Chapter 9: Design via Root Locus

ELG 3155 : Introduction to Control Systems

- Objective:
  - Using RL to design compensators to improve the closed-loop system's
    - steady-state error
    - transient response
  - Learning how to physically implement various types of compensators

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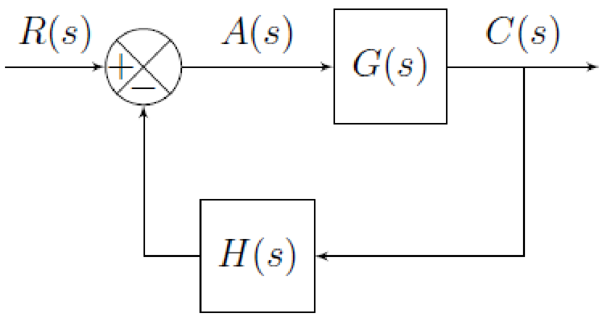
Outline

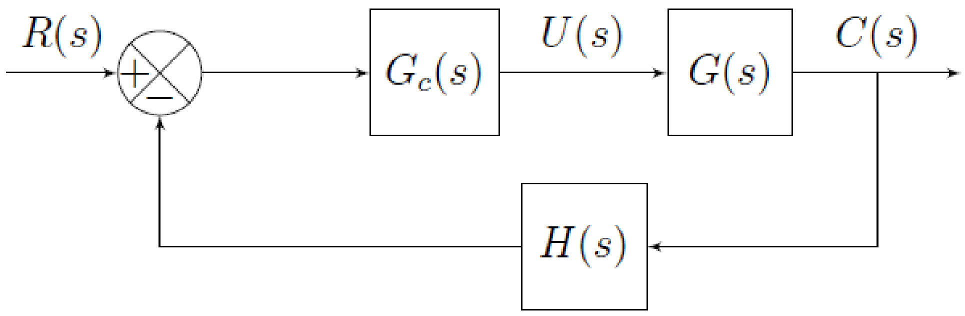
1. Introduction
2. Improving Steady-State Error via Cascade Compensation
    1. Adding a Compensator Without Changing the System’s Operating Point
    2. PI Compensator
    3. Phase-Lag Compensator
3. Improving Transient Response via Cascade Compensation
    1. PD Compensator
    2. Phase-Lead Compensator
4. Improving Transient Response and e_{ss}
    1. PID Compensator
    2. Phase Lead-Lag Compensator
5. Physical Realization of Compensation
    1. Active Circuit Realization
    2. Passive Circuit Realization

1. Introduction

- A compensator is added to a closed-loop system to modify its behavior (by changing its operating point) and force it to satisfy certain desired criteria (e.g., steady-state error, overshoot, settling time, etc.)
- A compensator can be added to the forward path (cascade compensator) and/or to the feedback loop (feedback compensator).



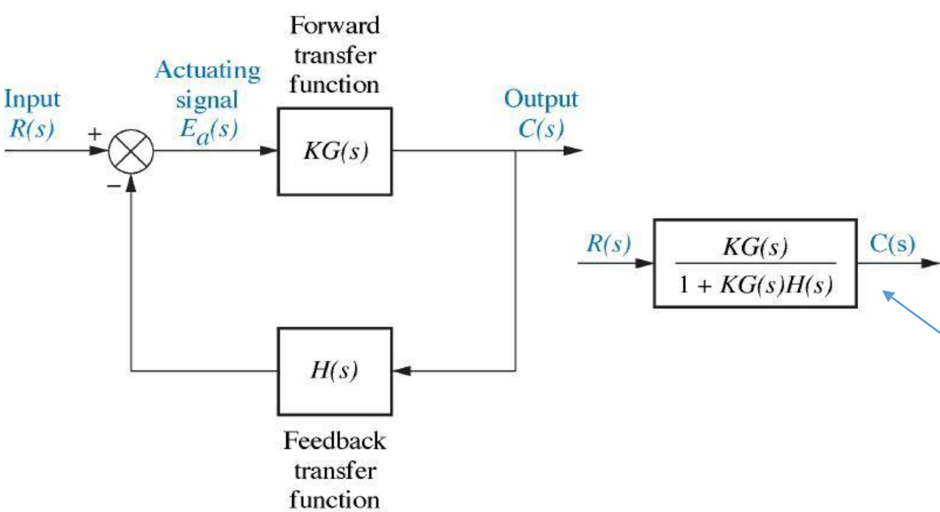
Uncompensated system



Compensated system

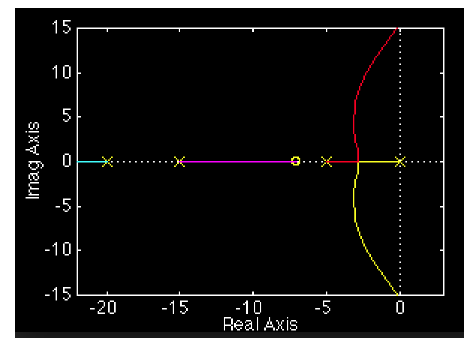
Root Locus Techniques

- Root locus is a graphical presentation of the closed-loop poles as a system parameter k is varied.
- The graph of all possible roots of this equation (K is the variable parameter) is called the root locus.
- The root locus gives information about the stability and transient response of feedback control systems.



(a) Closed-loop system

(b) Equivalent Transfer Function



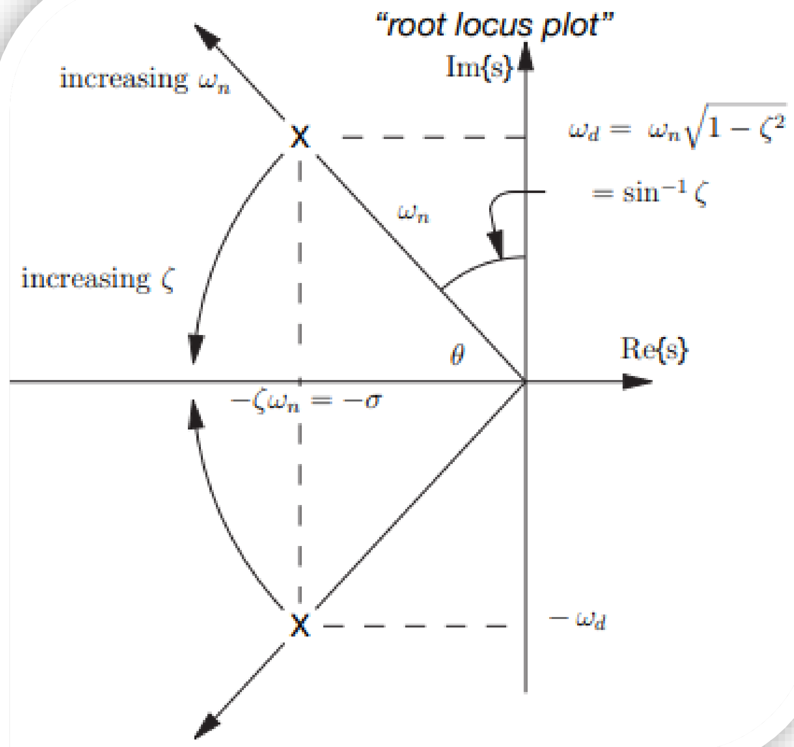
Rout-locus (poles motion graph)

Zeros of T(s) are zeros of G(s) and poles of H(s).

Poles of T(s) depends on gain K

CLCF is a function of K.
Root Locus graphically shows poles of T(s) as *K* varies

Recall: Damping ratio and pole location



Recall 2^{nd}–order underdamped sustem

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Complex poles

$$s_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$
$$= \underbrace{-\zeta\omega_n}_{-\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta} \Rightarrow \boxed{\cos\theta = \zeta.}$$

The angle θ that a complex pole subtends to the origin of the *s*-plane determines the damping ratio ζ of an underdamped 2^{nd} order system.

Design Specifications

- Relative stability
- Steady-state accuracy
- Transient response characteristics
- ….

Compensation Configurations

- Cascade compensation
- Feedback compensation.

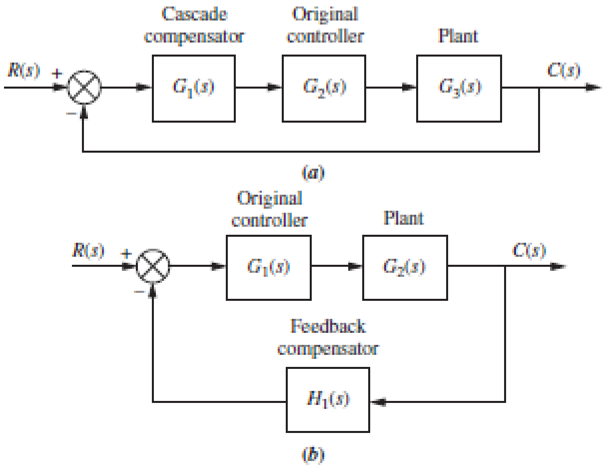


FIGURE 9.2 Compensation techniques: **a.** cascade; **b.** feedback

2. Improving Steady-State Error via Cascade Compensation

Mission

1. Design a compensator to eliminate (or at least improve) the steady-state error
2. without changing the closed-loop system’s operating point (poles).

2.1 Adding a Compensator Without Changing the System's Operating Point

- Imagine a compensator $G_c(s) = \frac{s+zc}{s+pc}$ with a zero $-z_c$ and pole $-pc$, is added in cascade to an uncompensated system to form a compensated system (Fig. 1).

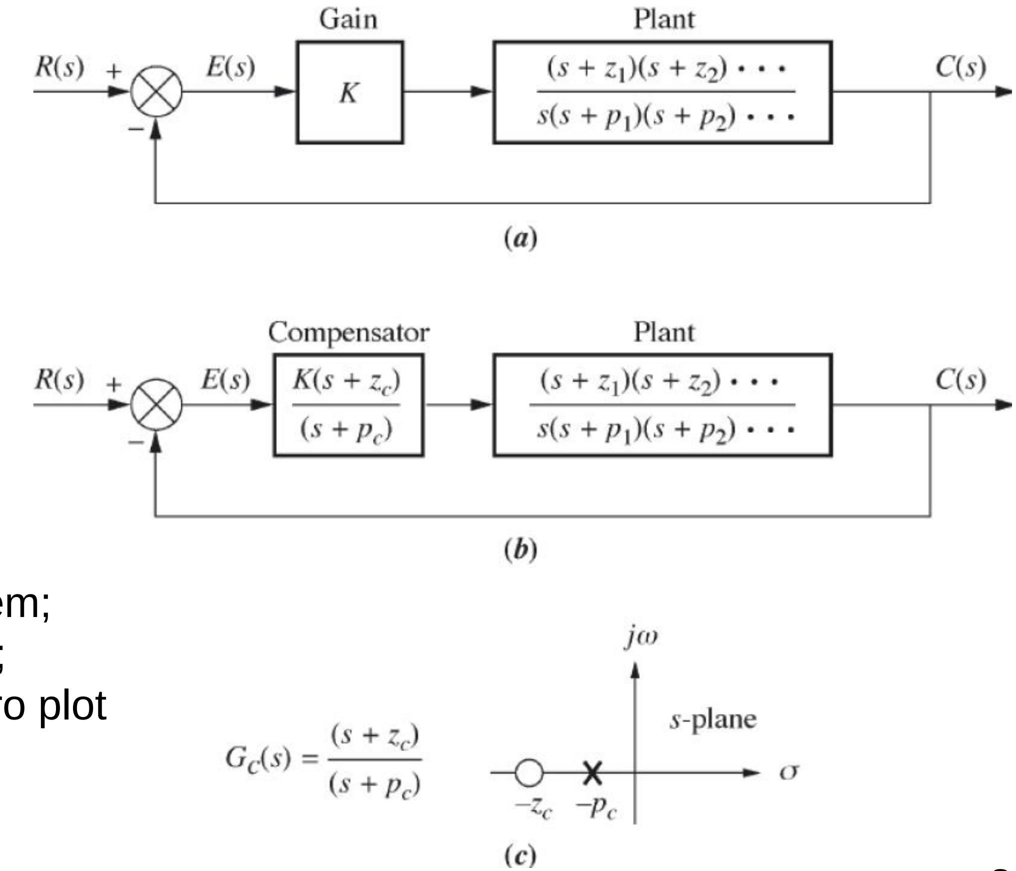


Figure 1: (a) Uncompensated system;
(b) Compensated system;
(c) Compensator pole-zero plot
(Fig. 9.9 of [1]).

- Then, the amplitude and phase conditions for any point s on the root loci of both systems are:

Uncompensated (Original) System

1. $$\frac{\prod_i |s+z_i|}{|s|\prod_i |s+p_i|} = \frac{1}{|K|}$$
2. $$\left(\sum_i \angle s+z_i - \sum_i \angle s+p_i\right) = \angle -1/K$$

Compensated System

1. $$\frac{\prod_i |s+z_i|}{|s|\prod_i |s+p_i|} \cdot \frac{|s+z_c|}{|s+p_c|} = \frac{1}{|K|}$$
2. $$\left(\sum_i \angle s+z_i - \sum_i \angle s+p_i\right) + \left(\angle s+z_c - \angle s+p_c\right) = \angle -1/K$$

- So that the compensator does not change the root locus of the system, we must have $\left|\frac{s+z_c}{s+p_c}\right| = 1$ and $\angle s+z_c - \angle s+p_c) = 0$ for every point s (P in Fig. 2) on the root locus of the compensated system.

- The only solution to this problem is $z_c = p_c$.
- However, if $z_c = p_c$, then $G_c(s) = \frac{s+zc}{s+pc} = 1$, and so the compensator will have no effect at all on the system which then remains uncompensated (not what we want!)
- A practical solution is to have $z_c \approx p_c$ (but not equal).
- This way, the root locus of the compensated system remains practically the same as that of the uncompensated system (although slightly different in theory).

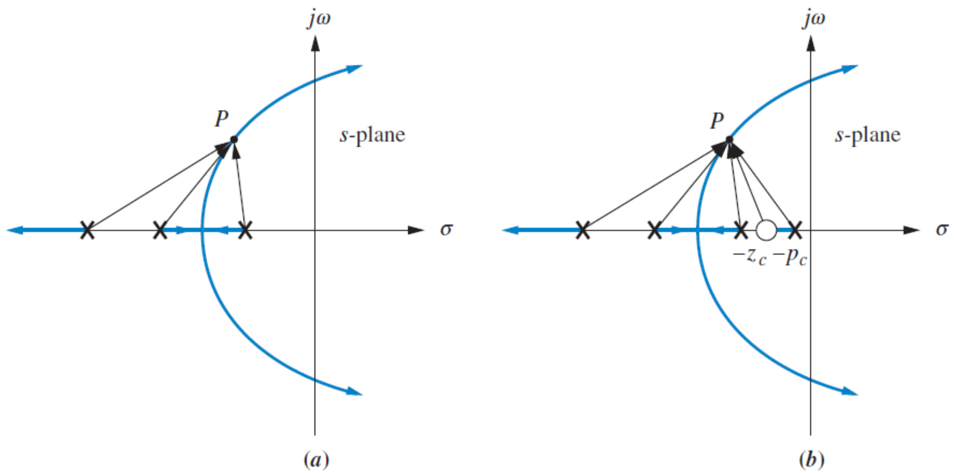


Figure 2: Root locus of (a) uncompensated system; and (b) compnsated system (Fig. 9.10 of [1]).

2.2 PI Compensator

The PI compensator's transfer function is of the form

$$G_c(s) = K_p + K_i \frac{1}{s} = \frac{K_p s + K_i}{s} \equiv K \frac{(s - z_c)}{s}$$

Where $K = K_p$, and $z_c = -K_i/K_p$ is the compensator's zero.

- The PI controller can eliminate (or at least improve) the steady-state error by adding a pole $p_c = 0$ (at zero) to the OLTF which increases the system's type.
- Since this pole $p_c = 0$ is independent of the controller's gains K_p and K_i , any nonzero values of those gains will do.
- However, in order not to change the system's transient response (operating point), the controller's zero $z_c = -K_i/K_p$ must be almost equal to the pole pc = 0 (as seen earlier).
- Thus, K_p and K_i must be chosen so that the zero $z_c = -K_i/K_p$ is very close to the pole $p_c = 0$ (on its left). Why on the left?

Example 1

Consider the following uncompensated system,

where $G_c(s) = K$, $G(s) = \frac{1}{(s+1)(s+2)(s+10)}$ and $K > 0$

- Design a P-controller $G_c(s) = K$ so that the system overshoot is 57.4%?
- Compute the closed-loop poles, transient characteristics, and the steady-state error for a unit-step input.
- Now, design a PI-controller $G_c(s) = K\frac{(s-z_c)}{s}$ to annihilate the actuation error while maintaining the overshoot at 57.4%.
- Re-compute the closed-loop poles, transient characteristics, and the steady-state error for a unit-step input.

Solution

- The design of a P compensator (uncompensated system) may be done in two ways:
  1. Numerically: using Matlab's sisotool, for instance; by varying K until reaching the desired criteria. This is the most accurate method.
  2. Manually: either graphically or analytically.

First manual method (graphical)

1. This method is based on finding the intersection between the root locus and the line(s) of interest (the one(s) corresponding to the desired criteria). This leads to the desired operating point, as illustrated by a few examples in Fig. 3.
2. Then, graphically measure the coordinates of the operating point (or/and the natural frequency ωn).
3. Finally, graphically calculate the value of the gain K by measuring the distance between the operating point and each of the open-loop pole and zero, and applying the magnitude condition (refer to rule 10 of sketching the root locus in Chapter 8).

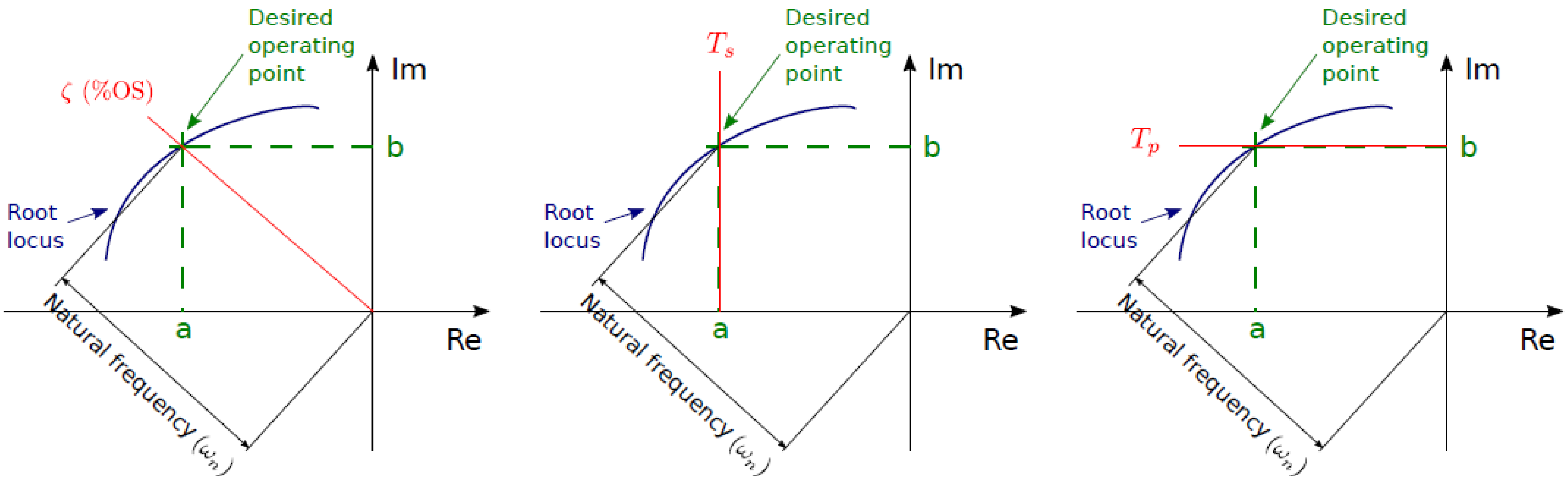


Figure 3: Examples of compensators' graphical design strategies

Second manual method (analytical)

1. This method is based on expressing the coordinates of the desired operating point parametrically using the desired criteria.
2. Then, factorize the characteristic polynomial in terms of the closed-loop poles, equate the coefficients, and solve for the unknowns.
3. Finally, the value of the gain K can be calculated using any of the methods seen in Chapter 8.

- Table 1 summarizes the characteristics of the numerically-designed uncompensated system of Example 1 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed uncompensated system shows that this design was quite close from satisfying the design objective ($\%OS = 57.4\%$).
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is a valid assumption, as can be deduced from the closed-loop poles.
- Figs. 4 and 5 show the operating points and the unit-step responses of the analytically- and numerically-designed uncompensated systems of Example 1.

Table 1: Uncompensated system characteristics of Example 1

	Analytical design	Numerical validation (`sisotool`)	Numerical design (`sisotool`)
Compensator	K	K	K
K	164.8	164.8	179.5
Conjugate poles	$-0.694 \pm j3.93$	$-0.693 \pm j3.93$	$-0.639 \pm j4.08$
Other poles	-11.6	-11.6	-11.7
ω_n (rad/s)	3.99	3.99	4.13
ζ	0.174	0.174	0.155
%OS	57.4	54.1	57.4
T_p (s)	0.80	0.89	0.86
T_s (s)	5.76	5.14	5.69
Position constant	8.24	-	-
$e(\infty)$	0.11	0.11	0.10
2^{nd}-ordrer approximation	OK	confirmed	-

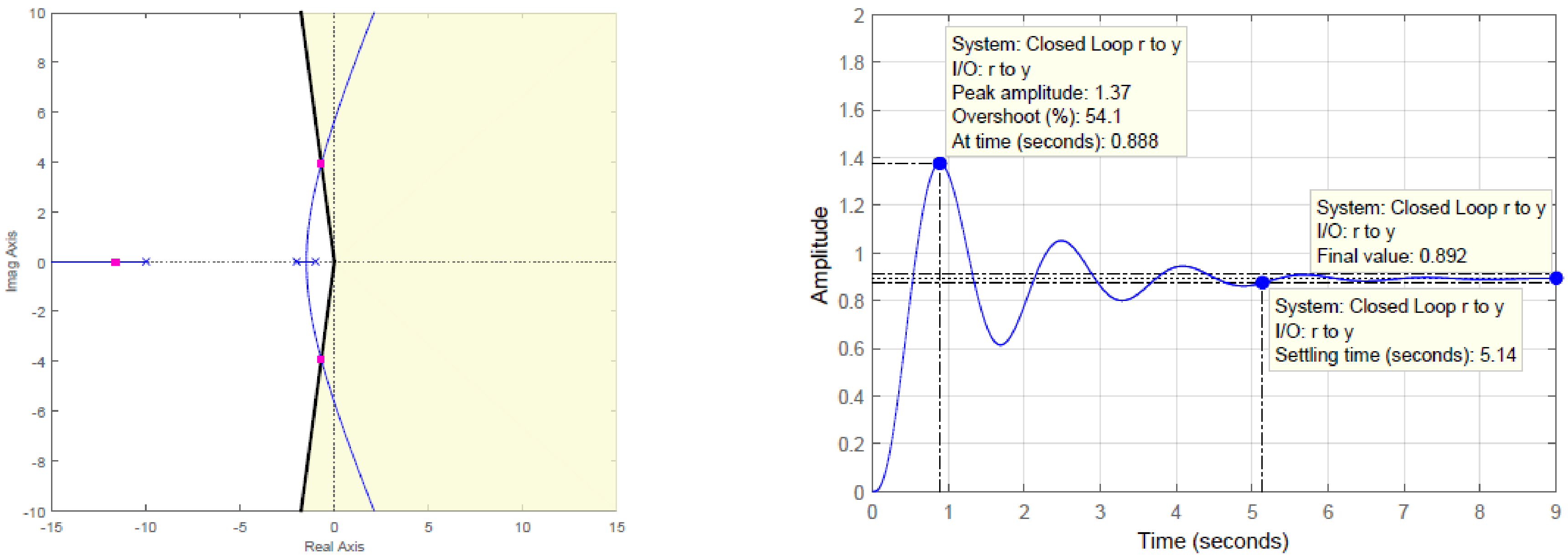
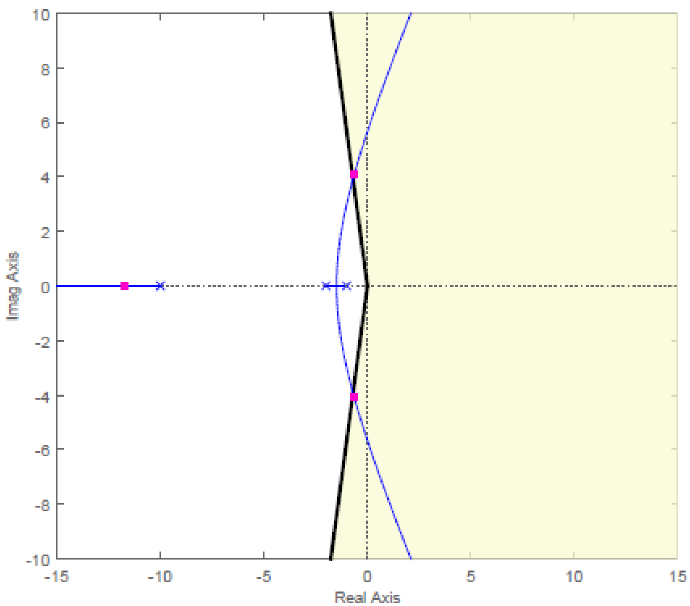


Figure 4: Operating point and step response of the analytically-designed uncompensated system in Example 1



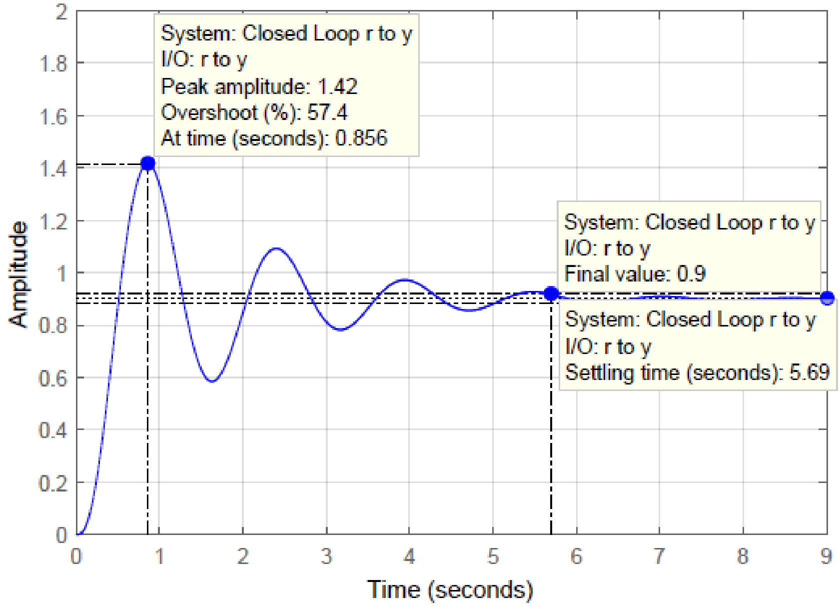


Figure 5: Operating point and step response of the numerically-designed uncompensated system in Example 1

- The design of a PI compensator $G_c(s) = K\frac{(s-z_c)}{s}$ that minimizes the changes on the transient response is often performed following the following steps:
  1. Choose a compensator's zero as close as possible to its pole.
  2. Find the gain K using one of the methods described to design a P compensator.
  3. If the design criteria are not met, then move the compensator's zero closer to repeat the procedure.

- Table 2 summarizes the characteristics of the numerically-designed compensated system of Example 1 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed compensated system shows that this design failed in satisfying the design objective $(\%OS = 57.4\%)$.
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is not the case.
- Figs. 6 and 7 show the operating points and the unit-step responses of the analytically- and numerically-designed compensated systems of Example 1.

Table 2: Compensated system characteristics of Example 1

	Analytical design	Numerical validation (`sisotool`)	Numerical design (`sisotool`)
Compensator	$K(s+0.1)/s$	$K(s+0.1)/s$	$K(s+0.1)/s$
K	145.8	145.8	227.5
Conjugate poles	$-0.656 \pm j3.71$	$-0.725 \pm j3.7$	$-0.435 \pm j4.49$
Other poles	$-11.5, -0.09$	$-11.5, -0.09$	$-12, -0.09$
ω_n (rad/s)	3.77	3.77	4.51
ζ	0.174	0.192	0.097
%OS	57.4	35.8	57.4
T_p (s)	0.85	0.94	0.78
T_s (s)	6.1	18.8	14.3
Position constant	∞	-	-
$e(\infty)$	0	0	0
2^{nd}-ordrer approximation	NO	confirmed	-

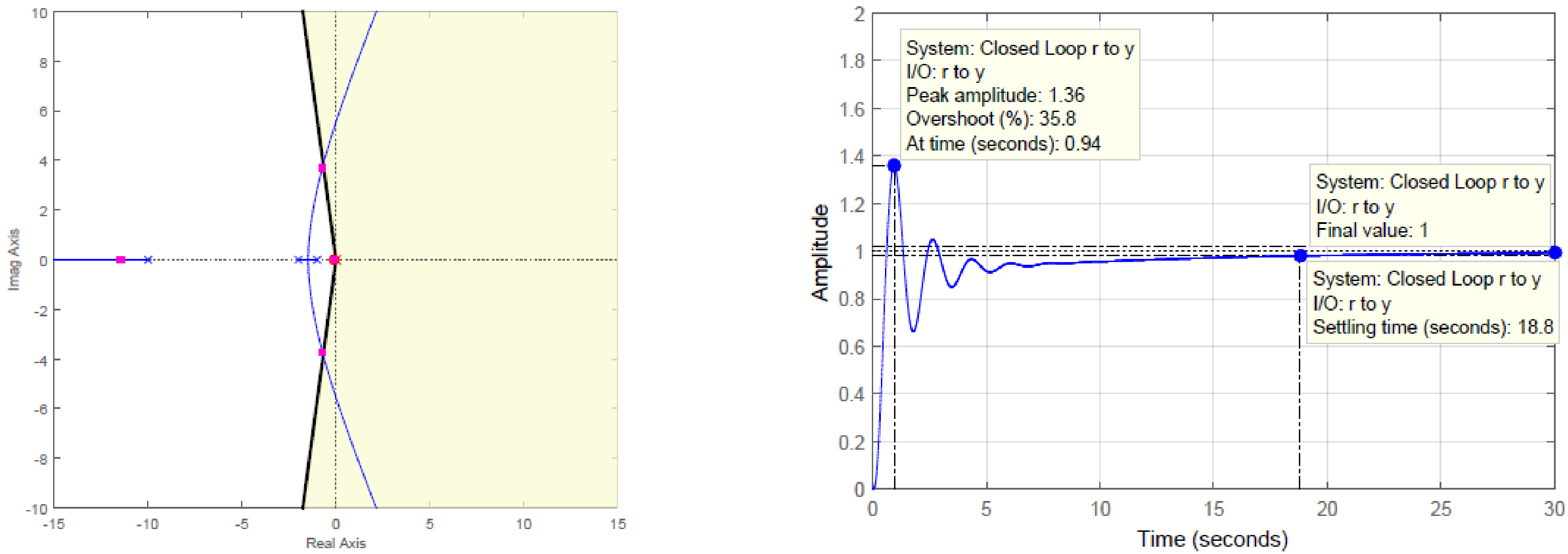


Figure 6: Operating point and step response of the analytically-designed compensated system in Example 1

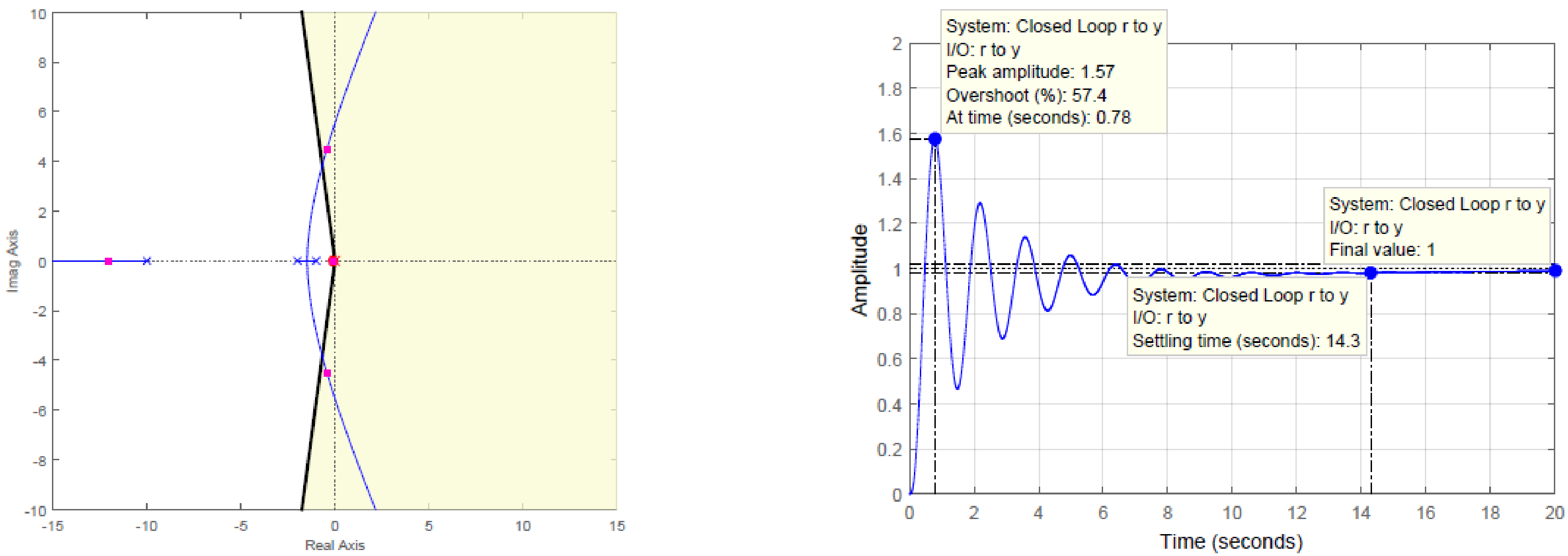


Figure 7: Operating point and step response of the numerically-designed compensated system in Example 1

Exercise

Solve Skill-Assessment Exercise 9.1 with a PI compensator.

Note that this type of design problems do not have a unique solution.

Implementation

- A PI compensator can only be implemented using an active circuit (with an op-amp).
- This is a disadvantage of the PI compensator.

2.3. Phase-Lag Compensator

The phase-lag compensator's transfer function is of the form

$$G_c(s) = \frac{s+z_c}{s+p_c}, \text{ Where } p_c \neq 0 \text{ and } \frac{|z_c|}{|p_c|} \gg 1$$

- If the pole $-p_c$ is chosen to be very close to 0 (on its left), and
- the zero $-z_c$ is chosen to be very close to the pole $-p_c$ (on its left) such that $\frac{|z_c|}{|p_c|} \gg 1$ (as in Fig. 1(c)), then
- the lag compensator can have the same practical effect as the PI controller.

Example

Placing the pole $-p_c = -0.001$ very close to 0 and the zero $-z_c = -0.01$ very close to the pole $-p_c$ (on its left)

- This indeed is a lag compensator since $-p_c \neq 0$ and $\frac{|z_c|}{|p_c|} \gg 1$ (10 in this case)
- $-p_c$ is so close to $-z_c \Rightarrow$ Almost no practical effect on the system's operating point and so the system's transient response.

Remarks

- The lag compensator does not change the type of the system.
- Hence, it cannot completely annihilate the steady-state error.
- Instead, it can reduce the error to practically nil ($e_{ss} \approx 0$).

Example

Solve Example 9.2 of Nise.

Note that this type of design problems may not have a unique solution.

Exercise

Solve Skill-Assessment Exercise 9.1 with a lag compensator.

Note that this type of design problems do not have a unique solution.

Implementation

- A lag compensator can be implemented using either an active circuit or a passive one.
- This is an advantage of the lag compensator over the PI type.

3. Improving Transient Response via Cascade Compensation

Mission

1. Design a compensator to force the system's transient response to satisfy certain criteria (e.g., new overshoot, settling time, etc.)
2. This is accomplished by changing the closed-loop system's operating point (pole locations).

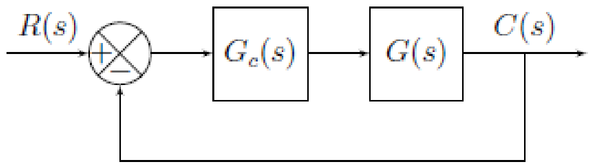
3.1 PD Compensator

- This mission can be accomplished by a PD compensator.
- The PD compensator adds a zero to the OLTF which changes the system's root locus and so its operating point.
- As a result, this type of compensation changes the transient response of the system by changing the locations of the closed-loop poles.

PD Compensator

The PD compensator's transfer function is of the form

$G_c(s) = K_p + K_d s = K_d(s - z_c)$ where $z_c = -K_p/K_d$ is the compensator's zero



Example 3

Consider the following uncompensated system,
where $G_c(s) = K$, $G(s) = 1/s(s+4)(s+6)$ and $K > 0$.

- Design a P compensator $G_c(s) = K$ to yield a 16% overshoot.
- Compute the closed-loop poles and the output transient characteristics for a step input.
- Now, design a PD compensator $G_c(s) = K_d(s - z_c)$ to
  1. maintain the overshoot at 16% and
  2. obtain a settling time of 1.107 s.
- Re-compute the closed-loop poles and the output transient characteristics for a step input.

Solution

- As in Example 1, the design of a P compensator (uncompensated system) may be done in two ways:
  1. Numerically: using Matlab's sisotool, for instance; by varying K until reaching the desired criteria. This is the most accurate method.
  2. Manually: either graphically or analytically.

- Table 3 summarizes the characteristics of the numerically-designed uncompensated system of Example 3 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed uncompensated system shows that this design was quite close from satisfying the design objective ($\%OS = 16\%$).
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is valid assumption, as can be deduced from the closed-loop poles.
- Figs. 8 and 9 show the operating points and the unit-step responses of the analytically- and numerically-designed uncompensated systems of Example 3.

Table 3: Uncompensated system characteristics of Example 3

	Analytical design	Numerical validation (`sisotool`)	Numerical design (`sisotool`)
Compensator	K	K	K
K	43.35	43.35	44.7
Conjugate poles	$-1.21 \pm j2.06$	$-1.20 \pm j2.06$	$-1.19 \pm j2.11$
Other poles	-7.59	-7.59	-7.62
ω_n (rad/s)	2.39	2.39	2.42
ζ	0.504	0.504	0.491
%OS	16	15.1	16
T_p (s)	1.53	1.68	1.63
T_s (s)	3.33	3.47	3.48
2^{nd}-ordrer approximation	OK	confirmed	-

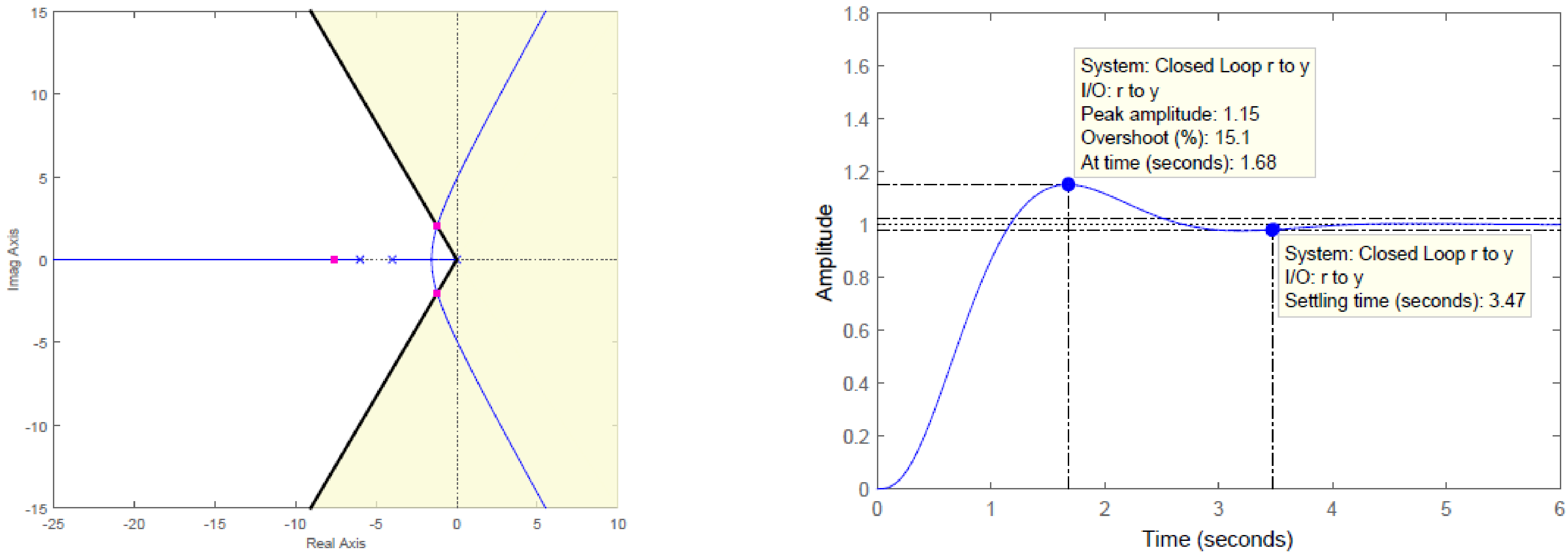
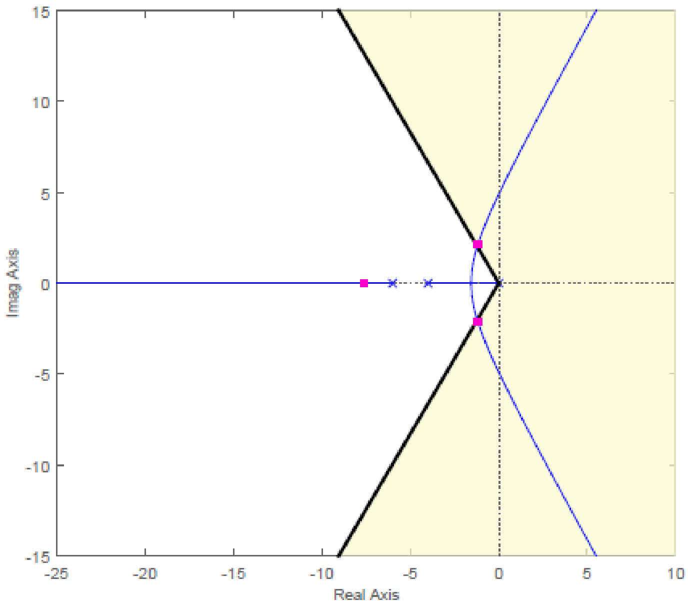


Figure 8: Operating point and step response of the analytically-designed uncompensated system in Example 3



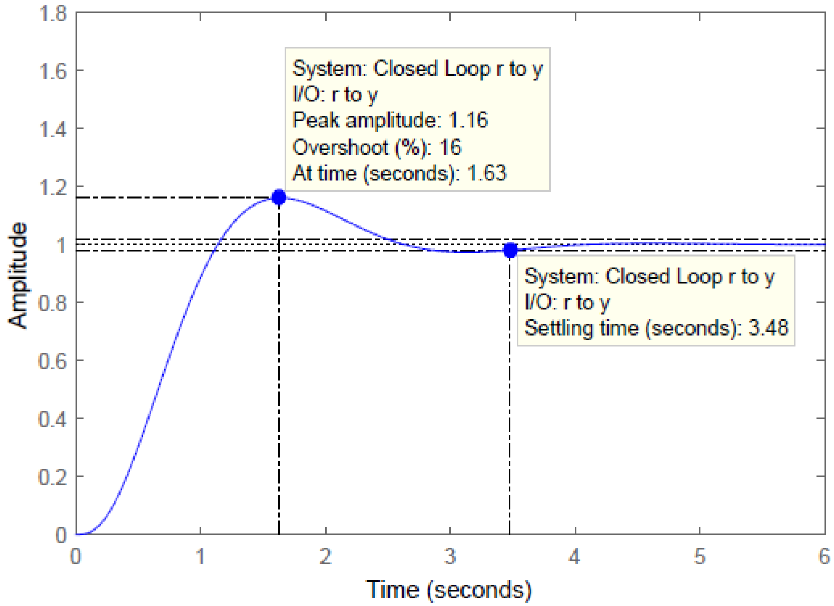


Figure 9: Operating point and step response of the numerically-designed uncompensated system in Example 3

- The design of a PD compensator is often performed following the steps below:
  1. Find the system's desired operating point.
  2. Calculate the compensator's zero corresponding to this operating point.
  3. Find the gain K using one of the methods described to design a P compensator.
  4. If the design criteria are not met, then slightly slide the compensator's zero left or right and repeat the procedure.

- Table 4 summarizes the characteristics of the numerically-designed compensated system of Example 3 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed compensated system shows that this design failed in satisfying the design objectives ($\%OS = 16\%$ and $Ts = 1.1$ s).
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is not the case.
- Figs. 10 and 11 show the operating points and the unit-step responses of the analytically- and numerically-designed compensated systems of Example 3.

Table 4: Compensated system characteristics of Example 3

	Analytical design	Numerical validation (`sisotool`)	Numerical design (`sisotool`)
Compensator	$K(s+3)$	$K(s+3)$	$K(s+3)$
K	47.45	47.45	57.33
Conjugate poles	$-3.613 \pm j6.193$	$-3.62 \pm j6.19$	$-3.59 \pm j6.94$
Other poles	-2.78	-2.77	-2.81
ω_n (rad/s)	7.17	7.17	7.82
ζ	0.504	0.504	0.46
%OS	16	11.9	16
T_p (s)	0.51	0.52	0.46
T_s (s)	1.1	1.15	1.1
2^{nd}-ordrer approximation	NO	confirmed	-

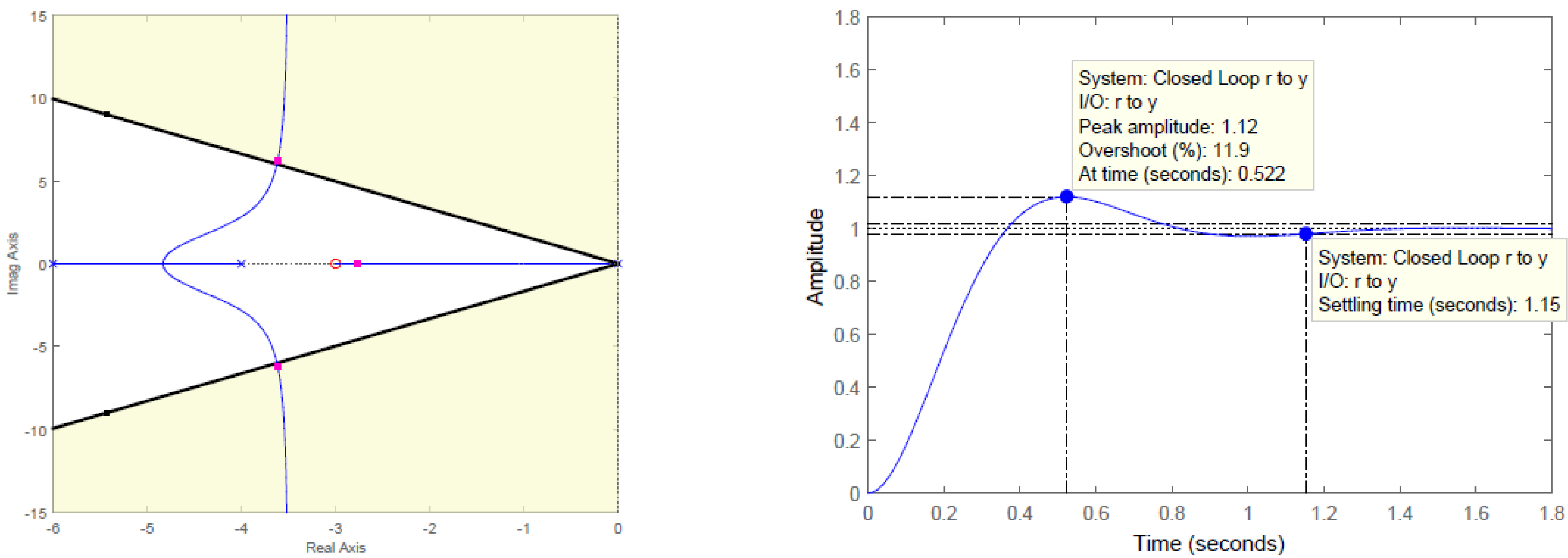


Figure 10: Operating point and step response of the analytically-designed compensated system in Example 3

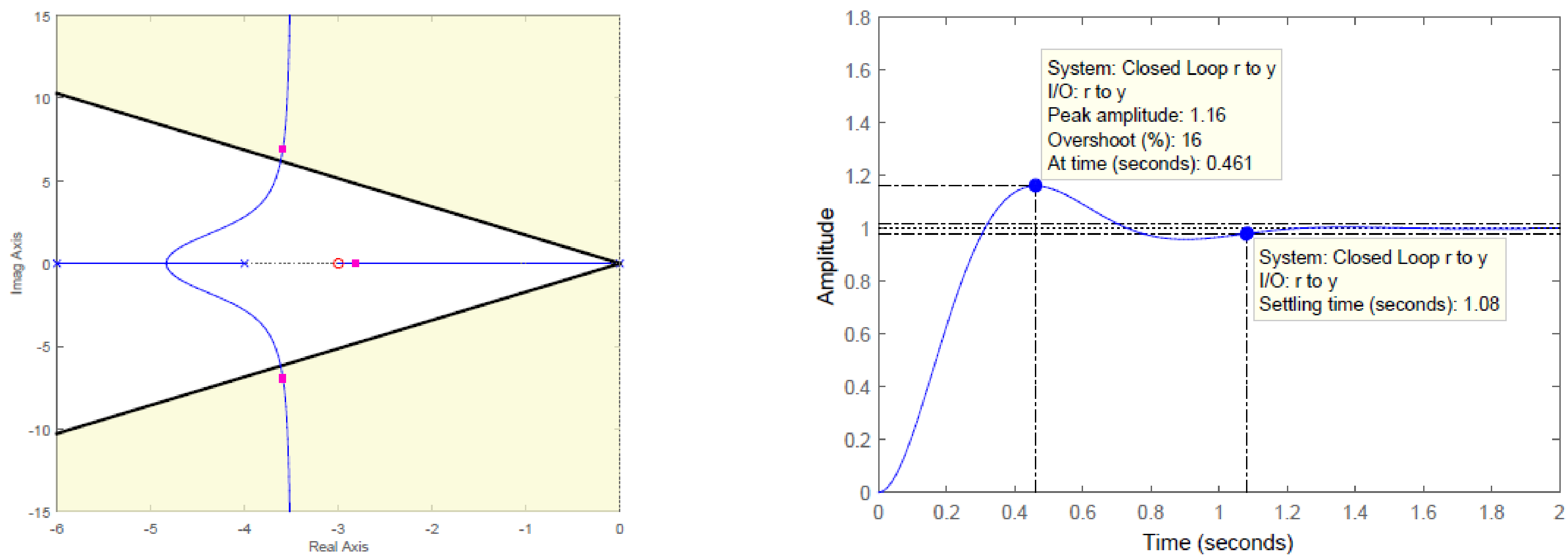


Figure 11: Operating point and step response of the numerically-designed compensated system in Example 3

Exercise

Solve Skill-Assessment Exercise 9.2 with a PD compensator.

Remarks

The PD compensator suffers from two main drawbacks:

1. It can only be implemented using an active circuit.
2. Its derivative term $K_d s$ amplifies the noise at its input terminal.

3.2 Phase-Lead Compensator

- A PD controller can be approximated by a lead controller.

Phase-Lead Compensator

The phase-lead compensator's transfer function is of the form

$$G_c(s) = \frac{s + z_c}{s + p_c}, \text{where } p_c \neq 0 \text{ and } |z_c| < |p_c|$$

Example

Placing the pole $-p_c = -14$

and the zero $-z_c = -10$ (on the right of the pole)

Example 9.4

Exercise

Solve Skill-Assessment Exercise 9.2 with a lead compensator.

Note that this type of design problems do not have a unique solution.

Implementation

- A lead compensator can be implemented using either an active circuit or a passive one.
- This is an advantage of the lead compensator over the PD type.

4. Improving Transient Response and e_{ss}

Mission

1. Design a compensator to force the system's transient response to satisfy certain criteria (e.g., new overshoot, settling time, etc.)
2. And eliminate (or at least improve) the steady-state error.

4.1 PID Compensator

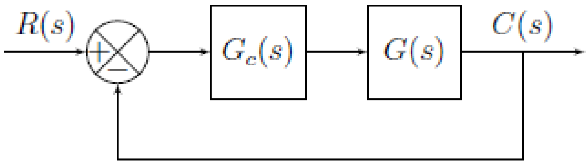
- This mission can be accomplished by a PID compensator.
- The PID controller adds a pole pc = 0 (at the origin) and two zeros to the OLTF.

PID Compensator

The PID compensator's transfer function is of the form

$$G_c(s) = K_p + K_d\, s + K_i \frac{1}{s} = \frac{K_d(s^2 + \frac{K_p}{K_d} s + \frac{K_i}{K_d})}{s} = K_d \underbrace{(s + z_{c1})}_{\text{PD}} \underbrace{\left(\frac{s + z_{c2}}{s}\right)}_{\text{PI}}$$

Example 5



Consider the following uncompensated system,

where $G_c(s) = K$, $G(s) \ = (s+8)/(s+3)(s+6)(s+10)$ and $K \ > 0$.

- Design a P compensator $G_c(s) = K$ to yield a 20% overshoot.
- Compute the closed-loop poles, the output transient characteristics, and the steady-state error for a unit-step input.
- Now, design a PID compensator $G_c(s) = K_d(s - z_{PD})(s - zPI)/s$ to
  1. maintain the overshoot at 20% ,
  2. obtain a peak time that is two-thirds that of the uncompensated system, and
  3. annihilate the steady-state error for a step input.

- Re-compute the closed-loop poles, the output transient characteristics, and the steady-state for a unit-step input.

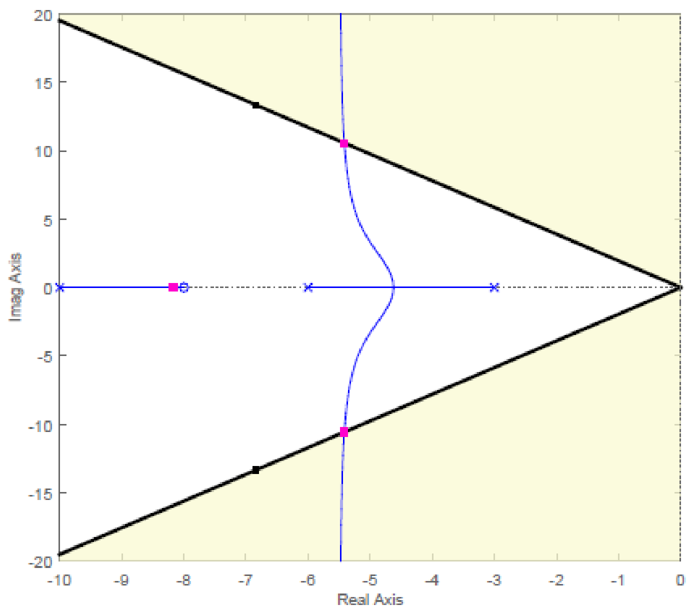
Solution

- As in Examples 9.1 and 9.3, the design of a P compensator (uncompensated system) may be done in two ways:
    1. Numerically: using Matlab's sisotool, for instance; by varying K until reaching the desired criteria. This is the most accurate method.
    2. Manually: either graphically or analytically.

- Table 5 summarizes the characteristics of the numerically-designed uncompensated system of Example 9.5 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed uncompensated system shows that this design was quite close from satisfying the design objective (%OS = 20%).
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is a valid assumption, as can be deduced from the closed-loop poles.
- Figs. 12 and 13 show the operating points and the unit-step responses of the analytically- and numerically-designed uncompensated systems of Example 9.5.

Table 5: Uncompensated system characteristics of Example 9.5

	Analytical design	Numerical validation (sisotool)	Numerical design (sisotool)
Compensator	K	K	K
K	121.5	121.5	116.5
Conjugate poles	$-5.415 \pm j10.57$	$-5.42 \pm j10.6$	$-5.41 \pm j10.3$
Other poles	-8.169	-8.17	-8.18
Zeros	-8	-8	-8
ω_n (rad/s)	11.9	11.9	11.7
ζ	0.456	0.456	0.464
%OS	20	20.7	20
T_p (s)	0.297	0.298	0.298
T_s (s)	0.74	0.70	0.71
Position constant	5.4	-	-
$e(\infty)$	0.16	0.16	0.16
2^{nd}-ordrer approximation	OK	confirmed	-



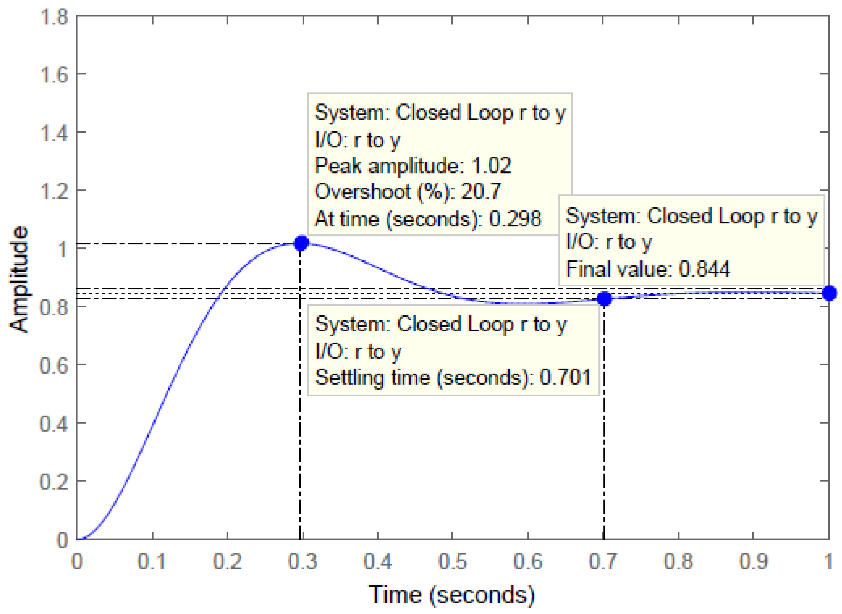


Figure 12: Operating point and step response of the analytically-designed uncompensated system in Example 9.5

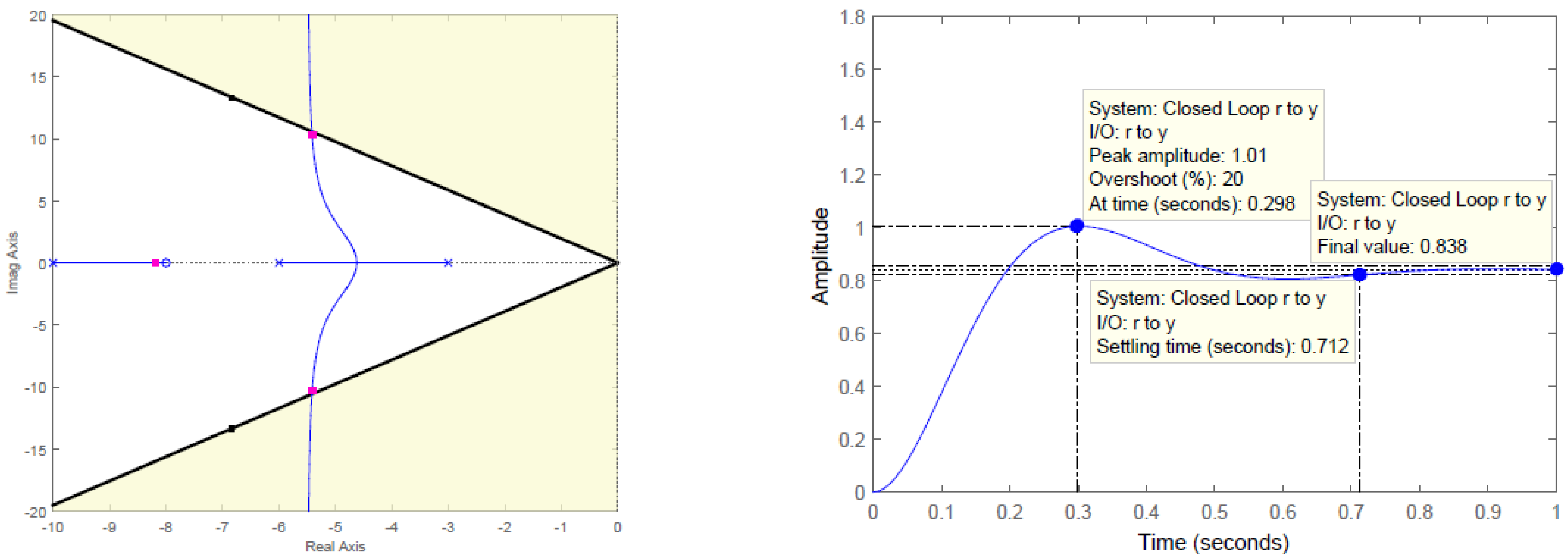


Figure 13: Operating point and step response of the numerically-designed uncompensated system in Example 9.5

- The design of a PID compensator is often performed following the steps below:
  1. Use the target transient characteristics to find the system's desired operating point.
  2. Design a PD controller to satisfy the target transient characteristics (as done for Example 9.3).
  3. Design a PI controller in cascade with the already designed PD controller to form a PID controller.

     The zero of the PI controller must be chosen to have as little effect as possible on the transient response not to change the one achieved by the PD controller.

- Table 6 summarizes the characteristics of the numerically-designed PD-compensated system of Example 9.5 along with the analytically-designed one and its validation.
- Note how the numerical simulation of the analytically-designed compensated system shows that this design is quite close to satisfying the design objectives (%OS = 20% and Tp = 0:2 s).
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is a valid assumption, as can be noticed from the closed-loop poles and zeros.
- Figs. 14 and 15 show the operating points and the unit-step responses of the analytically- and numerically-designed PD-compensated systems of Example 9.5.

Table 6: PD-Compensated system characteristics of Example 9.5

	Analytical design	Numerical validation (`sisotool`)	Numerical design (`sisotool`)
Compensator	$K(s+55.92)$	$K(s+55.92)$	$K(s+55.92)$
K	5.34	5.34	4.2
Conjugate poles	$-8.13 \pm j15.87$	$-8.11 \pm j15.8$	$-7.56 \pm j14.1$
Other poles	-8.08	-8.08	-8.10
Zeros	$-8, -55.92$	$-8, -55.92$	$-8, -55.92$
ω_n (rad/s)	17.8	17.8	16.0
ζ	0.456	0.456	0.473
%OS	20	21.7	20
T_p (s)	0.20	0.18	0.20
T_s (s)	0.49	0.45	0.5
Position constant	13.27	-	-
$e(\infty)$	0.07	0.07	0.09
2^{nd}-ordrer approximation	OK	confirmed	-

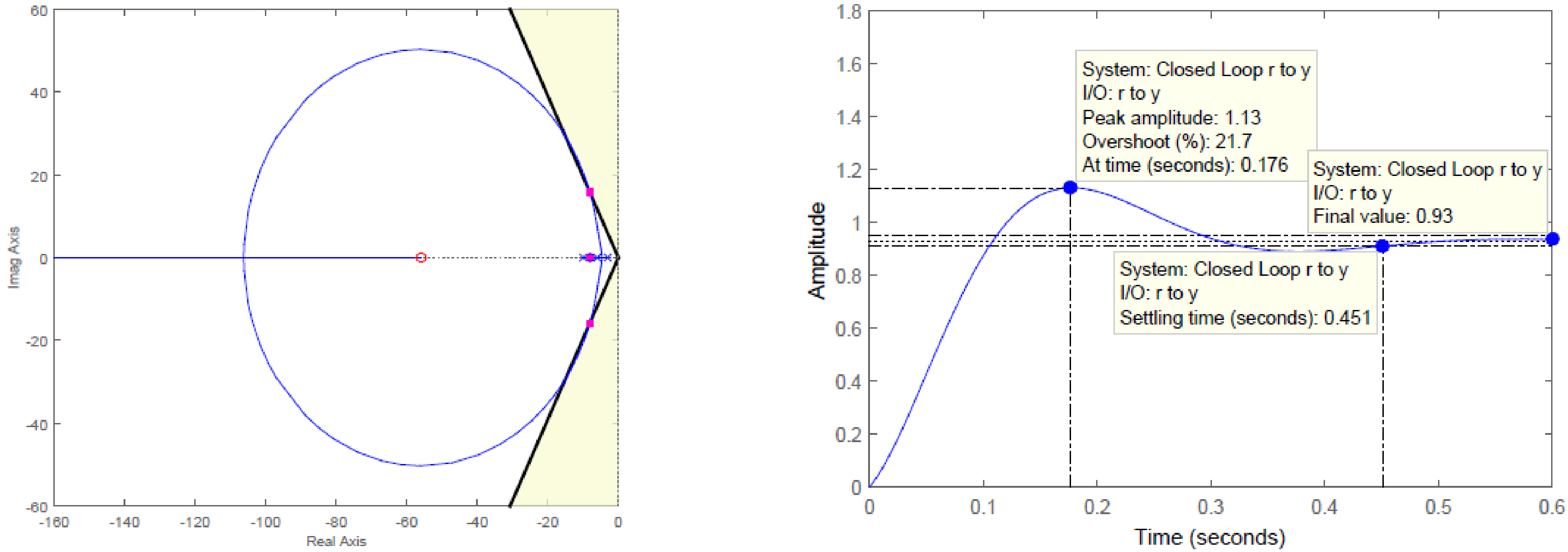
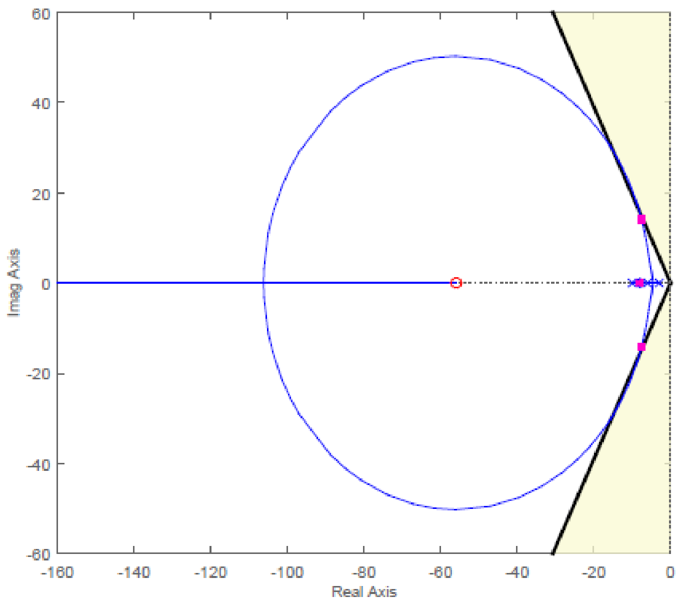


Figure 14: Operating point and step response of the analytically-designed PD-compensated system in Example 9.5



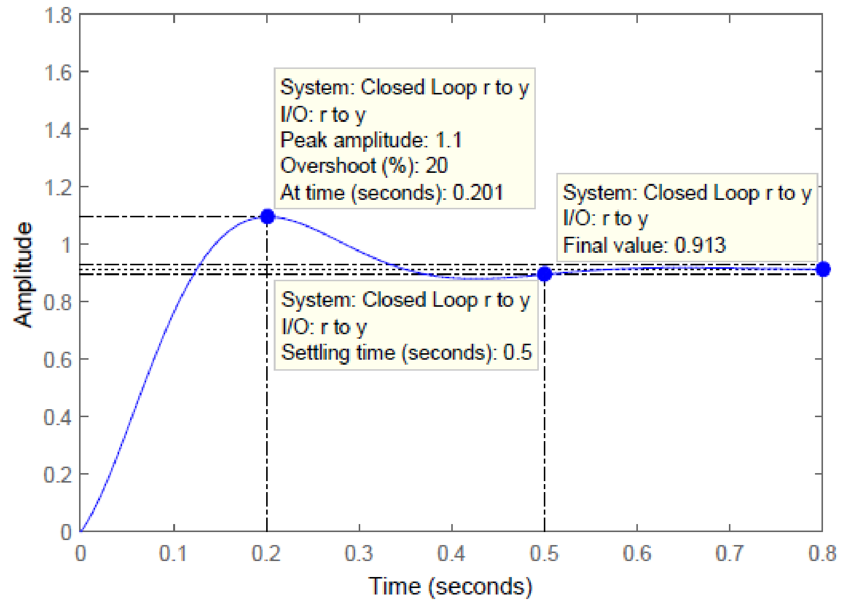


Figure 15: Operating point and step response of the numerically-designed PD-compensated system in Example 9.5

- Table 7 summarizes the characteristics of the analytically-designed PID-compensated system of Example 9.5, its numerical validation, and the numerically adjusted design (where K is numerically adjusted to reach a better compromise between the target design objectives).
- Note how the numerical simulation of the analytically-designed compensated system shows that this design failed in satisfying one of the design objectives ($\%OS = 20\%$) despite successfully achieving $Tp = 0.2\ s$.
- This is because the computations were based on the assumption that the system can be approximated by a second-order model, which is not the case, as can be noticed from the closed-loop poles and zeros.
- Figs. 16 and 17 show the operating points and the unit-step responses of the analytically- and numerically-adjusted PID-compensated systems of Example 9.5.

Table 7: PID-Compensated system characteristics of Example 9.5

	Analytical design	Numerical validation (`sisotool`)	Numerical gain adjustment (`sisotool`)
Compensator	$K(s+55.92)(s+0.5)/s$	$K(s+55.92)(s+0.5)/s$	$K(s+55.92)(s+0.5)/s$
K	4.6	4.6	5.7
Conjugate poles	$-7.516 \pm j14.67$	$-7.51 \pm j14.6$	$-8.08 \pm j16.4$
Other poles	$-8.1, -0.468$	$-8.1, -0.468$	$-8.08, -0.474$
Zeros	$-8, -55.92$	$-8, -55.92$	$-8, -55.92$
ω_n (rad/s)	16.5	16.4	18.3
ζ	0.456	0.456	0.442
%OS	20	14.1	17
T_p (s)	0.21	0.20	0.17
T_s (s)	0.53	2.57	2.08
Position constant	∞	-	-
$e(\infty)$	0	0	0
2^{nd}-ordrer approximation	NO	confirmed	-

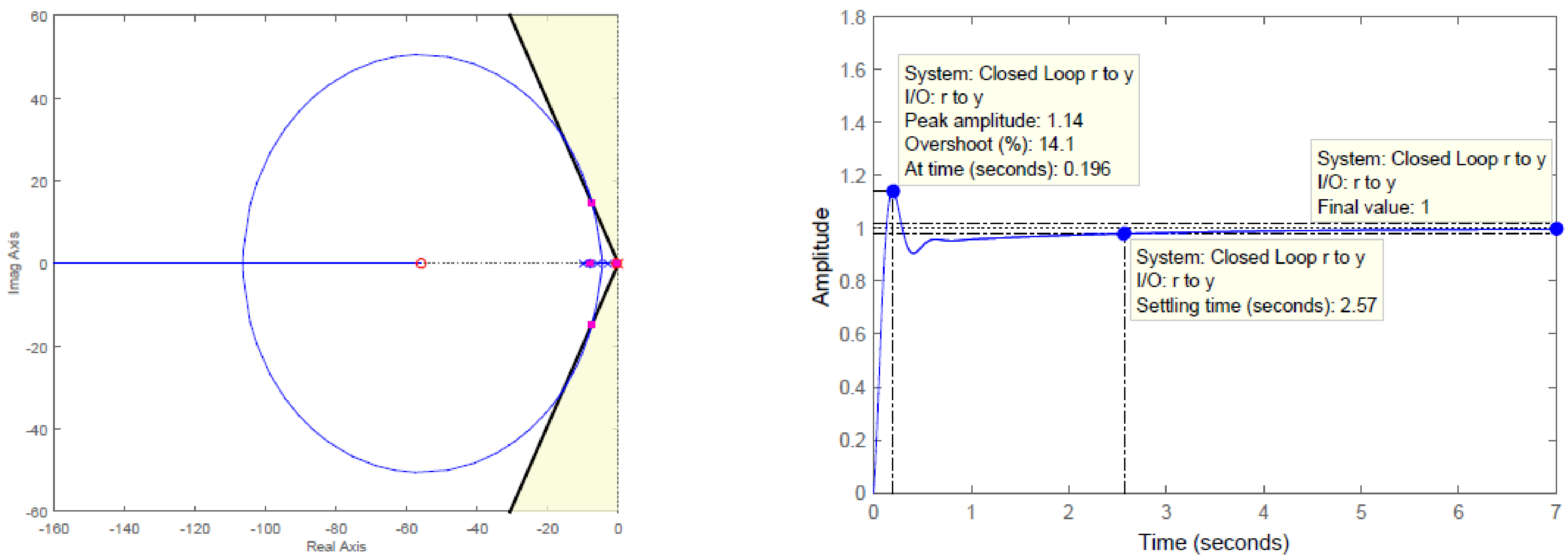
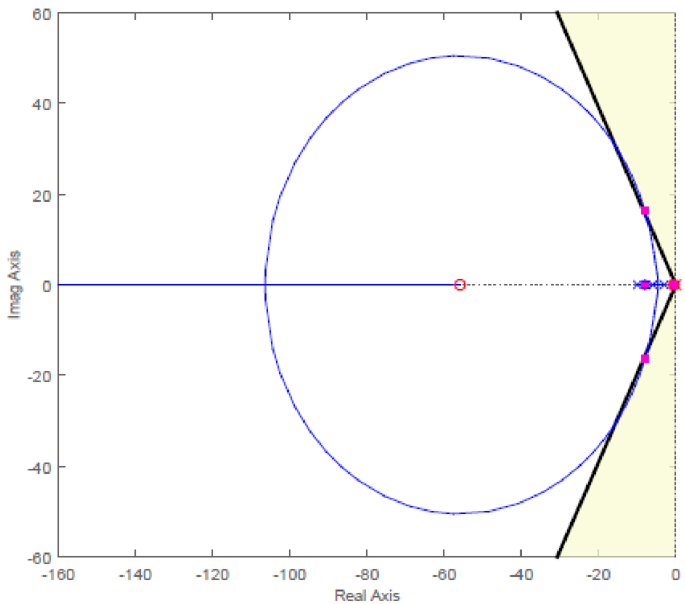


Figure 16: Operating point and step response of the analytically-designed PID-compensated system in Example 9.5



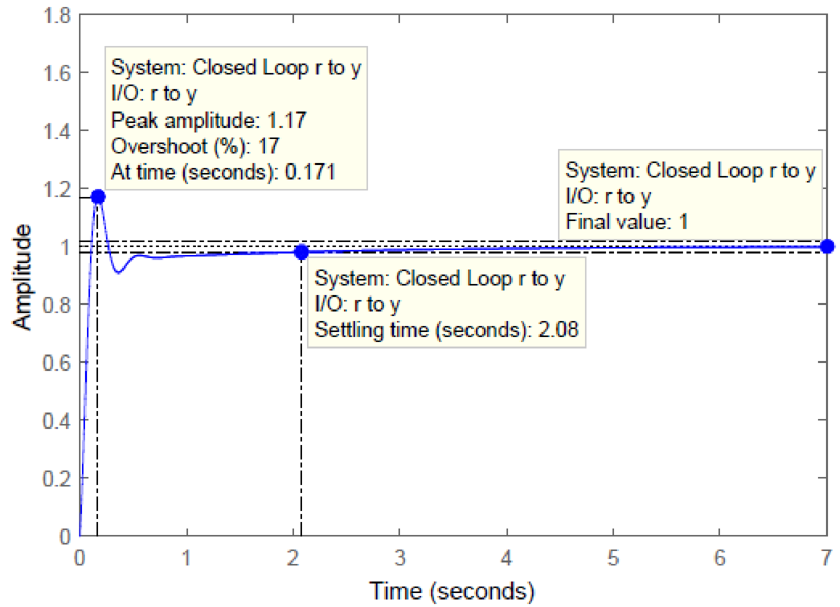


Figure 17: Operating point and step response of the numerically-adjusted PID-compensated system in Example 9.5

Exercise

Solve Skill-Assessment Exercise 9.3 with a PID compensator.

Note that this type of design problems do not have a unique solution.

Remarks

The PID compensator suffers from two main drawbacks:

1. It can only be implemented using an active circuit.
2. Its derivative term $K_d s$ amplifies the noise at its input terminal.

4.3 Phase Lead-Lag Compensator

- A PID controller can be approximated by a lead-lag controller.
- Just like the PID controller which may be implemented by combining a PD and a PI controller, a lead-lag compensator may be implemented by combining a lead and a lag controller.

Phase-Lead Compensator

The phase-lead compensator's transfer function is of the form

$$G_c(s) = K \underbrace{\left(\frac{s+z_{c1}}{s+p_{c1}}\right)}_{\text{Lag}} \underbrace{\left(\frac{s+z_{c2}}{s+p_{c2}}\right)}_{\text{Lead}}$$

Example 9.6.

Exercise

Solve Skill-Assessment Exercise 9.3 with a lead-lag compensator. Note that this type of design problems do not have a unique solution.

Implementation

- A lead-lag compensator can be implemented using either an active circuit or a passive one.
- This is an advantage of the lead-lag compensator over the PID type.

Summery

A summary of all of these compensators is provided in Table 9.7 of the textbook (Table 8 of the notes).

Table 8: Types of cascade compensators (Table 9.7 of [1]).

Function	Compensator	Transfer function	Characteristics
Improve steady-state error	PI	$K\dfrac{s+z_c}{s}$	**1.** Increases system type. **2.** Error becomes zero. **3.** Zero at $-z_c$ is small and negative. **4.** Active circuits are required to implement.
Improve steady-state error	Lag	$K\dfrac{s+z_c}{s+p_c}$	**1.** Error is improved but not driven to zero. **2.** Pole at $-p_c$ is small and negative. **3.** Zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. **4.** Active circuits are not required to implement.
Improve transient response	PD	$K(s+z_c)$	**1.** Zero at $-z_c$ is selected to put design point on root locus. **2.** Active circuits are required to implement. **3.** Can cause noise and saturation; implement with rate feedback or with a pole (lead).
Improve transient response	Lead	$K\dfrac{s+z_c}{s+p_c}$	**1.** Zero at $-z_c$ and pole at $-p_c$ are selected to put design point on root locus. **2.** Pole at $-p_c$ is more negative than zero at $-z_c$. **3.** Active circuits are not required to implement.

Table 8: Types of cascade compensators (Table 9.7 of [1]).

Improve steady-state error and transient response	PID	$K\dfrac{(s+z_{\text{lag}})(s+z_{\text{lead}})}{s}$	**1.** Lag zero at $-z_{\text{lag}}$ and pole at origin improve steady-state error. **2.** Lead zero at $-z_{\text{lead}}$ improves transient response. **3.** Lag zero at $-z_{\text{lag}}$ is close to, and to the left of, the origin. **4.** Lead zero at $-z_{\text{lead}}$ is selected to put design point on root locus. **5.** Active circuits required to implement. **6.** Can cause noise and saturation; implement with rate feedback or with an additional pole.
Improve steady-state error and transient response	Lag-lead	$K\dfrac{(s+z_{\text{lag}})(s+z_{\text{lead}})}{(s+p_{\text{lag}})(s+p_{\text{lead}})}$	**1.** Lag pole at $-p_{\text{lag}}$ and lag zero at $-z_{\text{lag}}$ are used to improve steady-state error. **2.** Lead pole at $-p_{\text{lead}}$ and lead zero at $-z_{\text{lead}}$ are used to improve transient response. **3.** Lag pole at $-p_{\text{lag}}$ is small and negative. **4.** Lag zero at $-z_{\text{lag}}$ is close to, and to the left of, lag pole at $-p_{\text{lag}}$. **5.** Lead zero at $-z_{\text{lead}}$ and lead pole at $-p_{\text{lead}}$ are selected to put design point on root locus. **6.** Lead pole at $-p_{\text{lead}}$ is more negative than lead zero at $-z_{\text{lead}}$. **7.** Active circuits are not required to implement.

5.1. Active Circuit Realization

- The active circuit in Fig. 18 can be used to implement different types of compensators.
- The compensator to be implemented depends on how the impedances $Z_1(s)$ and $Z_2(s)$ are configured, as listed in Table 9.

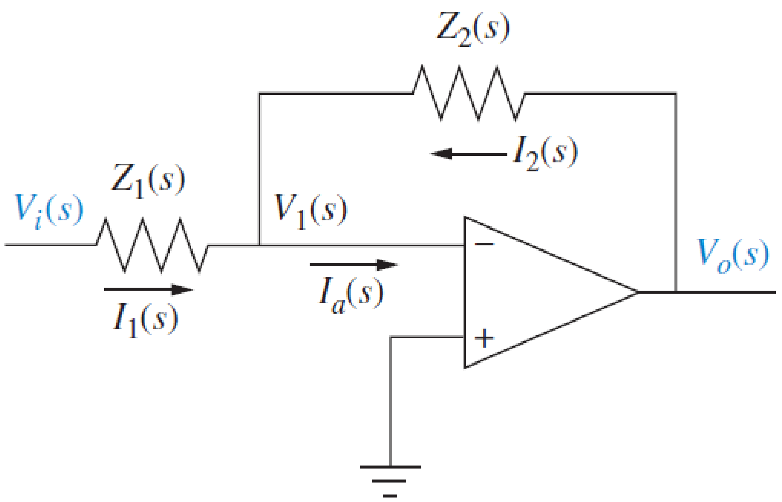


Figure 18: Active Circuit (Fig. 9.60 of [1]).

Table 9: Active realizations of compensators (Table 9.10 of [1]).

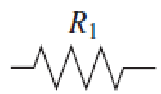
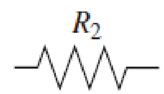
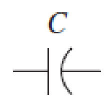
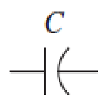
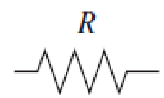
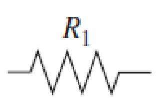
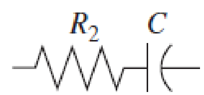
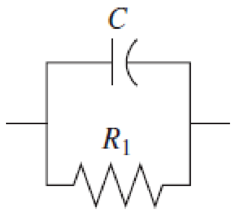
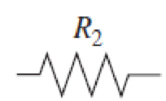
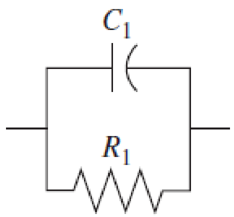
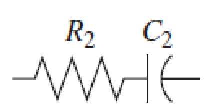
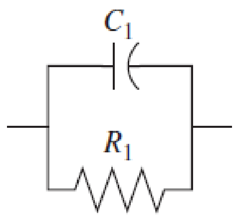
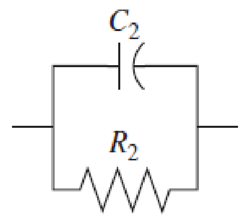
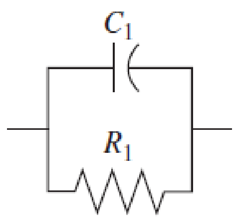
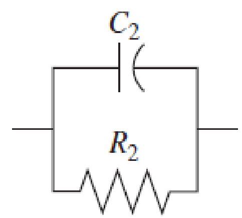
Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain			$-\frac{R_2}{R_1}$
Integration			$-\frac{\frac{1}{RC}}{s}$
Differentiation			$-RCs$
PI controller			$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller			$-R_2C\left(s+\frac{1}{R_1C}\right)$
PID controller			$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$

Table 9: Active realizations of compensators (Table 9.10 of [1]).

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Lag compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

5.2. Passive Circuit Realization

- The passive implementation of different types of compensators may be accomplished using the passive circuits listed in Table 10.

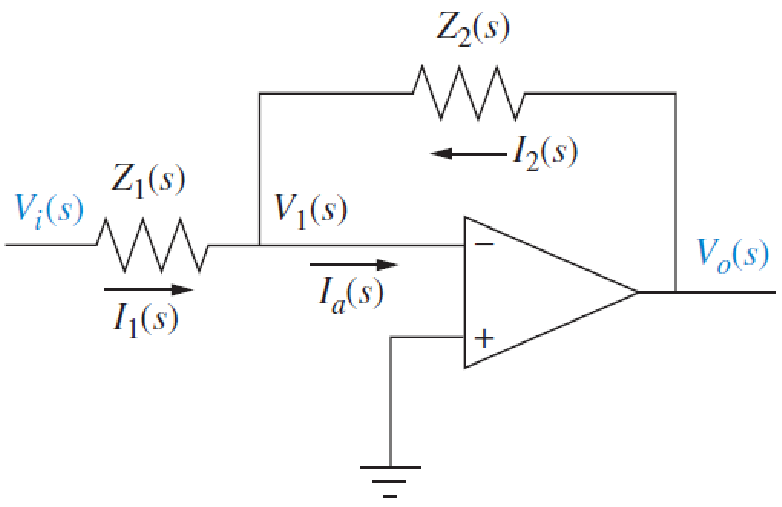


Figure 18: Active Circuit (Fig. 9.60 of [1]).

Table 10: Passive realizations of compensators (Table 9.11 of [1]).

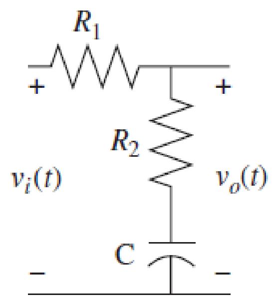
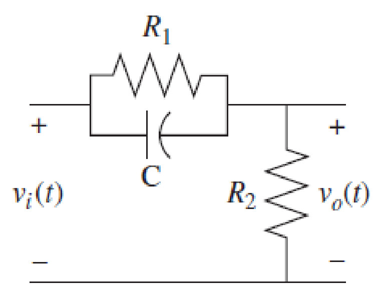
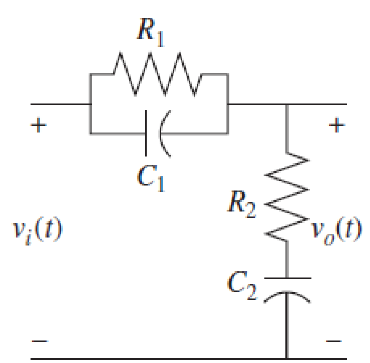
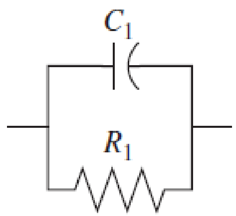
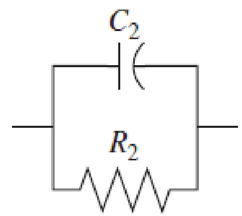
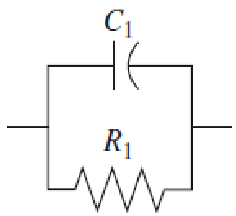
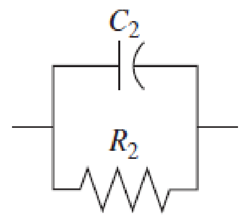
Function	Network	Transfer function, $\dfrac{V_o(s)}{V_i(s)}$
Lag compensation		$\dfrac{R_2}{R_1+R_2}\,\dfrac{s+\dfrac{1}{R_2C}}{s+\dfrac{1}{(R_1+R_2)C}}$
Lead compensation		$\dfrac{s+\dfrac{1}{R_1C}}{s+\dfrac{1}{R_1C}+\dfrac{1}{R_2C}}$
Lag-lead compensation		$\dfrac{\left(s+\dfrac{1}{R_1C_1}\right)\left(s+\dfrac{1}{R_2C_2}\right)}{s^2+\left(\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_2}+\dfrac{1}{R_2C_1}\right)s+\dfrac{1}{R_1R_2C_1C_2}}$

Table 9: Active realizations of compensators (Table 9.10 of [1]).

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Lag compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

References

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Control Systems Engineering.
Wiley, 7 edition.

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