Your **PRINTED FULL NAME**

SOLUTIONS

Your **STUDENT ID NUMBER**

Number of additional sheets

1. No computers, no tablets, no connected device (phone etc.)
2. Pocket calculator allowed
3. Closed book, closed notes, closed internet
4. Allowed: 1 page (double sided) Chi Chi
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| Part | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Score | 9 | $9+2$ | 22 | $10+2$ |

1. Laplace Transforms

   (a) (4pts) Use Laplace transform tables to derive the Laplace transform for the following time function

$$e^{-at}\sin(wt)\cos(wt)$$

**Answer:** $\dfrac{w}{(s+a)^2+4w^2}$

Using the trigonometric properties

$$e^{-at}\sin(wt)\cos(wt) = \frac{1}{2}e^{-at}\sin(2wt) \quad \textbf{(1pt)}$$

Using the frequency shift property and table of laplace transforms

$$e^{-at}f(s) = f(s+a) \quad \textbf{(1pt)}$$

$$\sin(2wt) = \frac{2w}{s^2+4w^2} \quad \textbf{(1pt)}$$

Combining the two we get,

$$\begin{aligned}\frac{1}{2}e^{-at}\sin(2wt) &= \frac{1}{2}\frac{2w}{(s+a)^2+(2w)^2} \\ &= \frac{w}{(s+a)^2+4w^2} \quad \textbf{(1pt)}\end{aligned}$$

   (b) (5pt)Use Laplace transforms to solve the following ODE. Assume all forcing functions are zero prior to $t = 0^-$.

$$\frac{d^2x}{dt^2} = 4e^{-t}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)$$

$$x(0) = 2,\ x'(0) = 3$$

**Answer:** $4t + 1 + e^{-t}\cos(t)$

Taking the Laplace transform of the ODE then solving for X(s),

$$s^2X(s) - sx(0) - x'(0) = \frac{2}{(s+1)^2+1} \quad \textbf{(1pt)}$$

$$\begin{aligned}s^2X(s) &= 2s + 3 + \frac{2}{(s+1)^2+1} \\ &= \frac{2s^3+7s^2+10s+8}{s^2+2s+2}\end{aligned}$$

$$X(s) = \frac{2s^3+7s^2+10s+8}{s^2(s^2+2s+2)} \quad \textbf{(1pt)}$$

Doing partial fraction decomposition we get,

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{s^2 + 2s + 2} \quad (\textbf{1pt})$$

$$= \frac{(B + C)s^3 + (A + 2B + D)s^2 + (2A + 2B)s + 2A}{s^2(s^2 + 2s + 2)}$$

Solving for coefficients A,B,C,D **(1pt)**

$$A = \left| \frac{2s^3 + 7s^2 + 10s + 8}{s^2(s^2 + 2s + 2)} \right|_{s=0} = 8/2 = 4$$

$$(2A + 2B) = 10$$

$$B = \frac{10 - 2A}{2} = 1$$

$$B + C = 2$$

$$C = 1$$

$$A + 2B + D = 7$$

$$D = 1$$

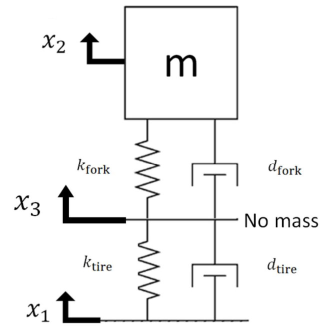
**Taking the laplace transform with the coefficients,**

$$X(s) = \frac{4}{s^2} + \frac{1}{s} + \frac{s + 1}{s^2 + 2s + 2}$$

$$= \frac{4}{s^2} + \frac{1}{s} + \frac{s + 1}{(s + 1)^2 + 1}$$

$$= 4t + 1 + e^{-t}\cos(t) \quad (\textbf{1pt})$$

2. Consider a simplified model of a front mountain bike suspension. The input is the position $x_1(t)$ of the rocky terrain and the output is the position $x_2(t)$ of the person with mass $m$. The spring and damping constants are defined as seen below in the figure. **Ignore the effect of gravity.**




(a) (4pts)Derive the transfer function $T(s) = \frac{X_2(s)}{X_1(s)}$ in terms of $k_{\text{fork}}, k_{\text{tire}}, d_{\text{fork}}, d_{\text{tire}}$ Note $T(s)$ is the transfer function from position $x_1$ to position $x_2$, derive the transfer function accordingly.

**Answer:** $T(s) = \frac{(d_t s+k_t)(d_f s+k_f)}{ms^2[(d_f+d_t)s+k_f+k_t]+(d_f s+k_f)(d_t s+k_t)}$
In order to solve this system we created a variable $x_3$ at the interface between the tire and fork

Writing the system of equations for the mass and at the tire fork interface we get,

for mass

$$m\ddot{x_2} + d_f\dot{x_2} + k_f x_2 = d_f\dot{x_3} + k_f x_3 \quad \textbf{(1pt)}$$

for the tire fork interface

$$(d_f + d_t)\dot{x_3} + (k_f + k_t)x_3 = k_f x_2 + d_f\dot{x_2} + k_t x_1 + d_t\dot{x_1} \quad \textbf{(1pt)}$$

Converting to the Laplace domain

$$(ms^2 + d_f s + k_f)x_2 = (d_f s + k_f)x_3$$

$$[(d_f + d_t)s + k_t + k_f]x_3 = (k_f + d_f s)x_2 + (k_t + d_t s)x_1 \quad \textbf{(1pt)}$$

Using substitution,

$$T(s) = \frac{(d_t s + k_t)(d_f s + k_f)}{ms^2[(d_f + d_t)s + k_f + k_t] + (d_f s + k_f)(d_t s + k_t)} \quad \textbf{(1pt)}$$

(b) (2pts) Make the approximation $k_{\text{tire}} = \infty$ and $d_{\text{tire}} = 0$ and derive the new transfer function $T(s) = \frac{X_2(s)}{X_1(s)}$.

Hint: this is equivalent to ignoring the dynamics of the tire.

**Answer:**

$$T(s) = \frac{(d_f s + k_f)}{ms^2 + d_f s + k_f}$$

This is the same as asking for the transfer function from $x_1$ to $x_3$(the fork tire interface) Using inspection or substituting

$$T(s) = \frac{(d_f s + k_f)}{ms^2 + d_f s + k_f} \quad \textbf{(2pt)}$$

(c) (3pt)Using the second order transfer function obtained from part (b) determine the settling time and percent overshoot for a step response given $k_{\text{fork}} = 200e3$ N/m $d_{\text{fork}} = 10e3$ Ns/m and $m = 10$ kg

**Answer:**

$$T_s = .008\text{s and } \%\text{OS} = 98.9\%$$

Since $k_f >> d_f$ we are able to make the second order approximations used previously in class

The second order approximation is as follows,

$$s^2 + 2\xi w_n s + w_n^2 = s^2 + \frac{d_f}{m}s + \frac{k_f}{m} \quad \textbf{(1pt)}$$

$$w_n = \sqrt{20e3}$$

$$\xi = \frac{d_f}{2mw_n} = 3.535$$

$$T_s = \frac{4}{\xi w_n} \quad \textbf{(1pt)}$$

$$T_s = \frac{8m}{d_f} = \frac{8}{1000} = .008\text{s}$$

$$\%\text{OS} = 0 \quad \text{system is heavily overdamped} \quad \textbf{(1pt)}$$

(d) (2pts) **BONUS:** Write the condition which makes the second order approximation used in part (c) valid.

The form for the second order transfer function used for making second order approximations is:

$$\frac{k}{s^2 + 2\xi w_n + w_n^2}$$

In the case of the simplified bike model the numerator is:

$$k + ds$$

Thus if $k$ is much greater than d, **2pts**

$$k + ds => k$$

3. Consider the following system:

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad with \qquad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \; B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \; C = \begin{bmatrix} 0 & 1 \end{bmatrix} \; x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

(a) (2pts) Compute the eigenvalues of $A$

**Answer:** 3,-1

To find the eigenvalues, solve the equation $\det(\lambda I - A) = 0$ for all $\lambda$ $\quad$ (**1pt**)

$$\begin{vmatrix} \lambda & -1 \\ -3 & \lambda - 2 \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1 \quad (\textbf{1pt})$$

(b) (3pts) Compute the associated eigenvectors of $A$

**Answer:** $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigenvectors and eigenvalues satisfy the property $(A - \lambda_i I)v_i = 0$ $\quad$ (**1pt**)

For $\lambda_1 = 3$ :

$$\begin{bmatrix} 3 & -1 \\ -3 & 3-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$3v_{11} = v_{12}$$

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad (\textbf{1pt})$$

For $\lambda_2 = -1$ :

$$\begin{bmatrix} -1 & -1 \\ -3 & -1-2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{21} = -v_{22}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (\textbf{1pt})$$

(c) (2pts) Find the $P$ and $P^{-1}$ matrices and use them to diagonalize $A$

**Answer:** $P = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$, $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$P$ is constructed by combining the two eigenvectors

$$P = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Taking the inverse of $P$ gives us

$$P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \quad (\textbf{1pt})$$

The diagonalized $D = P^{-1}AP$

$$D = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad (\mathbf{1pt})$$

(d) (3pts) Find $e^{At}$

**Answer:** $e^{At} = \frac{1}{4}\begin{bmatrix} e^{3t} + 3e^{-t} & e^{3t} - e^{-t} \\ 3e^{3t} - 3e^{-t} & 3e^{3t} + e^{-t} \end{bmatrix}$

$$e^{At} = Pe^{Dt}P^{-1} \quad (\mathbf{1pt})$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \quad (\mathbf{1pt})$$

$$= \frac{1}{4}\begin{bmatrix} e^{3t} + 3e^{-t} & e^{3t} - e^{-t} \\ 3e^{3t} - 3e^{-t} & 3e^{3t} + e^{-t} \end{bmatrix} \quad (\mathbf{1pt})$$

(e) (5pts)Compute the output $y(t)$ for a unit step $u(t)$. Use the results from part (d) to determine a soultion for $x(t)$, then find $y(t)$ **Answer:**

$$y(t) = -1 + e^{3t}$$

The general solution to a state space is,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \quad (\mathbf{1pt})$$

Substituting in the initial condition and $e^{At}$

$$x(t) = \int_0^t \frac{1}{4}\begin{bmatrix} e^{3(t-\tau)} + 3e^{-(t-\tau)} & e^{3(t-\tau)} - e^{-(t-\tau)} \\ 3e^{3(t-\tau)} - 3e^{-(t-\tau)} & 3e^{3(t-\tau)} + e^{-(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\mathrm{d}\tau \quad (\mathbf{1pt})$$

$$= \int_0^t \frac{1}{4}\begin{bmatrix} 4e^{3(t-\tau)} \\ 12e^{3(t-\tau)} \end{bmatrix}\mathrm{d}\tau$$

$$= \begin{matrix} -\frac{1}{3}e^{3(t-\tau)} \\ -e^{3(t-\tau)} \end{matrix}\Bigg|_0^t$$

$$= \begin{bmatrix} -\frac{1}{3} + \frac{1}{3}e^{3t} \\ -1 + e^{3t} \end{bmatrix} \quad (\mathbf{1pt})$$

$$y(t) = Cx(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} + \frac{1}{3}e^{3t} \\ -1 + e^{3t} \end{bmatrix} \quad (\mathbf{1pt})$$

$$= -1 + e^{3t} \quad (\mathbf{1pt})$$

(f) (2pts) Given the matrices $A$,$B$, and $C$ above determine the transfer function $G(s)$ for the state space system. How do the poles of the transfer function compare to the eigenvalues found in part (a)? **Answer:** $=$

$$G(s) = \frac{3(s+1)}{s^2 - 2s - 3}$$

$$G(s) = C(sI - A)^{-1}B \quad \textbf{(1pt)}$$

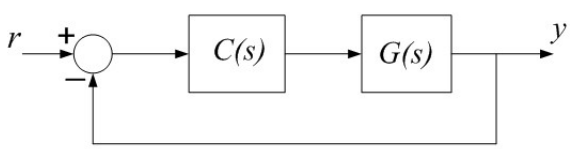
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -3 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{s^2 - 2s - 3} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s-2 & 1 \\ 3 & s \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{s^2 - 2s - 3} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 \\ 3s+3 \end{bmatrix}$$

$$= \frac{3(s+1)}{s^2 - 2s - 3} \quad \textbf{(1pt)}$$

(g) (3pts) Consider the following LTI system:




where $C(s) = \frac{K}{s+4}$ and $G(s) = \frac{3s+3}{s^2-2s-3}$

Determine for what values of $K$ the system is stable and unstable.
**Answer:** Stable for $K > 4$ Unstable for $K < 4$
The closed loop transfer function was first found

$$T = \frac{CG}{1+CG} = \frac{K(3s+3)}{s^3 + 2s^2 + (3k-11)s + (3k-12)} \quad \textbf{(1pt)}$$

Writing out the Routh Hurwitz table

$$\begin{array}{c|cc}
s^3 & 1 & 3K-11 \\
s^2 & 2 & 3K-12 \\
s^1 & -\dfrac{\begin{vmatrix} 1 & K-11 \\ 2 & 3K-12 \end{vmatrix}}{2} = \dfrac{3K-10}{2} & \\
s^0 & \dfrac{\begin{vmatrix} 2 & 3K-12 \\ \dfrac{3K-10}{2} & 0 \end{vmatrix}}{\dfrac{3K-10}{2}} = 3K-12 &
\end{array}$$

**(1pt)**

Since the signs of all the items in the first column need to be the same for the system to be stable,

$$\frac{3K-10}{2} > 0 \implies K > -\frac{10}{3}$$
$$3K - 12 > 0 \implies K > 4$$

The range of $K$ where both these constraints are satisfied is $K > 4$

**(1pt)**

(h) (2pts)What is the steady-state error of the closed-loop system for a unit step input $u(t)$ as a function of $K$ (for values of $K$ that stabilize the system)?

**Answer:**

$$e_{ss}^{closed} = \frac{4}{4-K}$$

The general equation for the closed loop steady state error is,

$$e_{ss}^{closed} = \frac{1}{1+K_p}\textbf{(1pt)}$$

where $K_p = \lim_{s\to\infty} G(s)$

$$K_p = -\frac{K}{4}$$
$$e_{ss}^{closed} = \frac{4}{4-K}\textbf{(1pt)}$$

4. Consider the following LTI system:

$$G(s) = \frac{(s+2)}{(s+.1)(s+1)(s+15)} \tag{2}$$

Plot the root locus for the system

(a) (3pts) Determine the number of branches and any asymptotes (position and angle) that exist.
**Answer:**

$$Branches = 3 \quad Asymptotes = 2 \qquad \text{center:} - 7.05 \quad \text{angle:} \frac{\pi}{2}, \frac{3\pi}{2}$$

The number of branches is equal to the number of open loop poles, 3

$$(\mathbf{1pt})$$

The asymptotes can be determined by the following,

$$\#of Asymptotes = \#OLPoles - \#OLZeros = 3 - 1 = 2 \quad (\mathbf{1pt})$$

$$center = \frac{(-.1 - 1 - 15) - (-2)}{2} = -7.05$$

$$angle = \frac{(2k+1)\pi}{2} \text{ for } k = 0, 1 \implies angle = \frac{\pi}{2}, \frac{3\pi}{2} \quad (\mathbf{1pt})$$

(b) (3pts) How many break in/break away points are there? Be sure to determine the equation that governs the break away points, but note you do not need to explicity find the points.
**Answer:**

$$\#breakawaypoints = 3 \quad 2\sigma^3 + 22.1\sigma^2 + 64.4\sigma + 31.7 = 0$$

The breaks out can be determined by solving the following equation

$$\frac{dK}{d\sigma} = 0 \quad (\mathbf{1pt})$$

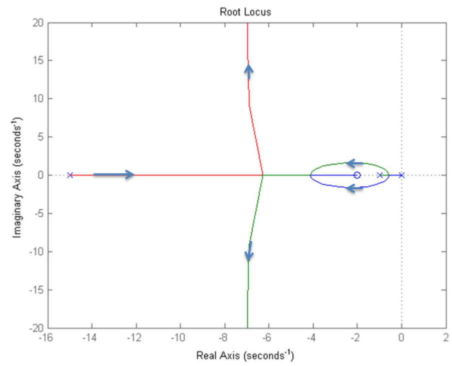
where K is defined as follows,

$$K(\sigma) = -\frac{1}{G(\sigma)} = -\frac{(\sigma+.1)(\sigma+1)(\sigma+15)}{(\sigma+2)} == -\frac{\sigma^3 + 16.1\sigma^2 + 16.6\sigma + 1.5}{(\sigma+2)} \quad (\mathbf{1pt})$$

$$\frac{dK}{d\sigma} = \frac{2\sigma^3 + 22.1\sigma^2 + 64.4\sigma + 31.7}{(\sigma+2)^2} = 0$$

$$2\sigma^3 + 22.1\sigma^2 + 64.4\sigma + 31.7 = 0 \quad (\mathbf{1pt})$$

(c) (4pts) Sketch the root locus. Make sure to label the asymptotes you found in part (a) and the direction of the root locus for increasing $K$ (For ease of plotting assume the breakaways occur at -6,-4,and -.5). **Answer:**




(**1pt** arrows)

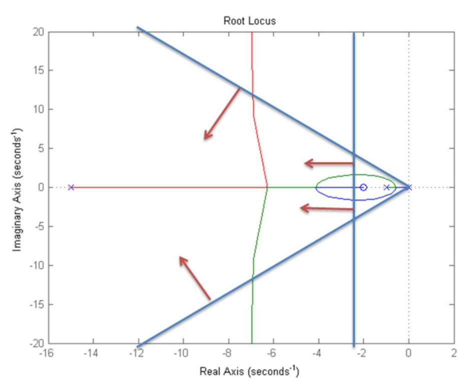
(**1pt** zeros/poles)

(**1pt** asymptotes)

(**1pt** correct figure)

All poles and zeros are located on the real axis. Note how the asymptotes centered at -7.05 affect the root locus. The direction of the root locus is away from the poles and towards the zeros for increasing K.

(d) (2pts) **BONUS:**Plot the feasible region for the poles if the design requirment is to have a minimum damping ratio of 0.5 and a settling time less than 1.6 seconds. (You may use the second order approximation for settling time). Given the region you drew, is it feasible to meet the design requirments (explain)? **Answer:** It is feasible to meet the design requirment!



**(2pt)**

The red arrows indicate the acceptable region to meet the design specs.
Using the second order approximations,

$$\xi = cos(\theta), \theta \text{ is the angle from the negative real axis}$$

$$\theta = 60\,^{\circ}$$

$$T_s = \frac{4}{\xi w_n} = \frac{4}{Re(s)}$$

$$w_n = \frac{4}{\xi 1.6} = 5 \qquad Re(s) = 2.5$$

| $f(t)$ | $F(s)$ |
| --- | --- |
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $tu(t)$ | $\frac{1}{s^2}$ |
| $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ |
| $\sin(\omega t)u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)u(t)$ | $\frac{s}{s^2+\omega^2}$ |

Table 1: Laplace transforms of common functions

| | |
| --- | --- |
| $\sin(2\theta)$ | $2\cos(\theta)\sin(\theta)$ |
| $\cos(2\theta)$ | $\cos(\theta)^2 - \sin(\theta)^2$ |
| $\tan(2\theta)$ | $\dfrac{2\tan(\theta)}{1-\tan(\theta)^2}$ |

Table 2: Trigonometric functions