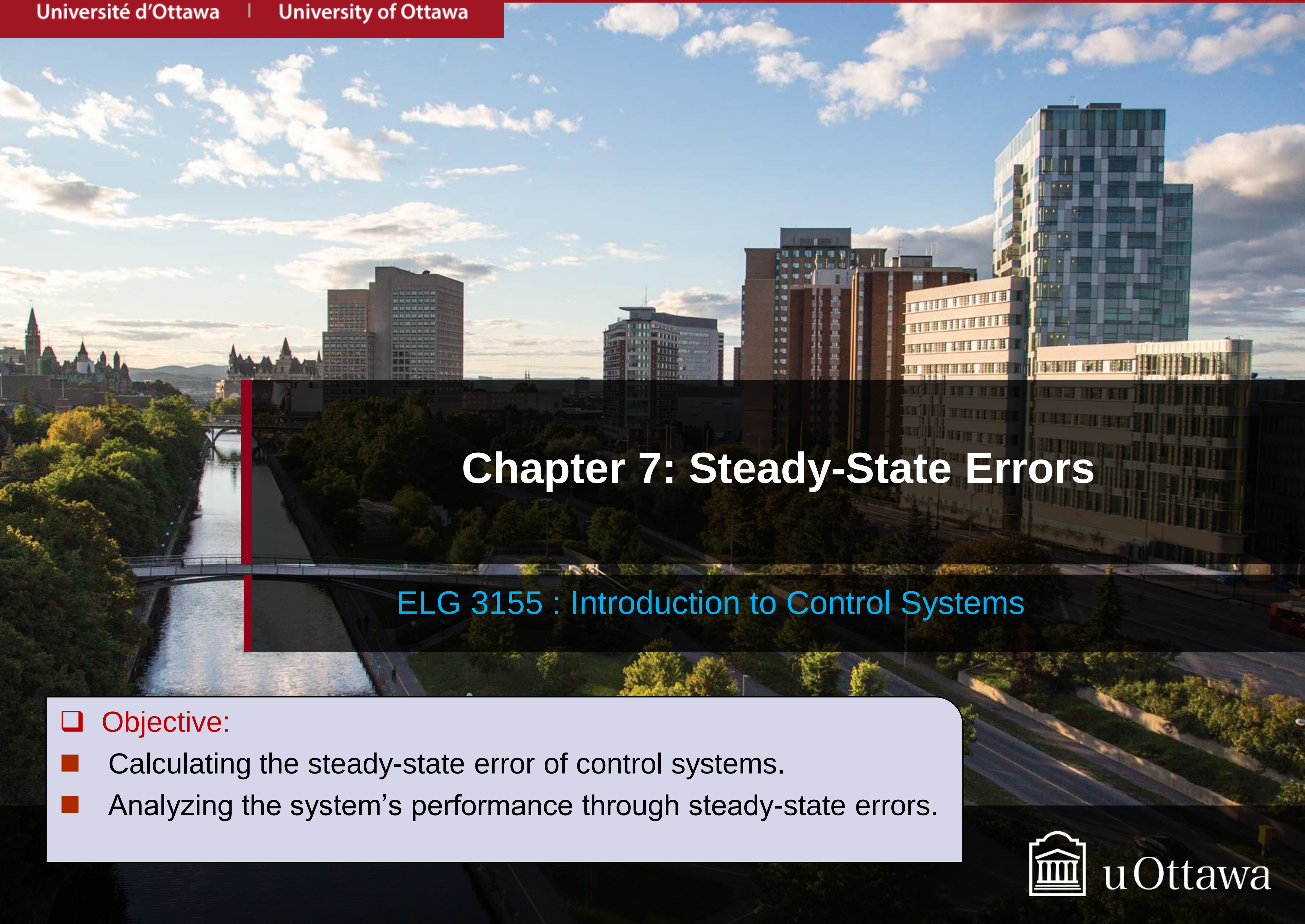


Chapter 7: Steady-State Errors

ELG 3155 : Introduction to Control Systems

- Objective:
  - Calculating the steady-state error of control systems.
  - Analyzing the system's performance through steady-state errors.

uOttawa

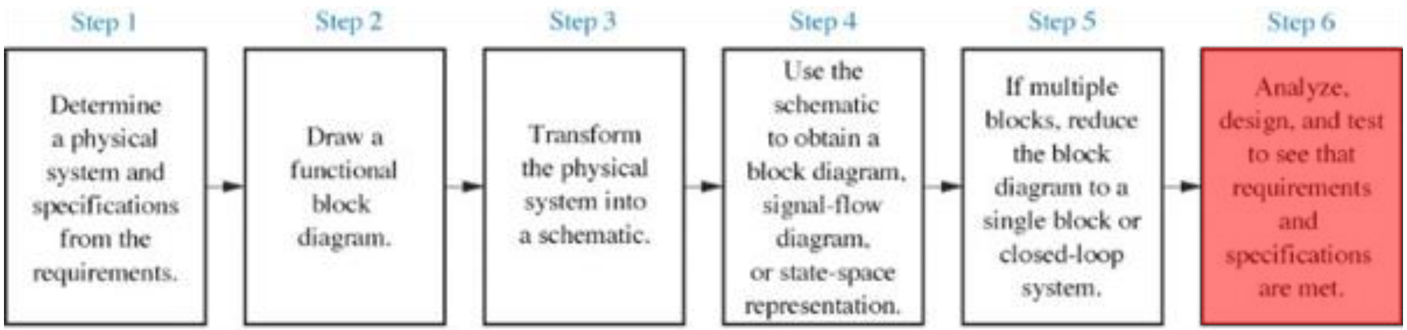
Hitham Jleed

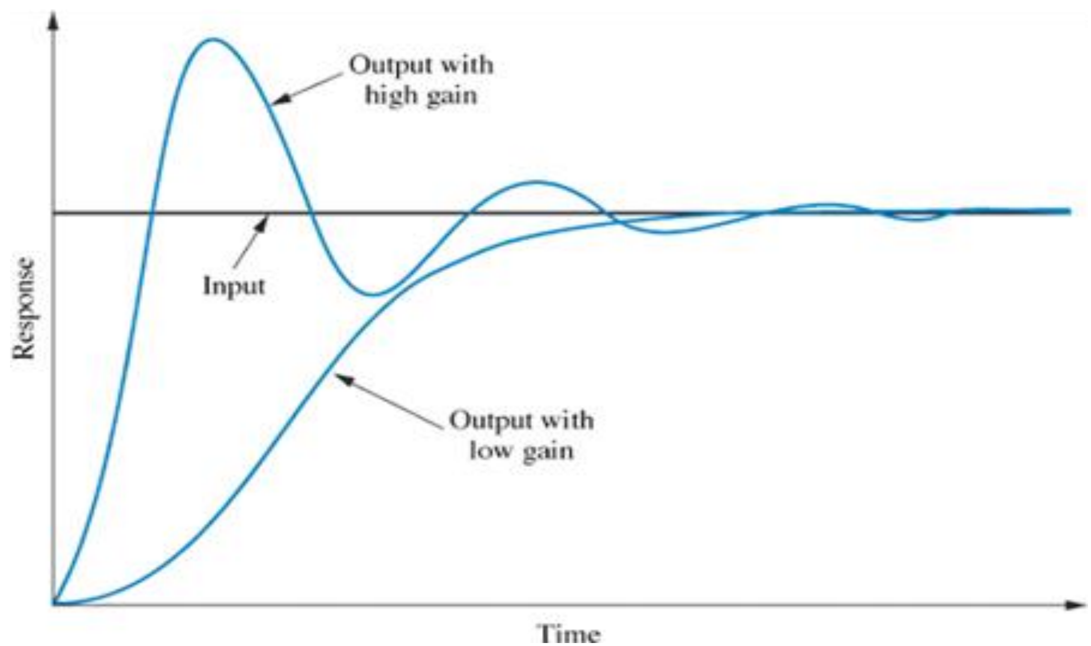
Outline

- 7 Steady-State Errors
  - 7.1 Introduction
  - 7.2 Steady-state error for unity feedback systems
  - 7.3 Static error constants and system type
  - 7.4 Steady-state error specifications
  - 7.5 Steady-state error for disturbances
  - 7.6 Steady-state error for non-unity feedback systems
  - 7.7 Sensitivity
  - 7.8 Steady-state error for systems in state space

Introduction

Background: Design Process



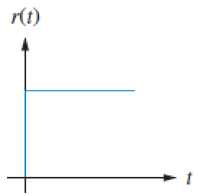
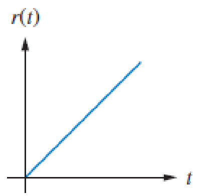
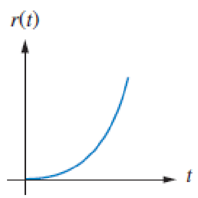


Definition and Test Inputs

Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$. Test inputs used for steady-state error analysis and design are summarized in Table 7.1.

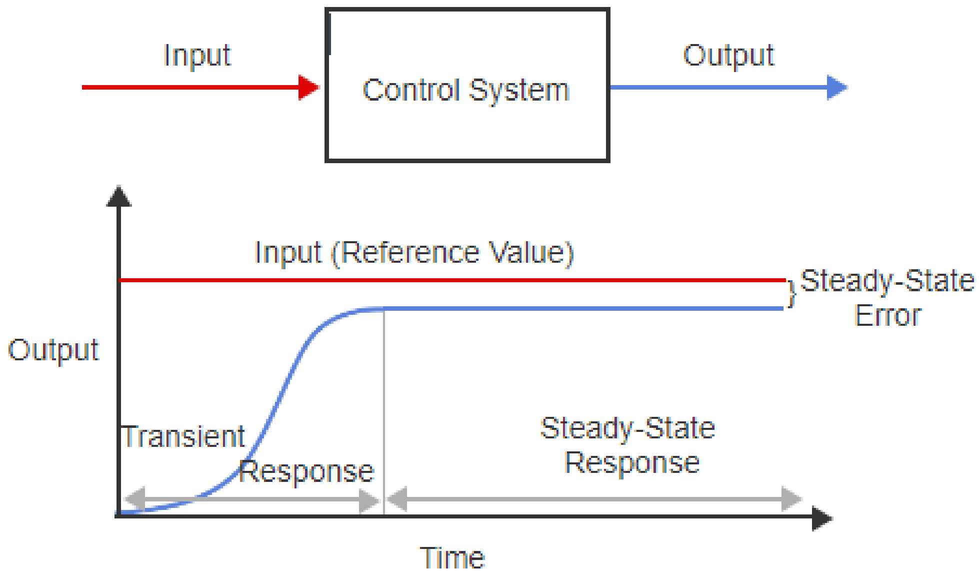
$$\boxed{e_{ss} = \lim_{t \to \infty} e(t)}$$

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Evaluating Steady-State Errors

Step Input:
Output 1 : No Steady-State Error
Output 2 : Constant Steady-State Error of e_2



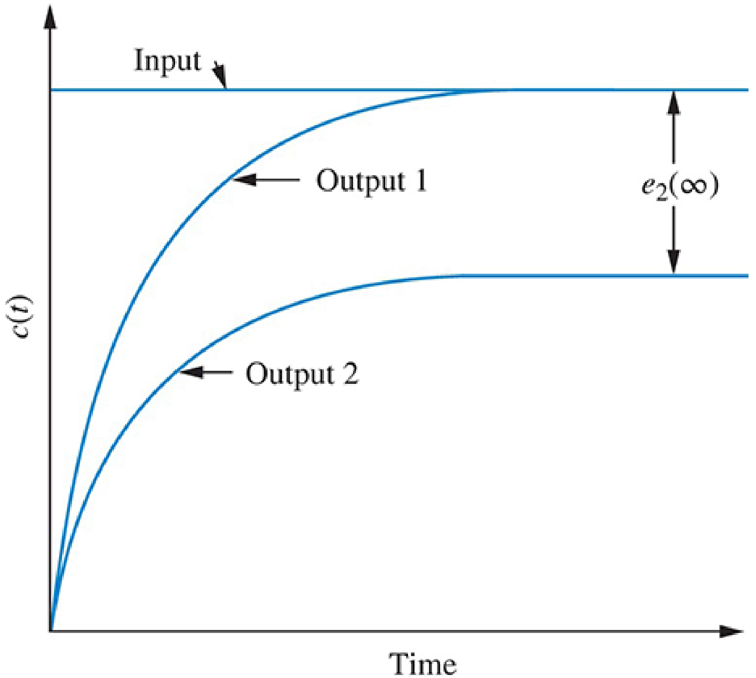


FIGURE 7.2 (Norman 2015)

Steady-state error in terms of CL TF, T(s), [1, p. 339]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

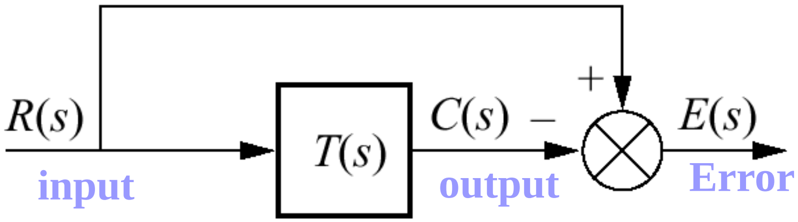
$$C(s) = R(s)T(s)$$

$$E(s) = R(s)[1 - T(s)]$$

Applying the final value theorem

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e(\infty) = \lim_{s\to 0} sR(s)[1 - T(s)]$$



Closed-loop control system error

FIGURE 7.3-a [Norman,2015,, P-338]

Steady-state error in terms of G(s), [1, p. 340]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

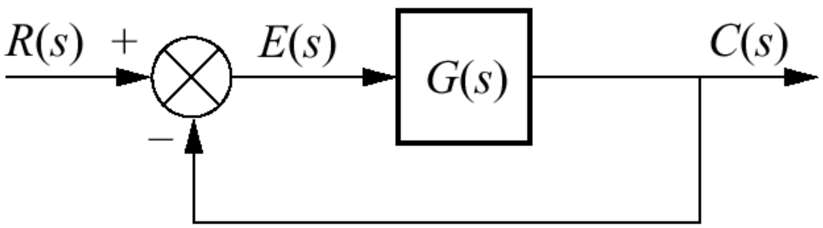
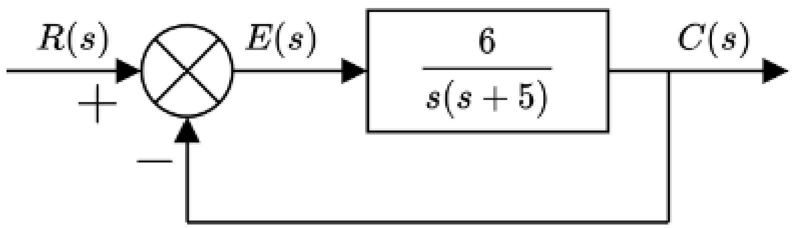


FIGURE 7.3-b [Norman,2015,, P-338]

Applying the final value theorem

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Assessment (a)



1) The closed-loop transfer function $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{L}{s^2 + 5s + L}$

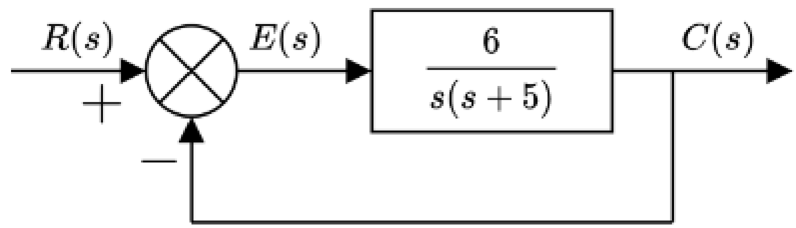
$L =$ ____.

Answer

6

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{6}{s^2+5s}}{1+\frac{6}{s^2+5s}} \quad T(s) = \frac{6}{s^2+5s+6}$$

Assessment (b)



2) For a step input, the error $E(s) = \dfrac{s^2 + M}{s(s^2+5s+6)}$

$M =$ ____.

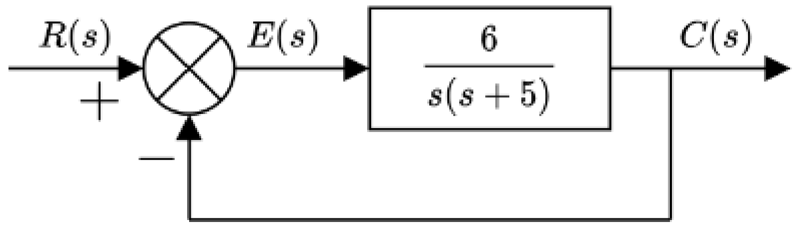
Answer

5s

For a step input, $R(s) = \dfrac{1}{s}$, and

$$E(s)=R(s)/[1+G(s)] = \frac{1}{s}\Big/\left[1+\frac{6}{s^2+5s}\right] = \frac{s^2+5s}{s(s^2+5s+6)}$$

Assessment (c)



3) For a step input, steady-state error $e(\infty) =$ ____.

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Answer

0

Because the system is stable, by the final value theorem,

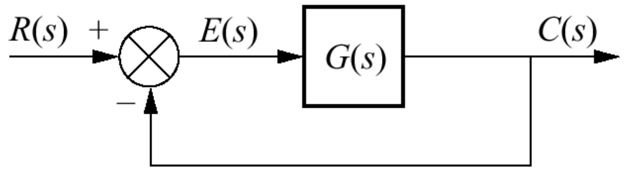
$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s^2 + 5s}{s^2 + 5s + 6}$$

Plugging in $s = 0$

$$e(\infty) = \frac{0}{6} = 0$$

Evaluating Steady-State Errors$_2$

- Forward transfer function $G(s)$ with a unity feedback.



$$E(s) = R(s) - C(s), \quad \text{and} \quad C(s) = E(s)G(s)$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

$$\boxed{e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}}$$ [Final-value theorem]

Step input: *u*(t)

$$\boxed{e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}}$$

For zero steady-state error, $\lim_{s\to 0} G(s) = \infty$

Ramp input: *tu*(t)

$$\boxed{e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}}$$

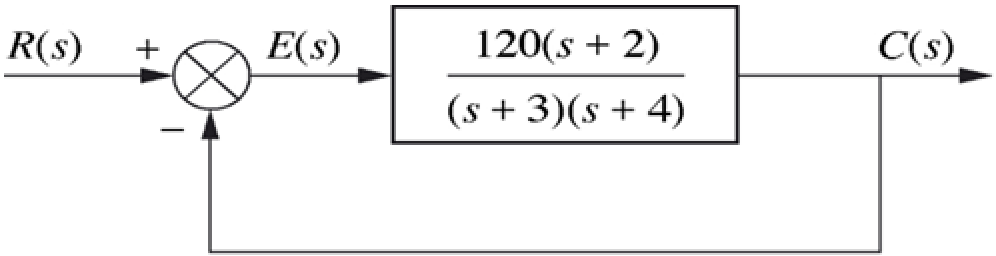
For zero steady-state error, $\lim_{s\to 0} sG(s) = \infty$

Parabolic input: (1/2)t^2*u*(t)

$$\boxed{e(\infty) = e_{parabolic}(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}}$$

For zero steady-state error, $\lim_{s\to 0} s^2G(s) = \infty$

Example: Steady-State Error for Unity Feedback



Find the steady-state errors for inputs of 5u(t), 5tu(t), and 5t^2u(t). The function u(t) is the step function.

Note Laplace transforms:

1. $5u(t) \rightarrow \frac{5}{s}$
2. $5tu(t) \rightarrow \frac{5}{s^2}$
3. $5t^2u(t) \rightarrow \frac{10}{s^3}$

1. $$e(\infty) = \frac{5}{1 + \lim_{s\to 0} \frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{1+20}$$

2. $$e(\infty) = \frac{5}{\lim_{s\to 0} s\left(\frac{120(s+2)}{(s+3)(s+4)}\right)} = \frac{5}{0} = \infty$$

3. $$e(\infty) = \frac{10}{\lim_{s\to 0} s^2 \frac{120(s+2)}{(s+3)(s+4)}} = \frac{10}{0} = \infty$$

Steady-State Errors for Systems with One Integration

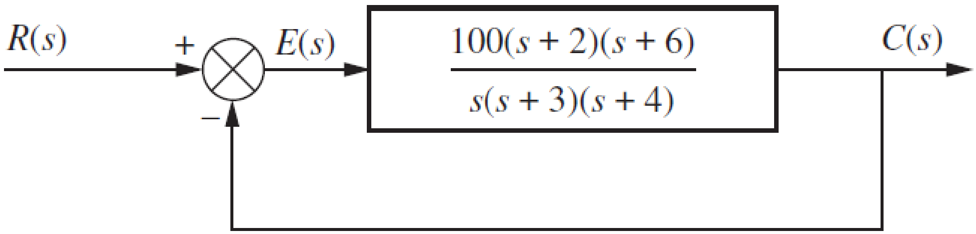


FIGURE 7.6 Feedback control system for Example 7.3

$$e(\infty) = e_{\text{step}}(\infty) = \frac{5}{1 + \lim_{s\to 0} G(s)} = \frac{5}{\infty} = 0$$

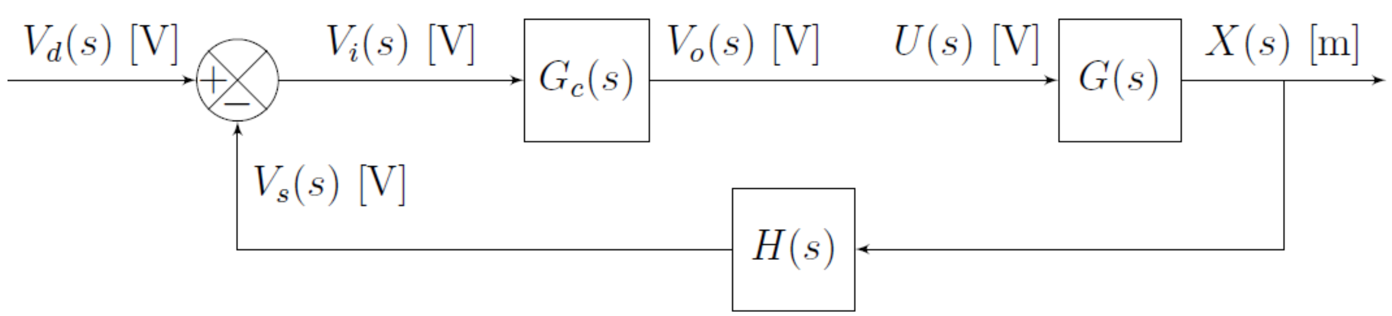
$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Solution

- To force the error to tend to zero, so that x converges to x_d , one must change the system's transfer function.
- The joint and the sensor are pre-fabricated devices.

  ⇒ Their transfer functions cannot be modified.
- The solution is to implement an electric circuit (called the compensator) and integrate it in the control system right before the plant.
- Let the input and output voltages of the compensator be v_i and v_o. Then, the trick is to design a compensator circuit whose transfer function $Gc(s) = Vo(s)/Vi(s)$ helps to modify the system's transfer function in such a way as to annihilate the error.



- If, for example, the compensator circuit is designed such that its transfer function is

$$G_c(s) = \frac{V_o(s)}{V_i(s)} = 2 + \frac{5}{s} + 5s = \frac{5s^2 + 2s + 5}{s}$$

Then, the overall system's transfer function is

$$T(s) = \frac{G_c G(s)}{1 + G_c G H(s)} = \frac{2.5s^2 + s + 2.5}{s^3 + 3.1s^2 + 2s + 2.5}$$

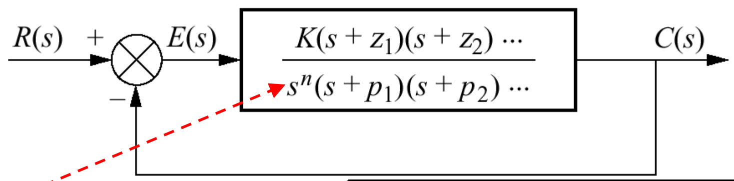
which clearly forces the error to decay to zero and so the joint displacement x converges to its desired values $x_d = 1\ m$.

Definition (Compensator)

A compensator (in this context) is an electric circuit that is integrated in a closed-loop control system for the purpose of changing its behavior.

Type of a Control System

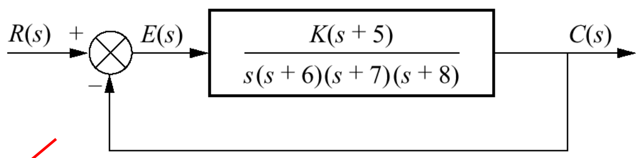
we define *system type* to be the number of pure integrations (s^n) in the forward path.



Type 0: if $n = 0$; (no integration)

Type 1: if $n = 1$; (one integration)

Type 2: if $n = 2$; (two integrations)



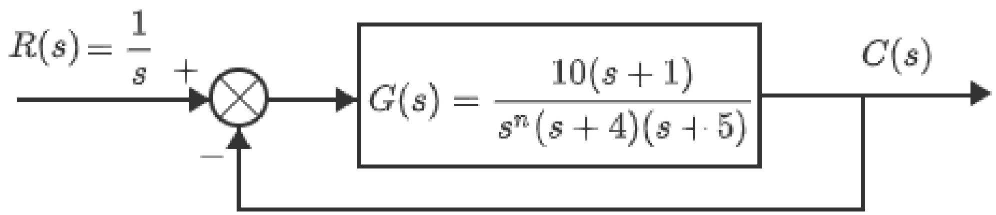
Problem: Find the value of K so that there is 10% error in the steady state.

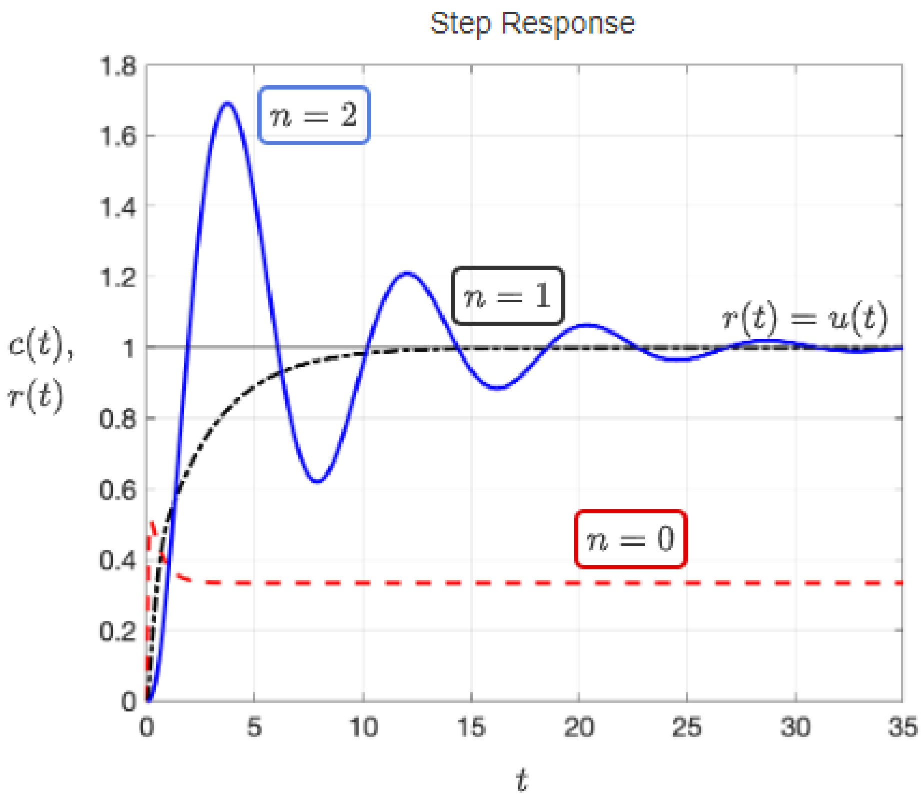
Solution: Type 1.

Input should be ramp, because only ramp yields a finite error in Type 1 system.

$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$\Rightarrow K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8} = \frac{5 \times K}{336} \qquad \Rightarrow K = 672$$





$\boxed{n = 0}$

$$G(s) = \frac{10(s+1)}{s^0(s+4)(s+5)}$$

$\boxed{n = 1}$

$$G(s) = \frac{10(s+1)}{s^1(s+4)(s+5)}$$

$\boxed{n = 2}$

$$G(s) = \frac{10(s+1)}{s^2(s+4)(s+5)}$$

Static Error Constants and Error Specifications

The steady-state errors of control systems can be specified by three static error constants.

For a step input, $u(t)$,

$$e(\infty) = e_{\text{step}}(\infty) = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

Definition (Static error constants):

- Position constant : $K_p = \lim_{s\to 0} GH(s)$
- Velocity constant : $K_v = \lim_{s\to 0} sGH(s)$
- Acceleration constant : $K_a = \lim_{s\to 0} s^2 GH(s)$

For a ramp input, $tu(t)$,

$$e(\infty) = e_{\text{ramp}}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)}$$

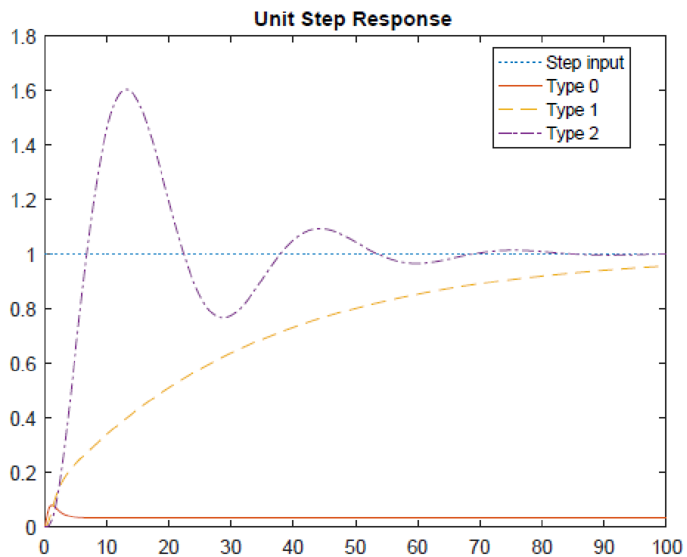
For a parabolic input, $\frac{1}{2}t^2u(t)$,

$$e(\infty) = e_{\text{parabola}}(\infty) = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

The relationship between the error, the system's type, and the form of the input, is summarized in Table 1 and illustrated in Figures 1, 2, and 3.

Table 1: Relationships between input, system type, static error constants, and steady-state errors (Table. 7.2 of [Nise, 2015]).

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$



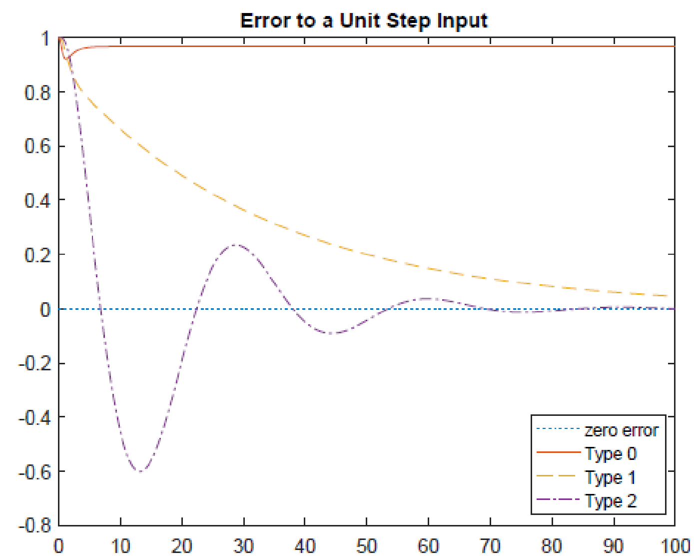


Figure 1: Response and error signals of systems with different types to a unit-step input.

Steady-State Error for Disturbances

Output

$$C(s) = E(s)G_1(s)G_2(s)+D(s)G_2(s)$$

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

After substitution

$$E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)$$

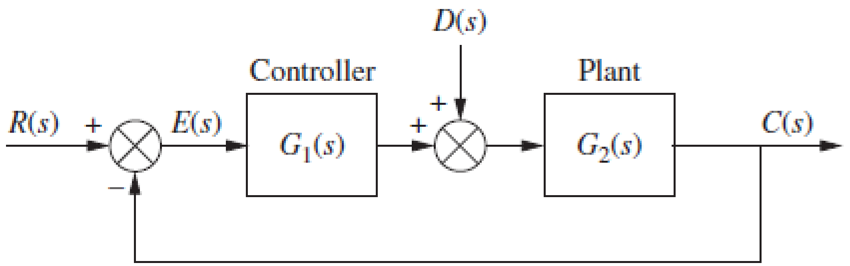


FIGURE 7.11 Feedback control system showing disturbance

Steady-State Error for Disturbances

Final value theorem

$$e(\infty) = \lim_{s \to 0} sE(s) = e_R(\infty) + e_D(\infty)$$

Steady-state error due to reference

$$e_R(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

Steady-state error due to disturbance

$$e_D(\infty) = -\lim_{s \to 0} \frac{s\,G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

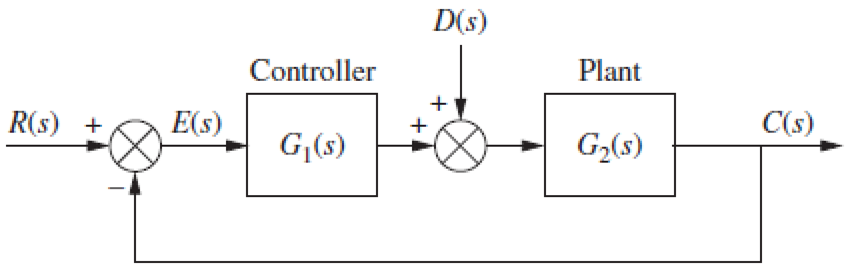


FIGURE 7.11 Feedback control system showing disturbance

Steady-state error due to step disturbance, [1, p. 352]

Steady-state error due to *step* disturbance

$$e_D(\infty) = \frac{1}{\lim\limits_{s\to 0} \frac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

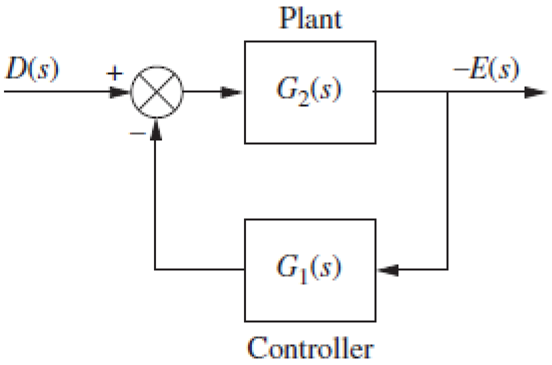


FIGURE 7.12 Figure 7.11 system rearranged to show disturbance as input and error as output, with $R(s) = 0$

Example 7.7

Steady-State Error Due to Step Disturbance

PROBLEM: Find the steady-state error component due to a step disturbance for the system of Figure 7.13.

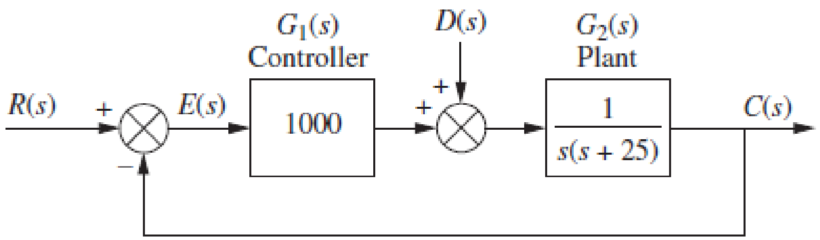


FIGURE 7.13 Feedback control system for Example 7.7

SOLUTION: The system is stable.

$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\frac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)} = -\frac{1}{0+1000} = -\frac{1}{1000}$$

The result shows that the steady-state error produced by the step disturbance is inversely proportional to the dc gain of $G_1(s)$. The dc gain of $G_2(s)$ is infinite in this example.

References

Nise, N. S. (2015).
Control Systems Engineering.
Wiley, 7 edition.