Chapter3: Modeling in the Time Domain

ELG 3155 : Introduction to Control Systems

- Objective:
  - state-space representation,
  - Model electrical and mechanical systems in state space.
  - Convert a transfer function to state space.
  - Convert a state-space representation to a transfer function.
  - Linearize a state-space representation.

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Outline

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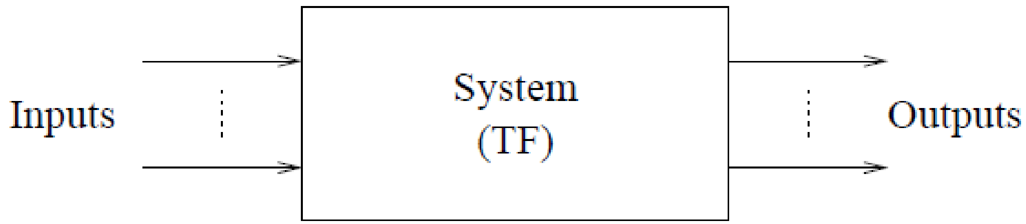
Outlets

- Learn how to find a state-space representation of an LTI system
- Learn how to convert between the transfer function and a state space of a system

1. Introduction

There are two fundamental mathematical models to represent any given system.

- An INPUT-OUTPUT model, using the transfer function to describe the system (see Chapter 2).



This is known as an external description of the system.

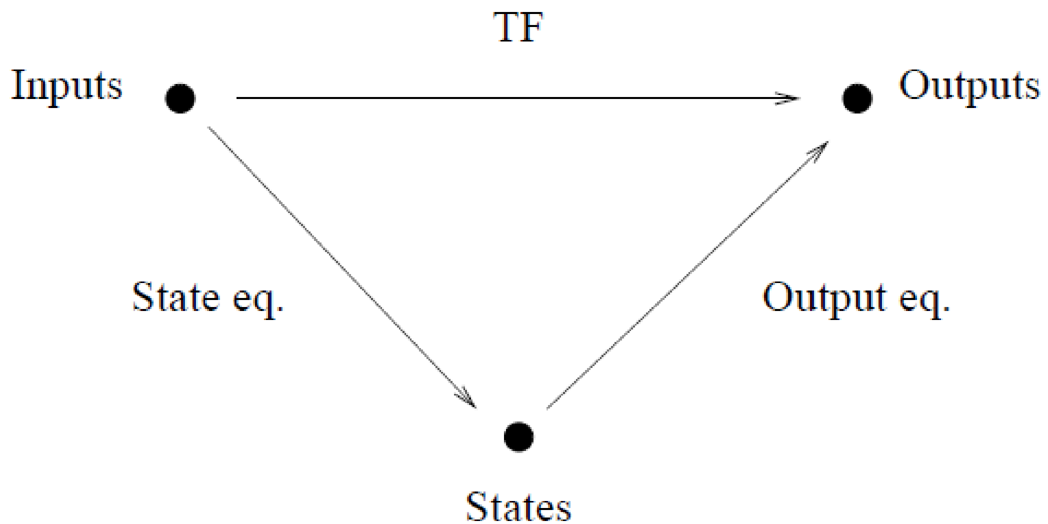
This type of modeling can be applied to linear systems only.

- State space: provides an internal description of the system.

Such type of modeling is more general (inclusive) than input-output models as it can be applied to linear and nonlinear systems.

2. Concept of a State

The concept of a state is based on the possibility of obtaining the future behavior of a system knowing its current behavior and all its input signals.



Definition (State)

The state of a system at instant t_0 is the minimum amount of information on its past ($t < t_0$) necessary to define its future behavior ($t \geq t_0$) knowing its input value $u(t)$; $t \geq t_0$.

SS representation

Procedure

1. System variables: Select a subset of all possible system variables as states and determine inputs & outputs.

2. State differential equations: Write n simultaneous, first-order DEs of the states in terms of the states and inputs for an nth-order system.

3. Initial conditions: If we know the initial conditions of all the states at t_0 as well as the inputs for $t \geq t_0$, we can solve the simultaneous Des for the states for $t \geq t_0$.

4. Output-state relation equations: Write linear relations of the outputs in terms of the states and inputs for $t \geq t_0$.

5. State space (SS) representation: The state and output equations represent a viable representation of the system.

3. The General State-Space Representation

In general, a state-space representation of a system is in the form

$$\begin{cases} \dot{x}(t) = g\big(x(t), u(t), t\big) & \leftarrow \quad \textit{state eq.} \\ y(t) = h\big(x(t), u(t), t\big) & \leftarrow \quad \textit{output eq.} \end{cases}$$

for some linear or nonlinear functions *g* and *h*, where

$x \equiv$ state vector of dimension n ($n =$ system order)

$\dot{x} \equiv \dfrac{dx}{dt}$

$u \equiv$ input vector (input to the system to control)

$y \equiv$ output vector

For a linear system, the functions g and h are linear, and the state-space model takes the following form:

$$\begin{cases} \dot{x}(t) = g\big(x(t), u(t), t\big) = Ax(t) + Bu(t) & \leftarrow \quad \textit{state eq.} \\ y(t) = h\big(x(t), u(t), t\big) = Cx(t) + Du(t) & \leftarrow \quad \textit{output eq.} \end{cases}$$

Nice Linear representation

g => nx1 vector (system equation).
h => px1 vector (system equation).

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- *State equation:* A set of n simultaneous, first-order DEs that expresses the time derivatives of the n states of a system as linear combinations of the states and inputs.

$$\dot{x} = Ax + Bu$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}; A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}; B = \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,m} \end{bmatrix}$$

- *Output equation:* An equation that expresses the measured output variables of a system as linear combinations of the states and inputs.

$$y = Cx + Du$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}; C = \begin{bmatrix} c_{1,1} & \dots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{p,1} & \dots & c_{p,n} \end{bmatrix}; D = \begin{bmatrix} d_{1,1} & \dots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{p,1} & \dots & d_{p,m} \end{bmatrix}$$

$\dot{x} \in \mathbb{R}^n$	time derivative of state vector
$x \in \mathbb{R}^n$	state vector
$u \in \mathbb{R}^m$	control input vector
$y \in \mathbb{R}^p$	measured output vector
$A \in \mathbb{R}^{n \times n}$	system matrix
$B \in \mathbb{R}^{n \times m}$	input matrix
$C \in \mathbb{R}^{p \times n}$	output matrix
$D \in \mathbb{R}^{p \times m}$	feedforward matrix

Definition (State of equilibrium)

A state of equilibrium (steady state) is a state where $\dot{x}(t) = 0$

A linear system with the above state-space representation, can be represented by the block diagram in Fig. 1.

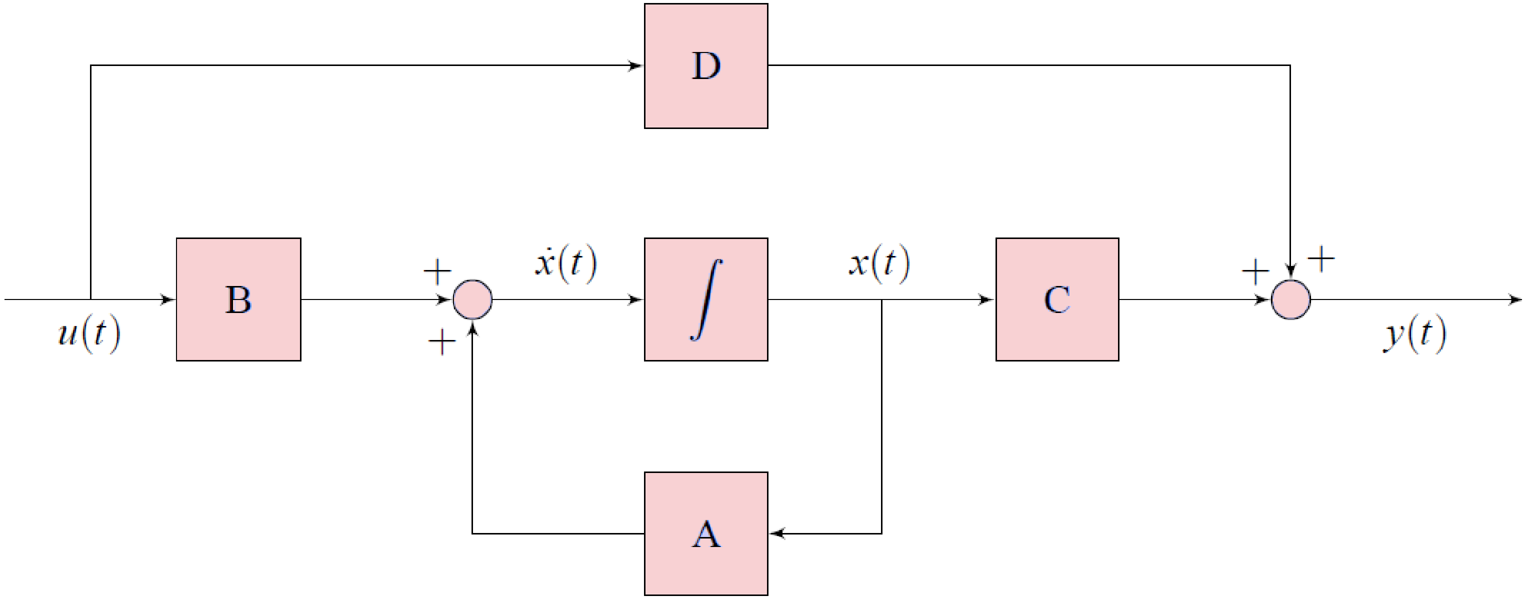


Figure 1: State space block diagram.

Remarks

- The state equation describes the system's dynamics. It is independent of the signals chosen to observe as outputs. In other words, choosing other signals as the system's outputs may change the output equation but not the state equation.
- For an electric system, the state may be defined by the current in each electric component as well as the voltage across it (or some equivalent combinations).
- For a mechanical system, the state may be the set of positions and velocities corresponding to each degree of freedom.
- Although a transfer function is unique for a given system, the choice of a states, and so the state space model, is not.
- The order of the system is equal to the number of linearly independent states. This is the minimum number of states necessary to represent the system.

Minimum number of state variables

Capacitor C

$i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$ $\Rightarrow$ may be convenient to choose $v_C(t)$ as a state!

Inductor L

$v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$ $\Rightarrow$ may be convenient to choose $i_L(t)$ as a state!

- In most cases, the minimum number of state variables in an electric circuit systems is the same as the number of independent energy-storage elements in the circuit (i.e., capacitors and inductors).

- Like that, one may choose the voltages across the capacitors and the currents through the inductors (preferably those which are linearly independent) as states.

Example 3.1

Representing an Electrical Network

Find a state space model for the circuit in Fig. 3 where v and i_R are the system's input and output, respectively.

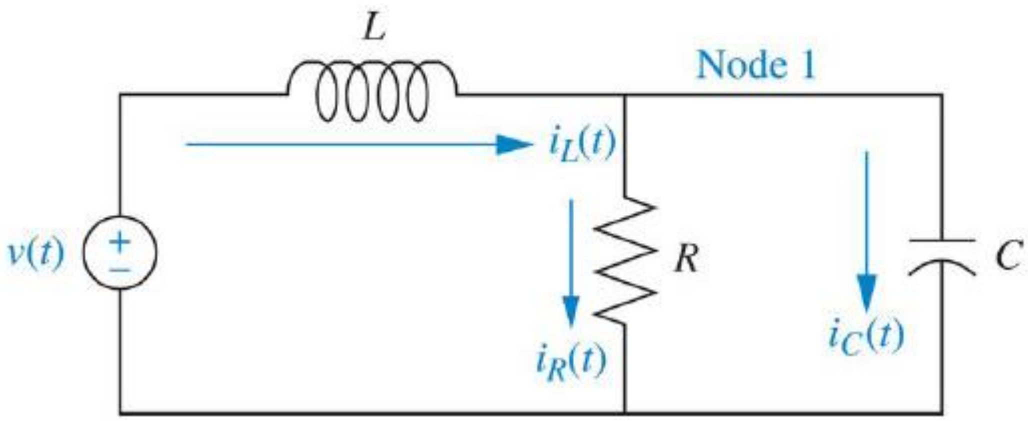


Figure 3: Circuit of example 3.1 [Nise, 2015].

1. Label all of the branch currents in the network. These include i_L, i_R, and i_C
2. Select the state variables by writing the derivative equation for all energy-storage elements, that is, the inductor and the capacitor.

$$C\frac{dv_C}{dt} = i_C \qquad L\frac{di_L}{dt} = v_L$$

Since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$.

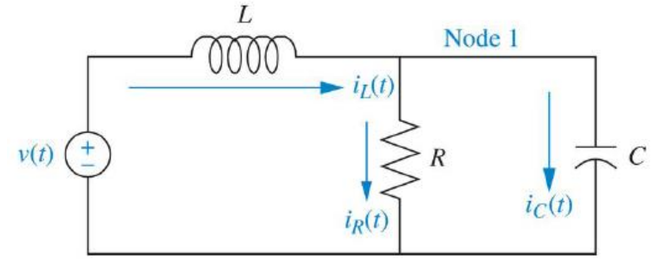
3. Apply network theory, such as Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L.

$$C\frac{dv_C}{dt} = i_C$$

$$L\frac{di_L}{dt} = v_L$$

$$i_C = -i_R + i_L$$

$$\boxed{i_C = -\frac{1}{R}v_C + i_L}$$



Around the outer loop:

$$\boxed{v_L = -v_C + v(t)}$$

4. Substitute the results above to obtain the following state equations:

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L$$

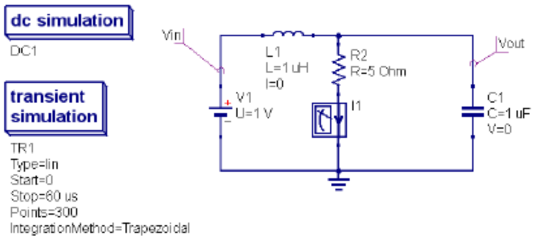
$$L\frac{di_L}{dt} = -v_C + v(t)$$

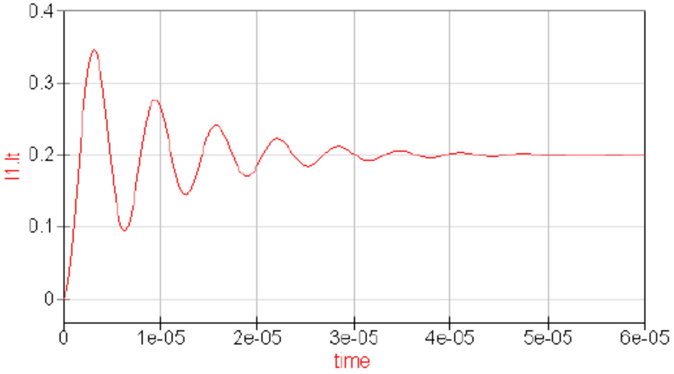
5. The final result for the state-space representation in vector-matrix form:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Verification

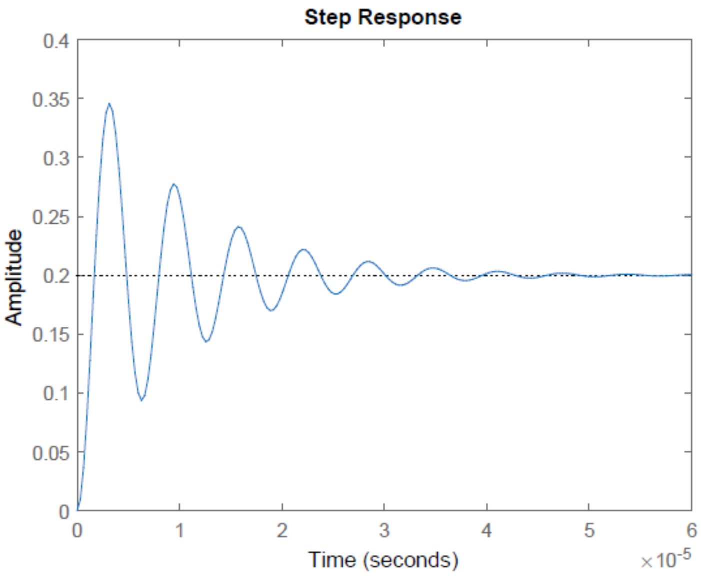




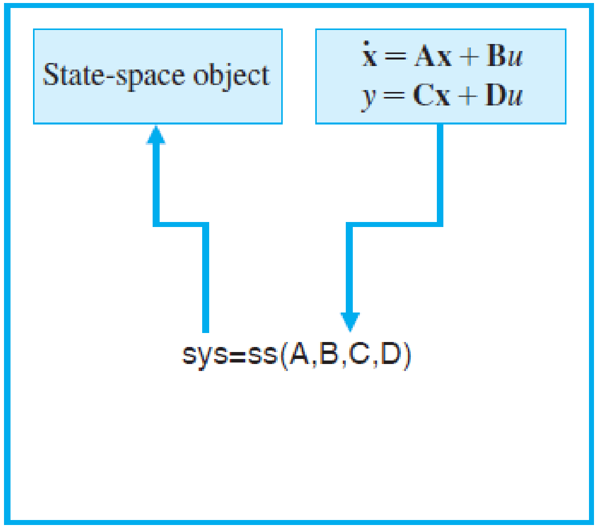
Listing 2: Matlab Code

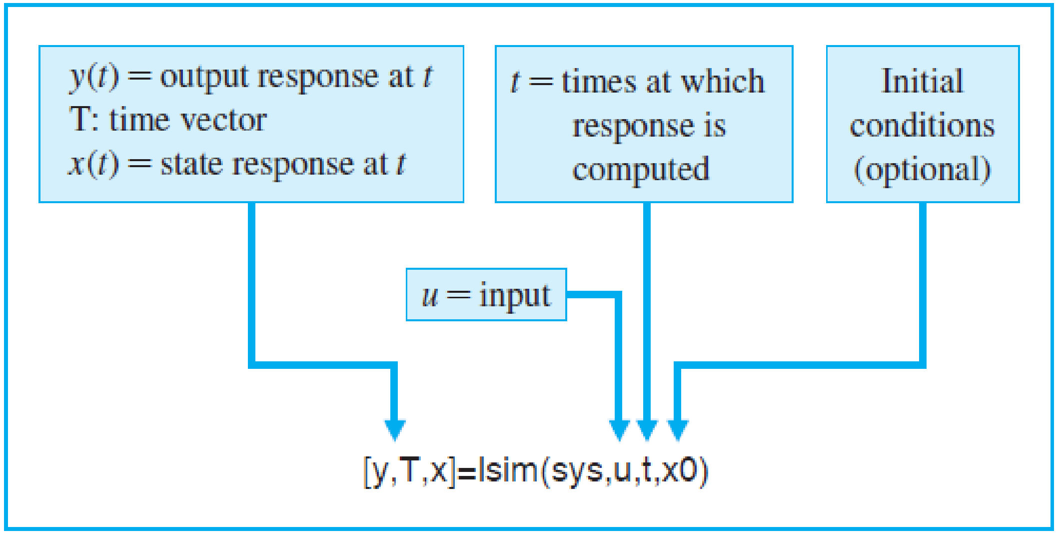
```
L=1e-6; R=5; C=1e-6;
A = [-1/(R*C)  1/C;  -1/L  0];
B = [0; 1/L];
C = [1/R  0];
D = 0;

sys=ss(A,B,C,D)
step(sys);
```



MATLAB Hints





Exercise 1

Find a state space model for the circuit in Fig. 4 using: (i) mesh analysis; then (ii) nodal analysis techniques; where v_i and v_o are the system's input and output, respectively.

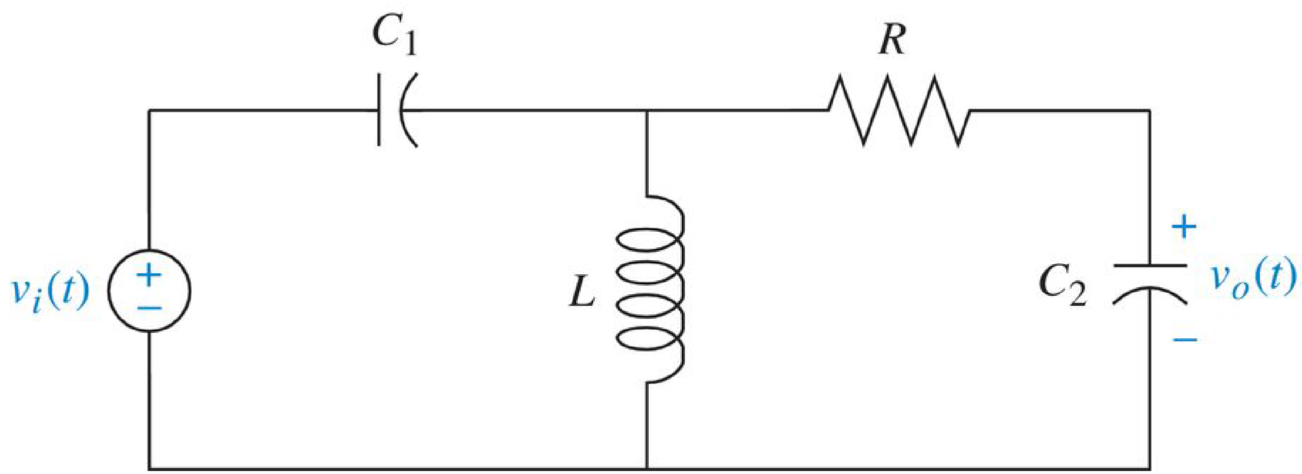


Figure 4: Circuit of Skill-Assessment Exercise 3.1 [Nise, 2015].

4. Converting a Transfer Function to State Space

Consider the system represented by the following equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u \ ,$$

where $y \in \mathbb{R}$, $u \in \mathbb{R}$, and $a_{n-1}, \ldots, a_0, b_0 \in \mathbb{R}$.

It is a system of order $n \Rightarrow$ a minimum of n state variables is required (the state vector x has n elements).

A convenient way to obtain its n state variables is by choosing

$$\underbrace{x_1 = y, x_2 = \frac{dy}{dt}, \ldots, x_n = \frac{d^{n-1} y}{dt^{n-1}}}_{n \text{ elements}}$$

Like this, we get

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\boxed{\dot{x}_n} &= -a_0 y - a_1 \dot{y} - \cdots - a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + b_0 u \\
&= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + b_0 u
\end{aligned}$$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Hence, the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & \ddots & \\ 0 & & \cdots\cdots & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & & \cdots & -a_{n-1} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}}^{B} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

With matrix $\boldsymbol{A}$ in this form, we say that the state space is in phase-variable form.

- Although a linear system has a unique transfer function, it can have several possible state-space representations.
- Hence, to convert a system's transfer function to a state-space representation (phase-variable form), it is sufficient to:
  1. Obtain the system's differential equations from the transfer function assuming zero initial conditions.
  2. Then, continue like we did in the earlier example.

Example 4

A system is governed by the following differential equation:

$\ddot{y} + a_1\dot{y} + a_0 y = b_0 u$; where $u \equiv$ input, $y \equiv$ output. Find a state-space model for it.

Matlab

The Matlab codes ch3p1 to ch3p4 in Appendix B of the textbook (and the student companion site) illustrates how to convert a transfer function to a state-space representation in Matlab.

Student Companion Site (Free) [click following link] [Nise: Control Systems Engineering, 7th Edition - Student Companion Site (wiley.com)]

Example 5 (a constant term in the numerator)

Find a state-space representation in phase-variable form of the transfer function

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Solution

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Select the state variables. Choosing the state variables as successive derivatives, we get

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

Derivation both sides,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

output $y = c = x_1$

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

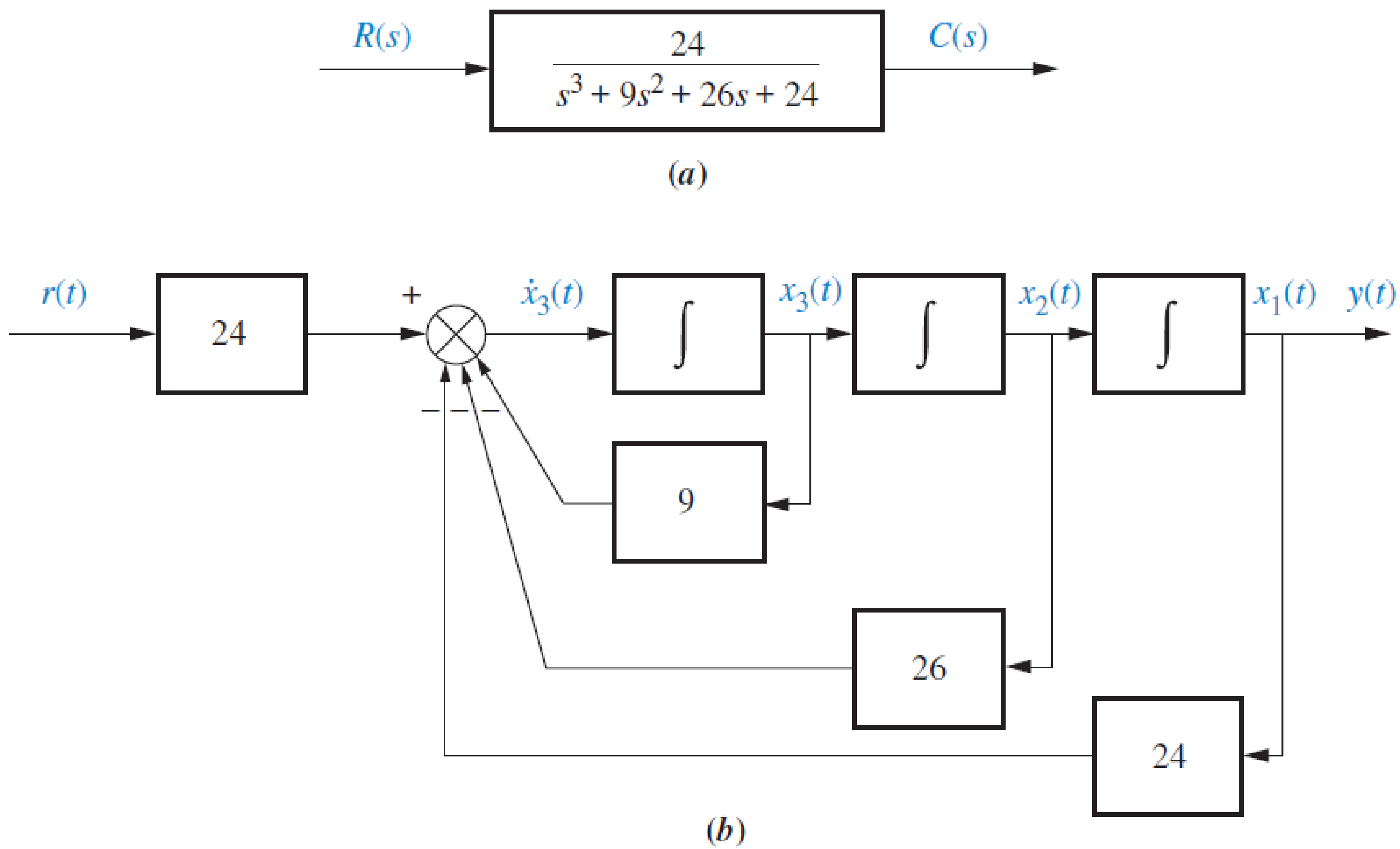


Figure 5: **a.** Transfer function; **b.** equivalent block diagram showing phase variables Note: $y(t) = c(t)$. [Nise, 2015].

Example 6 (a polynomial in the numerator)

Find a state-space representation in phase-variable form of the transfer function

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution

1. Separate the system into two cascaded blocks, as shown in Figure 6(*b*). The first block contains the denominator and the second block contains thenumerator.
2. Find the state equations for the block containing the denominator.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

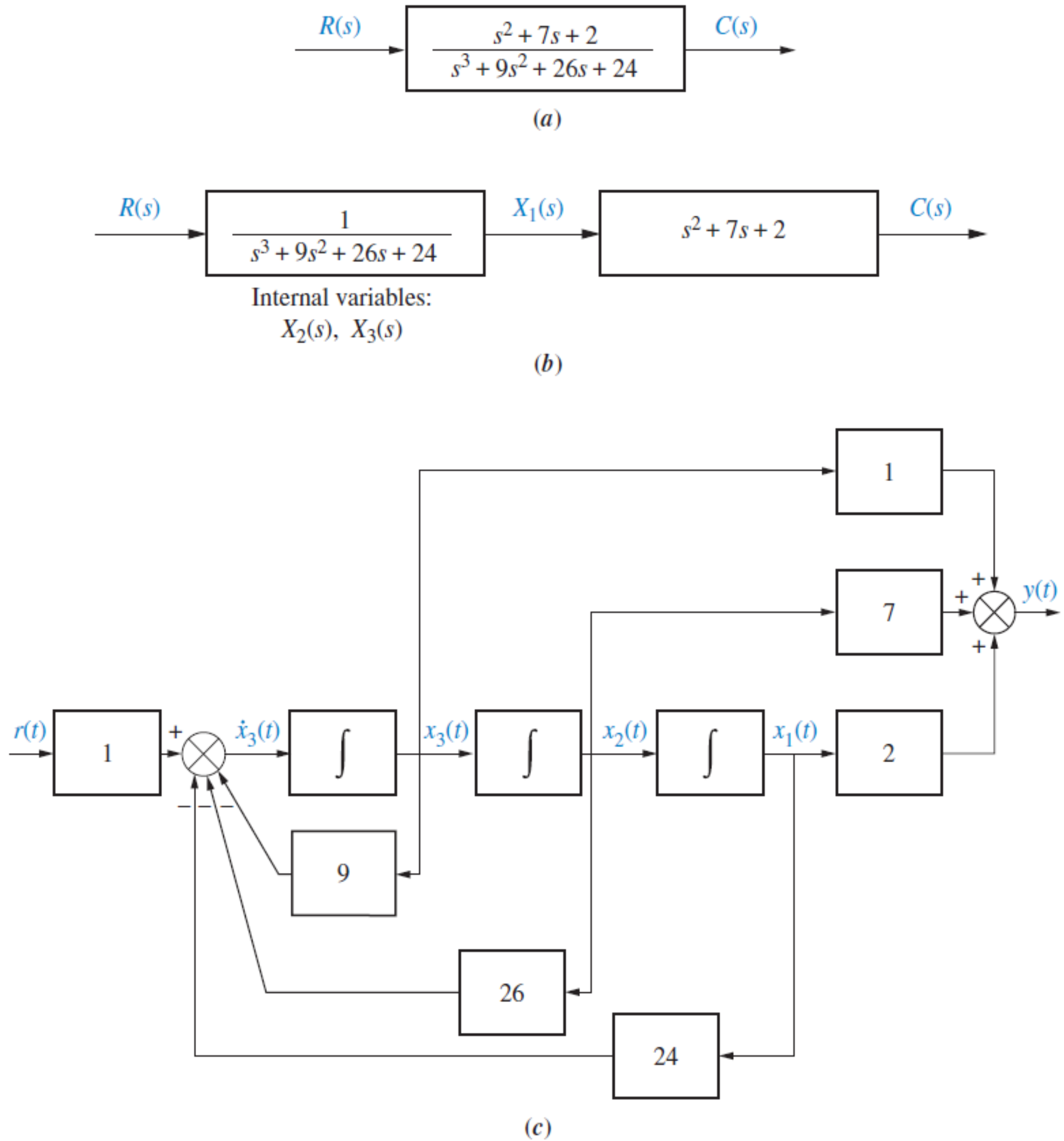


Figure 6: Example 3-5 [Nise, 2015].

3. Introduce the effect of the block with the numerator. The second block of Figure 6(*b*), states that

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

inverse LT $$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

But

$$x_1 = x_1$$
$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

Hence

$$y = c(t) = x_3 + x_2 + 2x_1$$

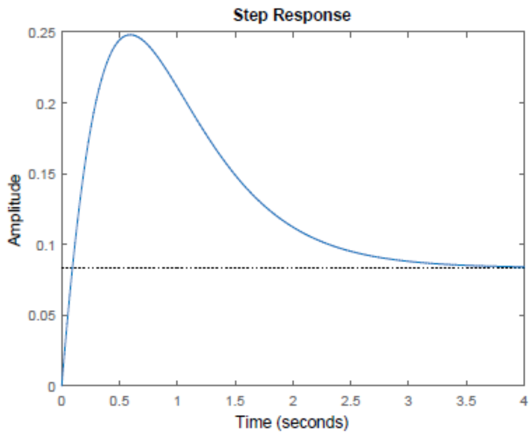
$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Verification

Listing 4: Matlab Code

```
s = tf([1 0],[1]);
sys1 = (s^2+7*s+2) / (s^3+9*s^2+26*s+24);
figure(1); step(sys1);

A = [0 1 0; 0 0 1; -24 -26 -9];
B = [0; 0; 1];
C = [2 7 1];
D = 0;
sys2=ss(A,B,C,D);
figure(2); step(sys2);
tf(sys2)
```



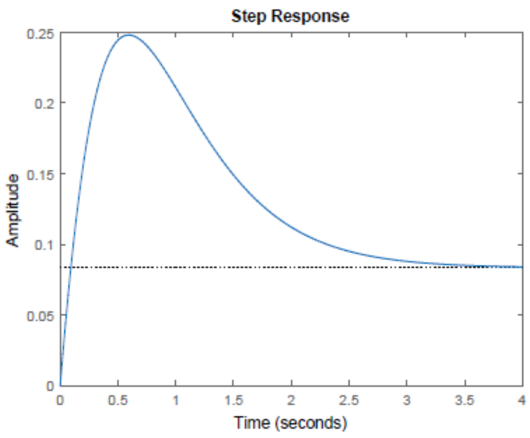
Listing 6: Matlab Output

```
>> mat_code

ans =

      s^2 + 7 s + 2
  ----------------------
  s^3 + 9 s^2 + 26 s + 24

Continuous-time transfer function.
```



Example: Consider a dynamical system represented by the following differential equation

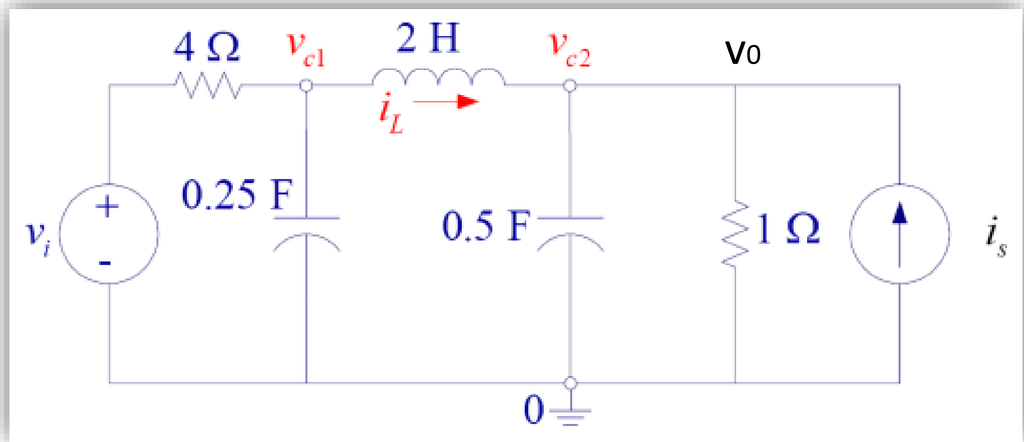
$$y^{(6)} + 6y^{(5)} - 2y^{(4)} + y^{(2)} - 5y^{(1)} + 3y = 7u^{(3)} + u^{(1)} + 4u$$

where $y^{(i)}$ stands for the ith derivative, i.e. $y^{(i)} = d^i y/dt^i$. According to (3.9) and (3.14), the state space model of the above system is described by the following matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 5 & -1 & 0 & 2 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [4 \quad 1 \quad 0 \quad 7 \quad 0 \quad 0], \quad \mathbf{D} = 0$$

Exercise1



Write the state equation for the network shown in Figure

Define the state variables as current through the inductor and voltage across the capacitors. Write two node equations containing capacitors and a loop equation containing the inductor. The state variables will be v_{c1}, v_{c2}, and i_L.

Node equations:

$$0.25\frac{dv_{c1}}{dt} + i_L + \frac{v_{c1} - v_i}{4} = 0 \Rightarrow \dot{v}_{c1} = -v_{c1} - 4i_L + v_i$$

$$0.5\frac{dv_{c2}}{dt} - i_L + v_{c2} - i_s = 0 \Rightarrow \dot{v}_{c2} = 2i_L - 2v_{c2} + 2i_s$$

i=v/R(1)

Loop equation:

$$2\frac{di_L}{dt} + v_{c2} - v_{c1} = 0 \Rightarrow \dot{i}_L = 0.5v_{c1} - 0.5v_{c2}$$

Equivalently, in matrix form:

$$\begin{bmatrix} \dot{v}_{c1} \\ \dot{v}_{c2} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -2 & 2 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_L \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_s \end{bmatrix}$$

$$vo = v_{c2} = [0\ 1\ 0] \begin{vmatrix} v_{c1} \\ v_{c2} \\ i_s \end{vmatrix}$$

Exercise 2

Find a state-space representation in phase-variable form of the transfer function

$$\frac{C(s)}{R(s)} = \frac{2s+1}{s^2+7s+9}$$

5. Converting from State Space to a Transfer Function

Problem Definition

Given the state and output equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

How to compute the transfer function? $\left(\frac{Y(s)}{U(s)} = ?\right)$

Matrix Inverse

The inverse of matrix Q is $Q^{-1} = \dfrac{\mathrm{adj}(Q)}{\det(Q)}$ ists if $\det(Q) \neq 0$)

Computing the transfer function

$$\dot{x} = Ax + Bu \overset{\mathscr{L}}{\Longrightarrow} sX(s) - x(0^-) = AX(s) + BU(s)$$
$$\Rightarrow (sI - A)X(s) = BU(s) + x(0^-)$$
$$\Rightarrow \ X(s) = [sI - A]^{-1}BU(s) + [sI - A]^{-1}x(0^-)$$

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C[sI - A]^{-1}BU(s) + C[sI - A]^{-1}x(0^-) + DU(s) \\ &= \underbrace{\left[C[sI - A]^{-1}B + D\right]}_{Transfer\ Function} U(s) + C[sI - A]^{-1}x(0^-) \end{aligned}$$

Remarks

- Although a system can have multiple state-space representations, the transfer function matrix (or simply the transfer function in the case of a SISO system) is unique.
- $\det(sI - A)$ is the system's characteristic polynomial.
- The poles of the system are the roots of $\det(sI - A)$, which are the eigenvalues of the system matrix A.

Matlab

- The code ch3p5 in Appendix B, shows how to convert a state-space representation to a transfer function in Matlab.

- A small tutorial on Matlab Symbolic Math Toolbox is given in ch3p1 in Appendix E.

Example 7

Given the system defined by the following state space representation, find the transfer function $T(s) = Y(s) / U(s)$, where u and y are the system's input and output signals, respectively.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Solution

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2 + 3s + 2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

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Explaining algebra

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \begin{matrix} s & -1 \\ 0 & s \\ 1 & 2 \end{matrix}$$

Delta= $s^3 + 3s^2 + 2s + 1$

----------------Cofactor matrix ----------------------

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} .* \left| \begin{matrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} & -\begin{vmatrix} 0 & -1 \\ 1 & s+3 \end{vmatrix} & \begin{vmatrix} 0 & s \\ 1 & 2 \end{vmatrix} \\ -\begin{bmatrix} -1 & 0 \\ 2 & s+3 \end{bmatrix} & \begin{bmatrix} s & 0 \\ 1 & s+3 \end{bmatrix} & -\begin{bmatrix} s & -1 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ s & -1 \end{bmatrix} & -\begin{bmatrix} s & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} \end{matrix} \right|$$

=[Cofactor matrix]

Adjoint =Transpose[cofactor matric]

$$\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Exercise 3

Given the system defined by the following state space representation, find the transfer function $T(s) = Y(s)/U(s)$, where u and y are the system's input and output signals, respectively.

$$\dot{x} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} x$$

$$G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

Answer

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6. Linearization

- A state-space model can be either linear or nonlinear depending on the type of the system represented.
- However, linear state-space representations are usually easier to study.
- A nonlinear state-space model can be approximated by a linear model using the same linearization technique covered in Chapter 2 (Modeling in the Frequency Domain).

Example 8

The pendulum system in Fig. 8 is the basis of many real-world applications, such as robotic arms, humanoid robots, etc. The dynamics of such a system is governed by the following equation:

$$J\frac{d^2\theta}{dt^2}+\frac{MgL}{2}\sin\theta = T$$

Where

$J \equiv$ moment of inertia

$M \equiv$ mass of pendumum

$g \equiv$ gravitational acceleration

$\theta \equiv$ angle from the vertical

$L \equiv$ length of pendulum

$T \equiv$ applied torque

1. Using θ and $\dot{\theta}$ as the states, find a state-space representation of the system, taking T as the input and θ as the output.
2. Find a linear approximation of the state-space model for small excursions around the state of equilibrium $(\theta, \dot{\theta}) = (0, 0)$.
3. Find the system transfer function around the state of equilibrium $(\theta, \dot{\theta}) = (0, 0)$.

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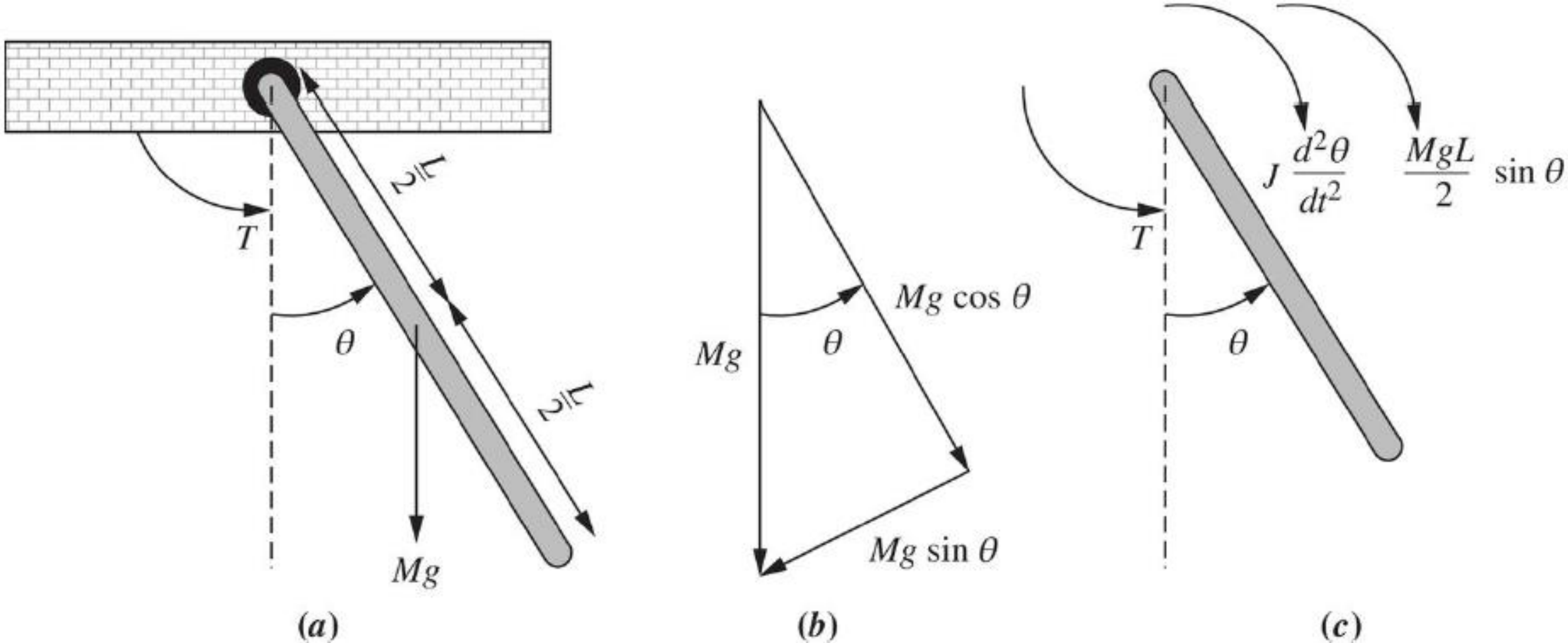


Figure 8: Pendulum system [Nise, 2015].

Solution

Letting $x1 = \theta$ and $x2 = d\theta/dt$,

$$J\ddot{\theta} + \tfrac{1}{2}MgL\sin\theta = T$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{MgL}{2J}\sin x_1 + \frac{T}{J}$$

$$x_1 = 0 + \delta x_1$$

$$x_2 = 0 + \delta x_2$$

$$\sin x_1 = \sin 0 + \left.\frac{d(\sin x_1)}{dx_1}\right|_{x_1=0} \delta x_1 = \delta x_1$$

$$\sin x_1 = \delta x_1$$

from which

$$\delta\dot{x}_1 = \delta x_2$$

$$\delta\dot{x}_2 = -\frac{MgL}{2J}\delta x_1 + \frac{T}{J}$$

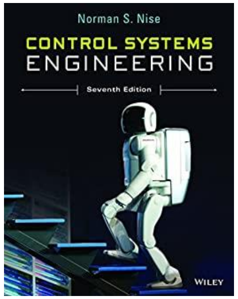
Exercise 4

Given the system defined by the following state space representation, where u and y are the system's input and output signals, respectively; and the state $x(t) \in \mathbb{R}$

$$\dot{x}(t) = \left(4x(t) - \sin^2 x(t)\right) u(t)$$
$$y(t) = x(t)$$

Find the transfer function $T(s) = Y(s) \,/\, U(s)$, for small excursions of the state around 0.

References



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