Tutorial 1 : Social Choice, Inequality and Poverty

(i) A population consists of H individuals distinguished by pretax income y_i and unobserved characteristics a_i, $i = 1, \ldots, H$. Mean income is $\bar{y} = \frac{1}{H}\sum_i y_i$. The government can redistribute income by imposing a proportional tax on income at rate $t \in [0, 1]$ and paying out a uniform grant G. However administrative costs mean that resources are lost in the process so that the value of the grant affordable is only $\phi t \bar{y}$ where $\phi < 1$. After-tax incomes are therefore

$$x_i \;=\; y_i(1-t) + \phi t \bar{y} \qquad i = 1, \ldots, H.$$

Individual preferences over tax rates are summarised in utility functions

$$U_i(t) = V(x_i, a_i)$$

where nothing is known about the form of dependence on a_i and it is known only that V depends positively on x_i for all values of a_i.

Government wishes to base decisions about which tax rate to implement on a social preference ordering which is based on individual preferences as captured in these utilities $U_i(t)$, $i = 1, \ldots, H$.

(a) i. Explain what it means for preferences to be *single-peaked* and show that individual preferences are single-peaked over tax rates $t \in [0, 1]$.

ii. Majority voting regards one tax rate t^A as weakly socially preferred to another t^B if and only if no fewer individuals prefer t^A to t^B than prefer t^B to t^A. Given that individual preferences are single peaked, explain why majority voting gives a social preference ordering that is complete, transitive, nondictatorial and satisfies both the Pareto principle and independence of irrelevant alternatives.

iii. Explain what a *Condorcet winner* is and why there is a Condorcet winning tax rate in this economy. What is the Condorcet winning tax rate?

(b) Suppose instead that it is known that individual utilities do not depend on characteristics a_i so that $U_i(t) = V(x_i)$, but that it is still true that nothing is known about dependence on x_i except that V is increasing. Explain why it is now possible, unlike before, to base social preference on the utility of the worst-off person, $W_\infty(t) = (1-t)y_{min} + \phi t\bar{y}$. Show that the optimal tax rate according to such a social preference ordering depends critically on a comparison of ϕ and $y_{min}/\bar{y}$.

(c) i. Suppose instead that individual utilities are known to have the form

$$U_i(t) = \lambda V(x_i) + F(a_i)$$

where λ is an unknown positive constant which is the same for all individuals, F is some unknown function of individual characteristics but V is a *known* increasing, concave function of x_i. Explain why it is now feasible to base social judgment on the Benthamite social welfare function

$$W(t) \;=\; \sum_i U_i(t).$$

ii. Explain what it means for a social welfare function to be *Schur-concave* in after-tax incomes and why this might be desirable. Show that $W(t)$ is Schur-concave.

iii. Suppose that $V(x_i) = \ln x_i$. Show that $W'(0)$ depends on

$$\mathcal{I} \;=\; 1 - \frac{\hat{y}}{\bar{y}}$$

where

$$\hat{y} = \frac{H}{\sum_i (1/y_i)}$$

is the harmonic mean income. Interpret $\mathcal{I}$ and discuss why it might be relevant to tax setting.

(d) Discuss alternative assumptions on cardinality and comparability of utilities and their relationship to possible social welfare functions.

(e) Suppose that the government's objective is to minimise the proportion of the population with posttax income falling below a poverty line z. What now is the optimum tax rate? How sensible an objective is this if the government is concerned about poverty?

Tutorial 2 : Commodity Taxation

Answer either question

(i) Individuals in a population supply labour for wages w and consume two goods, food q_1 and clothing q_2. They have preferences captured in indirect utility functions

$$V(w, p_1, p_2, Y) = \ln\left[\frac{wp_1}{(p_1 - w)\, p_2}\right] + \frac{Y}{w}$$

where p_1 and p_2 are the prices of the two goods and Y is non-labour income. You should assume throughout that values of w, p_1, p_2 and Y are such that all individuals demand positive values of both goods and supply positive hours of labour (so, in particular, $p_1 > w$ for all individuals).

(a) Explain why quantities demanded of the two goods are

$$q_1 = \frac{w^2}{p_1\,(p_1 - w)} \qquad q_2 = \frac{w}{p_2}$$

and therefore shares of total spending on the two goods are given by

$$\frac{p_1 q_1}{p_1 q_1 + p_2 q_2} = \frac{w}{p_1} = 1 - \frac{p_2 q_2}{p_1 q_1 + p_2 q_2}.$$

Suppose that wages in the population are identical. For administrative reasons, the government can raise funds only by proportional taxes at rates t_1 and t_2 on food and fuel spending. Pretax prices of both goods are equal to one so $p_1 = 1 + t_1$, $p_2 = 1 + t$.

(b) Show that revenue raised R satisfies

$$\frac{R}{w} = \frac{t_1 w}{(1 + t_1)(1 + t_1 - w)} + \frac{t_2}{1 + t_2}$$

(c) Suppose that the government wants to maximise consumer welfare subject to raising revenue $\bar{R}$. Explain why the problem can be represented as one with Lagrangean

$$\ln(1+t_1) - \ln(1+t_1-w) - \ln(1+t_2) \\ - \lambda\left[\frac{\bar{R}}{w} - \frac{t_1 w}{(1+t_1)(1+t_1-w)} - \frac{t_2}{1+t_2}\right]$$

(d) Show that optimum tax rates t_1^* and t_2^* satisfy

$$\frac{t_2^*}{1+t_2^*} = \frac{t_1^*}{1+t_1^*}\left[1+\frac{1+t_1^*}{1+t_1^*-w}\right]$$

and therefore $t_2^* > t_1^*$ unless $t_2^* = t_1^* = 0$ and $\bar{R} = 0$. Discuss in terms of the elasticities of demands for the two goods.

Suppose instead that individuals differ in wages.

(e) Discuss how this would alter considerations relevant to the choice of which good to tax more heavily.

(ii) The government is concerned about excessive consumption of sugar, particularly among low income households. It is considering introducing a tax on sugar and preliminary studies using aggregate data have suggested that the price elasticity for sugar is substantial. However it is not yet convinced that price responses at the lower end of the income distribution are as high as those at higher incomes and is unsure therefore whether the policy will be effective in reducing sugar consumption of those households which give it greatest concern. Further research is commissioned and you are engaged to advise.

It is proposed to estimate a demand system based on the following specification of the indirect utility function:

$$v(y, \mathbf{p}) = \frac{\ln(y/m(\mathbf{p}))}{g(\mathbf{p})}$$

where y is total budget and $m(\cdot)$ and $g(\cdot)$ are appropriate functions of prices $\mathbf{p} = (p_0, p_1, \ldots, p_n)$, there being $n+1$ goods and p_0 denoting the price of sugar.

(a) What sort of demand specification is this and what homogeneity properties are required of the $m(\cdot)$ and $g(\cdot)$ functions? Show that uncompensated budget shares $w(y, \mathbf{p})$ are

$$w_i(y, \mathbf{p}) = \eta_i(\mathbf{p}) + \epsilon_i(\mathbf{p})\ln(y/m(\mathbf{p})) \qquad i = 0, 1, \ldots, n$$

where $\eta_i(\mathbf{p}) = \partial \ln m(\mathbf{p})/\partial \ln p_i$ and $\epsilon_i(\mathbf{p}) = \partial \ln g(\mathbf{p})/\partial \ln p_i$ are elasticities of the $m(\cdot)$ and $g(\cdot)$ functions.

(b) It is proposed to adopt forms for the $m(\cdot)$ and $g(\cdot)$ functions as follows:

$$\begin{aligned}
\ln m(\mathbf{p}) = A \quad &+ \quad \sum_{i=0}^{n} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \beta_{ij} \ln p_i \ln p_j \\
\ln g(\mathbf{p}) = \quad & \quad \sum_{i=0}^{n} \gamma_i \ln p_i + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \delta_{ij} \ln p_i \ln p_j
\end{aligned}$$

Derive budget share equations and explain the restrictions on parameters required to ensure adding up and homogeneity.

(c) Since the budget share equations are not linear in parameters, suggest a method for estimating the parameters of the demand system.

(d) Given that

$$\frac{\partial \ln q_i}{\partial \ln p_i} = -1 + \frac{1}{w_i} \frac{\partial w_i}{\partial \ln p_i}$$

derive restrictions on parameters necessary for the uncompensated own price elasticity of demand for sugar to be independent of total budget y. Comment therefore on the importance of the terms in your budget share equations for addressing the government's concerns about variation of price elasticities with income.

Tutorial 3 : Income Taxation

(i) Consider a population consisting of individuals of two equally numerous types, A and B. The labour of high productivity types L^A produces output of one unit for each unit of labour supplied, $w^A = 1$, while labour of low productivity types L^B produces output of $\alpha < 1$ units per unit of labour supplied, $w^B = \alpha$. Individuals have identical preferences. Someone supplying L units of labour and consuming c has utility given by

$$U \;=\; \ln c + \ln(1 - L).$$

(a) Suppose there is no taxation and there are no sources of income other than labour. Show that both types choose to supply half a unit of labour, $L^A = L^B = \frac{1}{2}$.

(b) Suppose that a linear tax is introduced at rate τ on labour income, funding a uniform lump sum grant of G. Consumption is therefore given by

$$c^h \;=\; w^h(1-\tau)L + G \qquad h = A,\ B.$$

i. Show that each type chooses to supply

$$L^h \;=\; \frac{1}{2} - \frac{1}{2}\frac{G}{w^h(1-\tau)} \qquad h = A,\ B$$

and that utilities are therefore

$$U^h \;=\; 2\ln\left(w^h(1-\tau) + G\right) - \ln\left(w^h(1-\tau)\right) - 2\ln 2 \qquad h = A,\ B$$

ii. If the government budget balances so that

$$G \;=\; \frac{1}{2}\tau\left(w^A L^A + w^B L^B\right)$$

then show that

$$G \;=\; \frac{\frac{1}{2}\bar{w}\tau(1-\tau)}{1 - \frac{1}{2}\tau}$$

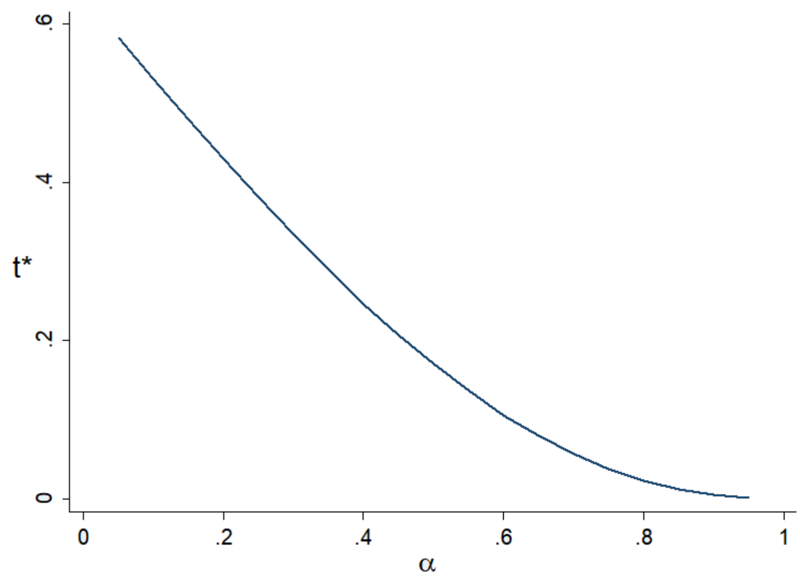
where $\bar{w} = \left(w^A + w^B\right)/2 = (1+\alpha)/2$. Comment.

iii. Suppose the government wants to maximise utilitarian social welfare $\Omega = U^A + U^B$. Explain why this is the same as choosing τ to solve

$$\max_{\tau}\left\{\frac{1}{2}\ln(1-\tau) - \ln\left(1-\frac{1}{2}\tau\right)\right.$$
$$\left.+\frac{1}{2}\ln\left(1+\frac{1}{2}\left(\frac{\bar{w}}{w^A}-1\right)\tau\right)+\frac{1}{2}\ln\left(1+\frac{1}{2}\left(\frac{\bar{w}}{w^B}-1\right)\tau\right)\right\}$$

Comment on Figure 3.1 which shows how optimal tax rate τ^* varies with α.

Figure 3.1: Optimum linear income tax rate



(c) Suppose now that the government can set an arbitrary nonlinear tax on labour income.

i. Explain why the Lagrangean for the government's problem can now be written

$$\begin{aligned}&\ln c^A + \ln c^B + \ln\left(1-L^A\right) + \ln\left(1-L^B\right)\\ &\quad - \lambda\left[c^A + c^B - L^A - \alpha L^B\right]\\ &\quad + \mu\left[\ln c^A - \ln c^B + \ln\left(1-L^A\right) - \ln\left(1-\alpha L^B\right)\right]\end{aligned}$$

with controls c^A, c^B, L^A and L^B.

ii. At the optimum, marginal rates of substitution between consumption and leisure are

$$\frac{c^A}{1-L^A} = 1 \qquad \frac{c^B}{1-L^B} = \alpha\frac{1-\mu-\alpha(1-\mu)L^B}{1-\mu\alpha-\alpha(1-\mu)L^B} < \alpha < 1.$$

Interpret and assess the significance of this.

(d) Discuss how far these results are indicative of general results for economies with many different labour types.

Tutorial 4 : Public Goods

(i) Individuals in a population of N individuals, $i \in I$, have endowments of a private good, bread, ω_i. They have preferences over consumption of bread q_i and of a public good, circuses, Q, as represented by Cobb-Douglas utility functions

$$u_i = \alpha_i \ln q_i + (1 - \alpha_i) \ln Q$$

where α_i are individual-specific preference parameters. There is a constant-returns-to-scale technology linking the two goods with one unit of bread needed to produce p units of circuses.

(a) Suppose production of circuses is by private contribution so that $Q = p \sum_i g_i$ where $q_i = \omega_i - g_i$. Consider the Nash equilibrium in which each individual contribution $\hat{g}_i$ is chosen optimally given the chosen contribution of every other individual. Let $C = \{i \mid \hat{g}_i > 0\}$ be the set of contributors in equilibrium and $Y^C = p \sum_{i \in C} \omega_i$ be the value of their collective endowment. Establish expressions for each individual's consumption of bread and the collective circus provision. Show that equilibrium circus provision is independent of the distribution of endowments among contributors. What must be true of the distribution of endowments for everyone to contribute?

(b) Suppose now that the government decides to provide the circuses with funds raised through taxation. Say first that it divides the bread cost of provision Q/p equally across all voters and that provision is decided by exhaustive pairwise majority voting. Explain why there is a Condorcet winning provision and what it is.

(c) Say instead that the government is elected with a mandate to implement a social consensus on maximising a particular social welfare function

$$W = \sum_{i \in I} \beta_i \left[\alpha_i \ln (\omega_i - T_i) + (1 - \alpha_i) \ln Q\right]$$

by choice of Q and individual tax payments T_i with the constraint that

$$p \sum_{i \in I} T_i \geq Q.$$

The welfare weights β_i sum to unity, $\sum_{i \in I} \beta_i = 1$. Find the government's optimal circus provision and the tax payments to fund it.

(d) What are the Lindahl prices at this social optimum? Show that the government's optimum coincides with a notional competitive equilibrium reached by imposing an appropriate set of lump sum transfers and individual-specific public good prices set at these Lindahl prices. Discuss.

(e) Discuss issues raised by the problems of identifying variation across individuals in the preference parameters α_i.

Tutorial 1 : Social Choice, Inequality and Poverty

(i) A population consists of H individuals distinguished by pretax income y_i and unobserved characteristics a_i, $i = 1, \ldots, H$. Mean income is $\bar{y} = \frac{1}{H}\sum_i y_i$. The government can redistribute income by imposing a proportional tax on income at rate $t \in [0, 1]$ and paying out a uniform grant G. However administrative costs mean that resources are lost in the process so that the value of the grant affordable is only $\phi t \bar{y}$ where $\phi < 1$. After-tax incomes are therefore

$$x_i \;=\; y_i(1-t) + \phi t \bar{y} \qquad i = 1, \ldots, H.$$

Individual preferences over tax rates are summarised in utility functions

$$U_i(t) = V(x_i, a_i)$$

where nothing is known about the form of dependence on a_i and it is known only that V depends positively on x_i for all values of a_i.

Government wishes to base decisions about which tax rate to implement on a social preference ordering which is based on individual preferences as captured in these utilities $U_i(t)$, $i = 1, \ldots, H$.

(a) i. Explain what it means for preferences to be *single-peaked* and show that individual preferences are single-peaked over tax rates $t \in [0, 1]$.

Ans: Preferences are single peaked if the policy is unidimensional, each individual has a most-preferred policy and prefers policies which are closer to their most preferred whenever comparing policies on the same side of it. In this case that is so since individual post-tax income $(1-t)y_{min} + \phi t \bar{y}$ *is either increasing or decreasing everywhere depending upon whether* $\phi \bar{y} > y$ *or* $\phi \bar{y} < y$.

ii. Majority voting regards one tax rate t^A as weakly socially preferred to another t^B if and only if no fewer individuals prefer t^A to t^B than prefer t^B to t^A. Given that individual preferences are single peaked, explain why majority voting gives a social preference ordering that is complete, transitive, nondictatorial and satisfies both the Pareto principle and independence of irrelevant alternatives.

Ans: Preferences are complete since there is always a weak majority one way or the other. They are nondictatorial since no one individual can decide social preference against a majority in the other direction. They satisfy the Pareto principle since one tax rate has a majority over another if everyone prefers it. They satisfy IIA because social preference between two tax rates depends only on individual preferences between those two tax rates. The problem with majority voting is that, in general, transitivity is not guaranteed because of the possible existence of Condorcet cycles. However, single peakedness of preferences rules out the possibility of Condorcet cycles since the middle tax rate in any triple can never be anyone's least preferred.

iii. Explain what a *Condorcet winner* is and why there is a Condorcet winning tax rate in this economy. What is the Condorcet winning tax rate?

Ans: A Condorcet winner is a policy that has a majority over all others in pairwise voting. A Condorcet winner exists if preferences are single-peaked as they are here. The Condorcet winner in such a case is the preference of the voter with median income y_{med} so is either 1 or 0 depending upon whether $\phi\bar{y} > y_{med}$ or $\phi\bar{y} < y_{med}$.

(b) Suppose instead that it is known that individual utilities do not depend on characteristics a_i so that $U_i(t) = V(x_i)$, but that it is still true that nothing is known about dependence on x_i except that V is increasing. Explain why it is now possible, unlike before, to base social preference on the utility of the worst-off person, $W_\infty(t) = (1-t)y_{min} + \phi t\bar{y}$. Show that the optimal tax rate according to such a social preference ordering depends critically on a comparison of ϕ and $y_{min}/\bar{y}$.

Ans: The worst-off individual is identifiable as the person with lowest income y_{min}. $W_{-\infty}$ is linear in t and either increasing or decreasing depending upon whether $\phi\bar{y} - y_{min}$ is positive or negative. If $\phi > y_{min}/\bar{y}$ then optimum t is 1 whereas if $\phi < y_{min}/\bar{y}$ optimum t is 0. Note that $\mathcal{I}_{-\infty} = 1 - y_{min}/\bar{y}$ is an Atkinson inequality index for Rawlsian social preferences so the optimum depends on a comparison of administrative losses to inequality.

(c) i. Suppose instead that individual utilities are known to have the form

$$U_i(t) = \lambda V(x_i) + F(a_i)$$

where λ is an unknown positive constant which is the same for all individuals, F is some unknown function of individual characteristics but V is a *known* increasing, concave function of x_i. Explain why it is now feasible to base social judgment on the Benthamite social welfare function

$$W(t) \ = \ \sum_i U_i(t).$$

Ans: Even though there is still an unobserved individual component to utility $F(a_i)$ it disappears in *comparisons of social welfare.*

$$\sum_i U_i(t^A) - \sum_i U_i(t^B) = \lambda \sum_i \left[V(y_i(1-t^A)+\phi t^A \bar{y}) - V(y_i(1-t^B)+\phi t^B \bar{y})\right]$$

and although λ is unknown it is positive and therefore doesn't affect the direction of social preference.

ii. Explain what it means for a social welfare function to be *Schur-concave* in after-tax incomes and why this might be desirable. Show that $W(t)$ is Schur-concave.

Ans: Schur concavity means that social welfare increases with transfers from richer to poorer individuals. This is true if $[\partial W/\partial y_i - \partial W/\partial y_j](y_i - y_j) < 0$ which is so here because of the concavity of V.

iii. Suppose that $V(x_i) = \ln x_i$. Show that $W'(0)$ depends on

$$\mathcal{I} = 1 - \frac{\hat{y}}{\bar{y}}$$

where

$$\hat{y} = \frac{H}{\sum_i (1/y_i)}$$

is the harmonic mean income. Interpret $\mathcal{I}$ and discuss why it might be relevant to tax setting.

Ans: $W_0'(0) = \int (\phi\bar{y}/y - 1)\ \mathrm{d}F(y)$ which is positive only if $\phi > \frac{1}{\bar{y}}/\int \frac{1}{y}\ \mathrm{d}F(y)$. Note that $\mathcal{I}$ is another Atkinson inequality index.

(d) Discuss alternative assumptions on cardinality and comparability of utilities and their relationship to possible social welfare functions.

Ans: Cardinal Ratio-Scale comparability would allow more general homothetic social welfare aggregates. Numerical Full Comparability would allow general social welfare functions.

(e) Suppose that the government's objective is to minimise the proportion of the population with posttax income falling below a poverty line z. What now is the optimum tax rate? How sensible an objective is this if the government is concerned about poverty?

Ans: If $\phi\bar{y} \geq z$ then setting $t = 1$ can eliminate poverty. In fact it is only necessary to set $t = (z - y_{min})/(\phi\bar{y} - y_{min})$ to achieve this. However if $\phi\bar{y} < z$ the number of posttax poor is minimised at $t = 0$. This rather sharp discontinuity is a consequence of the crude way of measuring poverty. Minimising the posstax shortfall or a more sophisticated poverty index would accept the transfer of income from people above z to reduce the poverty gaps of the poorest individuals as something that could be traded off against the increase in the number of posttax poor as a consequence of positive t.

Tutorial 2 : Commodity Taxation

Answer either question

(i) Individuals in a population supply labour for wages w and consume two goods, food q_1 and fuel q_2. They have preferences captured in indirect utility functions

$$V(w, p_1, p_2, Y) = \ln \left[\frac{wp_1}{(p_1 - w)\, p_2} \right] + \frac{Y}{w}$$

where p_1 and p_2 are the prices of the two goods and Y is non-labour income. You should assume throughout that values of w, p_1, p_2 and Y are such that all individuals demand positive values of both goods and supply positive hours of labour (so, in particular, $p_1 > w$ for all individuals).

(a) Explain why quantities demanded of the two goods are

$$q_1 = \frac{w^2}{p_1\,(p_1 - w)} \qquad q_2 = \frac{w}{p_2}$$

and therefore shares of total spending on the two goods are given by

$$\frac{p_1 q_1}{p_1 q_1 + p_2 q_2} = \frac{w}{p_1} = 1 - \frac{p_2 q_2}{p_1 q_1 + p_2 q_2}.$$

Ans: Use Roy's identity. $V_1 = 1/p_1 - 1/(p_1 - w)$, $V_2 = -1/p_2$, $V_w = 1/w + 1/(p_1 - w) - Y/w^2$ *and* $V_Y = 1/w$. *So* $q_1 = -V_1/V_Y = w^2/p_1(p_1 - w)$, $q_2 = -V_2/V_Y = w/p_2$ *and* $L = V_w/V_Y = p_1/(p_1 - w) - Y/w$. *Total spending on goods is* $p_1q_1 + p_2q_2 = wL + Y = wp_1/(p_1 - w)$ *from which the budget shares follow. Food is a higher share and fuel a lower share for those with higher wages.*

Suppose that wages in the population are identical. For administrative reasons, the government can raise funds only by proportional taxes at rates t_1 and t_2 on food and fuel spending. Pretax prices of both goods are equal to one so $p_1 = 1 + t_1$, $p_2 = 1 + t$.

(b) Show that revenue raised R satisfies

$$\frac{R}{w} = \frac{t_1 w}{(1+t_1)(1+t_1-w)} + \frac{t_2}{1+t_2}$$

Ans: Revenue (given the unit pretax prices) is $R = t_1 q_1 + t_2 q_2 = t_1 w^2/(1+t_1)(1+t_1-w) + t_2 w/1 + t_2$.

(c) Suppose that the government wants to maximise consumer welfare subject to raising revenue $\bar{R}$. Explain why the problem can be represented as one with Lagrangean

$$\ln(1+t_1) - \ln(1+t_1-w) - \ln(1+t_2) \\ -\lambda\left[\frac{\bar{R}}{w} - \frac{t_1 w}{(1+t_1)(1+t_1-w)} - \frac{t_2}{1+t_2}\right]$$

Ans: Utility is

$$V(w, 1+t_1, 1+t_2, Y) = \ln w + \ln(1+t_1) - \ln(1+t_1-w) - \ln(1+t_2) + \frac{Y}{w}$$

from which the first and last terms can be ignored since they do not depend on t_1 *or* t_2. *The remainder of the Lagrangean reflects the revenue constraint.*

(d) Show that optimum tax rates t_1^* and t_2^* satisfy

$$\frac{t_2^*}{1+t_2^*} = \frac{t_1^*}{1+t_1^*}\left[1 + \frac{1+t_1^*}{1+t_1^*-w}\right]$$

and therefore $t_2^* > t_1^*$ unless $t_2^* = t_1^* = 0$ and $\bar{R} = 0$. Discuss in terms of the elasticities of demands for the two goods.

Ans: The first order condition with respect to t_2 *is*

$$\frac{1}{1+t_2} - \frac{\lambda}{(1+t_2)^2} = 0$$

emphfrom which we infer $\lambda = 1 + t_2$. The first order condition with respect to t_1 is more complicated:

$$\frac{w(\lambda-1)}{(1+t_1)(1+t_1-w)} - \lambda\frac{t_1 w}{(1+t_1)(1+t_1-w)}\left[\frac{1}{1+t_1} + \frac{1}{1+t_1-w}\right] = 0$$

but dividing through by $w/(1+t_1)(1+t_1-w)$ *and substituting* $\lambda = 1+t_2$ *gives an expression which can be rearranged into that given. Snce* $t_2/(1+t_2)$ *equals* $t_1/(1+t_1)$ *multiplied by factor greater than one* $t_2 > t_1$ *unless* $t_2 = t_1 - 0$. *Since individuals are identical there is no redistributive purpose to taxation. The optimal tax rates raise the required revenue with least deadweight loss. As is well known from the Ramsey analysis, this involves equal discouragement to compensated demands of the two goods which will tend to imply heavier taxes on less elastic demands. The own price elasticity of fuel is* -1 *whereas that of food is more negative than* -1 *so the greater tax rate of fuel is in accord with this.*

Suppose instead that individuals differ in wages.

(e) Discuss how this would alter considerations relevant to the choice of which good to tax more heavily.

Ans: Redistributive considerations now enter. Because food is a greater share of the budget of those with high wages there will be reason to tax food more heavily to balance against the deadweight loss considerations (in line with the Diamond-Mirrlees extension of the Ramsey rules).

(ii) The government is concerned about excessive consumption of sugar, particularly among low income households. It is considering introducing a tax on sugar and preliminary studies using aggregate data have suggested that the price elasticity for sugar is substantial. However it is not yet convinced that price responses at the lower end of the income distribution are as high as those at higher incomes and is unsure therefore whether the policy will be effective in reducing sugar consumption of those households which give it greatest concern. Further research is commissioned and you are engaged to advise.

It is proposed to estimate a demand system based on the following specification of the indirect utility function:

$$V(y, \mathbf{p}) = \frac{\ln (y/m(\mathbf{p}))}{g(\mathbf{p})}$$

where y is total budget and $m(\cdot)$ and $g(\cdot)$ are appropriate functions of prices $\mathbf{p} = (p_0, p_1, \ldots, p_n)$, there being $n+1$ goods and p_0 denoting the price of sugar.

(a) What sort of demand specification is this and what homogeneity properties are required of the $m(\cdot)$ and $g(\cdot)$ functions? Show that uncompensated budget shares $w(y, \mathbf{p})$ are

$$w_i(y, \mathbf{p}) = \eta_i(\mathbf{p}) + \epsilon_i(\mathbf{p}) \ln (y/m(\mathbf{p})) \qquad i = 0, 1, \ldots, n$$

where $\eta_i(\mathbf{p}) = \partial \ln m(\mathbf{p})/\partial \ln p_i$ and $\epsilon_i(\mathbf{p}) = \partial \ln g(\mathbf{p})/\partial \ln p_i$ are elasticities of the $m(\cdot)$ and $g(\cdot)$ functions.

Ans: These are PIGLOG preferences. In order for $V(y, \mathbf{p})$ to be homogeneous of degree zero in y and $\mathbf{p}$ we require $m(\mathbf{p})$ to be homogeneous of degree one and $g(\mathbf{p})$ to be homogeneous of degree zero.

Budget shares can be found using Roy's identity.

$$\begin{aligned} w_i(y, \mathbf{p}) &= -\frac{\partial V/\partial \ln p_i}{\partial V/\partial \ln y} \\ &= -\left[-\frac{1}{g}\frac{\partial \ln m}{\partial \ln p_i} - \ln(y/m)\frac{1}{g^2}\frac{g}{\ln p_i}\right] \Big/ \frac{1}{g} \end{aligned}$$

(b) It is proposed to adopt forms for the $m(\cdot)$ and $g(\cdot)$ functions as follows:

$$\begin{aligned}\ln m(\mathbf{p}) = A \quad + \quad & \sum_{i=0}^{n} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \beta_{ij} \ln p_i \ln p_j \\ \ln g(\mathbf{p}) = \qquad & \sum_{i=0}^{n} \gamma_i \ln p_i + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \delta_{ij} \ln p_i \ln p_j\end{aligned}$$

Derive budget share equations and explain the restrictions on parameters required to ensure adding up and homogeneity.

Ans: The elasticities of the price indices $m(\mathbf{p})$ and $gt(\mathbf{p})$ take forms conveniently linear in logs of prices

$$\begin{aligned}\frac{\partial \ln m}{\partial \ln p_i} &= \alpha_i + \sum_j \beta^*_{ij} \ln p_j \\ \frac{\partial \ln g}{\partial \ln p_i} &= \gamma_i + \sum_j \delta^*_{ij} \ln p_j\end{aligned}$$

where $\beta^*_{ij} = \frac{1}{2}[\beta_{ij} + \beta_{ji}]$ and $\delta^*_{ij} = \frac{1}{2}[\delta_{ij} + \delta_{ji}]$. Hence

$$w_i(y, \mathbf{p}) = \alpha_i + \sum_j \beta^*_{ij} \ln p_j + \left(\gamma_i + \sum_j \delta^*_{ij} \ln p_j \right) \ln(y/m(\mathbf{p}))$$

If $\delta^*_{ij} = 0$ for all i and j then this is the Almost Ideal demand system.

Adding up requires $\sum_i w_i = 1$ for all y and $\mathbf{p}$ which is satisfied if $\sum \alpha_i = 1$, $\sum_i \beta^*_{ij} = 0$, $\sum_i \gamma_i = 0$ and $\sum_i \delta^*_{ij} = 0$.

Homogeneity requires $w_i(\lambda y, \lambda \mathbf{p}) = w_i(y, \mathbf{p})$ which is satisfied if $\sum_j \beta^*_{ij} = 0$, $\sum_j \delta^*_{ij} = 0$ and $m(\mathbf{p})$ is homogeneous of degree one.

(c) Since the budget share equations are not linear in parameters, suggest a method for estimating the parameters of the demand system.

The demand system can be estimated iteratively. The nonlinearity comes only through the role of the total budget deflator $m(\mathbf{p})$. Begin with an approximation to $m(\mathbf{p})$, such as a Stone price index $\ln m_0(\mathbf{p}) = \sum_i \bar{w}_i \ln p_i$, estimate the parameters linearly and use the estimates to update $m(\mathbf{p})$ iteratively until convergence. Adding up and homogeneity can be imposed by omitting the budget share equation for one good and expressing all nominal variables relative to the price of this good. Slutsky symmetry requires $\beta_{ij}* = \beta^*_{ji}$ and $\delta^*_{ij} = \delta^*_{ji}$ which can be imposed as a cross-equation restriction during estimation or afterwards by using, say, minimum distance techniques.

(d) Given that

$$\frac{\partial \ln q_i}{\partial \ln p_i} = -1 + \frac{1}{w_i} \frac{\partial w_i}{\partial \ln p_i}$$

derive restrictions on parameters necessary for the uncompensated own price elasticity of demand for sugar to be independent of total budget y. Comment therefore on the importance of the terms in your budget share equations for addressing the government's concerns about variation of price elasticities with income.

Ans: Using the budget share equation

$$\frac{\partial \ln q_i}{\partial \ln p_i} = -1 + \frac{\beta^*_{ii} - \left(\gamma_i \ + \ \sum_j \delta^*_{ij} \ln p_j\right) \partial \ln m / \partial \ln p_i + \delta^*_{ii} \ln(y/m)}{\alpha_i \ + \ \sum_j \beta^*_{ij} \ln p_j + \left(\gamma_i \ + \ \sum_j \delta^*_{ij} \ln p_j\right) \ln(y/m)}$$

The price elasticity is independent of y only if $\gamma_i = \delta^*_{ij} = 0$ which is to say that preferences are homothetic. If $\gamma_i \neq 0$ but $\delta^*_{ij} = 0$ (ie preferences are Almost Ideal) then the elasticity depends on y because the budget share varies with y but it is either everywhere price elastic or everywhere price inelastic and either everywhere increasing or everywhere decreasing according to whether the good is a necessity or a luxury. Allowing $\delta^*_{ij} \neq 0$ allows much greater freedom in dependence of price elasticities on y.

Tutorial 3 : Income Taxation

(i) Consider a population consisting of individuals of two equally numerous types, A and B. The labour of high productivity types L^A produces output of one unit for each unit of labour supplied, $w^A = 1$, while labour of low productivity types L^B produces output of $\alpha < 1$ units per unit of labour supplied, $w^B = \alpha$. Individuals have identical preferences. Someone supplying L units of labour and consuming c has utility given by

$$U = \ln c + \ln(1 - L).$$

(a) Suppose there is no taxation and there are no sources of income other than labour. Show that both types choose to supply half a unit of labour, $L^A = L^B = \frac{1}{2}$.

Ans: With no other source of income $U = \ln(w^h L) + \ln(1 - L)$, $h = A, B$. *Maximisation requires* $1/L = 1/(1 - L)$ *and therefore* $L = \frac{1}{2}$ *whatever* w^h.

(b) Suppose that a linear tax is introduced at rate τ on labour income, funding a uniform lump sum grant of G. Consumption is therefore given by

$$c^h = w^h(1 - \tau)L + G \qquad h = A,\ B.$$

i. Show that each type chooses to supply

$$L^h = \frac{1}{2} - \frac{1}{2}\frac{G}{w^h(1 - \tau)} \qquad h = A,\ B$$

and that utilities are therefore

$$U^h = 2\ln\left(w^h(1 - \tau) + G\right) - \ln\left(w^h(1 - \tau)\right) - 2\ln 2 \qquad h = A,\ B$$

Ans: With the tax and grant $U = \ln(w^h(1 - \tau)L^h + G) + \ln(1 - L^h)$, $h = A, B$. *First order condition is* $w^h(1-\tau)/(w^h(1-\tau)L^h+G) = 1/(1-L^h)$ *and therefore* $2w^h(1-\tau)L^h = w^h(1-\tau)-G$ *from which the labour supply expression follows. Hence* $c^h = w^h(1 - L^h) = \frac{1}{2}[w^h(1 - \tau) + G]$ *and substituting these into the utility function gives the expression for indirect utility.*

ii. If the government budget balances so that

$$G = \frac{1}{2}\tau\left(w^A L^A + w^B L^B\right)$$

then show that

$$G = \frac{\frac{1}{2}\bar{w}\tau(1-\tau)}{1-\frac{1}{2}\tau}$$

where $\bar{w} = \left(w^A + w^B\right)/2 = (1+\alpha)/2$. Comment.

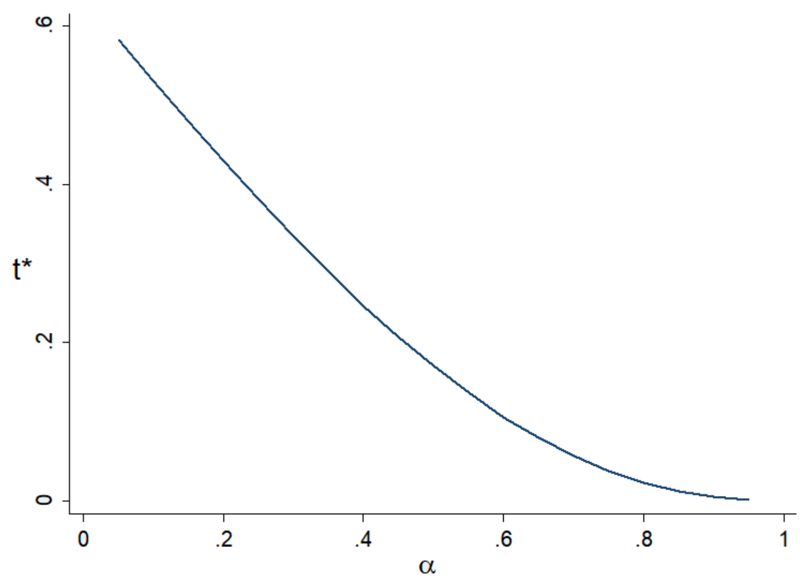
Ans: Revenue raised from each type is $\tau w^h L^h = \frac{1}{2}\tau[w^h - \frac{G}{1-\tau}]$. *Equating the average revenue gained to the grant disbursed means* $\frac{1}{2}\tau[\bar{w} - \frac{G}{1-\tau}] = G$. *Collecting the terms in* G *an rearranging gives the expression for* G. *The grant affordable is increasing with the tax rate but reaches a maximum for some* t *between* $\frac{1}{2}$ *and* 1.

iii. Suppose the government wants to maximise utilitarian social welfare $\Omega = U^A + U^B$. Explain why this is the same as choosing τ to solve

$$\max_{\tau}\left\{\frac{1}{2}\ln(1-\tau) - \ln\left(1-\frac{1}{2}\tau\right) + \frac{1}{2}\ln\left(1+\frac{1}{2}\left(\frac{\bar{w}}{w^A}-1\right)\tau\right) + \frac{1}{2}\ln\left(1+\frac{1}{2}\left(\frac{\bar{w}}{w^B}-1\right)\tau\right)\right\}$$

Comment on Figure ?? which shows how optimal tax rate τ^* varies with α.

Figure 3.1: Optimum linear income tax rate



Ans: Substituting the expression for the grant into that for utility

$$\begin{aligned} U^h = & \quad 2\ln\left(w^h(1-\tau)+\frac{\frac{1}{2}\bar{w}\tau(1-\tau)}{1-\frac{1}{2}\tau}\right)-\ln w^h-\ln(1-\tau)-2\ln 2 \\ = & \quad \ln w^h+\ln(1-\tau)+2\ln\left(1-\frac{1}{2}\tau+\frac{\frac{1}{2}\bar{w}\tau}{w^h}\right)-\ln\left(1-\frac{1}{2}\tau\right)-2\ln 2 \end{aligned}$$

The component $\ln w^h - 2\ln 2$ *is unaffected by the chosen tax rate so can be ignored if trying to maximise the sum of utilities. The expression proposed for maximisation is then proportional to the sum of what is left.*

The first order condition for maximising,

$$\frac{1}{2}\left\{-\frac{1}{1-\tau}+\frac{1}{1-\frac{1}{2}\tau}-\frac{\beta^A}{1+\beta^A\tau}+\frac{\beta^B}{1+\beta^B\tau}\right\} \;=\; 0$$

where $\beta^h = \frac{1}{2}\left(\frac{\bar{w}}{w^B}-1\right)$*, is a cubic in* τ *which is solved by the values illustrated in Figure* **??***. The optimum is zero if there is no inequality,* $\alpha = 1$*, and increases as* α *decreases.*

(c) Suppose now that the government can set an arbitrary nonlinear tax on labour income.

i. Explain why the Lagrangean for the government's problem can now be written

$$\begin{aligned} & \ln c^A+\ln c^B+\ln\left(1-L^A\right)+\ln\left(1-L^B\right) \\ & \quad -\lambda\left[c^A+c^B-L^A-\alpha L^B\right] \\ & \quad +\mu\left[\ln c^A-\ln c^B+\ln\left(1-L^A\right)-\ln\left(1-\alpha L^B\right)\right] \end{aligned}$$

with controls c^A, c^B, L^A and L^B.

Ans: $c^A+c^B-L^A-\alpha L^B \leq 0$ *is simply the government budget constraint. The additional constraint* $\ln c^A - \ln c^B + \ln\left(1-L^A\right) - \ln\left(1-\alpha L^B\right) \geq 0$ *is an incentive compatibility constraint specifying that the more able should have no incentive to pose as less able.*

ii. At the optimum, marginal rates of substitution between consumption and leisure are

$$\frac{c^A}{1-L^A} \;=\; 1 \qquad \frac{c^B}{1-L^B} \;=\; \alpha\frac{1-\mu-\alpha(1-\mu)L^B}{1-\mu\alpha-\alpha(1-\mu)L^B} \;<\; \alpha \;<\; 1.$$

Interpret and assess the significance of this.

Ans: The first expression says that the MRS for the more able equals their wage, $w^A = 1$*. The labour supply of the more able is therefore undistorted at the optimum. The second expressions says that the MRS of the less able is below their wage,* $w^B = \alpha$*. The consequence of incentive compatibility is that the labour supply of the less able is distorted.*

(d) Discuss how far these results are indicative of general results for economies with many different labour types.

Ans: Adding further labour types adds further incentive compatibility constraints. It remains true that the labour supply of the most able is undistorted at the optimum. In the limit as we approach a continuum of types we get a problem analysable using optimum control techniques in the fashion of Mirrlees.

Tutorial 4 : Public Goods

(i) Individuals in a population of N individuals, $i \in I$, have endowments of a private good, bread, ω_i. They have preferences over consumption of bread q_i and of a public good, circuses, Q, as represented by Cobb-Douglas utility functions

$$u_i = \alpha_i \ln q_i + (1 - \alpha_i) \ln Q$$

where α_i are individual-specific preference parameters. There is a constant-returns-to-scale technology linking the two goods with one unit of bread needed to produce p units of circuses.

(a) Suppose production of circuses is by private contribution so that $Q = p \sum_i g_i$ where $q_i = \omega_i - g_i$. Consider the Nash equilibrium in which each individual contribution $\hat{g}_i$ is chosen optimally given the chosen contribution of every other individual. Let $C = \{i \mid \hat{g}_i > 0\}$ be the set of contributors in equilibrium and $Y^C = p \sum_{i \in C} \omega_i$ be the value of their collective endowment. Establish expressions for each individual's consumption of bread and the collective circus provision. Show that equilibrium circus provision is independent of the distribution of endowments among contributors. What must be true of the distribution of endowments for everyone to contribute?

Ans: Individual choices solve

$$\max_{g_i} \alpha_i \ln [\omega_i - g_i] + (1 - \alpha_i) \ln \left(p \sum_{j \in C} g_j \right)$$

which is the same as

$$\max_{q_i} \alpha_i \ln q_i + (1 - \alpha_i) \ln \left(Y^C - p \sum_{j \in C} q_j \right)$$

First order conditions require

$$pq_i = \frac{\alpha_i}{1 - \alpha_i} \left(Y^C - p \sum_{j \in C} q_j \right)$$

Summing over the set of M contributors gives

$$p \sum_{j \in C} q_j = \frac{A_C}{1 + A_C} Y^C$$

where $A_C = \sum_{j \in C} \frac{\alpha_i}{1-\alpha_i}$ so that

$$\begin{aligned} Q &= \frac{1}{1 + A_C} Y^C & \\ pq_i &= \frac{\alpha_i}{(1-\alpha_i)(1 + A_C)} Y^C & i \in C \\ pq_i &= p\omega_i & i \notin C \end{aligned}$$

Notice that Q and $q_i,\ i \in C$ depend only upon Y^C so that the distribution of endowments among contributors has no bearing on the equilibrium quantities so long as the set of contributors stays the same.

Individuals contribute if the value of their endowment exceeds the value of their equilibrium private consumption if a contributor. Hence all contribute if

$$p\omega_i > \frac{\alpha_i}{(1+\alpha_i)(1 + A_I)} Y$$

where Y is the economy-wide endowment $Y = p \sum_{i \in I} \omega_i$, which will be true if the smallest value of $\omega_i(1-\alpha_i)/\alpha_i$ exceeds $Y/p(1 + A_I)$

(b) Suppose now that the government decides to provide the circuses with funds raised through taxation. Say first that it divides the bread cost of provision Q/p equally across all voters and that provision is decided by exhaustive pairwise majority voting. Explain why there is a Condorcet winning provision and what it is.

Ans: Each individual's preferred level solves

$$\max_Q \alpha_i \ln(\omega_i - Q/pN) + (1-\alpha_i) \ln Q$$

which has an optimum $Q = (1-\alpha_i)p\omega_i N$. The Condorcet winning level is the choice of the voter with median value of $(1-\alpha_i)\omega_i$.

(c) Say instead that the government is elected with a mandate to implement a social consensus on maximising a particular social welfare function

$$W = \sum_{i \in I} \beta_i \left[\alpha_i \ln(\omega_i - T_i) + (1-\alpha_i) \ln Q\right]$$

by choice of Q and individual tax payments T_i with the constraint that

$$p \sum_{i \in I} T_i \geq Q.$$

The welfare weights β_i sum to unity, $\sum_{i\in I}\beta_i = 1$. Find the government's optimal circus provision and the tax payments to fund it.

Ans: The government's problem is

$$\max_{T_1,\ldots,T_N} \sum_{i\in I} \beta_i\alpha_i \ln(\omega_i - T_i) + (1-\alpha_i)\ln\left(p\sum_{i\in I} T_i\right)$$

This is a straightforward Cobb-Douglas maximisation with solutions

$$\begin{aligned} Q &= \sum_i \beta_i(1-\alpha_i)\,Y \\ pq_i &= \beta_i\alpha_i\,Y \end{aligned}$$

(d) What are the Lindahl prices at this social optimum? Show that the government's optimum coincides with a notional competitive equilibrium reached by imposing an appropriate set of lump sum transfers and individual-specific public good prices set at these Lindahl prices. Discuss.

Ans: Lindahl prices, π_i, equal the private good price times the marginal rates of substitution

$$\begin{aligned} \frac{\pi_i}{p} &= \frac{\partial u_i/\partial Q}{\partial u_i/\partial q_i} \\ &= \frac{(1-\alpha_i)/\sum_j \beta_j(1-\alpha_j)Y}{\alpha_i p/\beta_i\alpha_i Y} \;=\; \frac{\beta_i(1-\alpha_i)}{p\sum_j \beta_j(1-\alpha_j)} \end{aligned}$$

In the Lindahl equilibrium, endowments need to be such that individuals demand the socially optimal quantities at the Lindahl prices. It is therefore necessary to impose lump sum taxes δ_i such that

$$\begin{aligned} (1-\alpha_i)p(\omega_i-\delta_i)/\pi_i &= (1-\alpha_i)(\omega_i-\delta_i)/p\sum_j \beta_j(1-\alpha_j)/\beta_i(1-\alpha_i) \\ &= \sum_j \beta_j(1-\alpha_j)\,Y \\ \alpha_i p(\omega_i-\delta_i) &= \beta_i\alpha_i\,Y \end{aligned}$$

which are simultaneously satisfied by $\delta_i = \omega_i - \beta_i Y/p$. As a competitive equilibrium, the Lindahl equilibrium is Pareto efficient and public provision satisfies the Samuelson condition since $\sum_{i\in I}\pi_i = 1$.

(e) Discuss issues raised by the problems of identifying variation across individuals in the preference parameters α_i.

Ans: Implementation of Lindahl equilibrium would require knowledge of the Lindahl prices which in turn would require knowledge of preference parameters α_i. But individuals have no incentive to reveal these if they expect to be charged higher for declaring a stronger preference for the public good. Mechanisms such as Clarke-Groves schemes are designed to incentivise truthful revelation. Individuals' declared preferences *affect the quantity of the public good provided but contributions depend on the excess of cost of provision over other individuals' declared willingness to pay.*