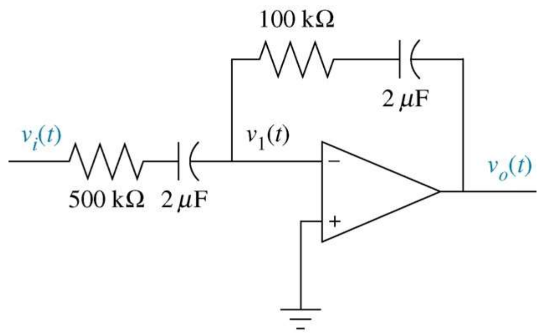
Homework 2 - Solutions

1. Transfer Functions of Electrical Networks with Operational Amplifiers
   Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each operational amplifier circuit shown in the Figures below.

   (a) **Solution:**



(a)

Calculating the feedback and feedforward impedances.

$$Z_{feedback}(s) = Z_{fb}(s) = 100 \times 10^3 + \frac{1}{2 \times 10^{-6} s}$$

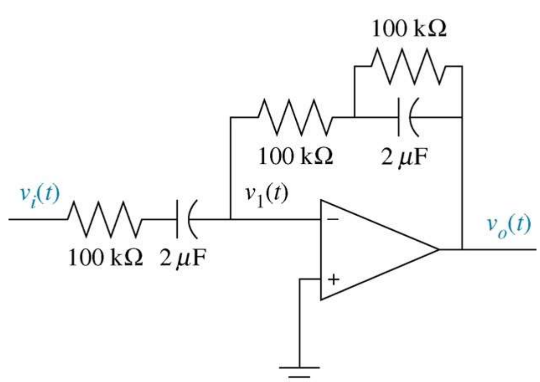
$$Z_{feedforward}(s) = Z_{ff}(s) = 500 \times 10^3 + \frac{1}{2 \times 10^{-6} s}$$

Since the system is in the inverted amplifier configuration, we know that the gain is given by

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_{fb}(s)}{Z_{ff}(s)} = -\frac{10^5(2 \times 10^{-6} s) + 1}{5 \times 10^5(2 \times 10^{-6} s) + 1}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{0.2s + 1}{s + 1}$$

   (b) **Solution:**



(b)

Calculating the feedback and feedforward impedances.

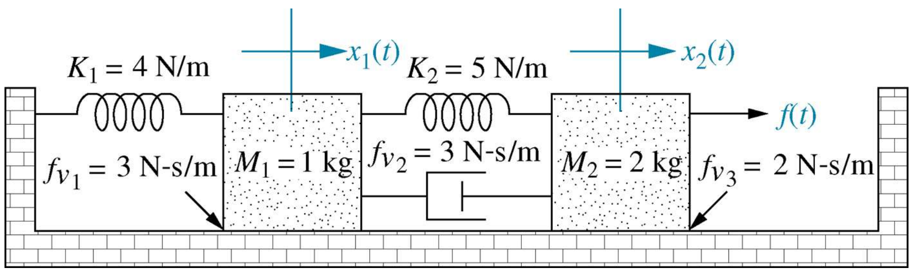
$$Z_{feedback}(s) = Z_{fb}(s) = 10^5\left(1 + \frac{5}{s+5}\right) = 10^5\frac{(s+10)}{(s+5)}$$

$$Z_{feedforward}(s) = Z_{ff}(s) = 10^5\left(\frac{5}{s} + 1\right) = 10^5\frac{(s+5)}{s}$$

Since the system is in the inverted amplifier configuration, we know that the gain is given by

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_{fb}(s)}{Z_{ff}(s)} = -\frac{s(s+10)}{(s+5)^2}$$

2. Transfer Functions of Translational Mechanical Systems
For the system shown below, find the transfer function, $G(s) = X_1(s)/F(s)$.



Solution: Equation of motion (in s-domain) for mass M_1 is given by

$$(s^2 + 6s + 9)X_1(s) - (3s+5)X_2(s) = 0 \tag{1}$$

and equation of motion (in s-domain) for mass M_2 is given by

$$-(3s+5)X_1(s) + (2s^2 + 5s + 5)X_2(s) = F(s)$$

$$\Rightarrow X_2(s) = \frac{F(s) + (3s+5)X_1(s)}{2s^2 + 5s + 5} \tag{2}$$

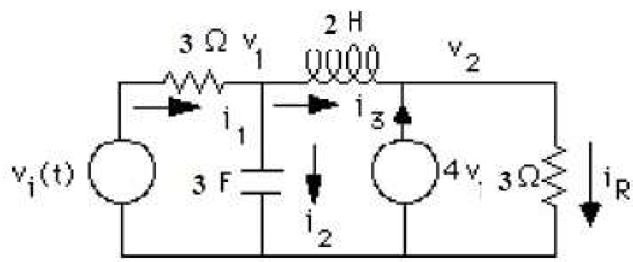
Plugging Equation (2) in Equation (1) and rearranging yields

$$X_1(s) = \frac{(3s+5)F(s)}{2s^4 + 17s^3 + 44s^2 + 45s + 20}$$

$$\Rightarrow \frac{X_1(s)}{F(s)} = \frac{3s+5}{2s^4 + 17s^3 + 44s^2 + 45s + 20}$$

3. State Space Representation of Electrical Networks

(a) Represent the electrical network shown below in state space, where $i_R(t)$ is the output.



Solution: Write the differential equations for each energy storage element.

$$\frac{dv_1}{dt} = \frac{i_2}{3}$$

$$\frac{di_3}{dt} = \frac{v_L}{2}$$

Therefore the state vector is $\mathbf{x} = \begin{bmatrix} v_1 \\ i_3 \end{bmatrix}$. Now obtain v_L and i_2 in terms of the state variables,

$$v_L = v_1 - v_2 = v_1 - 3i_R = v_1 - 3(i_3 + 4v_1) = -11v_1 - 3i_3$$

$$i_2 = i_1 - i_3 = \frac{1}{3}(v_i - v_1) - i_3 = -\frac{1}{3}v_1 - i_3 + \frac{1}{3}v_i$$

Also, the output is

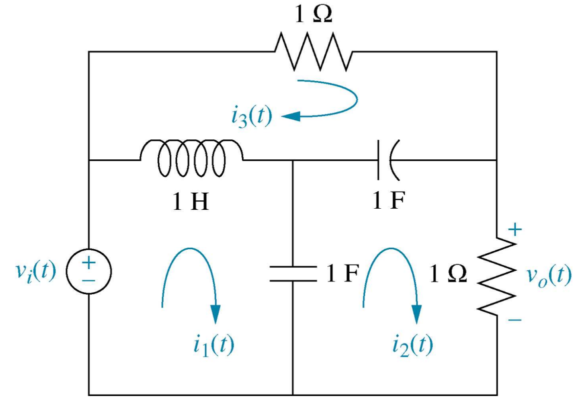
$$y = i_R = 4v_1 + i_3$$

Hence,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{3} \\ -\frac{11}{2} & -\frac{3}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{9} \\ 0 \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 4 & 1 \end{bmatrix} \mathbf{x}$$

(b) Find the state space representation of the network shown below if the output is $v_o(t)$.



Solution: Let C_1 be the grounded capacitor and C_2 be the other. Now, writing the equations for the energy storage components yields,

$$\frac{di_L}{dt} = v_i - v_{C_1} \tag{3}$$
$$\frac{dv_{C_1}}{dt} = i_1 - i_2 \tag{4}$$
$$\frac{dv_{C_2}}{dt} = i_2 - i_3 \tag{5}$$

Thus the state vector is $\mathbf{x} = \begin{bmatrix} i_L \\ v_{C_1} \\ v_{C_2} \end{bmatrix}$. Now, find the three loop currents in the terms of state variables and the input. Writing KVL around Loop 2 yields

$$v_{C_1} - v_{C_2} = i_2$$

Writing KVL around the outer loop yields

$$i_3 = v_i - i_2 = v_i - v_{C_1} + v_{C_2}$$

Also, $i_1 - i_3 = i_L$. Hence,

$$i_1 = i_L + i_3 = i_L + v_i - v_{C_1} + v_{C_2}$$

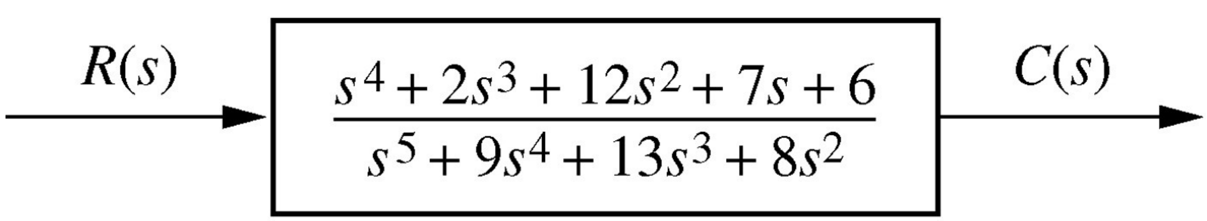
Substituting the loop currents in Equations (3), (4) and (5), yields the results in vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v_i$$

Since $v_0 = i_2 = v_{C_1} - v_{C_2}$, the output equation is

$$y = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \mathbf{x}$$

4. Transfer Function to Phase Variable Representation
For the system shown below, write the state equations and the output equation for the phase-variable representation.



Solution: Using the standard form derived in the textbook,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -13 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = \begin{bmatrix} 6 & 7 & 12 & 2 & 1 \end{bmatrix} \mathbf{x}$$