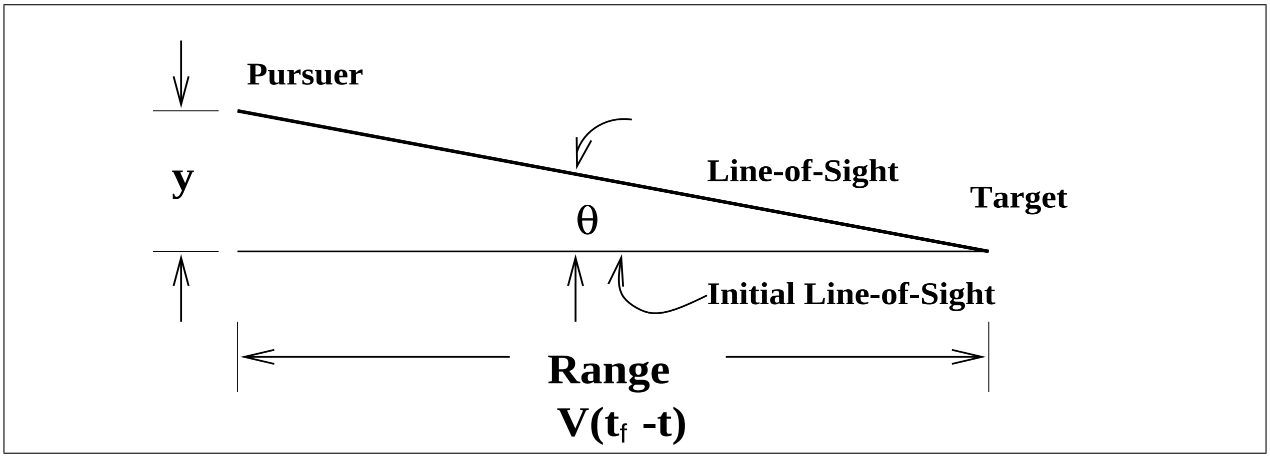
MAE 271B Project
Missile State Estimation

Due Date: March 10, 2021

Consider estimating the relative state of a missile intercept (see Figure).



Missile Intercept Illustration

The dynamics of the problem are

$$\begin{aligned}\dot{y} &= v \\ \dot{v} &= a_p - a_{\scriptscriptstyle T}.\end{aligned}$$

where a_p, the missile acceleration, is known and assumed here to be zero. The input, $a_{\scriptscriptstyle T}$, is the target acceleration and is treated as a random forcing function with an exponential correlation,

$$E\left[a_{\scriptscriptstyle T}\right] = 0$$

$$E\Big[a_{\scriptscriptstyle T}(t)a_{\scriptscriptstyle T}(s)\Big] = E[a_{\scriptscriptstyle T}^2]e^{\frac{-|t-s|}{\tau}}.$$

The scalar, τ, is the correlation time. The initial lateral position, $y(t_0)$, is zero by definition. The initial lateral velocity, $v(t_0)$, is random and assumed to be the result of launching error:

$$E\left[y(t_0)\right] = 0 \qquad E\left[v(t_0)\right] = 0$$

$$E\Big[y(t_0)^2\Big] = 0 \qquad E\Big[y(t_0)v(t_0)\Big] = 0 \qquad E\Big[v(t_0)^2\Big] = \text{given}.$$

The measurement, z, consists of a line-of-sight angle, θ. For $|\theta| \ll 1$,

$$\theta \approx \frac{y}{V_c(t_f - t)}.$$

It will also be assumed that z is corrupted by fading and scintillation noise so that

$$z = \theta + n$$
$$E\Big[n(t)\Big] = 0$$
$$E\Big[n(t)n(\tau)\Big] = V\delta(t-\tau) = \left[R_1 + \frac{R_2}{(t_f - t)^2}\right]\delta(t-\tau).$$

The process noise spectral density, W, is

$$W = GE\left[a_T^2\right]G^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E[a_T^2] \end{bmatrix}.$$

Consider the case where the parameters of the problem are

$$V_c = 300\,\frac{\text{ft}}{\text{sec}},\; E[a_T^2] = \left[100\,\text{ft}\,\text{sec}^{-2}\right]^2, \qquad t_f = 10\text{ sec}, \qquad R_1 = 15\times 10^{-6}\text{ rad}^2\,\text{sec},$$
$$R_2 = 1.67\times 10^{-3}\text{ rad}^2\,\text{sec}^3, \qquad \tau = 2\text{ sec},$$

The initial covariance is

$$P(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (200\,\text{ft}\,\text{sec}^{-1})^2 & 0 \\ 0 & 0 & (100\,\text{ft}\,\text{sec}^{-2})^2 \end{bmatrix}.$$

which may be the parameters for a Falcon or Sparrow guided missile.

1. Compute the error variance of the error in the state and the associate Kalman filter gains. See Figures 9.6 and 9.7 in the book.
2. Develop a simulation for the dynamics and the measurement.
3. Show by a Monte Carlo simulation that the actual error variance computed in the simulation matches the *a priori* error variance used in the Kalman filter gains.
4. Show that the residual process is uncorrelated in time, i.e. a white noise process.
5. The Gauss-Markov model is an approximation to the random telegraph signal model that is more realistic. In this model the value a_T changes sign at random times given by a Poisson probability. We assume that $a_T(0) = \pm a_T$ with probability .5 and $a_T(t)$ changes polarity at Poisson times. The probability of k sign changes in a time interval of length T, $P(k(T))$, is $P(k(T)) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}$ where λ is the rate. The probability of an even number of sign changes in T, $P(even\#\,\text{in}\,T)$, is

$$P(even\#\,\text{in}\,T) = \sum_{\substack{k=0 \\ k\text{ even}}}^{\infty} \frac{(\lambda T)^k e^{-\lambda T}}{k!} = e^{-\lambda T}\sum_{k=0}^{\infty}\frac{(1+(-1)^k)(\lambda T)^k}{2k!}$$

Since $e^{\lambda T} = \sum_{k=0}^{\infty}\frac{(\lambda T)^k}{k!}$, then

$$P(even\#\,\text{in}\,T) = e^{-\lambda T}\left[\sum_{k=0}^{\infty}\frac{(\lambda T)^k}{2k!} + \sum_{k=0}^{\infty}\frac{(-\lambda T)^k}{2k!}\right] = \frac{1}{2}e^{-\lambda T}\left[e^{\lambda T} + e^{-\lambda T}\right] = \frac{1}{2}\left[1 + e^{-2\lambda T}\right]$$

By a similar process, $P(odd\#\,\text{in}\,T) = \frac{1}{2}\left[1 - e^{-2\lambda T}\right]$. Then the probability that $a_T(t)$ is positive is

$$\begin{aligned}
P(a_T(t) = a_T) &= P(a_T(t) = a_T | a_T(0) = a_T) P(a_T(0) = a_T) \\
&+ P(a_T(t) = a_T | a_T(0) = -a_T) P(a_T(0) = -a_T) \\
&= \frac{1}{2} P(even\#\,\text{in}\,T = [0,\, t]) + \frac{1}{2} P(odd\#\,\text{in}\,T = [0,\, t]) \\
&= \frac{1}{2}\left\{\frac{1}{2}\left[1 + e^{-2\lambda t}\right] + \frac{1}{2}\left[1 - e^{-2\lambda t}\right]\right\} = \frac{1}{2}
\end{aligned}$$

Then, the mean acceleration, $\bar{a}_T$ is

$$\bar{a}_T = a_T P(a_T(t) = a_T) - a_T P(a_T(t) = -a_T) = 0$$

The autocorrelation, $R_{a_T a_T}(t_1,\, t_2) = E[a_T(t_1) a_T(t_2)]$ is

$$\begin{aligned}
R_{a_T a_T}(t_1,\, t_2) &= a_T^2 P(a_T(t_2) = a_T(t_1)) - a_T^2 P(a_T(t_2) \neq a_T(t_1)) \\
&= a_T^2 \frac{1}{2}\left[1 + e^{-2\lambda |t_2 - t_1|}\right] - a_T^2 \frac{1}{2}\left[1 - e^{-2\lambda |t_2 - t_1|}\right] = a_T^2 e^{-2\lambda |t_2 - t_1|}
\end{aligned}$$

If $\frac{1}{\tau} = 2\lambda$ then the autocorrelation of the Gauss-Markov process is the same as the random telegraph signal $(t_2 - t_1) = (t - s)$ and the means of both are zero.

We need a method of generating the random switching times. Let $T = t_{n+1} - t_n$ be the time between two switch times. The probability that the switch occurred after t_{n+1} is

$$P(T' > T | t = t_n) = 1 - P(T' \leq T | t = t_n)$$

Now the probability that no switch occurred in T, but occurred after t_{n+1}, is

$$P(T' > T | t = t_n) = P(\#\,\text{of sign changes in T is zero}) = e^{-\lambda T}$$

Then,

$$P(T' \leq T | t = t_n) = 1 - e^{-\lambda T}$$

which is the probability that at least one change occurred. To produce the random time t_{n+1}, set $P(T' \leq T | t = t_n)$ equal to U, the output of $[0,\, 1]$ from a uniform density function. Then,

$$\begin{aligned}
1 - e^{-\lambda T} = U \quad &\Rightarrow \quad e^{-\lambda T} = 1 - U \\
&\Rightarrow \quad -\lambda T = \ln(1 - U) \\
&\Rightarrow \quad T = \frac{-1}{\lambda} \ln(1 - U) \\
&\Rightarrow \quad t_{n+1} = t_n - \frac{1}{\lambda} \ln(1 - U)
\end{aligned}$$

Since $1 - U$ is also a uniform density function, then

$$t_{n+1} = t_n - \frac{1}{\lambda} \ln(U)$$

The objective of the above is to use the random telegraph signal in the simulation rather than the Gauss-Markov process that was used to ensure that the Kalman filter was implemented correctly. By running a Monte Carlo analysis using the random telegraph signal with $a_T = 100\,\text{ft}\,\text{sec}^{-2}$ and $\lambda = .25/\text{sec}$, determine the actual error variance.