Advanced Micro Theory: Part 2

Lecture 1

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Introductions

Me: Duarte Gonçalves ("Doo-art")

- Fields: Information, Game Theory, Behavioral, Experimental, Political Economy
- Topics: info acquisition, belief formation, strategic uncertainty, learning, conflict
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This course:

- Extensive-form games: refinements of Nash equilibrium, repeated games
- Applications: bargaining, conflict, reputation, career concerns, collusion

Logistics

Lectures

- Thursday, 9:00-11:00
- Cruciform Building B404 - LT2 (in person)

2 Quizzes: end of week 3 and 5

Problem sets

- Don't count toward final grade, but important!
  - Keep up with material, get feedback, apply concepts, applications
- Available on Friday, due Thursday before class

Overview

1. Why Extensive-Form Games?
   - Normal-Form Games
   - Limitations of Normal-Form Games

2. Extensive-Form Games
   - Definition
   - Game Trees
   - Strategies

3. Nash Equilibria in Extensive-Form Games

4. Subgame Perfection
   - Credibility
   - Subgames
   - Subgame-Perfect Nash Equilibrium

5. Applications
   - Alternating Bargaining
   - Centipede

Normal-Form Games and Extensive-Form Games

Normal-Form Games: $\Gamma = \langle I, S, u \rangle$

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$-i := I \setminus \{i\}$, players other than player i (player i's "opponents")

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What each player can do: S_i, set of feasible strategies for each player;

$s_i \in S_i$, one particular strategy of player i;

$S := \times_{i \in I} S_i$ set of feasible strategy profiles; a particular strategy profile $s \in S$,

$s = (s_i)_{i \in I}$, specifying one strategy for each player

$s_{-i} \in S_{-i} := \times_{j \in -i} S_j$ a specific strategy profile of player i's opponents

A strategy profile s determines an outcome of the game.

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What are each player's incentives: $u_i : S \to \mathbb{R}$, player i's payoff function, determining

how player i evaluates a specific outcome/strategy profile s;

$u = (u_i)_{i \in I}$, all players' payoff functions

Strategies in Normal-Form Games

Strategies: pure $s_i \in S_i$ and mixed $\boldsymbol{\sigma}_i \in \Sigma_i := \Delta(S_i)$

$\Delta(A)$: probability distributions over set A

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$\Delta(A)$: probability distributions over set A

Extending payoff functions to mixed strategies

von-Neumann–Morgenstern expected utility representations:

$u_i(\boldsymbol{\sigma}) := \sum_{s \in S} (\times_{j \in I} \boldsymbol{\sigma}_j(s_j)) u_i(s)$

Example: $I = \{1, 2\}$ and finitely many pure strategies,

$u_1(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = \sum_{s=(s_1,s_2) \in S} \boldsymbol{\sigma}_1(s_1) \boldsymbol{\sigma}_2(s_2) u_1(s_1, s_2)$

Limitations of Normal-Form Representations

Normal-form representation of games: simple, useful, but lacks notion of time

Some players may be able to observe opponents' choices before making their own

Examples:

- Employers may known which courses students chose to take
- Banks observe central bank's monetary policy before deciding on loans
- Firms may observe their competitors' pricing decisions before making theirs
- Employers and employees first sign contracts, then employees decide on how much effort to put in, and firms later decide on bonuses and/or promotions
- Firms make choices about which technologies to invest in prior to start producing

Information Matters

Not only that actions may be dynamic, but how dynamics interacts with information

- Two competing firms set prices for the following day
- If neither can observe their competitor's price in advance, then whether one chooses the price before or after the other is of no consequence
- But if a firm learns its competitor's pricing decision in advance, then it can condition its own pricing stratgy on the opponent's price

Crucial to capture what players know when they make their decisions; otherwise model predictions could be very much at odds with the data

Normal-Form Games and Extensive-Form Games

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- Crucial to capture what players know when they make their decisions; otherwise model predictions could be very much at odds with the data

We need a different way to model games to account for the fact that

(1) strategic interaction unfolds over time, and

(2) what players know when they make their choices matters

Extensive-Form Games

Definition

An **extensive-form game** is given by a tuple $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ where

1. I denotes a *set of players*; nature or chance is represented by $\mathbf{0} \notin I$
2. $\mathcal{A}$ denotes the overall *set of actions*
3. H denotes the *set of histories*
4. $\mathcal{I} := \{\mathcal{I}_i\}_{i \in I \cup \{\mathbf{0}\}}$, where $\mathcal{I}_i$ denotes player i's *information sets* or *information partition*
5. ρ is a function that associates each information set of nature $I_{\mathbf{0}} \in \mathcal{I}_{\mathbf{0}}$ with a probability measure over feasible actions following any history $h \in I_{\mathbf{0}}$, $\quad \rho(I_{\mathbf{0}}) \in \Delta(A(I_{\mathbf{0}}))$
6. $u := (u_i)_{i \in I}$, where each u_i represents player i's payoff function, $u_i : T \to \mathbb{R}$

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I *players*; $\mathcal{A}$ *actions*; H *histories*; $\mathcal{I}_i$ *i's info sets*; ρ *nature's move*; u_i *payoffs*

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- As before, I is the set of players
- Nature to represent randomness (whether or not nature $\in I$ is just convention)
- E.g. Firms decide on investment decisions; with some prob. a pandemic will start

$\mathcal{A}$ denotes the overall *set of actions*

- All the actions that some player or nature can take at some point

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H denotes the *set of histories* , satisfying the following properties

(i) The **empty history** $\emptyset$ is a member of H

(the 'starting point' of the game)

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(ii) A **nonempty history** $h \in H$ consists of a (possibly infinite) sequence of actions,

$h = (a^1, \ldots, a^t) \in \mathcal{A}^t$ for some $t \in \mathbb{N} \cup \{\infty\}$

(what has happened thus far)

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(iii) If, for $n \in \mathbb{N} \cup \{\infty\}$, $(a^\ell)_{\ell=1}^n \in H$, then, for any positive integer $m < n$, $(a^\ell)_{\ell=1}^m \in H$

A (*proper*) **subhistory** h' of history $h = (a^1, ..., a^t)$ is a sequence of actions

$h' = (a'^1, ..., a'^s)$ such that $s \leq (<) t$ and $a^n = a'^n$ for $n = 1, ..., s$

(if a given seq of n actions is a feasible history, then so are its subhistories)

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(iv) If $(a^\ell)_{\ell=1}^\infty$ is such that, for every $n \in \mathbb{N}$, $(a^\ell)_{\ell=1}^n \in H$, then $(a^\ell)_{\ell=1}^\infty \in H$

(if all finite subhistories are feasible histories, then so is the history)

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The **set of terminal histories** is denoted by $T \subset H$

A history which is not terminal ($h \in H \setminus T$) is called a **nonterminal history**

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(Q: why don't we just do H = all possible sequences of actions from $\mathcal{A}$?)

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The **set of feasible actions** following nonterminal history h is defined as
$A(h) := \{a \in \mathcal{A} \mid (h, a) \in H\}$

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(ii) The set of all players information sets (including nature) $\cup_{i \in I \cup \{0\}} \mathcal{I}_i$ determines a partition over the set of all nonterminal histories

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(ii) The set of all players information sets (including nature) $\cup_{i \in I \cup \{0\}} \mathcal{I}_i$ determines a partition over the set of all nonterminal histories i.e.

(a) any two information sets are disjoint ($\tilde{I} \cap \hat{I} = \emptyset, \forall \tilde{I}, \hat{I} \in \cup_{i \in I \cup \{0\}} \mathcal{I}_i$), and

(b) the union of all information sets of all players (including nature) corresponds to the set of nonterminal histories ($H \setminus T = \cup_{i \in I} \{\tilde{I} \in \mathcal{I}_i\}$)

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In general, nature's information sets are singletons, corresponding to a single history

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(ii) The set of all players information sets (including nature) $\cup_{i \in I \cup \{0\}} \mathcal{I}_i$ determines a partition over the set of all nonterminal histories

(iii) For any two histories belonging to the same information set, $h, h' \in I_i \in \mathcal{I}_i$, the set of feasible actions is the same, $A(h) = A(h') =: A(I_i)$

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Idea: $\mathcal{I}_i$ represents what player i knows

Two histories are in the same info set = player i cannot distinguish between them

After sequence of actions $h = (a^1, a^2, ..., a^t) \in I_i$, *player i knows some history in I_i was played*, but cannot observe which

That is why player i has to choose the same action following all histories in the same info set $h \in I_i$

When does each player move?: Player i moves following each history h that belongs to some information set $I_i \in \mathcal{I}_i$

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We write $A(I_i) := A(h)$ to denote the set of feasible actions following any history in information set I_i

If following two different histories belonging to the same information set player i had different actions available, then they would be able to distinguish between them

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Game is of (*im*)**perfect information** if (*not*) all information sets are singletons

Game is of **perfect recall** if players don't forget (i) what they know nor (ii) which actions they take. Formally,

(1) If $h \in I_i$, then for any proper subhistory h' of h, $h' \notin I_i$

(2) Let $h, h' \in I_i$, and take any $\tilde{h}, \tilde{h}' \in \tilde{I}_i$ that subhistories of h and h', respectively, belonging to the same informantion set of player i

Then $(\tilde{h}, a)$ is a subhistory of h if and only if $(\tilde{h}', a)$ is a subhistory of h'

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ρ is a function that associates each information set of nature $I_0 \in \mathcal{I}_0$ with a probability measure over the set of feasible actions following any history $h \in I_0$, $\rho(I_0) \in \Delta(A(I_0))$

Nature moves following each history h that belongs to some information set $I_0 \in \mathcal{I}_0$

ρ determines what nature does at each information set

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$u := (u_i)_{i \in I}$, where each u_i represents player i's payoff function, $u_i : T \to \mathbb{R}$

- Payoffs realize after terminal histories
- We will assume that u_i corresponds to a von-Neumann–Morgenstern utility function (Bernoulli index) representing preferences of player i over terminal histories

Representing Extensive-Form Games

Game tree: nodes, edges, and information sets

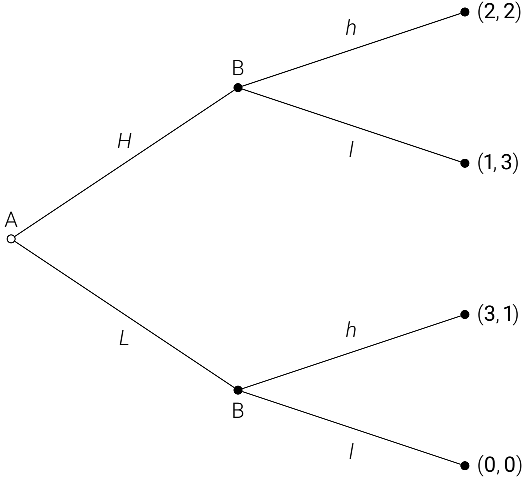
- **Nodes**: each node corresponds to a different history
- **Root**: empty history, 'starting point' of the game
- **Terminal nodes**: terminal histories; typically labeled with players' payoffs
- **Non-Terminal nodes**: correspond to non-terminal histories; nodes/histories at which a player makes a choice

  Only one player makes a choice at any given node/following any given history
- **Edges**: correspond to different actions the player choosing may take; typically labeled with the name of the corresponding actions
- **Information sets**: correspond to histories a given player is unable to distinguish between; typically labeled with the name of the player that is choosing/active

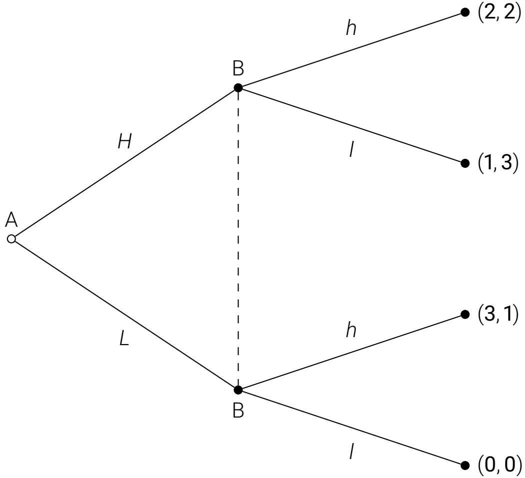
  Represented by grouping of non-terminal nodes (circling them, dashed lines)

  The same player choosing at any node/history in the same information set

Perfect Information



Imperfect Information



Strategies in Extensive-form Games

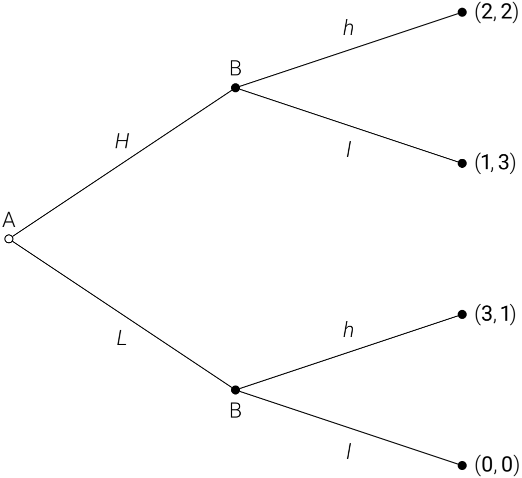
At each $I_i \in \mathcal{I}_i$, player i's feasible **actions** are $A(I_i)$

(Pure) Strategy for player i: $s_i : \mathcal{I}_i \to \mathscr{A}$ such that $s_i(I_i) \in A(I_i)$

A (pure) strategy of player i specifies a full contingent plan: which feasible action player i chooses at each information set

Think about it as delegating decision to a representative

What are the histories/info sets/strategies/actions?



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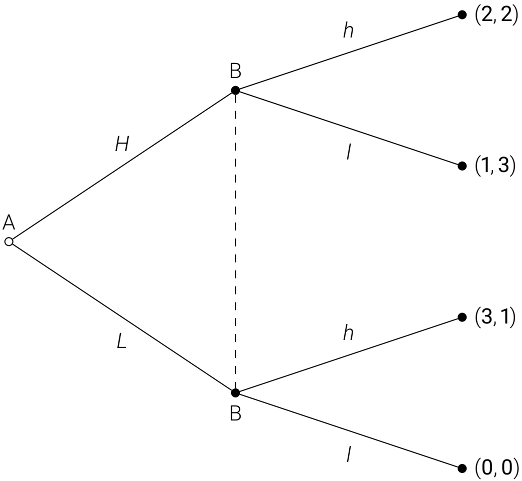

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In games without nature moves,

a pure strategy profile $(s_i)_{i \in I}$ induces a unique terminal history

(multiple pure strategy profiles may induce the same terminal history)

In general (with nature moves, randomness),

a pure strategy profile induces a distribution over terminal histories

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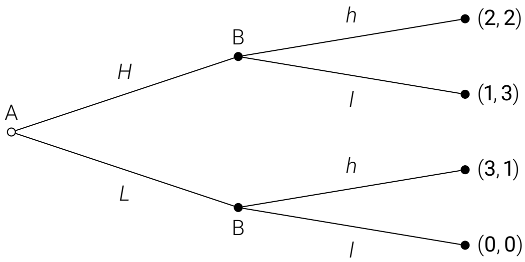


Strategies: $S_A = \{H, L\}$, $S_B = \{(h|H, h|L), (l|H, h|L), (l|H, h|L), (l|H, l|L)\}$

Both $(H, (h|H, h|L))$ and $(H, (h|H, l|L))$ induce terminal history Hh

Extensive-Form Games

Strategies in Extensive-form Games

Mixed Strategy for player *i*: distribution over pure strategies, $\sigma_i \in \Delta(S_i) =: \Sigma_i$

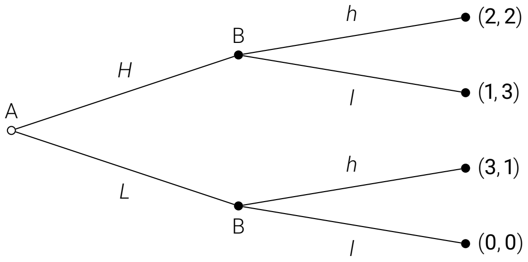
Behavioral Strategy for player *i*: distribution over actions at each information set,

$\lambda_i : \mathcal{I}_i \to \Delta(\mathcal{A})$ such that $\lambda_i(I_i)(a) = 0 \; \forall a \notin A(I_i)$

(Can only randomize over strategies that are feasible at I_i)

Both mixed and behavioral strategies induce distributions over terminal histories

What are the histories/info sets/strategies/actions?



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$\sigma_A(H) = 1/3$, $\sigma_B((h|H, h|L)) = 1/4$, $\sigma_B((l|H, h|L)) = 3/4$

$$P_\sigma(Hh) = \sigma_A(H)(\sigma_B((h|H, h|L)) + \sigma_B((h|H, l|L))) = 1/3(1/4 + 0) = 1/12$$

$$P_\sigma(Hl) = \sigma_A(H)(\sigma_B((l|H, h|L)) + \sigma_B((l|H, l|L))) = 1/3(0 + 3/4) = 3/12$$

$$P_\sigma(Lh) = \sigma_A(L)(\sigma_B((h|H, h|L)) + \sigma_B((l|H, h|L))) = 2/3(1/4 + 3/4) = 2/3$$

$$P_\sigma(Ll) = \sigma_A(L)(\sigma_B((h|H, l|L)) + \sigma_B((l|H, l|L))) = 2/3(0 + 0) = 0$$

$\lambda_A(\emptyset)(H) = 1/3$, $\lambda_B(H)(h) = 1/4$, $\lambda_B(L)(h) = 1$

Extensive-Form Games

Strategies in Extensive-form Games

Mixed Strategy for player i: distribution over pure strategies, $\sigma_i \in \Delta(S_i)$

Behavioral Strategy for player i: distribution over actions at each information set,

$\lambda_i : \mathcal{I}_i \to \Delta(\mathcal{A})$ such that $\lambda_i(I_i)(a) = 0 \; \forall a \notin A(I_i)$

(Can only randomize over strategies that are feasible at I_i)

Both mixed and behavioral strategies induce distributions over terminal histories

Kuhn's Theorem

For finite extensive-form games with perfect recall, every mixed strategy of a player has an outcome-equivalent behavioral strategy and vice-versa

To an extent, can use mixed and behavioral strategies interchangeably

Note: What OR call Kuhn's theorem (Prop 99.2) is typically known as Zermelo's theorem (later)

Nash Equilibria in Extensive-Form Games

A strategy profile $\boldsymbol{\sigma} = (\sigma_i)_{i \in I}$ maps to a distribution over terminal histories $h \in T$

Although $u_i : T \to \mathbb{R}$, we can unambiguously write $u_i : S \to \mathbb{R}$

(just as in normal-form games)

We also extend payoffs to mixed strategy profiles as before, $u_i(\boldsymbol{\sigma}) := \sum_{s \in S} (\times_{j \in I} \sigma_j(s_j)) u_i(s)$

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Definition

A **Nash equilibrium** of an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ is a strategy profile $\boldsymbol{\sigma} \in \Sigma$ such that for every player $i \in I$

$$u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_{-i}) \quad \forall \boldsymbol{\sigma}'_i \in \Sigma_i$$

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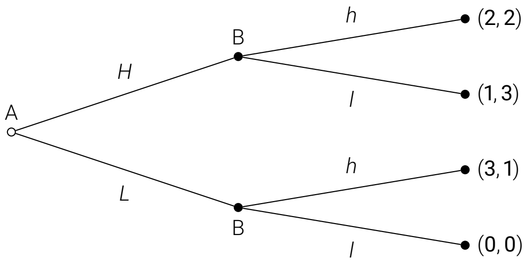
Definition

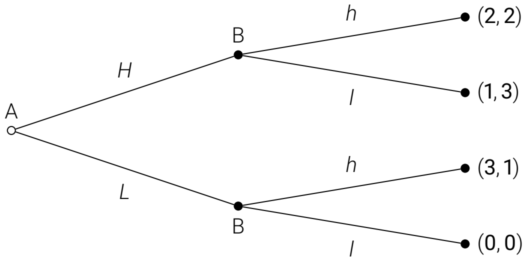
Every finite ($|H| < \infty$) extensive-form game has a Nash equilibrium, possibly in mixed strategies.

From Extensive- to Normal-form



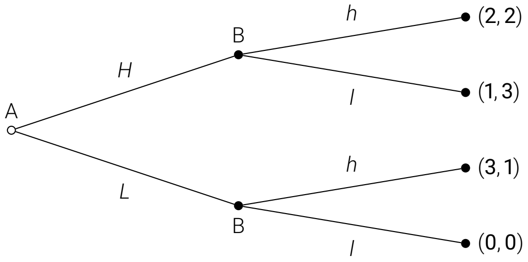
		Player *B*			
		h|H, h|L	l|H, h|L	h|H, l|L	l|H, l|L
Player *A*	H	2,2	1,3	2,2	1,3
	L	3,1	3,1	0,0	0,0

The Problem of Credibility



		Player *B*			
		h|H, h|L	l|H, h|L	h|H, l|L	l|H, l|L
Player *A*	H	2,2	1,**3**	**2**,2	**1**,**3**
	L	**3**,**1**	**3**,**1**	0,0	0,0

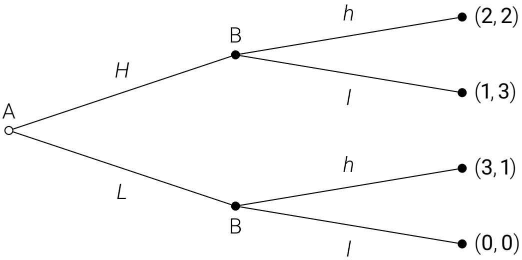
The Problem of Credibility



		Player B			
		h|H, h|L	l|H, h|L	h|H, l|L	l|H, l|L
Player A	H	2,2	1,**3**	**2**,2	**1**,**3**
	L	**3**,**1**	**3**,**1**	0,0	0,0

PSNE: $\{(H, (l|H, l|L)),\ (L, (h|H, h|L)),\ (L, (l|H, h|L))\}$

The Problem of Credibility

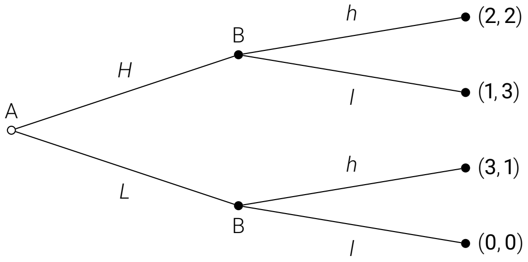


		Player B			
		h|H, h|L	l|H, h|L	h|H, l|L	l|H, l|L
Player A	H	2,2	1,**3**	**2**,2	**1**,**3**
	L	**3**,**1**	**3**,**1**	0,0	0,0

PSNE: $\{(H,(l|H,l|L)),\ (L,(h|H,h|L)),\ (L,(l|H,h|L))\}$

(PSNE *outcomes*: (Hl, Lh))

The Problem of Credibility



		Player B			
		h|H, h|L	l|H, h|L	h|H, l|L	l|H, l|L
Player A	H	2,2	1,**3**	**2**,2	**1**,**3**
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PSNE: $\{(H, (l|H, l|L)),\ (L, (h|H, h|L)),\ (L, (l|H, h|L))\}$

(PSNE *outcomes*: (Hl, Lh))

But... Hl is supported by B's threat of choosing l if A chooses L

Is this really credible? Not really

Subgames

Definition

A **subgame** of an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ is another extensive-form game $\Gamma^h = \langle I, \mathcal{A}, H^h, \mathcal{I}(h), \rho^h, u^h \rangle$ such that

(i) $\exists I_i = \{h\} \in \mathcal{I}_i$ such that for any $H^h = \{h' \mid (h, h') \in H\}$; (ii) $\mathcal{I}_i^h \subseteq \mathcal{I}_i$ for all $i \in I$;

(iii) $\rho^h(I_0) = \rho(I_0)$ for all $I_0 \in \mathcal{I}_0$; and (iv) $u_i^h(h') = u_i(h')$ for all $h \in T^h$

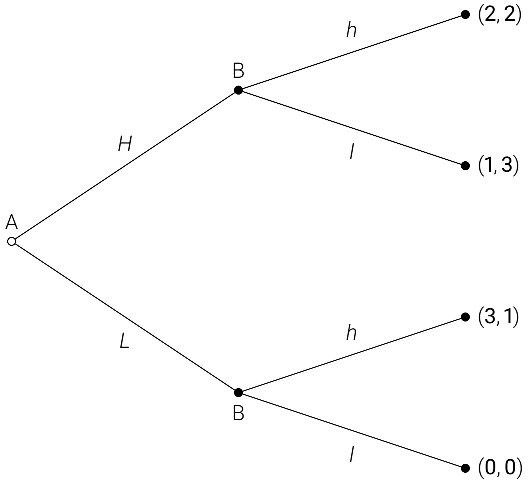
(i) states that we start a subgame starts at a singleton information set of the game and includes all histories 'starting from there'; this implies that $T^h = T \cap \{(h, h') \mid h' \in H^h\}$

(ii) implies that subgames don't 'cut across' information sets
(players know they are playing the subgame)

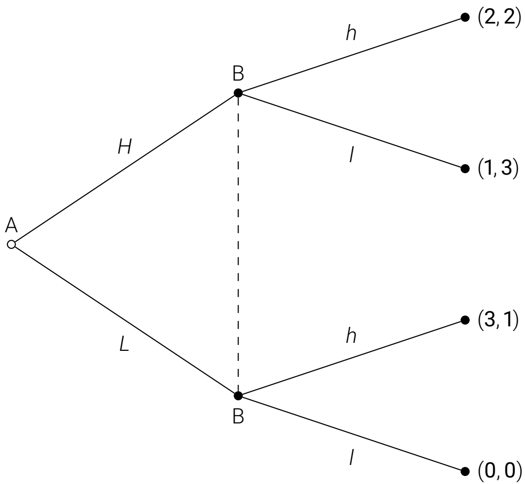
(iii) say that nature moves the same way in the subgame as in the original game, and

(iv) means that payoffs over the subgame's terminal histories are the same as in the original game

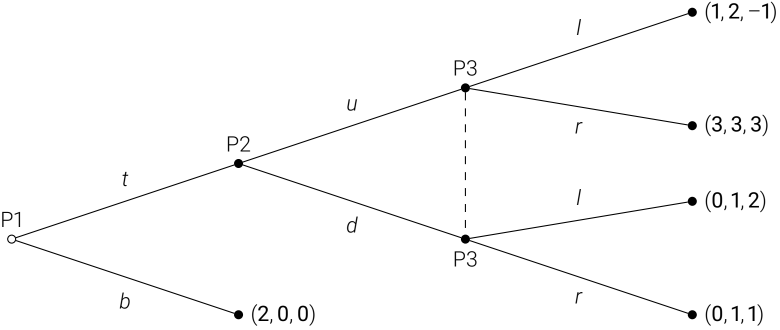
How Many Subgames?



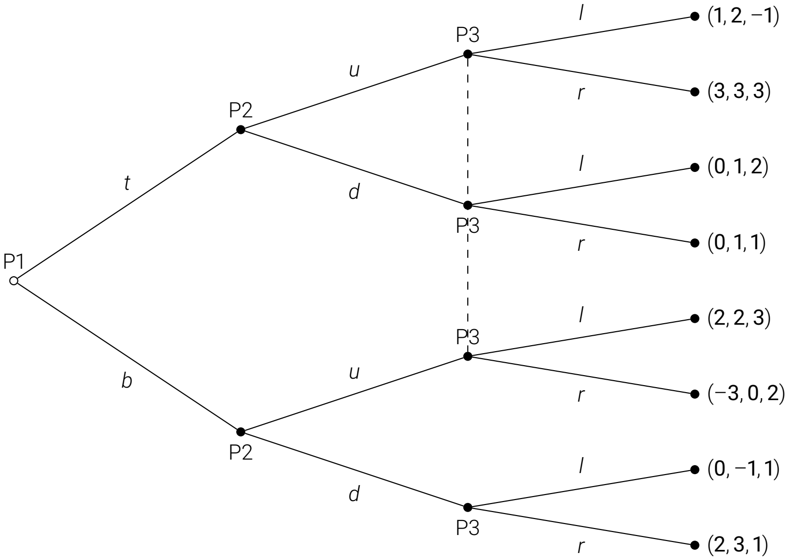
How Many Subgames?



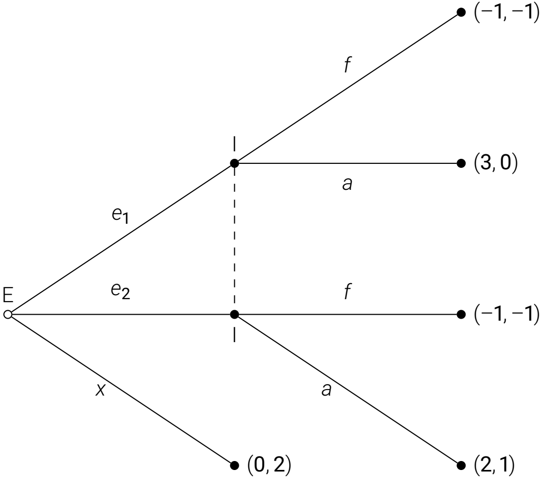
How Many Subgames?



How Many Subgames?



How Many Subgames?



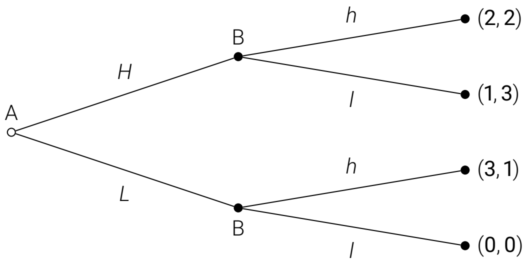
Refining Nash Equilibria in Extensive-Form Games: Subgame Perfection

Definition

A **subgame-perfect Nash equilibrium** (SPNE) of an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ is a strategy profile σ that induces a Nash equilibrium in every subgame of Γ.

The whole game is a subgame of itself $\Longrightarrow$ an SPNE is an NE

Backward Induction



Backward Induction:

Main gist: start with terminal nodes/histories, pick payoff-maximizing actions, and work your way backward

PSNE: $\{(H, (l|H, l|L)),\ (L, (h|H, h|L)),\ (L, (l|H, h|L))\}$

PSNE of subgame starting at H: l; PSNE of subgame starting at L: h

PS-SPNE: $\{(L, (l|H, h|L)\}$

Refining Nash Equilibria in Extensive-Form Games: Subgame Perfection

Definition

A **subgame-perfect Nash equilibrium** (SPNE) of an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u\rangle$ is a strategy profile σ that induces a Nash equilibrium in every subgame of Γ.

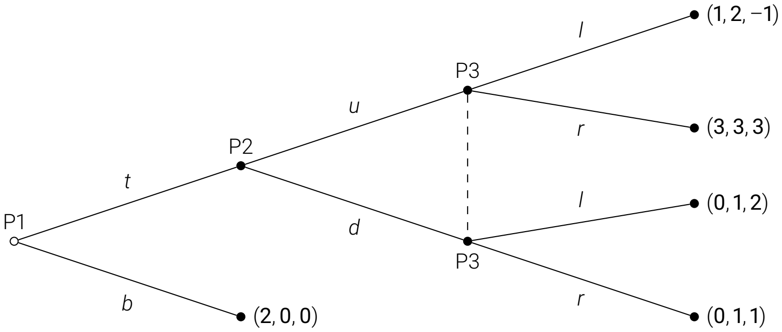
Zermelo's Theorem

Every finite ($|H| < \infty$) extensive-form game of perfect information has a pure-strategy subgame-perfect Nash equilibrium.

A pure strategy profile is a subgame perfect Nash equilibrium of a finite extensive-form game of perfect information if and only if it can be obtained by backward induction.

Furthermore, if no player has the same payoffs at any two terminal histories, then there is a unique Nash equilibrium coinciding with the unique pure strategy subgame perfect Nash equilibrium.

Generalized Backward Induction



Generalized Backward Induction:

Main gist: start with subgames 'closest' to terminal nodes/histories, pick a NE in the subgame, and work your way backward

In this case, (u, r) is the unique NE of the only subgame

Refining Nash Equilibria in Extensive-Form Games: Subgame Perfection

Definition

A **subgame-perfect Nash equilibrium** (SPNE) of an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ is a strategy profile σ that induces a Nash equilibrium in every subgame of Γ.

Theorem

Every finite ($|H| < \infty$) extensive-form game has a subgame-perfect Nash equilibrium (possibly in mixed strategies).

A pure strategy profile is a subgame perfect Nash equilibrium of a finite extensive-form game ~~of perfect information~~ if and only if it can be obtained by **generalized** backward induction.

Alternating Bargaining

Setup

Two players, 1 and 2, bargain over the split of £$v > 0$.

- There are up to T periods of bargaining, where T is odd, and both players discount payoffs at a rate $\delta \in (0, 1)$ per period.
- Conditional on bargaining continuing up to period t, Player i gets to propose a split $b_t \in [0, v]$, which the opponent can accept or reject, where $i = 1$ if t is odd and $i = 2$ if otherwise.
- If the opponent accepts, the game ends, with the proposer — Player i — getting $\delta^{t-1}(v - b_t)$, and the opponent $\delta^{t-1} b_t$.
- If the opponent rejects, the game moves on to the next period $t + 1$ if $t < T$, or it ends if $t = T$, in which case both players get zero.

Strategies are complicated, as they can depend on the whole observed history, but as we will see, the SPNE is quite straightforward

Alternating Bargaining

Backward Induction

If $T = 1$, this is just an ultimatum game; same for $t = T$

At period T, Player 2 accepts if $b_T > 0$; if $b_T = 0$, Player 2 is indifferent.

The unique SPNE in any subgame that reached period T is to have Player 1 proposing $b_T = 0$ and Player 2 accepting iff $b_T \geq 0$.

For any $b_T > 0$, Player 2 strictly prefers accepting over rejecting, and, given this, Player 1 strictly prefers proposing to get a higher share $\frac{1}{2}b_T$.

Then, they accrue payoffs $\delta^{T-1}(v, 0)$.

Alternating Bargaining

Backward Induction

Then, at any subgame starting at period $T-1$, Player 1 is willing to accept b_{T-1} iff

$$\delta^{T-2} b_{T-1} \geq \delta^{T-1} v \Longleftrightarrow b_{T-1} \geq \delta v$$

Otherwise they would prefer to reject and move to the next period and get the chance to propose themselves.

By a similar argument, the unique SPNE in this subgame is to have Player 2 offering exactly

$$b_{T-1} = \delta v$$

and payoffs are then

$$(\delta^{T} v, \delta^{T-1}(1-\delta) v).$$

Alternating Bargaining

Backward Induction

At any subgame starting at period $T-2$, the unique SPNE in the subgame will have Player 1 proposing a split that Player 2 accepts while indifferent between accepting and rejecting, and thus

$$\delta^{T-2} b_{T-2} = \delta^{T-1}(1-\delta)v \iff b_{T-2} = \delta(1-\delta)v,$$

resulting in equilibrium payoffs

$$\delta^{T-2}((1-\delta+\delta^2)v, (\delta-\delta^2)v).$$

Iterating backward, we have that, at any $t \in [T-1]$, the proposer suggests a split

$$b_{T-t} = \sum_{\ell=1}^{t} (-1)^{\ell-1}\delta^{\ell} v = v\delta \frac{1-(-1)^t\delta^t}{1+\delta}$$

Backward Induction

Iterating backward, we have that, at any $t \in [T-1]$, the proposer suggests a split

$$b_{T-t} = \sum_{\ell=1}^{t} (-1)^{\ell-1} \delta^{\ell} v = v\delta \frac{1-(-1)^t \delta^t}{1+\delta}$$

and, the opponent accepts iff

$$b_{T-t} \geq v\delta \frac{1-(-1)^t \delta^t}{1+\delta}$$

SPNE payoffs for the whole game are then

$$(v-b_1, b_1) = v\left(1-\delta\frac{1-\delta^T}{1+\delta}\ ,\ \delta\frac{1-\delta^T}{1+\delta}\right) = v\left(\frac{1+\delta^{T+1}}{1+\delta}\ ,\ \frac{\delta-\delta^{T+1}}{1+\delta}\right)$$

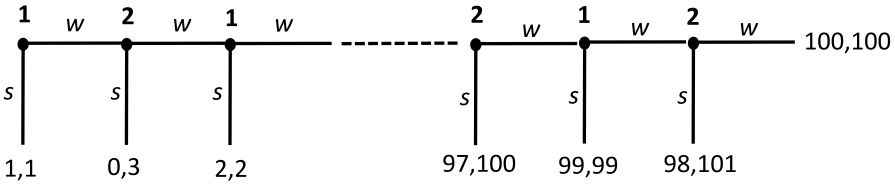
Alternating Bargaining

Insights

1. No delay: a solution is reached immediately
2. First and last propose confers advantage: Player 1 gets a larger share of the fixed resource

   As $T \to \infty$, equilibrium payoffs are given by $\left(v\frac{1}{1+\delta}, v\frac{\delta}{1+\delta}\right)$
3. Patience pushes in favor of the last proposer; impatience, of the first (problem set)

Centipede Game

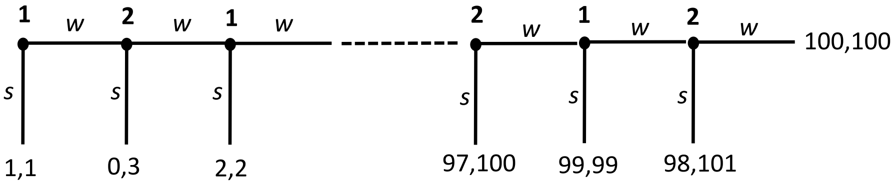


Setup

Two players 1 and 2 take turns in choosing whether to continue or to stop

- Player 1 moves first; Player 2 moves after Player 1 provided Player 1 decided to continue, and vice-versa
- The game reaches a terminal node if either player decides to stop, or after each player decided to continue T times

Centipede Game



Setup

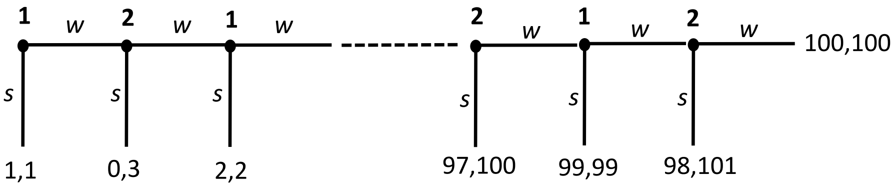
Payoffs are given as follows:

Each player start with £1 in their pile

Every time each player decides to continue, £1 is deducted from their pile and £2 are added to their opponents

Their payoff equals the amount of money they have in their pile at the time they reach a terminal node

Centipede Game



Setup

$A_i := \{0, 1\}$, $S_i := A_i^T$, $s_i = (a_{i,t})_{t \in [T]}$

Let $a_{1,0} = a_{2,0} = 1$.

$$u_1(s_1, s_2) := 1 + 2 \sum_{t \in [T]} \left(\prod_{\ell \in [t]} a_{1,\ell} a_{2,\ell} \right) - \sum_{t \in [T]} \left(\prod_{\ell \in [t]} a_{1,\ell} a_{2,\ell-1} \right).$$

$$u_2(s_1, s_2) := 1 + 2 \sum_{t \in [T]} \left(\prod_{\ell \in [t]} a_{1,\ell} a_{2,\ell-1} \right) - \sum_{t \in [T]} \left(\prod_{\ell \in [t]} a_{1,\ell} a_{2,\ell} \right).$$

Centipede Game

Analysis

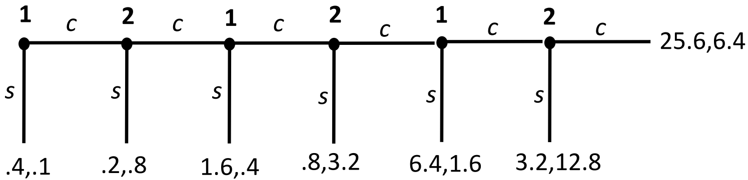
The last subgame has Player 2 can either decides between continuing and getting £$1 + T - 1$ and stopping and getting £$1 + T$; then, by backward induction, $a_{2,T} = 0$.

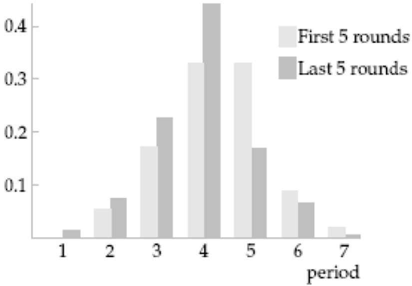
Then, as Player 2 stops in the last subgame, Player 1 prefers to stop and get £$1 + T - 1$, rather then continuing and get £$1 + T - 1 - 1$.

Iterating backward, we'll find that the unique subgame perfect equilibrium has both players always stopping and getting £1!

Zermelo's theorem: no two terminal histories with the same payoff, hence unique NE = unique SPNE, obtained by backward induction

Centipede Game





Experimental results (McKelvey and Palfrey, 1992)

McKelvey and Palfrey (1992)

Centipede Game

Why?

- Monetary payoffs don't capture how players evaluate the outcome
  - This doesn't dent at the theory then: we just have the wrong payoff function
- How soon they stop depends on their beliefs on their opponent's strategic sophistication
  - E.g. chess players stop earlier when playing other chess players, and the earlier the higher their ranking (Palacios-Huerta and Volij, 2009)
  - Stop later the less strategically sophisticated they perceive their opponent
- People may have limited foresight (inability to reason many steps ahead) and rely on heuristics
  - Forward-looking behavior often requires considering many contigencies, making issues fairly complicated