

# HW 11 (Optional) - Continuous time LQR



Consider the mechanical system given in Figure 1, where  $\xi_1, \xi_2$  are displacements of the masses,  $u_1, u_2$  are forces acting on the masses. Take the masses to be  $m_1 = m_2 = 1$  kg, the spring stiffness to be  $k_1 = k_2 = k_3 = 1$  N/m, and the damping coefficient to be  $\nu = 1$  Ns/m.

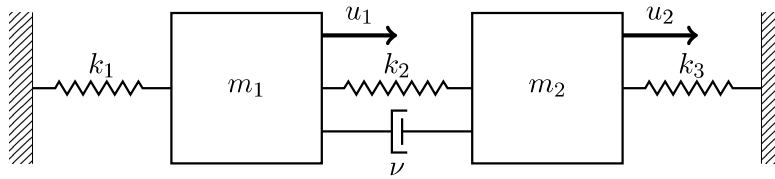


Figure 1: Simple spring-damper system

1. Write the equations of motion of the system and put them into state space form, using as state vector  $x(t) = [\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2]' \in \mathbb{R}^4$ .
2. Consider the infinite horizon quadratic cost

$$J = \int_0^\infty (\xi_1(t)^2 + \xi_2(t)^2 + u_1(t)^2 + u_2(t)^2) dt$$

Recall that in the infinite horizon case, the Riccati solution  $P_t$  to the Riccati equation is constant (does not depend on time), and it is a solution to the Algebraic Riccati Equation (ARE) given by

$$A^T P + P A - P B R^{-1} B^T P + Q = 0$$

Solve the ARE for the cost function  $J$  using MATLAB. Given the matrices  $A, B, Q, R$ , you can use the `care` command in Matlab to solve the ARE.

3. Using the solution  $P$ , give the expression of the optimal control in the form  $u^*(t) = Kx(t)$ . Verify that you obtain the same  $K$  matrix using the `lqr` command in Matlab.
4. By plugging the optimal control  $u^*(t)$  into the system dynamics  $\dot{x}(t) = Ax(t) + Bu(t)$ , write the dynamics of the closed-loop system in the form  $\dot{x}(t) = \tilde{A}x(t)$ .
5. Plot the trajectory  $\xi_1(t), \xi_2(t)$  of the system, on the same figure, for the following cases
  - open-loop system with zero input ( $u(t) = 0$ ) and initial condition  $x(0) = [1, 0, 0, 0]^T$ . Plot for time  $t \in [0, 15]$ .
  - closed-loop system (optimal control  $u^*(t) = Kx(t)$ ) with initial condition  $x(0) = [1, 0, 0, 0]^T$ .

Plot for time  $t \in [0, 10]$ .

*Hint:* You can use the `initial` command in Matlab.

For the following questions, we will always consider the closed-loop system with optimal control and initial condition  $x_0 = [1, 0, 0, 0]^T$ .

6. Plot the optimal control inputs  $u_1^*(t), u_2^*(t)$ .
7. Assume that the actuator on the first mass is limited, and can only deliver a force of magnitude less than 0.2. Observe that the current optimal control  $u_1^*(t)$  does not satisfy this constraint. We will modify the cost function  $J$  in order to satisfy the constraint. Consider the cost function

$$J_\alpha = \int_{t=0}^{\infty} (\xi_1(t)^2 + \xi_2(t)^2 + \alpha u_1^2(t) + u_2^2(t)) dt$$

Write a loop in Matlab that finds the minimal value for  $\alpha$  (up to a precision of  $10^{-1}$ ) for which the optimal control satisfies  $\sup_t |u_1^*(t)| \leq 0.2$ . Plot  $\xi(t)$  and  $u^*(t)$  of the closed-loop system for this value of  $\alpha$ . Compare to the trajectories in the nominal case  $\alpha = 1$ . Attach your code for this question.

8. Suppose now that we would like to limit the velocity of the first mass to be always less than 0.5 in absolute value. First, plot the velocities  $\dot{\xi}(t)$  for the nominal system (cost function  $J$ ), and observe the constraint is not satisfied. Now consider the modified cost function

$$J_\beta = \int_{t=0}^{\infty} (\xi_1(t)^2 + \xi_2(t)^2 + \beta \dot{\xi}_1(t)^2 + u_1^2(t) + u_2^2(t)) dt$$

Write a loop that finds the minimal value of  $\beta$  (up to a precision of  $10^{-1}$ ) such that the resulting trajectory satisfies  $\sup_t |\dot{\xi}_1(t)| \leq .5$ . For this value of  $\beta$ , plot  $\xi_1(t), \xi_2(t), \dot{\xi}_1(t), u_1(t), u_2(t)$ . Compare the system's behavior to the trajectories of the nominal case ( $\beta = 0$ ). In particular, comment on changes of the control input.