HW 11 Solutions

1. Consider the mechanical system given in Figure 1, where ξ_1, ξ_2 are displacements of the masses, u_1, u_2 are forces acting on the masses. Take the masses to be $m_1 = m_2 = 1$ kg, the spring stiffness to be $k_1 = k_2 = k_3 = 1$ N/m, and the damping coefficient to be $\nu = 1$ Ns/m.

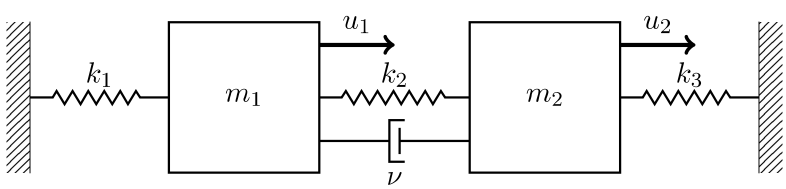

Figure 1: Simple spring-damper system

2. Write the equations of motion of the system and put them into state space form, using as state vector $x(t) = [\xi_1, \xi_2, \dot{\xi}_1, \dot{\xi}_2]' \in \mathbb{R}^4$.
Answer:

Writing the equations of motion of the first mass, we have

$$\ddot{\xi}_1 = -\xi_1 + (\xi_2 - \xi_1) + (\dot{\xi}_2 - \dot{\xi}_1) + u_1$$

similarly for the second mass, we have

$$\ddot{\xi}_2 = -\xi_2 + (\xi_1 - \xi_2) + (\dot{\xi}_1 - \dot{\xi}_2) + u_2$$

putting the system in state space form, we have

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

```
A = [0  0  1  0;
     0  0  0  1;
     -2 1  -1 1;
     1 -2  1 -1];

B = [0 0; 0 0; 1 0; 0 1];

C = [eye(2) zeros(2)];
D = [];
```

3. Consider the infinite horizon quadratic cost

$$J = \int_0^\infty (\xi_1(t)^2 + \xi_2(t)^2 + u_1(t)^2 + u_2(t)^2)\, dt$$

Recall that in the infinite horizon case, the Riccati solution P_t to the Riccati equation is constant (does not depend on time), and it is a solution to the Algebraic Riccati Equation (ARE) given by

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Solve the ARE for the cost function J using MATLAB. Given the matrices A, B, Q, R, you can use the `care` command in Matlab to solve the ARE.

Answer:

Using the form of the cost function $J = \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt$, the matrices Q and R are given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we can solve for the P matrix using the command

```
Q = blkdiag(eye(2), zeros(2));
R = [1 0; 0 1];

care(A, B, Q, R)
```

$$P = \begin{bmatrix} 0.9317 & 0.3555 & 0.2882 & 0.1260 \\ 0.3555 & 0.9317 & 0.1260 & 0.2882 \\ 0.2882 & 0.1260 & 0.4949 & 0.4153 \\ 0.1260 & 0.2882 & 0.4153 & 0.4949 \end{bmatrix}$$

4. Using the solution P, give the expression of the optimal control in the form $u^*(t) = Kx(t)$. Verify that you obtain the same K matrix using the `lqr` command in Matlab.

Answer:

$$u^*(t) = K_{care}x(t)$$

where

$$K_{care} = R^{-1}B^TP = \begin{bmatrix} 0.2882 & 0.1260 & 0.4949 & 0.4153 \\ 0.1260 & 0.2882 & 0.4153 & 0.4949 \end{bmatrix}$$

```
Kcare = inv(R)*B'*P
```

5. By plugging the optimal control $u^*(t)$ into the system dynamics $\dot{x}(t) = Ax(t) + Bu(t)$, write the dynamics of the closed-loop system in the form $\dot{x}(t) = \tilde{A}x(t)$.
Answer:

$$\dot{x}(t) = Ax(t) + Bu^*(t) = (A + BK_{care})x(t) = \tilde{A}x(t)$$

where

$$\tilde{A} = A + BK_{care} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.7118 & 1.1260 & -0.5051 & 1.4153 \\ 1.1260 & -1.7118 & 1.4153 & -0.5051 \end{bmatrix}$$

6. Plot the trajectory $\xi_1(t), \xi_2(t)$ of the system, on the same figure, for the following cases
   - open-loop system with zero input ($u(t) = 0$) and initial condition $x(0) = [1, 0, 0, 0]^T$. Plot for time $t \in [0, 15]$.
   - closed-loop system (optimal control $u^*(t) = Kx(t)$) with initial condition $x(0) = [1, 0, 0, 0]^T$. Plot for time $t \in [0, 10]$.

Hint: You can use the `initial` command in Matlab.
Answer:

```
% initial condition for both systems
x0 = [1 0 0 0]';

% open-loop
[Y, T, X] = initial(sys,x0,10);
figure;
plot(T,X(:,1),T,X(:,2))
title('Trajectories of \xi_1 and \xi_2 of the open loop system with no control')


% closed-loop
clsys = ss(A-B*Kcare,B,C,D);
[Y,T,X] = initial(clsys,x0,10);
U = (-Kcare*X')';

figure;
```

```
subplot(2,1,1);
title('Trajectories of \xi_1 and \xi_2 under nominal LQR')
```

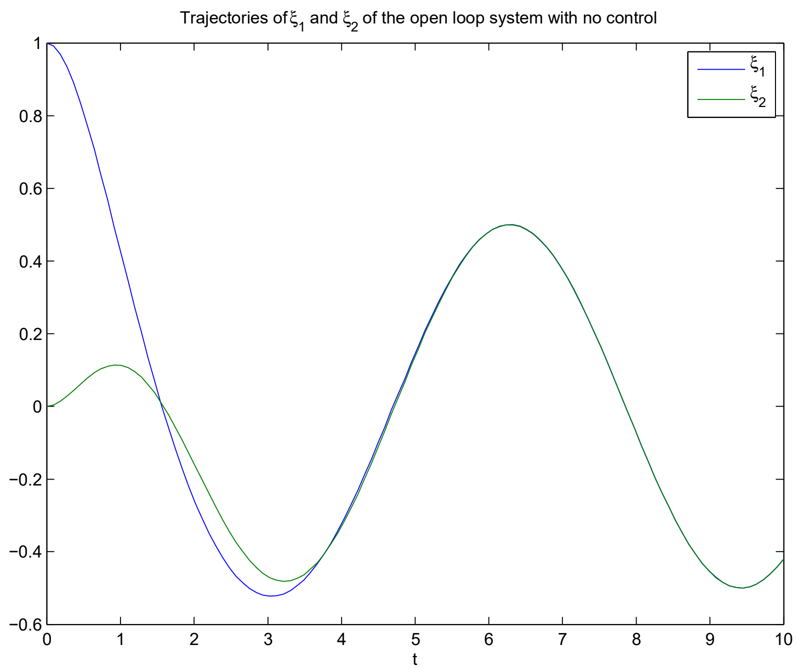

Figure 2: trajectory of the open-loop system with zero control and initial condition x_0. We observe that the two masses are synchronized in the limit, due to the presence of the damper between the two masses

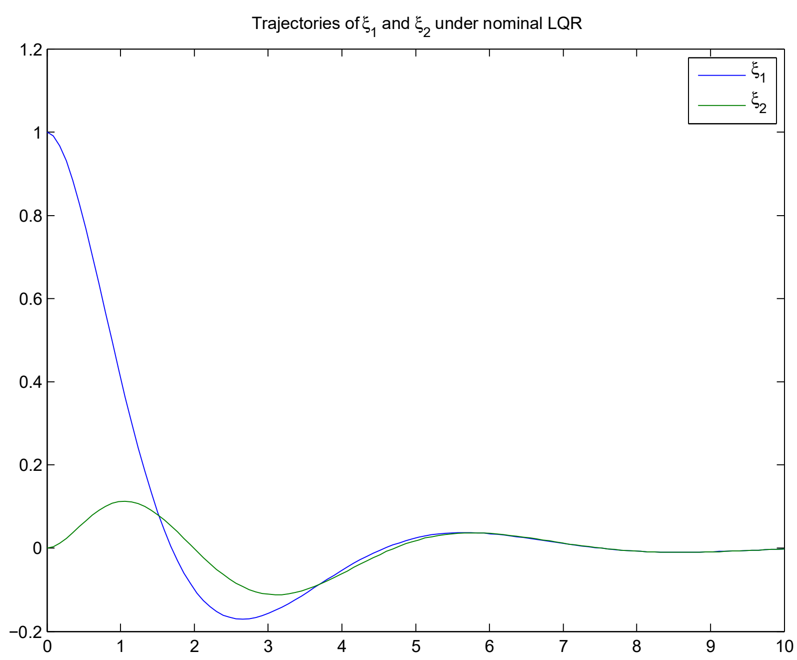

Figure 3: trajectory of the closed-loop system with optimal control u^* and initial condition x_0. The states converge to zero as expected.

For the following questions, we will always consider the closed-loop system with optimal control and initial condition $x_0 = [1, 0, 0, 0]^T$.

7. Plot the optimal control inputs $u_1^*(t), u_2^*(t)$.
Answer:

```
U = (-Kcare*X')';

figure;
plot(T,U(:,1),T,U(:,2))
xlabel('t')
legend({'u^*_1', 'u^*_2'})
title('Control inputs u^*_1 and u^*_2 under nominal LQR')
```

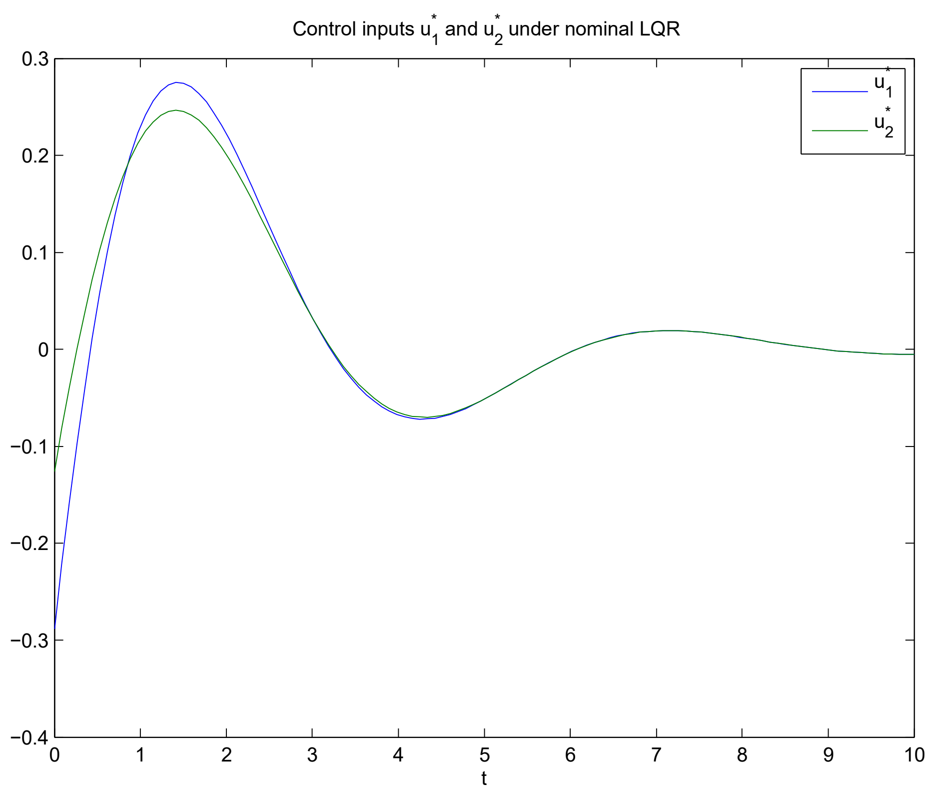

Figure 4: trajectory of the closed-loop system with optimal control u^* and initial condition x_0

8. Assume that the actuator on the first mass is limited, and can only deliver a force of magnitude less than 0.2. Observe that the current optimal control $u_1^*(t)$ does not satisfy this constraint. We will modify the cost function J in order to satisfy the constraint. Consider the cost function

$$J_\alpha = \int_{t=0}^{\infty} (\xi_1(t)^2 + \xi_2(t)^2 + \alpha u_1^2(t) + u_2^2(t))\, dt$$

Write a loop in Matlab that finds the minimal value for α (up to a precision of 10^{-1}) for which the

optimal control satisfies $\sup_t |u_1^*(t)| \leq 0.2$. Plot $\xi(t)$ and $u^*(t)$ of the closed-loop system for this value of α. Compare to the trajectories in the nominal case $\alpha = 1$. Attach your code for this question.
Answer:

The minimal value of α is $\alpha = 1.64$. Comparing the trajectories to the nominal case, we observe that the convergence to 0 of ξ_1 is slightly slower, and the control u_1 is smaller in absolute value (Figure 6)

```
%% 5) LQR Increase weight on u1
Q = blkdiag(eye(2), zeros(2));
R = [1.64 0; 0 1];
K = lqr(A,B,Q,R);
clsys = ss(A-B*K,B,C,D);
[Y,T,X] = initial(clsys,x0,10);
U = (-K*X')';

figure;
subplot(2,1,1);
plot(T,X(:,1),T,X(:,2))
title('Trajectories of \xi_1 and \xi_2 under LQR')
subplot(2,1,2);
plot(T,U(:,1),T,U(:,2))
title('Control inputs u_1 and u_2 under LQR')
```

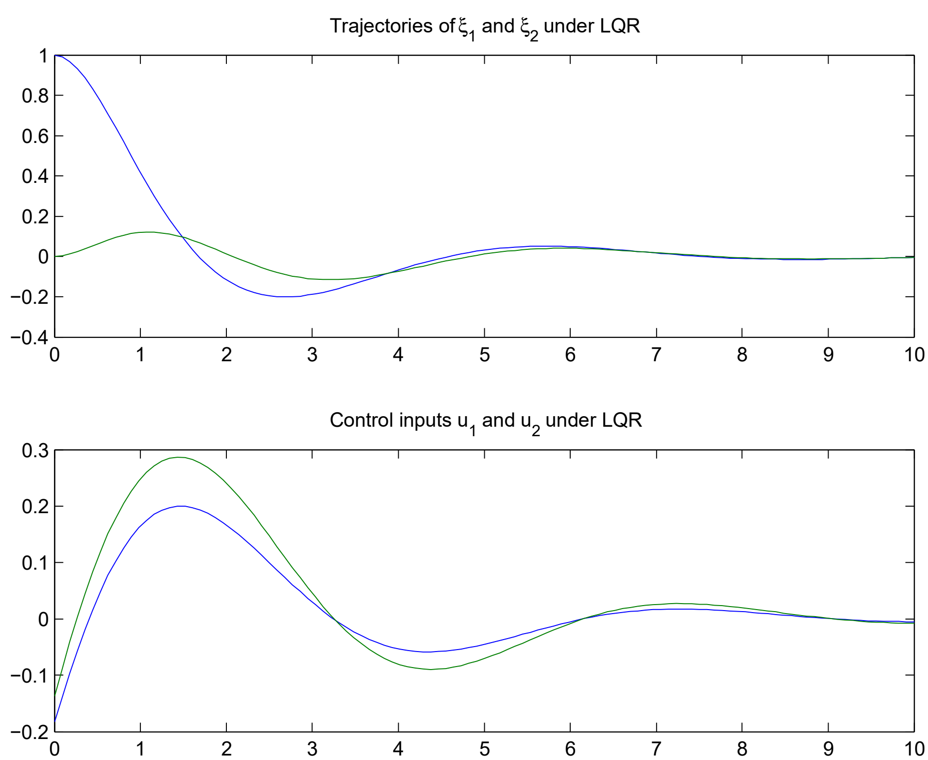

Figure 5: trajectory of the closed-loop system with optimal control u^* and initial condition x_0

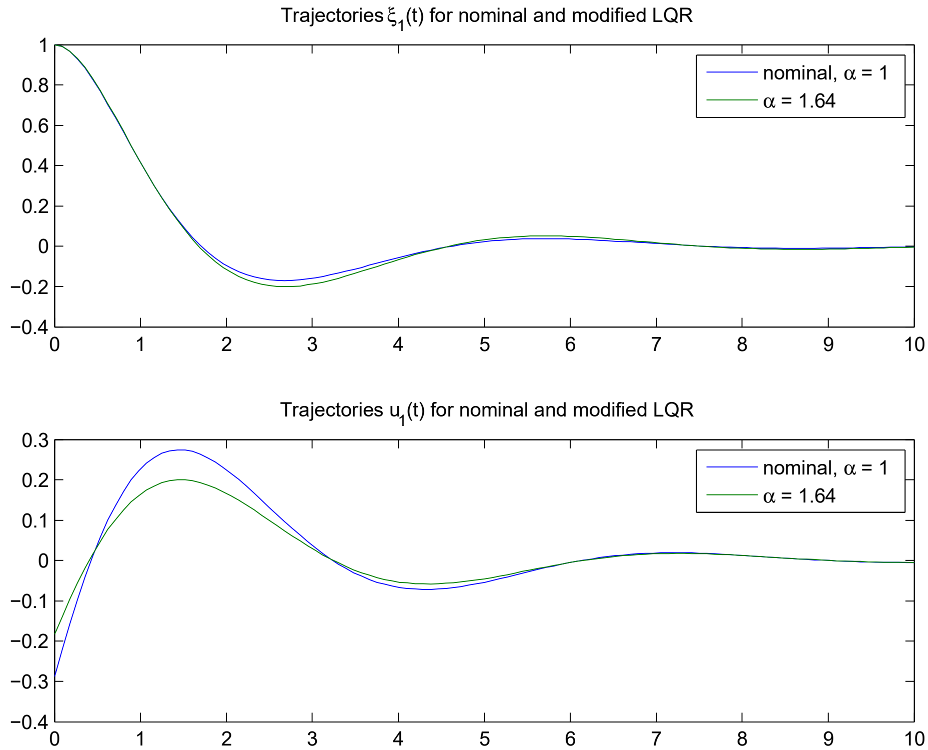

Figure 6: Compare the trajectory of ξ_1 for the nominal LQR ($\alpha = 1$) and modified LQR ($\alpha = 1.64$)

9. Suppose now that we would like to limit the velocity of the first mass to be always less than 1.5 in absolute value. First, plot the velocities $\dot{\xi}(t)$ for the nominal system (cost function J), and observe the constraint is not satisfied. Now consider the modified cost function

$$J_\beta = \int_{t=0}^{\infty} (\xi_1(t)^2 + \xi_2(t)^2 + \beta \dot{\xi}_1(t)^2 + u_1^2(t) + u_2^2(t))\, dt$$

Write a loop that finds the minimal value of β (up to a precision of 10^{-1}) such that the resulting trajectory satisfies $\sup_t |\dot{\xi}_1(t)| \leq .5$. For this value of β, plot $\xi_1(t), \xi_2(t), \dot{\xi}_1(t), u_1(t), u_2(t)$. Compare the system's behavior to the trajectories of the nominal case ($\beta = 0$). In particular, comment on changes of the control input.

Answer:

The minimal value of β is $\beta = 6.7$

```

%% 6) LQR Introduce weight on x1dot
Q = blkdiag(eye(2), zeros(2));
Q(3,3) = 6.7;
R = [1 0; 0 1];
K = lqr(A,B,Q,R);
clsys = ss(A-B*K,B,C,D);
[Y,T,X] = initial(clsys,x0,10);
U = (-K*X')';
```

```
figure;
subplot(3,1,1);
plot(T,X(:,1),T,X(:,2))
title('Trajectories of \xi_1 and \xi_2 under LQR')
subplot(3,1,2);
plot(T,X(:,3),T,X(:,4))
title('Trajectories of \dot{x}_1 and \dot{x}_2 under LQR')
subplot(3,1,3);
plot(T,U(:,1),T,U(:,2))
title('Control Inputs under LQR')
```

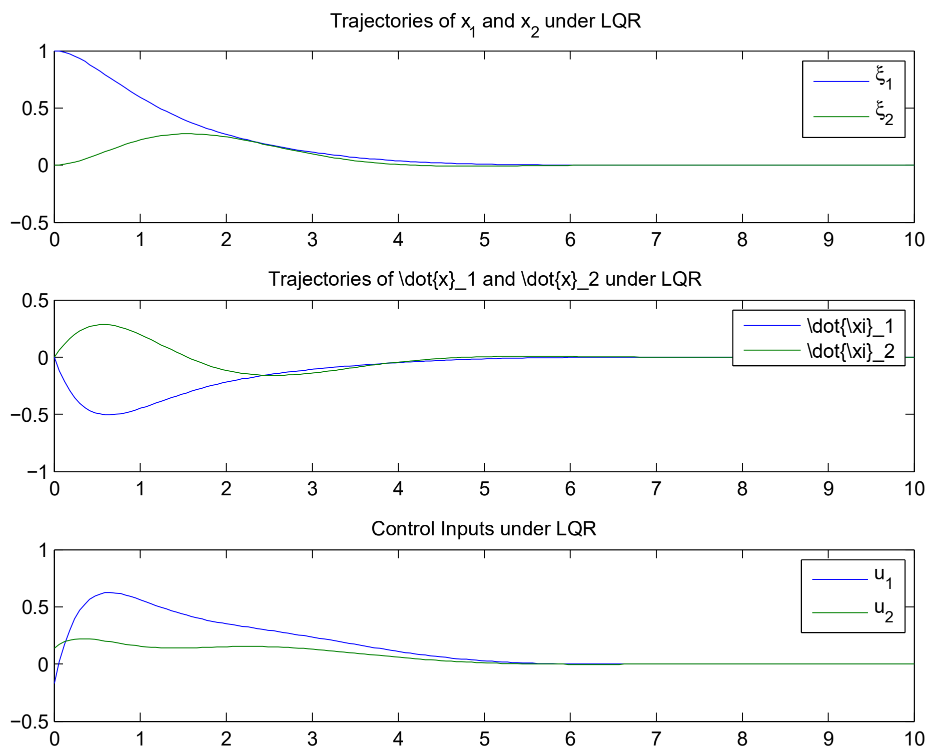

Figure 7: trajectory of the closed-loop system with optimal control u^* and initial condition x_0

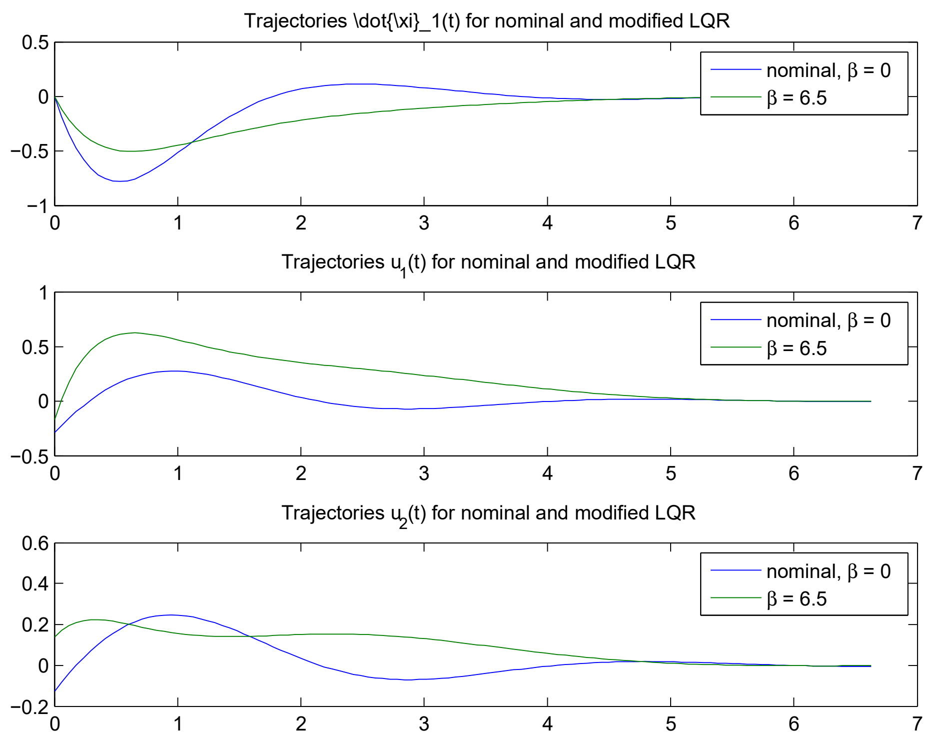

Figure 8: Compare the optimal controls u_1 and u_2 for the nominal LQR ($\beta = 0$) and modified LQR ($\beta = 6.5$)