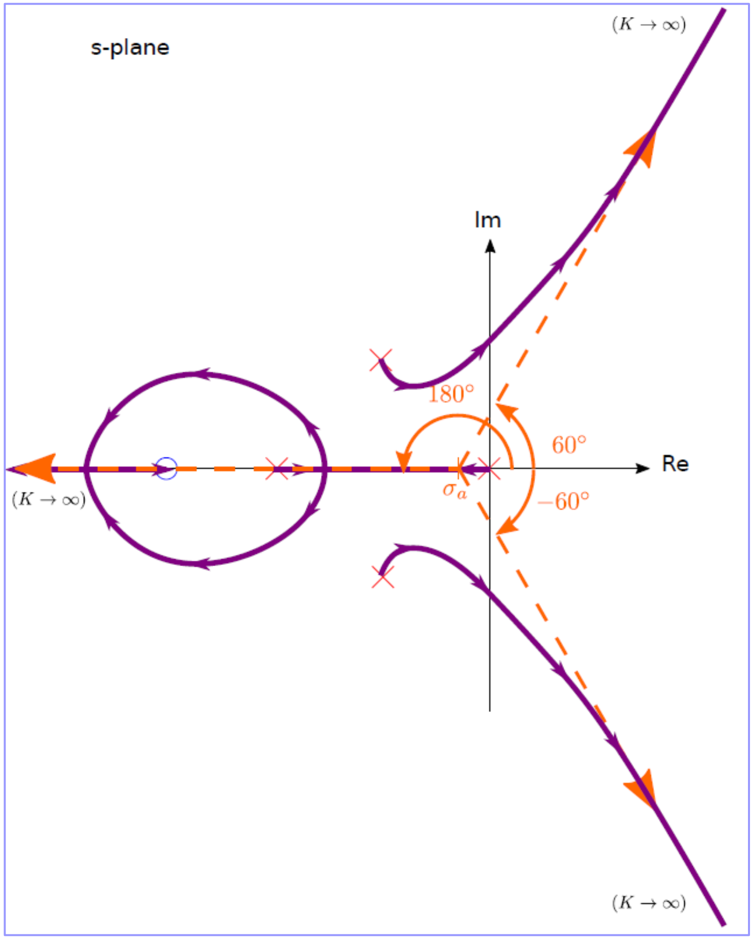
$$\theta_a = \frac{(2q+1)180^\circ}{(n-m)}, \quad q = 0, \pm 1, \pm 2, \ldots$$

= 60	when q=0
=180	when q=1
=-60	when q=2

6-Break in/out point
7-$j\omega$-axis crossings
8-Angles of dep& arriv
9-9-Plotting & calibrating

$$\frac{dk}{d\sigma} = \frac{1}{\frac{\sigma+3}{\sigma(\sigma+2)(\sigma^2+2\sigma+2)}} = 0$$

Graph each side of the equation.
$\sigma \approx -3.64872899, -1.53909438$



6-Break in/out point
7-$j\omega$-axis crossings
8-Angles of dep& arriv
9-9-Plotting & calibrating

$$\boxed{\frac{20-k}{4}s^2 + 3k = 0}$$

S=$\pm\sqrt{1.55}$

$$T(s) = \frac{k}{s^4 + 4s^3 + 6s^2 + (4+k)s + 3k}$$

s^4	1	6	3k
s^3	4	4+k	
s^2	(20-k)/4	3k	
s	$\frac{-k^2-32k+20}{20-k}$		

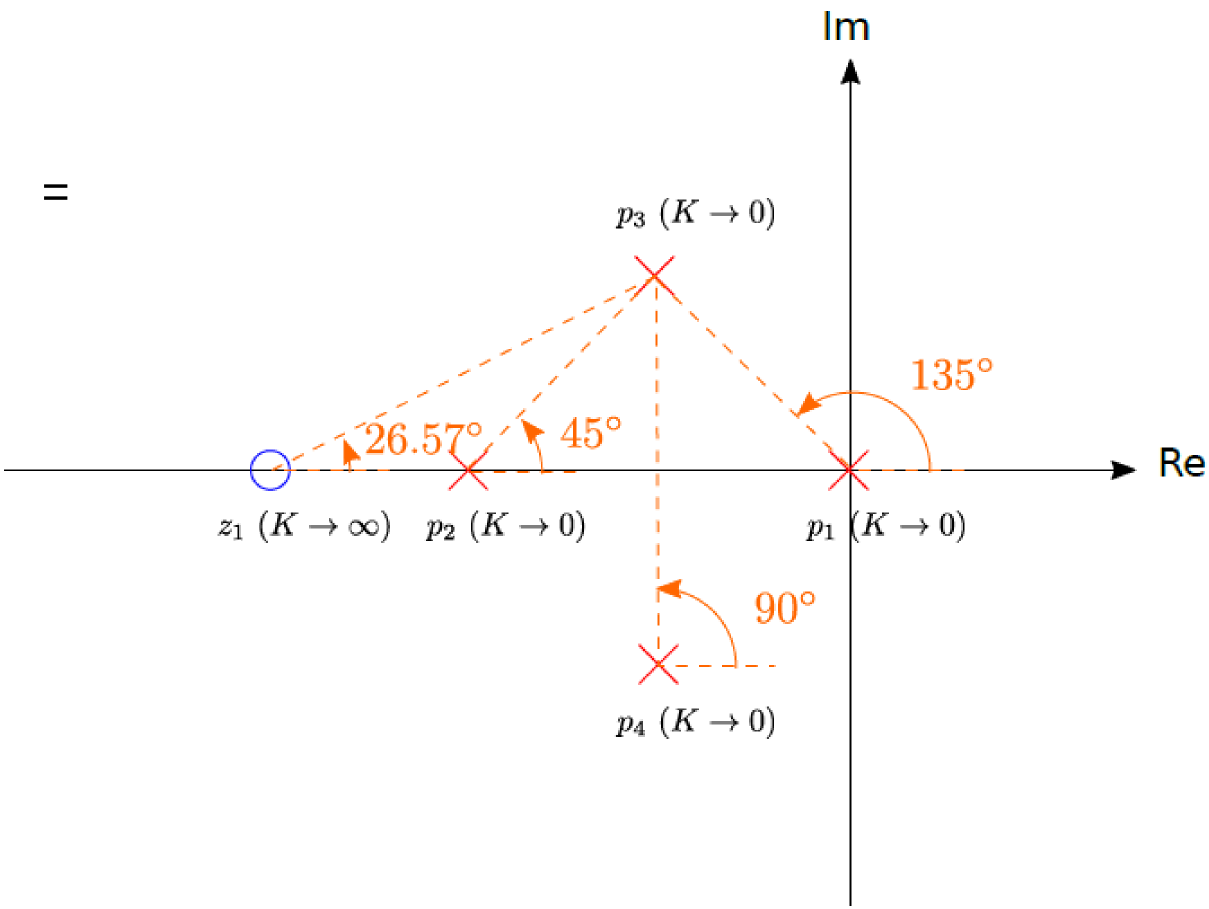
$$-k^2 - 32k + 20 = 0$$

K= 2.3 or -34.32 So we choose the positie

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) - 90 - \tan^{-1}\left(\frac{1}{1}\right) - \left(180 - \tan^{-1}\left(\frac{1}{1}\right)\right) - 180$$

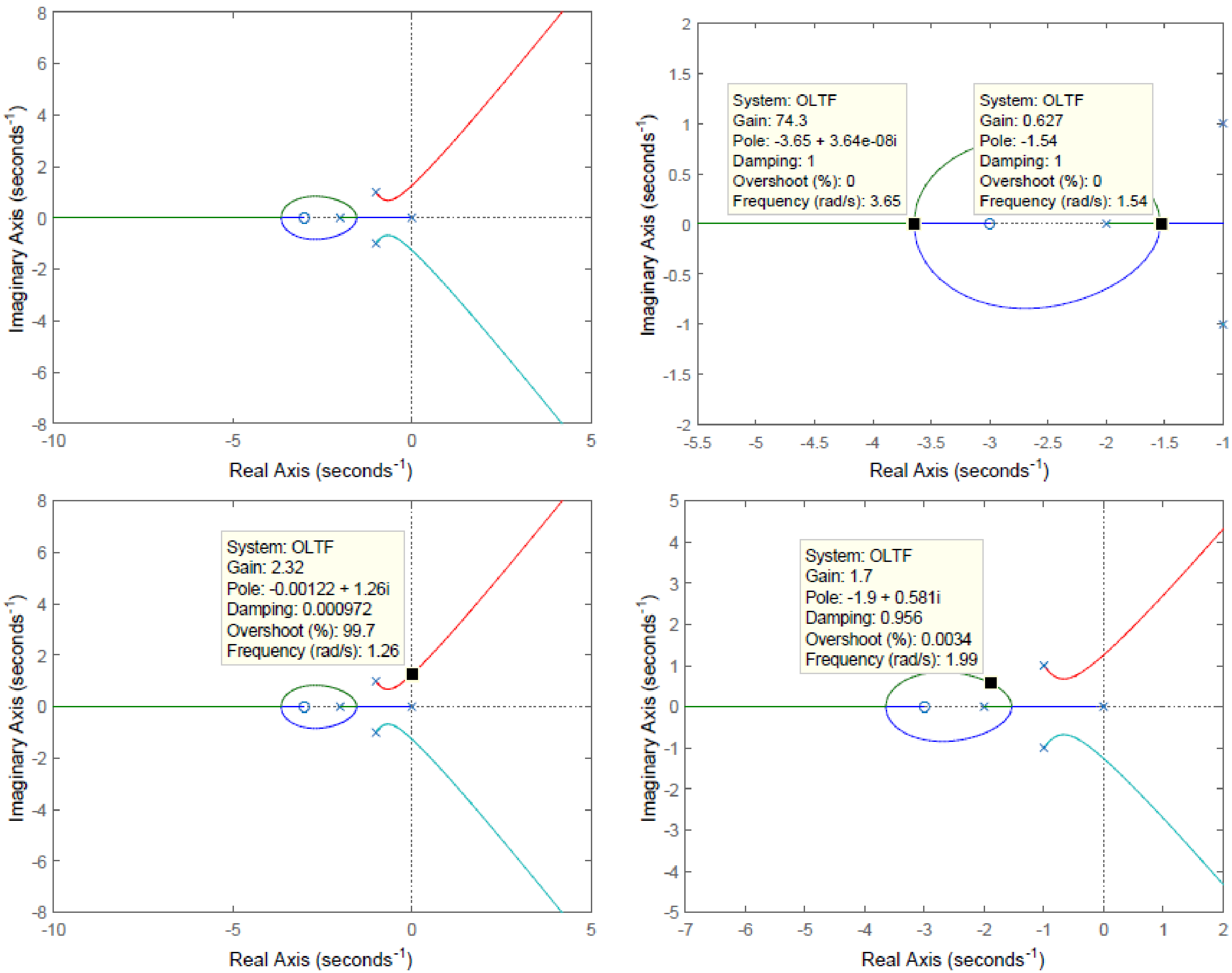
$\theta =$ 26.57-90-45-135-180 =

$=296.57^0$

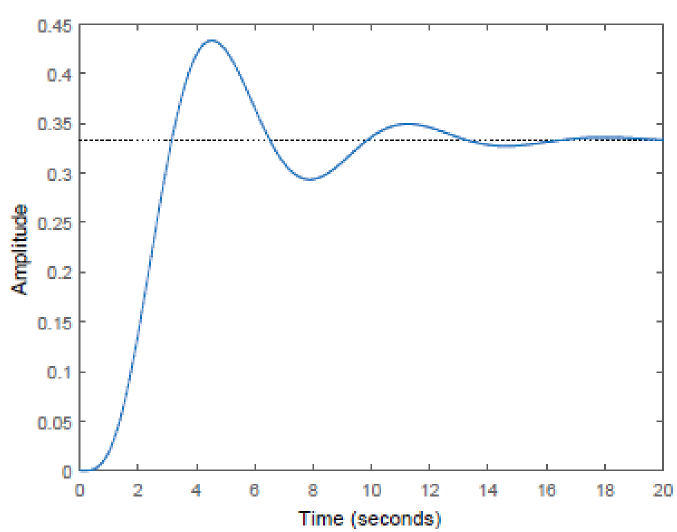


Verification: Figures 6 and 7 confirm the validity of the manually-drawn root locus.

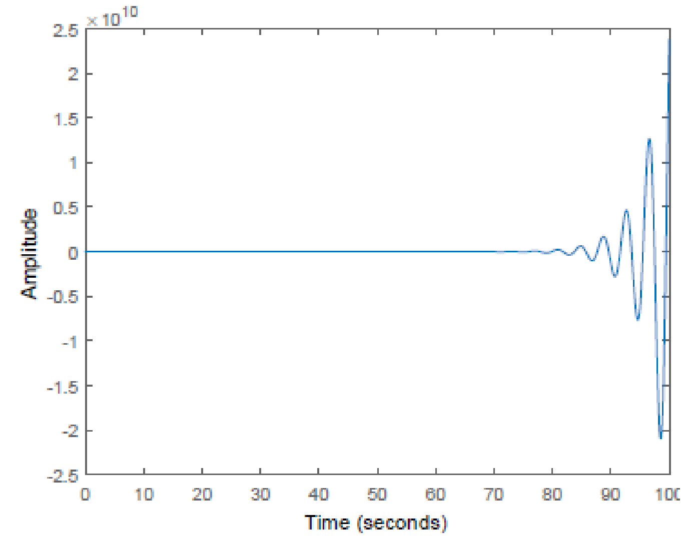
Figure 6: (upper-left) exact root locus plot, (upper-right) break-in and break-away points, (lowerleft) intersection with the imaginary axis, (lower-right) gain calculation.



Verification (cont'd)

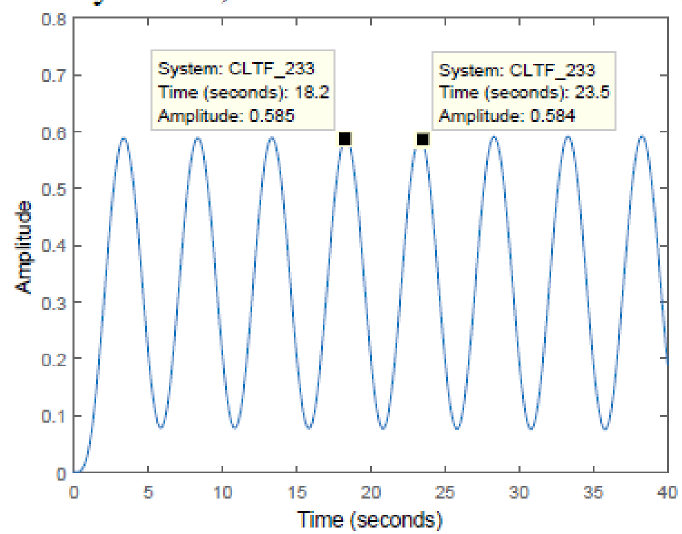


$K = 1$ (asymptotically stable)



$K = 5$ (unstable)

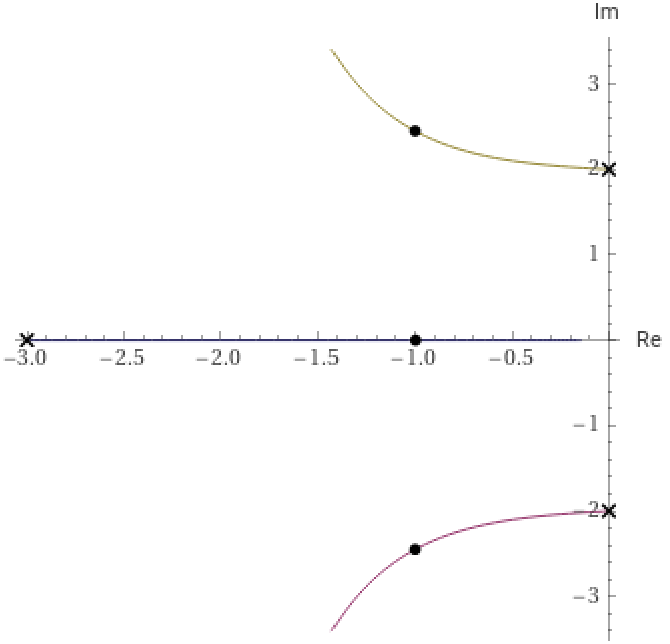
Figure 7: Unit step responses with various gains (showing system stability).



$K = 2.33$ (marginally stable)

Problem 2. For each of the following OPLTF, sketch the root loci for the system. Find for each the (i) number of branches in the root locus, (ii) number of asymptotes, and give the angles and centers, (iii) the break-in or break-away points, if any, and (iv) angles of departure from poles above the real axis.

(a) $\dfrac{s-1}{(s^2+4)(s+3)}$ (b) $\dfrac{1}{s^2+3}$ (c) $\dfrac{(s+4)(s+3)}{(s+5)(s+8)}$



(shown for gain between 0 and 10)

Solution

(a) Poles= {-3, ±j2}, zeroes={1}

(i) 3 branches. $\max\{n, m\}$

(ii) 2 asymptotes, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, centered\ at\ -2$

(iii) no break-in/break-away points

(iv) angle of departure from pole at $s = 2i$ is $172.9°$

The sum of the angles from all poles and zeros to a point near the complex pole must add up to odd multiples of π. $(180° - \tan^{-1}(2/1)) - 90° - \tan^{-1}(2/3) - \theta = 180°$ so $\theta = -187.1° = 172.9°$ (mod $180°$).

(b)

(b) $\dfrac{1}{s^2+3}$

Poles=$\{\pm j\sqrt{3}\}$

(i) 2 branches

(ii) 2 asymptotes, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, centered at 0

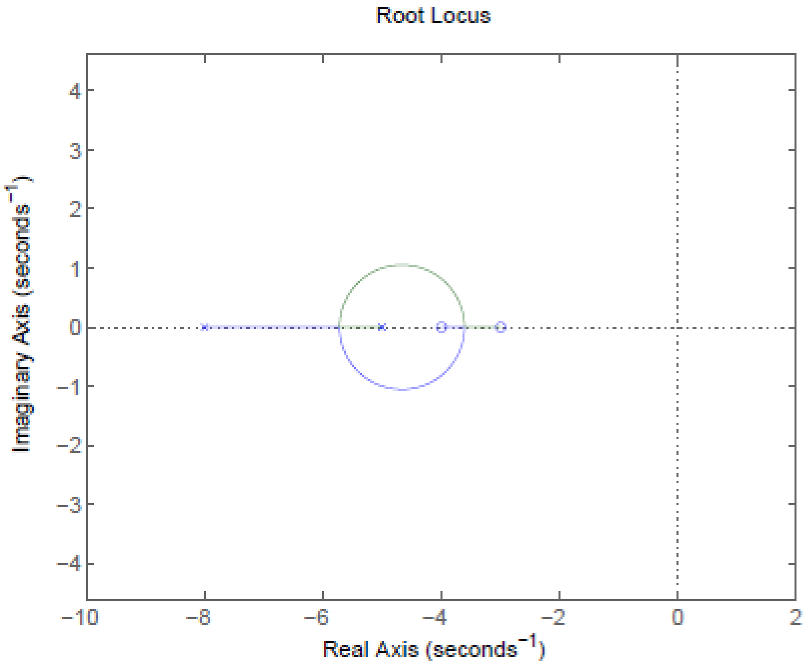
(iii) no break-in/break-away points

(iv) angle of departure from pole at $s = \sqrt{3}i$ is 90°
$-90^\circ - \theta = 180^\circ$ so $\theta = -270^\circ = 90^\circ \pmod{180^\circ}$.

(c)

(c) $\dfrac{(s+4)(s+3)}{(s+5)(s+8)}$

Poles={-5,-8}
Zeros={-4,-3}



(i) 2 branches

(ii) no asymptotes

(iii) break-away at $s = -5.72$, break-in at $s = -3.61$

Break-away occurs when $K(\sigma) = -\dfrac{1}{G(\sigma)} = -\dfrac{(s+5)(s+8)}{(s+4)(s+3)}$ is maximum, that is when $\dfrac{\mathrm{d}K}{\mathrm{d}\sigma} = 0.$

(iv) no departure from complex poles

Problem 8 chapter8 [Nise, p 425]

8. The characteristic polynomial of a feedback control system, which is the denominator of the closed-loop transfer function, is given by $s^3+2s^2+(20K+7)s+100K$. Sketch the root locus for this system. [Section: 8.8]

Solution

$$s^3+2s^2+(20K+7)s+100K=0$$

When this CE is put into "root locus" for versus parameter K, it is

$$1+K\frac{20(s+5)}{s\underbrace{\left(s^2+2s+7\right)}_{s+1\pm j2.45}}=0$$

Real-axis branches. As shown on the sketch, there is a real axis branch between the pole at $s = 0$ and the zero at $s = -5$.

Asymptote intersection and angle. Since there are 3 poles and 1 zero, there are 2 asymptotes. The asymptote angle and intersection are:

$$\boxed{\theta_a = \frac{\pm 180^\circ}{3-1} = \pm 90^\circ, \quad \sigma_a = \frac{-1-1-(-5)}{3-1} = \frac{3}{2} = 1.5}$$

Imaginary axis crossing point and corresponding K. Substituting $s = j\omega$ into the characteristic equation

$$j\omega(-\omega^2 + j2\omega + 7) + K[20(j\omega + 5)] = 0.$$

so

$$j\omega^3 - 2\omega^2 + j7\omega + j20K\omega + 100K = 0.$$

yields a Real equation and an Imag equation:

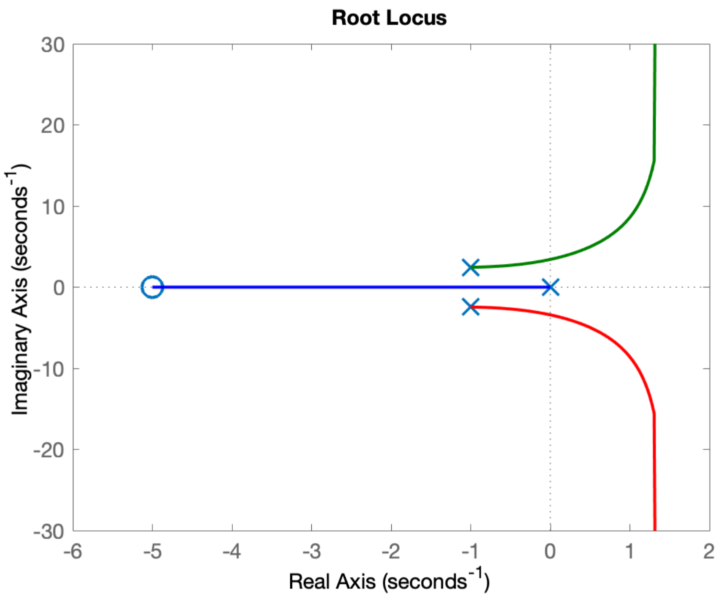
$$\begin{aligned} \text{Real:} &\quad -2\omega^2 + 100K = 0 \\ \text{Imag:} &\quad -\omega^3 + 7\omega + 20K\omega = 0 \end{aligned}$$

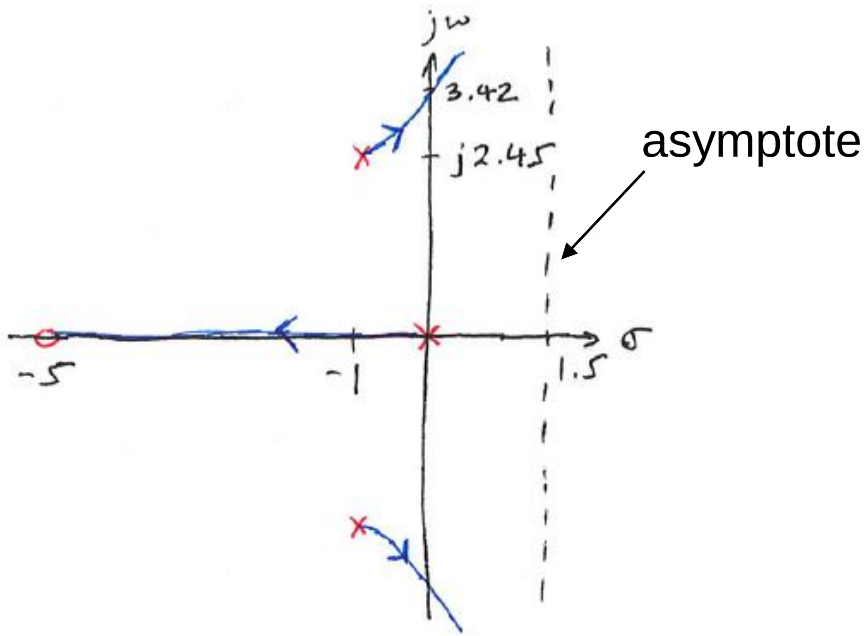
simultaneously (not hard), you find that

$$\boxed{K = \frac{7}{30} = 0.2333, \quad \omega = \sqrt{50K} = 3.42}$$

Departure angle from complex poles. Let $s \approx -1 + j2.45$ (near the upper complex pole), and evaluate the angle condition (angle of vectors from poles to s minus angle of vectors from zeros to s):

$$\boxed{112.2^\circ + \theta + 90^\circ - 31.5^\circ = \pm 180^\circ \quad \Longrightarrow \quad \theta = 9.3^\circ}$$





References

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