# Chapter 5: Reduction of Multiple Subsystems

## ELG 3155 : Introduction to Control Systems

- Objective:
- Reduce a block diagram of multiple subsystems.
- Represent in signal-flow graph a system consisting of multiple subsystems

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# Outline

5 Reduction of Multiple Subsystems.

# 1. Introduction

- Finding the Transfer function of a system represented by a block diagram is not always straightforward.
- For example, find the transfer function of the system in Figure.

- Reducing (i.e., simplifying) the block diagram is one way to derive the transfer function of such a system.

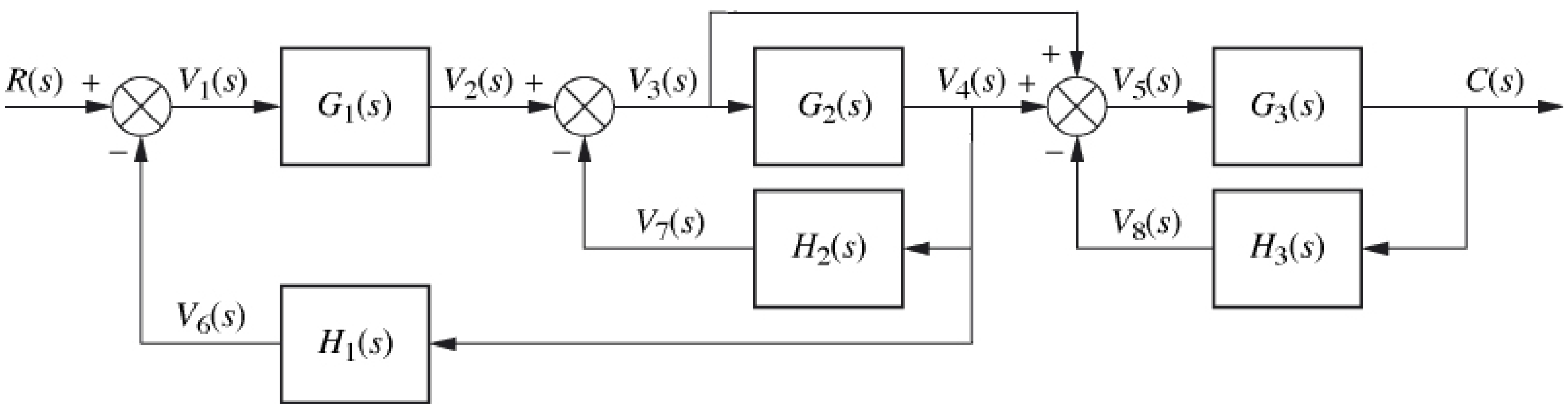
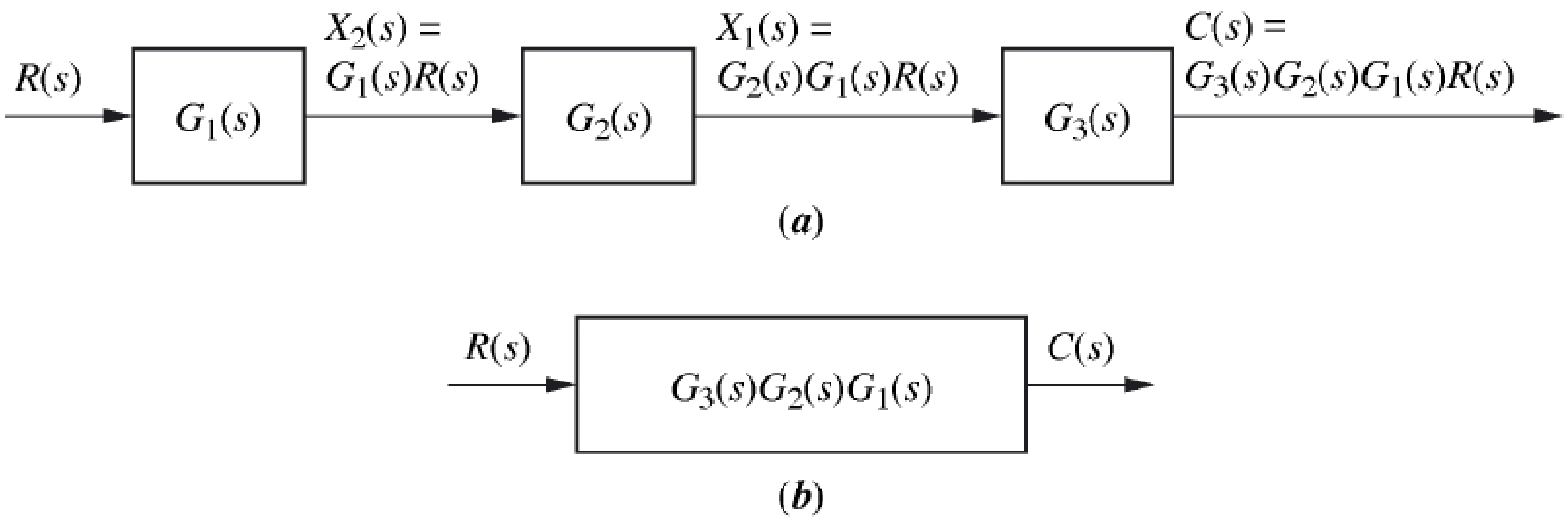


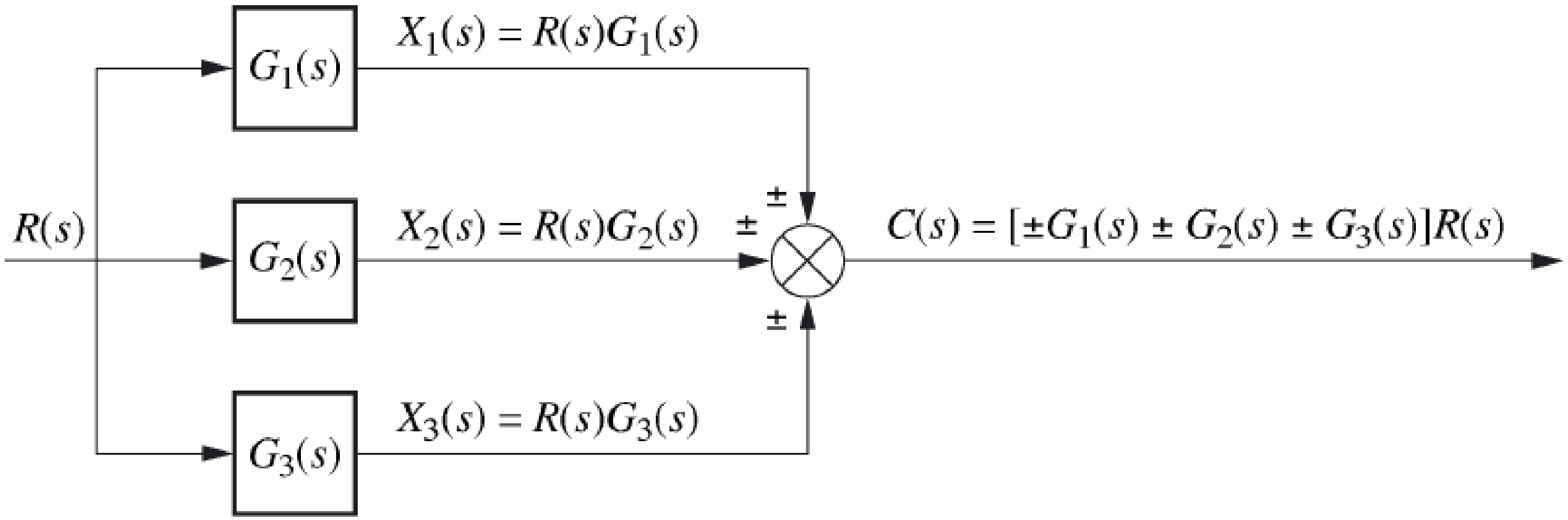

Figure 1: (Fig. 5.11 of [Nise, 2015])

# 2. Block Diagram Reduction

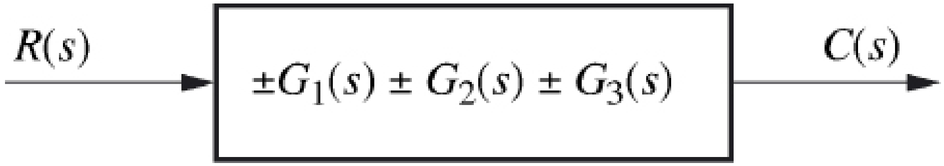
## 1. Cascade Form



## 2. Parallel form

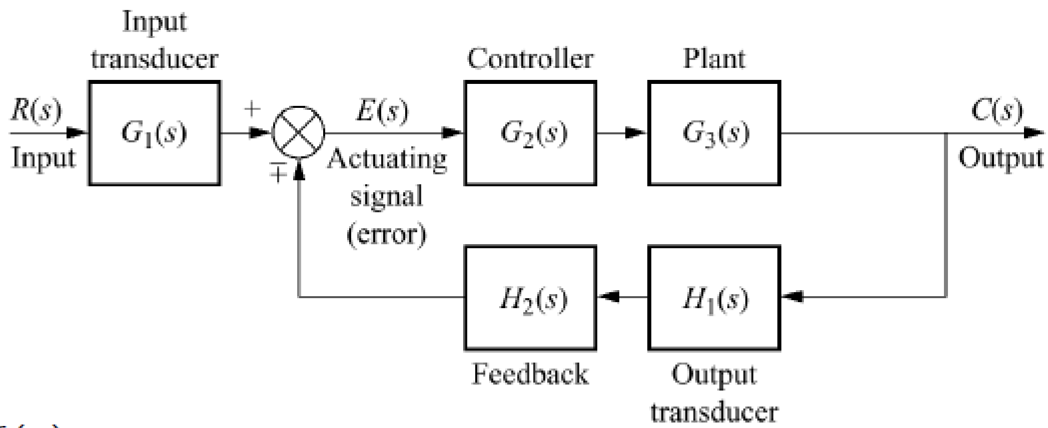



(*a*)



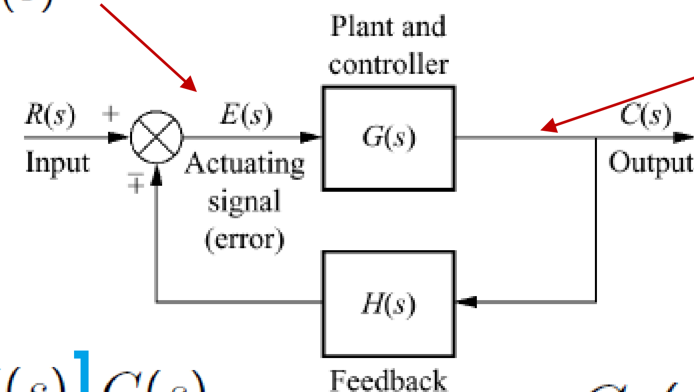

(*b*)

# 3. Feedback Form




$$E(s) = R(s) \mp C(s)H(s)$$

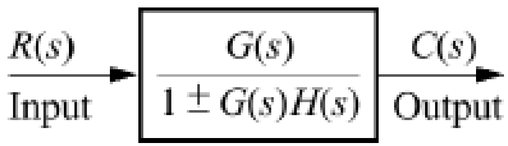
$$C(s) = E(s)G(s)$$

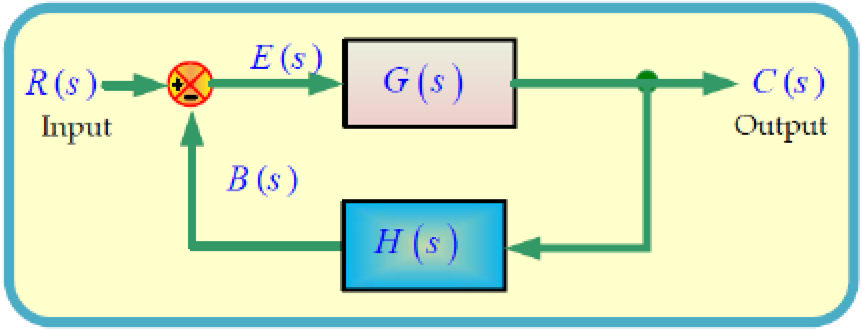



$$C(s) = \left[R(s) \mp C(s)H(s)\right]G(s)$$

$$G_e(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

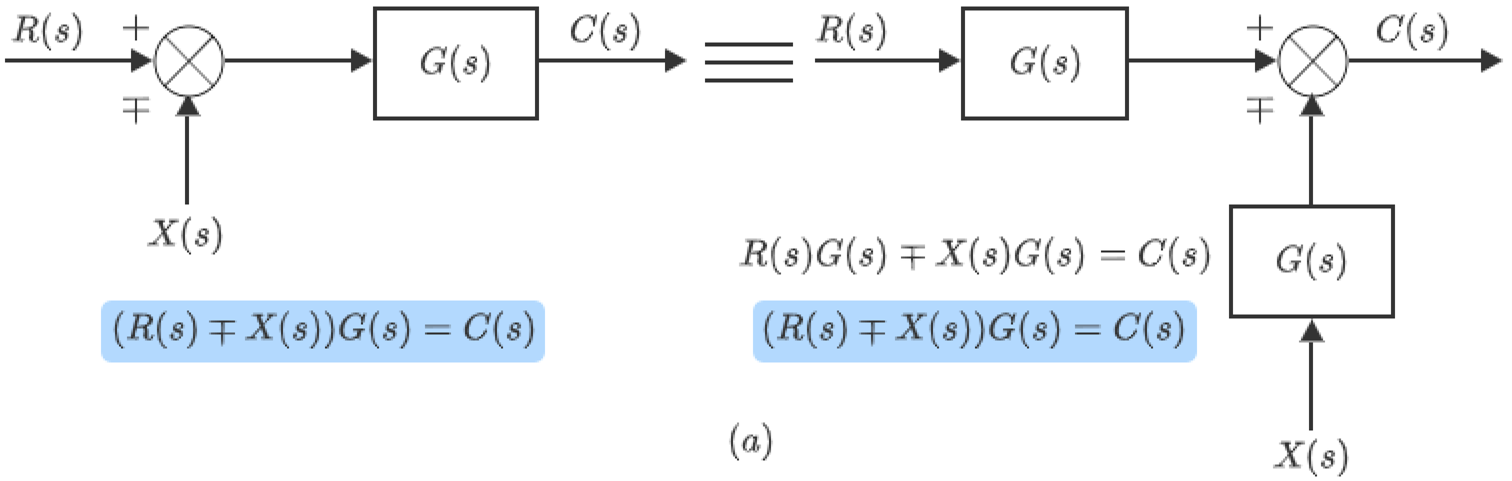
We call $G(s)H(s)$ the open loop transfer function or loop gain.



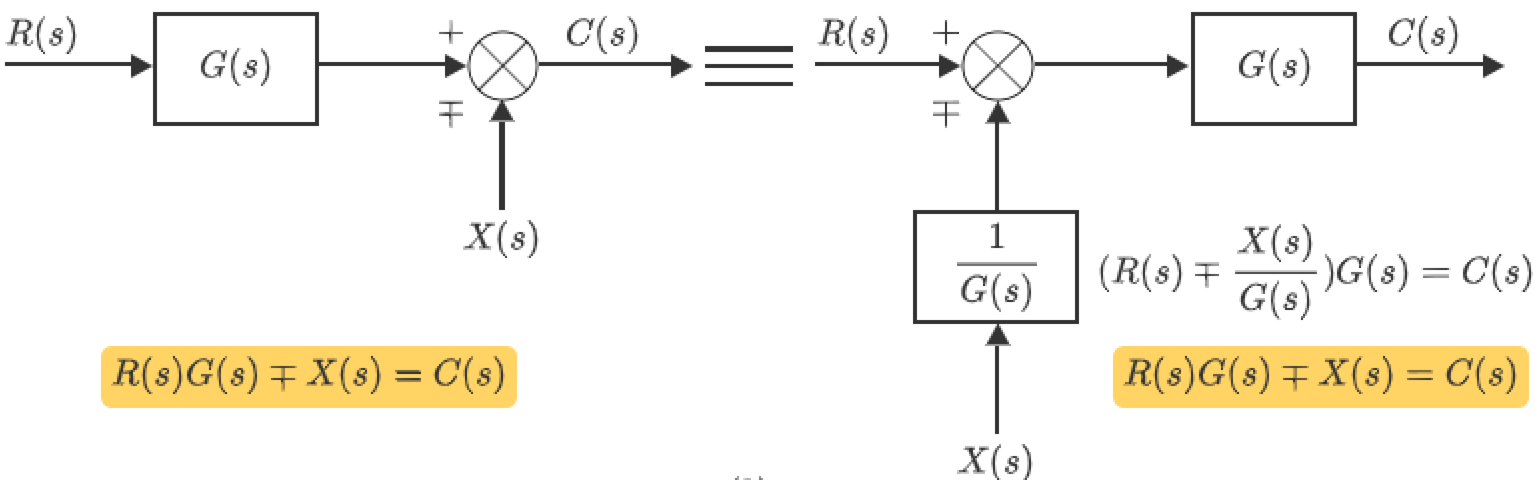



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

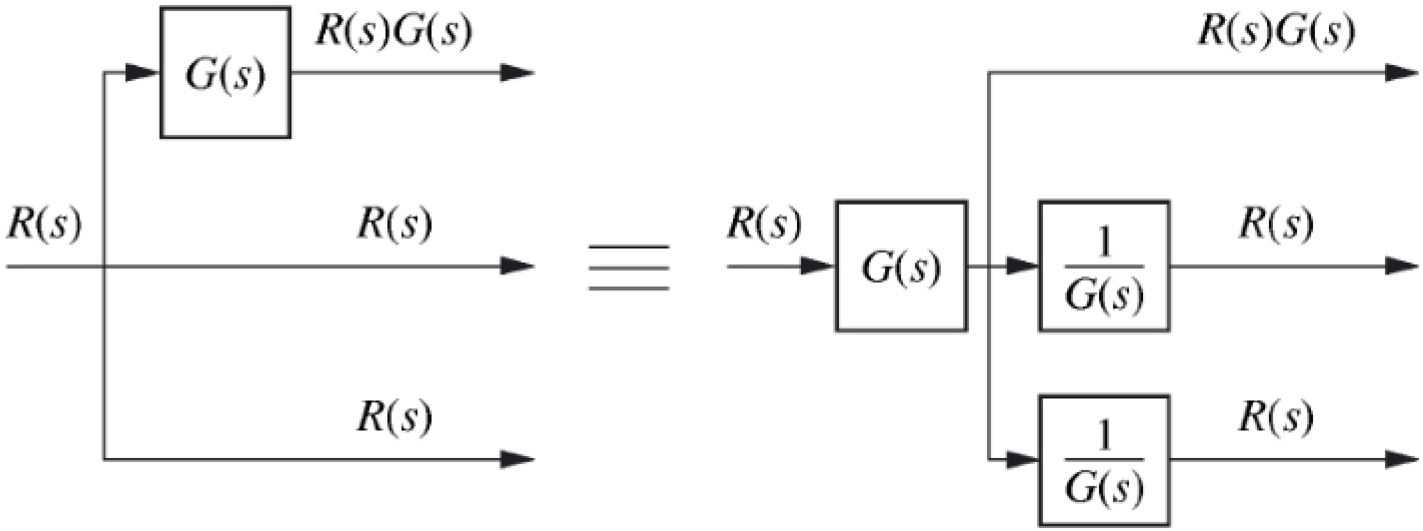
## Moving Blocks to create Familiar Forms



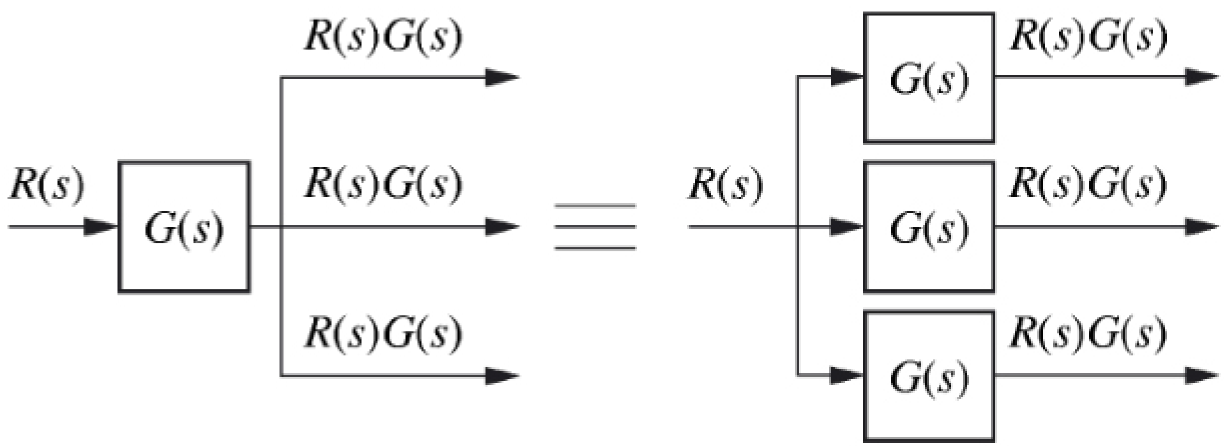

$(a)$




$(b)$


R(s)G(s)
G(s)
R(s)
R(s)
R(s)
R(s)G(s)
R(s)
G(s)
1/G(s)
R(s)
1/G(s)
R(s)

(a)


R(s)G(s)
R(s)
G(s)
R(s)G(s)
R(s)G(s)
R(s)
G(s)
R(s)G(s)
G(s)
R(s)G(s)
G(s)
R(s)G(s)

(b)

# Definitions

In reference to Fig. 2,

$$\begin{aligned}
\text{Feedforward gain} &= G_1G_2\cdots G_n(s)\\
\text{Feedback gain} &= H_1H_2\cdots H_m(s)\\
\text{Open-loop transfer function (OLTF)} &= \text{feedforward} \times \text{feedback}\\
&= G_1G_2\cdots G_nH_1H_2\cdots H_m\\
\text{Closed-loop transfer function (CLTF)} &= \text{System's transfer function}\\
&= \text{feedforward}/(1+OLTF)\\
&= \frac{G_1G_2\cdots G_n}{1+G_1G_2\cdots G_nH_1H_2\cdots H_m}\\
\text{For a positive feedback, CLTF} &= \text{feedforward}/(1-OLTF)\\
&= \frac{G_1G_2\cdots G_n}{1-G_1G_2\cdots G_nH_1H_2\cdots H_m}
\end{aligned}$$

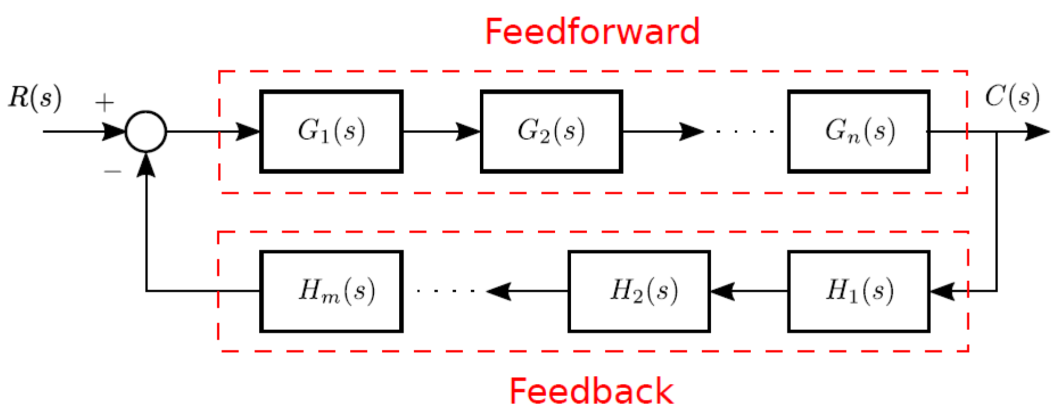
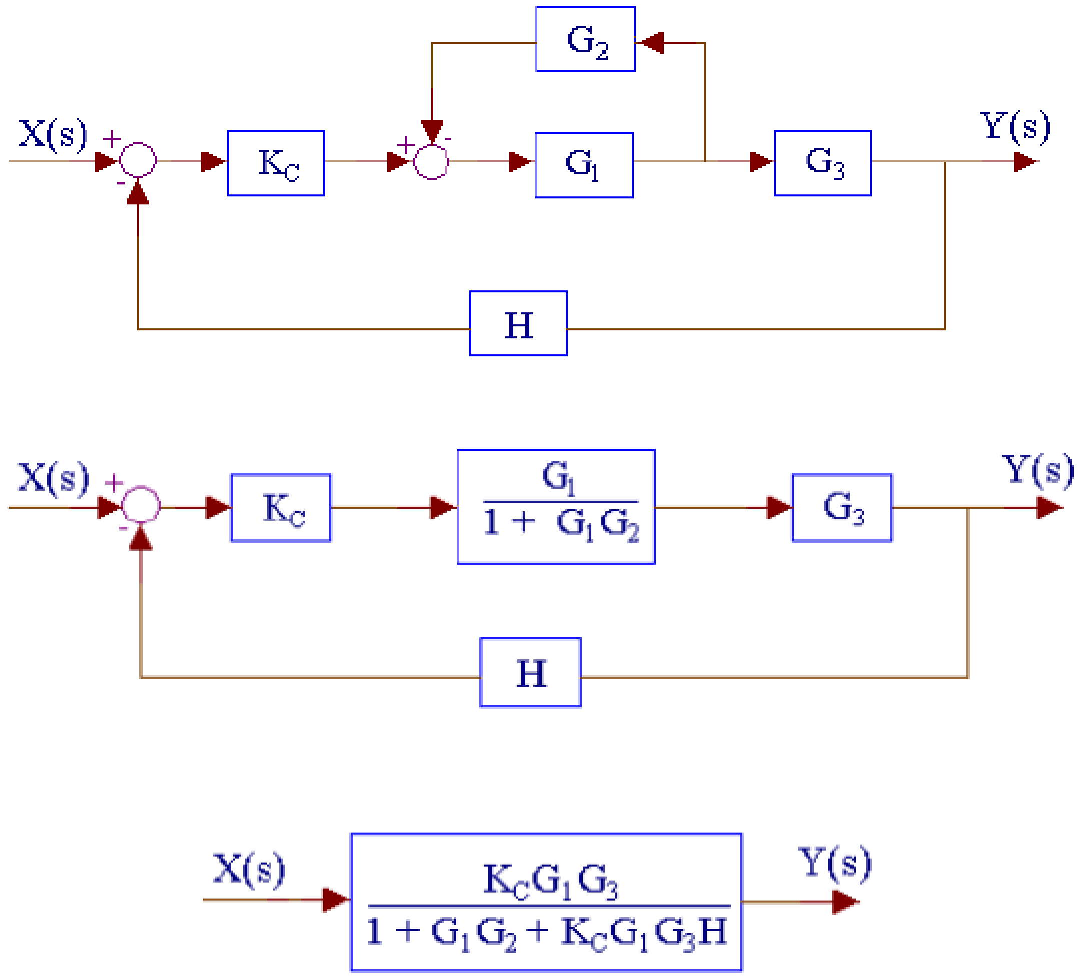



Figure 2: Block diagram of a system with a negative feedback

# Example 1



X(s)
+
-
$K_C$
+
-
$G_2$
$G_1$
$G_3$
Y(s)
H
X(s)
+
-
$K_C$
$\frac{G_1}{1 + G_1 G_2}$
$G_3$
Y(s)
H
X(s)
$\frac{K_C G_1 G_3}{1 + G_1 G_2 + K_C G_1 G_3 H}$
Y(s)

## Example 5.2: Find the transfer function of the system in Fig

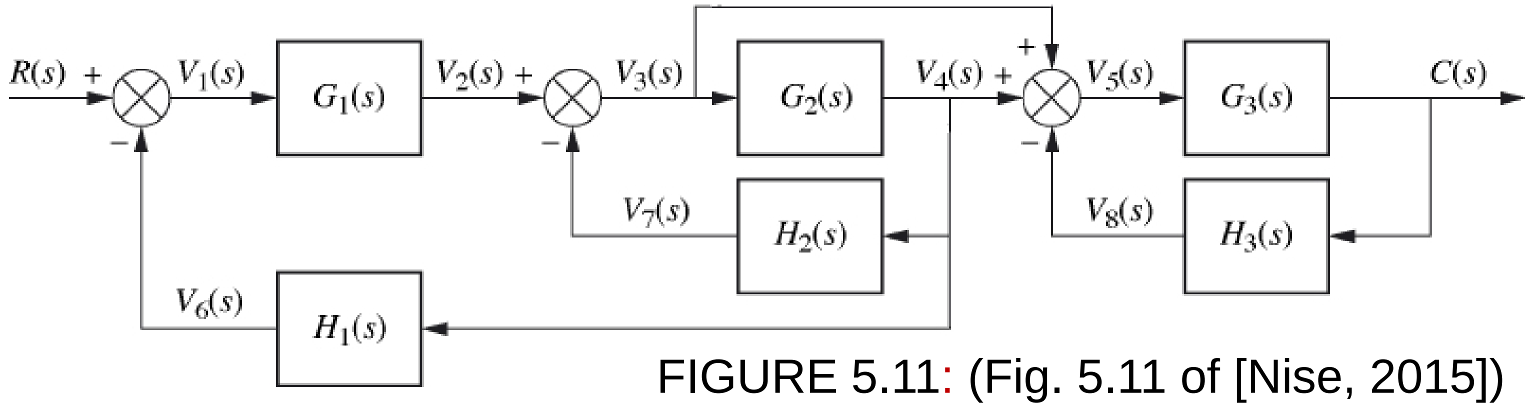
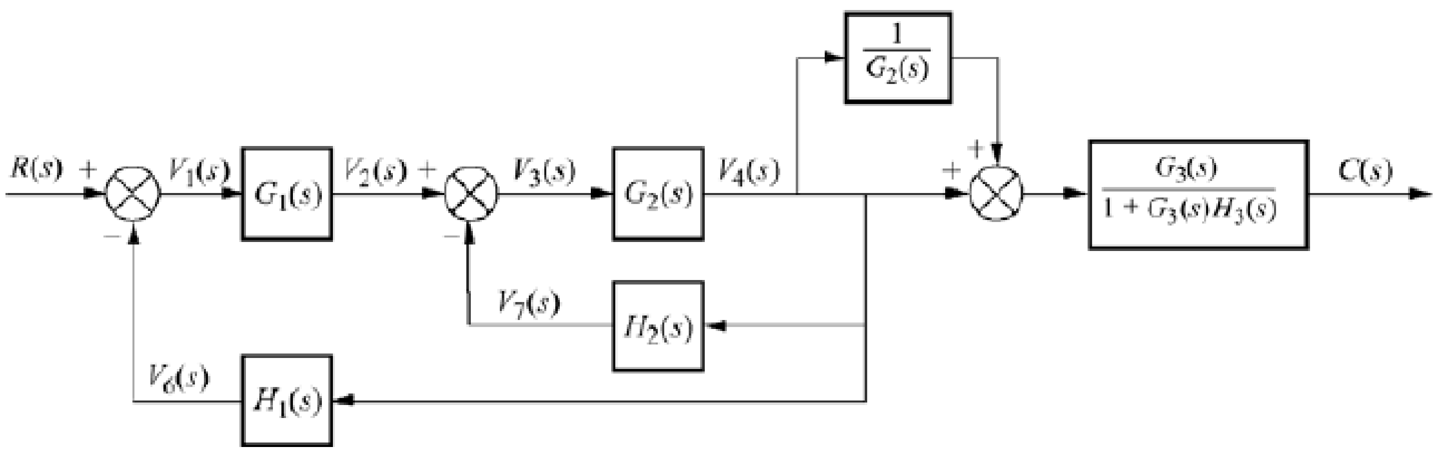
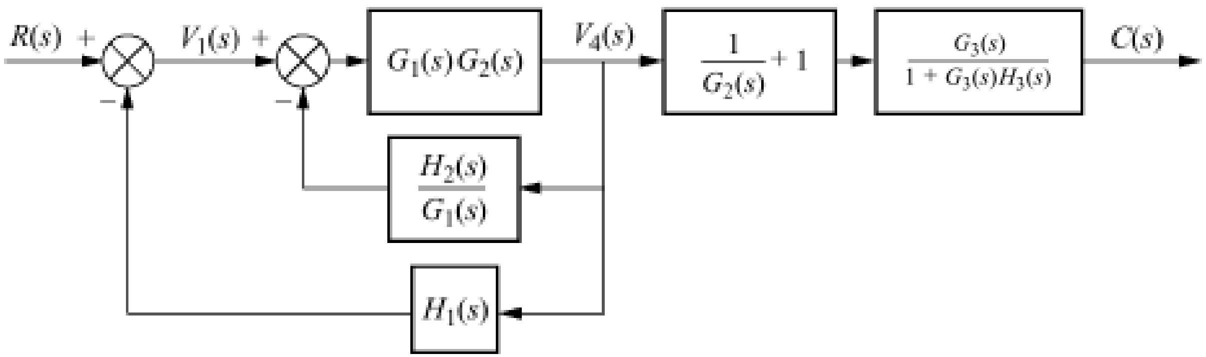



FIGURE 5.11: (Fig. 5.11 of [Nise, 2015])

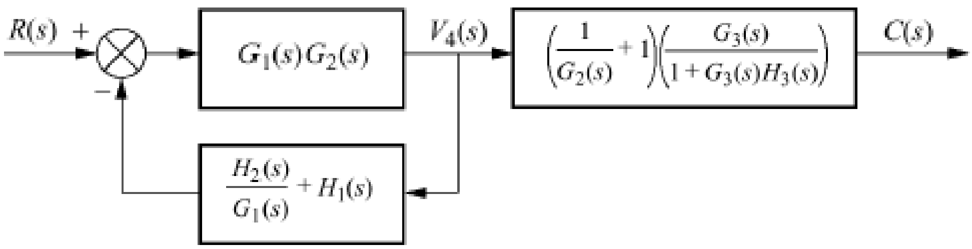
1. Move $G_2$ to left of pickoff point creating parallel form.
2. Reduce feedback system ($G_3$,$H_3$).



3. Reduce parallel form containing $\frac{1}{G_2(s)}$ and unity.
4. Push $G_1(s)$ to the right past summing junction. Creates parallel form ($H_1$ and $[\frac{1}{G_1}, H_2]$).
5. Combine serial forms ($G_1$, $G_2$) and ($\frac{1}{G_1}$, $H_2$).

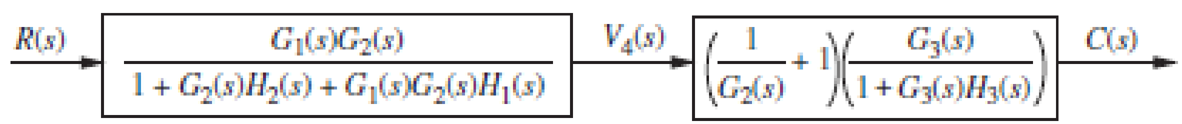



6. Collapse summing junctions, and combine parallel form.
7. Combine serial form on right.



8. Collapse feedback form.

9. Combine the two cascade blocks.



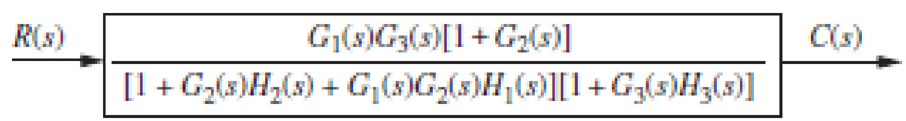




**FIGURE 5.12** Steps in the block diagram reduction for Example 5.2

## Exercise 5.1 Find the equivalent transfer function,.

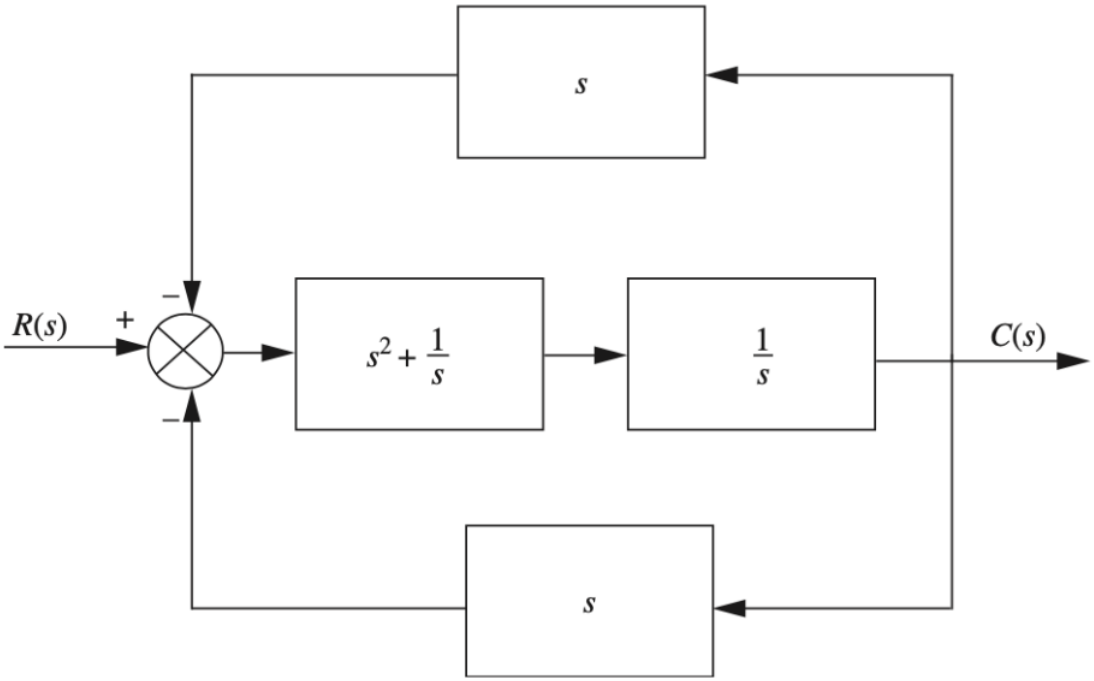


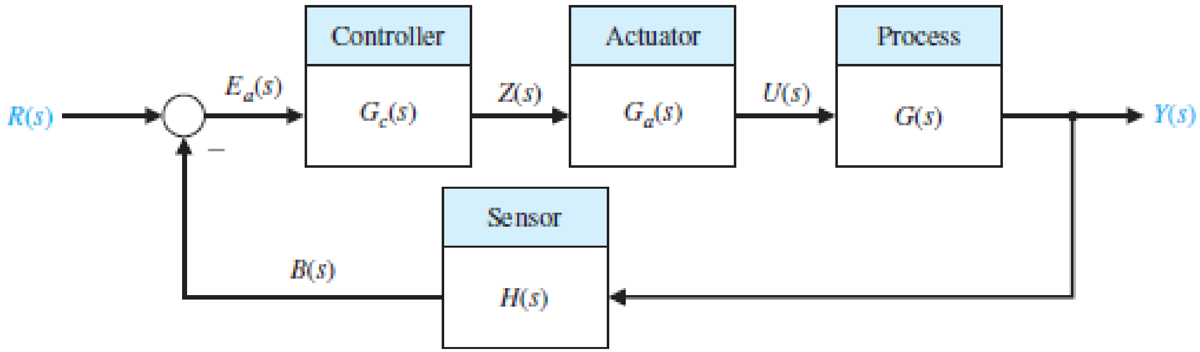

FIGURE 5.13 Block diagram for Skill-Assessment Exercise 5.1[Nise, 2015]

## Answer

Combine the parallel blocks in the forward path. Then, push $\frac{1}{s}$ to the left past the pickoff point.

$$T(s) = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$$

# 3. Analysis and Design of Feedback Systems




$$\frac{Y(s)}{R(s)} = \frac{G(s)G_a(s)G_c(s)}{1 + G(s)G_a(s)G_c(s)H(s)}.$$

Example5.3: It explains analysis and design of feedback systems that reduce to second-order systems. Percent overshoot, settling time, peak time, and rise time can then be found from the equivalent transfer function. **(a=5)**

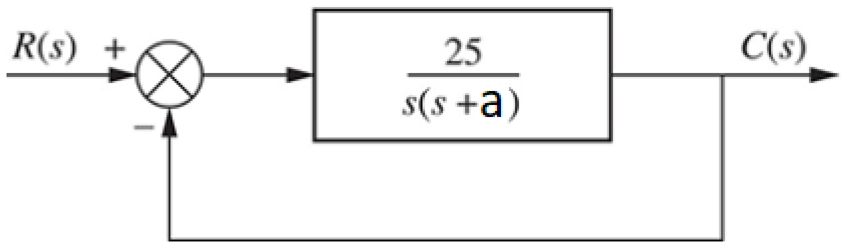



FIGURE 5.15 Feedback system for Example 5.3 [Nise, 2015]

$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5$$

$$2\zeta\omega_n = 5 \Rightarrow \zeta = 0.5$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ second}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ seconds}$$

```
clear; close all;
a=5;    % change a to yield desired %OS
G=tf(25,poly([0,-a]));
T=feedback(G,1);
[numt, dent]=tfdata(T,'v');
wn=sqrt(dent(3))   % natural frequency
z=dent(2)/(2*wn)   % zeta
Ts=4/(z*wn)
Tp=pi/(wn*sqrt(1-z^2))
percent_OS=exp(-z*pi/sqrt(1-z^2))*100
Tr=(1.76*z^3-0.417*z^2 + 1.039*z + 1)/wn
step(T),grid
```

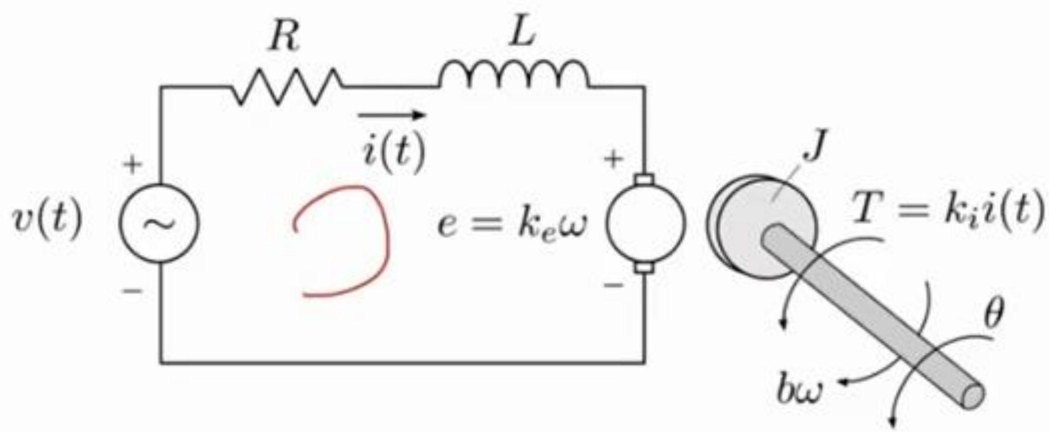
# Block diagram of a DC motor

$$I(s) = \frac{V(s) - V_b(s)}{Ls + R}$$ e

$$\omega(s) = \frac{T(s) - T_d(s)}{Js + b}$$

$$T(s) = k_i I(s)$$

$$V_m(s) = k_m \omega(s)$$




$$V(s) = (R + Ls)I(s) + \omega(s)k_m \rightarrow I(s) = \frac{V(s) - \omega(s)k_m}{\cancel{Ls} + R}$$

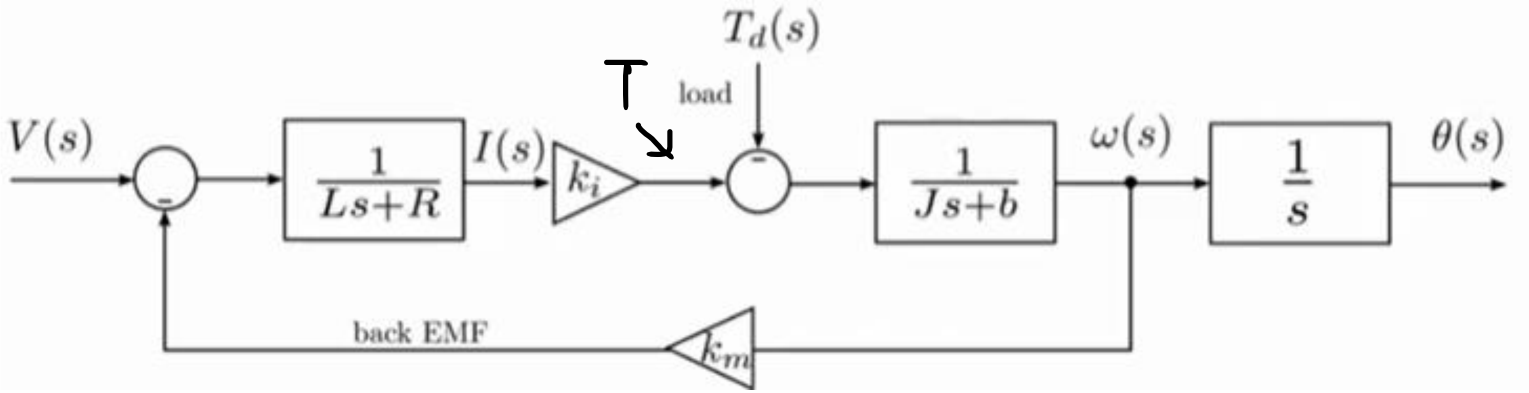
$$T(s) = (Js + b)\omega(s) + T_d(s) \rightarrow \omega(s) = \frac{I(s)k_i - T_d(s)}{\cancel{Js} + b}$$

Under steady-state condition:

$$\omega = \frac{k_i V - RT_d}{k_i k_m + Rb}, \qquad T = \frac{k_i(Vb + k_m T_d)}{k_m k_i + Rb}$$

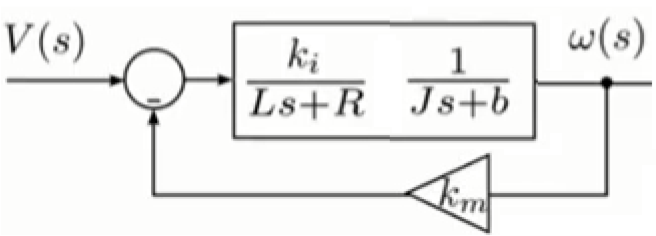
We have two equations. One describes the electrical and one describes the mechanical.

# Transfer Function




**Speed** to voltage transfer function for $T_d(s) = 0$

$$\boxed{S(s) = \frac{\omega(s)}{V(s)}}$$

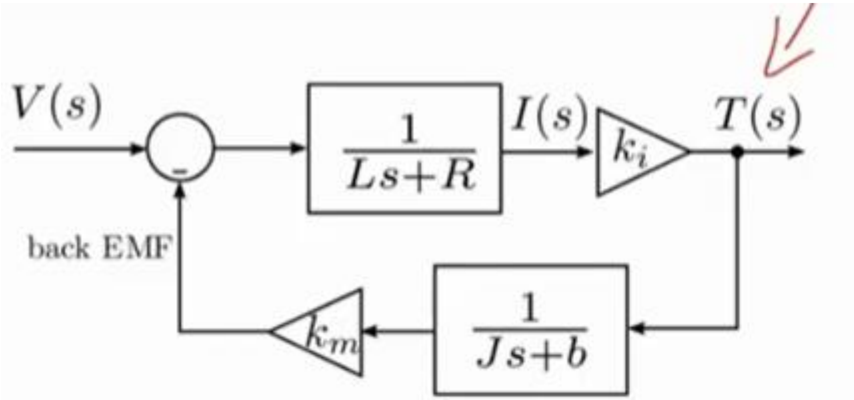



$$S(s) = \frac{\omega(s)}{V(s)} = \frac{k_i}{(Ls+R)(Js+b)+k_i k_m}$$

**Position** to voltage transfer function for $T_d(s) = 0$

$$P(s) = \frac{\theta(s)}{V(s)} = \frac{k_i}{s[(Ls+R)(Js+b)+k_i k_m]}$$

# Transfer function ….




| | | |
|---|---|---|
| Torque constant | $k_i$ | mNm/A |
| Speed constant | $k_m$ | mV/(rad/s) |
| Resistance | $R$ | Ω |
| Inductance | $L$ | H |
| Rotor inertia | $J$ | kg·m$^2$ |
| Friction constant | $b$ | Nm/(rad/s) |

**Torque** to voltage transfer function for $T_d(s) = 0$.

$$M(s) = \frac{T(s)}{V(s)} = \frac{\frac{k_i}{Ls+R}}{1 + \frac{k_m}{Js+b}\frac{k_i}{Ls+R}}$$

$$M(s) = \frac{T(s)}{V(s)} = \frac{k_i(Js+b)}{(Js+b)(Ls+R)+k_i k_m} = S(s)\,(Js+b)$$

# 4. Signal-Flow Graphs

- Extracting the transfer function from the system's block diagram is not a systematic procedure.

  ⇒ cannot be automated or programmed using a computer.

- The transfer function of a system with a complicated block diagram can be derived in a systematic fashion by transforming the block diagram into a signal-flow graph ; by applying Mason's formula.

- A block diagram can be transformed into a signal-flow graph by :
  1. transforming every signal (link) in the block diagram to a node in the signal-flow graph.
  2. Transforming each block (local TF) in the block diagram to a directed link in the signal-flow graph.

# Definitions

- A source node is a node from which signals flow only away from it (has no incoming signal).

  In a signal-flow graph, source nodes correspond to the system inputs and initial conditions.

- A sink node is a node that has only incoming signals (branches) and no departing signals.

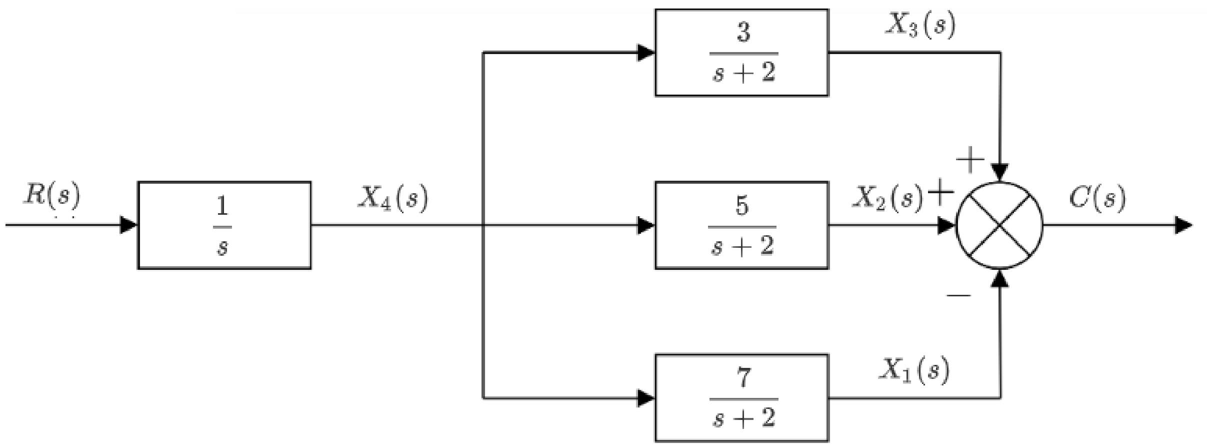
- A path is a continuous connection of branches from one node to another following the signal flow direction.

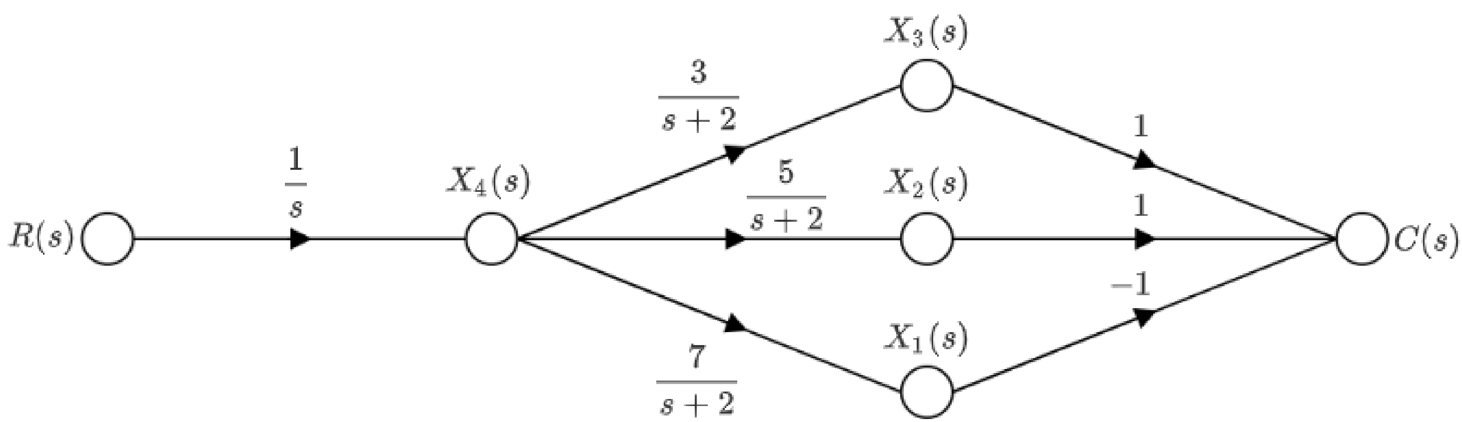
- A forward path is path that connects a source node to a sink node without visiting any node more than once.

- Two paths are said to be nontouching if they have no node in common.

- A path gain is the product of the transfer functions of all the branching forming the path.

- A loop is a closed path in which no node is visited more than once (departure node = arrival node).


$R(s)$
$\frac{1}{s}$
$X_4(s)$
$\frac{3}{s+2}$
$X_3(s)$
$\frac{5}{s+2}$
$X_2(s)$
$\frac{7}{s+2}$
$X_1(s)$
+
+
−
$C(s)$


$R(s)$
$\frac{1}{s}$
$X_4(s)$
$\frac{3}{s+2}$
$X_3(s)$
$\frac{5}{s+2}$
$X_2(s)$
$\frac{7}{s+2}$
$X_1(s)$
1
1
−1
$C(s)$

# 5.5 Mason's Rule

## Definitions

- *Loop gain*. The product of branch gains found by traversing a path that starts at a node and ends at the same node

  1. $G_2(s)H_1(s)$
  2. $G_4(s)H_2(s)$
  3. $G_4(s)G_5(s)H_3(s)$
  4. $G_4(s)G_6(s)H_3(s)$

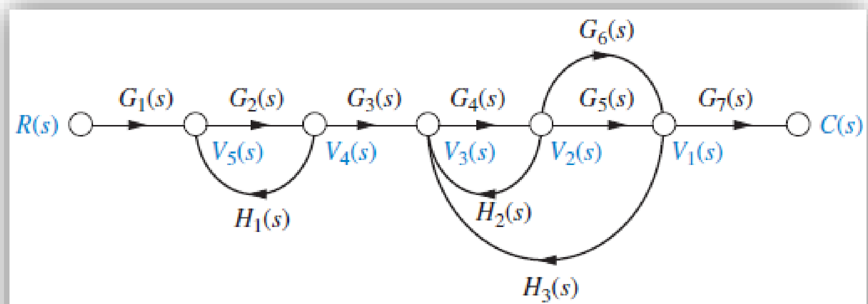



**FIGURE 5.20** Signal-flow graph for demonstrating Mason's rule

- *Forward-path gain*. The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal flow.

  1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
  2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

- *Nontouching loops*. Loops that do not have any nodes in common.

  $G_2(s)H_1(s)$ does not touch loops

  $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$.

- *Nontouching-loop gain*. The product of loop gains from nontouching loops taken two, three, four, or more at a time.

  1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
  2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
  3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

# Mason's Rule

- The transfer function between a source $R(s)$ and a sink $C(s)$ in a signal-flow graph is formulated as :

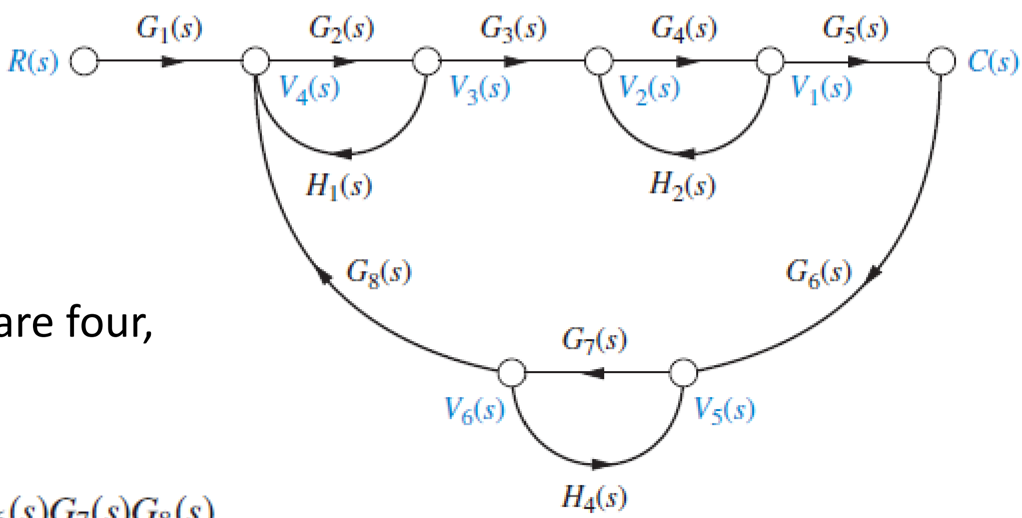
$$\boxed{G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}}$$

$$\begin{aligned}
k &\equiv \text{number of forward paths between } R(s) \text{ and } C(s) \\
T_k &\equiv \text{gain of forward path number } k \\
\Delta =\ & 1 - (\textstyle\sum \text{loop gains}) \\
& + (\textstyle\sum \text{nontouching loop gains taken 2 at a time}) \\
& - (\textstyle\sum \text{nontouching loop gains taken 3 at a time}) \\
& + \cdots
\end{aligned}$$

$\Delta_k$ is calculated in the same way as $\Delta$ but with dropping the loops that touch the forward path number k (those having at least one common node with this path).

## Example 5.7 Page : 252

Find the transfer function, $C(s)/R(s)$, for the signal-flow graph




(Fig. 5.21 of [Nise, 2015])

Solution

First, identify the *forward-path gains*

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

Second, identify the *loop gains*. There are four, as follows:

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_7(s)H_4(s)$
4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Third, identify the *nontouching loops taken two at a time*.

Loop 1 and loop 2: $G_2(s)H_1(s)G_4(s)H_2(s)$

Loop 1 and loop 3: $G_2(s)H_1(s)G_7(s)H_4(s)$

Loop 2 and loop 3: $G_4(s)H_2(s)G_7(s)H_4(s)$

Finally, the *nontouching loops taken three at a time* are as follows:

Loops 1, 2, and 3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Now, we form $\Delta$ and $\Delta_k$. Hence,

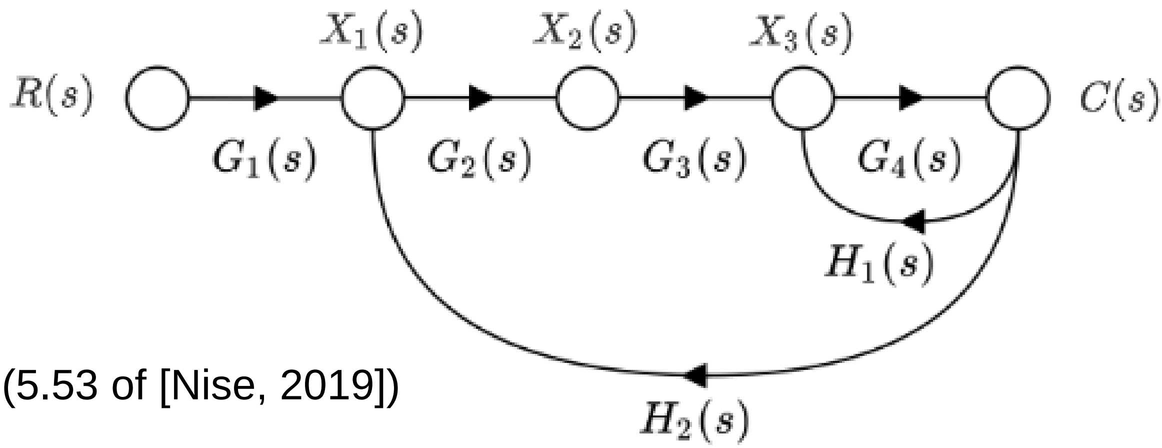
$$\begin{aligned}\Delta = 1 &-[G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) \\ &\qquad + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ &+[G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\ &\qquad + G_4(s)H_2(s)G_7(s)H_4(s)] \\ &-[G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

We form $\Delta_k$ by eliminating from $\Delta$ the loop gains that touch the $k$th forward path:

$$\Delta_1 = 1 - G_7(s)H_4(s)$$

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta}$$

PARTICIPATION ACTIVITY | 5.5.3: Application of Mason's rule.




(5.53 of [Nise, 2019])

Forward-path gain

$$G_1(s)G_2(s)G_3(s)G_4(s)$$

Loop gain 1

$$G_2(s)G_3(s)G_4(s)H_2(s)$$

Loop gain 2

$$G_4(s)H_1(s)$$

Mason's Rule

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

$$k = \text{forward paths} = 1$$

$$T_k = T_1 = G_1(s)G_2(s)G_3(s)G_4(s)$$

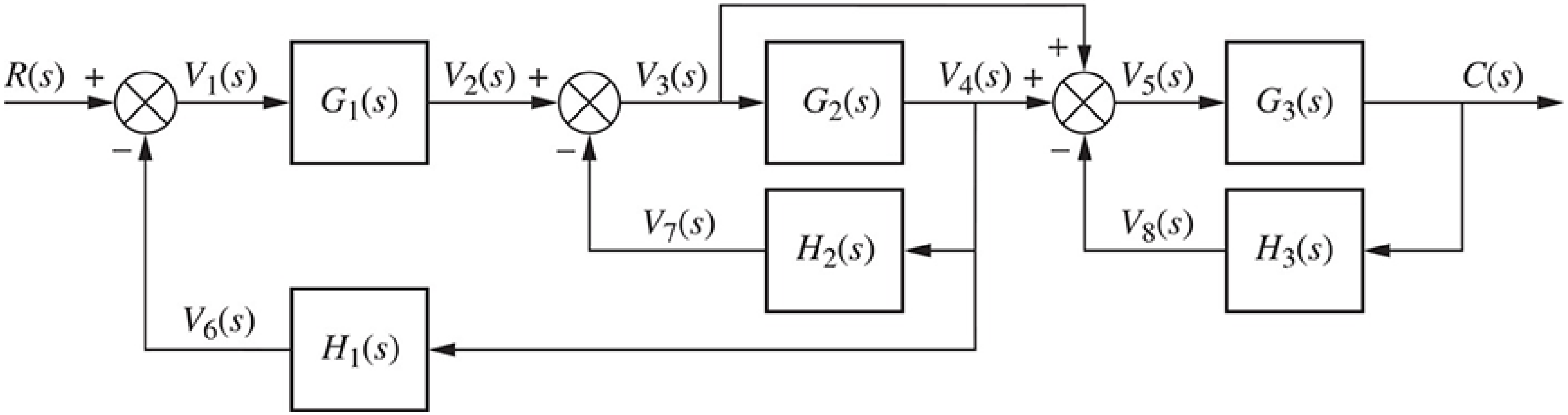
$$\Delta = 1 - \Sigma(\text{loop gains}) + 0$$

$$= 1 - (G_2(s)G_3(s)G_4(s)H_2(s) + G_4(s)H_1(s))$$

$\Delta_k = \Delta_1 =$ eliminating from $\Delta$ the loop gains touching the forward path $= 1$

$$\boxed{G(s) = \frac{G_1(s)G_2(s)G_3(s)G_4(s)(1)}{1 - (G_2(s)G_3(s)G_4(s)H_2(s) + G_4(s)H_1(s))}}$$

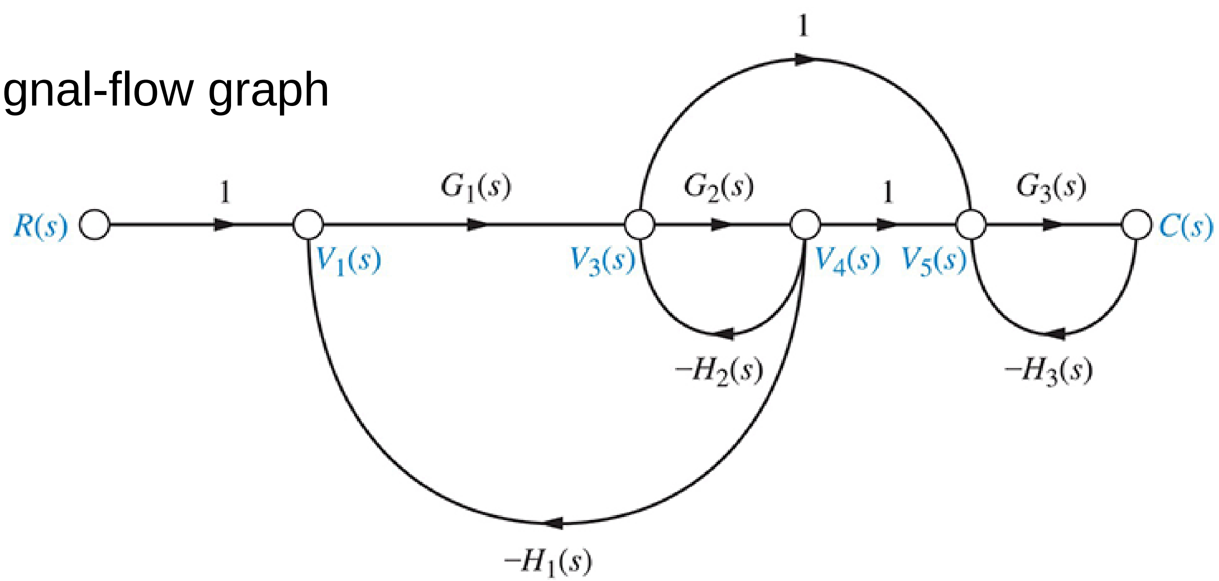
Example. Find the transfer function of the system using Mason's formula.

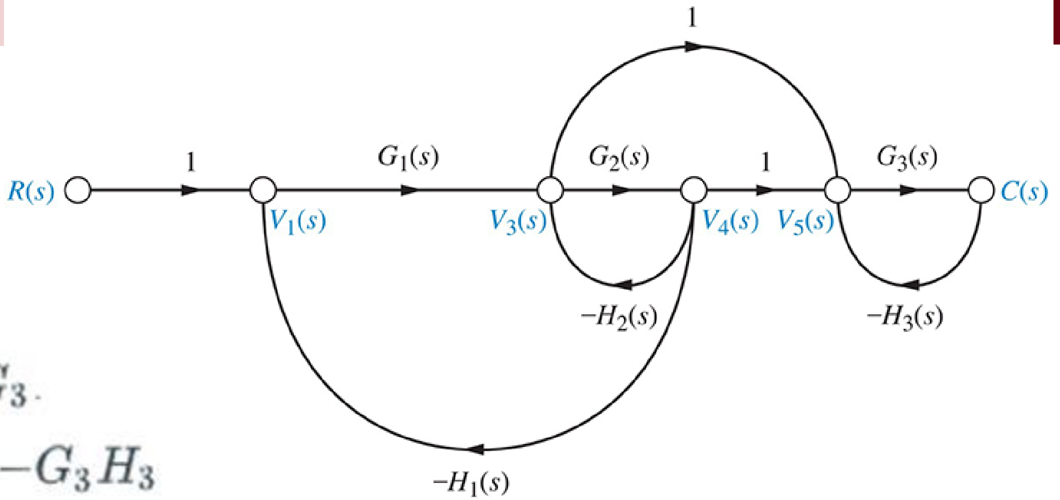



(Fig. 5.11 of [Nise, 2015])

Solution.

First convert it to signal-flow graph






Forward-path gains are $G_1G_2G_3$ and $G_1G_3$.

Loop gains are $-G_1G_2H_1$, $-G_2H_2$, and $-G_3H_3$

Nontouching loops are $[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$ and $[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3$.

$\Delta = 1 - \Sigma$ loop gains $+ \Sigma$ nontouching-loop gains taken two at a time $- \Sigma$ nontouching-loop gains taken three at a time $+ \Sigma$ nontouching-loop gains taken four at a time $- \ldots$

$$\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3$$

Finally, $\Delta_1 = 1$ and $\Delta_2 = 1$.

Substituting these values into

$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum T_k \Delta_k}{\Delta}$$

$$\boxed{T(s) = \frac{G_1(s)G_3(s)\left[1 + G_2(s)\right]}{\left[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)\right]\left[1 + G_3(s)H_3(s)\right]}}$$

## Exercise 2

Find the transfer function of the system in Fig. 6 using Mason's formula.

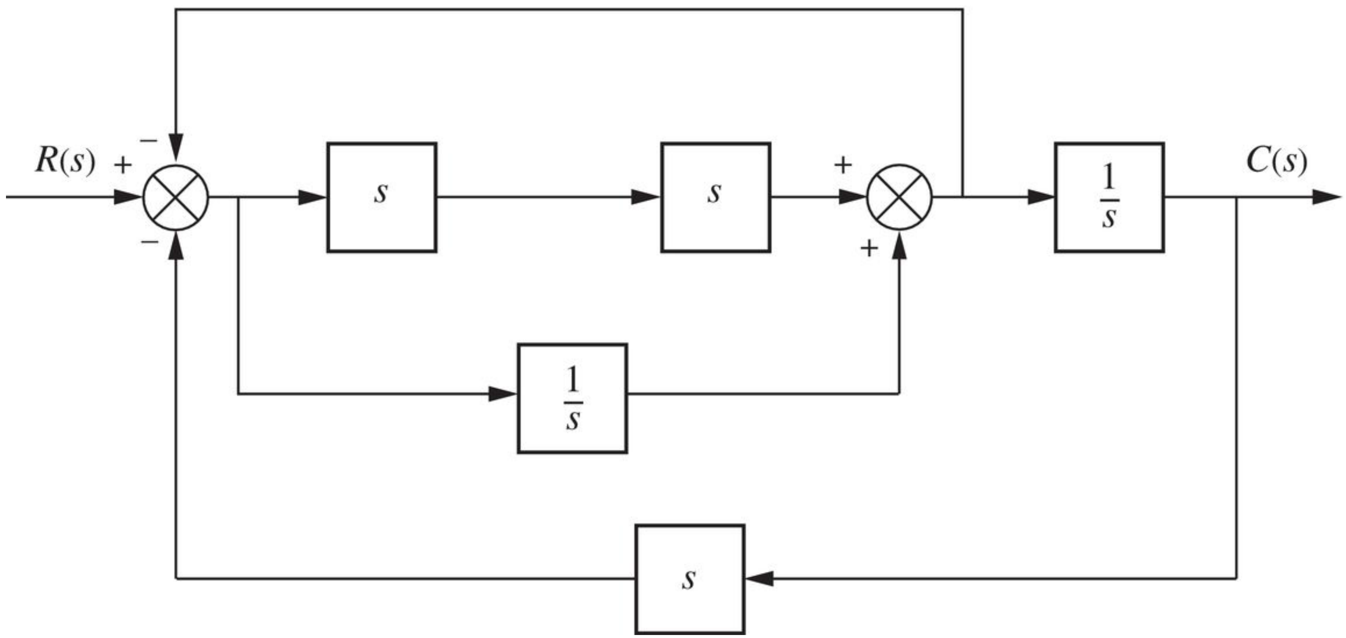



Figure 6: (Fig. 5.13 of [Nise, 2015])

## Answer

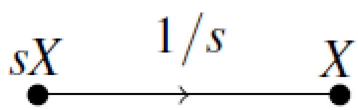
$$T(s) = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$$

# 6. Signal-Flow Graphs of State Equations

## Question

Knowing the state space representation of a system, how to draw its signal-flow graph ?

1. Create a node for each state variable.
2. To the left of each state variable node, create a node that corresponds to its time-derivative.
3. Connect each pair of these nodes with links of the following form :




4. Create nodes corresponding to the system's inputs and outputs.
5. Use the state and output equations to interconnect the nodes of the graph.

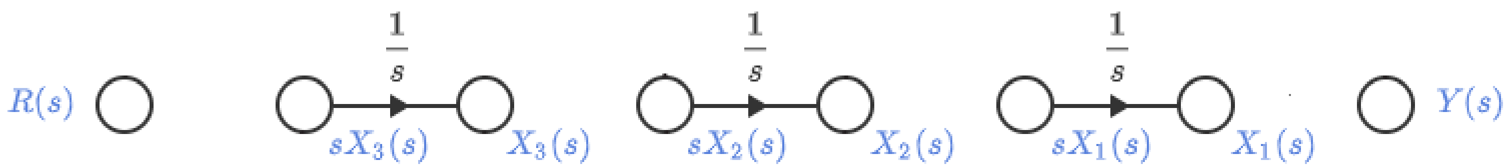
## Example 5

An LTI system is represented by the following state-space model :

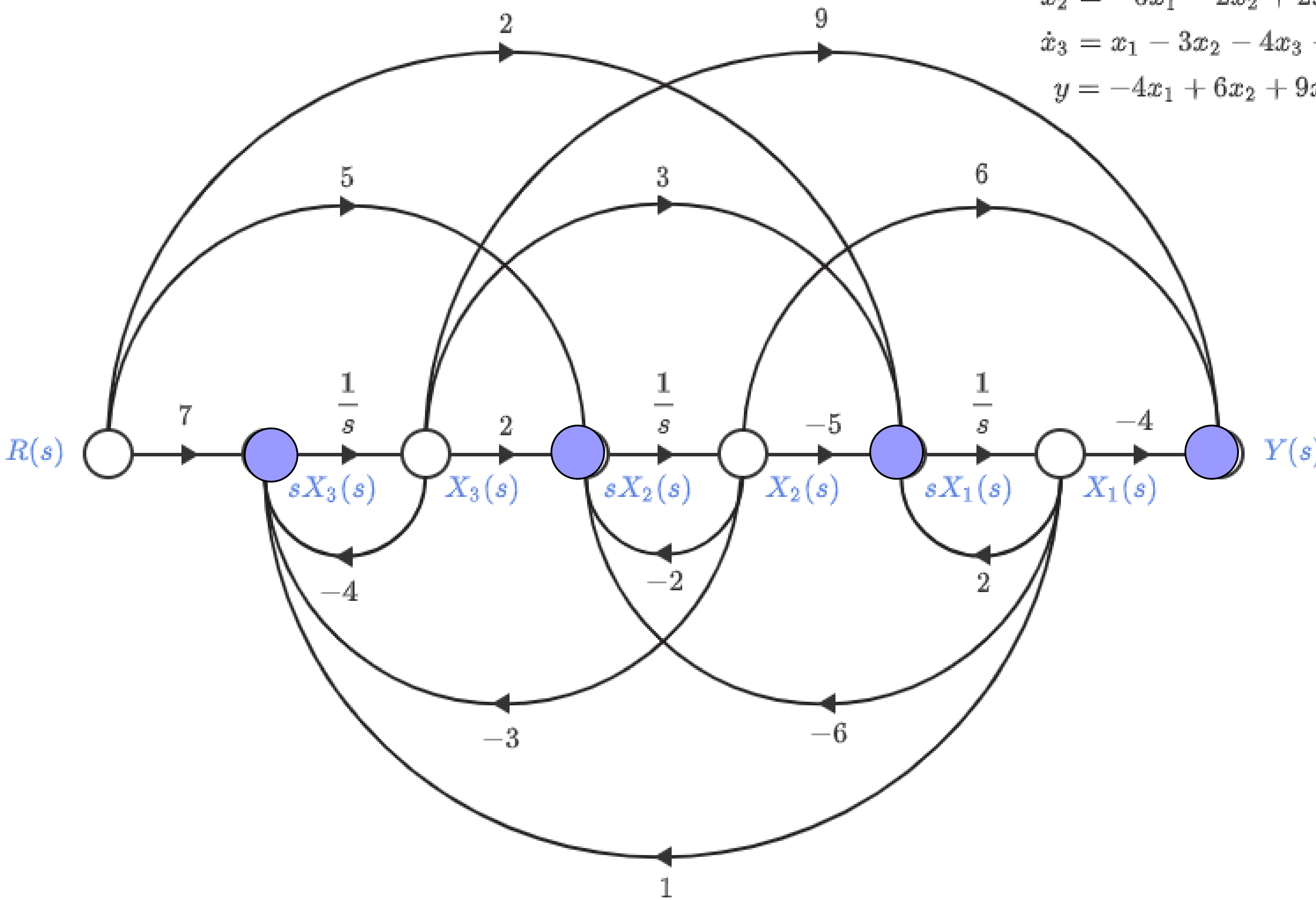
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

Find the system's transfer function using Mason's formula.

1.For the state equations above, nodes are drawn for state variables x1,x2 and x3 in the frequency domain. A node s$X_k(s)$ for each derivative $\dot{x}_k$ is inserted to the left of the state variable. Finally, Input R(s) and output Y(s) nodes are added.



$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$



## Exercise 3

Draw the signal-flow graph corresponding to the following state space model :

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

# 7. Alternative Representations in State Space

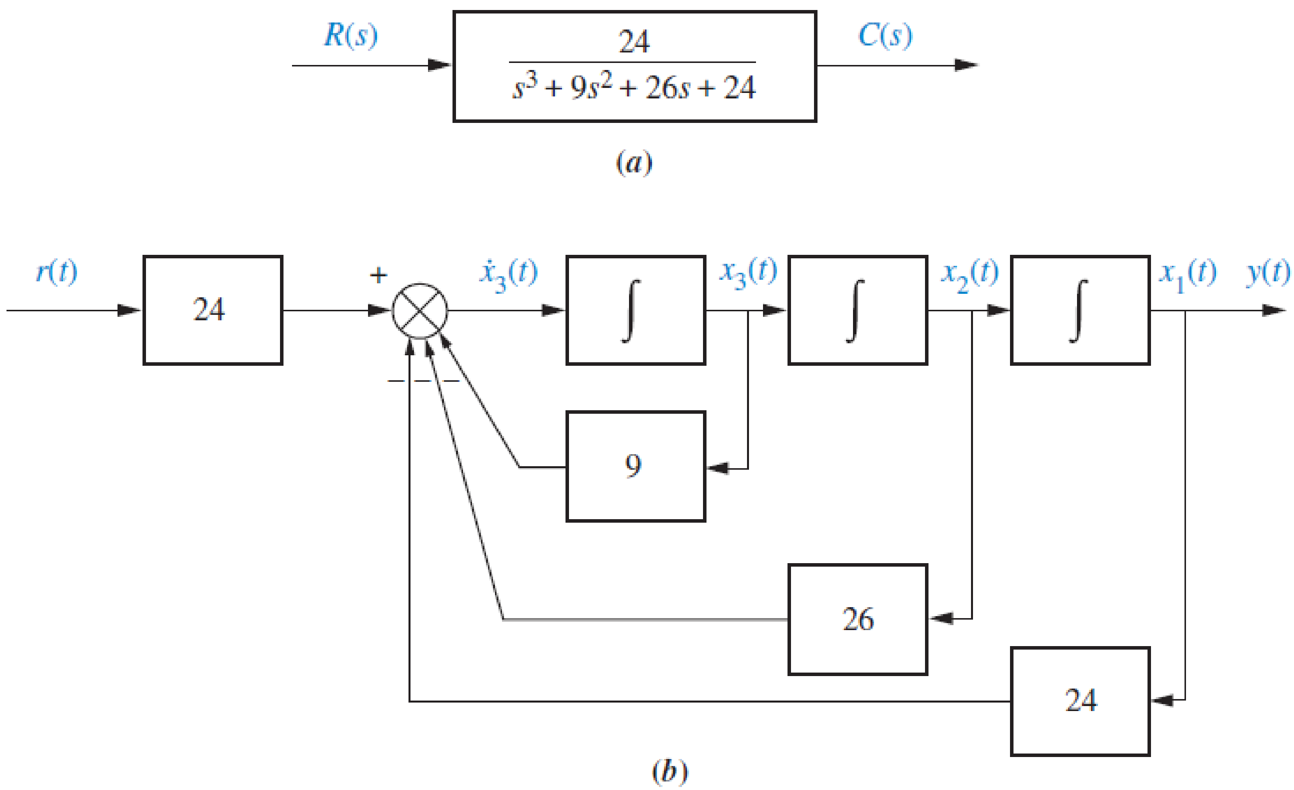



Figure 5: (Fig. 3.10 of [Nise, 2015])

Problem 5.25 (modified)

An LTI system has the following transfer function :

$$\frac{C(s)}{R(s)} = \frac{20}{s(s-2)(s+5)(s+8)}$$

Express the system in a state-space representation (phase-variable form) and draw its corresponding signal-flow graph.

## Exercise 4

- In Chapter 2, the transfer function $G(s) = VL(s)/V(s)$ of the circuit in Fig. was derived straight from the system's dynamic equations using (i) mesh analysis, and (ii) nodal analysis techniques. It was found to be :

$$G(s) = \frac{V_L(s)}{V(s)} = \frac{(Ls+R)^2}{L^2s^2+5RLs+2R^2} \qquad L = 1\,\mathrm{H}\,,\ R = 1\,\Omega$$

- Repeat the exercise by : deriving a block diagram for the system → Signal flow graph → Mason's formula.

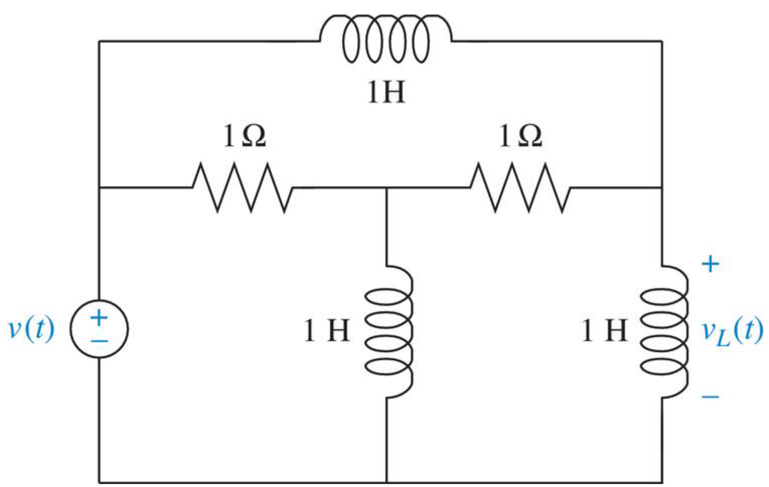



Figure: Circuit of Skill-Assessment Exercise 2.6 [Nise, 2015].

## Summary of the modeling techniques

Fig. shows a summary of the modeling techniques covered so far.

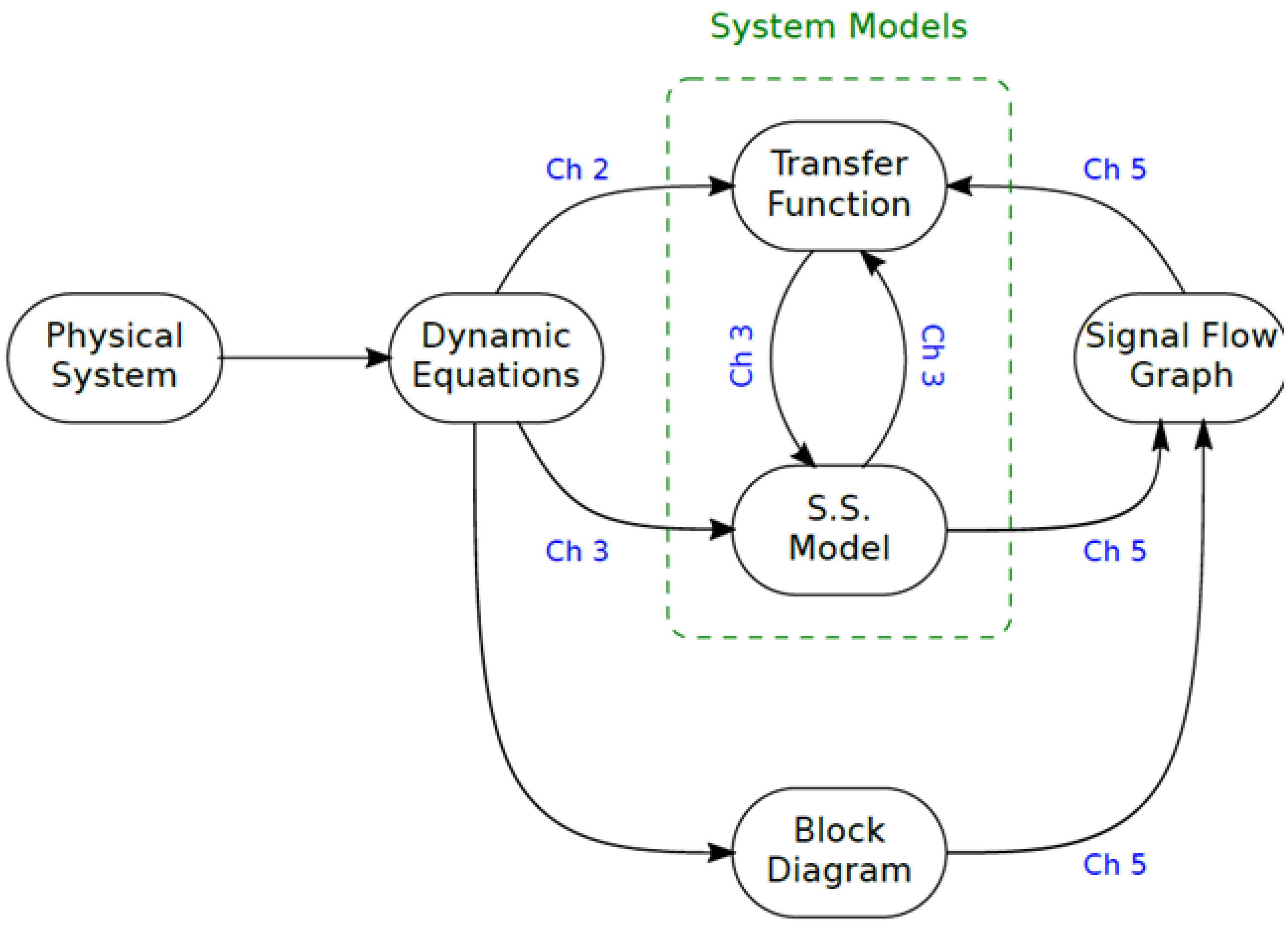



Figure : Summary of the modeling techniques

# References

Nise, N. S. (2015).
*Control Systems Engineering.*
*Wiley, 7 edition.*