Chapter2: Modeling in the Frequency Domain

ELG 3155 : Introduction to Control Systems

❑Objective:

- Review of the Laplace transform
- Mathematical modeling of LTI systems in the Laplace domain.

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Hitham Jleed

Outline

1. Introduction
2. Laplace Transform
   1. Inverse Laplace Transform
   2. Properties of LT
   3. Partial-Fraction Expansion
3. Transfer Function
4. Electrical Network Transfer Function
5. Translational Mechanical System Transfer Functions
6. Rotational Mechanical System Transfer Functions
7. Electromechanical System Transfer Functions
8. Linearization

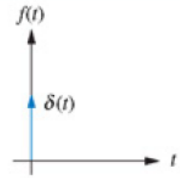
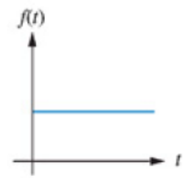
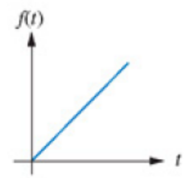
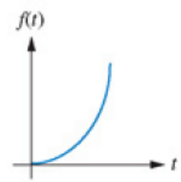
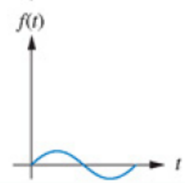
Outlets

After completing this lecture, the student will be able to:

- Find the Laplace transform of time functions and the inverse Laplace transform (Sections 2.1–2.2)
- Find the transfer function from a differential equation and solve the differential equation using the transfer function (Section 2.3).
- Find the transfer function for linear, time-invariant electrical networks (Section 2.4)
- Linearize a nonlinear

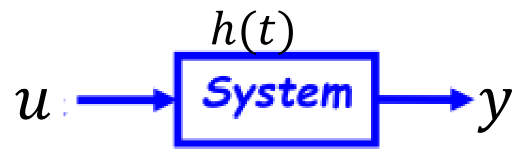
Recap

Control systems are often analyzed using well defined excitation signals called test waveforms.

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

* The table brought from Norman S. Nise. Control Systems Engineering. Wiley, 7 edition, 2015.

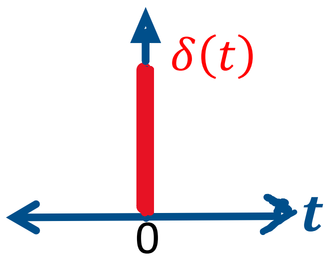
Example: Impulse response



Consider the system $\dot{y}(t) + ky(t) = u(t)$

✓ **Find the impulse response of this system?**

Solution:



We have:

$$\dot{y}(t) + ky(t) = u(t) \quad \textbf{(1)}$$

$$u(t) = \delta(t) \rightarrow y(t) = h(t) =? \quad \textbf{(2)}$$

$t=0 \Longrightarrow h(t)=1$

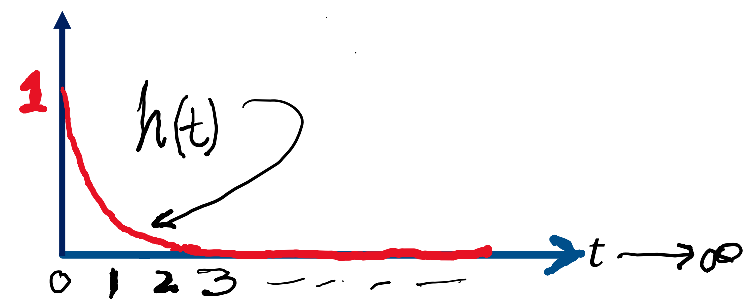
$t=1 \Longrightarrow h(t)=0.36$

$t=2 \Longrightarrow h(t)=0.13$

$t=3 \Longrightarrow h(t)=0.049$

$\vdots \qquad \vdots$

$t=\infty \Longrightarrow h(t)= 0$



$$h(t) = e^{-kt} \text{ for } t > 0$$

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- 2.10 Nonlinearities
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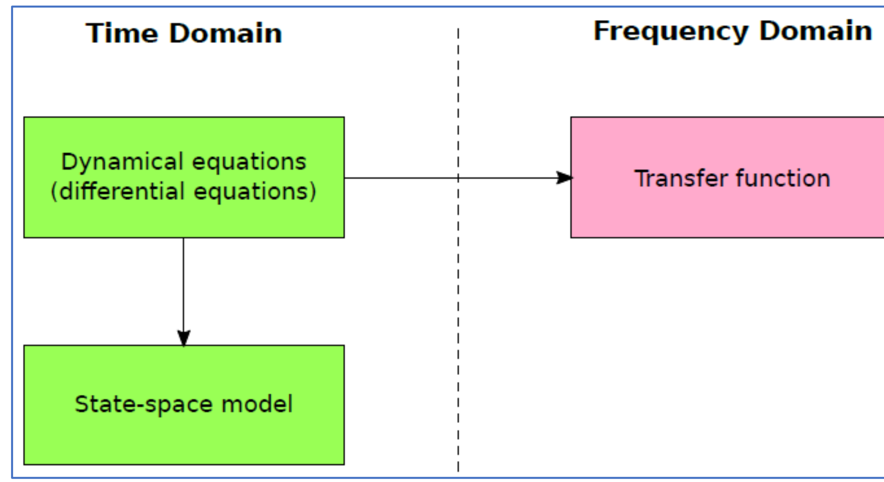

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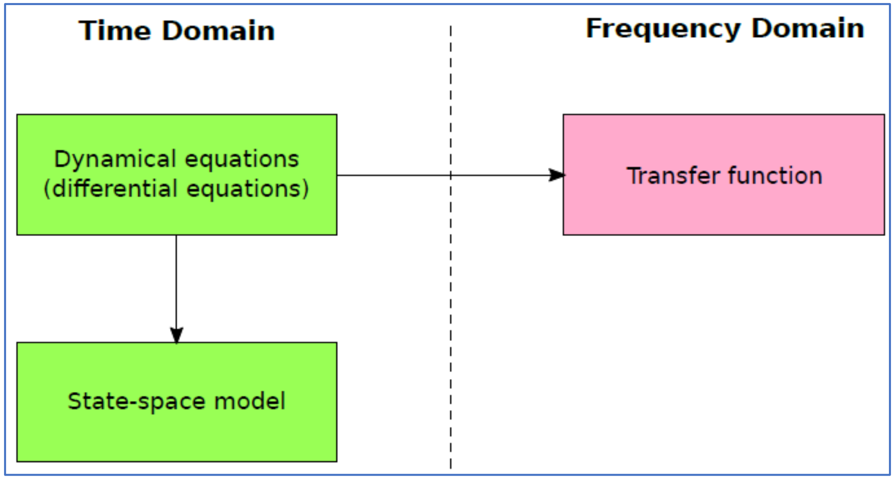
dynamical equations

System Models

- In general, a system can be described by a set of equations which govern the system's behavior. They are called dynamical equations.
- For instance, these can be circuit equations in the case of an electric system.
- The dynamical equations can be converted to a system model in the form of either a state-space model or a transfer function (Fig 1).
    - **State-space model (time-domain):** A state-space model is usually a convenient model to study the system in the time domain.
    - **Transfer function (frequency-domain):** A transfer function is a convenient model to study the system in the frequency domain.



Remarks

- Deriving the dynamical equations of a system, requires some expertise in the system type. For instance, deriving circuit equations requires expertise in circuit theory.

- However, converting the dynamical equations to a state-space model or to a transfer function, does not require such expertise. It is an algebraic process.

Differential equation

- The dynamical equations of a system can be converted to a differential equation relating the system's outputs to its inputs, or vice versa.
- This is often referred to as the external description of the system.

$$r(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \cdots + b_1 \frac{dr}{dt} + b_0 r(t)$$

as a convolution integral as:

$$\boxed{y(t) = \int_{-\infty}^{t} r(\tau) h(t-\tau) d\tau} \qquad (1)$$

where h is the system's impulse response; i.e, $h(t) = y(t)$ when the system is excited with a unit impulse input.

Disadvantage:

- Usually, it is difficult to study a system that is modeled by a differential equation.
- It is often more convenient to model the system using either a transfer function (frequency domain) or a state-space model (time domain).

Laplace Transform

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TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \tag{2.1}$$

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t) \tag{2.2}$$

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Laplace Transform

- The Laplace transform (LT) allows to represent the input, output, and the system itself, as separate entities.

Definition

The LT of a $f(t)$, for $t > 0$, where $f(t) = 0 \ \forall t < 0$, is given by

$$\boxed{\mathscr{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt} \quad (2)$$

where $\mathscr{L}$ is the operator of the LT.

The variable $s = \sigma + j\omega$ is a complex variable referred to as the Laplace variable

Inverse Laplace Transform

The inverse LT of a complex function $F(s)$ is defined by:

$$\boxed{\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}ds = f(t)u(t) \equiv \begin{cases} f(t) & \text{, if } t \geq 0 \\ 0 & \text{, if } t < 0 \end{cases}} \quad (3)$$

The inverse LT is useful to find $f(t)$ given $F(s)$.

Properties of LT

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

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Theorem (Final-value theorem)

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathscr{L}(f(t)) = \lim_{s\to 0} sF(s)}$$

The final-value theorem is applicable ONLY if both of the following conditions are satisfied:

- *No more than one root of the denominator of $F(s)$ is at $s = 0$; and*
- *All the other roots (if any) must have negative real parts (< 0).*

Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$
$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \qquad (2.3)$$

Example 2.2

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

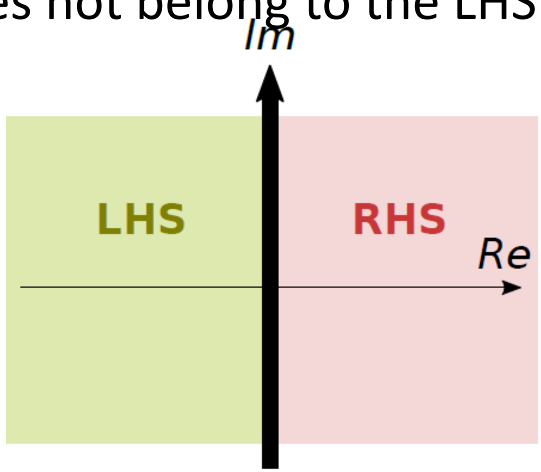
Recale: s-Plane partition

Throughout this course, we will assume the following partition of the s-plane (Fig. 2) :

$$\text{Imaginary axis} : \{j\omega \mid \forall \omega \in \mathbb{R}\}$$
$$\text{Left-hand side (LHS)} : \{\sigma + j\omega \mid \forall \sigma < 0 \text{ and } \omega \in \mathbb{R}\}$$
$$\text{Right-hand side (RHS)} : \{\sigma + j\omega \mid \forall \sigma > 0 \text{ and } \omega \in \mathbb{R}\}$$

Remark

Note that the imaginary axis does not belong to the LHS nor to the RHS of the s-plane.



Partial-Fraction Expansion

To compute the LT of a complicated function, it is better converted to a sum of simpler terms whose respective LTs are known (or easier to compute). The result is called a partial-fraction expansion.

Let

$$F_1(s) = \frac{N(s)}{D(s)}.$$

- If the order of $N(s)$ < order of $D(s)$ $\Rightarrow$ partial-fraction expansion is possible.
- Otherwise, $F1(s)$ must be first converted to the form of {polynomial(s) + $N2(s)$ / $D2(s)$}, where the order of $N2(s)$ < order of $D2(s)$, then apply the partial-fraction expansion to $N2(s)/D2(s)$.

Example (p37)

Partial-fraction expansion of

Solution

$$F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Taking the inverse Laplace transform

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

Let students finish it

Case 1: Roots of denominator are real and distinct

Example (p38 in 7th)

Partial-fraction expansion of $F(s) = \frac{2}{(s+1)(s+2)}$

Solution

We can write the partial-fraction expansion as a sum of terms,

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

$$K_1 = 2, \quad K_2 = -2$$

Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Example 2.3 (textbook)

Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

Solution

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \frac{32}{(s+4)(s+8)}\bigg|_{s\to 0} = 1$$

$$K_2 = \frac{32}{s(s+8)}\bigg|_{s\to -4} = -2$$

$$K_3 = \frac{32}{s(s+4)}\bigg|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

Matlab

Run and examine the Matlab codes ch2p1 to ch2p8 (posted on Brightspace).

Case 2: Roots of denominator are real and repeated

Example (p40)

Partial-fraction expansion of $F(s) = \frac{2}{(s+1)(s+2)^2}$

Solution

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K}{(s+1)} + \frac{K_1}{(s+2)^2} + \frac{K_2}{(s+2)}$$

The general expression for K_1 through K_r for the multiple roots is

$$K_i = \frac{1}{(i-1)!} \frac{d^{\,i-1} F_1(s)}{ds^{\,i-1}} \bigg|_{s \to -p_1} \qquad i = 1, 2, \ldots, r; \quad 0! = 1$$

$$K_1 = 2, \quad K_2 = -2, \quad K_3 = -2$$

Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2te^{-2t} - 2e^{-2t})u(t)$$

Case 3: Roots of denominator are complex

Example 7

Partial-fraction expansion of $F(s) = \frac{3}{s(s^2+2s+5)}$

Solution

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + j2)(s + 1 - j2)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 1 + j2} + \frac{K_3}{s + 1 - j2}$$

$$K_2 = \frac{3}{s(s + 1 - j2)}\bigg|_{s \to -1-j2} = -\frac{3}{20}(2 + j1)$$

Similarly, K_3 is found to be the complex conjugate of K_2, and K_1 is found as previously described. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2 + j1}{s + 1 + j2} + \frac{2 - j1}{s + 1 - j2}\right)$$

$$f(t) = \frac{3}{5} - \frac{3}{20}\left[(2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t}\right]$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t}+e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t}+e^{-j2t}}{2j}\right)\right]$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right)u(t)$$

Skill-Assessment Exercise 2.1

PROBLEM: Find the Laplace transform of $f(t) = te^{-5t}$.

ANSWER: $F(s) = 1/(s+5)^2$

Skill-Assessment Exercise 2.2

PROBLEM: Find the inverse Laplace transform of $F(s) = 10/[s(s+2)(s+3)^2]$.

ANSWER: $f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$

Transfer Function

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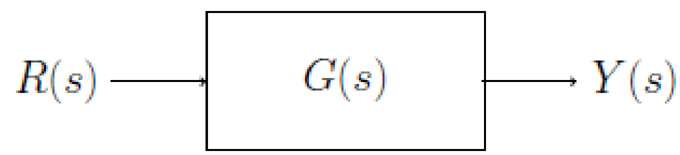
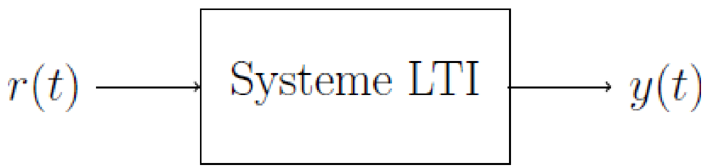


Transfer Function

- The concept of transfer function of a system provides a convenient framework to study the system in the frequency (Laplace) domain.
- An LTI system is characterized by a differential equation that relates its output to the input.



$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \cdots + b_1 \frac{dr}{dt} + b_0 r(t)$$

Definition (System order)

The order of the system is $\max(m, n)$.

Taking the LT of this equation,

$$y(t) = \int_{-\infty}^{t} r(\tau)h(t-\tau)d\tau$$

$$[a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0]Y(s) + \text{I.C. (i.e., initial condition)}$$
$$= [b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s^1 + b_0]R(s) + \text{I.C. (i.e., initial condition)}$$

$$\Rightarrow G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s^1 + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0} \equiv \frac{N(s)}{D(s)}$$

where the initial conditions are considered as zero.

Definition (Transfer function)

The function $G(s)$ is called the system's transfer function. It is the Laplace transformation of the system's impulse response $h(t)$ in (1).

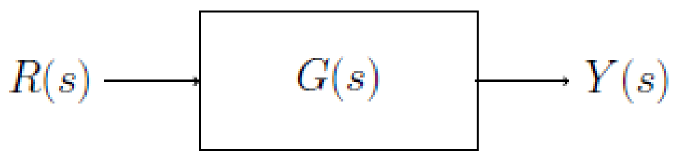
Definition (Characteristic polynomial/equation)

- The polynomial $D(s)$ is called the system's Characteristic polynomial.
- The equation $D(s) = 0$ is called the system's Characteristic equation.

Definition (Zeros and poles)

- The roots of $N(s)$ are called the system's zeros.
- The roots of $D(s)$ are called the system's poles.

- The transfer function $G(s)$ relates the system's output $Y(s)$ to its input $R(s)$ by $Y(s) = G(s)R(s)$ (assuming zero initial conditions).
- It assumes zero initial conditions.
- A system can be represented by a block diagram through its transfer function.



- The transfer functio ... n's input and output.

Example 8

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

where r and c are the system's input and output, respectively.

Then, solve for $c(t)$ when the system is excited with a unit step input, assuming zero initial conditions.

Solution

Taking the LT $$sC(s) + 2C(s) = R(s)$$

The TF: $$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

To solve for $c(t)$, we have $r(t) = u(t)$. $R(s) = 1/s$ and assuming zero ICs

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

PFE ⇒ $$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Inverse LT ⇒ $$c(t) = \left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)u(t)$$

Exercise 1

Find the unit ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

Answer:

$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

Matlab

Run and examine the Matlab codes ch2p9 to ch2p12 (posted on Brightspace).

Definition (DC gain)

The steady-state value of a unit-step response (with zero initial conditions) is called the system's DC gain.

$$\text{DC gain} = \lim_{t \to \infty} y(t) \quad , \quad \text{for } r(t) = u(t)$$

- If the conditions of the final-values theorem are satisfied (which is equivalent to the condition of having all the system poles on the left-hand side of the s-plane), then the DC gain can also be determined by applying the final-value theorem.

$$\text{DC gain} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sR(s)G(s) = \lim_{s \to 0} s(1/s)G(s) = \lim_{s \to 0} G(s),$$

where $G(s)$ is the system's TF.

Example 9

Compute the DC gain of the system in the Ex. 8 and Exercise 1.

Exercises

SOLVING DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Solve the following differential equation using Laplace transforms. Assume all forcing functions are zero prior to $t = 0^-$. (*Hint*: you will need to use partial fraction decomposition)

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 5\sin 3t$$

$$x(0) = 4, x'(0) = 1$$

TRANSFER FUNCTION REVIEW

Write the corresponding differential equation for the following transfer function:

$$\frac{X(s)}{F(s)} = \frac{s+3}{s^3 + 11s^2 + 12s + 18}$$

2.4 Electrical Network Transfer Function

- 2. Modeling in the Frequency Domain
  - 2.1 Introduction
  - 2.2 Laplace Transform Review
  - 2.3 The Transfer Function
  - 2.4 Electrical Network Transfer Functions
  - 2.5 Translational Mechanical System Transfer Functions
  - 2.6 Rotational Mechanical System Transfer Functions
  - 2.7 Transfer Functions for Systems with Gears
  - 2.8 Electromechanical System Transfer Functions
  - 2.9 Electric Circuit Analogs
  - 2.10 Nonlinearities
  - 2.11 Linearization
  - Case Studies
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Electrical Network Transfer Function

We will apply the concept of transfer function to model electrical circuits:

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Attention

- Capacitor: $v(t) = \frac{1}{c}\int_0^t i(\tau)d\tau.$
- Inductor $i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau.$

Simple Circuits via Mesh Analysis

Kirchhoff's voltage law

The algebraic sum of voltages in each electric loop is zero.

Kirchhoff's current law

The algebraic sum of currents entering a node is equal to the sum of currents coming out of it.

- Modeling electrical circuits can be realized by these laws.

Example 2.6

Transfer Function—Single Loop via the Differential Equation

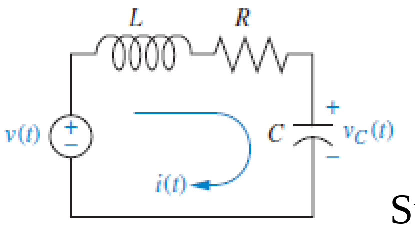

FIGURE 2.3 *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION:

Summing the voltages around the loop, assuming zero initial conditions, yields,

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \quad (2.61)$$

Changing variables from current to charge using ***i(t)= dq(t)/dt*** yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \quad (2.62)$$

From the voltage-charge relationship for a capacitor in Table 2.3, $q(t) = Cv_C(t) \quad (2.63)$

Substituting Eq. (2.63) into Eq. (2.62) yields

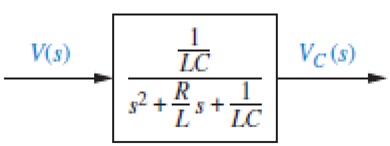

FIGURE 2.4 Block diagram of series *RLC* electrical network

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \quad (2.64)$$

Taking the Laplace transform (IC=0) $(LCs^2 + RCs + 1)V_C(s) = V(s) \quad (2.65)$

Solving TF, *vc(s)/v(s)*, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Other solutions

- Transfer Function—**Single Loop via Transform Methods.** (use transform methods to bypass writing the differential equation).

capacitor	resistor	inductor
$V(s) = \frac{1}{Cs} I(s)$	$V(s) = RI(s)$	$V(s) = LsI(s)$

- Transfer Function—**Single Node via Transform Methods** (using Kirchhoff's current law and summing currents flowing from nodes)

- Transfer Function—**Single Loop via Voltage Division**

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Repeat the example: Transfer Function—**Single Loop via Transform Methods** using mesh analysis and transform methods without writing a differential equation.

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

Solving for I(s)/V(s),

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

But

$$V_C(s) = I(s)\frac{1}{Cs}$$

The TF

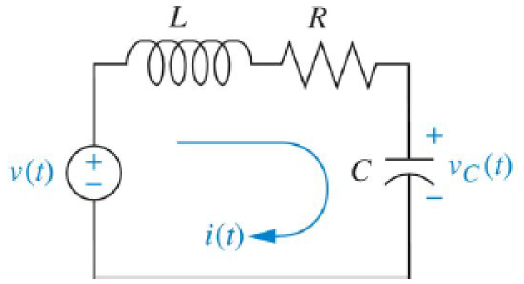
$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Repeat: Transfer Function—**Single Node via Transform Methods** using nodal analysis and without writing a differential equation.

The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Fig 8. We assume that currents leaving the node are positive and currents entering the node are negative.

The TF

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0$$

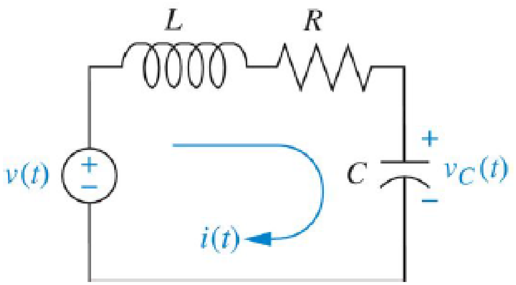


$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

repeat: Transfer Function—**Single Loop via Voltage Division**

using voltage division and the transformed circuit.

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

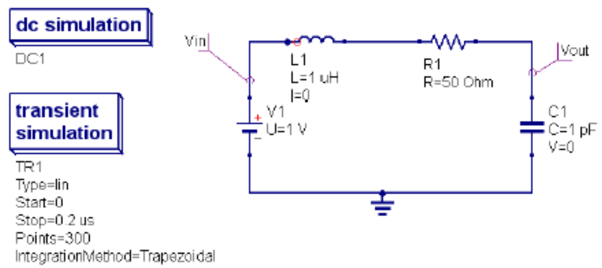


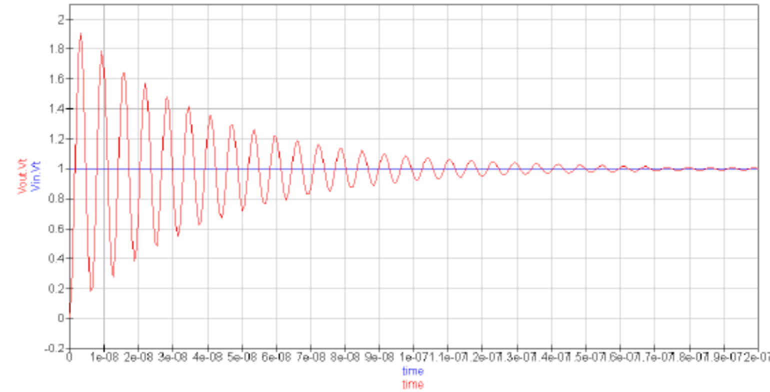
The TF

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Which method is the easiest?

Verification



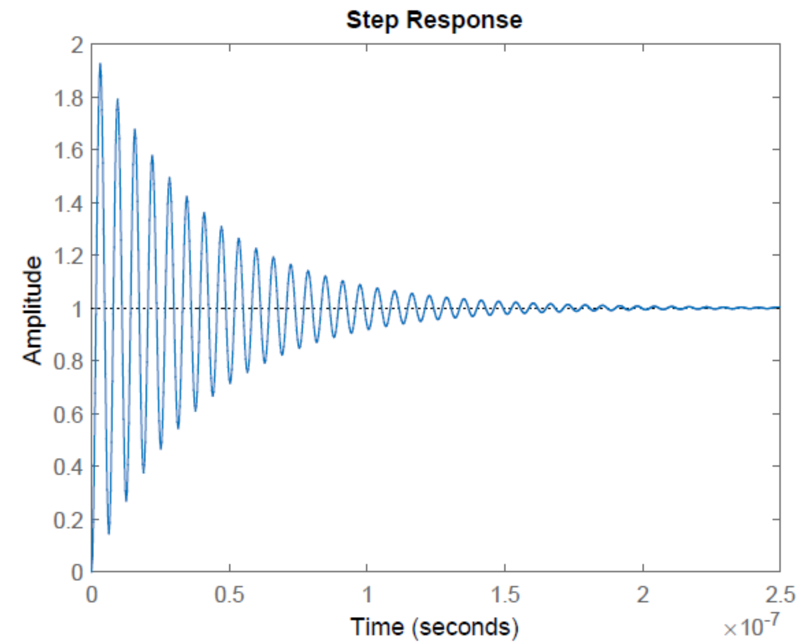


$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Listing 2: Matlab Code

```
L=1e-6;
R=50;
C=1e-12;

sys=tf([1] , [L*C  R*C  1]);
step(sys);
```



Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.

2. Replace all sources and time variables with their Laplace transform.

3. Assume a transform current and a current direction in each mesh.

4. Write Kirchhoff's voltage law around each mesh.

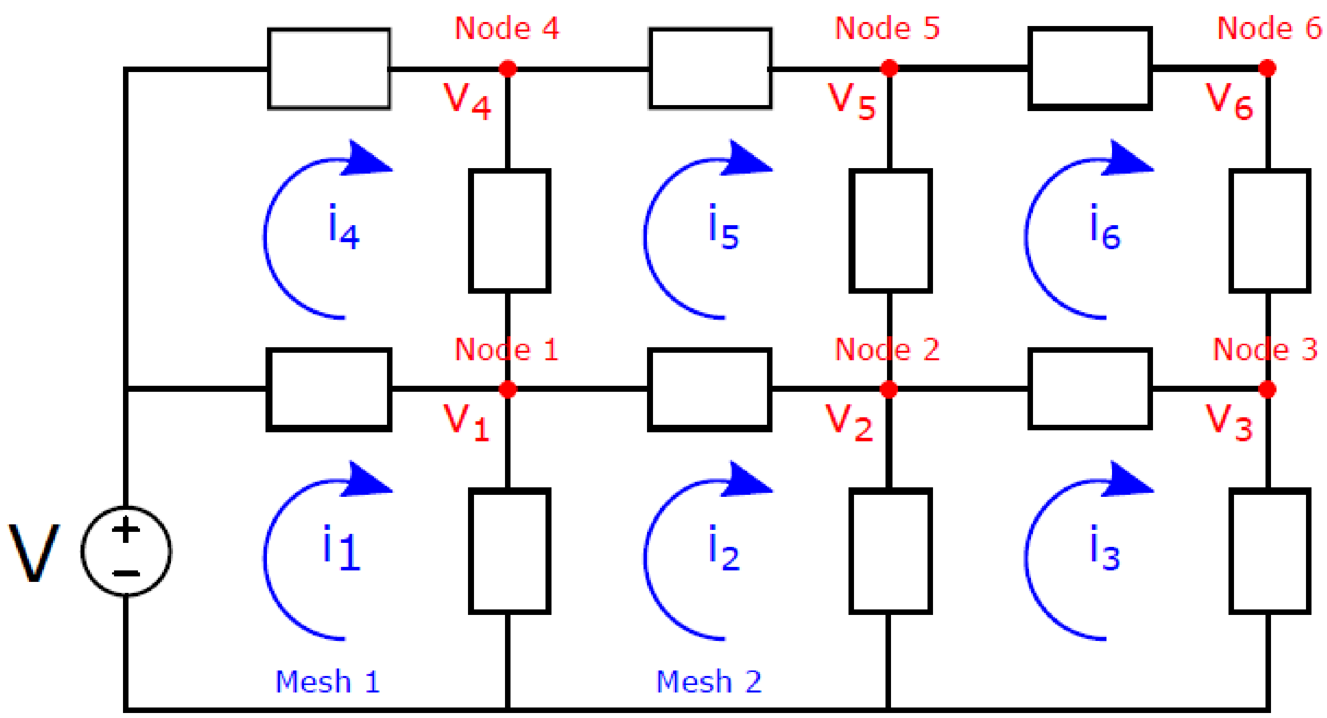
5. Solve the simultaneous equations for the output.

6. Form the transfer function.

Complex Circuits

Drawback of Kirchoff's laws

Kirchoff's laws may be convenient to model simple circuits (with 1 or 2 loops), but they may not be so for complex circuits (with multiple loops).



Example Find the transfer function $I_2(s)/V(s)$ of the circuit in Fig. 10(a) using mesh analysis method.

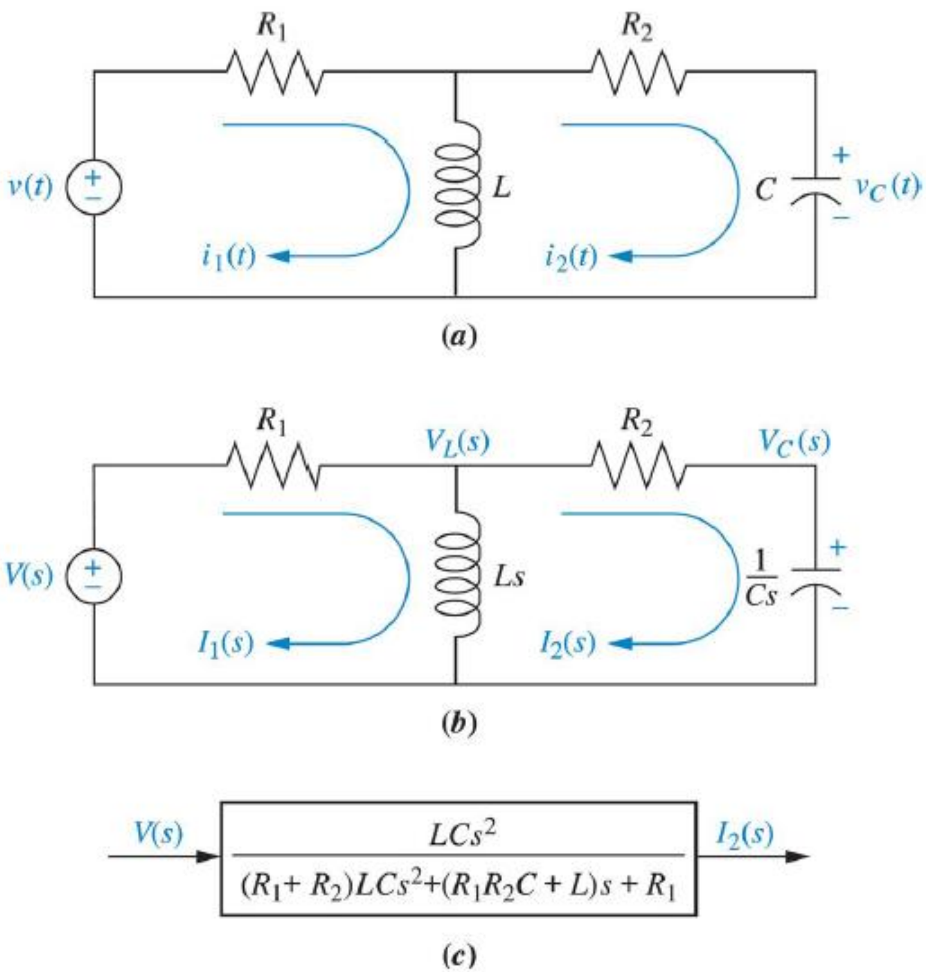


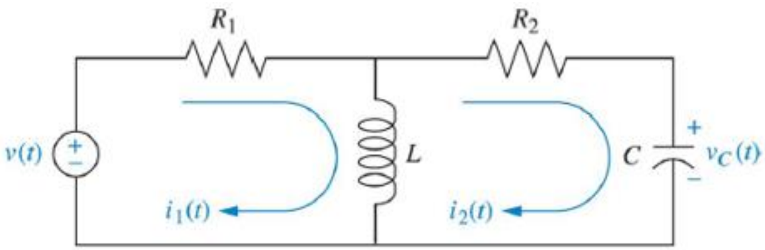
Figure : Circuit of Example 2.10 [Nise, 2015].

Solution

The circuit requires two simultaneous equations to solve for the transfer function.

$$(R_1 + Ls)I_1(s) \; - \; LsI_2(s) = V(s) \quad \Rightarrow \quad I_1(s) = \frac{V(s)+Ls\,I_2(s)}{(R_1+Ls)} \quad (1)$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \quad (2)$$



Substitute (1) in (2)

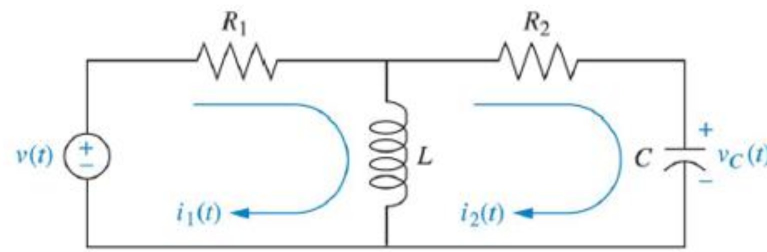
Forming the transfer function, *G*(*s*), yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

OR, We can use Cramer's rule

$$(R_1 + Ls)I_1(s) \quad - \quad LsI_2(s) = V(s)$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0$$



Where

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \quad \Rightarrow \quad \Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \dfrac{1}{Cs}\right) \end{vmatrix}$$

Forming the transfer function, *G*(*s*), yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Remark

$I_j(s)$ is the current unique to loop j (not in a branch shared with another loop).

Cramer's rule

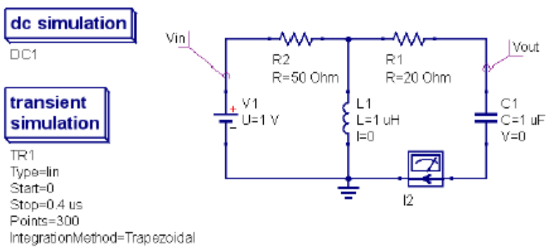
Let $\mathbf{Ax} = \mathbf{b}$, where

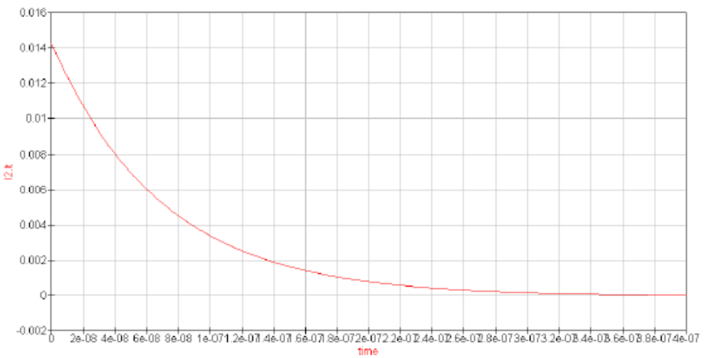
$$\mathbf{A} \in \mathbb{R}^{n \times n} \equiv \text{matrix}, \qquad \mathbf{x} \in \mathbb{R}^n \equiv [x_1 \ \cdots \ x_n]^T, \qquad \mathbf{b} \in \mathbb{R}^n \equiv [b_1 \ \cdots \ b_n]^T$$

Then, $$x_j = \frac{\det(\mathbf{A_j})}{\det(\mathbf{A})}, \quad j = 1, \ldots, n$$

where $\mathbf{A_j}$ is the matrix formed by replacing the *j-th* column of $\mathbf{A}$ by the column vector $\mathbf{b}$.

Verification



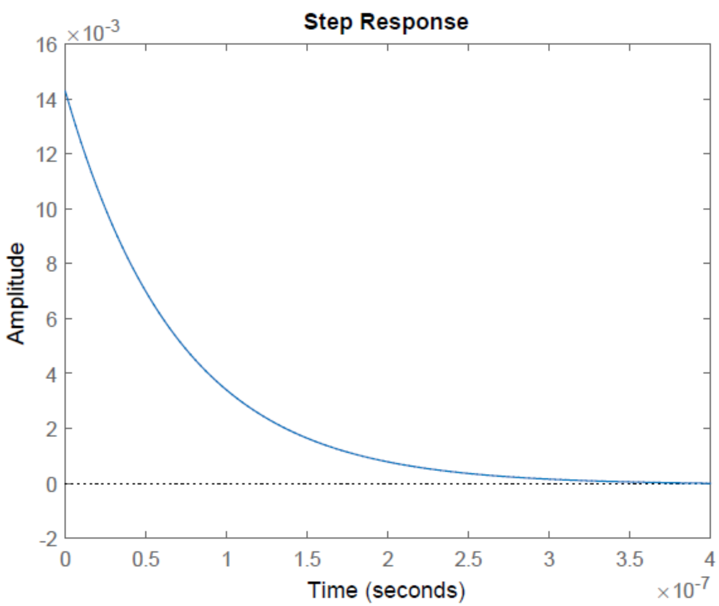


$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2 + (R_1R_2C+L)s + R_1}$$

Listing 4: Matlab Code

```
L=1e-6;
R1=50;
R2=20;
C=1e-6;

sys=tf([L*C 0 0] , [(R1+R2)*L*C  R1*R2*C+L  R1]);
step(sys);
```



Operational Amplifiers (op-amp)

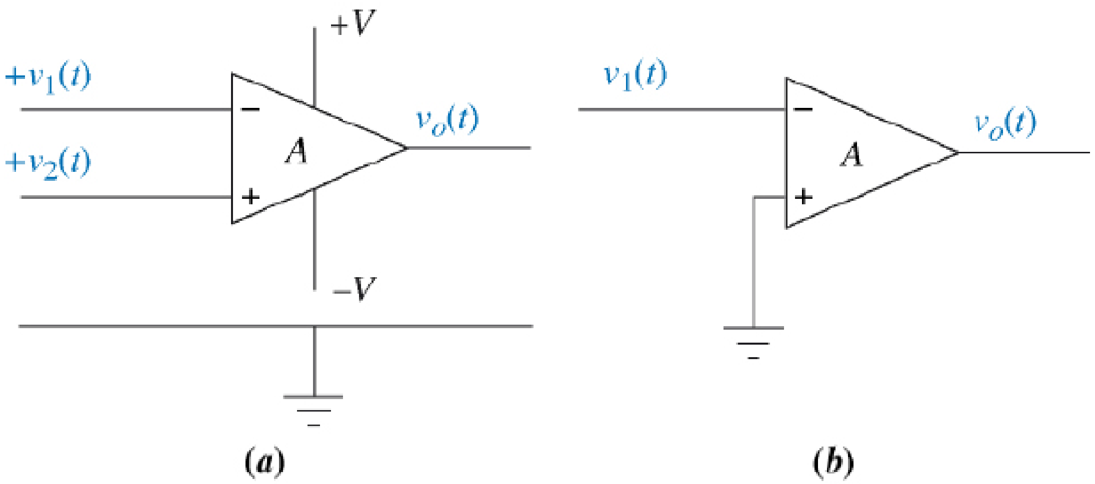
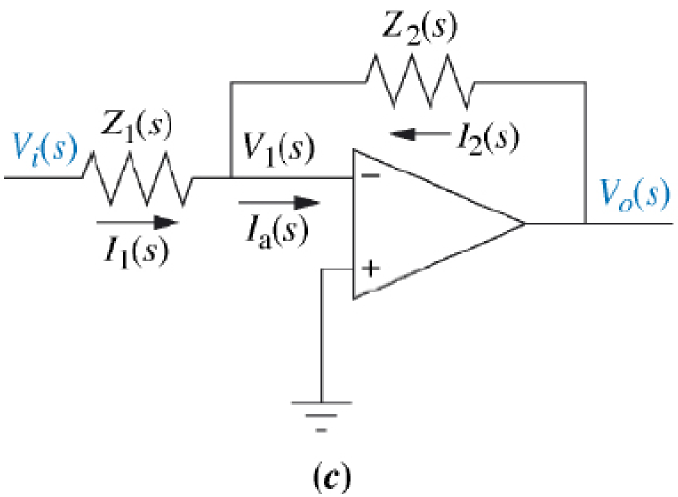


Figure 4: (a): An op-amp;
(b) and (c): inverting op-amp [Nise, 2015].



Characteristics of an ideal op-amp

1. Differential input: $V_2(t)$ - $V_1(t)$
2. High input impedance: $Z_i = \infty$
3. Low output impedance: $Z_o = 0$
4. High constant amplification gain: $A = \infty$

Inverting Op-amp

The transfer function of an inverting op-amp (Fig. 4(c)) is:

$$\boxed{G(s) = \frac{V_o(s)}{V_i(s)} = \frac{-Z_2(s)}{Z_1(s)}}$$

Noninverting Op-amp

The transfer function of a noninverting op-amp (Fig. 5) is:

$$\boxed{G(s) = \frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}}$$

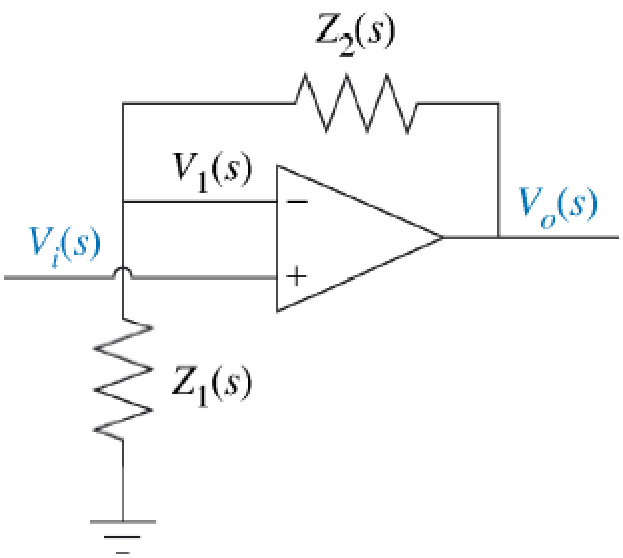


Figure 5: Noninverting op-amp circuit [Nise, 2015].

Remarks

- The equivalent impedance, Z_e, of two impedances, Z_1 and Z_2, placed in series, is $Z_e = Z_1 + Z_2$.
- The equivalent impedance, Z_e, of two impedances, Z_1 and Z_2, placed in parallel, is defined by

$$\frac{1}{Z_e} = \frac{1}{Z_1} + \frac{1}{Z_2}.$$

Example

Find the transfer function $V_o(s)/\, Vi(s)$ for the system in Fig.

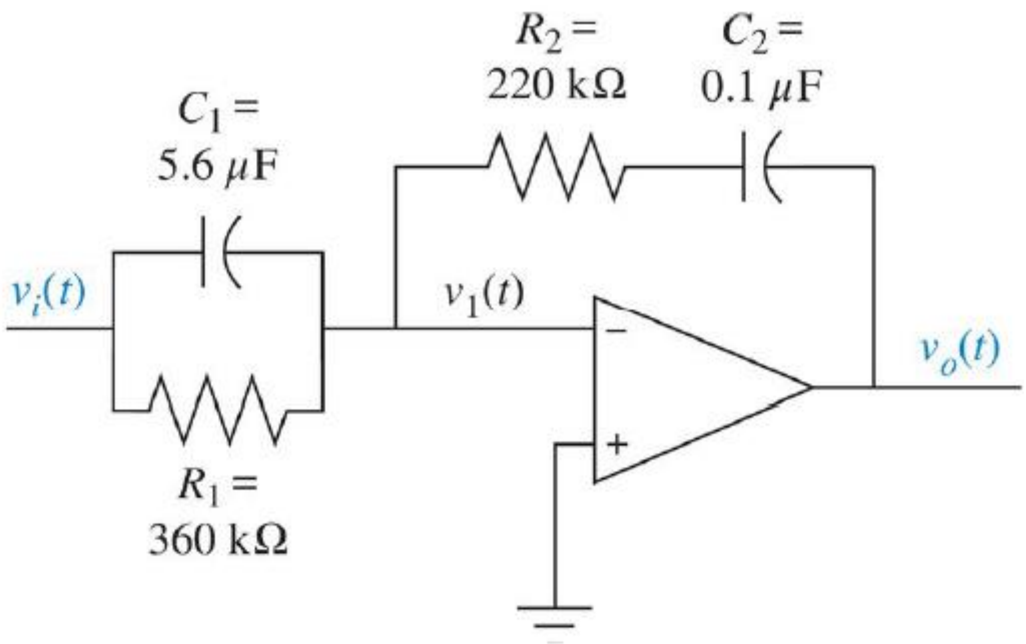


Figure : Op-amp circuit for Example 2.14 [Nise, 2015].

SOLUTION: The transfer function of the operational amplifier circuit is given by Eq. (2.97). Since the admittances of parallel components add, $Z_1(s)$ is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 + 10^{-6}s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1}$$

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

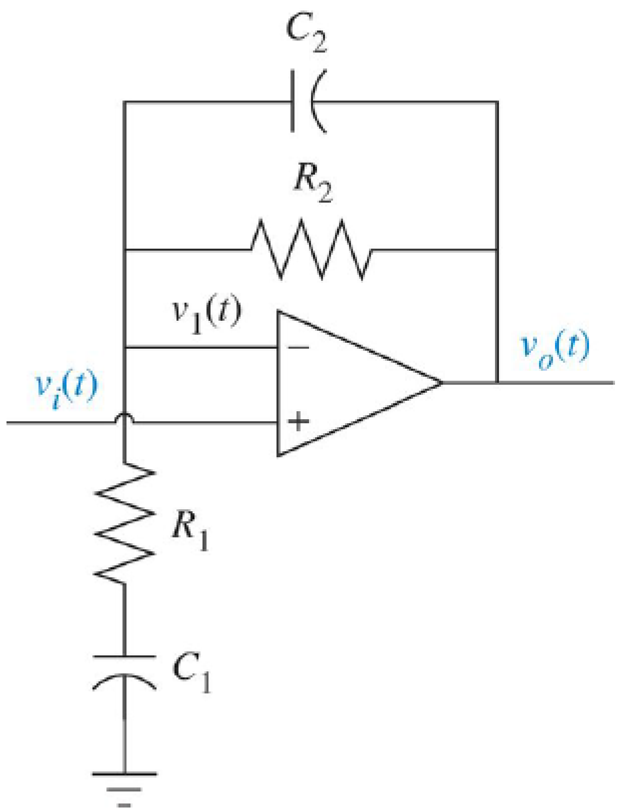
$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s}$$

Try to solve Z1(s) by computing impedance (R//C) and compare the result.

The resulting circuit is called a PID controller and can be used to improve the performance of a control system.

Example 12

Find the transfer function $V_o(s)/\ Vi(s)$ for the system in Fig.



Op-amp circuit for Example 2.15 [Nise, 2015] p59.

Solution

$$V_o(s) = A(V_i(s) - V_1(s))$$

By voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$

$$\boxed{Z_1(s) = R_1 + \frac{1}{C_1 s}}$$

$$\boxed{Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)}}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1}}$$

2.5 Translational mechanical system TFs, [1, p. 61]

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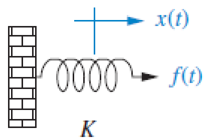
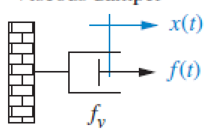
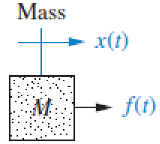
TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

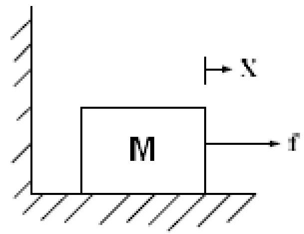
Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

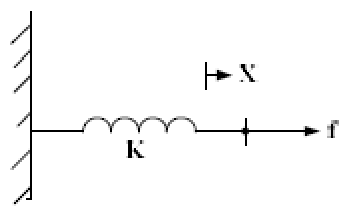
Translational mechanical system

There are three basic elements in a translational mechanical system, i.e. (a) mass, (b) spring and (c) damper.

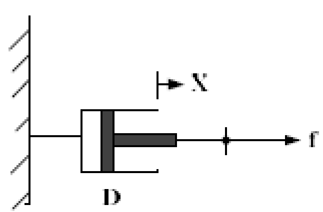
(a) **Mass:** A mass is denoted by M. If a force f is applied on it and it displays distance x, then $f = M\frac{d^2x}{dt^2}$ as shown in Fig.4.1.



(b) **Spring:** A spring is denoted by K. If a force f is applied on it and it displays distance x, then $f = Kx$ as shown in Fig.4.3.



(c) **Damper:** A damper is denoted by D. If a force f is applied on it and it displays distance x, then $f = D\frac{dx}{dt}$ as shown in Fig.4.5.



Translational mechanical system TFs, [1, p. 63]

Example (Translational inertia-spring-damper system)

- *Problem:* Find the TF relating the position, $X(s)$, to the input force, $F(s)$
- *Solution:*

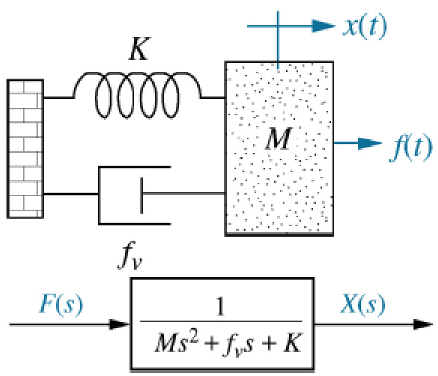
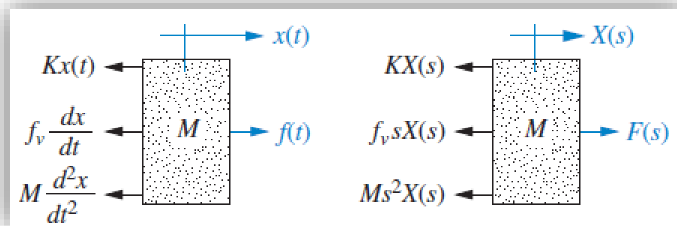


Figure: Physical system; block diagram

We assume the mass is traveling toward the right

Only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$



Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

or

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

2.6 Rotational Mechanical System Transfer Functions

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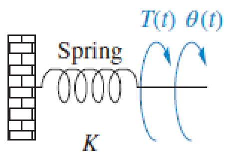
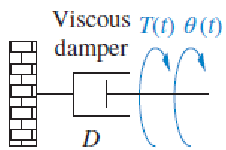
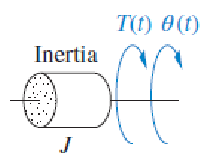
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad (radians), $\omega(t)$ – rad/s (radians/second), K – N-m/rad (newton-meters/radian), D – N-m-s/rad (newton-meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Rotational mechanical system

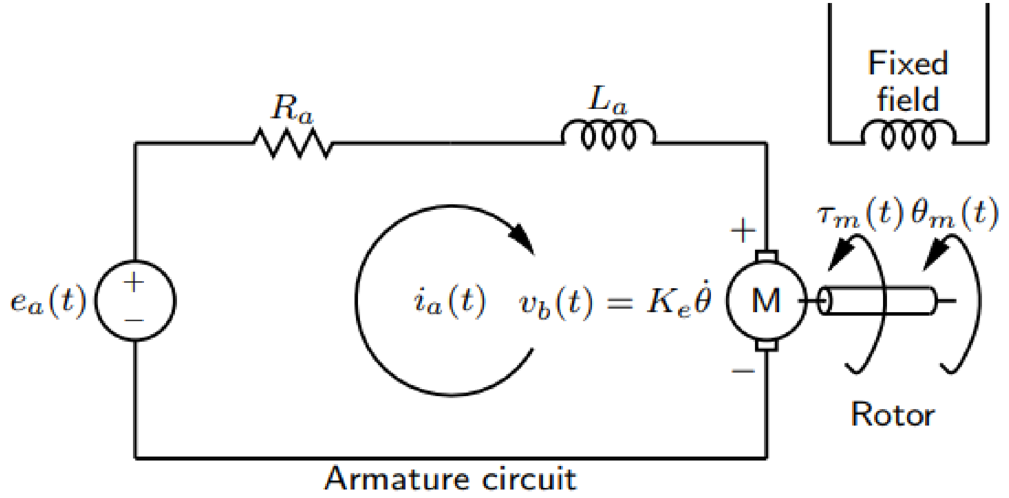
There are three basic elements in a Rotational mechanical system, i.e. (a) inertia, (b) spring and (c) damper.

(a) **Inertia:** A body with aninertia is denoted by J. If a torque$T$ is applied on it and it displays distanceΘ, then $T = J\frac{d^2\theta}{dt^2}$. If a torque$T$ is applied on a body with inertia J and it displays distance Θ_1 in the direction of T and distance Θ_2 in the opposite direction, then $T = J\left(\frac{d^2\theta_1}{dt^2} - \frac{d^2\theta_2}{dt^2}\right)$.

(b) **Spring:** A spring is denoted by K. If a torque$T$ is applied on it and it displays distanceΘ, then $T = K\theta$. If a torque$T$ is applied on a body with inertia J and it displays distance Θ_1 in the direction of T and distance Θ_2 in the opposite direction, then $T = K(\theta_1 - \theta_2)$.

(c) **Damper:** A damper is denoted by D. If a torque$T$ is applied on it and it displays distanceΘ, then $T = D\frac{d\theta}{dt}$. If a torque$T$ is applied on a body with inertia J and it displays distance Θ_1 in the direction of T and distance Θ_2 in the opposite direction, then $T = D\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right)$.

2.8 Electromechanical System Transfer Functions



- $v_b(t) = K_b \dfrac{d\theta_m(t)}{dt}$, where $v_b(t)$ is the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant.
- The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$ is

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) = e_a(t)$$

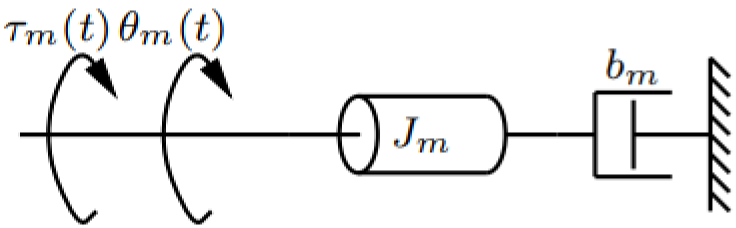
- The torque developed by the motor is proportional to the armature current; thus

$$\tau_m(t) = K_t i_a(t),$$

where $\tau_m(t)$ is the torque developed by the motor, and K_t is a constant for proportionality, called the motor torque constant.

- Taking the Laplace transform of both relationship and substituting $I_a(s)$ into the mesh equation, we have

$$\frac{(R_a + L_a s)\hat{\tau}_m(s)}{K_t} + K_b s \Theta_m(s) = E_a(s)$$



The figure shows a typical equivalent mechanical loading on a motor. We have

$$\hat{\tau}_m(s) = \left(J_m s^2 + b_m s\right)\Theta_m(s)$$

Substituting $\hat{\tau}_m(s)$ into the armature equation yields

$$\frac{(R_a + L_a s)\left(J_m s^2 + b_m s\right)\Theta_m(s)}{K_t} + K_b s \Theta_m(s) = E_a(s)$$

Assuming that the armature inductance, L_a is small compared to the armature resistance, R_a, the equation become

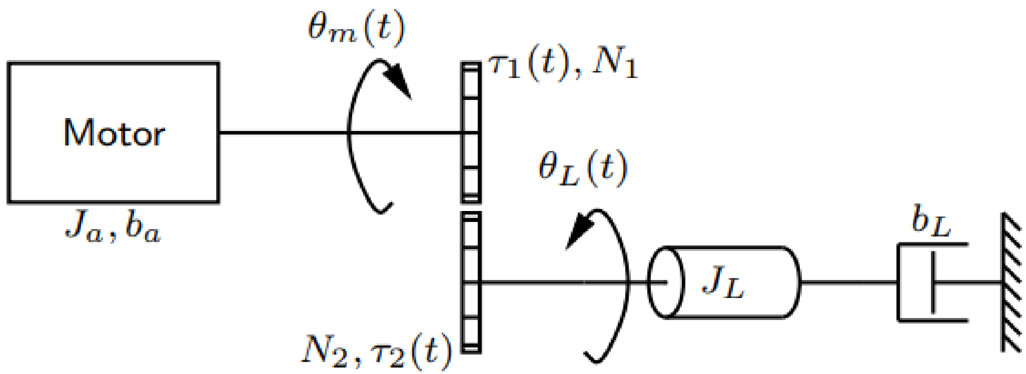
$$\left[\frac{R_a}{K_t}(J_m s + b_m) + K_b\right] s\Theta_m(s) = E_a(s)$$

After simplification, the desired transfer function is

$$\boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}}$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)}$$

DC motor driving a rotational mechanical load



A motor with inertia J_a and damping b_a at the armature driving a load consisting of inertia J_L and damping b_L. Assuming that J_a, J_L, b_a, and b_L are known. Then, we have

$$\left(J_a s^2 + b_a s\right) \Theta_m(s) + \hat{\tau}_1(s) = 0.$$

At the load side, we obtain

$$\left(J_L s^2 + b_L s\right) \Theta_L(s) = \hat{\tau}_2(s)$$
$$\left(J_L s^2 + b_L s\right) \frac{N_1}{N_2} \Theta_m(s) = \frac{N_2}{N_1} \hat{\tau}_1(s)$$

Substituting the $\hat{\tau}_1(s)$ back to the motor side equation, the equivalent equation is

$$\left[\left(J_a + J_L\left(\frac{N_1}{N_2}\right)^2\right)s^2 + \left(b_a + b_L\left(\frac{N_1}{N_2}\right)^2\right)s\right]\Theta_m(s) = 0$$

Or the equivalent inertial, J_m, and the equivalent damping, b_m, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \qquad b_m = b_a + b_L\left(\frac{N_1}{N_2}\right)^2$$

Next step, we going to find the electrical constants by using a *dynamometer* test of motor. This can be done by measuring the torque and speed of a motor under the condition of a constant applied voltage. Substituting $V_b(s) = K_b s\Theta_m(s)$ and $\hat{\tau}_m(s) = K_t I_a(s)$ in to the Laplace transformed armature circuit, with $L_a = 0$, yields

$$\frac{R_a}{K_t}\hat{\tau}_m(s) + K_b s\Theta_m(s) = E_a(s)$$

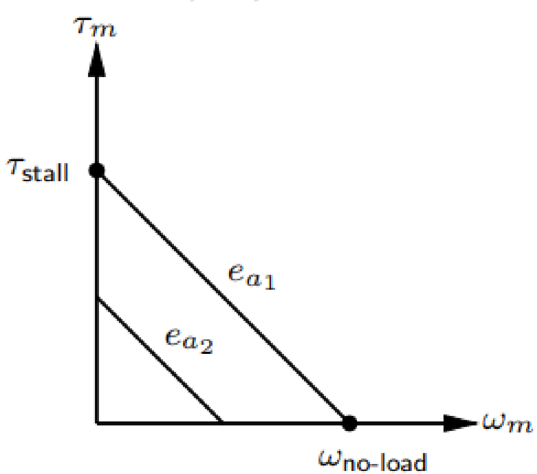
Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}\tau_m(t) + K_b\omega_m(t) = e_a(t)$$

If $e_a(t)$ is a DC voltage, at the steady state, the motor should turn a a constant speed, ω_m, with a constant torque, τ_m. With this, we have

$$\frac{R_a}{K_t}\tau_m + K_b\omega_m = e_a \quad \Rightarrow \quad \tau_m = -\frac{K_t K_b}{R_a}\omega_m + \frac{K_t}{R_a}e_a$$

Torque-peed curve



- $\omega_m = 0$, the value of torque is called the *stall torque*, τ_{stall}. Thus

$$\tau_{\text{stall}} = \frac{K_t}{R_a}e_a \Rightarrow \frac{K_t}{R_a} = \frac{\tau_{\text{stall}}}{e_a}$$

- $\tau_m = 0$, the angular velocity becomes *no-load speed*, $\omega_{\text{no-load}}$. Thus

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \Rightarrow K_b = \frac{e_a}{\omega_{\text{no-load}}}$$

Example 2.23 Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(*a*) and (*b*), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION:

Find $\Theta_L(s)/E_a(s)$ from the given system and torque-speed curve. The total inertia and the total damping at the armature of the motor are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$b_m = b_a + b_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

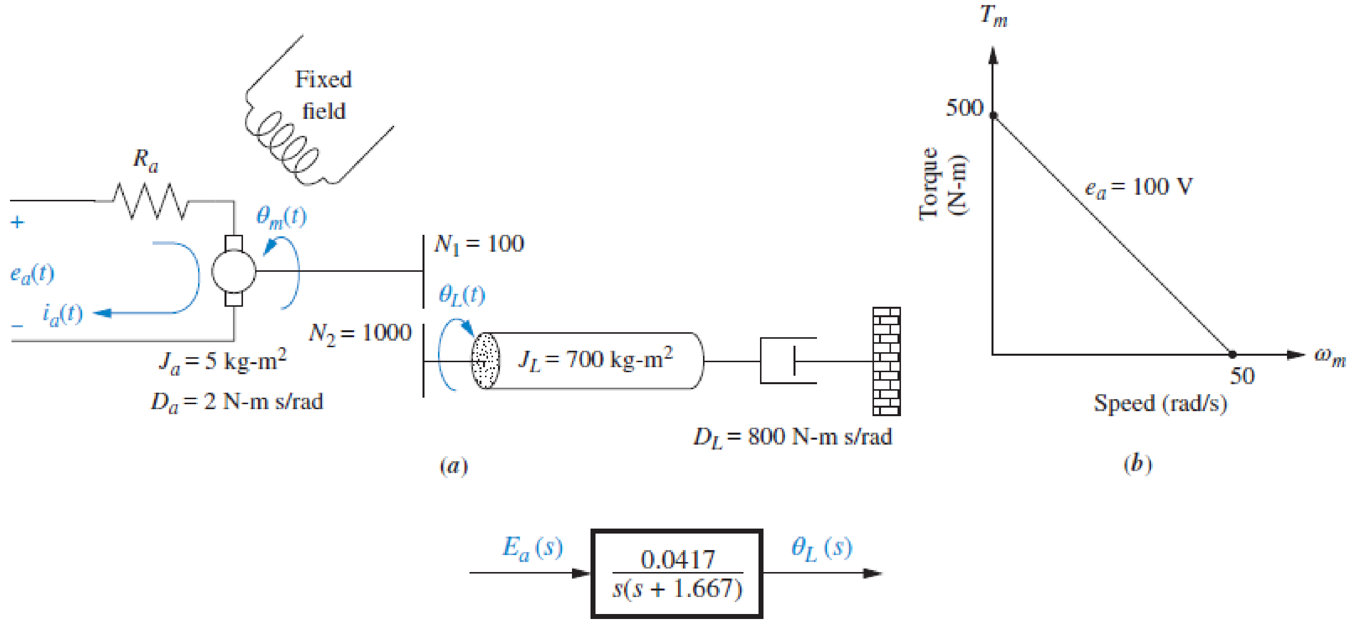


FIGURE 2.39 **a.** DC motor and load; **b.** torque-speed curve; **c.** block diagram

Next, we find the electrical constants K_t/R_a and K_b from the torque-speed curve. Hence,

$$\frac{K_t}{R_a} = \frac{\tau_{\text{stall}}}{e_a} = \frac{500}{100} = 5$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2$$

We have

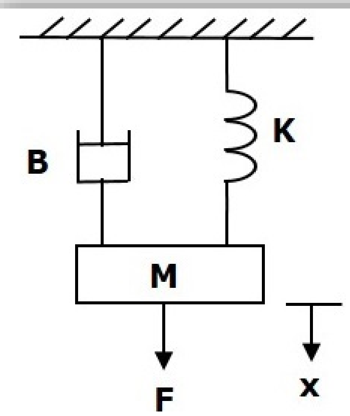
$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(b_m + \frac{K_t K_b}{R_a}\right)\right]} = \frac{0.417}{s(s+1.667)}.$$

Using the gear ratio, $N_1/N_2 = 0.1$

$$\frac{\Theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$$

2.9 Electric Circuit Analogs

Force ⇔ Voltage Analogy

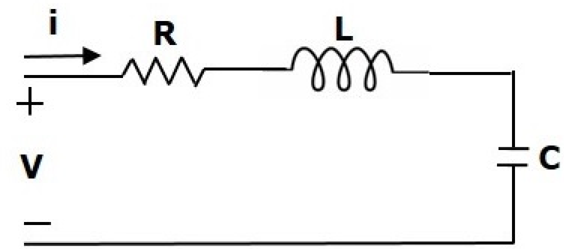


The **force balanced equation** for this system is

$$F = F_m + F_b + F_k$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

Force ⇔ Voltage
Spring ⇔ capacitor
viscous Damper ⇔ Resistor
Mass ⇔ Inductor



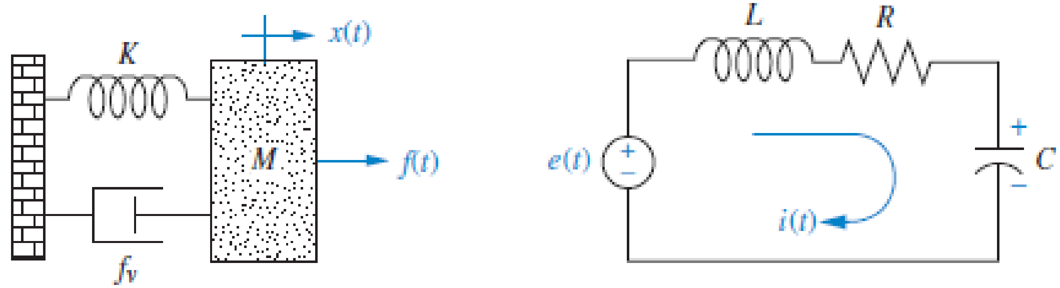
Mesh equation for this circuit is

$$V = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$$

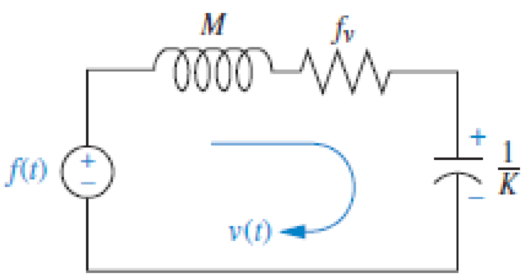
Substitute, $i = \frac{dq}{dt}$

$$V = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

$$\Rightarrow V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \left(\frac{1}{c}\right)q$$



(a) (b)



(c)

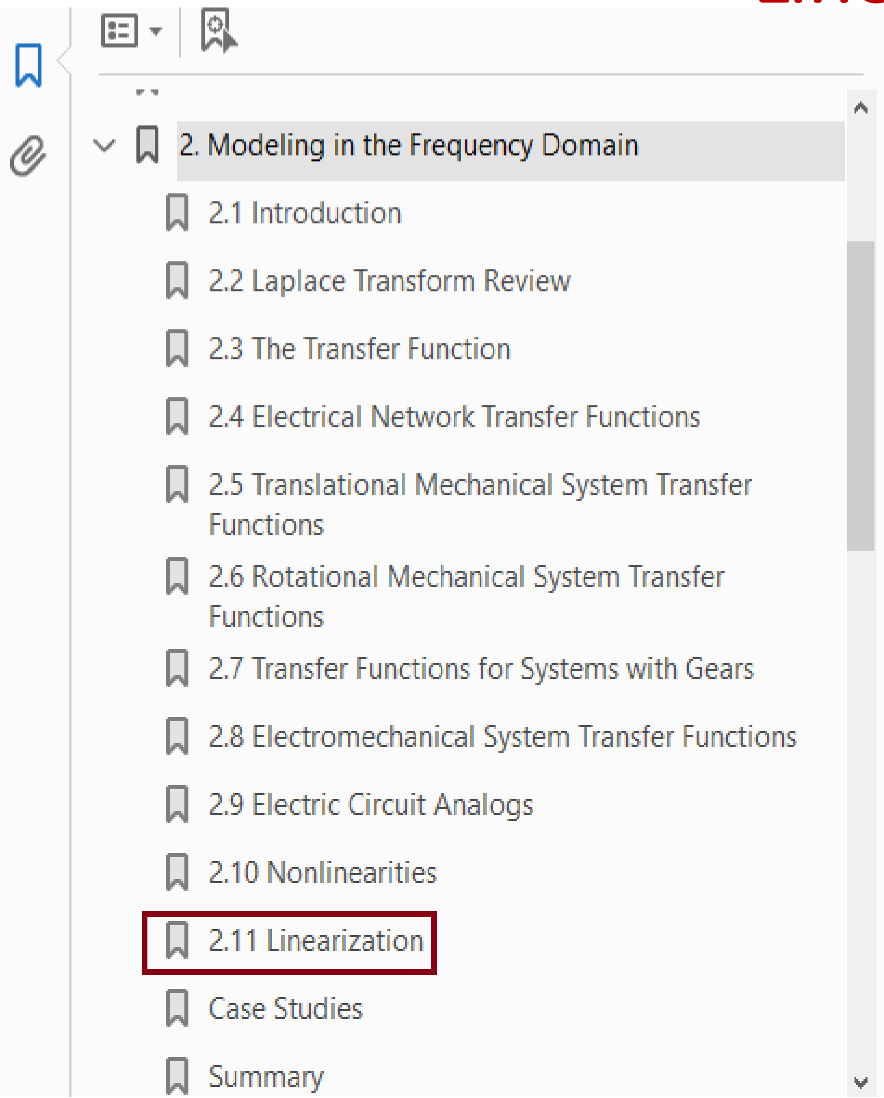
mass = M	$\longrightarrow$	inductor	=	M henries
viscous damper = f_v	$\longrightarrow$	resistor	=	f_v ohms
spring = K	$\longrightarrow$	capacitor	=	$\frac{1}{K}$ farads
applied force = $f(t)$	$\longrightarrow$	voltage source	=	$f(t)$
velocity = $v(t)$	$\longrightarrow$	mesh current	=	$v(t)$

(d)

operating on Eq. (2.173) to convert displacement to velocity by dividing and multiplying the left-hand side by s, yielding

$$\frac{Ms^2 + f_v s + K}{s} sX(s) = \left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s) \qquad (2.175)$$

Linearization



Motivation

I Must linearize a NL system into a LTI DE before we can find a TF.

Linearization procedure

1. Recognize the NL component and write the NL DE.
2. Linearize the NL DE into an LTI DE.
3. Laplace transform of LTI DE assuming zero initial conditions.
4. Separate input and output variables.
5. Form the TF.

Recap linearity

. In general, a **necessary condition** for a linear system can be determined in terms of an excitation $x(t)$ and a response $y(t)$. When the system at rest is subjected to an excitation $x_1(t)$, it provides a response $y_1(t)$. Furthermore, when the system is subjected to an excitation $x_2(t)$, it provides a corresponding response $y_2(t)$. For a linear system, it is necessary that the excitation $x_1(t) + x_2(t)$ result in a response $y_1(t) + y_2(t)$. This is the **principle of superposition**.

Furthermore, the magnitude scale factor must be preserved in a **linear system**. Again, consider a system with an input $x(t)$ that results in an output $y(t)$. Then the response of a linear system to a constant multiple β of an input x must be equal to the response to the input multiplied by the same constant so that the output is equal to $\beta y(t)$. This is the property of **homogeneity**.

A linear system satisfies the properties of superposition and homogeneity.

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A system characterized by the relation $y(t) = x^2(t)$ is not linear, because the superposition property is not satisfied. A system represented by the relation $y(t) = mx(t) + b$ is not linear, because it does not satisfy the homogeneity property.

One can often linearize nonlinear elements assuming small-signal conditions.

operating point x_0, y_0 for small changes Δx and Δy. When $x(t) = x_0 + \Delta x(t)$ and $y(t) = y_0 + \Delta y(t)$, we have

$$y(t) = mx(t) + b$$

or

$$y_0 + \Delta y(t) = mx_0 + m\Delta x(t) + b.$$

Therefore, $\Delta y(t) = m\Delta x(t)$, which satisfies the necessary conditions.

Linearization

Function approximation

- A transfer function does not exist for a nonlinear system.
- If a system is nonlinear, it can
  - either be modeled with a state-space model (in the time domain); or
  - be linearized around an operating point, and so the system's model can be approximated by a transfer function whose accuracy degrades as the system moves away from that point.

Taylor series (Fig. 12)

The Taylor series of a real or complex-valued function *f(x)* that is infinitely differentiable at a real or complex number x_0 is the power series

$$f(x) = f(x_0) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \underbrace{\left.\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \cdots}_{\text{High-Order Terms (H.O.T)}}$$

A linear approximation of *f(x)* around point $(x_0, f(x_0))$ truncates the H.O.T.

$$f(x) \equiv f(x_0+\delta x) \approx f(x_0) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0} (x-x_0) = f(x_0) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0} \delta x$$

$$\delta x \equiv x - x_0.$$

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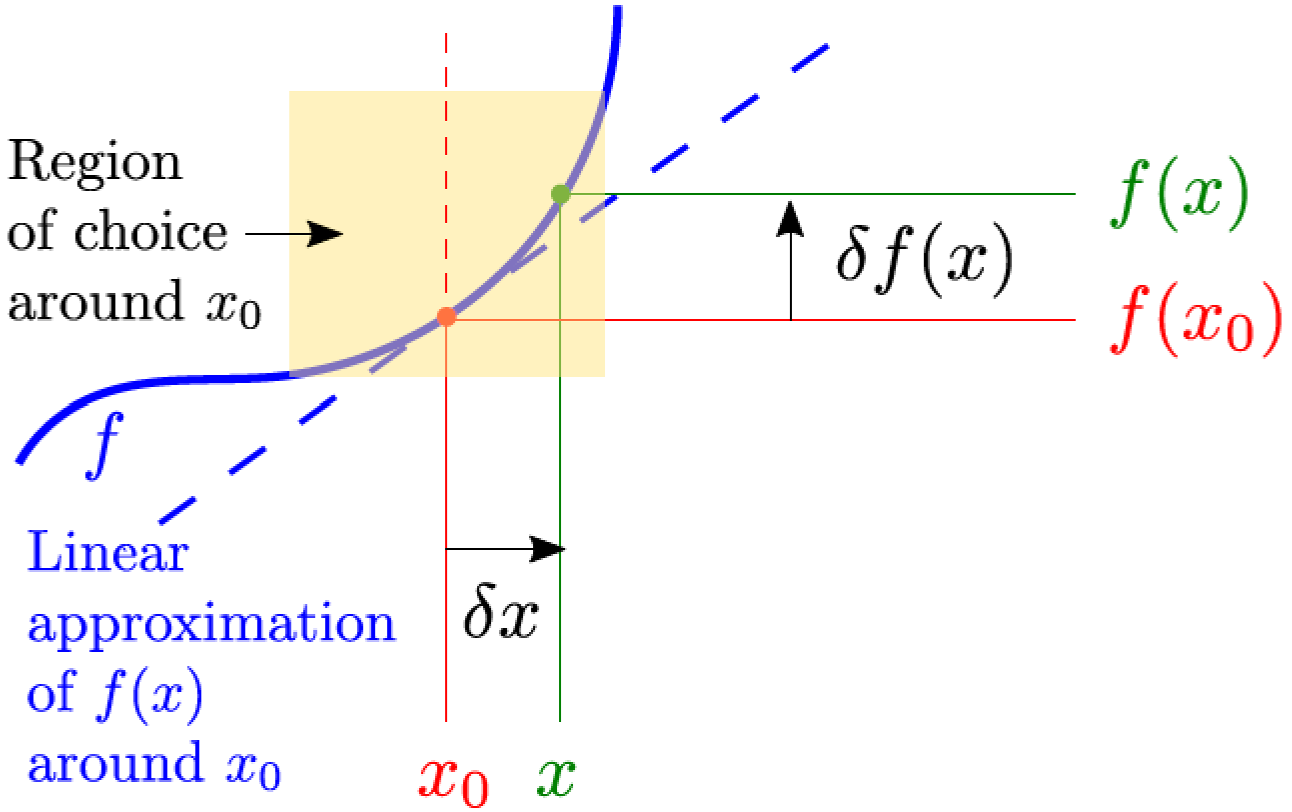


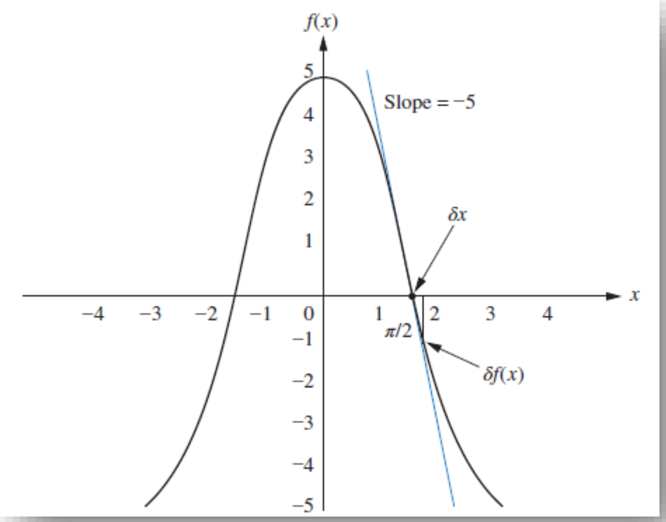
Figure : Linearization of a nonlinear function around point (x_0 , $f(x_0)$)

Example 2.26

Linearizing a Function

PROBLEM: Linearize $f(x) = 5\cos x$ about $x = \pi/2$.

SOLUTION: We first find that the derivative of $f(x)$ is $df/dx = (-5\sin x)$. At $x = \pi/2$, the derivative is -5. Also $f(x_0) = f(\pi/2) = 5\cos(\pi/2) = 0$. Thus, from Eq. (2.180), the system can be represented as $f(x) = -5\,\delta x$ for small excursions of x about $\pi/2$. The process is shown graphically in Figure 2.48, where the cosine curve does indeed look like a straight line of slope -5 near $\pi/2$.



Start with find the slope at $x = x_0 = \pi/2$.

$$m = \frac{df}{dx} = -5\sin x\Big|_{x=\frac{\pi}{2}} = -5$$

Since $f(x_0) = 5\cos\frac{\pi}{2} = 0$, then

$$f(x) = f(x_0) + m\delta x = -5\delta x.$$

This system can be represented as

$$f(x) = -5\delta x,$$

for small excursion of x.

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Example 2.27

Linearizing a Differential Equation

PROBLEM: Linearize Eq. (2.184) for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \cos x = 0 \tag{2.184}$$

SOLUTION: The presence of the term $\cos x$ makes this equation nonlinear. Since we want to linearize the equation about $x = \pi/4$, we let $x = \delta x + \pi/4$, where δx is the small excursion about $\pi/4$, and substitute x into Eq. (2.184):

the nonlinear term $\cos x$ can be linearized as follow:

$$\cos x = \cos\frac{\pi}{4} + \left.\frac{d\cos x}{dx}\right|_{x=\frac{\pi}{4}} \delta x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta x$$

Substituting $x = x_0 + \delta x$ and the linearized of $\cos x$ to the system, we have

$$\frac{d^2\left(\frac{\pi}{4} + \delta x\right)}{dt^2} + 2\frac{d\left(\frac{\pi}{4} + \delta x\right)}{dt} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta x = 0$$

$$\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} - \frac{\sqrt{2}}{2}\delta x = -\frac{\sqrt{2}}{2}$$

Exercise

Linearize the following equation for small excursions about 0.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + \sin x = 0$$

Answer $\frac{d^2\delta x}{dt^2} + 2\frac{d\delta x}{dt} + \delta x = 0$

Example 2.28

Transfer Function—Nonlinear Electrical Network

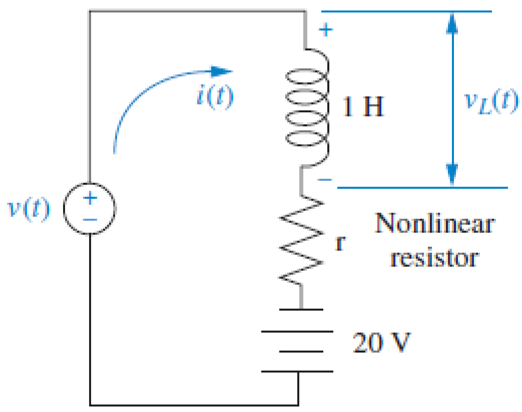

FIGURE 2.49 Nonlinear electrical network

PROBLEM: Find the transfer function, $V_L(s)/V(s)$, for the electrical network shown in Figure 2.49, which contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where i_r and v_r are the resistor current and voltage, respectively. Also, $v(t)$ in Figure 2.49 is a small-signal source.

SOLUTION: We will use Kirchhoff's voltage law to sum the voltages in the loop to obtain the nonlinear differential equation, but first we must solve for the voltage across the nonlinear resistor. Taking the natural log of the resistor's current-voltage relationship, we get $v_r = 10\ln\frac{1}{2}i_r$. Applying Kirchhoff's voltage law around the loop, where $i_r = i$, yields

$$L\frac{di}{dt} + 10\ln\frac{1}{2}i - 20 = v(t) \qquad (2.191)$$

1. When $v(t) = 0$, prove that the steady state of the current is $i_0 = 14.78$ A.
2. When the voltage source v(t) fluctuates around $v_0 = 0$, the current fluctuates around $i_0 = 14.78$ A. Find the transfer function $G(s) = V_L(s) / V(s)$ for small current deviations around i_0.

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Solution

Taking the natural log of the nonlinear relation:

$$v_r = 10 \ln \tfrac{1}{2} i_r$$

Applying Kirchhoff's voltage law around the loop, where $i_r = i$, yields

$$L\frac{di}{dt} + 10 \ln \frac{1}{2} i - 20 = v(t)$$

1. The small-signal source, $v(t)$, equal to zero. The circuit consists of a 20 V battery in series with the inductor and nonlinear resistor. In the steady state, the voltage across the inductor will be zero, since $v_L(t) = Ldi/dt$ and di/dt is zero in the steady state, given a constant battery source. Hence, the resistor voltage, v_r , is 20 V. Using the characteristics of the resistor, $i_r = 2e^{0.1vr}$, we find that $i_r = i =$ 14.78 amps
2. The current

$$i = i_0 + \delta i$$

$$L\frac{d(i_0 + \delta i)}{dt} + 10 \ln \frac{1}{2}(i_0 + \delta i) - 20 = v(t)$$

To linearize $\ln\frac{1}{2}(i_0+\delta i)$, ət

$$\ln\frac{1}{2}(i_0+\delta i)=\ln\frac{1}{2}i_0+\frac{d\left(\ln\frac{1}{2}i\right)}{di}\Big|_{i=i_0}\delta i=\frac{1}{i}\Big|_{i=i_0}\delta i=\frac{1}{i_0}\delta i$$

or

$$\ln\frac{1}{2}(i_0+\delta i)=\ln\frac{i_0}{2}+\frac{1}{i_0}\delta i$$

the linearized equation becomes

$$L\frac{d\delta i}{dt}+10\left(\ln\frac{i_0}{2}+\frac{1}{i_0}\delta i\right)-20=v(t)$$

Letting $L=1$ and $i_0=14.78$, the final linearized differential equation is

$$\frac{d\delta i}{dt}+0.677\delta i=v(t)$$

Taking the LT with zero initial conditions

$$\delta i(s) = \frac{V(s)}{s+0.677} \text{ .)}$$

But the voltage across the inductor about the equilibrium point is

$$v_L(t) = L\frac{d}{dt}(i_0 + \delta i) = L\frac{d\delta i}{dt}$$

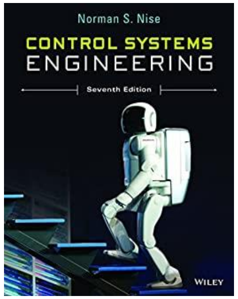
Taking the Laplace transform, $$V_L(s) = Ls\delta i(s) = s\delta i(s)$$

Substitute in (1) $$V_L(s) = s\frac{V(s)}{s+0.677}$$

The final TF is $$\frac{V_L(s)}{V(s)} = \frac{s}{s+0.677}$$

for small excursions about $i = 14.78$ or, equivalently, about $v(t) = 0$.

References



Nise, N. S. (2015).
Control Systems Engineering.
Wiley, 7 edition.