Econ0059 Final Exam 2021 SOLUTIONS

Question 1.

Two profit-maximizing traders in a market for a financial asset choose their market positions. The traders' choices are independent. The position of each trader i, denoted x_i, can take any value in the interval [-1,1]. Note that negative positions are allowed, representing short selling. The asset's net market value is a function of the traders' positions, given by $v = -(x_1 + x_2)$. Note that the net value can be negative as well. A trader's profit is $x_i \cdot v$.

(a) Formulate the interaction as a strategic game.

(b) Is the game strictly competitive?

No. When $x_1=x_2=0$, both players get 0. But when $x_1,x_2>0$, both players get negative payoffs.

(c) Find each trader's max-min strategy and max-min payoff, both without and with randomization.

The game is symmetric. Focus on trader 1, w.l.o.g. Her expected payoff from a lottery p over x_1 is

$$-\sum_{x_1} \boldsymbol{p(x_1)x_1(x_1+x_2) = -E(x_1)^2 - x_2 \cdot E(x_1)}$$

where expectations are taken according to p. Therefore, if $E(x_1)>0$, the worst-case scenario is $x_2=1$; and if $E(x_1)<0$, the worst-case scenario is $x=-1$. In both cases, trader 1's payoff is strictly negative. In contrast, if $x_1=0$, the trader ensure a payoff of 0. Therefore, this is the max-min.

(d) Characterize the set of Nash equilibria in the game (allowing for mixed strategies).

This is just like a Cournot game with linear demand P = A - Q, where A=0. The FOC gives us $x_1=x_2=0$, hence this is the unique Nash eq.

(e) Characterize the set of rationalizable outcomes in the game.

The game is symmetric. Focus on trader 1, w.l.o.g. The derivative w.r.t x_1 is $-2x_1-x_2$. We can

see that it is negative for every x_1>0.5 regardless of x_2, and it is positive for every x_1<-0.5, regardless of x_2. Therefore, all actions outside [-0.5,0.5] are strictly dominated and therefore eliminated in the first round. In the second round, using the same logic, all actions outside [-0.25,0.25] are eliminated. In this manner, we can see that after an infinitely number of elimination rounds, the only rationalizable action is x=0.

Now suppose that when each trader *i* chooses x_i, her actual realized position is $\theta_i x_i$, where $\theta_i \in \{0,1\}$ is an exogenous variable that indicates whether the trader's orders are fed into the market's clearinghouse. The two values of θ_i are equally likely, independently for each trader. The asset's net value is $v = -(\theta_1 x_1 + \theta_2 x_2)$. Before trader *i* makes her choice, she learns the value of θ_i and receives a binary signal regarding the realization of θ_j, j≠i. The signal's precision is q∈(½,1) – for every θ_j, the signal is equal to θ_j with probability q.

(f) Formulate the interaction as a Bayesian game and characterize the set of pure-strategy Nash equilibria in it.

The information structure is as follows. A state of the world is (θ_1,s_1,θ_2,s_2). Trader i's signal in the state is (θ_i,s_i). The prior is:

p(1,1,1,1)=p(0,0,0,0)=p(1,1,0,0)=p(0,0,1,1)=0.25q^2

p(1,0,1,0)=p(0,1,0,1)=p(1,0,0,1)=p(0,1,1,0)=0.25$(1\text{-}q)^2$

p(1,1,1,0)=p(1,0,1,1)=p(0,0,1,0)=p(0,1,1,1)=p(0,1,0,0)=p(0,0,0,1)=p(0,1,0,0) =p(1,1,0,1)=0.25q(1-q)

When trader 1 learns that θ_1=0, her action doesn't matter and therefore every action is consistent with Nash eq. Now suppose the trader learns that θ_1=1 and some signal regarding θ_2. Denote this composite signal by t_1. The trader forms a belief regarding the opponent's expected realized position, denoted E($\theta_2 x_2|t_1$). She chooses x_1 to maximize – x_1[x_1+E($\theta_2 x_2|t_1$)]. Her best-reply is x^*_1 = -0.5E($\theta_2 x_2|t_1$). Likewise, player 2's best-reply when she learns θ_2=1 and some signal regarding θ_1 is x^*_2 = -0.5E($\theta_1 x_1|t_2$). Now, E($\theta_1 x_1|t_2$) is some

weighted average of x^*_1 and zero, and likewise, $E(\theta_2 x_2|t_1)$ is some weighted average of x^*_2 and zero. Therefore, the only possible solution is $x^*_1=x^*_2=0$. Therefore, in Nash equilibrium, traders choose $x=0$ when $\theta=1$, and what they do when $\theta=0$ is indeterminate.

Question 2.

Consider the following signalling game. The seller of a car observes its quality θ, which can be low ($\theta = L$) or high ($\theta = H$). The seller offers a price $p \in [0, \infty)$. The buyer observes the offered price but not the quality of the car. The buyer chooses either "accept" or "reject." If the buyer accepts, the seller's payoff is p and the buyer's payoff is $v_\theta - p$. If the buyer rejects, the seller's payoff is w_θ and the buyer's payoff is 0. There is a commonly known probability $q \in (0,1)$ that the car is of high quality, and v_H, v_L, w_H, w_L are known constants.

First consider the case where $v_H > v_L > w_H > w_L > 0$.

(a) For which values of p is there a pooling Perfect Bayesian Equilibrium (PBE) in which price p is offered? For each of these values of p, fully characterize at least one such PBE.

(b) Is there a separating PBE? If so, fully characterize at least one. If not, explain why.

Now consider the case where $v_H > w_H > v_L > w_L > 0$.

(c) For which values of p is there a pooling PBE where price p is offered? For each of these values of p, fully characterize at least one such PBE.

(d) Now suppose $v_H = 10$, $w_H = 6$, $v_L = 4$, $w_L = 2$, and $q = \frac{1}{2}$. Fully characterize the set of PBE in which (i) the seller of one quality always offers price $p = z$; (ii) the seller of the other quality offers the same price z with probability $t \in (0,1)$ but a higher price z' with probability $1 - t$; (iii) offer z is always accepted; and (iv) offer z' is accepted with probability $h \in (0,1)$ and rejected otherwise.

Solutions:

(a) There is such a PBE for $p \in [v_L, qv_H + (1-q)v_L]$. For each of these values there is an equilibrium in which the seller believes that the car is of low quality out of equilibrium path. For $p < v_L$ there is no pooling PBE, as the seller can always offer v_L; this offer will be accepted and is profitable. For $p > qv_H + (1-q)v_L$ the offer will be rejected; thus switching to $p = v_L$ will again be a profitable deviation for the seller.

(b) No. First, note that there is never an offer with a price below v_L in equilibrium because, if there was, the seller would switch to v_L and have that offer accepted.
Now, if both offers are accepted in equilibrium, the type that proposes a lower offer will switch to the other one. If one offer is accepted and the other is not, the seller would always prefer to switch to the offer which is accepted. Finally, if neither is accepted in equilibrium, the seller will prefer to switch to $p = v_L$, which is always accepted.

(c) If $w_H \leq qv_H + (1-q)v_L$, there is such a PBE for $p \in [w_H, qv_H + (1-q)v_L]$; otherwise there is none. Indeed, for $p < w_H$ the seller of a high-quality car can always choose not to sell (by setting a very high price) and get a payoff of w_H. For $p > qv_H + (1-q)v_L$ the offer will be rejected; thus switching to $p = v_L$ will be a profitable deviation for the seller for the low-quality car seller. For $p \in [w_H, qv_H + (1-q)v_L]$ it is easy to see that this is an equilibrium if the buyer believes that the car is of low quality after observing any other offer.

(d) We solve for the equilibria in which the high type (i.e. seller of high-quality cars) randomizes.

- After seeing z', the buyer knows it's a high-quality car. Since the buyer randomizes between accepting and rejecting, she must be indifferent between those options, and thus $z' = 10$.
- The high-type seller is indifferent between getting *z* for sure and getting 10 with probability h or 6 with probability $1-h$. Thus, $z = 10h + 6(1-h) = 6 + 4h$.
- The low-type seller should weakly prefer getting *z* for sure rather than 10 with probability h or 2 with probability $1-h$. Clearly, the latter option is worse for this buyer than it is for the high type who is indifferent. Thus, the low-type seller indeed prefers *z*. (By the same logic there are no equilibria in which the low type randomizes.)
- By the Bayes rule, the belief $\mu = Pr(H \mid z) = \frac{0.5t}{0.5t+0.5} = \frac{t}{t+1}$.
- The buyer needs to be weakly better off accepting *z* than rejecting it. Thus, $6 + 4h \leq \frac{t}{t+1} \cdot 10 + \frac{1}{t+1} \cdot 4$, which yields $h \leq \frac{4t-2}{4t+4}$. This can be satisfied at $h > 0$ only for $t > \frac{1}{2}$.
- Suppose as usual that after seeing any offer that is not z or z', the buyer believes it's a low-quality car valued at 4, and would accept only offers at or below 4. Then no seller would want to switch to those prices since they can guarantee a payoff of $z > 4$.

In sum, we have a set of PBE with the high-quality seller randomizing between $z = 6 + 4h$ and $z' = 10$, the low-quality seller offering *z*; with $t > \frac{1}{2}$ and $0 < h \leq \frac{4t-2}{4t+4}$; with beliefs following the Bayes rule on equilibrium path and the belief of a low type with probability 1 out of the equilibrium path; and the buyer accepting 10 with probability *h*; accepting *p*=*z* or any price $p \leq$

4 always but rejecting otherwise.

Question 3.

Consider the following two-player, two-stage stage. In the first stage, players 1 and 2 play the stage game on the left side below, with player 1 picking a row and player 2 simultaneously picking a column. The chosen actions are then revealed, and players proceed to play the stage game on the right-hand side below, with player 1 again choosing a row and player 2 choosing a column. The payoffs in this two-stage game are the sum of the payoffs in the two stages with no discounting. Consider pure strategies only throughout the question.

	C	D
A	2, 2	-1, 3
B	3, -1	0, 0

	G	H
E	0, 0	3, 1
F	1, 3	0, 0

(a) Are there subgame-perfect equilibria in which (A,C) is chosen in the first stage? If so, *fully* characterize *all* such equilibria. If not, explain why.

(b) Are there subgame-perfect equilibria in which (B,C) is chosen in the first stage? If so, *fully* characterize *all* such equilibria. If not, explain why.

In the rest of the problem consider the following modification of the game: after choosing and observing the first-stage actions (but before choosing the second-stage actions), the two players together roll one die and thus observe one random variable *x* (same for both players) that takes values 1, …, 6 with equal probabilities. Players maximize expected total payoff.

(c) Describe the set of each player's pure strategies in this game.

(d) Are there subgame-perfect equilibria in which (A,C) is chosen in the first stage? If so, *fully* characterize *at least one* such equilibrium. If not, explain why.

Solutions:
(a) No. The stage-game Nash equilibria are (B,D) in the first stage and (F,G) and (E,H) in the second stage.
In the second stage of any SPE, a Nash equilibrium has to be played. Suppose there was an SPE with (A,C) played in the first stage. On the equilibrium path, one of the players is earning 1 in the second stage. This player then wants to deviate in the first stage: she would earn +1 in the first stage and cannot lose anything in the second stage.

(b) Yes. First, there are no equilibria in which (E,H) in played in the second stage on the equilibrium path. This is because player 2 would want to deviate in period 1: she would earn +1 in the first stage and would not lose anything in the second stage.
Now consider equilibria in which (F,G) is played in the second stage on the equilibrium path, i.e. after observing (B,C). For player 1 not to want to deviate to A in the first stage (losing 1 unit of payoff), (F,G) should be played after (A,C) as well. For player 2 not to want to deviate to D (earning +1), (E,H) should be played after (B,D). Either (F,G) or (E,H) can be played after (A,D) since this does not matter, according to the one-shot deviation principle. We have therefore described two equilibria.

(c) Strategies of player 1 are combinations of:

- A or B
- E or F for each tuple (p,q,x), where p=A,B, q=C,D, and x=1,…,6.

Strategies of player 2 are symmetric.

(d) Yes. Now different second-stage Nash equilibria can be played for different values of x. Here is one SPE:

- Play (A,C) in the first-stage
- After observing (A,C), play (F,G) if x=1,…,3 and (E,H) if x=4,…,6.
- After observing (B,C), play (F,G) regardless of x
- After observing (A,D), play (E,H) regardless of x
- After observing (B,D), play some combination of Nash equilibria across values of x (doesn’t matter which one).

On equilibrium path, both players have expected second-stage payoff of 2 if they play (A,C), but are punished by 1 if they unilaterally deviate, and that is just enough to deter deviations.