Chapter 6. Stability

ELG 3155 : Introduction to Control Systems

- Objective:
  - to determine a system's stability from its transfer function without solving for the poles.
  - to determine a system's parameter to guarantee its stability.

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Outline

1. Introduction
2. Routh-Hurwitz Criteria
3. Routh-Hurwitz Table
4. Routh-Hurwitz Criteria: Special Cases
    1. Case 1: a zero component at the beginning of a nonzero row
    2. Case 2: an entire row of zeros (ROZ)
5. Routh-Hurwitz Criteria: Supplementary Examples
6. Stability in State Space

1. Introduction

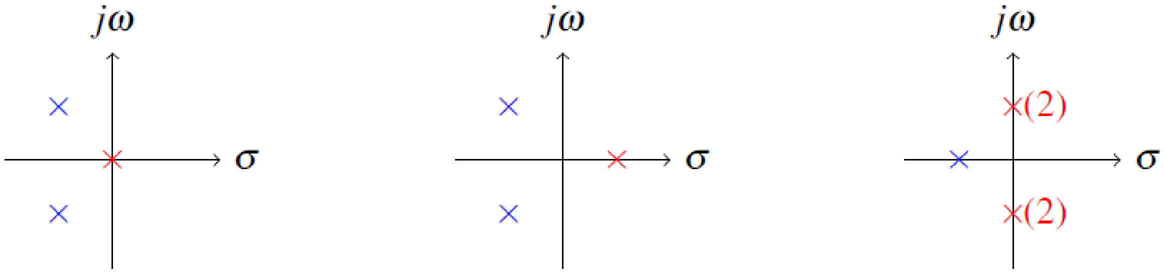
- Stability is the most important feature of a control system.
- If a system is unstable, studying its transient response becomes uninteresting.

Definition (Instability)

A system is said to be unstable if and only if there exists at least one bounded input that leads to an unbounded output. Otherwise, it is stable.

Poles of an unstable LTI system

An LTI system is unstable if at least one of its poles is at the origin, on the RHS of the s-plane, or a pair of conjugate poles on the $j\omega$-axis is repeated twice or more.



Definition (Stability)

A system is said to be stable if and only if every bounded input leads to a bounded output. A stable system is either asymptotic stable or marginally (critically) stable.

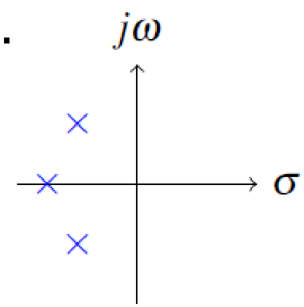
Definition (Asymptotic stability)

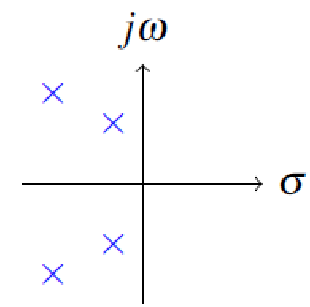
A system is said to be asymptotically stable if its natural response tends to zero as $t \to \infty$.

$$c(t) = c_f(t) + c_n(t)\ , \text{ where } \lim_{t\to\infty} c_n(t) = 0.$$

Poles of an asymptotically stable LTI system

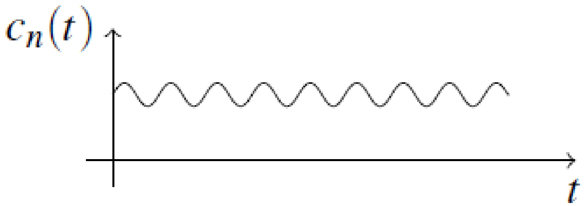
An LTI system is asymptotically stable if and only if all of its poles are on the LHS of the s-plane.





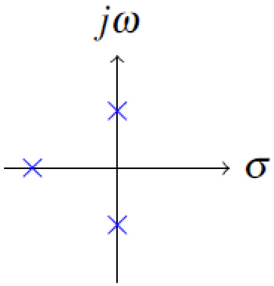
Definition (Marginal stability)

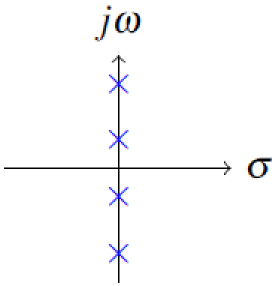
A system is marginally stable (or critically stable) when its natural response $c_n(t)$ is oscillatory with a constant magnitude.



Poles of a critically stable LTI system

An LTI system is critically stable if and only if it is stable and has at least one conjugate pair of poles are on the imaginary axis of the s-plane.





Figs. 1 and 2 show a graphical illustration of these concepts.

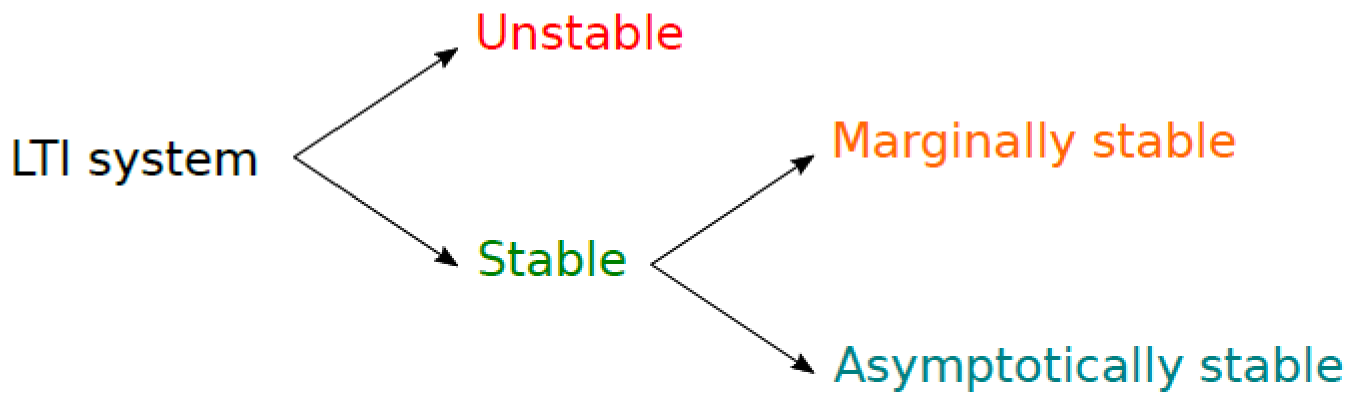


Figure 1: The different types of stability for an LTI system.

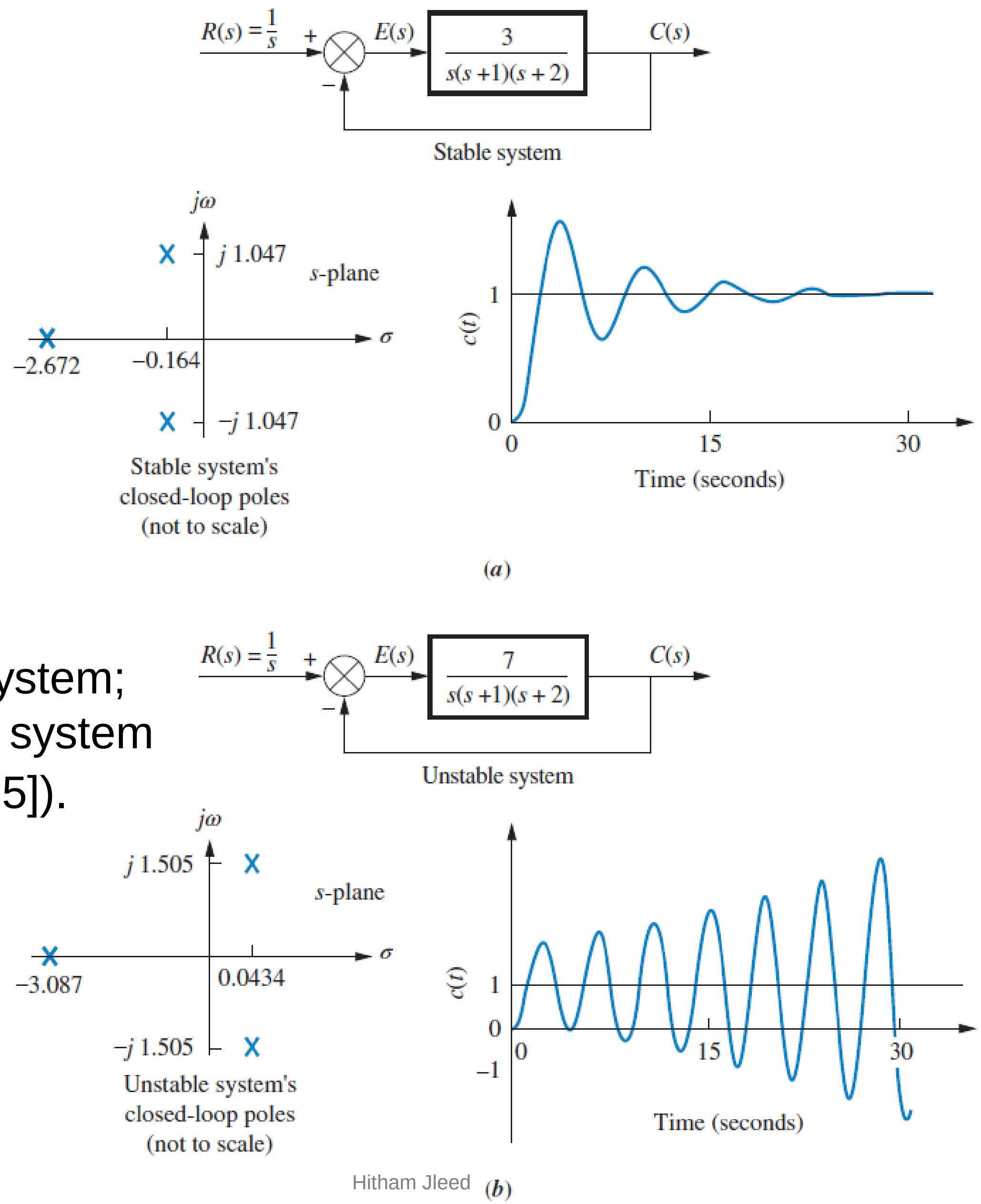


Figure 2: (a) Stable system;
(b) unstable system
(Fig. 6.1 of [Nise, 2015]).

stability conditions

1. All the coefficients of the equation q(s) should have the same sign. The main reason why we get a coefficient of a different sign is because of having a pole in the right half plane.
2. There should be no missing term. This means that there should not be any coefficient with zero value between the highest power of s and lowest power of s in the characteristic equation.

Examples of characteristic equations satisfying this condition (the system may or may not be stable)

- $2s^9 + 10s^7 + 66s^5 + 9s^3 + 4 = 0$
- $-30s^8 - 5s^7 - 12s^4 - 3s^2 - 1 = 0$

Examples of characteristic equations violating this condition (the system is then unstable)

- $2s^9 + 10s^7 - 66s^5 + 9s^3 + 4 = 0$
- $-30s^8 - 5s^7 - 12s^4 - 3s^2 - s = 0$ Missing term

2. Routh-Hurwitz Criteria

- The Routh-Hurwitz criteria are a set of rules to determine the number of the roots of an algebraic equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

which have positive (> 0) real parts without solving the equation.

Routh-Hurwitz Criteria (necessary and sufficient condition)

The number of roots of a polynomial which are on the RHS of the s-plane is equal to the number of sign changes in the first column of the Routh-Hurwitz table.

Routh-Hurwitz Table

Building the RH table

- Consider a polynomial of the form: $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0$
- The top two rows of the RH table correspond to s^4 and s^3. The coefficients of these two rows are extracted directly from the polynomial.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

$$b_1 = -\frac{\det\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3}$$

Routh-Hurwitz Table

- The rest of the table is calculated as shown below.

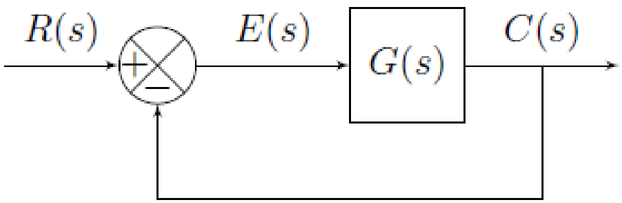
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example

Determine the stability of the following system,

Where,

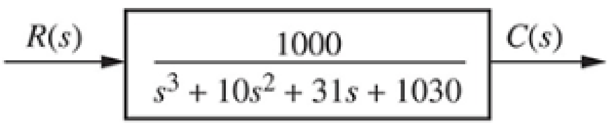
$$G(s)=\frac{1000}{(s+2)(s+3)(s+5)}.$$



Solution

The Routh-Hurwitz criterion will be applied to this denominator.

s^3	1	31	0
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103	0
s^1			
s^0			



Note: For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below

s^3	1	31	0
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103	0
s^1	$\frac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1}=-72$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$	0
s^0	$\frac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72}=103$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72}=0$	0

Theorem 1.

The number of sign changes in the first column of the Routh table equals the number of roots of the polynomial in the Closed Right Half-Plane (CRHP).

There are two sign changes: 1 → −72 and −72 → 103 I

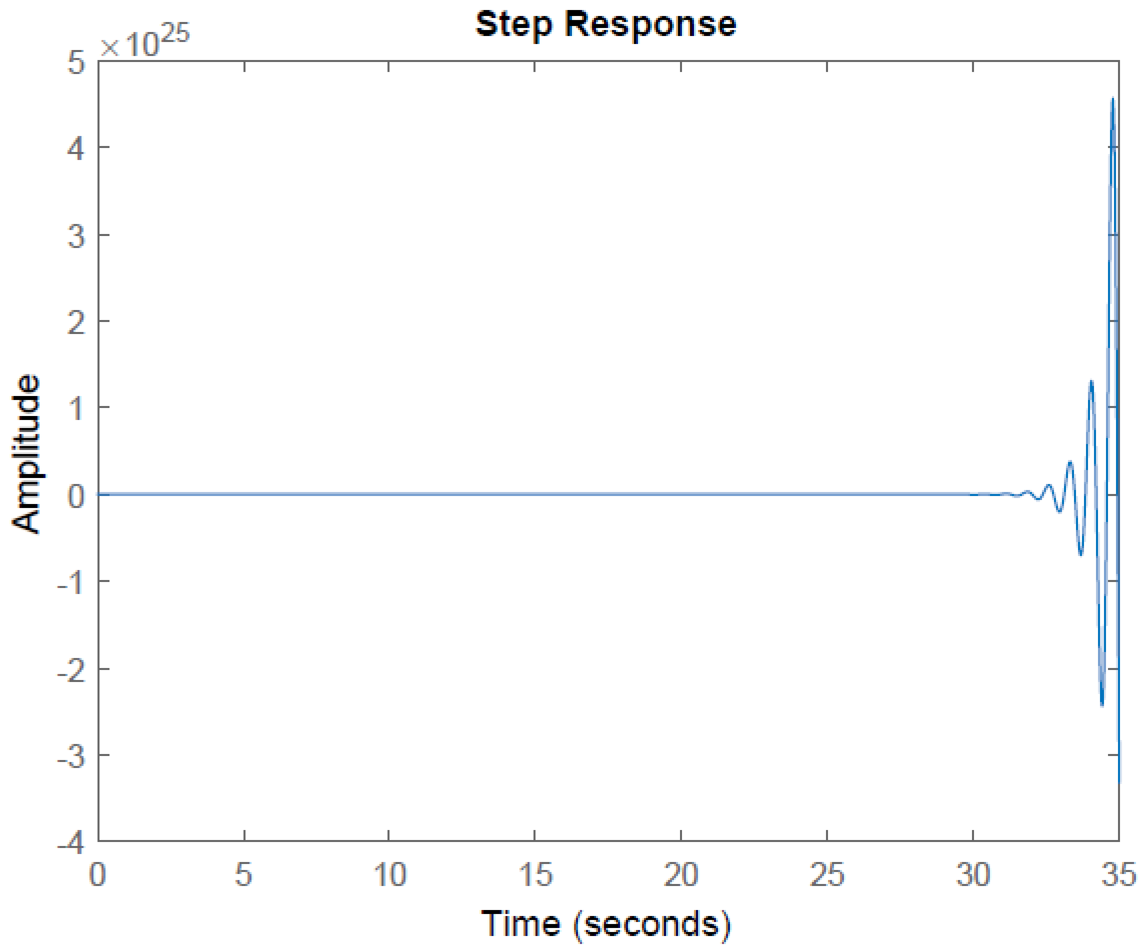
Two poles in the CRHP.

=>Unstable

Verification

Fig. 3 shows the step response of this system. It is clear that the response goes out of bound, hence the system is unstable.

Figure 3: Step response of the system in Example 1



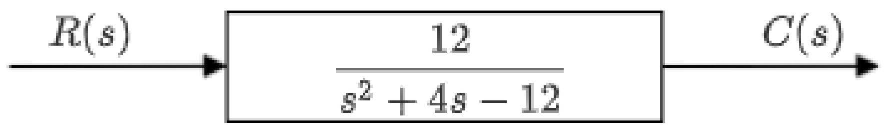
System poles:

-13.4136 and

$1.7068 \pm j8.595$

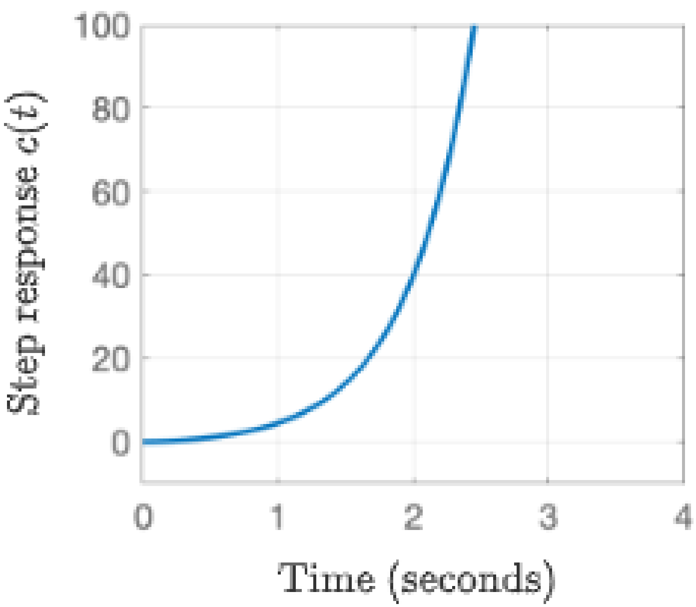
Example

Determine the stability of the following system,



Routh table

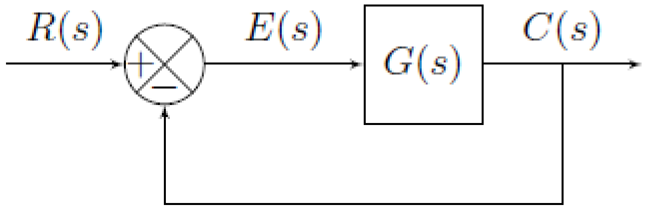
s^2	1	-12
s^1	4	0
s^0	$\dfrac{-\begin{vmatrix}1 & -12\\ 4 & 0\end{vmatrix}}{4} = -12$	$\dfrac{-\begin{vmatrix}1 & 0\\ 4 & 0\end{vmatrix}}{4} = 0$



One sign change $\longrightarrow$ One pole on the right half-plane. $\longrightarrow$ System is unstable

Example

Consider the following system, where $G(s) = \frac{K}{(s+2)(s+3)(s+5)}$ and $K \in \mathbb{R}$. For what values of K is the system stable?



Solution

The Closed-Loop Transfer Function is

$$\frac{k}{s^3 + 10s^2 + 31s + 30 + k}$$

$$\frac{k}{s^3 + 10s^2 + 31s + 30 + k}$$

a_n =1	a_{n-2} =31	a_{n-4} =0
a_{n-1} =10	a_{n-3} =30+k	0
$b_1 = \frac{310 - 30 - k}{10}$	b_2 =0	0
c_1=30+k	c_2=0	0

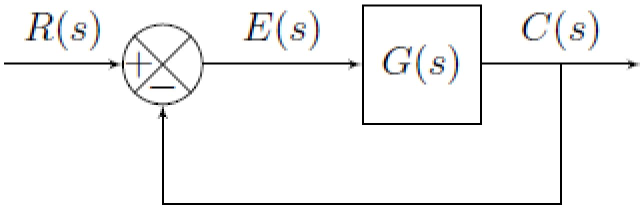
The leading edge cannot change signs for the system to be stable. Therefore, the following conditions must be met:

$$a_n = 1 > 0$$
$$a_{n-1} = 10 > 0$$
$$b_1 = \frac{310-30-k}{10} > 0$$
$$c_1 = 30 + k > 0$$

The positive constraint on b_1 leads to K<280. The positive constraint on c_1 means that K>-30. Therefore, the following ranges are acceptable for the controller stability. -30<k<280

Example 2

Consider the following system, where $G(s) = \frac{K}{(s+1)(s^2+2s+1)}$ and $K \in \mathbb{R}$. For what values of K is the system stable?



Solution

The Closed-Loop Transfer Function is

$$G(s) = \frac{K}{s^3 + 3s^2 + 3s + 1\text{+k}}$$

$$G(s) = \frac{K}{s^3 + 3s^2 + 3s + 1\text{+k}}$$

a_n =**1**	a_{n-2} =**3**	a_{n-4} =**0**
$a_{n-1} = 3$	$a_{n-3} = $ 1+k	0
$b_1 = \frac{9-1-k}{3}$	b_2 =0	0
c_1=1+k	c_2=0	0

The leading edge cannot change signs for the system to be stable. Therefore, the following conditions must be met:

a_n =1>0
$a_{n-1} = $ 3>0
$b_1 = \frac{8-k}{3}$ >0
c_1=1+k >0

The positive constraint on b_1 leads to K<8. The positive constraint on c_1 means that K>-1. Therefore, the following ranges are acceptable for the controller stability. -1<k<8

Verification

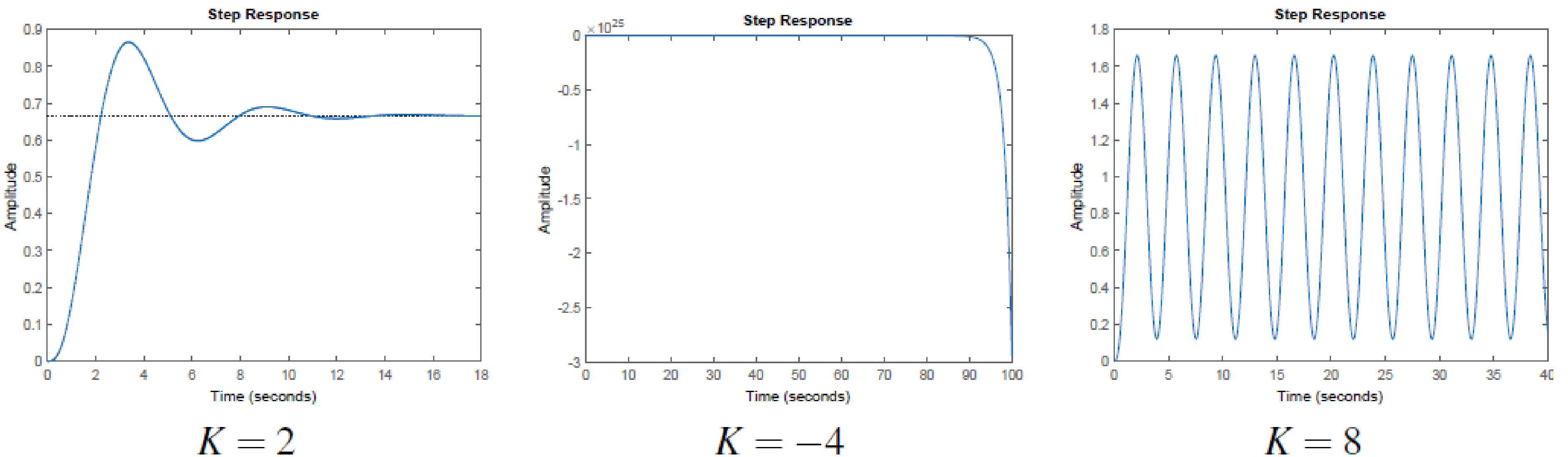


Figure 4: Stability verification with different values of K

4. Routh-Hurwitz Criteria: Special Cases

Case 1: a zero component at the beginning of a nonzero row

If the first element of a row is zero, division by zero would be required to form the next row. To avoid this phenomenon, an epsilon, ϵ is assigned to replace the zero in the first column. The value ϵ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

Special Cases:

Stability via Epsilon Method

Example 6.2. Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}$ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Determining signs in first column of a Routh table with zero as first element in a row

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}\quad \epsilon$	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

Hence, the system is unstable and has two poles in the right half-plane.

Verification

Fig. 5 shows the step response of this system. It is clear that the response goes out of bound, hence the system is unstable.

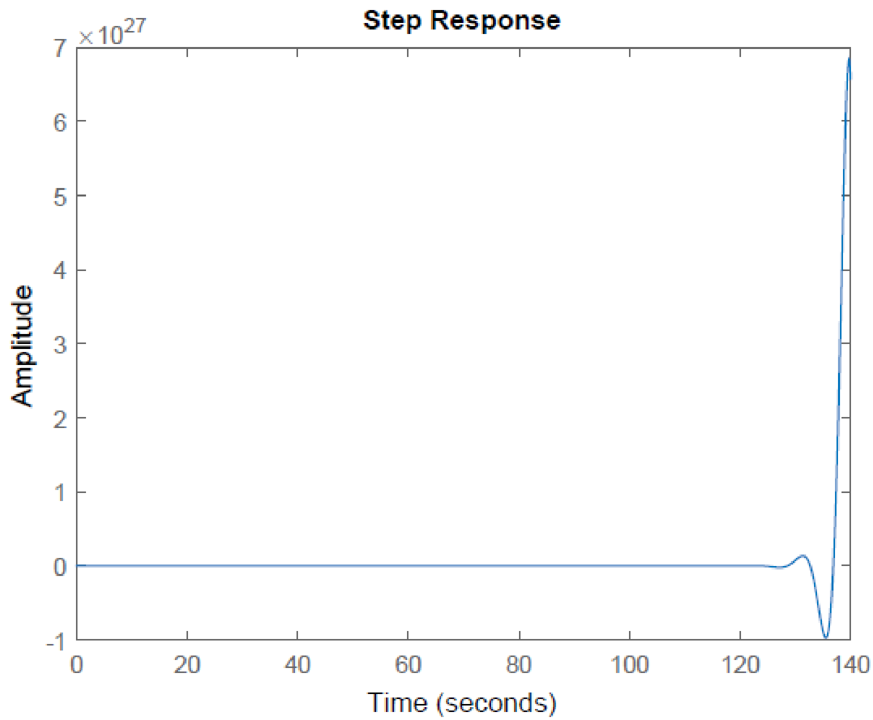


Figure 5: Stability verification with different values of *K*

System poles: $-0.7927 \pm j\,0{:}4292$ and $0.2927 \pm j\,0.7278$

Example: Consider the transfer function

$$\hat{G}(s) = \frac{1}{s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10}$$

				$\epsilon = +$	$\epsilon = -$
s^5	1	2	11		
s^4	2	4	10		
s^3	ϵ	6	0		
s^2	$\frac{4\epsilon - 12}{\epsilon}$	$\frac{10\epsilon}{\epsilon}$	0		
s^2	$\frac{-12}{\epsilon}$	10	0		
s	$\frac{10\epsilon^2+72}{12}$	0	0		
s	6	0	0		
1	10	0	0		

So there are two sign changes

- Two unstable poles

4. Routh-Hurwitz Criteria: Special Cases

Case 2: an entire row of zeros (ROZ)

Example 5

Determine the stability of the following system, where

$$T(s) = \frac{1}{s^3+5s^2+6s+30}.$$

s^3	1	6
s^2	5	30
s^1	0	0

a row of zeros

The entire third row becomes zero and due to this, we can't proceed further and form the fourth row. So, what do we do now?

The second row is taken to form the auxiliary polynomial.

$$A(s) = 5s^2 + 30$$

$$\frac{dA(s)}{ds} = 10s + 0$$

Now, replace the zeros of the third row with 10 and 0 and proceed as usual.

s^3	1	6	
			No sign change
s^2	5	30	
			No sign change
s^1	10	0	
			No sign change
s^0	30		

the first column have the same sign, a pair of imaginary roots is indicated.

If entire row of element is zero then there is a chance that system is marginally stable it means some root lie on imaginary axis in conjugate form.

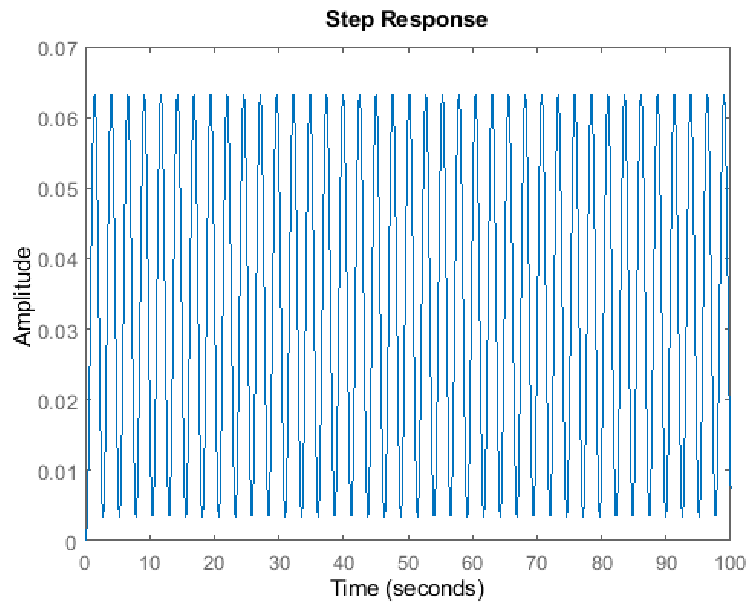
<u>**The system is either stable or marginally stable.**</u>

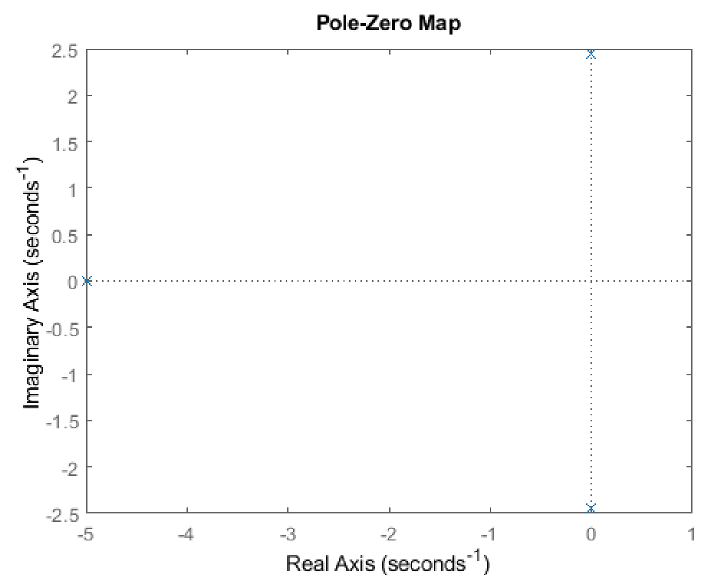
Plot with MATLAB

```
numg=1;
deng=[1 5 6 30];
T=tf(numg,deng)
'Poles of T(s)'
pole(T)
step(T)
figure, pzmap(T)
```

```
T =
            1
  ----------------------
  s^3 + 5 s^2 + 6 s + 30
Continuous-time transfer function.

ans =
    'Poles of T(s)'
ans =
  -5.0000 + 0.0000i
   0.0000 + 2.4495i
   0.0000 - 2.4495i
```

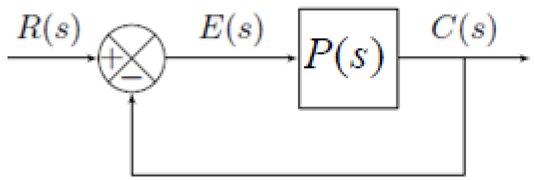




Example: The Routh array for $P(s) = s^5 + 6s^3 + 5s^2 + 8s + 15$ and k=5

$$T(s) = \frac{5}{s^5 + 6s^3 + 5s^2 + 8s + 20}$$

s^5	1	6	8
s^4	$\cancel{0}\ \varepsilon$	5	20
s^3	$-\frac{5}{\varepsilon}$	$-\frac{20}{\varepsilon}$	
s^2	5	20	
s	$\cancel{0}\ 10$		
1	20		



The auxiliary polynomial for row 4 is

$Q(s) = 5s^2 + 20$, with $Q'(s) = 10\,s$

so there are two roots on the $j\omega$-axis.

The first column shows two sign changes so there are two roors on the right half-plane.

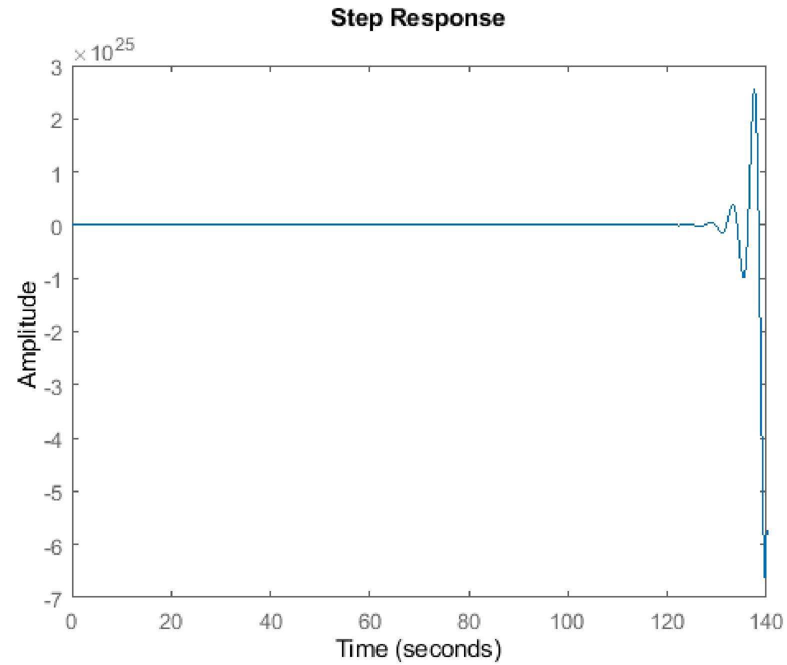
Verification

```
numg=5;
deng=[1 6 5 8 15];
'P(s)'
P=tf(numg,deng)
'T(s)'
T=feedback(P,1)
'Poles of T(s)'
pole(T)
```

```
Command Window
T =
                 5
  ------------------------------
  s^4 + 6 s^3 + 5 s^2 + 8 s + 20
Continuous-time transfer function.
ans =
    'Poles of T(s)'
ans =

  -5.1906 + 0.0000i
  -1.6881 + 0.0000i
   0.4394 + 1.4455i
   0.4394 - 1.4455i
>>
```

```
step(T)
```



Auxiliary equation

- A system's auxiliary equation exits only when there is a ROZ in the RH table. The auxiliary equation is the equation corresponding to the row in the RH table that is right above the ROZ.
- An auxiliary polynomial is always even.

Remarks

- A ROZ exists in the RH table if (not iff) the characteristic equation has one or more pairs of complex conjugate roots in the $j\omega$-axis. In this case,
  - The characteristic polynomial, say $Q(s)$, can be factorized into a product of an even auxiliary polynomial, say $P(s)$, and a non-even polynomial, say $P'(s)$. That is,

$$Q(s) = P(s)\,P'(s)$$

    Thus, the roots of $P(s)$ and $P'(s)$, are also roots of $Q(s)$.

  - The distribution of the roots of the auxiliary polynomial $P(s)$ on the s-plane is determined from the sign changes of the coefficients in the first column of the RH table corresponding to the rows of s^p down to s^0, where p is the order of $P(s)$. See Fig. 7.
  - The distribution of the roots of the remaining (non-even) polynomial $P'(s)$ is determined from the sign changes of the coefficients in the first column of the RH table corresponding to the rows of s^n down to s^{p}, where n is the order of the control system (and of the characteristic polynomial).

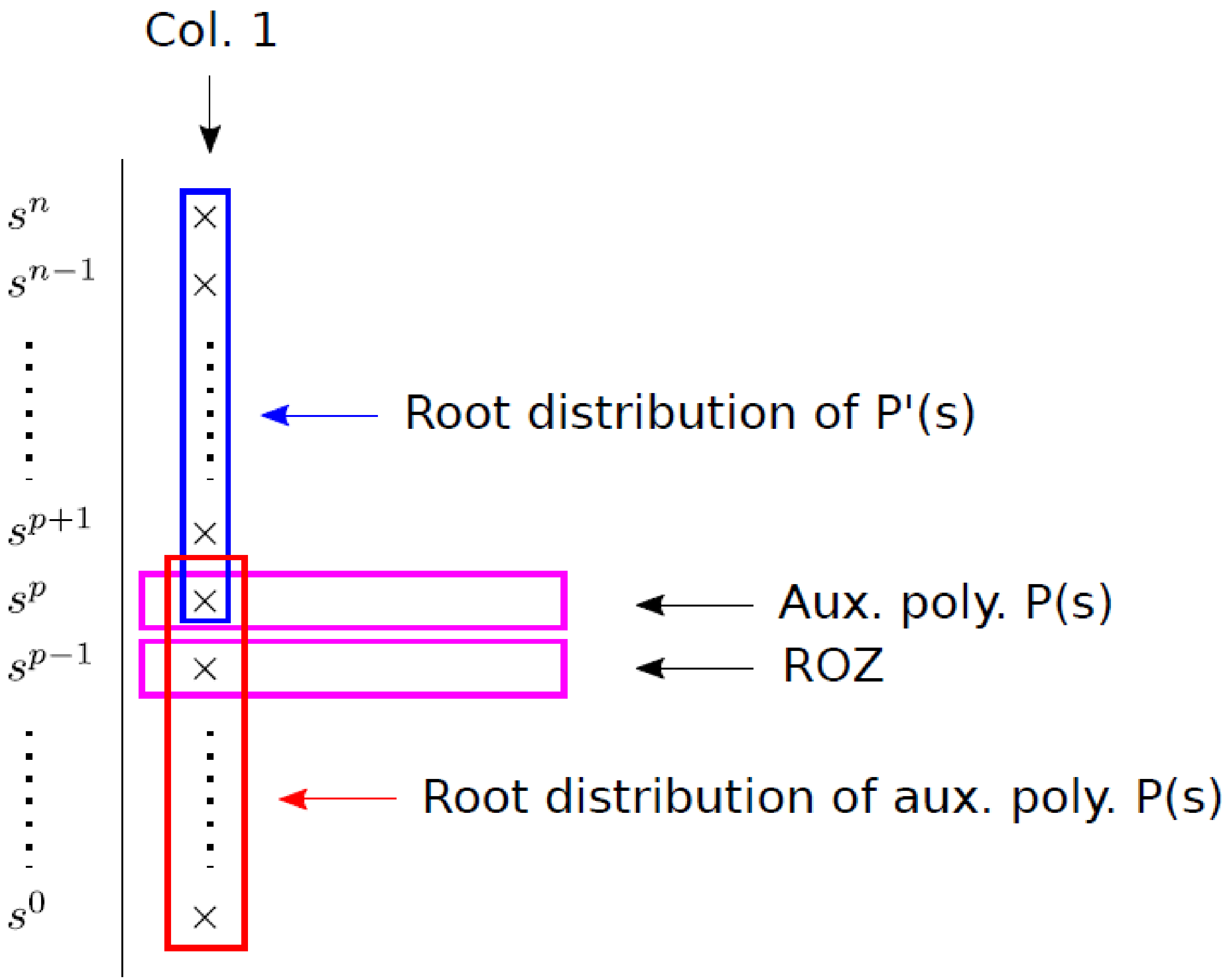


Figure 7: Pole distribution on the s-plane in the existence of an auxiliary equation.

Definition (Critical frequency)

The critical frequency is defined when a system is critically stable. It is the oscillation frequency of the system’s natural response. It can be computed from the auxiliary equation.

Definition (Critical gain)

The critical gain is the value of this gain which makes the system critically stable.

Example 6

Find the distribution of the poles on the s-plane for a system whose transfer function is

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}.$$

TABLE 6.7 Routh table for Example 6.4

s^5	1	6	8
s^4	$\cancel{7}\ \ 1$	$\cancel{42}\ \ 6$	$\cancel{56}\ \ 8$
s^3	$\cancel{0}\ \ \cancel{4}\ \ 1$	$\cancel{0}\ \ \cancel{12}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Verification

Fig. 8 shows the step response of this system. At steady state, the response is oscillatory with a fixed amplitude and frequency, hence the system is critically stable.

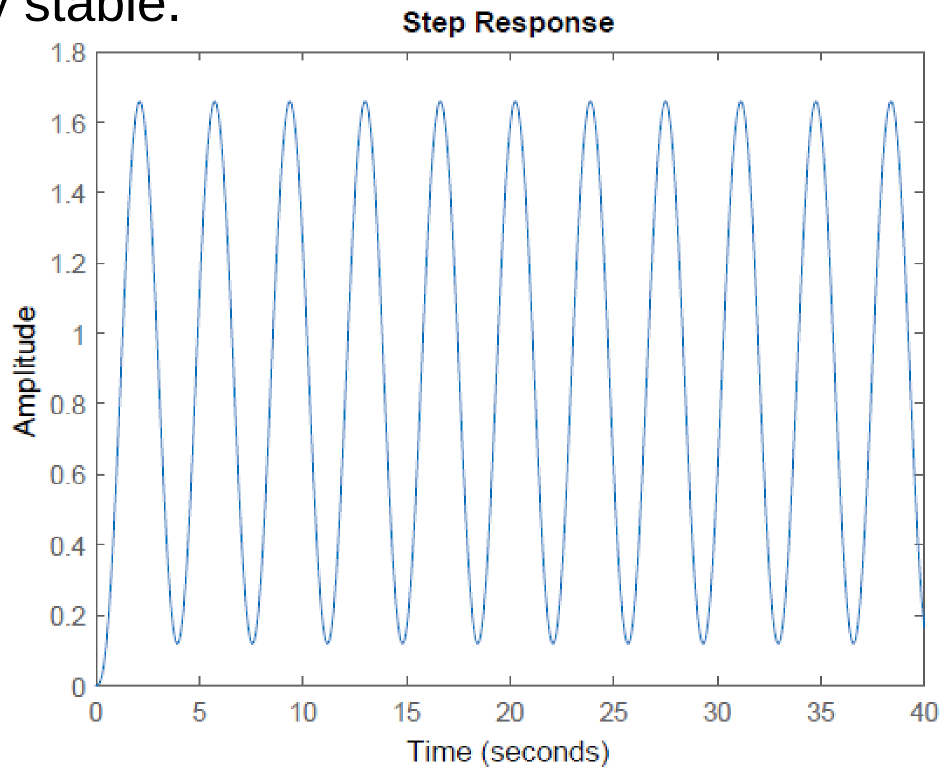


Figure 8: Step response of the system

System poles: $-7, \ \pm j2$ and $\pm j1.4142$

Remarks

- If a system is marginally stable, then it must have a ROZ in its RH table.
- However, a ROZ doesn't necessary mean that the system is marginally stable (see Example 6 and Problem 6.19, for instance).
- A ROZ simply means that the characteristic polynomial can be factorized as a product of two polynomials, one of which is even.

Example 7

Find the distribution of the poles on the s-plane for a system whose transfer function is

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}.$$

TABLE 6.8 Routh table for Example 6.5

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ \ 1$	$\cancel{20}\ \ 2$	0
s^5	$\cancel{20}\ \ 1$	$\cancel{60}\ \ 3$	$\cancel{40}\ \ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \ \cancel{4}\ \ 2$	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ \ 3$	$\cancel{2}\ \ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Verification

- Fig. 9 shows the step response of this system. It is clear that the response goes out of bound, hence the system is unstable.

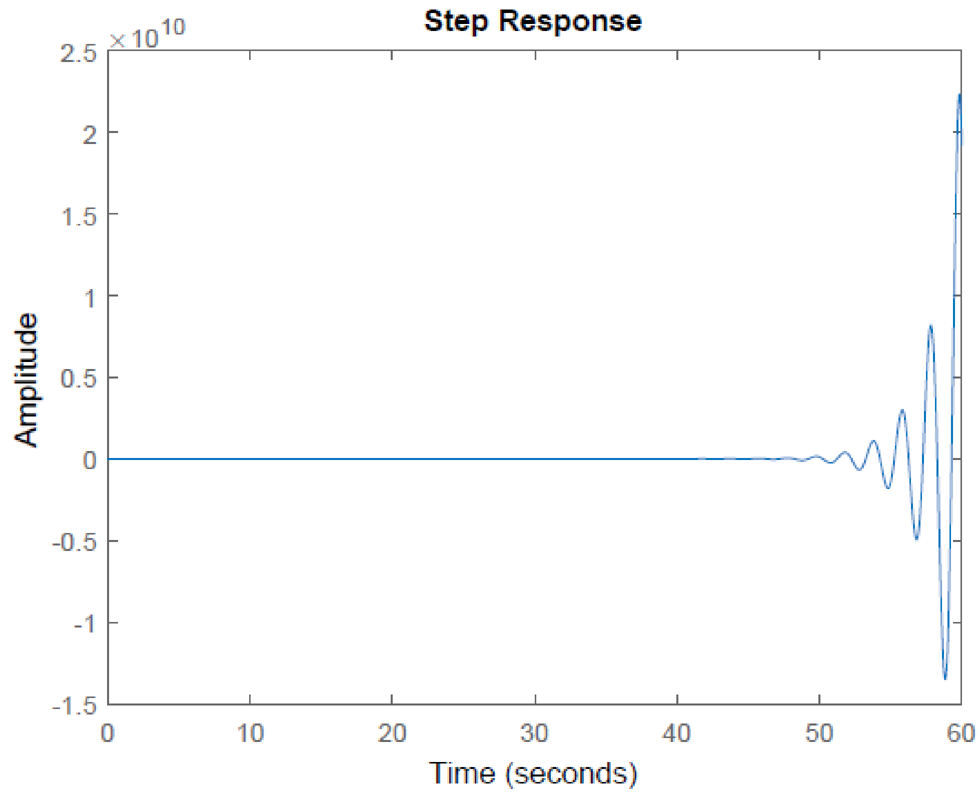


Figure 9: Step response of the system

System poles: $0.5 \pm j3.1225,\ \pm j1.4142,\ \pm j$, and two poles at -1

5. Routh-Hurwitz Criteria: Supplementary Examples

Example 8

Consider the following system, where $G(s) = \dfrac{K}{s(s+7)(s+11)}$ and $K \in \mathbb{R}$. For what values of K is the system stable, unstable , and marginally stable?

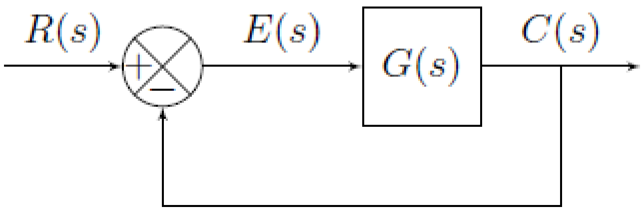
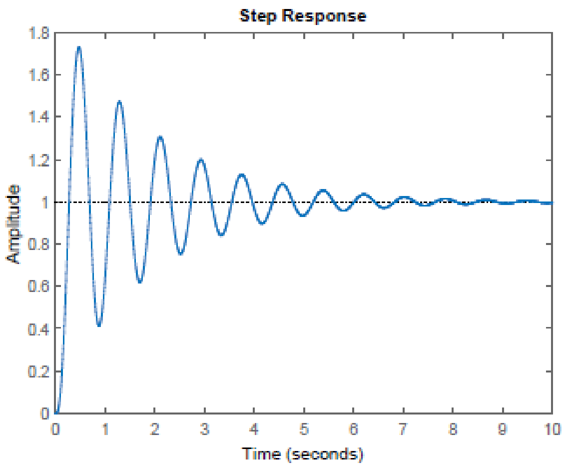


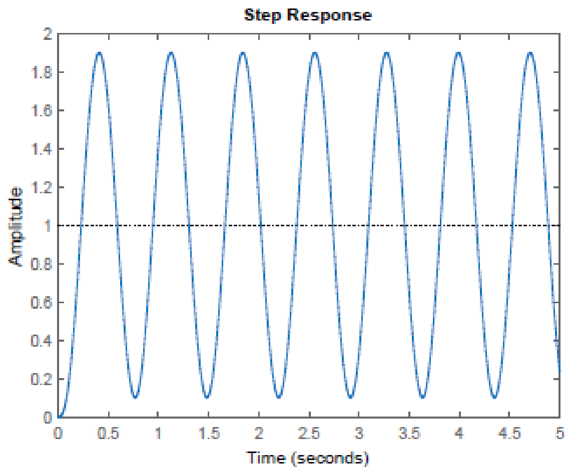
TABLE 6.15 Routh table for Example 6.9

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386-K}{18}$	
s^0	K	

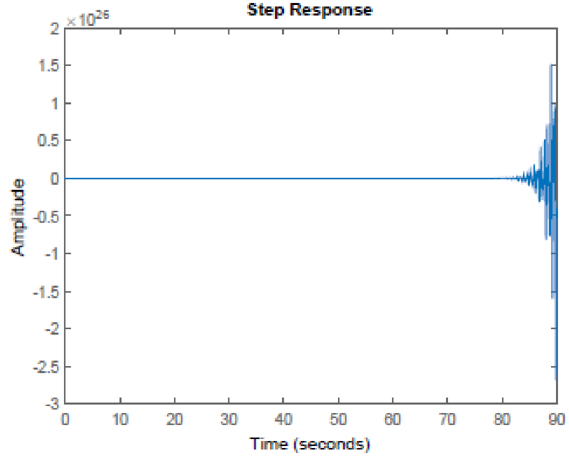
Verification



$K = 1000$



$K = 1386$



$K = 2000$

Figure 10: Stability verification with different values of K

Attention!

- In many examples in the textbook, the author jumps to the conclusion that the system is stable simply by showing that there are no sign changes in the first column of the RH table.
- This is not a proper proof that a system is stable.
- The number of sign changes in the first column of the RH table only indicates the number of roots in the right-hand side of the s plane.
- You still need to show that there are no poles at the origin of the plane and that there are no repeated complex conjugate roots on the imaginary axis to conclude that a system is stable.

Matlab

Ch6sp1 (Appendix F) shows how to derive the Routh-Hurwitz table even for systems with variable gain.

Examine ch6p1, ch6p2, and ch6p3 (Appendix B), and ch6sp2 (Appendix F).

Student Companion Site (Free) [html] Nise: Control Systems Engineering, 7th Edition - Student Companion Site (wiley.com)

References

Nise, N. S. (2015).
Control Systems Engineering.
Wiley, 7 edition.