ECON0106: **Microeconomics**

1. Choice, Preferences, Utility*

Duarte Gonçalves†

University College London

1. Overview

Economic theory studies the behavior of agents: to predict what we expect there to happen, to explain why we observe a particular regularity, to recommend a particular course of action. At its core we find a model, a stylized but informative representation of the situation being studied. Our goal is to develop building blocks that can and have been used to model a wide variety of questions, e.g. consumer demand and firm pricing, student applications to university, voting, technology adoption, hospital residency program management, etc.

The three main approaches that have been taken are to represent an agent's behavior by means of their choices, their preferences, or a utility function. We often work directly with the assumption that agents choices are described by utility maximization: agents choose an alternative x from a set of feasible alternatives S to maximize their utility u.

However, utility functions are not directly observable: we just observe their choices. How then can we make sure that the utility function we are using is the right one? By assuming a particular utility function, we are implicitly making assumptions on how agents behave, on their preferences, that may or may not be reasonable assumptions, depending on the application at hand. Hence, we will be paying some attention on how properties of utility functions relate to properties of the agents preferences, and how those, in turn, relate to properties of their choices. This will enable us to *test* our models, as we can *identify* the assumptions underlying them. On a more pragmatic level, while assumptions are often for the sake of tractability — a model is, after all, a simplified description of reality — studying their properties and their *empirical content* allows us to better understand the limitations of our models.

*Last updated: 11 October 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk.

2. Choice and Preferences

We will start by fixing a finite set of alternatives X and consider all possible subsets $2^X := \{A \mid A \subseteq X\}$. The agent can then choose from a subset of alternatives, $A \in 2^X$, which we model via a choice function:

Definition 1. A **choice function** is a function $C : 2^X \to 2^X$ such that $C(A) \subseteq A \ \forall A \in 2^X$. We further require choice functions to be **nonempty**, that is, $\forall A \neq \emptyset$, $C(A) \neq \emptyset$.[1]

In short, a choice function determines the agent's choices in every possible situation.

Another way to model behavior is by considering preference relations on X. We say that $\succsim \subseteq X \times X$ is a **binary relation** on X and if $(x, y) \in \succsim$, we will often write $x \succsim y$ (or $y \precsim x$). Let us introduce some properties that binary relations can satisfy.

Definition 2. We say that a binary relation $\succsim$ on X is

- **reflexive** if, $\forall x \in X$, $x \succsim x$;
- **transitive** if, $\forall x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$;
- **negatively transitive** if, $\forall x, y, z \in X$, $x \succsim y$, then $x \succsim z$ or $z \succsim y$;
- **complete**[2] if, $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$;
- **antisymmetric** if, $\forall x, y \in X$, $x \succsim y$ and $y \succsim x$ implies $x = y$;
- **symmetric** if, $\forall x, y \in X$, $x \succsim y$ implies $y \succsim x$;
- **asymmetric** if, $\forall x, y \in X$, $x \succsim y$ implies $\neg(y \succsim x)$.

The binary relation is then given different names when it satisfies different properties:

Definition 3. A binary relation $\succsim$ is called

(i) a **preorder** if it is reflexive and transitive;

(ii) a **partial order** if it is reflexive, transitive, and antisymmetric (an antisymmetric preorder);

(iii) a **linear order** (or total order) if it reflexive, transitive, antisymmetric, and complete (a complete partial order).

[1] This does not mean that we don't allow the agent to "choose/do nothing"; rather, that we will make "choose/do nothing" an element of X.

[2] In order theory, especially outside economics, you may also find this property being called (strongly) connected, total, or connex.

In each of those cases, $(X, \succsim)$ is called (i) a preordered set, (ii) a partially ordered set, and (iii) and linearly or totally ordered set. Some examples for $(X, \succsim)$: (i) people in a room and their height (having the same height does not mean they are the same person), (ii) $\mathbb{R}^n$ and the natural product order $x \geq y \Longleftrightarrow x_i \geq y_i, i = 1, .., n$, (iii) $\mathbb{R}$ and the natural order.

Throughout, we will assume that **preference relations** are complete and transitive and when $x \succsim y$ we say that x is *weakly preferred* to y. We allow the agent to be *indifferent* between two alternatives x and y, that is, $x \succsim y$ and $y \succsim x$, in which case we write $x \sim y$. Note that $x \sim y$ does not imply $x = y$: the agent may be indifferent between an apple and a banana, but that does not mean that they are the same element (this is why we don't require that $\succsim$ be antisymmetric). We say that x is *strictly preferred* to y if $x \succsim y$ and $\neg(y \succsim x)$, and we write $x \succ y$.[3] Often, $\succ$, the asymmetric or strict part of $\succsim$, is called a strict preference relation, whereas $\sim$ is called an indifference relation, corresponding to the symmetric part of $\succsim$. Note that the asymmetric ($\succ$) and symmetric ($\sim$) parts of $\succsim$ can be defined for any binary relation $\succsim \subseteq X \times X$, and that $\succsim = \succ \cup \sim$.

The next proposition shows that if we are given a strict preference relation, we can recover the original preferences:

Proposition 1. A binary relation $\succsim \subseteq X \times X$ is complete and transitive if and only if its asymmetric part, $\succ \subseteq X \times X$, is asymmetric and negatively transitive.

Exercise 1. (i) If you are given a strict preference relation $\succ$, how do you recover the weak preference relation $\succsim$?

(ii) Prove Proposition 1.

3. Revealed Preference

For a preference relation $\succsim$ on X, define, for every $A \in 2^X$, $\arg\max_{\succsim} A := \{x \in A \mid x \succsim y \text{ for all } y \in A\}$, the set of maximizers in A, that is, the most preferred elements in A. We want to understand when can we represent an agent's choices as being driven by preference maximization.

Let us note some properties of $\arg\max_{\succsim}$:

Proposition 2. Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{\succsim} A$ and $y \in \arg\max_{\succsim} B$, $x \succsim y$.

(ii) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{\succsim} A$, then $x \in \arg\max_{\succsim} B$.

[3] We will also equivalently use the expressions "x (weakly/strictly) dominates y".

(iii) For any nonempty $A \subseteq X$, $\arg\max_{\succsim} A \neq \emptyset$.

(iv) For $x, y \in A \subseteq X$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{\succsim} A$.

Proof. (i) As $x \in \arg\max_{\succsim} A \iff x \succsim z \, \forall z \in A$, and $y \in B \subseteq A$, the result follows.

(ii) As $x \in \arg\max_{\succsim} A \iff x \succsim z \, \forall z \in A$ and $B \subseteq A$, then $x \succsim z \, \forall z \in B \iff x \in \arg\max_{\succsim} B$.

(iii) As X is finite, A is finite. For any $A \in 2^X$ such that $|A| = 1$, then $A = \arg\max_{\succsim} A$ as $x \succsim x$ (by completeness), and therefore $x \sim x$. For the purpose of induction, suppose that for any $B \in 2^X$ such that $B \neq \emptyset$ and $|B| = n \geq 1$, $\arg\max_{\succsim} B \neq \emptyset$. Take any $A \in 2^X$ such that $|A| = n + 1$; we want to show that $\arg\max_{\succsim} A \neq \emptyset$. By definition, $A = B \cup \{x\}$, where $|B| = n$, and, for any $y, z \in \arg\max_{\succsim} B \neq \emptyset$, by completeness, $y \succsim x$ or $x \succsim y$. If the former, then we have that $y \in \arg\max_{\succsim} A$, as $y \succsim z \;\; \forall z \in B$ and $y \succsim x$. If the latter, then as $x \succsim y$ and $y \in \arg\max_{\succsim} B \iff y \succsim z \;\; \forall z \in B$, by transitivity, $x \succsim z \;\; \forall z \in B$, and hence $x \in \arg\max_{\succsim} A$.

(iv) Let $\{x, y\} \subseteq A$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$. Without loss of generality, suppose $x \in \arg\max_{\succsim} A$. As $y \sim x \implies y \succsim x \succsim z \;\; \forall z \in A$, by transitivity $y \succsim z \, \forall z \in A \iff y \in \arg\max_{\succsim} A$. For the other direction, if $\{x, y\} \subseteq \arg\max_{\succsim} A$, then by definition of $\arg\max_{\succsim}$, $x \succsim y$ and $y \succsim x$.

$\square$

Claim (i) in Proposition 2 states that when the set of feasible alternatives expands, the agent is always weakly better off. This is understandable, as whatever they could choose before is still available. Claim (ii) tells us that if a $\succsim$-maximizer of a set A is also a $\succsim$-maximizer of any of its subsets. This is commonly referred to a **independence of irrelevant alternatives**. Claim (iii) is showing that if we consider a finite set, then there is always one element that is weakly preferred to every element in the set — a claim that does not necessarily hold if the set is not finite. Finally, property (iv) says not only that the agent must be indifferent between any two $\succsim$-maximizers, but also that, if the agent is indifferent between two elements, either they are both $\succsim$-maximizers or neither is.

Exercise 2. Show that, if $B \subseteq A$, then $B \cap \arg\max_{\succsim} A \subseteq \arg\max_{\succsim} B$.

Exercise 3. For a finite set X and a binary relation $\succ$ on X, let the set of **maximal** elements of subset $A \subseteq X$ be defined as those for which there is no element that dominates them $\text{MAX}_{\succ} A := \{x \in A \mid \nexists y \in A : y \succ x\}$.

(i) Show that if $\succsim$ is a preference relation and $\succ$ its asymmetric part, then $\arg\max_{\succsim} A = \text{MAX}_{\succ} A \ \forall A \in 2^X$.

(ii) Now suppose that $\succsim$ is reflexive and transitive, but not necessarily complete. What is the relation between $\arg\max_{\succsim} A$ and $\text{MAX}_{\succ} A \ \forall A \in 2^X$?

(iii) Prove that $\text{MAX}_{\succ} : 2^X \to 2^X$ is a choice function if and only if $\succ$ is an **acyclic** binary relation on X, i.e. there is no sequence $x_1, x_2, ..., x_n \in X$ such that $x_1 \succ x_2 \succ \cdots \succ x_n \succ x_1$.

Let us introduce the following property on choice functions due to Houthakker (1956):

Definition 4. A choice function $C : 2^X \to 2^X$ satisfies **Houthakker's Axiom of Revealed Preference** (HARP) if $\forall x, y \in X$, $\{x, y\} \subseteq A \cap B$, $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$.

You will find that HARP is oftentimes called the *weak axiom of revelead preference*.

As the next result shows, if an agent always chooses their preferred elements in the feasible set, then their choices satisfy HARP. But if an agent's choices satisfy HARP, we can interpret these choices as maximizing a preference relation; and, importantly, we can recover their preferences by observing their choices.

Theorem 1. Let X be a finite set. A choice function $C : 2^X \to 2^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \ \forall A \in 2^X$.

Proof. $\Longrightarrow$: (only if) Define $\succsim \subseteq X \times X$ as follows: $\forall x, y \in X$, $x \succsim y$ if $\exists A \in 2^X$ such that $x, y \in A$ and $x \in C(A)$. Completeness of $\succsim$ follows from the fact that, $\forall x, y \in X$, as $C(\{x, y\})$ is nonempty and a subset of $\{x, y\}$, then $x \in C(\{x, y\}) \Longrightarrow x \succsim y$ or $y \in C(\{x, y\}) \Longrightarrow y \succsim x$.

To show transitivity, let $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$; we want to show $x \succsim z$. By definition of $\succsim$, $\exists A \ni x, y$ and $B \ni y, z$ such that $x \in C(A)$ and $y \in C(B)$. Now we want to find a set $E \ni x, z$ and show that $x \in C(E) \Longrightarrow x \succsim z$ (by definition of $\succsim$). Take $\{x, y, z\}$. If $x \in C(\{x, y, z\})$, we are done. If $y \in C(\{x, y, z\})$, as $x \in C(A)$ and $x, y \in A \cap \{x, y, z\}$, by HARP we have that $x \in C(\{x, y, z\})$ and the result follows. And if $z \in C(\{x, y, z\})$, as $y \in C(B)$ and $y, z \in B \cap \{x, y, z\}$, HARP implies that $y \in C(\{x, y, z\})$ and we are back to the previous case, where we showed that $x \in C(\{x, y, z\})$.

We then need to show that $C(A) = \arg\max_{\succsim} A$, $\forall A \in 2^X$. By definition of $\succsim$, $x \in C(A) \Longrightarrow x \succsim y \ \forall y \in A$, which, by definition of $\arg\max_{\succsim} A$ implies that $x \in \arg\max_{\succsim} A$; hence $C(A) \subseteq$

$\arg\max_{\succsim} A$. Now, we show that $\arg\max_{\succsim} A \subseteq C(A)$ to conclude that $\arg\max_{\succsim} A = C(A)$. Take $x \in \arg\max_{\succsim} A$ $(\subseteq A)$. This implies that $A \neq \emptyset$ and thus that $\exists y \in C(A)$ (as choice functions on nonempty sets are nonempty). As $x \in \arg\max_{\succsim} A$ and $y \in A$ implies that $x \succsim y$, then, by how $\succsim$ was defined, $\exists B \in 2^X$ such that $x, y \in B$ and $x \in C(B)$. As $x, y \in A \cap B$, $x \in C(B)$ and $y \in C(A)$, by HARP, $x \in C(A)$; that is, $x \in \arg\max_{\succsim} A \Longrightarrow x \in C(A)$.

$\Longleftarrow$: (if) For some preference relation $\succsim \subseteq X \times X$, define $C : 2^X \to 2^X$ such that $C(A) = \arg\max_{\succsim} A$ $\forall A \in 2^X$. By definition of $\arg\max_{\succsim} A$, $C(A) \subseteq A$; and by Proposition 2(ii), $A \neq \emptyset \Longrightarrow C(A) = \arg\max_{\succsim} A \neq \emptyset$. Hence, C is a choice function.

Now we show it satisfies HARP. Take any x, y such that $\{x, y\} \subseteq A \cap B$, $x \in C(A)$, and $y \in C(B)$. As $y \in A$ and $x \in C(A) = \arg\max_{\succsim} A$, then $x \succsim y$; a symmetric argument shows that $y \succsim x$. By Proposition 2(iii), $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} E \Longleftrightarrow \{x, y\} \subseteq \arg\max_{\succsim} E = C(E)$, which applies to $E = A, B$; this concludes the proof. $\square$

Another way to state HARP is by decomposing it in two properties of choice functions $C : 2^X \to 2^X$. The first is Sen's (1971) α:

Property α. If $x \in B \subseteq A \subseteq X$ and $x \in C(A)$, then $x \in C(B)$.

The intuition behind this axiom can be illustrated as follows: if you choose raspberry jam when you can choose between {raspberry, strawberry, blueberry, orange}, then you choose it too when you only {raspberry, strawberry} are available. Note that property α — which refers to the independent of irrelevant alternatives for choice functions — is the counterpart for choice functions to the analogous property we showed for $\arg\max_{\succsim}$ in Proposition 2(i).[4]

The second property is Sen's (1971) β, also called *expansion consistency*:

Property β. If $B \subseteq A \subseteq X$, $x, y \in C(B)$, and $y \in C(A)$, then $x \in C(A)$.

Exercise 4. (i) Show that Sen's α is equivalent to the following property: if $B \subseteq A$, then $B \cap C(A) \subseteq C(B)$.

(ii) Show that Sen's β is equivalent to the following property: if $B \subseteq A$ and $C(A) \cap C(B) \neq \emptyset$, then $C(B) \subseteq C(A)$.

(iii) Let $C : 2^X \to 2^X$ be a choice function. Prove that HARP is equivalent to Sen's α and β. Conclude on the properties that $\arg\max_{\succsim}$ satisfies, where $\arg\max_{\succsim} A := C(A)$, $\forall A \in 2^X$.

[4]While a compeling property, it is also easy to entertain situations where it may fail. For instance, if decision-makers fail to consider all possible alternatives but instead consider only a subset of the available elements, called their consideration set. This is indisputably the case: e.g. Amazon sells over 12 million items and it is unrealistic to think consumers consider all of them. For a conceptualization of consideration sets see e.g. Masatlioglu et al. (2012).

4. Preferences and Utility

We have seen in the previous section necessary and sufficient conditions to interpret an agent's choices as being driven by preference maximization. In this section, we are going to understand in which circumstances we can think of agents' behavior as though they are maximizing a utility function.

Definition 5. A utility function $u : X \to \mathbb{R}$ represents $\succsim \subseteq X \times X$ if $x \succsim y \iff u(x) \geq u(y)$, $\forall x, y \in X$.

For $\succsim \subseteq X \times X$ and its asymmetric part $\succ$ let us define, for any subset $A \subseteq X$,

(i) $A_{\succsim x} := \{y \in A \mid y \succsim x\}$;

(ii) $A_{\succ x} := \{y \in A \mid y \succ x\}$;

(iii) $A_{x \succsim} := \{y \in A \mid x \succsim y\}$; and

(iv) $A_{x \succ} := \{y \in A \mid x \succ y\}$.

These sets can be understood as the alternatives in A that are (i) weakly preferred to x, (ii) strictly preferred to x, (iii) weakly less preferred than x, and (iv) strictly less preferred than x.

Proposition 3. Let X be finite. $\succsim \subseteq X \times X$ is a preference relation if and only if it admits a utility representation u.

Proof. The "if" part is straightforward. For the "only if" part, define $u(x) := |X_{x \succsim}|$. Note that for any $x \succsim y$, $X_{y \succsim} \subseteq X_{x \succsim}$ and therefore $u(x) \geq u(y)$. If $\neg(x \succsim y)$, by completeness we have that $y \succ x$. Transitivity yields $X_{x \succsim} \subseteq X_{y \succsim}$. But $y \in X_{y \succsim}$, as $y \succsim y$, but $y \notin X_{x \succsim}$, and therefore $X_{x \succsim} \subsetneq X_{y \succsim}$, and then $u(y) > u(x)$. □

Are utility representations unique? The answer is no: for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, if u represents a preference relation $\succsim$ on X, then $v := f \circ u$ does too. However,

Proposition 4. If $\succsim$ and $\hat{\succsim}$ are two different preference relations on X, then they cannot be represented by the same utility function u.

Exercise 5. Prove Proposition 4. Conclude that utility representations are unique up to positive monotone transformations.

Can we go beyond finite set of alternatives?

Proposition 5. Let X be countable. $\succsim \subseteq X \times X$ is a preference relation if and only if it admits a utility representation u.

Proof. Again we focus on the "only if" part. Since X is countable, let us fix an order on $X = \{x_1, x_2, \ldots\}$. Because it is not necessarily finite, it can be the case that $|X_{x\succsim}| = |X_{y\succsim}| = \infty$, even if $x \succ y$ (i.e. not $y \succsim x$). Define

$$u(x) := \sum_{n \in \{m \mid x_m \in X_{x\succsim}\}} 2^{-n}.$$

As X is countable, u is well-defined as the sum is finite.
Let $x \succsim y$. Then, $X_{y\succsim} \subseteq X_{x\succsim}$ (transitivity) $\Longrightarrow u(x) \geq u(y)$. If $\neg(x \succsim y)$, then $y \succ x$ and $X_{x\succsim} \subseteq X_{y\succsim}$ $\Longrightarrow u(y) \geq u(x)$. As $y = x_m$ for some finite $m \in \mathbb{N}$, then $u(y) \geq u(x) + 2^{-m} > u(x)$. □

What if X is not countable?

Example 1. The canonical example is with lexicographic preferences in $X = \mathbb{R}^2$, where the agent considers the first dimension and only in case of a tie do they resort to the second dimension: $x \succsim y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. While $\succsim$ is a preference relation on X it admits no utility representation. To see this, suppose it did, $u : X \to \mathbb{R}$. Then, for any $r \in \mathbb{R}$, we have that $u(r,1) > u(r,0)$ as $(r,1) \succ (r,0)$. Moreover, for any $r' > r$, $u(r',0) > u(r,1)$. Then $\{(u(r,0), u(r,1)) \mid r \in \mathbb{R}\}$ is an uncountable collection of nonempty and disjoint open intervals. However, for $r \in \mathbb{R}$, $(u(r,0), u(r,1))$ is a nonempty open interval, and as $\mathbb{Q}$ is dense in $\mathbb{R}$,[5] we can find a rational number $q \in (u(r,0), u(r,1))$. As the set of rational numbers is countable, we obtain a contradiction.

The main intuition for why a utility representation is not possible in Example 1 is that there are 'too many' "indifference sets": every point in $\mathbb{R}^2$ is an indifference set and we want to represent every indifference set with a real number.

Definition 6. Let $\succsim \subseteq X \times X$. A subset $X^* \subseteq X$ is **order-dense** in X if for every $x, y \in X : x \succ y$, there is $z \in X^*$ such that $x \succsim z \succ y$.

As $\mathbb{R}$ has a countable $\geq$-dense subset — e.g. that of the rational numbers — we cannot have more than a countable number of "indifference sets." The next result shows that this is an if and only if condition:

[5] That is, for any $x \in \mathbb{R}$ and any $\epsilon > 0$, $B_\epsilon(x) \cap \mathbb{Q} \neq \emptyset$.

Theorem 2. $\succsim \subseteq X \times X$ is a preference relation and there is a countable order-dense $X^* \subseteq X$ if and only if $\succsim$ admits a utility representation u.

Proof. $\Longrightarrow$: (only if)
Fix an order on $X^* = \{x_1^*, x_2^*, \ldots\}$. Define

$$u(x) := \sum_{n \in \{m \mid x_m \in X_{x\succsim} \cap X^*\}} 2^{-n}.$$

As X^* is countable, u is well-defined as the sum is finite.
Let $x \succsim y$. Then, $X_{y\succsim} \subseteq X_{x\succsim}$ (transitivity) $\Longrightarrow X_{y\succsim} \cap X^* \subseteq X_{x\succsim} \cap X^* \Longrightarrow u(x) \geq u(y)$. If $\neg(x \succsim y)$, then $y \succsim x$ (completeness) $\Longrightarrow X_{x\succsim} \cap X^* \subseteq X_{y\succsim} \cap X^*$. As $\neg(x \succsim y)$ and $y \succsim x$, $y \succ x$ and, as X^* is order-dense in X, there is $x_m^* \in X_{y\succsim} \cap X^*$ and $x_m^* \notin X_{x\succsim} \cap X^*$. We then conclude $u(y) \geq u(x) + 2^{-m} > u(x)$.

$\Longleftarrow$: (if)
That $\succsim \in X \times X$, defined as $x \succsim y$ if and only if $u(x) \geq u(y)$, is complete and transitive is straightforward to verify. Then, let us construct our countable, order-dense $X^* \subseteq X$.
Let $u(X) := \{u(x) \in \mathbb{R} \mid x \in X\}$.
For every $(p,q) \in \mathbb{Q}^2$ such that $p < q$ and $(p,q) \cap u(X) \neq \emptyset$, take one $x_{p,q} \in X$ such that $u(x_{p,q}) \in (p,q)$, and let $X_{p,q} := \{x_{p,q}\}$.
And for every $p \in \mathbb{Q}$ such that $\exists x \in X : u(x) = \inf([p,\infty) \cap u(X))$, take one x_p such that $u(x_p) = \inf([p,\infty) \cap u(X))$, and define $X_p := \{x_p\}$.
By construction, $\cup_{(p,q) \in I\mathbb{Q}} X_{p,q}$ and $\cup_{p\in\mathbb{Q}} X_p$ are countable subsets of X and therefore so is $X^* := \left(\cup_{p\in\mathbb{Q}} X_p\right) \cup \left(\cup_{(p,q)\in\mathbb{Q} \mid p<q} X_{p,q}\right)$. To see that X^* is order-dense in X take any $x, y \in X$ such that $x \succ y$. If $\exists z \in X : x \succ z \succ y \Longleftrightarrow u(x) > u(z) > u(y)$, then

$$\begin{aligned}
u(x) > u(z) > u(y) &\Longrightarrow \exists p, q \in \mathbb{Q} : u(x) \geq q \geq u(z) \geq p > u(y), \quad \text{and } p < q \\
&\Longrightarrow (p,q) \cap u(X) \neq \emptyset \\
&\Longrightarrow \exists x_{p,q} \in X^* \subseteq X : u(x) \geq u(x_{p,q}) > u(y) \\
&\Longrightarrow x \succsim x_{p,q} \succ y.
\end{aligned}$$

If $\nexists z \in X : x \succ z \succ y$, then there is $p \in \mathbb{Q} : u(x) > p > u(y)$. Moreover, as $u(x) = \inf([p,\infty) \cap u(X))$, $\exists x_p \in X^* : u(x_p) = u(x)$. Hence, $u(x) = u(x_p) > u(y) \Longleftrightarrow x \succsim x_p \succ y$. □

Note that Theorem 2 subsumes the previous utility-representation result, Proposition 5, as, for any preference relation $\succsim$ on countable X, X is already an order-dense subset of itself.

In the next exercise, we will try to see how restrictive our model is by considering procedures other than utility maximization.

Exercise 6. A consumer is choosing between books from a finite set X. They have a utility function, $u : X \to \mathbb{R}$, and a 'threshold utility' $\bar{u}$. The bookseller sets the books in a given fixed ordering S which is complete, transitive, and antisymmetric. E.g. alphabetically by title (assuming no two books have the same title). Then, in any set of books $A \subseteq X$ in display, the consumer starts searching according to S in decreasing order (e.g. alphabetically), and chooses the first book for which the utility is equal or exceeds $\bar{u}$. If there is no such book in A, then they just go with the one with the highest utility.

(i) Does this procedure satisfy α, β, both, or neither?

(ii) Let $\succsim$ be such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Can the bookseller learn the consumer's preferences $\succsim$? If so, how? If not, why?

(iii) Discuss the statement: If choices are consistent with HARP, we are sure that consumers are choosing their most preferred items.

4.1. Choice Theory and Optimization

To conclude this section, note that the results we proved earlier allow us to derive a number of useful properties for optimization without needing to know much about the function or set over which we are optimizing.

Let $f : X \to \mathbb{R}$ be a real-valued function on X. Define, for every $A \in 2^X$,

$$\max_{x \in A} f(x) := \{f(x) \mid x \in A \text{ and } f(x) \geq f(y), \forall y \in A\};$$
$$\arg\max_{x \in A} f(x) := \{x \in A \mid f(x) \geq f(y), \forall y \in A\};$$
$$\min_{x \in A} f(x) := \{f(x) \mid x \in A \text{ and } f(x) \leq f(y), \forall y \in A\};$$
$$\arg\min_{x \in A} f(x) := \{x \in A \mid f(x) \leq f(y), \forall y \in A\}.$$

Note that $\min_{x \in A} f(x) = -\max_{x \in A} -f(x)$ (why?).

Then, reusing the results for preference relations we can deduce the following:

Proposition 6. The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{z \in A} f(z)$ and $y \in \arg\max_{z \in B} f(z)$, $f(x) \geq f(y)$.

(ii) For any nonempty $A \subseteq X$ and X is finite, $\arg\max_{x \in A} f(x) \neq \emptyset$.

(iii) For $x, y \in A \subseteq X$, $f(x) = f(y)$ and $\{x, y\} \cap \arg\max_{z \in A} f(z) \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{z \in A} f(z)$.

(iv) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{z \in A} f(z)$, then $x \in \arg\max_{z \in B} f(z)$.

Exercise 7. Prove Proposition 6 by making use of the results derived above.

5. Limited Observability (*)

Suppose that you want to test whether an agent's choice function admit a preference representation. Technically speaking, you would need to observe a mapping $C : 2^X \to 2^X$. This is a lot of data: if $|X| = 20$ we need to observe choices from over 1 million different subsets of X.

On the other hand, consider the following example:

Example 2. Suppose that $X = \{x, y, z\}$ and you only observe $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. HARP (as well as Sen's α and β) is trivially satisfied, but no preference relation $\succsim$ exists that is consistent with $C(A) = \arg\max_{\succsim} A$ for $A \in \{\{x, y\}, \{y, z\}, \{x, z\}\}$.

We need to, somehow, generalize HARP. In particular, we want to be able to infer that if the data tells us that $x \succ y$ and $y \succ z$, then we should infer that $x \succ z$. For that purpose we first need the following definition:

Definition 7. Let $\succsim$ be a binary relation on X. $T(\succsim)$ is the **transitive closure** of $\succsim$ if (i) $T(\succsim)$ is a transitive binary relation on X, (ii) $\succsim \subseteq T(\succsim)$, that is, $x \succsim y \Longrightarrow xT(\succsim)y$, (iii) any binary relation $\hat{\succsim} \subsetneq T(\succsim)$ is either intransitive or $\neg(\succsim \subseteq \hat{\succsim})$.

We can obtain the transitive closure when X is finite as follows: First, for any binary relation on X, $\succsim$, let $f(\succsim)$ such that if $x \succsim y$ and $y \succsim z$, then $xf(\succsim)z$. Define $\succsim_1 := f(\succsim)$ and, for $n > 1$, $\succsim_n := f(\succsim_{n-1})$. Then, $\succsim_{|X|} = T(\succsim)$. The next theorem ensures that $T(\succsim)$ is well-defined:

Theorem 3. For any $\succsim$ binary relation on X, $\exists T(\succsim)$.

We will see the proof for it later on in the course.

Now that we know that we can render a binary relation transitive, we need to make it complete.

It is easy to complete a binary relation $R \subseteq X^2$: for any $x, y \in X$ such that $\neg(x \succsim y), \neg(y \succsim x)$, we could simply add either (x, y) or (y, x) or both to the original binary relation R. Following this *completion* procedure we would indeed end up with a complete binary relation that contains R. However, we will need to be a bit more careful, as we need to complete the binary relation in a way such that the resulting binary relation is transitive *and* its strict part contains the strict part of the original binary relation R. That is, a completion is not enough.

For our purposes, we want to rely on the concept of an extension:

Definition 8. Let $\succsim$ be a preorder on X. An **extension** of $\succsim$ is a complete preorder[6] $\trianglerighteq$ on X such that $\succsim \subseteq \trianglerighteq$ and $\succ \subseteq \triangleright$, where $\succ$ and $\triangleright$ are the asymmetric parts of $\succsim$ and $\trianglerighteq$, respectively.

The following is a version of Sziplrajn's theorem that says that an extension always exists:

Theorem 4. **(Sziplrajn)** For any nonempty set X and preorder $\succsim$ on X, there is an extension of $\succsim$.

Let us pursue our intuition that we should be able to infer preferences about two elements even when they are never available at the same time. First, given a subset $Y \subseteq 2^X$ and a choice function $C : Y \to Y$, let us define a binary relation R^D on X such that $xR^D y$ if there is $A \in Y$ for which $y \in A$ and $x \in C(A)$, in which case we say that x is **directly revealed preferred to** y. We then define the revealed preference relation R as the transitive closure of R^D, that is $R := T(R^D)$, and we say that x is **revealed preferred to** y and if $x \succsim y$.

Finally, we need to preserve strict preferences. If we were to define the revealed preference relation from Example 2, we would get that the agent would be indifferent with respect to any element. Instead, we want a restrictive interpretation of the data. This is given by the concept of revealed strict preference: Given the same choice function on Y, we say that x is **revealed strictly preferred to** y — and write xSy — if there is $A \in Y$ such that $y \in A \setminus C(A)$ and $x \in C(A)$. That is, if x was chosen and y was not chosen but could have been chosen, then we understand that it cannot be the case that y is weakly preferred to x.

Definition 9. Let $Y \subseteq 2^X$ and let $C : Y \to Y$ be a choice function. C satisfies the **Generalized Axiom of Revealed Preference** (GARP) if $\nexists x, y \in X$ such that x is revealed preferred to y and y is revealed strictly preferred to x.

[6] That is, a preference relation.

The main result of this section is the following:

Theorem 5. Let $Y \subseteq 2^X$. A choice function $C : Y \to Y$ satisfies GARP if and only if there is a preference relation $\succsim \subseteq X^2$ such that $C(A) = \arg\max_{\succsim} A$ for any $A \in Y$.

Proof. The 'if' part is straightforward to show; we focus on the 'only if' part. By Theorem 3, R is well defined. By GARP, S is a subset of the asymmetric part of R. Note that $\tilde{R} := R \cup \{(a,a) \mid a \in X\}$ is a preorder on X. Let $\succsim$ be an extension of $\tilde{R}$ such that $\succsim$ is a complete preference relation on X; by Theorem 4, $\succsim$ exists. By definition of an extension, the result follows: $C(A) = \arg\max_{\succsim} A$ for any $A \in Y$. □

While revealed preference is a powerful way to model behavior, as it enable us to use optimization to describe behavior, Exercise 6 recommends caution when using behavior that is consistent with "preference maximization" to make inferences about how well-off an agent is.

Exercise 8. Suppose that the decision-maker has a preference relation $\succsim$ on X and their choices at any subset $A \in 2^X$ are given by $C(A) := \arg\max_{\succsim} A$.

If, instead of observing a dataset $(A_t, C(A_t))_t$, we observe $(A_t, x_t)_t$, where $x_t \in C(A_t)$, what can only say about $\succsim$?

(Extra) Suppose $X = \{x_n\}_{n\in[10]}$, where each x_n represents an ice-cream flavor. You observe the following data:

A	$C(A)$
$\{x_5, x_7\}$	$\{x_5\}$
$\{x_1, x_7\}$	$\{x_7\}$
$\{x_4, x_8, x_{10}\}$	$\{x_4, x_{10}\}$
$\{x_1, x_2, x_3, x_6\}$	$\{x_1, x_3, x_6\}$
$\{x_3, x_9, x_{10}\}$	$\{x_3, x_{10}\}$
$\{x_2, x_8, x_9\}$	$\{x_8, x_9\}$

Write a program (in Python/Julia/R) to test whether the dataset satisfies GARP and, if so, provide a preference relation $\succsim \in X \times X$ such that $C(A) = \arg\max_{\succsim} A$.

Suggestion: To derive preference relations, when the dataset satisfies GARP, create a $|X| \times |X|$ matrix M of zeros and replace the ij-th coordinate whenever $x_j \in A$ and $x_i \in C(A)$. Then, obtain the transitive closure of M (which you can do easily with matrix multiplication).

6. References

Houthakker, Hendrik S. 1956. "Chapter 11 – On the Logic of Preference and Choice." In *Contributions to Logic and Methodology in Honor of J. M. Bochenski*, edited by Tymieniecka, Anna-Teresa 193–207, North-Holland, . 10.1016/B978-1-4832-3159-4.50017-4. 5

Masatlioglu, Yusufcan, Daisuke Nakajima, and Erkut Y. Ozbay. 2012. "Revealed Attention." *American Economic Review* 102 (5): 439–443. 10.1257/aer.102.5.2183. 6

Sen, Amartya K. 1971. "Choice Functions and Revealed Preference." *The Review of Economic Studies* 38 (3): 307–317. 10.2307/2296384. 6

ECON0106: **Microeconomics**

2. Structural Properties of Preferences and Utility Representations*

Duarte Gonçalves†

University College London

1. Overview

In many circumstances, economic research involves taking a stance on the form of utility functions that drive agents' behavior. This is true in applied research — where models are then used to analyze policies with structural estimation and counterfactuals, or to identify a particular effect — as it is in theory. Often, making simplifying assumptions are needed to make progress, but it is important to keep in mind what each assumption implies, and what it is ruling out. Our goal will be to develop a better understanding about common restrictions that specific functional form assumptions impose on behavior to be able to better evaluate the limitations of any given model.

1.1. Notation

For simplicity, assume throughout that (X,d) is a metric space. For $\epsilon > 0$ and $x \in X$, we denote by $B_\epsilon(x) := \{y \in X \mid d(x,y) < \epsilon\}$ for an open ϵ-neighborhood of x in X. For a set S, we denote its closure by $\overline{S}$.

*Last updated: 19 October 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

2. Continuous Utility Representation

Why do we care about continuity[1] of our utility representation? Because, whenever the feasible set is compact,[2] we are guaranteed the existence of a maximizer by Weierstrass extreme value theorem:

Theorem 1. (Weierstrass Extreme Value Theorem) Let (X, d_X) and (Y, d_Y) be two metric spaces. If $f: X \to Y$ is a continuous function and S a compact set in (X, d_X), then f attains a maximum and a minimum in S: $\arg\max_{x \in S} f(x) \neq \emptyset$ and $\arg\min_{x \in S} f(x) \neq \emptyset$.

We want to guarantee that utility maximizing choices are well-defined ($\arg\max_{x \in A} u(x) \neq \emptyset$) in order to ensure that an agent's behavior can be represented as the outcome of utility maximization. And, in general, one way to do so is by restricting to compact feasible sets and continuous utility functions.

In this section we will relate continuity properties of the utility representation to underlying continuity properties of preferences and choices.

Definition 1. A preference relation $\succsim$ on X is **continuous** if for any two converging sequences, $\{x_n\}_n, \{y_n\}_n \subseteq X$, $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n$ $\forall n$, we have $x \succsim y$.

This next lemma is particularly useful to characterize continuity of preference relations:

Lemma 1. Let $\succsim$ be a preference relation on X, and $\succ$ its asymmetric part. The following statements are equivalent:

(i) $\succsim$ is continuous;

(ii) for any $x \in X$, $X_{x\succsim}$ and $X_{\succsim x}$ are closed sets;

(iii) for any $x \in X$, $X_{x\succ}$ and $X_{\succ x}$ are open sets;

(iv) for any $x, y \in X$ such that $x \succ y$, there is $\epsilon > 0$ such that $\forall x' \in B_\epsilon(x), y' \in B_\epsilon(y)$, $x' \succ y'$.

[1] Recall the definition: Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f: X \to Y$ is **continuous** at $x_0 \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in \{x' \in X : d_X(x', x_0) < \delta\}$, $d_Y(f(x), f(x_0)) < \epsilon$. Equivalently, f is continuous at $x_0 \in X$ if for every sequence $\{x_n\}_n \subseteq X$ such that $x_n \to x_0$, $f(x_n) \to f(x_0)$. A function is continuous if it is continuous at all $x_0 \in X$.

[2] When (X, d_X) is a metric space, a set S is **compact** if and only if it is **sequentially compact**, that is, if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_m\}_m \subseteq \{x_n\}_n$ such that $x_m \to x^* \in S$. Note that if $S \subseteq Y \subseteq X$, S is closed, and Y is compact, then S is compact. Moreover, if S is a compact set, then S is both closed and bounded. The **Heine–Borel Theorem** shows that, in Euclidean spaces, the converse holds: if S is closed and bounded, then S is compact.

Exercise 1. Prove Lemma 1. *(Hint: a standard way to go about it is to show (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).)*[3]

The main result in this section is Debreu's (1954; 1964) Theorem:

Theorem 2. **(Debreu's Theorem)** Let $\succsim$ be a preference relation on X, and suppose that X admits a countable, order-dense subset Z. Then, $\succsim$ is continuous if and only if $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$.

We won't prove the theorem in all its generality, but rather a more modest version of it in which we will assume that X is convex, that is, $\forall x, y \in X$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in X$:

Theorem 3. Let (X, d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha) y, y) \leq d(x, y)$. Let $\succsim$ be a preference relation on a convex set X, and suppose that X admits a countable, order-dense subset Z. Then, $\succsim$ is continuous if and only if $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$.

Proof. $\Longleftarrow$: (if)

Take any $\{x_n\}_n, \{y_n\}_n \subseteq X$ such that $x_n \to x$, $y_n \to y$, and $x_n \succsim y_n$. Then, $u(x_n) - u(y_n) \geq 0, \forall n$ and, by continuity of u, $\lim_{n\to\infty} u(x_n) - u(y_n) = u(x) - u(y) \geq 0 \Longrightarrow x \succsim y$.

$\Longrightarrow$: (only if)

Assume that $\exists x, y \in X : x \succ y$ — otherwise we can just set $u(x) = c$ for any constant c.
We will prove this part in three steps:

1. Show that $\forall x, y \in X : x \succ y$, there is $z \in Z$ such that $x \succ z \succ y$.
2. Construct a utility function $u : X \to \mathbb{R}$ such that $u(Z)$ is dense in $[0,1]$.
3. Show that u is continuous.

Lemma 2. Let (X, d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha) y, y) \leq d(x, y)$. Let $\succsim$ be a continuous preference relation on X, and $\succ$ its asymmetric part. Suppose that Z is a countable, order-dense subset of X. For any $x, y \in X : x \succ y$, there is $z \in Z$ such that $x \succ z \succ y$.

[3] You can think about $\Longrightarrow$, i.e. "implies", as a preorder on the set of all statements, which is why if $A \Longrightarrow B \Longrightarrow C \Longrightarrow A$, we have, by transitivity, $A \Longrightarrow C$, $C \Longrightarrow B$, and $B \Longrightarrow A$, and, hence, that A, B, and C are equivalent.

Proof. First we show existence of $x' \in X : x \succ x' \succ y$.

For $\alpha \in [0,1]$, let $x_\alpha := \alpha x + (1-\alpha) y \in X$ (by convexity of X). Define $A := \{\alpha \in [0,1] \mid x_\alpha \succsim x\}$. By completeness of $\succsim$, $1 \in A$ and, by assumption, as $x \succ y$, $0 \notin A$. Then, as A is nonempty and bounded below, define $\alpha := \inf A$.

We now want to show that $x_\alpha \sim x$. Suppose that this is not the case.

$$\begin{aligned} \text{If } x_\alpha \succ x \Longrightarrow & \exists \epsilon > 0 : x_{\alpha-\epsilon} \succ x && \text{by Lemma 1} \\ \Longrightarrow & \alpha - \epsilon \in A \\ \Longrightarrow & \alpha \neq \inf A, \end{aligned}$$

a contradiction. If instead,

$$\begin{aligned} x \succ x_\alpha \Longrightarrow & \exists \epsilon' > 0 : x \succ x_{\alpha+\epsilon},\ \forall \epsilon \leq \epsilon' && \text{by Lemma 1} \\ \Longrightarrow & \alpha + \epsilon \leq \inf A \\ \Longrightarrow & \alpha \neq \inf A, \end{aligned}$$

again a contradiction. As $x \sim x_\alpha \succ y$, this means that, again, $\exists \epsilon > 0 : x_{\alpha-\epsilon} \succ y$. As $\alpha - \epsilon < \alpha \Longrightarrow \alpha - \epsilon \notin A$, and, therefore, $x \succ x_{\alpha-\epsilon} \succ y$.

Now, we find a $z \in Z$ such that $x \succ z \succ y$: as Z is order-dense in X, then $\exists z \in Z : x \succ x_{\alpha-\epsilon} \succsim z \succ y$. □

Lemma 3. Let (X,d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha)y, y) \leq d(x,y)$. Let $\succsim$ be a continuous preference relation on X, and Z a countable order-dense subset of X. There is a utility representation of $\succsim$ $u : X \to \mathbb{R}$ such that $u(X)$ is dense in $[0,1]$.

Proof. Without loss of generality, suppose that the maximal and minimal elements of X are not in Z, i.e. $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$. Fix an order on $Z = \{z_1, z_2, \ldots\}$ and, for $n \geq 2$, let $Z_n := \{z_1, \ldots, z_{n-1}\}$. We define u on Z by induction. Let $u(z_1) = 1/2$. If $\exists z_m \in Z_n$ such that $z_n \sim z_m$, set $u(z_n) = u(z_m)$. If $z_n \succ z_m$ (resp. $z_m \succ z_n$) for all $z_m \in Z_n$, then set $u(z_n) := (1 + \max_{z \in Z_n} u(z))/2$ (resp. $u(z_n) := (0 + \min_{z \in Z_n} u(z))/2$). Finally, if $\exists z_\ell, z_m \in Z_n$ such that $z_\ell \succ z_n \succ z_m$, then set $u(z_n) := (\min_{z \in Z_n : z \succ z_n} u(z) + \max_{z \in Z_n : z_n \succ z} u(z))/2$. Note that, by

Lemma 2, $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$. This implies that for any two elements $z_n, z_m \in Z$ such that $z_n \succ z_m$, there is $\ell, \ell', \ell'' > n, m$ such that $z_\ell \succ z_n \succ z_{\ell'} \succ z_m \succ z_{\ell''}$, where z_ℓ and $z_{\ell''}$ exist because we removed the maximal and minimal elements of X from Z.
By construction, the set $u(Z)$ corresponds to the set of dyadic numbers in $(0,1)$, i.e. the set of numbers that can be represented as $m/2^n$ for $m, n \in \mathbb{N}$ and $m < 2^n$, which is dense in $[0,1]$.
If $\arg\max_{\succsim} X$ is nonempty, we know that $\forall x, y \in \arg\max_{\succsim} X$, we have that $x \sim y$, and then we can assign $u(x) = u(y) = 1$; and analogously for $x \in \arg\min_{\succsim} X$ assign $u(x) = 0$.

Now we extend u to X by setting $u(x) := \sup\{u(z) \mid z \in Z \text{ and } x \succ z\} = \sup_{z \in Z_{x\succ}} u(z)$ and check that it represents $\succsim$. That $x \sim y \implies u(x) = u(y)$ is immediate from the definition. To see that $x \succ y \implies u(x) > u(y)$, note that, by Lemma 2, $\exists z, z' \in Z$ such that $x \succ z \succ z' \succ y \implies u(x) \geq u(z) > u(z') \geq u(y)$. As $u(Z) \subseteq u(X) \subseteq [0,1]$ and $u(Z)$ is dense in $[0,1]$, then $u(X)$ is dense in $[0,1]$. □

Finally, the last step: showing continuity of u as defined in the proof of Lemma 3. Take any $x \in X \setminus (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X)$. By Lemmata 2 and 3, for any $\epsilon > 0$, there are $z, z' \in Z$ such that $u(x) - \epsilon < u(z) < u(x) < u(z') < u(x) + \epsilon$. By Lemma 1, we then have that $\exists \delta > 0$ such that $\forall x' \in B_\delta(x)$, $u(x) - \epsilon < u(z) < u(x') < u(z') < u(x) + \epsilon$. To show that u is continuous at $x \in \arg\max_{\succsim} X$, note that by Lemmata 2 and 3, for any $\epsilon > 0$, there is $z \in Z$ such that $u(x) - \epsilon < u(z) < u(x)$ and by Lemma 1, $\exists \delta > 0$ such that $\forall x' \in B_\delta(x)$, $u(x) - \epsilon < u(z) < u(x') \leq u(x)$ (the proof for continuity of u at $x \in \arg\min_{\succsim} X$ is symmetric). □

Exercise 2. Prove that, if X is a convex subset of $\mathbb{R}^k$ and $\succsim$ is a continuous preference relation on X, then X admits a countable, order-dense subset. Conclude about the existence of a continuous utility representation.

Note that even if $\succsim$ is a continuous preference relation on X, it does not mean that any utility representation of $\succsim$ is continuous. For instance, suppose that $\succsim$ is a continuous preference relation on $\mathbb{R}$ such that $x \geq y \iff x \succsim y$. Clearly, $u(x) := x$ is a possible utility representation, but so is $v(x) := x$ if $x < 1$, $v(x) := 3x$ if $x > 1$, and $v(1) := 2$, and v is not continuous.

3. Convexity

If the former section dealt with having the agent's choices well-defined for arbitrary compact sets, this section will provide sufficient conditions for choices to be uniquely defined (i.e. a singleton). For that, we will study a property of interest for a utility representation — quasiconcavity — and the conditions on preferences that guarantee it.

Definition 2. A u real-valued function on a convex set X is **(strictly) quasiconcave** if $\forall x, y \in X$ and $\forall \lambda \in [0,1]$ (resp. $\lambda \in (0,1)$), $u(\lambda x + (1-\lambda y)) \geq (>) \min\{u(x), u(y)\}$.

Definition 3. We say that a preference relation $\succsim$ on a convex set X is **convex** if for any $x \succsim y$ and any $\lambda \in [0,1]$, we have that $\lambda x + (1-\lambda)y \succsim y$. It is said to be **strictly convex** if $\forall x \succsim y$, $x \neq y$, and any $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \succ y$.

This property can be interpreted as prefering mixtures to extremes (when indifferent with respect to both): when the agent is indifferent between two elements (say "apple" and "banana") then they prefer to have a convex combination (λ"apple"$+(1-\lambda)$"banana"$=:$"fruit salad") to either of them.

In fact, quasiconcavity of the utility representation is equivalent to convexity of preferences:

Proposition 1. Let $\succsim$ be a preference relation on a convex set X and let $u : X \to \mathbb{R}$ be a utility representation. The following statements are equivalent:

(i) $\succsim$ is convex;

(ii) $X_{\succsim y}$ is convex $\forall y \in X$;

(iii) u is quasiconcave;

(iv) $\{x \in X \mid u(x) \geq \overline{u}\}$ is convex $\forall \overline{u} \in \mathbb{R}$.

Moreover, $\succsim$ is strictly convex if and only if u is strictly quasiconcave.

Proof. (i)$\Longrightarrow$(ii): Take any $x, x' \in X_{\succsim y}$ and let, without loss of generality (by completeness), $x \succsim x'$. Then $\lambda x + (1-\lambda)x' \succsim x' \succsim y$ $\forall \lambda \in [0,1]$ (by convexity and transitivity).
(i)$\Longleftarrow$(ii): By completeness, $y \in X_{\succsim y}$. As $X_{\succsim y}$ is convex, then $\forall x \in X_{\succsim y}$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \succsim y$.

(i)$\Longleftrightarrow$(iii): Take any $x, y \in X$ such that $x \succsim y \Longleftrightarrow u(x) \geq u(y)$, and any $\lambda \in [0,1]$.

$$\begin{aligned}\succsim \text{ convex } &\Longleftrightarrow \lambda x + (1-\lambda)y \succsim y \\ &\Longleftrightarrow u(\lambda x + (1-\lambda)y) \succsim u(y) = \min\{u(x), u(y)\} \\ &\Longleftrightarrow u \text{ quasiconcave.}\end{aligned}$$

For the strict convexity of $\succsim$ and strict quasiconcavity of u, replace $\succsim$ and $\geq$ with $\succ$ and $>$.

(iii)$\Longrightarrow$(iv): $\forall x, y \in X : u(x), u(y) \geq \overline{u},\ u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \geq \overline{u},\ \forall \lambda \in [0,1]$ (by quasiconcavity of u).

(iii)$\Longleftarrow$(iv): $\forall x, y \in X$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in \{z \in X \mid u(z) \geq \min\{u(x), u(y)\}\}$ by convexity of $\{z \in X \mid u(z) \geq \min\{u(x), u(y)\}\}$ and the fact that $u(x), u(y) \geq \min\{u(x), u(y)\}$; then $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$. □

Theorem 4. Let $\succsim$ be a convex preference relation on a convex set X. Then, for any convex $A \in 2^X$, $\arg\max_{\succsim} A$ is convex. If in addition $\succsim$ is strictly convex, then $\arg\max_{\succsim} A$ contains at most one element.

Combining Proposition 1 and Theorem 4, we learn that the set of maximizers of a quasiconcave function u on a convex set A, $\arg\max_{x \in A} u(x)$, is convex and, if u is strictly quasiconcave, it has at most one element. This gives us a lot of structure on the agent's choices and allows us to make better predictions. In fact, pooling results we have shown so far, we can say that if preferences are continuous and strictly convex, the agent will choose exactly one element out of any compact and convex set of alternatives.

Exercise 3. Prove Theorem 4.

4. Monotonicity and Insatiability

A very natural property when $X \subseteq \mathbb{R}^k$ is that of monotonicity, capturing the principle that 'more is better.' We define three notions of monotonicity:

Definition 4. (i) $\succsim$ is **monotone** if $x \geq y \Longrightarrow x \succsim y$;

(ii) $\succsim$ is **strongly monotone** if $x \geq (\gg) y \Longrightarrow x \succsim (\succ) y$;[4]

(iii) $\succsim$ is **strictly monotone** if $x > y$ (i.e. $x \geq y$ and $x \neq y$) $\Longrightarrow x \succ y$.

[4]It is to be understood that $x \gg y$ stands for $x_i > y_i$ for all $i \in [k]$, whereas $x > y$ simply denotes the asymmetric part of $\geq$, $x \geq y$ and $\neg(y \geq x)$. That is, $x > y$ if $x_i \geq y_i$ for all i and $x_j > y_j$ for some j.

A simple result ensues:

Proposition 2. Let $\succsim$ be a preference relation on $X \subseteq \mathbb{R}^k$ and $u : X \to \mathbb{R}$ a utility representation of $\succsim$.

(i) $\succsim$ is **monotone** if and only if $x \geq y \implies u(x) \geq u(y)$;

(ii) $\succsim$ is **strongly monotone** if and only if $x \geq (\gg) y \implies u(x) \geq (>) u(y)$;

(iii) $\succsim$ is **strictly monotone** if and only if $x > y$ $(x \geq y$ and $x \neq y) \implies u(x) > u(y)$.

A related property is that of insatiability, the sense in which, for any alternative x, there is always some other alternative y that is strictly preferred.

Definition 5. (i) We say that $\succsim$ is **globally non-satiated** if for any $x \in X$, there is $y \in X$ such that $y \succ x$.

(ii) It is **locally non-satiated** if for any $x \in X$, and any $\epsilon > 0$, $\exists y \in B_\epsilon(x)$ such that $y \succ x$.

Then we have that strict monotonicity $\implies$ strong monotonity $\implies$ monotonicity and, if, say, $X = \mathbb{R}^k$, monotonity $\implies$ local non-satiation $\implies$ global non-satiation.[5] While non-satiation does not easily translate into properties of utility representations, we will see later on that it plays an important role in consumer theory.

5. Homotheticity

Definition 6. A preference relation $\succsim$ on $X = \mathbb{R}^k$ is **homothetic** if $x \succsim y \implies \alpha x \succsim \alpha y$, $\forall \alpha \geq 0$.

Homotheticity of preferences will allow us to show that one can interpret aggregate demand as choices by a representative consumer. We will defer on that result and instead focus on its implications for utility representation.

[5] In general, monotonicity need not imply non-satiation: consider the trivial case where X is a singleton.

Proposition 3. Let $\succsim$ be a continuous, homothetic, and strongly monotone preference relation on $X = \mathbb{R}^k$. Then, it admits a continuous utility representation $u : X \to \mathbb{R}$ that is homogeneous of degree one.[6]

Exercise 4. Prove Proposition 3 by following the following steps:

(1) Show that, for any $x \in X$ there is an $\alpha, \alpha' \in \mathbb{R}$ such that $\alpha \mathbf{1} \succsim x \succsim \alpha' \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^k$ is a vector of ones.

(2) Show then that, for any $x \in X$, there is a unique $\beta_x \in \mathbb{R}$ such that $x \sim \beta_x \mathbf{1}$.

(3) Define $u(x) := \beta_x$ and show that u is continuous and homogeneous of degree one.

6. Separability

It is often the case that alternatives have different features. For instance, consider subscribing to a gym, and you consider the location and the open hours. It may be the case that you consider these two features separately, and then simply trade-off between the two. Or it may be the case that if the gym is open and not too crowded after work hours, you prefer it to be close to work as you can go with friends from work; but if it is usually too crowded, you actually prefer it to be close to home. When preferences are separable, the problem in a sense becomes simply a matter of assigning a value to each dimension and think about how you trade these off. This is exactly what we are going to show.

Let $X := \times_{i \in [n]} X_i \times \overline{X}$, where each X_i is a dimension and $[n] = \{1, \ldots, n\}$. We write $x_{-i} \in X_{-i} := \times_{j \in [n] \setminus \{i\}} X_j \times \overline{X}$ and $x = (x_i, x_{-i})$.

Definition 7. A preference relation on X is said to be **weakly separable** in $\times_{i \in [n]} X_i$ if, $\forall i \in [n]$, for every $x_i, y_i \in X_i$ and every $x_{-i}, y_{-i} \in X_{-i}$, we have that $(x_i, x_{-i}) \succsim (y_i, x_{-i}) \iff (x_i, y_{-i}) \succsim (y_i, y_{-i})$.

Theorem 5. Let $\succsim$ be a preference relation on $X = \times_{i \in [n]} X_i \times \overline{X}$ that admits a utility representation $u : X \to \mathbb{R}$. Then, $\succsim$ is weakly separable in $\times_{i \in [n]} X_i$ if and only if there are $v, \{u_i\}_{i \in [n]}$, where $u_i : X_i \to \mathbb{R}$, and $v : \times_{i \in [n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ such that $u(x) = v(u_1(x_1), \ldots, u_n(x_n), \overline{x})$ and v is strictly increasing in its first n arguments.

[6] That is, $u(\alpha x) = \alpha u(x)$, $\forall \alpha \geq 0$.

Proof. $\Longleftarrow$: (if) Follows immediately from the fact that v is strictly increasing in its first n arguments.

$\Longrightarrow$: (only if) We break the proof into steps:

(1) Define u_i: Fix $x^* \in X$. For $i \in [n]$, let $u_i(x_i) := u(x_i, x^*_{-i})$.

(2) Show that, for any $x, y \in X$ such that $\overline{x} = \overline{y}$, if $u_i(x_i) \geq u_i(y_i)$ $\forall i \in [n]$, then $u(x) \geq u(y)$:

$$\begin{aligned}
u_i(x_i) \geq u_i(y_i)\, \forall i \in [n] &\Longleftrightarrow u(x_i, x^*_{-i}) \geq u(y_i, x^*_{-i})\, \forall i \in [n] \\
&\Longleftrightarrow (y_1, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots, x_n, \overline{x}) \succsim (y_1, \ldots, y_{i-1}, y_i, x_{i+1}, \ldots, x_n, \overline{x})\, \forall i \in [n] \\
&\qquad \text{by weak separability} \\
&\Longrightarrow x \succsim y \\
&\qquad \text{by transitivity} \\
&\Longrightarrow u(x) \geq u(y).
\end{aligned}$$

Moreover, it is also the case that if for some i $u_i(x_i) > u_i(y_i)$, then $u(x) > u(y)$.

(3) Define v: For any $r \in \mathbb{R}^n$ such that $r_i \in u_i(X_i)$ $\forall i \in [n]$, pick any $x_i \in X_i$ such that $u_i(x_i) = r_i$. For any $\overline{x} \in \overline{X}$, and for any $r \in \times_{i \in [n]} u_i(X_i)$, let $v(r, \overline{x}) := u(x)$. By (2), v is strictly increasing in r.

□

Note that weak separability *does not* deliver additive separability, that is, it does not guarantee that we can write $u(x) = \sum_{i \in [n]} u_i(x_i)$. For that we need preferences to be **strongly separable** on $X = \times_{i \in [n]} X_i$:

Definition 8. A preference relation $\succsim$ on $X = \times_{i \in [n]} X_i$ is **strongly separable** if $\forall I \subsetneq [n]$, $\forall x_I, y_I \in \times_{i \in I} X_I$ and $\forall x_{-I}, y_{-I} \in \times_{i \in [n] \setminus I} X_i =: X_{-I}$, we have that $(x_I, x_{-I}) \succsim (y_I, x_{-I}) \Longleftrightarrow (x_I, y_{-I}) \succsim (y_I, y_{-I})$.

In essence, strongly separable preferences are those that are separable not only in each dimension but in each group of dimensions.

We will also need a further definition: $i \in [n]$ is an **essential component** if $\exists x_i, y_i \in X_i$ and $x_{-i} \in X_{-i}$ such that $(x_i, x_{-i}) \succ (y_i, x_{-i})$. The result — which we state without proof — is then as follows:

Theorem 6. **(Debreu 1960)** Let $\succsim$ be a continuous preference relation on a connected set $X := \times_{i\in[n]} X_i$, such that $\succsim$ admits a preference relation $u : X \to \mathbb{R}$, and there are at least three essential components. If $\succsim$ is strongly separable, then there are $\{u_i\}_{i\in[n]}$, where $u_i : X_i \to \mathbb{R}$, such that $u(x) = \sum_{i\in[n]} u_i(x_i)$.

7. Quasilinearity

One of the most widely used functional forms is **quasilinear utility**: $u : Y \times \mathbb{R} \to \mathbb{R}$ such that $u(y,m) = v(y) + m$ for some $v : Y \to \mathbb{R}$. The first argument y is interpreted as an item, while the second argument m is taken to be money (available to acquire other items). As it is a recurrently assumed functional form — e.g. in contract theory, auctions, and mechanism design — it is particularly important to understand precisely what assumptions on preference relations are necessary to obtain this type of representation.

Theorem 7. Let $\succsim$ be a preference relation on $Y \times \mathbb{R}$. $\succsim$ admits a quasi-linear utility representation if and only if it satisfies the following properties:

(1) $m' \geq m \iff (y,m') \succsim (y,m)$, $\forall y \in Y$, $m, m' \in \mathbb{R}$ (money is good);

(2) $(y,m) \succsim (y',m') \iff (y, m + m'') \succsim (y', m' + m'')$, $\forall y, y' \in Y$, $m, m', m'' \in \mathbb{R}$ (no wealth effects);

(3) $\forall y, y' \in Y$, $\exists m, m' \in \mathbb{R}$ such that $(y,m) \sim (y',m')$ (money can compensate).

Note that property (2) is needed in some way or another: after all, quasilinear preferences are weakly separable.

Proof. $\implies$ (only if):

(1) $m' \geq m \iff v(y) + m' \geq v(y) + m \iff (y,m') \succsim (y,m)$, $\forall y \in Y$, $m, m' \in \mathbb{R}$ (money is good);

(2) $(y,m) \succsim (y',m) \iff v(y) + m \geq v(y') + m \iff v(y) + m' \geq v(y') + m' \iff (y,m') \succsim (y',m')$, $\forall y, y' \in Y$, $m, m' \in \mathbb{R}$ (no wealth effects);

(3) $\forall y, y' \in Y$, $\exists m, m' \in \mathbb{R}$ such that $v(y) - v(y') = m' - m \iff v(y) + m = v(y') + m' \iff (y, m) \sim (y', m')$ (money can compensate).

$\Longleftarrow$ (if):

Fix $(y^*, m^*) \in Y \times \mathbb{R}$.

Step 1: there is a unique $\rho : Y \to \mathbb{R}$ such that $(y, \rho(y)) \sim (y^*, m^*)$.

By (3), $\exists m(y), m'(y) \in \mathbb{R} : (y, m(y)) \sim (y^*, m'(y))$. By (2), $(y, m(y) - m'(y) + m^*) \sim (y^*, m^*)$. Let $\rho(y) := m(y) - m'(y) + m^*$. Suppose there is $v : Y \to \mathbb{R}$ such that $(y, v(y)) \sim (y^*, m^*)$ $\forall y \in Y$, but $v(y') \neq \rho(y')$ for some $y' \in Y$. Then $v(y') < (>) \rho(y')$ implies by (1) that $(y', v(y')) \sim (y^*, m^*) \sim (y, \rho(y')) \succ (\prec) (y', v(y'))$, a contradiction of reflexivity of $\succsim$.

Step 2: Show that we can define $v(y) := -\rho(y)$.

$$
\begin{aligned}
(y, m) \succsim (y', m') &\iff (y, m - m' + \rho(y')) \succsim (y', \rho(y')) \sim (y^*, m^*) && \text{by Step 1 and (2)} \\
&\iff m - m' + \rho(y') \geq \rho(y) && \text{by (1)} \\
&\iff -\rho(y) + m \geq -\rho(y') + m' \\
&\iff v(y) + m \geq v(y') + m'.
\end{aligned}
$$

□

8. Indifference Curves

What are indifference curves? At their very core, indifference curves are indifference sets: $[x]_\sim := \{y \in X \mid y \sim x\}$. If $X = \mathbb{R}^2_+$, let's define the 'indifference curve' that goes through y as a function $I(\cdot, y) : \mathbb{R} \to \mathbb{R}$ such that $I(x_1, y) := \{x_2 \in \mathbb{R} \mid (x_1, x_2) \in [y]_\sim\}$.

Exercise 5. Suppose that $X = \mathbb{R}^2_+$. Which properties on $\succsim$ guarantee that indifference curves satisfy each of the following:

(i) have an empty interior;

(ii) are continuous;

(iii) are downward sloping;

(iv) are convex.

Exercise 6. Let $\succsim$ be a preference relation on X. Suppose that theorem A states that if $\succsim$ satisfies property P_A, then there is a utility representation that satisfies property Q_A and that, similarly, theorem B states that if $\succsim$ satisfies property P_B, then there is a utility representation that satisfies property Q_B. Now suppose that you find that $\succsim$ satisfies both P_A and P_B. Can we guarantee that there is a utility representation satisfying both Q_A and Q_B?

9. References

Debreu, Gérard. 1954. "Representation of a Preference Ordering by a Numerical Function." In *Decision Processes*, edited by Thrall, M., R.C. Davis, and C.H. Coombs 159–165, New York, NY: John Wiley and Sons. 3

Debreu, Gérard. 1960. "Topological Methods in Cardinal Utility Theory." In *Mathematical Methods in the Social Sciences*, edited by Arrow, Kenneth J., Samuel Karlin, and Patrick Suppes 16–26, Stanford, CA: Stanford University Press. 11

Debreu, Gérard. 1964. "Continuity properties of Paretian utility." *International Economic Review* 5 (3): 285–293. 10.2307/2525513. 3

ECON0106: **Microeconomics**

3. Optimal Choice and Consumer Theory*

Duarte Gonçalves†

University College London

1. Overview

A classic problem studied in economic theory is that of consumer demand. One standard approach is that consumers as choosing bundles of goods, $x \in \mathbb{R}^k_+$, and are faced with a budget constraint $B(p,w)$ determined by their income $w \geq 0$ and the vector of prices they face, $p \in \mathbb{R}^k_{++}$. Specifically, $B(p,w) := \{x \in \mathbb{R}^k_+ \mid p \cdot x \leq w\}$. We assume that they have preferences over the goods, $\succsim$, and we study the properties of their demand $x(p,w) := \arg\max_{\succsim} B(p,w)$. This is perhaps the model of economics that is most used outside academia.

2. Utility Maximization Problem

As we have seen in previous lectures, when we impose some consistency properties on choices (here, demand), we can actually represent preferences $\succsim$ by way of a utility function u, and characterize preference-maximizing choices as utility-maximizing ones. This motivates the term utility maximization problem to denote the consumer's problem:

$$x(p,w) := \arg\max_{x \in B(p,w)} u(x) \tag{UMP}$$

$$v(p,w) := \sup_{x \in B(p,w)} u(x)$$

This section shows how we can use structural properties of preference relations to derive properties on consumer demand.

*Last updated: 21 October 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

2.1. General Implications

We recall that regardless of which utility representation we choose, insofar as it represents the same preference relation, we have the same set of maximizers.

Proposition 1. Let $\succsim$ be a preference relation on $\mathbb{R}^k_+$ and let u and v be two utility representations of $\succsim$. Then, $x(p,w) = \arg\max_{\succsim} B(p,w) = \arg\max_{x\in B(p,w)} u(x) = \arg\max_{x\in B(p,w)} v(x)$.

Proof. Follows by definition: for any utility representation u of $\succsim$, $x \in \arg\max_{\succsim} B(p,w) \iff x \succsim y\ \forall y \in B(p,w) \iff u(x) \geq u(y)\ \forall y \in B(p,w) \iff x \in \arg\max_{x\in B(p,w)} u(x)$. $\square$

In the sequel, to avoid repetition, $\succsim$ will denote a preference relation on $\mathbb{R}^k_+$, $u : \mathbb{R}^k_+ \to \mathbb{R}$ a utility representation of $\succsim$, $v : (p,w) \mapsto v(p,w) \in \mathbb{R}$ the indirect utility,[1] and $x : (p,w) \mapsto x(p,w) \subseteq B(p,w)$ the set of maximizers.

As one would expect, indirect utility is increasing in income and decreasing in prices. The next proposition shows that it is also quasiconvex in prices and income:

Proposition 2. $v(p,w)$ is quasiconvex[2] in (p,w), weakly decreasing in p, and weakly increasing in w.

Proof. To show quasiconvexity, take any $(p,w),(p',w') \in \{(p,w) \mid v(p,w) \leq \overline{v}\}$. We want to show that $v(\lambda(p,w)+(1-\lambda)(p',w')) \leq \max\{v(p,w), v(p',w')\}$, for any $\lambda \in [0,1]$. $\forall x'' \in X$ such that $(\lambda p + (1-\lambda)p')\cdot x'' \leq \lambda w + (1-\lambda)w'$, we have that (i) $x'' \in B(p,w)$ or (ii) $x'' \in B(p',w')$. (Supposing otherwise means that $p\cdot x'' - w > 0$ and $p'\cdot x'' - w' > 0$ and, doing a convex combination of these, we get a contradiction.) The result follows.

As for the monotonicity properties, note that $p \geq p', w \leq w' \implies B(p,w) \subseteq B(p',w') \implies v(p,w) \leq v(p',w')$ (where is this last implication coming from?). $\square$

[1] This is a slight abuse of terminology given we defined v as the supremum instead of the maximum, as the latter need not be well defined, i.e. $x(p,w)$ can be empty. Given $B(p,w)$ is compact, these will be the same if $\succsim$ is continuous, as shown later on.

[2] A function $f : X \to \mathbb{R}$ is quasiconvex if $-f$ is quasiconcave. This is equivalent to having that $\{x \in X \mid f(x) \leq \alpha\}$ is convex for any $\alpha \in \mathbb{R}$.

If you scale up prices and income, then the consumer is able to afford exactly the same bundles. This implies that both the indirect utility and the set of maximizers remain the same.

Proposition 3. $v(p,w)$ and $x(p,w)$ are homogeneous of degree zero in (p,w): $\forall \lambda > 0$, $v(\lambda p, \lambda w) = v(p,w)$ and $x(\lambda p, \lambda w) = x(p,w)$.

Proof. As $B(\lambda p, \lambda w) = B(p,w)$, then $\arg\max_{\succsim} B(p,w) = \arg\max_{\succsim} B(\lambda p, \lambda w)$. □

2.2. Implications of Continuity

We will apply a result we derived earlier to show that consumer demand is nonempty when preferences are continuous.

Proposition 4. If $\succsim$ is continuous, then $x(p,w)$ is nonempty.

Correspondences

We will need to take a small detour to introduce correspondences in order to make use of a very powerful result in optimization: Berge's Maximum Theorem.

Definition 1. A **correspondence** F from X to Y is a mapping that associates with each element $x \in X$ a subset $A \subseteq Y$. This is typically denoted by $F : X \rightrightarrows Y$ or $F : X \to 2^Y$, with $F(x) \subseteq Y$. For $A \subseteq X$, we define the image of F as $F(A) := \cup_{x \in A} F(x)$.

We will introduce two notions of continuity of correspondences in metric spaces:

Definition 2. Let (X, d_X) and (Y, d_Y) be metric spaces and $F : X \rightrightarrows Y$.

(i) F is **upper hemicontinuous (uhc) at** $x_0 \in X$ if for any open set $U \subseteq Y$, such that $F(x_0) \subseteq U$, $\exists \epsilon > 0$ such that $F(B_\epsilon(x_0)) \subseteq U$;

(ii) F is **upper hemicontinuous (uhc)** if it is upper hemicontinuous at any $x_0 \in X$;

(iii) F is **lower hemicontinuous (lhc) at** $x_0 \in X$ if for any open set $U \subseteq Y$, such that $F(x_0) \cap U \neq \emptyset$, $\exists \epsilon > 0$ such that $F(x) \cap U \neq \emptyset$, for any $x \in B_\epsilon(x_0)$;

(iv) F is **lower hemicontinuous (lhc)** if it is lower hemicontinuous at any $x_0 \in X$;

(v) F is **continuous at** $x_0 \in X$ if it is both uhc and lhc at x_0;

(vi) F is **continuous** if it is both uhc and lhc.

The next proposition provides sequential characterizations of correspondences that may be easier to interpret:

Proposition 5. Let (X,d_X) and (Y,d_Y) be metric spaces and $F: X \rightrightarrows Y$.

(i) F is lhc at x_0 if and only if for any sequence $\{x_n\}_n \subseteq X$ converging to x_0 and any $y_0 \in F(x_0)$, there is an N and a sequence $\{y_n\}_{n>N}$ with $y_n \in F(x_n)$, such that $y_n \to y_0$.

(ii) F is uhc (and *compact-valued*) at x_0 if (and only if) for any sequence $\{x_n\}_n \subseteq X$ converging to x_0 and any sequence $\{y_n\}_n$ such that $y_n \in F(x_n)$, there is some subsequence $\{y_m\}_m \subseteq \{y_n\}_n$ such that y_m converges to some $y_0 \in F(x_0)$.

Put loosely, part (i) of Proposition 5 shows that lhc is equivalent to stating that every point $y_0 \in F(x_0)$ can be reached by some sequence $y_n \in F(x_n)$. And part (ii) that uhc and compact-valuedness are equivalent to having that the limit y_0 of converging sequences $y_n \in F(x_n)$ is also a point in the limit set $F(x_0)$.

These concepts are difficult to diggest; it is very strongly recommended that you develop your understanding with the following:

Exercise 1. (i) Read the lecture notes on correspondences.

(ii) Watch a brief (10min) lecture by Rajiv Sethi on upper and lower hemicontinuity: https://youtu.be/OJfzJhsC3Rc.

One of the main results that you will reencounter later on to prove other fundamental results is Berge's maximum theorem:

Theorem 1. (Berge's Maximum Theorem) Let X and Θ be metric spaces, $f: X \times \Theta \to \mathbb{R}$ be a continuous function, and $B: \Theta \rightrightarrows X$ be a non-empty and compact-valued correspondence. Let $f^*(\theta) := \sup_{x \in B(\theta)} f(x,\theta)$ and $X^*(\theta) := \arg\sup_{x \in B(\theta)} f(x,\theta)$. If B is continuous at $\theta \in \Theta$, then f^* is continuous at θ and X^* is uhc, nonempty, and compact-valued at θ.

This is very useful theorem that we can then apply off-the-shelf in many circumstances. One of such applications is the following:

Proposition 6. If $\succsim$ is continuous, then $v(p,w)$ is continuous and $x(p,w)$ is upper hemicontinuous, nonempty- and compact-valued in (p,w).

Exercise 2. Prove Proposition 6 by showing the following steps:

(i) B is nonempty-valued;

(ii) B is closed-valued and bounded for any $(p,w) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$. Appeal to Heine–Borel theorem to show it is compact-valued;

(iii) B is uhc at any (p_0,w_0) *(The sequential characterization from Proposition 5 is probably easiest)*;

(iv) B is lhc at any (p_0,w_0) *(Take any open $U \subseteq \mathbb{R}^k_+ : B(p_0,w_0) \cap U \neq \emptyset$ and construct an $\epsilon > 0$ such that $B(p,w) \cap U \neq \emptyset$, $\forall (p,w) \in B_\epsilon((p_0,w_0))$)*;

(v) Argue that there is a continuous utility representation of $\succsim$;

(vi) Show that you can apply Berge's maximum theorem.

2.3. Implications of Convexity

The next properties are obtained by applying results we have seen in the previous set of lecture notes:

Proposition 7. If $\succsim$ is convex, then $x(p,w)$ is convex. If $\succsim$ is strictly convex, then $x(p,w)$ contains at most one element.

And, now, we combine both Berge's maximum theorem and the preceeding result to obtain:

Corollary 1. If $\succsim$ is continuous and strictly convex, then $x(p,w)$ is continuous in (p,w).

Exercise 3. Let (X,d_X) and (Y,d_Y) be metric spaces and $F : X \rightrightarrows Y$. Prove that if F is singleton-valued and uhc, then F is continuous.

2.4. Implications of Local Non-Satiation

Local nonsatiation, which we defined in the previous lecture, is exactly the condition that we need to show that the consumer always exhausts their budget. This is known as the Walras's law:

Proposition 8. **(Walras's Law)** If $\succsim$ is locally non-satiated, then for any $x \in x(p,w)$, and any $(p,w) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$, $p \cdot x = w$.

Proof. Let $x \in x(p,w)$ and suppose that $p \cdot x < w$. Then, $\exists \epsilon > 0$ such that $\forall x' \in B_\epsilon(x)$, $p \cdot x' < w$. By local nonsatiation, $\exists x'' \in B_\epsilon(x)$ such that $x'' \succ x$. As $x'' \in B(p,w)$, then $x \notin \arg\max_{\succsim} B(p,w)$. $\square$

Proposition 9. If $\succsim$ is continuous and locally nonsatiated, then $v(p,w)$ is strictly increasing in w.

Proof. $w < w' \Longrightarrow B(p,w) \subsetneq B(p,w')$. Take any $x \in x(p,w)$ and $x' \in x(p,w')$, which exist, by continuity. As $x \in x(p,w) \subseteq B(p,w)$, then $p \cdot x \leq w < w'$, and therefore it violates Walras's Law. Hence, $x \notin \arg\max_{\succsim} B(p,w') \ni x' \Longrightarrow x' \succ x \Longleftrightarrow v(p,w') = u(x') > u(x) = v(p,w)$. $\square$

2.5. Implications of Homotheticity

As mentioned in the last lecture, homothetic preferences ensure that a representative consumer exists. That is, if all consumers face the same prices and share the same preferences (but not necessarily the same incomes), then we can treat aggregate demand — the sum of all individual demands — as the choices of an agent that shares the same preferences and whose income is the sum of individual incomes. While this is not true in general, it holds when preferences are homothetic:

Proposition 10. Let every consumer $i \in I$ have income $w_i \geq 0$ and identical preferences $\succsim$. If $\succsim$ is continuous, homothetic and strictly convex, then $\sum_{i \in I} x(p,w_i) = x(p, \sum_{i \in I} w_i)$.

Proof. As $\succsim$ is homothetic, $x \in x(p,1) \Longleftrightarrow w \cdot x \in x(p,w)$. As $\succsim$ is strictly quasiconcave, $x(p,w)$ is at most a singleton. Continuity of $\succsim$ implies $x(p,w)$ is nonempty. Combining these results, we get that $\sum_{i \in I} x(p,w_i) = \sum_{i \in I} w_i x(p,1) = x(p, \sum_{i \in I} w_i)$. $\square$

3. Expenditure Minimization Problem

The consumer's utility maximization problem has a "dual problem": given a utility level u, the consumer chooses a bundle to minimize the expenditure incurred, subject to the requirement of attaining at least the prespecified utility threshold. More formally, let $\succsim$ be a preference relation on $X := \mathbb{R}^k_+$ and suppose that it admits a utility representation u. Define $U := \mathrm{co}(u(X))$, where $\mathrm{co}(A)$ denotes the **convex hull** of set A, i.e. the smallest convex set that contains A. For any $u \in U$, the consumer's expenditure minimization problem is given by

$$h(p,u) := \arg\min_{x \in X \,|\, u(x) \geq u} p \cdot x \tag{EMP}$$

$$e(p,u) := \inf_{x \in X \,|\, u(x) \geq u} p \cdot x$$

The set of minimizers $h(p,u)$ is called the **Hicksian demand**.

3.1. General Implications

A few properties follow without needing any further assumption. We start with a simple observation that mimicks Proposition 3:

Proposition 11. h is homogeneous of degree zero in p and e is homogeneous of degree one in p.

Proof. It follows by definition that $\forall \lambda > 0$, $h(\lambda p, u) = h(p,u)$ and $e(\lambda p, u) = \lambda e(p,u)$. □

While v was shown to be quasiconvex in (p,w), we find that e is concave in p, which will allow us to derive further properties:

Proposition 12. e is concave in p.

Proof. This follows from the fact that the if $f_i : X \to \mathbb{R}$ is concave for every $i \in I$, then $\inf_{i \in I} f_i$ is also concave in X.[3] But let us prove this directly in our case: Take any $p, p' \in \mathbb{R}^k_{++}$, $u \in U$, and $\lambda \in [0,1]$. Let $A := \{x \in X \mid u(x) \geq u\}$. For any $x \in A$, by definition, $p \cdot x \geq \inf_{x \in A} p \cdot x =:$

[3]Equivalently, the supremum over a family of convex functions $f_i : X \to \mathbb{R}$ is convex in X.

$e(p,u)$ and, similarly, $p' \cdot x \geq e(p',u)$. Hence, for any $x \in A$, $(\lambda p + (1-\lambda)p') \cdot x \geq \lambda e(p,u) + (1-\lambda)e(p',u)$. Then, $e(\lambda p + (1-\lambda)p', u) := \inf_{x \in A} (\lambda p + (1-\lambda)p') \cdot x \geq \lambda e(p,u) + (1-\lambda)e(p',u)$. □

Now we leverage concavity of e in p. For that, we need to introduce the concept of a supergradient.

Definition 3. A **supergradient** of $f : X \to \mathbb{R}$ at $x_0 \in X$ is an element $c \in \mathbb{R}^k$ such that $f(y) \leq f(x_0) + c \cdot (y - x_0)$, for all $y \in X$. We denote the set of supergradients of f at x_0 by $\partial f(x_0)$.

Theorem 2. Let $X \subseteq \mathbb{R}^k$ be a convex set and f be a real-valued function on X. f is concave if and only if $\forall x \in X$, $\partial f(x) \neq \emptyset$.

The intuition is as follows: pick $x, y, z \in X$. For $c \in \partial f(x)$, $f(y) \leq f(x) + c \cdot (y - x)$ and $f(z) \leq f(x) + c \cdot (z - x)$. By a convex combination of the two, with $\lambda \in (0,1)$, $\lambda f(y) + (1-\lambda) f(z) \leq f(x) + c(\lambda y + (1-\lambda)z - x)$. Choosing $x = \lambda y + (1-\lambda)z$ delivers concavity of f.

A supergradient — also called superderivative — is generalizing the notion of derivative to functions that are not necessarily differentiable everywhere. For instance, take the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$ if $x \leq 0$ and $f(x) = -x$ if $x > 0$. This is a concave function and its derivative exists everywhere — $f'(x) = 1$ if $x < 0$ and $f'(x) = -1$ for $x > 0$ — but at zero, where it has a kink. Its supergradient, however, is well-defined everywhere: $\partial f(x) = \{1\}$ for $x < 0$, $\partial f(x) = \{-1\}$ for $x > 0$ and $\partial f(0) = [-1, 1]$.

Given a convex $X \subseteq \mathbb{R}^k$ and a concave function $f : X \to \mathbb{R}$, we can say a lot about it:[4]

(i) For any $x \in \text{int}X$, $\partial f(x)$ is nonempty, convex, and compact.

(ii) For any $c \in \partial f(x)$ and $c' \in \partial f(x')$, $(c' - c) \cdot (x' - x) \leq 0$.

(iii) The supergradient $\partial f(x)$ is a singleton if and only if f is differentiable at x. In this case, $f'(x) = c \in \partial f(x) = \{c\}$.

(iv) f'' exists almost everywhere in $\text{int}X$.[5]

(v) If $k = 1$, at any $x \in \text{int}X$, $\partial f(x) = [f'_+(x), f'_-(x)]$, where f'_-, f'_+ denote the left- and right-derivatives of f.

[4]We can also have counterparts of all of these results for convex functions, as if f is concave, $-f$ is convex. Two references for the future: Boyd and Vandenberghe (2004) and the less well-known Niculescu and Persson (2018).

[5]This is called **Alexandrov theorem**.

Back to consumer demand. The next theorem is called the compensated law of demand and it says that the Hicksian demand is weakly decreasing in prices. We shall prove this result by showing that Hicksian demand is a supergradient of expenditure and then using the properties of supergradients.

Lemma 1. If $x_0 \in h(p_0, u)$, then x_0 is a supergradient of $e(\cdot, u)$ at p_0.

Proof. As $p_0 \cdot x_0 = e(p_0, u)$ and $p \cdot x_0 \geq e(p, u)$ for any $p \in \mathbb{R}^k_{++}$, we have that $e(p, u) \leq e(p_0, u) + x_0 \cdot (p - p_0)$. □

Theorem 3. **(Compensated Law of Demand)** If $p' \geq p$, $x \in h(p, u)$, and $x' \in h(p', u)$, then $(p' - p) \cdot (x' - x) \leq 0$.

Proof. This is obtained immediately by combining property (ii) of concave functions as listed above and Lemma 1. □

Note that, if p' equals p in every dimension except one, say dimension i for which $p'_i > p_i$, then the theorem is telling us that $x'_i \leq x_i$.

Finally, the last result that we can show without adding any assumption on preferences is a counterpart of the monotonicity properties in Proposition 2:

Proposition 13. e is weakly increasing in p and u.

Proof. Take any $u' \geq u$ and $p' \geq p$. For any $p'' \in \mathbb{R}^k_{++}$, we have that (by transitivity) $\{x \in X \mid u(x) \geq u\} \supseteq \{x \in X \mid u(x) \geq u'\} \implies e(p'', u) \leq e(p'', u')$. And, for any $u'' \in U$, $p \cdot x \leq p' \cdot x$ $\forall x : u(x) \geq u''$, which implies $e(p, u'') \leq e(p', u'')$. □

3.2. Implications of Continuity

We were sneaky in defining $e(p, u)$ as an infimum rather than a minimum, as the infimum will always be well defined (why?), but the minimum may not as $h(p, u)$ may be empty. As you may have anticipated, continuity of the utility function will solve this problem and provide some additional properties on Hicksian demand (courtesy of Berge's maximum theorem).

Proposition 14. If u is a continuous utility representation of $\succsim$, then $e(p, u)$ is continuous and $h(p, u)$ is nonempty, compact-valued, and uhc in (p, u).

Proof. Take an arbitrary point $x_0 \in X$ such that $u(x_0) \geq u$. As $u \in U$, x_0 must exist. Let $A := \{x \in \mathbb{R}^k_+ \mid p \cdot x \leq p \cdot x_0\}$

Claim: A is compact.

Note that A is closed. Let $\bar{x} \in X$ be such that all its coordinates are equal to the largest coordinate of x_0, denoted by $\bar{x}_0$. As $[0, \bar{x}_0]^k$ is a compact subset of $\mathbb{R}^k_+$ (why?) and $A \subseteq [0, \bar{x}_0]^k$ — as $p \cdot x \leq p \cdot x_0 \leq p \cdot \bar{x}_0$ — we have that A is compact.

Claim: $B := A \cap \{x \in X \mid u(x) \geq u\}$ is compact.

To see this, note that by continuity of u,[6] $\{x \in X \mid u(x) \geq u\}$ is closed. Hence, B is a closed subset of $[0, \bar{x}_0]^k$ and therefore compact.

Claim: $\min_{x \in B} p \cdot x = \inf_{x \in B} p \cdot x = \inf_{x \in X \mid u(x) \geq u} p \cdot x$.

The first equality is due to Weierstrass extremum theorem; the second equality is due to the fact that $\forall x \in B$, $y \in \{x \in X \mid u(x) \geq u\} \setminus B$, y induces a higher expenditure than x $p \cdot y > p \cdot x_0 \geq p \cdot x$, and both attain utility weakly higher than u.

Last step: The remainder of the proof follows by constructing a continuous, nonempty- and compact-valued correspondence that does not entail greater expenditure and then applying Berge's maximum theorem. □

Exercise 4. Complete the proof of Proposition 14.

In fact, with continuity we get that the lower bound on the utility is actually attained:

Lemma 2. If u is a continuous utility representation of $\succsim$, then $\forall x \in h(p, u)$, $u(x) = u$.

Proof. Suppose instead that $u(x) > u$. Then, continuity implies that $\exists \lambda \in [0, 1)$ such that $u(\lambda x) > u$, and as $p \cdot x > p \cdot \lambda x$, $x \notin h(p, u)$, a contradiction. □

This gives sense to the expression "compensated law of demand": by varying the prices p, $h(p, u)$ describes how the consumer substitutes across the different goods while attaining the same utility level. The "compensated" term comes from imagining that the consumer is given additional income to compensate the price changes. The next section — dealing with local non-satiation — makes this clearer.

[6] This is why we need to assume that u is a continuous utility representation and not just that $\succsim$ is continuous.

3.3. Implications of Local Non-Satiation

Theorem 4. Let $\succsim$ be locally nonsatiated and u be a continuous utility representation of $\succsim$. Then

(i) $h(p, v(p,w)) = x(p,w)$ and $e(p, v(p,w)) = w$;

(ii) $h(p,u) = x(p, e(p,u))$ and $u = v(p, e(p,u))$.

Exercise 5. Prove Theorem 4.

This equivalence between Marshallian ($x(p,w)$) and Hicksian demand ($h(p,u)$) casts light onto why the compensated demand $h(p,u)$ is called compensated. If by increasing prices $v(p,w)$ decreases, in order to keep $v(p,w) = u$ we need to compensate the consumer by increasing income w.

3.4. Implications of Convexity

Finally, to conclude the overview of the properties of the expenditure minimization program, we note some implications of convexity of preferences.

Proposition 15. (i) If $\succsim$ is convex, then $h(p,u)$ is convex.

(ii) If $\succsim$ is strictly convex and u is a continuous utility representation, then $h(p,u)$ is a singleton, continuous in (p,u), and $h(p,u) = e'_p(p,u)$.

Proof. For (i) take any $x, x' \in h(p,u)$ and any $\lambda \in [0,1]$. Note that $p \cdot (\lambda x + (1-\lambda)x') = e(p,u)$ and that $u(\lambda x + (1-\lambda)x') \geq \min\{u(x), u(x')\} \geq u$. Hence, $\lambda x + (1-\lambda)x' \in h(p,u)$.
For (ii), we note that by Theorem 4, we have that $x(p, e(p,u)) = h(p,u)$, and by Proposition 7, $x(p, e(p,u))$ is a singleton. Continuity follows from Proposition 14. The last bit of (ii) follows by the fact that $h(p,u)$ is unique supergradient of $e(p,u)$. □

4. Solving Optimization Problems using Calculus

It is expected and assumed that you wil be able to handle constrained optimization problems using Lagrangians and the Karush-Kuhn-Tucker conditions — although it is unlikely you will need it in this course. If you are unfamiliar with optimization using calculus, a concise reference is the Mathematical Appendix in Mas-Colell et al. (1995); in this case, the directly relevant appendices are *M.J: Unconstrained Optimization* (pp. 954–56), *M.K: Constrained Optimization* (pp. 956–64), and *M.L: The Envelope Theorem* (pp. 964–66).

5. Afriat's Theorem (*)

Suppose we observe data in the form $\{(x_t, p_t)\}_{t\in[T]}$. We want to know when it is the case that our data can be rationalized by positing that the consumer is maximizing utility. That is, $\forall t \in [T]$, $x_t \in x(p_t, w_t)$ for some income w_t. One issue that is quickly resolved is that we don't observe income. Notice that, if we assume that the consumer's preferences are locally nonsatiated, we should have that $p_t \cdot x_t = w_t$.

Let's recall some definitions on revealed preference, adjusted to the case at hand. We say that x_t is **directly revealed preferred to** x_s if x_t was chosen and x_s was affordable, i.e. $p_t \cdot x_s \leq p_t \cdot x_t$. Bundle x_t is **revealed preferred** to x_s if there is a sequence of bundles $\{x_m\}_{m\in[M]}$ such that x_t is directly revealed preferred to x_1, x_1 to x_2, and so on, with x_M being directly revealed preferred to x_s.

To adjust the definition of revealed strict preference, we rely on local nonsatiation (how?): we say that x_t is **revealed strictly preferred to** x_s if it was strictly less expensive than x_t under p_t, that is, $p_t \cdot x_s < p_t \cdot x_t$. Finally, our data satisfies **Generalized Axiom of Revealed Preference** (GARP) if there is no pair of bundles x, y such that x is revealed preferred to y and y is revealed strictly preferred to x.

Our main result for this section is:

Theorem 5. **(Afriat's (1967) Theorem)** Let be $\{(x_t, p_t)\}_{t\in[T]}$ be a collection of bundles a x_t at prices p_t. The following statements are equivalent:

(i) The data can be rationalized by a locally nonsatiated preference relation $\succsim$ that admits a utility representation.

(ii) There is a continuous, concave, piecewise linear, strictly monotone utility function u that rationalized the data.

(iii) The data satisfies GARP.

(iv) There exist positive $\{u_t, \lambda_t\}_{t\in[T]}$ such that $u_s \le u_t + \lambda_t p_t \cdot (x_s - x_t)$, for all $t, s \in [T]$.

A proof, while certainly not beyond the scope of this course, is surely beyond its time contraints. However, some comments are in order. First, comparing (i) and (ii) we see that if we can rationalize the data with local nonsatiation, we might as well throw in continuity, concavity, piecewise linearity, and strict monotonicity, as these pose no additional constraint on the (finite) data. Second, less surprisingly, GARP (appropriately redefined) is still exactly what we need to rationalize the data as being originated by preference-maximizing behavior.

Third, while GARP is already saving computing time/cost — and certainly data requirements — when compared to HARP, condition (iv) above is far easier to check as it reduces to a simple linear programming problem. Where does this condition come from? Recall the concept of a supergradient: $\forall q_t \in \partial u(x_t)$ and $\forall x_s$, $u(x_s) \le u(x_t) + q_t \cdot (x_s - x_t)$. If u is concave, then supergradients always exist, and, as u is differentiable almost everywhere (by concavity), $\partial u(x) = \{u'(x)\}$ almost everywhere. So, almost everywhere we get $\forall x_s$, $u(x_s) \le u(x_t) + u'(x_t) \cdot (x_s - x_t)$. Now, to get intuition as to why we have $\lambda_t p_t$ in the stead of $u'(x_t)$, just consider that u is in fact differentiable. The Langragian for utility maximization problem $\max_{x\in B(p,w)} u(x)$ is then given by $u(x) + \lambda \cdot (w - p \cdot x)$, with first-order conditions for an interior optimum $u'(x) = \lambda p$.

6. References

Afriat, Sidney N. 1967. "The Construction of Utility Functions from Expenditure Data." *International Economic Review* 8 (1): 67–77. 10.2307/2525382. 13

Boyd, Stephen, and Lieven Vandenberghe. 2004. *Convex Optimization*. Cambridge University Press. 8

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory*. Oxford University Press. 12

Niculescu, Constantin P., and Lars-Erik Persson. 2018. *Convex Functions and Their Applications: A Contemporary Approach*. Springer. 8

ECON0106: **Microeconomics**

4. Monotone Comparative Statics of Individual Choices*

Duarte Gonçalves[†]

University College London

1. Overview

As the previous lectures have shown, a typical problem in economics regards constrained optimization, where an agent choose and action x from $S \subseteq X$ to maximize an objective function $f : X \to \mathbb{R}$. It is then of interest to understand how the agent's behavior, as given by

$$X(S; f) := \arg\max_{x \in S} f(x),$$

changes when their objective f or their feasible set S change. This is what is typically termed comparative statics.

Comparative statics are monotone when one makes claims that $X(S; f)$ "increases" in some sense when S or f also "increase." A canonical example is that — fixing output level — firm demand for an input decreases weakly in the price to that input. A related question is whether — fixing input prices — firm demand for inputs increases in the target output. Further, when there multiple cost-minimizing manners to organize production, how should we compare the two sets of optimal inputs?

A classical approach to this problem is to consider the use of calculus, relying on the Lagrangian and a calculus-based version of the envelope theorem. In this lecture we will learn how to be able to derive comparative statics results *without* using calculus.

*Last updated: 26 October 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

2. General Definitions

We will start by defining a way to order elements. Let $(X, \geq)$ be a **partially ordered set**, that is, $\geq$ is a binary relation on X that is reflexive, transitive, and anti-symmetric.

Two important notational definitions that are easy to mistake are those of a join ($\vee$) and a meet ($\wedge$). The **join** of two elements x, x' taken with respect to X, written $x \vee_X x'$ corresponds to the $\geq$-smallest element(s) in X that are simultaneously larger than both x and x': $x \vee_X x' := \inf\{y \in X : y \geq x \text{ and } y \geq x'\}$, where the infimum is taken with respect to $\geq$. The **meet** — denoted by $x \wedge_X x'$ — is symmetrically defined: the $\geq$-largest element in X that are simultaneously smaller than both x and x': $x \wedge_X x' := \sup\{y \in X : x \geq y \text{ and } x' \geq y\}$.

We then call $(X, \geq)$ different names depending on the properties it satisfies regarding joins and meets:

Definition 1. (i) A partially ordered set $(X, \geq)$ for which joins and meets exist for any pair of elements — i.e. $\forall x, x' \in X$, $x \vee_X x' \in X$ and $x \wedge_X x' \in X$ — is called a **lattice.**

(ii) A **complete lattice** is one where any subset attains its supremum and infimum in the set: $\forall S \subseteq X$, $\sup_X S \in X$ and $\inf_X S \in X$[1]

(iii) A sublattice of X is a subset $S \subseteq X$ where that includes the joins and the meets of any two of its elements, where the joins and meets are taken in X, i.e. S is a **sublattice** if $\forall x, x' \in S$, $x \vee_X x' \in S$ and $x \wedge_X x' \in S$.

(iv) A **complete sublattice** S is a sublattice of X for which the supremum and infimum of any of its subsets $S' \subseteq S$ is contained in S. That is, a sublattice is complete if any of its subsets attains its supremum and infimum in it — again the supremum and infimum are taken in X.

Below are some examples that can help gaining intuition:

[1] Sometimes you will also see the notation $\vee_X S$ and $\wedge_X S$ instead of $\sup_X S$ and $\inf_X S$.

Example 1. 1. $((0,1),\geq)$ is a lattice but not a complete lattice.

2. $(\mathbb{R}^k,\geq)$, where $\geq$ is the natural product order[2] is a lattice.

   For any sublattice $S \subseteq \mathbb{R}^k$, S is a complete sublattice if and only if it is compact; further $(S,\geq)$ is then also a complete lattice (Topkis 1998, Theorem 2.3.1.).

3. $(0,1) \subseteq \mathbb{R}$ is a sublattice of $(\mathbb{R},\geq)$ but not a complete sublattice, where $\geq$ is the natural order.

4. Under the natural product order, $\{(0,0),(1,0),(0,1),(2,2)\}$ is a complete lattice, but not a sublattice of $\mathbb{N}^2$.

5. Under the product order, $\{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\}$ is not a lattice.

3. Ordering Sets

As mentioned earlier, one of the main complications is how to order sets based on the given partial order $\geq$. The existing literature provides some different alternatives. The most conventional one is the **strong set order** $\geq_{ss}$ (Topkis 1979, 1998; Milgrom and Shannon 1994), where $\geq_{ss}$ is a binary relation on the powerset of some set X, 2^X. It is defined as follows:

Definition 2. We say that S' **strong set dominates** S (writing $S' \geq_{ss} S$) if $\forall x' \in S', x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

That is, a set S strong set dominates another set S' if, taking any one element from each set, their join belongs to the dominating set and their meet to the dominated set.

Exercise 1. For instance, recall our definition of budget sets $B(p,w) := \{x \in \mathbb{R}^k_+ \mid p \cdot x \leq w\}$, with $p \in \mathbb{R}^k_{++}$ and $w \in \mathbb{R}_+$. Suppose that $\succsim$ are strongly monotone and assume $k \geq 2$.

(i) Prove that $\neg(B(p,w) \geq_{ss} B(p,w))$ $\forall w > 0$.

(ii) Fix p and provide necessary and sufficient conditions on w', w so that $B(p,w') \geq_{ss} B(p,w)$.

[2] $x \geq (>) y$ if $x_i \geq (>) y_i$ for every $i \in [k]$.

The strong set order can be too demanding and therefore inapplicable to many situations; this is one possible motivation for the weak set order (Che et al. 2021) that we will study later on.

4. Ordering Functions

We now want good notions to compare functions. And the supply more than met demand.

Definition 3. Let f be a real-valued function on $X \times T$, where X, T are partially ordered sets, and joins and meets of elements in $X \times T$ are with respect to the product order. We say that

(i) f satisfies the **single-crossing property** (SCP) in $(x;t)$ if $\forall x, x' \in X,\ t, t' \in T$, such that $x' > x$ and $t' > t$, $f(x';t) - f(x;t) \geq (>) 0 \implies f(x';t') - f(x;t') \geq (>) 0$. It satisfies the ***strict* single-crossing property** if the last inequality is always strict.

(ii) f has **increasing differences** (ID) in $(x;t)$ if $\forall x, x' \in X,\ t, t' \in T$, such that $x' > x$ and $t' > t$, $f(x';t') - f(x;t') \geq f(x';t) - f(x;t)$. It has ***strict* increasing differences** if the last inequality is always strict.

(iii) f is **quasisupermodular** (QSM) in (x,t) if $\forall y, y' \in X \times T$, $f(y) - f(y \wedge y') \geq (>) 0 \implies f(y \vee y') - f(y') \geq (>) 0$.

(iv) f is **supermodular** (SM) in (x,t) if $\forall y, y' \in X \times T$, $f(y \vee y') - f(y') \geq f(y) - f(y \wedge y')$. f is **submodular** if $-f$ is supermodular.

Note that SM $\implies$ {QSM, ID} $\implies$ SCP, i.e. satisfying the single-crossing property in $(x;t)$ is weaker than satisfying quasisupermodularity or increasing differences and each of these is weaker than satisfying supermodularity.

Single-crossing and quasisupermodularity provide *ordinal* conditions on f that can be readily translated into restrictions on preference relations. In contrast, increasing differences and supermodularity are their respective *cardinal* counterparts. It is then very much surprising that Chambers and Echenique (2009) found that preference relations on a lattice have a weakly monotone and quasisupermodular utility representation if and only if they have a weakly monotone and supermodular utility representation.

Some useful properties of supermodular functions:

Exercise 2. Prove the following statements:

(i) If f and g are supermodular real-valued functions on X, then $\alpha f + \beta g$ are supermodular $\forall \alpha, \beta \geq 0$.

(ii) If $\exists$ strictly increasing $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is supermodular, then f is quasisupermodular.

(iii) If $f \in \mathscr{C}^2$ in $y \in Y \equiv X \times T$, then f is supermodular in y if and only if $\frac{\partial^2}{\partial y_i \partial y_j} f \geq 0$, $\forall i \neq j$.

(iv) If X and Y are partially ordered sets, $X \times Y$ is a lattice with respect to the product order, and $f : X \times Y \to \mathbb{R}$ is supermodular, then $g(x) := \sup_{y \in Y} f(x, y)$ is supermodular.

The above are properties of *a* function. But note that given we have $f(x,t)$, we can interpret it as a parameterized family of functions $f_t(x) := f(x,t)$. So, we adjust the above definitions to handle comparison of functions. We will focus on one of them, single-crossing.

Definition 4. Let v, u be two real-valued functions on X; v **single-crossing** dominates u ($v \geq_{sc} u$) if $\forall x, x' \in X$ such that $x' \geq x$, $u(x') - u(x) \geq (>) 0 \Longrightarrow v(x') - v(x) \geq (>) 0$.

5. Monotone Comparative Statics of Individual Choices

5.1. Strong Monotone Comparative Statics

Theorem 1. (Monotonicity; (Milgrom and Shannon 1994, Theorem 4)) Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.

Proof. $\Longrightarrow$: Take any $x \in X(S;u), x' \in X(S';v)$. As $S' \geq_{ss} S$, we have $x \wedge x' \in S$ and $x \vee x' \in S'$. Then

$$
\begin{aligned}
& x \in X(S;u) && \\
\Longrightarrow & u(x) - u(x \wedge x') \geq 0 && \text{optimality of } x \\
\Longrightarrow & u(x \vee x') - u(x') \geq 0 && \text{quasisupermodularity of } u \\
\Longrightarrow & v(x \vee x') - v(x') \geq 0 && v \geq_{sc} u \\
\Longrightarrow & x \vee x' \in X(S';v) && \text{optimality of } x';
\end{aligned}
$$

and

$$
\begin{aligned}
& x' \in X(S';v) && \\
\Longrightarrow & v(x \vee x') - v(x') \leq 0 && \text{optimality of } x' \\
\Longrightarrow & v(x) - v(x \wedge x') \leq 0 && \text{quasisupermodularity of } v \\
\Longrightarrow & u(x) - u(x \wedge x') \leq 0 && v \geq_{sc} u \\
\Longrightarrow & x \wedge x' \in X(S;u) && \text{optimality of } x.
\end{aligned}
$$

Hence $X(S';v) \geq_{ss} X(S;u)$.

$\Longleftarrow$:

To show necessity of quasisupermodularity, let $S = \{x, x \wedge x'\}$, $S' = \{x', x \vee x'\}$, $\neg(x' \geq x)$, and $u = v$. Clearly, $S' \geq_{ss} S$. Note that if we have $u(x) \geq (>) u(x \wedge x') \Longleftrightarrow x \in (=) X(S;u)$. As $X(S';u) \geq_{ss} X(S;u)$, then $x \in (=) X(S;u) \Longrightarrow x \vee x' \in (=) X(S';u) \Longrightarrow u(x \vee x') \geq (>) u(x')$.

To show necessity of single-crossing, let $S = \{x, x'\}$ with $x' > x$. As $X(S;v) \geq_{ss} X(S;u)$, $x' \in (=) X(S;u) \Longrightarrow x' \in (=) X(S;v)$. And then, $u(x') - u(x) \geq (>) 0 \Longrightarrow v(x') - v(x) \geq (>) 0$. □

Some other results that are easy to obtain by adjusting the proof above:

Corollary 1. (Milgrom and Shannon 1994, Corollary 1) Let X be a lattice and f a real-valued function on X. f is quasisupermodular if and only if, for $S' \geq_{ss} S$, $X(S';f) \geq_{ss} X(S;f)$.

Corollary 2. (Milgrom and Shannon 1994, Corollary 2) Let X be a lattice, S a sublattice, and f a real-valued function on X. If f is quasisupermodular, then $X(S;f)$ is a sublattice of S.

Corollary 3. (Monotone Selection; Milgrom and Shannon 1994, Theorem 4') Let X be a lattice, v,u be two real-valued functions on X, and $S' \geq_{ss} S$, with $S,S' \subseteq X$. If v and u are quasisupermodular and v strictly single-crossing dominates u, then $\forall x' \in X(S';v), x \in X(S;u)$, $x' \geq x$.

This last result is stronger than it might look like at first glance: it is saying that *any* maximizer in $X(S';v)$ is greater than *any* maximizer in $X(S;u)$.

This next exercise will guide you through an application of these results:

Exercise 3. Suppose a firm hires labor $l \in \mathbb{R}_+$ for a wage rate $w > 0$ and capital $k \in \mathbb{R}_+$ for $r > 0$, in order to produce a quantity $y \in \mathbb{R}_+$. The firm sells its product at a price $p > 0$. Prices p,r,w are taken as given. Their production function, mapping combinations of inputs to output quantities, is given by $F : \mathbb{R}^2_+ \to \mathbb{R}_+$, and we write $y = F(k,l)$. Their profit maximization problem is then

$$\max_{(k,l)\geq 0} \pi(k,l;p,r,w) = pF(k,l) - rk - wl.$$

Throughout we fix (and omit dependence on) (p,w), and denote the optimal levels of inputs by $k^*(r)$ and $l^*(r)$

(i) Show that $\pi(k,l;r)$ has strict increasing differences in $(-k,r)$ and strict single-crossing property in $(-k,r)$.

(ii) Show that $\bar{l}(k,r) := \arg\max_{l\geq 0} \pi(k,l;r)$ may depend on k but does not depend (directly) on r; we can then write $\bar{l}(k)$ instead of $\bar{l}(k,r)$.

(iii) Assume $\bar{l}$ is a function (or fix a selection) and prove that $\pi(k,\bar{l}(k);r)$ has strict increasing differences in $(-k,r)$ and strict single-crossing property in $(-k,r)$. Use this to show that $k^*(r) = \arg\max_{k\geq 0} \pi(k,\bar{l}(k);r)$ is nonincreasing in r. Conclude that the own-price effect for capital is negative on the firm's capital demand.

(iv) Now let's consider the effect of a change in r on l. Assume $k^*(r)$ is a function (or fix a selection) and define $l^*(r) := \arg\max_{l\geq 0} \pi(k^*(r),l;r)$. Show that if $F(k,l)$ has increasing differences in (k,l) (resp. $(-k,l)$), then (i) $\pi(k^*(r),l;r)$ has increasing differences in $(l,-r)$ (resp. (l,r)) and (ii) conclude $l^*(r)$ is nonincreasing (resp. nondecreasing) in r, i.e. that increasing the price of capital weakly decreases the optimal level of capital and consequently weakly decreases (increases) the amount of labor hired.

(v) Comment on the relation between increasing differences in (k,l) vs. $(-k,l)$ and factor complementarity/substitutability.

6. References

Chambers, Christopher P., and Federico Echenique. 2009. "Supermodularity and Preferences." *Journal of Economic Theory* 144 (3): 1004–1014. 10.1016/j.jet.2008.06.004. 4

Che, Yeon-Koo, Jinwoo Kim, and Fuhito Kojima. 2021. "Weak Monotone Comparative Statics." *Working Paper* 1–65, `https://arxiv.org/pdf/1911.06442.pdf`. 4

Milgrom, Paul, and Chris Shannon. 1994. "Monotone Comparative Statics." *Econometrica* 62 (1): 157—180. 10.1007/BF01215200. 3, 5, 6, 7

Topkis, Donald M. 1979. "Equilibrium Points in Nonzero-Sum n-Person Submodular Games." *SIAM Journal on Control and Optimization* 17 773—787. 10.1137/0317054. 3

Topkis, Donald M. 1998. *Supermodularity and Complementarity*. Princeton, NJ: Princeton University Press. 3

ECON0106: **Microeconomics**

5. Expected Utility*

Duarte Gonçalves†

University College London

1. Overview

So far we have looked at situations in which the outcomes of choices are deterministic, that is, where the decision-maker knows exactly all the consequences associated to all alternatives and makes the utility-maximizing choice.

Here is an example of what is missing in our model: Say a decision-maker sets out to buy a computer. It just so happens that new computers may or may not be faulty. The decision-maker would prefer the computer not to be faulty, so our model treats faulty computer and not faulty computer as two different elements. The issue is that, ex-ante, one may know how likely it is that a computer is faulty, but not know whether a computer is or is not faulty. How should we take that into account?

In this lecture we will address exactly this question: how to have a useful utility representation of preferences over probability distributions. We will work with the assumption that the agent is facing a situation in which these probabilities are *known* and *objective*. In the case above, this amounts to knowing the exactly probability that the computer is faulty. When probabilities over outcomes are known, we say the agent is facing **risk**. This is an important assumption: it allows a researcher to treat these probabilities as something observable.

There are of course cases in which probabilities over outcomes are unknown. For instance, people may disagree about the probability with which a given team wins a sports match. When probabilities are not known, we say the agent is facing **uncertainty**[1] and

*Last updated: 31 October 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

[1]The term "ambiguity" is also used in the literature.

their beliefs about the likelihood of the outcome are *subjective*. This is a crucial difference, as we cannot directly observe a person's beliefs. We defer a treatment of uncertainty to later on and focus first on choice under risk.

2. Setup

Let X finite **outcome space**. An element $x \in X$ entails a complete description of all relevant aspects of the environment. That is, in the example above of buying a computer, x would describe whether a particular computer is faulty or not. We denote the **set of all probability measures on** X by $\Delta(X)$, i.e. the set of all functions $p : X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$; we will occasionally call p a **lottery**. Equivalently, as X is finite — say $|X| = N$, you can think of p as a vector in a subset of $[0,1]^N$.[2] We let $\succsim$ be a preference relation on $\Delta(X)$, and we endow the set with the Euclidean metric.

A special case is that of **degenerate probability measures**, i.e. those assigning probability one to a particular outcome and we will write them differently whenever we want to emphasize that aspect. In particular, we write $\delta_x \in \Delta(X)$ to denote the probability measure that assigns probability 1 to x, where $\delta_x(x') = 1$ if $x' = x$ and zero if otherwise.

For any $\alpha \in [0,1]$, and any $p, p' \in \Delta(X)$, we write the **probability mixture** as $\alpha p + (1-\alpha)p'$ to denote the probability measure $\alpha p + (1-\alpha)p' \in \Delta(X)$ such that for any $x \in X$, $(\alpha p + (1-\alpha)p')(x) = \alpha p(x) + (1-\alpha)p'(x)$. Two things to note. First, that $\Delta(X)$ is convex with respect to mixtures. Second, that a probability mixture *is not* a probability distribution on probability distributions. That is, $\alpha p + (1-\alpha)p'$ *is not* the probability distribution that delivers p with probability α and p' with completementary probability. (in the same way that $\delta_x \neq x$). The former lives in $\Delta(X)$ (on which preferences are defined), whereas the latter lives in $\Delta(\Delta(X))$. We discuss this subtlety at the end of the notes.

[2] Or, given that it is a probability, in the $(N-1)$-dimensional simplex $\Delta^{N-1} := \{p \in [0,1]^{|X|-1} \mid \sum_{i=1}^{N-1} p_i \leq 1\}$.

3. Expected Utility

We want to have a sensible conditions that allow us to have a useful utility representation of the agent's preferences over $\Delta(X)$. From what we have already seen, we know that if, say, $\succsim$ is a continuous preference relation on $\Delta(X)$, then it has a continuous utility representation $U : \Delta(X) \to \mathbb{R}$, that is, a utility function such that $p \succsim p' \iff U(p) \geq U(p')$. This is *a* utility representation, but we want something more tractable.

Suppose we restrict X to be money amounts. One possibility is to have utility being equal to the expected value, $U(p) = \mathbb{E}_p[x]$, where $\mathbb{E}_p[x] = \sum_{x \in X} p(x)x$. Now take these two fair lotteries: p that assigns equal probability to £5 and -£5, and p' gives probability $1/2$ to both £5,000 and -£5,000. Both have expected value equal to zero; however, some people can and do disagree about which is better and have strict preferences for one over the other. It would then be too restrictive to simply assume that everyone is indifferent.

Now consider p'' such that is assigns equal probability to £5,000 and -£5. A reasonable assumption would be that everyone prefers p'' to both p (and p') as the worst outcome is the same and occurs with same probability in both, and the best outcome also occurs with the same probability in both p and p'' too, but it is far better in the p'' than in p. As $x \in X$ can be anything, the expected value approach also doesn't make much sense in many cases. So we are looking for something that relaxes the expected value assumption, but retains its appeal

In some sense, we want to capture this by having a utility representation that disentangles these two elements: objective probabilities p, and preferences over outcomes x. Expected utility does just this.

Definition 1. We say that $\succsim$ on $\Delta(X)$ has an **expected utility (EU) representation** if there is $u : X \to \mathbb{R}$ such that $\forall p, p' \in \Delta(X)$, $p \succsim p' \iff \mathbb{E}_p[u] \geq \mathbb{E}_{p'}[u]$.

The function u is called a *Bernoulli* or *von Neumann–Morgenstern utility*. As $\sum_{x \in X} p(x)u(x)$ is just an expectation with respect to p, we will use the more compact notation $\mathbb{E}_p[u]$ to denote it. Unfortunately, it is not the case that any continuous preference relation $\succsim$ on $\Delta(X)$ has a EU representation; we need something else.

3.1. Properties

Definition 2. We say that a preference relation $\succsim$ on $\Delta(X)$ satisfies **independence** if $\forall p, p' \in \Delta(X)$, $p \succsim (\succ) p'$ if and only if for any $p'' \in \Delta(X)$, and any $\alpha \in (0,1]$, $\alpha p + (1-\alpha)p'' \succsim (\succ) \alpha p' + (1-\alpha)p''$.

In essence, what independence buys us is linearity in the space of probability distributions: $p \sim p' \Longrightarrow \alpha p + (1-\alpha)p' \sim p'$. This is necessary if we want to have an expected utility representation because *expectations are linear in probabilities* in this sense: $\mathbb{E}_p[u] = \mathbb{E}_{p'}[u] \Longrightarrow \mathbb{E}_{\alpha p+(1-\alpha)p'}[u] = \mathbb{E}_p[u]$. On the other hand, this implies that we are ruling out strict preference for randomization — i.e. we cannot have $p \sim p'$ and $\alpha p + (1-\alpha)p' \succ p'$.

We consider two other properties:

Definition 3. A preference relation $\succsim$ on $\Delta(X)$

(i) has the **Archimedean property** if $\forall p, p', p'' \in \Delta(X)$ such that $p \succ p' \succ p''$, there is an $\alpha, \beta \in (0,1)$ for which $\alpha p + (1-\alpha)p'' \succ p' \succ \beta p + (1-\beta)p''$;

(ii) satisfies **vNM continuity**[3] if $\forall p, p', p'' \in \Delta(X)$ such that $p \succsim p' \succsim p''$, $\exists \gamma \in [0,1]$ for which $\gamma p + (1-\gamma)p'' \sim p'$.

We can see that if $\succsim$ has an expected utility representation, then it must be vNM continuous. To see this, note that if $\mathbb{E}_p[u] \geq \mathbb{E}_{p'}[u] \geq \mathbb{E}_{p''}[u]$, then there is $\gamma \in [0,1]$ such that $\gamma \mathbb{E}_p[u] + (1-\gamma)\mathbb{E}_{p''}[u] = \mathbb{E}_{p'}[u]$. Then, by linearity of the expectation operation in p, $\gamma \mathbb{E}_p[u] + (1-\gamma)\mathbb{E}_{p''}[u] = \mathbb{E}_{\gamma p+(1-\gamma)p''}[u]$.

This next exercise will help you develop intuition on what they imply:

Exercise 1. Let $\succsim$ be a preference relation on $\Delta(X)$.

(i) Prove or find a counterexample:

  (a) None of the three properties implies another.

  (b) If $\succsim$ satisfies independence and the Archimedean property, then it is vNM continuous.

[3] I am going to call it vNM continuity — where vNM stands for von-Neumann and M for Morgenstern — to distinguish it from our previous notion of preference continuity.

(c) If $\succsim$ satisfies independence and vNM continuity, then it has the Archimedean property.

(ii) Show that if $\succsim$ satisfies independence and the Archimedean property, then there are $\overline{p}, \underline{p} \in \Delta(X)$ such that $\overline{p} \succsim p \succsim \underline{p}$, for all $p \in \Delta(X)$.

(iii) How do these properties relate to continuity?

(a) Does continuity imply or is implied by vNM continuity or the Archimedean property?

(b) Does continuity imply or is implied by independence?

(c) Is continuity implied by independence and vNM continuity?

3.2. Expected Utility Representation Theorem

The main result for this lecture is von Neumann and Morgenstern's (1953) expected utility representation theorem:

Theorem 1. Let X be finite and let $\succsim$ be a preference relation over $\Delta(X)$.

(i) $\succsim$ satisfies independence and vNM continuity if and only if it admits an expected utility representation u.

(ii) If u and v are two expected utility representations of $\succsim$, then $\exists \alpha > 0$, $\beta \in \mathbb{R}$ such that $v = \alpha u + \beta$.

Proof. The "if" part of (i) was shown in the main text. For the "only if" part of (i) we break the proof into several small steps.

Step 1 As X is finite, $\exists \delta_{\overline{x}}, \delta_{\underline{x}} \in \Delta(X)$ such that $\forall \delta_x \in \Delta(X)$, $\delta_{\overline{x}} \succsim \delta_x \succsim \delta_{\underline{x}}$.

Step 2 $\forall \{p_i\}_{i \in [n]} \subseteq \Delta(X)$ and $\forall p, p' \in \Delta(X)$ such that $p \succsim p'$, we have that $\alpha_0 p + \sum_{i \in [n]} \alpha_i p_i \succsim \alpha_0 p' + \sum_{i \in [n]} \alpha_i p_i$, for any $\{\alpha_i\}_{i \in \{0\} \cup [n]} \in [0,1]^{n+1}$ such that $\sum_{i \in \{0\} \cup [n]} \alpha_i = 1$.

Proof. If $\alpha_0 \in \{0,1\}$, the claim is trivially satisfied. For $\alpha_0 \in (0,1)$, $1 - \alpha_0 = \sum_{i \in [n]} \alpha_i$, define $p'' := \sum_{i \in [n]} \frac{\alpha_i}{1-\alpha_0} p_i$ (which, by convexity of $\Delta(X)$ with respect to mixtures, belongs to $\Delta(X)$). Then, by independence,

$$\begin{aligned}\alpha_0 p + \sum_{i\in[n]} \alpha_i p_i &= \alpha_0 p + (1-\alpha_0)p'' \\ &\succsim \alpha_0 p' + (1-\alpha_0)p'' = \alpha_0 p' + \sum_{i\in[n]} \alpha_i p_i.\end{aligned}$$

□

Step 3 $\forall p \in \Delta(X)$, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Proof. Fix an order on $X = \{x_1, x_2, ..., x_n\}$, such that $x_1 = \overline{x}$ and $x_n = \underline{x}$. By Step 1 and repeated application of Step 2,

$$\begin{aligned}\delta_{\overline{x}} = \sum_{i=1}^{n} p(x_i)\delta_{\overline{x}} &\succsim p(x_1)\delta_{x_1} + \sum_{i=2}^{n} p(x_i)\delta_{\overline{x}} \\ &\succsim p(x_1)\delta_{x_1} + p(x_2)\delta_{x_2} + \sum_{i=3}^{n} p(x_i)\delta_{\overline{x}} \succsim \cdots \\ &\succsim \sum_{i=1}^{n} p(x_i)\delta_{x_i} = p \\ &\succsim p(x_1)\delta_{\underline{x}} + \sum_{i=2}^{n} p(x_i)\delta_{x_i} \succsim \cdots \\ &\succsim \sum_{i=1}^{n} p(x_i)\delta_{\underline{x}} = \delta_{\underline{x}}\end{aligned}$$

□

If $\delta_{\overline{x}} \sim \delta_{\underline{x}}$, set u equally constant to any number.

In the sequel, assume $\delta_{\overline{x}} \succ \delta_{\underline{x}}$.

Step 4 $\forall \alpha, \beta : 1 \geq \alpha > \beta \geq 0$, $\alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} \succ \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$.

Proof. By independence,

$$\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\overline{x}} + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}} \succ \left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\underline{x}} + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}} = \delta_{\underline{x}}.$$

Then, again by independence,

$$\alpha\delta_{\overline{x}}+(1-\alpha)\delta_{\underline{x}}=\beta\delta_{\overline{x}}+(1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\overline{x}}+\left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}}\right]$$
$$\succ\beta\delta_{\overline{x}}+(1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\underline{x}}+\left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}}\right]=\beta\delta_{\overline{x}}+(1-\beta)\delta_{\underline{x}}$$

□

Step 5 $\forall p\in\Delta(X)$, there is unique $\gamma(p)\in[0,1]$ such that $\gamma(p)\delta_{\overline{x}}+(1-\gamma(p))\delta_{\underline{x}}\sim p$.

Proof. By Step 3, $\delta_{\overline{x}}\succsim p\succsim\delta_{\underline{x}}$. vNM continuity ensures existence of a $\gamma\in[0,1]$. By Step 4, it must be unique. □

Step 6 Let $u:X\to\mathbb{R}$ be given by $u(x)=\gamma(\delta_x)$. $p\sim\left(\sum_{i\in[n]}p(x_i)\gamma(\delta_{x_i})\right)\delta_{\overline{x}}+\left(1-\sum_{i\in[n]}p(x_i)\gamma(\delta_{x_i})\right)\delta_{\underline{x}}$.

Proof. By repeated application of independence, Step 2, and definition of γ,

$$p=\sum_{i=1}^{n}p(x_i)\delta_{x_i}\sim\sum_{i=1}^{n}p(x_i)\left(\gamma(\delta_{x_i})\delta_{\overline{x}}+(1-\gamma(\delta_{x_i}))\delta_{\underline{x}}\right)$$
$$=\sum_{i=1}^{n}p(x_i)\left(\gamma(\delta_{x_i})\right)\delta_{\overline{x}}+\sum_{i=1}^{n}p(x_i)\left((1-\gamma(\delta_{x_i}))\right)\delta_{\underline{x}}$$

□

Step 7 Take any $p,p'\in\Delta(X)$. $p\succsim p'\iff\mathbb{E}_p[u]\geq\mathbb{E}_{p'}[u]$.

Proof. By Step 4 and Step 5, $\gamma(p)\delta_{\overline{x}}+(1-\gamma(p))\delta_{\underline{x}}\sim p\succsim p'\sim\gamma(p')\delta_{\overline{x}}+(1-\gamma(p'))\delta_{\underline{x}}$, if and only if $\gamma(p)\geq\gamma(p')$. By Step 5 and Step 6, it must be that $\mathbb{E}_p[\gamma]=\sum_{i\in[n]}p(x_i)\gamma(\delta_{x_i})=\gamma(p)$. By definition, $\mathbb{E}_p[u]=\mathbb{E}_p[\gamma]$. □

For (ii), take u as defined in (i) and let v be some other EU representation of $\succsim$.
Note that for any $p\in\Delta(X)$, it must be that $v(\overline{x})\geq\mathbb{E}_p[v]\geq v(\underline{x})$. Therefore, define $\phi(p)$ as the unique number such that $\phi(p)v(\overline{x})+(1-\phi(p))v(\underline{x})=\mathbb{E}_p[v]$.
As

$$\phi(p)v(\overline{x})+(1-\phi(p))v(\underline{x})=\mathbb{E}_{\phi(p)\delta_{\overline{x}}+(1-\phi(p))\delta_{\underline{p}}}[v],$$

we have that

$$\phi(p)\delta_{\overline{x}} + (1-\phi(p))\delta_{\underline{x}} \sim p \sim \gamma(p)\delta_{\overline{x}} + (1-\gamma(p))\delta_{\underline{x}}.$$

By Step 5, $\gamma(p) = \phi(p)$. Hence, $u = \frac{v - v(\underline{x})}{v(\overline{x}) - v(\underline{x})}$. □

The theorem above implies that a expected utility representation is unique up to affine transformations. This implies that u has a cardinal interpretation. But note that this is just the same as many structural properties of preferences we have seen: while strict monotone transformations of the utility representation are also representing the same preferences, they need not preserve additive separability or even continuity.

4. Concluding Remarks

4.1. Compound Lotteries

As promised a brief discussion on compound lotteries or lotteries over lotteries. First, what is a lottery over lotteries? Take again the two lotteries we used in our initial example: p that assigns equal probability to £5 and -£5, and p' gives probability 1/2 to both £5,000 and -£5,000. A compound lottery ℓ is for instance a lottery that gives you p with probability 1/2 and p' with complementary probability. This *is not* the same as the mixture of $p'' = 1/2p + 1/2p''$, which gives you -£5, £5, -£5,000, and £5,000 all with probability 1/4; p'' is a *reduction* of ℓ and, in fact, you may value them differently. Segal (1990) provides a discussion on how you can have EU representations for preferences on $\Delta(X)$ and $\Delta(\Delta(X))$ that treat the compound lottery and the reduced lottery differently — unless the compount lottery is degenerate, i.e. assigns probability one to a specific $p \in \Delta(X)$, in which case, one would argue, there is nothing to reduce.

4.2. Issues with Expected Utility

Over lunch, during a colloquium in Paris on choice under risk,[4] sometime between 12 and 17 May 1952, Maurice Allais arguing that EU was not a good descriptive theory asked J. Leonard Savage (who we will encounter later on) the following question:

[4]Which included some very famous people in the discipline, such as Kenneth Arrow, Bruno de Finetti, Milton Friedman, Ragnar Frisch, Jacob Marschak, besides the two main characters in the story.

1. Which of the following two gambles do you prefer?

    a) £2 million wp 1; or

    b) £2 million wp .89; £10 million wp .10; nothing wp .01.

Savage readily answer a). Allais had then a follow-up question:

2. Which of the following two gambles do you prefer?

    A) nothing wp .89; £2 million wp .11; or

    B) nothing wp .90; £10 million wp .10.

To which Savage replied B). Allais then told him that his choices could not be rationalized by EU. This became known as the Allais (1953) paradox, and the evidence supports that most people make the same choices.

Exercise 2. Show that if a person chooses a) and B) or b) and A), then their behavior cannot be rationalized by EU. That is, if $\succsim$ are such that a) $\succ$ (resp. $\prec$) b) and B) $\succ$ (resp. $\prec$) A), then $\succsim$ cannot admit a EU representation. Which property is this violating?

Should we just throw away the model? No. There are two reasons why we should not do that. One is if you — like Savage[5] — take a *normative* instead of descriptive stance and believe that this is a rational way to behave. In many respects, theory is also meant to provide advice on how to act (as does engineering on how to build a bridge).

Even if your inclinations are towards a more descriptive approach to modeling behavior (as are my own), this still does not mean you should throw away the model. All models will be wrong, as they are just that, simplified descriptions. In many domains, expected utility maximization provides a good enough approximation to describing behavior. For that it is important to understand the conditions under which it performs well and when it fails, so that we can improve on it.

There are two plausible explanations for the Allais paradox. One is that in question 1. there is the possibility of getting something good for sure, the so-called *certainty effect*. The other is that b) has the possibility of getting nothing with positive probability and there

[5]Savage later replied that he had acted *irrationally* and that he still thought that the properties were good characterizations of rational behavior (Heukelom 2015).

may be a natural aversion to getting nothing. These are naturally related and you cannot disentangle them from this question alone, you need to go beyond that.

To conclude, let's point out three possible ways (out of many) that extend expected utility and accommodate Savage's intuitive choices: One is **rank-dependent expected utility** (popularized by cumulative prospect theory) (Quiggin 1982), in which the small probabilities of the worst events loom larger than they are. A second one is **cautious expected utility** (Cerreia-Vioglio et al. 2015), which uses the following relaxion of independence: $\forall p, p' \in \Delta(X)$, $x \in X$, and $\alpha \in [0,1]$, if $p \gtrsim \delta_x$, then $\alpha p + (1-\alpha)p' \gtrsim \alpha\delta_x + (1-\alpha)p'$. A third way — **ordered reference dependent choice** (Lim 2021) — focuses on the fact that choices depend on context: in this case, on having both a sure-thing and the possibility of gaining nothing.

5. References

Allais, Maurice. 1953. "Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'École Américaine." *Econometrica* 21 (4): 503–546. 10.2307/1907921. 9

Cerreia-Vioglio, Simone, David Dillenberger, and Pietro Ortoleva. 2015. "Cautious Expected Utility and the Certainty Effect." *Econometrica* 83 (2): 693–728. 10.3982/ECTA11733. 10

Heukelom, Floris. 2015. "A history of the Allais paradox." *The British Journal for the History of Science* 48 (1): 147–169. 10.1017/S0007087414000570. 9

Lim, Xi Zhi. 2021. "Ordered Reference Dependent Choice." *Working Paper* 1–85, `https://arxiv.org/abs/2105.12915v2`. 10

von Neumann, John, and Oskar Morgenstern. 1953. *Theory of Games and Economic Behavior*. Princeton, NJ: Princeton University Press. 5

Quiggin, John. 1982. "A Theory of Anticipated Utility." *Journal of Economic Behavior and Organization* 3 (4): 323–343. 10.1016/0167-2681(82)90008-7. 10

Segal, Uzi. 1990. "Two-Stage Lotteries without the Reduction Axiom." *Econometrica* 58 (2): 349–377. 10.2307/2938207. 8

ECON0106: **Microeconomics**

6. Risk Attitudes[*]

Duarte Gonçalves[†]

University College London

1. Overview

In this lecture, we will introduce and study risk attitudes. Attitudes toward risk are of fundamental importance in understanding individuals' behavior in face of risk: how they constitute their financial portfolio, their behavior in the context of a pandemic, their purchasing decisions, their willingness to take up a job or continue searching for a better one, and they even relate to how people vote. By grounding our definitions in terms of primitives, we not only gain a better understanding on what properties of expected utility representations imply, we are also then able to identify and test statements based on data.

We will focus on the case where the decision-maker has preferences over gambles affecting their wealth. Why wealth? Well, this can be motivated by making use of, for instance, consumer theory. As we have seen, with continuous preferences on the set of bundles, $v(p,w) = \max_{x\in B(p,w)} u(x)$. With continuous preferences satisfying independence and the Archimedean property on the set of distributions over wealth, we get a utility representation that looks like $\mathbb{E}_F[v(p,\cdot)]$. And, with local nonsatiation, we also get that $v(p,\cdot)$, our Bernoulli utility, is strictly increasing in w.[1]

Our first task is to introduce and study behavioral notions of risk aversion, that is, definitions which can be falsified with data; in our case, choice data. After defining these, we show how these relate to properties of the Bernoulli utility function in an expected utility representation. We then provide a behavioral way to compare individuals in terms of their risk attitudes, even if they are not risk averse, and again show how this related

[*]Last updated: 5 November 2021.

[†] Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

[1]For details on induced preferences on wealth, see Kreps (2012, Section 6.2).

to strutural properties of their expected utility representations. Finally, we consider how attitudes toward risk can be affected by wealth.

2. Setup

Our agents will now have expected utility representations not just on probability distributions over a finite set of items, but over the real line.

Let X be a convex subset of $\mathbb{R}$ where $x \in X$ denotes the final wealth of the decision-maker (the outcome). Probabilities are now given by (cumulative) distributions functions $F : \mathbb{R} \to [0,1]$ such that F is nondecreasing, right-continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$ with support on X, i.e. $\mathbb{P}_F(X) = \int_X dF(x) = 1$. For any (cumulative) distribution function F, we denote the expectation operator by $\mathbb{E}_F[\cdot]$ and we define its mean as $\mu_F := \int_X x dF(x)$. We will focus on the set of distributions on X with finite mean, which we denote by $\mathscr{F}$.

We want $\succsim$ to have expected utility representation, that is, a $u : X \to \mathbb{R}$ such that $\forall F, G$, $F \succsim G \iff \mathbb{E}_F[u] \geq \mathbb{E}_G[u]$. For convenience, we will define $U(F) := \mathbb{E}_F[u]$. In order to make sure that we have a utility representation U, we will assume that $\succsim$ is a continuous preference relation on $\mathscr{F}$. For us to have an expected utility representation u, $\succsim$ has to also satisfy independence and the Archimedean property.[2] Note that this implies that $U(F)$ has to be finite, as otherwise it will not satisfy the Archimedean property. Finally, we will assume that the decision-maker prefers more money to less, i.e. $x > y \implies \delta_x \succ \delta_y$ ($\succsim$ is strictly monotone in $\{\delta_x, x \in X\}$). This will resut in having u be strictly increasing.

To simplify the statement of the results, we set this as an assumption that we maintain throughout this lecture:

Assumption 1. The preference relation $\succsim$ on $\mathscr{F}$ has an expected utility representation with $u : X \to \mathbb{R}$ strictly increasing.

3. Risk Attitudes

We want a concept that captures the idea of avoiding and seeking risk; this is main motivation for the definitions of risk averse/seeking preferences. For instance, an intuitive

[2]We are glossing over some subtleties here; see section 5.2 in Kreps (2012) for details, in particular Propositions 5.3 and 5.10.

definition of a person who is risk averse is someone who would decline to take fair gambles, say making or losing £1 with equal probability. Naturally, risk seeking would be defined as the opposite, and risk neutral as someone who is indifferent between taking and not taking the fair bet. Then, if you extend this notion to lotteries that do not have expected value of zero, you would have that a risk averse person is someone who prefers to get the expected value of a lottery to taking the lottery. This is exactly what our definition is saying:

Definition 1. A preference relation $\succsim$ on $\mathscr{F}$ is

(i) **risk averse** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \succsim F$;

(ii) **risk neutral** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \sim F$;

(iii) **risk seeking** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \precsim F$.

For every lottery, we assume that there is a value that makes the agent indifferent between taking that lottery and taking this fixed sure value. From this, we obtain the following concepts:

Definition 2. (i) The **certainty equivalent** of F for $\succsim$ is the real number $c(F, \succsim) \in X$ such that $\delta_{c(F,\succsim)} \sim F$.

(ii) The **risk premium** of F for $\succsim$ is the real number $R(F, \succsim) := \mu_F - c(F, \succsim)$.

Exercise 1. Show that if $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$, then $c(F, \succsim)$ is uniquely defined.

If is intuitive that if an agent is risk averse, then their certainty equivalent is lower then the expected value of a gamble: they would be willing to give up money to avoid risk. Our next result not only shows this, it also relates risk aversion with a structural property of our Bernoulli utility function:

Theorem 1. Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. The following statements are equivalent:

(i) $\succsim$ is risk averse (risk seeking).

(ii) $c(F,\succsim) \leq (\geq) \mu_F$, $\forall F \in \mathscr{F}$.

(iii) u is concave (convex).

Proof.

(i)$\Longleftrightarrow$(ii): $\delta_{\mu_F} \succsim F \Longleftrightarrow u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = u(c(F,\succsim))$, where we used monotonicity of u.

(i)$\Longrightarrow$(iii): $\forall x, x' \in X$, such that $x > x'$, and any $\alpha \in [0,1]$, let F deliver x with probability α and x' with complementary probability. Then $u(\alpha x + (1-\alpha)x') = u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = \mathbb{E}_F[u] = \alpha u(x) + (1-\alpha)u(x')$.

(i)$\Longleftarrow$(iii): Take the same F as defined above. Then, $U(\delta_{\mu_F}) = u(\mu_F) \geq \mathbb{E}_F[u] = U(F)$.

The proof for the equivalences for risk seeking preferences is symmetric. □

4. Comparing Risk Attitudes

It is not necessary that a person be consistently risk averse (or risk seeking). For instance, a person may be willing to take a bet for low stakes, but not to take it if the stakes are too high. If so, they are neither risk averse nor risk seeking. Still, we want to compare different people in what regards their risk attitudes:

Definition 3. $\succsim^a$ is said to be more risk averse than $\succsim^b$ if $F \succsim^a \delta_x \Longrightarrow F \succsim^b \delta_x$, $\forall F \in \mathscr{F}$, $\forall x \in X$.

That is, if whenever person b declines a bet in favor of some sure thing, a more risk averse person a declines too.

We will be able to capture this with an index that summarizes their risk attitudes:

Definition 4. For an expected utility representation $u \in C^2$ and $x \in X$, the **Arrow-Pratt coefficient of absolute risk aversion** is given by $r_A(x,u) := -\frac{u''(x)}{u'(x)}$.

What is the coefficient of absolute risk aversion measuring? The *rate* at which marginal utility of wealth changes. Why the rate and not just the curvature? The following exercise will help with that:

Exercise 2. Let $\succsim$ be a preference relation on $\mathscr{F}$, and $u \in \mathscr{C}^2$ a strictly increasing expected utility representation of $\succsim$.

1. Show that, $\succsim$ is risk-averse if and only if $r_A(x,u) \leq 0$.
2. If $v \in \mathscr{C}^2$ is another expected utility representation of $\succsim$, then what is the relationship between $r_A(x,v)$ and $r_A(x,u)$?

This coefficient will allow us, later on, to make statements on how risk attitudes change with wealth. Before that, we want to show that this indeed captures how attitudes toward risk of different individuals compare.

Theorem 2. Let $\succsim^a, \succsim^b$ be two preference relations on $\mathscr{F}$. Let u^a, u^b be strictly increasing expected utility representations of $\succsim^a, \succsim^b$, respectively. The following statements are equivalent:

(i) $\succsim^a$ is more risk averse than $\succsim^b$.

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b)$, $\forall F \in \mathscr{F}$.

(iii) If $u^b \in \mathscr{C}^0$,[3] then there is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

(iv) If $u^a, u^b \in \mathscr{C}^2$, then $r_A(x, u^a) \geq r_A(x, u^b)$ for any $x \in X$.

Proof.

(i)$\Longleftrightarrow$(ii): $\delta_{c(F,\succsim^a)} \sim^a F \precsim^b \delta_{c(F,\succsim^b)} \Longleftrightarrow c(F,\succsim^a) \leq c(F,\succsim^b)$.

(ii)$\Longrightarrow$(iii): As u^b is strictly increasing, then ${u^b}^{-1}$ is well-defined. As u^a is strictly increasing, then let $\phi := u^a \circ {u^b}^{-1}$. As X is convex and u^b is continuous and strictly increasing, then $u^b(X)$ (the domain of ϕ) is convex.[4] Note that $\phi(u^b(x)) = u^a({u^b}^{-1}(u^b(x))) = u^a(x)$.

We prove by contrapositive. Suppose that ϕ is not concave. Then, there are $x, x' \in X$,

[3]Continuity of u^b is needed to show that (ii)$\Longrightarrow$(iii), but not the converse.
[4]This is where continuity of u^b plays a role.

and $\alpha \in (0,1)$, such that $\phi(\alpha u^b(x) + (1-\alpha)u^b(x')) < \alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x'))$. Let F yield x with probability α and x' with complementary probability. Note that $\phi(\alpha u^b(x) + (1-\alpha)u^b(x')) = \phi(\mathbb{E}_F[u^b])$ and $\alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x')) = \mathbb{E}_F[\phi \circ u^b]$. Then

$$
\begin{aligned}
u^a(c(F, \succsim^a)) &= U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b] \\
&> \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F, \succsim^b)) = u^a(c(F, \succsim^b)).
\end{aligned}
$$

By monotonicity of u^a, we obtain that $c(F, \succsim^a) > c(F, \succsim^b)$.

(ii)$\Longleftarrow$(iii):

$$
\begin{aligned}
u^a(c(F, \succsim^a)) &= U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b] \\
&\leq \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F, \succsim^b))) = u^a(c(F, \succsim^b)),
\end{aligned}
$$

which, by strict monotonicity of u^a implies $c(F, \succsim^a) \leq c(F, \succsim^b)$.

(iii)$\Longleftrightarrow$(iv): As u^a, u^b are strictly increasing and differentiable, $u^{a\prime}, u^{b\prime} > 0$. As $\phi := u^a \circ u^{b^{-1}}$ and $u^a, u^b \in \mathscr{C}^2$, then $\phi' > 0$ and $\phi \in \mathscr{C}^2$. Moreover, $u^{a\prime\prime}(x) = \phi''(u^b(x))(u^{b\prime}(x))^2 + \phi'(u^b(x))u^{b\prime\prime}(x)$. Then,

$$
\begin{aligned}
r_A(x, u^b) = -\frac{u^{b\prime\prime}}{u^b} &\leq -\frac{\phi''(u^b(x))(u^{b\prime}(x))^2 + \phi'(u^b(x))u^{b\prime\prime}(x)}{\phi'(u^b(x))u^{b\prime}(x)} = r_A(x, u^a) \\
&= -\frac{\phi''(u^b(x))u^{b\prime}(x)}{\phi'(u^b(x))} - \frac{u^{b\prime\prime}(x)}{u^{b\prime}(x)} \\
\Longleftrightarrow \phi'' \leq 0,
\end{aligned}
$$

proving the equivalence.

$\square$

5. Risk Attitudes Changing with Wealth

It is popular wisdom that wealthier people are more risk seeking. Or equivalently, risk aversion decreases with wealth. This section will provide us the tool to express these ideas.

Let $F+w$ denote the distribution which arises from adding w to every outcome, i.e. $(F+w)(x) := F(x-w)$. For a preference relation $\succsim$ on $\mathscr{F}$, we will write $\succsim_w$ to denote the preference relation of the agent given additional wealth w, i.e. $F \succsim_w G \Longleftrightarrow F+w \succsim G+w$. Analogously, we define its expected utility representation $u_w(x) := u(x+w)$, and $U_w(F) = \mathbb{E}_F[u_w]$. Finally, to simplify the statements, let us just assume that $X = \mathbb{R}$.

Definition 5. We say that u exhibits **decreasing/constant/increasing absolute risk aversion** (DARA/CARA/IARA) if $r_A(x,u)$ is decreasing/constant/increasing in x.

Theorem 3. Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. The following statements are equivalent:

(i) If $u \in \mathscr{C}^2$, u exhibits DARA.

(ii) $\succsim_{w^a}$ is more risk averse than $\succsim_{w^b}$, $\forall w^a \leq w^b$.

(iii) $c(F,\succsim_{w^a}) \leq c(F,\succsim_{w^b})$, $\forall F \in \mathscr{F}$, $\forall w^a \leq w^b$.

(iv) $w^b - w^a \leq c(F+w^b,\succsim) - c(F+w^a,\succsim)$, $\forall F \in \mathscr{F}$, $\forall w^a \leq w^b$.

Proof.

(i)$\Longleftrightarrow$(ii): Follows from Theorem 2(i)$\Longleftrightarrow$(iv).

(ii)$\Longleftrightarrow$(iii): Follows from Theorem 2(i)$\Longleftrightarrow$(ii).

(iii)$\Longleftrightarrow$(iv): This follows from this next lemma.

Lemma 1. Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. Then, $c(F,\succsim_w) = c(F+w,\succsim) - w$.

Proof.

$$u(c(F,\succsim_w)+w) = u_w(c(F,\succsim_w)) = \mathbb{E}_F[u_w] = \int_X u_w(x)dF(x) = \int_X u(x+w)dF(x)$$
$$= \int_{X+w} u(x)dF(x-w) = \mathbb{E}_{F+w}[u] = u(c(F+w,\succsim)),$$

where $X+w := \{x+w \mid x \in X\}$. □

□

6. Two Functional Forms for Expected Utility

In this section we will see where two extremely common functional forms for Bernoulli utility come from. With this, we gain a better understanding of what we are assuming with adopting that functional form for a given model.

Proposition 1. For any $u \in \mathscr{C}^2$, such that $r_A(x,u) = \gamma$, $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $u(x) = -\alpha \operatorname{sign}(\gamma)\exp(-\gamma x) + \beta$ if $\gamma \neq 0$, and $u(x) = \alpha x + \beta$ if otherwise.

Proof.

$$r_A(x,u) = -\frac{u''(x)}{u'(x)} = \gamma \Longleftrightarrow \int \gamma dx = -\int \frac{u''(x)}{u'(x)} dx \Longleftrightarrow \ln u'(x) + k_1 = -\gamma x.$$

If $\gamma \neq 0$, then

$$\ln u'(x) + k_1 = -\gamma x \Longleftrightarrow u'(x) = \exp(-\gamma x + k_1) \Longleftrightarrow u(x) = -\frac{\exp(k_1)}{\gamma}\exp(-\gamma x) + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. If instead $\gamma = 0$,

$$0 = -\ln u'(x) + k_1 \Longleftrightarrow u'(x) = \exp(-k_1) \Longleftrightarrow u(x) = \exp(-k_1)x + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. □

That is, CARA preferences are completely pinned-down up to positive affine transformations, as with any expected utility representation.

Definition 6. For an expected utility representation $u \in C^2$ and $x \in X$, the **Arrow-Pratt coefficient of relative risk aversion** is given by $r_R(x,u) := -\frac{u''(x)}{u'(x)}x$.

Proposition 2. For any $u \in \mathscr{C}^2$, such that $r_R(x,u) = \gamma$, $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $u(x) = \alpha \frac{x^{1-\gamma}}{1-\gamma} + \beta$, if $\gamma \neq 1$, and $u(x) = \alpha \ln(x) + \beta$ if otherwise.

Proof.

$$r_R(x,u) = -\frac{u''(x)}{u'(x)}x = \gamma \Longleftrightarrow \int \gamma \frac{1}{x} dx = -\int \frac{u''(x)}{u'(x)} dx \Longleftrightarrow \ln u'(x) = -\gamma \ln x + k_1 \Longleftrightarrow u'(x) = \exp(k_1)x^{-\gamma}$$
$$\Longleftrightarrow u(x) = \exp(k_1)\frac{x^{1-\gamma}}{1-\gamma} + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. If $\gamma = 0$, then we can use l'Hôpital's rule:

$$\lim_{\gamma \to 1} u(x) = \lim_{\gamma \to 1} \exp(k_1) \frac{x^{1-\gamma}}{1-\gamma} + k_2 = \lim_{\gamma \to 1} \exp(k_1) \frac{-x^{1-\gamma} \ln x}{-1} + k_2 = \exp(k_1) \ln x + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. □

An interesting fact about CRRA preferences: it is actually *the only* class of utility functions that, in a Solow model with technological progress at rate g, delivers a balanced growth path, i.e. $\frac{k_{t+1}}{k_t} = \frac{c_{t+1}}{c_t} = 1 + g$.

7. Application

7.1. Buying and Selling Risky Assets

Exercise 3. Given a gamble $\tilde{x}$ (a real-valued random variable) and a strictly increasing, twice continuously differentiable, $u : \mathbb{R} \to \mathbb{R}$, let the *sale price* s of the gamble is defined by $\mathbb{E}[u(\tilde{x})] = u(s)$. Thus, s is the minimum amount of money the person with the utility function u must be given in order to induce them to give up the gamble $\tilde{x}$.
If instead they start with no money and no gample, the maximum price they would be willing to pay fo the gamble $\tilde{x}$ is its *buy price* b, defined as $\mathbb{E}[u(\tilde{x} - b)] = u(0)$.

1. Show that s and b are uniquely defined.

2. Show that if u exhibits constant absolute risk aversion, then $b = s$.

3. What is the relationship between b and s if u exhibits strictly decreasing absolute risk aversion? Prove your claim.

4. Consider another gamble $\tilde{y} = \tilde{x} + c$, where $c \in \mathbb{R}$ is a constant. Let s_y be the sale price of $\tilde{y}$.

   (a) What is the relationship between s_y and s if u exhibits constant absolute risk aversion? Prove your claim.

   (b) What is the relationship between s_y and s if u exhibits strictly decreasing absolute risk aversion? Prove your claim.

7.2. Demand for Risky Assets

Exercise 4. An agent has $w > 0$ pounds to invest in two assets. The first asset is risk free – for every pound invested, the agent receives $1+r$ pounds, with $r > 0$. The second asset is risky: for every pound invested, the agent gets a gross return of $\theta \in \Theta$, where θ is a real-valued random variable distributed according to F.

If the agent invests x pounds in the risky asset, their expected utility is given by $U(x) = \int_\Theta u(x\theta + (1+r)(w-x))dF(\theta)$. Assume that u is concave, strictly increasing, and differentiable.

The agent chooses $x^* \in \arg\max_{x\geq 0} U(x)$. We allow the agent to choose $x > w$, i.e. they can borrow at the risk free rate r.

1. Show that U is concave.
2. Suppose that $\mathbb{E}[\theta] < r$. Solve for the agent's optimal investment decision.
3. Suppose that $\mathbb{E}[\theta] = r$. Show that the investment decision found in part 2 is still optimal.
4. Suppose that $\mathbb{E}[\theta] > r$. Show that if a solution to the agent's problem exists, then the agent will always invest a strictly positive amount of money on the risky asset.

7.3. Compounding Risks

Exercise 5. Let an agent with preferences $\succsim$ over gambles on wealth. Suppose gamble F yields x and $-y$ with equal probability and $x > y$. Let $F^{(n)}$ be a gamble that has n copies of F and $G^{(n)}$ be n copies of a gamble G_n that yields x/n and $-y/n$ with equal probability. That is,

- a random variable $w \sim F^{(n)}$ is such that $w = \sum_{i=1}^n w_i$, where each w_i is independently and identically distributed according to F;
- a random variable $z \sim G^{(n)}$ is such that $z = \sum_{i=1}^n z_i$, where each z_i is independently and identically distributed according to G_n.

1. Prove or disprove:

   (a) If $F \gtrsim \delta_{\mu_F} \implies F^{(2)} \gtrsim \delta_{\mu_{F^{(2)}}}$.

   (b) If $F \gtrsim \delta_{\mu_F} \implies G^{(2)} \gtrsim \delta_{\mu_{G^{(2)}}}$.

2. Assume the agent is risk averse and has a twice continuously differentiable Bernoulli utility u. Prove or disprove:

   (a) If $F \gtrsim \delta_0 \implies F^{(2)} \gtrsim \delta_0$.

   (b) If $F \gtrsim \delta_0 \implies G^{(2)} \gtrsim \delta_0$.

   (c) If n is large enough, then $F^{(n)} \gtrsim \delta_0$.

   (d) If n is large enough, then $G^{(n)} \gtrsim \delta_0$.

8. References

Kreps, David M. 2012. *Microeconomic Foundations I. Choice and Competitive Markets*. Princeton, NJ: Princeton University Press. 1, 2

ECON0106: **Microeconomics**

7. Stochastic Orders*

Duarte Gonçalves†

University College London

1. Overview

In the previous lectures we considered how individuals evaluate distributions, e.g. of stock returns or lottery tickets prizes. We modeled their preferences of distributions and derived properties of the expected utility representation of a particular agent based on how they ranked them. In this lecture we take a different approach: we want to know how to rank distributions in an unambiguous manner among groups of individuals.

First, we look at a ranking on distributions with which *every* expected utility maximizer prefer one distribution would agree. While this is quite a strong requirement, we obtain a simple characterization based on how the (cumulative) distributions compare. We discuss the properties that this ordering has and a useful refinement.

Then, we look into riskiness. The idea is to have a well-grounded notion of what it means for a distribution to be riskier than another. Our strategy will be to require that *every* risk-averse expected utility maximizer would agree on which distribution is riskier. Again, this turns out to also have a simple characterization.

Finally, we briefly discuss a recent result on how background risks can affect the ranking of distributions.

2. First-Order Stochastic Dominance

Our first ranking on the space of distributions requires every expected utility maximizer to agree.

*Last updated: 8 November 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

Let $\mathscr{F}$ denote the set of all distributions on $X \subseteq \mathbb{R}$.

Definition 1. A distribution F **first-order stochastically dominates** (FOSD) a distribution G, denoted by $F \geq_{FOSD} G$ if, for all nondecreasing functions $u : X \to \mathbb{R}$, $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$.

This means that every expected utility maximizer with increasing Bernoulli utility would weakly prefer F to G. This is quite a strong requirement. The following theorem provides a simple characterization:

Theorem 1. For any distributions F, G on $\mathbb{R}$, $F \geq_{FOSD} G$ if and only if, $\forall x \in X$, $F(x) \leq G(x)$.

The first, more restrictive, version of this theorem first appeared in Hadar and Russell (1969).

Proof. $\Longrightarrow$: For any $a \in X$, define $u_a(x) := \mathbf{1}_{\{x \geq a\}}$, where $\mathbf{1}_A$ is the indicator function, taking the value 1 if A is true and 0 if otherwise. Note that u_a is nondecreasing. Then,

$$
\begin{aligned}
F \geq_{FOSD} G \Longrightarrow \mathbb{E}_F[u_a] \geq \mathbb{E}_G[u_a] &\Longleftrightarrow \int_X u_a(x) dF(x) \geq \int_X u_a(x) dG(x) \\
&\Longleftrightarrow \int_{x \geq a} 1 dF(x) \geq \int_{x \geq a} 1 dG(x), \qquad \forall a \in X \\
&\Longleftrightarrow 1 - F(a) \geq 1 - G(a) \Longleftrightarrow F(a) \leq G(a), \qquad \forall a \in X.
\end{aligned}
$$

$\Longleftarrow$: For this part we are going to make use of a result in statistics called the inverse transform method. For a cumulative distribution F of a real-valued random variable, define the **generalized inverse** — also called a **quantile function** — $Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}$, for every $\tau \in (0,1)$.[1]

The next result is also very useful in statistics, to simulate random variables given by difficult expressions:

Proposition 1. (Inverse Transform Method) Let F be the cumulative distribution of a real-valued random variable X. Then, X has the same distribution as $Q_F(U)$, $X \overset{d}{=} Q_F(U)$, where U is uniformly distributed in $(0,1)$.

[1] Why min and not inf? Because, as F is nondecreasing and right-continuous with left-limits, it is upper semi-continuous, and for any τ, $\{x \in \mathbb{R} \mid F(x) \geq \tau\}$ is closed and therefore contains its infimum.

The inverse transform method gives then a way to represent the distribution of X through a transformation of a standard uniformly distributed random variable. This is very convenient computationally as we know how to efficiently simulate uniformly distributed random variables. As we will see, this transformation is also helpful from a theoretical standpoint.[2] First, let's prove Proposition 1:

Proof. We want to show that $\mathbb{P}(Q_F(U) \leq x) = F(x)$. First note that Q_F is nondecreasing: As F is nondecreasing, $\forall \tau' \geq \tau$, $\{x \in \mathbb{R} \mid F(x) \geq \tau'\} \subseteq \{x \in \mathbb{R} \mid F(x) \geq \tau\} \implies Q_F(\tau) \leq Q_F(\tau')$.

Now take any $\tau \in (0,1)$ and x such that $\tau < F(x)$.

$$\tau < F(x) \implies Q_F(\tau) \leq Q_F(F(x)) \leq x$$

where the last inequality is due to $\tau < F(x) \implies x \in \{y \in \mathbb{R} \mid F(y) \geq \tau\}$ and, by definition, $Q_F(F(x)) \leq x$.

As we have that $Q_F(\tau) \leq x$ implies $\tau \leq F(x)$, we can order the following three events, recalling that U is uniformly distributed on $(0,1)$;

$$\begin{aligned}
&\{U < F(x)\} \subseteq \{Q_F(U) \leq x\} \subseteq \{U \leq F(x)\} \\
\iff &\mathbb{P}(U < F(x)) \leq \mathbb{P}(Q_F(U) \leq x) \leq \mathbb{P}(U \leq F(x)) \\
\iff &F(x) \leq \mathbb{P}(Q_F(U) \leq x) \leq F(x).
\end{aligned}$$

□

Let's finalize our proof of Theorem 1 by showing that $F(x) \leq G(x), \forall x \in X \implies F \geq_{FOSD} G$. Define Q_F and Q_G as the quantile functions of F and G. Then,

$$\begin{aligned}
F(x) \leq G(x), \forall x \in X \implies &(F(x) \geq \tau \implies G(x) \geq \tau) \\
\implies &\{x \in X \mid F(x) \geq \tau\} \subseteq \{x \in X \mid G(x) \geq \tau\} \\
\implies &Q_F(\tau) \geq Q_G(\tau).
\end{aligned}$$

[2] Another implication of Proposition 1 is that if $X \sim F$, with F continuous, then $F(X) \sim U(0,1)$.

As then we finally get

$$
\begin{aligned}
F(x) \leq G(x),\ \forall x \in X \implies & Q_F(z) \geq Q_G(z),\ \forall z \in (0,1) \\
\implies & u(Q_F(z)) \geq u(Q_G(z)),\ \forall z \in (0,1) && \text{as } u \text{ is nondecreasing} \\
\implies & \int_{[0,1]} u(Q_F(z))dz \geq \int_{[0,1]} u(Q_G(z))dz \\
\iff & \int_X u(x)dF(x) \geq \int_X u(x)dG(x) && \text{by inverse transform sampling} \\
\iff & \mathbb{E}_F[u] \geq \mathbb{E}_G[u].
\end{aligned}
$$

□

Exercise 1. Consider $\geq_{FOSD}$ on $\Delta([0,1])$.

1. Prove or disprove: $\geq_{FOSD}$ is (i) reflexive; (ii) transitive; (iii) antisymmetric; (iv) complete.
2. In light of 1, how would you classify $(\Delta([0,1]), \geq_{FOSD})$?
3. Is $(\Delta([0,1]), \geq_{FOSD})$ a lattice? Is it a complete lattice?

Exercise 2.

(i) Let $F, G, \hat{F}, \hat{G} \in \Delta(\mathbb{R})$. Show that if $\mathbb{E}_F[u] \geq \mathbb{E}_{\hat{F}}[u]$ and $\mathbb{E}_G[u] \geq \mathbb{E}_{\hat{G}}[u]$ for every nondecreasing Bernoulli utility function u, then $\alpha F + (1-\alpha)G \geq_{FOSD} \alpha\hat{F} + (1-\alpha)\hat{G}$.

(ii) For a distribution F on $\mathbb{R}$, let $G := (F + w)$, with $w > 0$. Show that $G \geq_{FOSD} F$.

Exercise 3. Suppose an agent is selling two lottery tickets, x and y, with $x \geq_{FOSD} y$. Which one should have a higher price?

3. Monotone Likelihood Ratio Order

One stochastic ordering that you will encounter almost surely is the monotone likelihood ratio order. For this section, we will restrict attention to distributions that either (i) admit a density or (ii) have discrete support. If the F is a distribution satisfying (i) f will denote its density, whereas if it satisfies (ii) we use f to denote its probability mass function.

Here's the definition of the monotone likelihood ratio order:

Definition 2. Let F,G two distributions on $\mathbb{R}$ and suppose that they (i) either both admit a density, or (ii) both have discrete support. F **monotone likelihood ratio** dominates G ($F \geq_{MLR} G$) if $f(x)/g(x)$ is nondecreasing in x.

One extremely convenient property of $\geq_{MLR}$ is that it is not only a partial order, but also a coarsening of $\geq_{FOSD}$ within this class of distributions:[3]

Theorem 2. Let F,G two distributions on $\mathbb{R}$ and suppose that they (i) either both admit a density, or (ii) both have discrete support. If $F \geq_{MLR} G$, then $F \geq_{FOSD} G$.

Proof. $f(x)g(y) \geq f(y)g(x) \, \forall x \geq y \implies f(x)G(x) - F(x)g(x) \geq 0, (1-F(x))g(x) - f(x)(1-G(x)) \geq 0 \, \forall x$. As $f(x)G(x) - F(x)g(x) \geq 0 \implies \frac{f(x)}{g(x)} \geq \frac{F(x)}{G(x)}$ and $(1-F(x))g(x) - f(x)(1-G(x)) \geq 0 \implies \frac{1-F(x)}{1-G(x)} \geq \frac{f(x)}{g(x)}$, we obtain $G(x) \geq F(x)$ for all x. □

One of the reasons for why this is such a convenient ordering is that it pairs very well with Bayesian updating (i.e. Bayes' rule).

Exercise 4. Suppose a coin toss flips heads ($x = 1$) with probability $\theta \in [0,1]$, and tails ($x = 0$) with complementary probability. Due to machine impression, θ is distributed according to a distribution F with density $f > 0$. You know f and want to estimate θ.

(i) Show that for any sequences $x_1, \dots, x_m$, $x'_1, \dots, x'_n$ such that $n \geq m$ and $\sum_i x_i \geq \sum_i x'_i$, you have $\theta | x_1, \dots, x_m \geq_{MLR} \theta | x'_1, \dots, x'_n$. Conclude that $\mathbb{E}_F[\theta | x_1, \dots, x_m] \geq \mathbb{E}_F[\theta | x'_1, \dots, x'_n]$.

(ii) Now suppose that there is another machine that produces coins, but with a different impression: $\theta \sim G$, where $g := G' > 0$. Show that if $f \geq_{MLR} g$, then, for any sequence of coin tosses $x_1, \dots, x_m$, $f \mid x_1, \dots, x_m \geq_{MLR} g \mid x_1, \dots, x_m$. Conclude that $\mathbb{E}_F[\theta | x_1, \dots, x_m] \geq \mathbb{E}_G[\theta | x_1, \dots, x_m]$.

4. Second-Order Stochastic Dominance

Since we observe that individuals are typically risk-averse, it may be useful the minimal requirements under which a lottery is preferred to another for any risk-averse expected

[3] Some pairs of distributions can be compared according to $\geq_{FOSD}$ but not according to $\geq_{MLR}$, i.e. $\geq_{MLR} \subseteq \geq_{FOSD}$.

utility maximizer. This provides us also with not only natural but also a sharper definition of what it means for a lottery to be riskier than another.

Definition 3. A distribution F **second-order stochastically dominates** (SOSD) a distribution G, denoted by $F \geq_{SOSD} G$ if $\mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$ for all nondecreasing, concave functions $u : \mathbb{R} \to \mathbb{R}$, such that $\mathbb{E}_F[u] - \mathbb{E}_G[u]$ is well-defined and $\int_{-\infty}^{0} u(x)dF(x), \int_{-\infty}^{0} u(x)dG(x) > -\infty$.

If we restrict F and G to have bounded support, $\int_{-\infty}^{0} u(x)dF(x), \int_{-\infty}^{0} u(x)dG(x) > -\infty$ is automatically satistied.

From the definitions, it should be immediate that $F \geq_{FOSD} G \implies F \geq_{SOSD} G$. That is, $\geq_{SOSD}$ is finer than $\geq_{FOSD}$ as it allows us to compare the same elements and more ($\geq_{FOSD} \subseteq \geq_{SOSD}$). The next theorem fully characterizes second-order stochastic dominance from the properties of the distributions alone:

Theorem 3. For any distributions F, G on $\mathbb{R}$, $F \geq_{SOSD} G$ if and only if, $\forall x \in X$, $\int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds$.

This result has had a troubled history; we follow the statement in Tesfatsion (1976).[4] Given that the proof of the statement in such generality is quite daunting, we will prove the theorem for distributions F, G with bounded support.

Proof. First, let us recall that, from integration by parts, $\int_a^b u(x)dF(x) = F(b)u(b) - F(a)u(a) - \int_a^b F(x)du(x)$. As we are assuming that F, G have bounded support, let $\overline{x}, \underline{x}$ be such that $F(\underline{x}) = G(\underline{x}) = 0$ and $F(\overline{x}) = G(\overline{x}) = 1$ and we assume u is defined on $(\underline{x} - \epsilon, \overline{x} + \epsilon)$, for some $\epsilon > 0$.

$\implies$: Let $u_a(x) = \mathbf{1}_{x \leq a}(x - a)$, a nondecreasing and concave function. From integration by parts, we have $\int_{\underline{x}}^{a} u_a(x)dF(x) - \int_{\underline{x}}^{a} u_a(x)dG(x) = (F(a) - G(a))(a - a) - (F(\underline{x}) - G(\underline{x}))u_a(\underline{x}) +$

[4] In case you find a weaker statement with a correct proof, do let me know; our restrictions are hidden in how we defined $\geq_{SOSD}$. Early versions of this theorem appeared in Hadar and Russell (1969) and Hanoch and Levy (1969). The first one imposed excessively restrictive assumptions: finite support and strictly increasing utility. The second imposed no restrictions — not even our condition — but is incorrect; a corrected version was given by Tesfatsion (1976), which we follow in our statement.

$\int_{\underline{x}}^{a}(G(x)-F(x))dx = \int_{\underline{x}}^{a}(G(x)-F(x))$. Then,

$$\begin{aligned}
\mathbb{E}_F[u_a]-\mathbb{E}_G[u_a]\geq 0, \quad \forall a &\iff \int_{x\leq a} u_a(x)dF(x) \geq \int_{x\leq a} u_a(x)dG(x), \quad \forall a\\
&\iff \int_{x\leq a} u_a(x)dF(x) - \int_{x\leq a} u_a(x)dG(x) \geq 0, \quad \forall a\\
&\iff \int_{x\leq a} (G(x)-F(x))dx \geq 0, \quad \forall a\\
&\iff \int_{\underline{x}}^{a} F(x)dx \leq \int_{\underline{x}}^{a} G(x)dx, \quad \forall a.
\end{aligned}$$

$\Longleftarrow$: Let us construct a linear interpolation of any concave nondecreasing u on $[\underline{x},\overline{x}]$.

For any $n\in\mathbb{N}$ let $x_i^n := \underline{x}+\frac{i}{n}(\overline{x}-\underline{x})$ for $i=0,\dots,n$. The set $\{x_i^n\}_{i=0}^n$ is an evenly spaced grid on $[\underline{x},\overline{x}]$, where $x_{i+1}^n - x_i^n = \frac{1}{n}(\overline{x}-\underline{x})$.

Now we want to construct u^n defined on $[\underline{x},\overline{x}]$ such that $u^n(x_i^n)=u(x_i^n)$, that is, that touches u at each point in the grid, and is a linear interpolation of u, which means that it will also be nondecreasing and concave. To do this, define $c_i^n := \frac{u(x_i^n)-u(x_{i-1}^n)}{x_i^n - x_{i-1}^n}$, which gives the slope of the line that connects $u(x_{i-1}^n)$ to $u(x_i^n)$. As u is nondecreasing, we must have that $c_i^n \geq 0$. Furthermore, as u is concave, we have that c_i^n is nonincreasing in i.

Exercise 5. Prove that c_i^n is nonnegative and nonincreasing in i.

For any $x\in[\underline{x},\overline{x}]$, by construction, $\exists i\in[n]$ such that $x\in[x_{i-1}^n, x_i^n]$, and then we define $u^n(x) := u(x_{i-1}^n)+c_i^n(x-x_{i-1})$.[5] Clearly, $u^n(x_i^n)=u(x_i^n)$ for every $i=0,\dots,n$ and $u(x)-u^n(x)\geq 0$.

Exercise 6. Prove that

(i) u^n is concave.

(ii) If $x\in[x_{i-1}^n, x_i^n]$, then $u(x_{i-1}^n)+c_i^n(x-x_{i-1}^n)\leq u(x)\leq u(x_{i-1}^n)+c_{i-1}^n(x-x_{i-1}^n)$. You may assume that u is differentiable. *(Hint: The mean value theorem may be of use.)*

Given (ii) from the exercise above, we have that

$$|u(x)-u(x_{i-1}^n)| \leq c_{i-1}^n(x-x_{i-1}^n) \leq c_0^n(x-x_{i-1}^n) \leq u(x_1^n)-u(\underline{x}),$$

[5] This is equivalent to defining $u^n(x) := \mathbf{1}_{x=\underline{x}}u(\underline{x})+\sum_{i\in[n]}\mathbf{1}_{x\in(x_{i-1}^n,x_i^n]}[u(x_{i-1}^n)+c_i^n(x-x_{i-1}^n)]$.

where we used that $c_i^n \geq 0$ and nonincreasing in i, and $(x - x_{i-1}^n)/(x_i^n - x_{i-1}^n) \leq 1$. As u is defined on an open interval and real-valued concave functions on an open interval are continuous, then $\lim_{n\to\infty} \sup_{x \in [\underline{x},\overline{x}]} |u^n(x) - u(x)| \leq \lim_{n\to\infty} u(x_1^n) - u(\underline{x}) = 0$, and we have that u^n converges uniformly to u.

Exercise 7. Recall our definition $u_a(x) := \mathbf{1}_{x \leq a}(x - a)$. Let $d_n^n := c_n^n$ and, for $i = 1, 2, \ldots, n-1$, let $d_i^n := c_i^n - c_{i+1}^n$.

(i) Prove that $u^n(x) = u(\overline{x}) + \sum_{i=1}^n d_i^n u_{x_i^n}(x)$. *(Hint: Either you show it by expanding the RHS; or you show that (a) the right-hand side equals u at points in our grid, (b) it is continuous and piecewise-linear, (c) its derivative is equal to c_i^n at any $x \in (x_{i-1}^n, x_i^n)$.)*

(ii) Use (i) to prove that if $\mathbb{E}_F[u_a] \geq \mathbb{E}_G[u_a]$, for every $a \in [\underline{x}, \overline{x}]$, then $\mathbb{E}_F[u^n] \geq \mathbb{E}_G[u^n]$, for every n.

Then,

$$\int_{\underline{x}}^{a} F(x)dx \leq \int_{\underline{x}}^{a} G(x)dx, \quad \forall a \Longrightarrow \mathbb{E}_F[u^n] \geq \mathbb{E}_G[u^n], \quad \forall n.$$

As u^n converges uniformly and is integrable, then

$$0 \leq \lim_{n\to\infty} \mathbb{E}_F[u^n] - \mathbb{E}_G[u^n] = \mathbb{E}_F[u] - \mathbb{E}_G[u].$$

$\square$

A more restrictive notion of a gamble F being "riskier" than another gamble G requires that both F and G have the same mean but G has higher variance.

Definition 4. Let F, G be distributions on $\mathbb{R}$. G is a **mean-preserving spread** of F if there are random variables X, Y, and ϵ, such that $Y \overset{d}{=} X + \epsilon$, $X \sim F$, $Y \sim G$, and $\mathbb{E}[\epsilon \mid X] = 0$.

Exercise 8. 1. Let $\geq_{MPS}$ be such that $F \geq_{MPS} G$ if G is a mean-preserving spread of F. Prove that $F \geq_{MPS} G \Longrightarrow F \geq_{SOSD} G$, but that the converse is not true in general.

2. Show that if $F \geq_{SOSD} G$, then $\mathbb{E}_F[x] \geq \mathbb{E}_G[x]$.
3. Show that if $F \geq_{MPS} G$, then $\mathbb{E}_F[x] = \mathbb{E}_G[x]$ and $\mathbb{V}_F[x] \leq \mathbb{V}_G[x]$.
4. Prove $F \geq_{FOSD} G \Longrightarrow F \geq_{SOSD} G$, but that the converse is not true in general.
5. Show that $\geq_{SOSD}$ and $\geq_{MPS}$ are partial orders.

4.1. Second-Order Stochastic Dominance in $\mathbb{R}^n$ (*)

The discussion above extends to more general spaces;[6] we focus on extending our characterization of SOSD to $\mathbb{R}^n$.

Definition 5. Let F and G be distributions on $\mathbb{R}^n$. We say that F **second-order stochastically dominates** G ($F \geq_{SOSD} G$) if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all nondecreasing concave $u : \mathbb{R}^n \to \mathbb{R}$, whenever both expectations exist.

The key result in this section, which we state for reference, is the following:

Theorem 4. (Strassen's (1965) Theorem) Let F and G be distributions on $\mathbb{R}^n$ with bounded support. Then, $F \geq_{SOSD} G$, if and only if there are $X \sim F$ and $Y \sim G$ such that $X \geq \mathbb{E}[Y \mid X]$ a.s.

In short, what this theorem is giving is a way to define a joint distribution $H(x, y)$ such that the marginals over x and y equal F and G, respectively, and $\int_{\mathbb{R}^n} yH(x, y)dy \leq x$.[7]

We can also adjust the definition of mean-preserving spreads as expected and obtain a useful corollary:

Corollary 1. Let F and G be distributions on $\mathbb{R}^n$ with bounded support. If G is a mean-preserving spread of F if and only if $F \geq_{SOSD} G$ and $\mathbb{E}_F[x] = \mathbb{E}_G[x]$.

[6] For a reference, see Müller and Stoyan (2002).

[7] Strassen (1965) proves far more general results; see Müller and Stoyan (2002, Theorem 2.6.8) for a recent proof.

5. Background Risks

When considering investing in stock X or Y, professional traders typically consider only their expected return, that is, whether $\mathbb{E}[X] > \mathbb{E}[Y]$, and potentially not so much the associated risk, captured, for instance, by $\mathbb{V}[X], \mathbb{V}[Y]$. While one may consider that they have risk attitudes that are very particular — risk aversion is a common finding, both empirical and experimentally — another possible explanation is that they are considering the existence of background risks. Stock X and Y are not the only stocks that traders invest in, and large traders are likely to have large amounts of money invested in a number of different stocks that add to background risk on their portfolio.

A recent paper by Pomatto et al. (2020) explores the connection between background risks and stochastic orders. In short, the result says that there are (independent!) background risks large enough that, when considered, can dwarf any riskiness considerations and make even the most risk-averse person to simply go with the gamble that yields the highest expected value. An abbreviated statement goes as follows:

Theorem 5. (Pomatto et al. 2020) Let X and Y be random variables with finite variance.

(i) If $\mathbb{E}[X] > \mathbb{E}[Y]$, then there is an independent random variable ϵ such that $X + \epsilon \geq_{FOSD} Y + \epsilon$.

(ii) If $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{V}[X] < \mathbb{V}[Y]$, then there is an independent random variable ϵ such that $X + \epsilon \geq_{SOSD} Y + \epsilon$.

6. References

Hadar, Josef, and William R. Russell. 1969. "Rules for Ordering Uncertain Prospects." *American Economic Review* 59 (1): 25–34. 10.2307/1811090. 2, 6

Hanoch, Giora, and Haim Levy. 1969. "The Efficiency Analysis of Choices Involving Risk." *Review of Economic Studies* 36 (3): 335–346. 10.2307/2296431. 6

Müller, Alfred, and Dietrich Stoyan. 2002. *Comparison Methods for Stochastic Models and Risks*. Wiley. 9

Pomatto, Luciano, Philipp Strack, and Omer Tamuz. 2020. "The Existence of Probability Measures with Given Marginals." *Journal of Political Economy* 128 (5): 1877–19900. 10.1086/705555. 10

Strassen, Volker. 1965. "The Existence of Probability Measures with Given Marginals." *Annals of Mathematical Statistics* 36 (2): 423–439. 10.1214/aoms/1177700153. 9

Tesfatsion, Leigh. 1976. "Stochastic Dominance and the Maximization of Expected Utility." *Review of Economic Studies* 43 (2): 301–315. 10.2307/2297326. 6

ECON0106: **Microeconomics**

8. Uncertainty*

Duarte Gonçalves†

University College London

1. Overview

How likely is it that it is going to rain today? Should I take the umbrella with me or not? Different websites show different probabilities, but do we know the true, objective probability that it rains today? (Is there such a thing?) While this is an uninteresting example if you are in London — everyone knows the answers: (i) it is going to rain a.s., (ii) always take the umbrella — it points to a fundamental issue that we need to address: agents may not know the probability that an event realizes.

More: from a frequentist perspective, it makes little sense to talk about the *objective* probabilities of singular, unrepeatable events. Instead, the modern conceptualization of probability, based on measure theory, enables us to talk about *subjective* probability.

In this lecture, we will study models of choice under *uncertainty*. This can be seen as a decision-theoretic foundation to the very notions of subjective probability. We will introduce the two main models of subjective expected utility, due to Savage (1954) and Anscombe and Aumann (1963), and we will focus on the latter. We then turn to understanding a choice-based representation of belief updating, and we conclude with a discussion on uncertainty aversion.

*Last updated: 18 November 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

2. Subjective Expected Utility

2.1. Anscombe–Aummann's Framework

The main ingredients of Anscombe and Aumann's (1963) framework are the following:

Ω: set of states of the world;

X: set of consequences or outcomes;

$f : \Omega \to \Delta(X)$: an act;

$\mathscr{F} := \Delta(X)^{\Omega}$: set of acts;

$\succsim \subseteq \mathscr{F}^2$: preference relation.

This framework differs from Savage's (1954) (which we will introduce later) by considering two sources of uncertainty: (i) subjective uncertainty (horse race) — which state $\omega \in \Omega$ will be realized — and (ii) objective uncertainty/risk (roulette wheel) — which consequence $x \in X$ will be realized in a lottery. That is, each can can be seen as a compound lottery: a potentially different objective lottery is triggered by each (uncertain) state of the world.

Our goal will be to characterize properties of $\succsim$ that enable us to recover not only a Bernoulli utility function on consequences, $u : X \to \mathbb{R}$, but also a probability measure $\mu \in \Delta(\Omega)$ such that

$$f \succsim g \quad \text{if and only if} \quad \mathbb{E}_{\mu}[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_{\mu}[\mathbb{E}_{g(\omega)}[u]],$$

where, for a given state ω, $f(\omega)$ is an objective probability distribution in $\Delta(X)$ and so $\mathbb{E}_{f(\omega)}[u]$ is our von Neumann – Morgenstern expected utility; and as μ represents the decision-maker's belief over the states, we then take expectations of vNM expected utility with respect to the beliefs μ to obtain the agent's subjective expected utility. Further, we want the representation to be sharp, in that beliefs should be uniquely pinned down from preferences.

For simplicity, we will assume that Ω and X are finite.

Definition 1. For acts $f, g \in \mathscr{F}$, we call their **mixture** given $\alpha \in [0,1]$ the act $\alpha f + (1-\alpha)g$ such that $(\alpha f + (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$.
For lottery $p \in \Delta(X)$ we will abuse notation and call $p \in \mathscr{F}$ a **constant act** such that $p(\omega) = p \in \Delta(X)$ for every $\omega \in \Omega$.

That is, a constant act p is the act that delivers lottery p in every state of the world.

We define **continuity** of $\succsim$ as we did before, i.e. for all sequences $\{f_n, g_n\}_n$ such that $f_n \succsim g_n$ for every n and $f_n \to f, g_n \to g$, we have that $f \succsim g$. Analogously to what we did with preferences on objective probability distributions, we now say that $\succsim$ satisfies **independence** if for any acts $f, g, h \in \mathscr{F}$ and any $\alpha \in (0,1]$, $f \succsim g \Longleftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

We then get an intermediate result, a state-dependent expected utility representation.

Theorem 1. (State-Dependent Expected Utility) A preference relation $\succsim$ on $\mathscr{F}$ satisfies continuity and independence if and only if there exists a utility function $u : X \times \Omega \to \mathbb{R}$ and a uniform probability measure $\mu \in \Delta(\Omega)$ such that, for any $f, g \in \mathscr{F}$, $f \succsim g$ if and only if $\mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]]$.

Is this it? Not quite. Note that $u : X \times \Omega \to \mathbb{R}$, i.e. we have $u(x, \omega)$, not $u(x)$. This may be a virtue or a vice, depending on how you perceive it, but the main issue is that it is capturing both preferences over consequences and beliefs about states. This is why we have a uniform belief/prior μ, void of any empirical content.

Proof. The if part, as usual, is a matter of verification, and we then focus on the only if part.

Before proving Theorem 1, a stepping stone toward our target representation, we introduce some notation and a lemma:

$E \subseteq \Omega$: an event

fEg: a 'conditional act,' where for acts f, g and event E, $fEg \in \mathscr{F}$ is such that $(fEg)(\omega) = f(\omega)$ is $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if otherwise;

Null event E: an event such that for any $f, g, h \in \mathscr{F}$ for which $f \succ g$, $fEh \sim gEh$;

$\tilde{x}$: a constant act, $\tilde{x}(\omega) = x, \forall \omega \in \Omega$.

Cleary null events are disregarded by the decision-maker and this, as we will see, has an important connection with their beliefs. Now, the lemma:

Lemma 1. Let $V : \mathscr{F} \to \mathbb{R}$ be affine and continuous. Then, for every $\omega \in \Omega$, there is an affine and continuous function $V_\omega : \Delta(X) \to \mathbb{R}$ such that $V(f) = \sum_\omega V_\omega(f(\omega))$, for all $f \in \mathscr{F}$.

Proof. Fix $f^* \in \mathscr{F}$. For any $f \in \mathscr{F}$, we can write

$$
\begin{aligned}
&\frac{1}{|\Omega|} f = \frac{1}{|\Omega|} f^* + \frac{1}{|\Omega|} \sum_\omega (f\{\omega\} f^* - f^*) \\
\Longleftrightarrow &\frac{1}{|\Omega|} f + \left(1 - \frac{1}{|\Omega|}\right) f^* = \frac{1}{|\Omega|} \sum_\omega (f\{\omega\} f^*) \\
\Longleftrightarrow &V\left(\frac{1}{|\Omega|} f + \left(1 - \frac{1}{|\Omega|}\right) f^*\right) = V\left(\frac{1}{|\Omega|} \sum_\omega (f\{\omega\} f^*)\right) \\
\Longleftrightarrow &\frac{1}{|\Omega|} V(f) + \left(1 - \frac{1}{|\Omega|}\right) V(f^*) = \frac{1}{|\Omega|} \sum_\omega V\left((f\{\omega\} f^*)\right) \\
\Longleftrightarrow &V(f) = \sum_\omega \left[V\left((f\{\omega\} f^*)\right) - \left(1 - \frac{1}{|\Omega|}\right) V(f^*) \right],
\end{aligned}
$$

where we used the fact that V is affine. Then, we can define $V_\omega : \Delta(X) \to \mathbb{R}$ such that $V_\omega(f(\omega)) := V((f\{\omega\} f^*)) - \left(1 - \frac{1}{|\Omega|}\right) V(f^*)$. As V is continuous and affine, so is V_ω. $\square$

Now, for every $f \in \mathscr{F}$, let $\rho_f \in \Delta(X \times \Omega)$ as $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x)$, for all $(x, \omega) \in X \times \Omega$. ρ_f is then a joint distribution over $X \times \Omega$ with a uniform marginal over Ω.

Let $R := \{\rho_f \mid f \in \mathscr{F}\} \subseteq \Delta(X \times \Omega)$ and define $\trianglerighteq \subseteq R^2$ as $\rho_f \trianglerighteq \rho_g$ if and only if $f \succsim g$. As $\succsim$ is a continuous preference relation, so is $\trianglerighteq$. Moreover, as $\succsim$ satisfies independence, then, for any $\alpha \in (0,1)$ and every $h \in \mathscr{F}$, $\rho_f \succsim \rho_g \Longleftrightarrow \rho_{\alpha f + (1-\alpha) h} \trianglerighteq \rho_{\alpha g + (1-\alpha) h}$. As

$$
\begin{aligned}
\rho_{\alpha f + (1-\alpha) h}(\{x\} \times \{\omega\}) &= (\alpha f + (1-\alpha) h)(\omega)(x) = \alpha f(\omega)(x) + (1-\alpha) h(\omega)(x) \\
&= \alpha \rho_f(\{x\} \times \{\omega\}) + (1-\alpha) \rho_h(\{x\} \times \{\omega\}),
\end{aligned}
$$

then we immediately have that $\trianglerighteq$ (defined on $\Delta(X)$) satisfies independence.

Thus, by the von Neumann – Morgenstern expected utility representation theorem, we have that there is a unique function (up to positive affine transformations) $v : X \times \Omega \to \mathbb{R}$ such that $\rho_f \trianglerighteq \rho_g$ if and only if $\sum_{x,\omega} \rho_f(\{x\} \times \{\omega\}) v(x, \omega) \geq \sum_{x,\omega} \rho_g(\{x\} \times \{\omega\}) v(x, \omega)$.

Define $V : \mathscr{F} \to \mathbb{R}$ such that $V(f) := \sum_{x,\omega} \rho_f(\{x\} \times \{\omega\}) v(x,\omega)$ and note that V is affine and continuous. Additionally, V represents $\succsim$ given that

$$f \succsim g \Longleftrightarrow \rho_f \trianglerighteq \rho_g \Longleftrightarrow V(f) \geq V(g).$$

By Lemma 1, there is a $V_\omega : \Delta(X) \to \mathbb{R}$ which is affine and continuous such that $V(f) = \sum_\omega V_\omega(f(\omega))$. Finally, define $u : X \times \Omega$ as $u(x,\omega) := V_\omega(\delta_x)|\Omega|$. Then, as for each ω, V_ω is affine, $V_\omega(p) = \sum_x p(x)\frac{1}{|\Omega|}u(x,\omega)$. □

In order to establish a state-independent subjective expected utility represetantion we will need $\succsim$ to satisfy a property which we call **separability**: $\forall p,q \in \Delta(X)$, all $h \in \mathscr{F}$, and all $\omega, \omega' \in \Omega$ such that $\{\omega\}$ and $\{\omega'\}$ are non-null events, $\tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h$ if and only if $\tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h$.

We then have what we wanted, a state-independent representation:

Theorem 2. (Anscombe–Aumann Subjective Expected Utility) Let $\succsim$ be a preference relation on $\mathscr{F}$. $\succsim$ satisfies continuity, independence, and separability if and only if there exists $\mu \in \Delta(\Omega)$ and $u : X \to \mathbb{R}$ such that

$$f \succsim g \quad \Longleftrightarrow \quad \mathbb{E}_\mu[\mathbb{E}_f[u]] \geq \mathbb{E}_\mu[\mathbb{E}_g[u]],$$

where, for all $f \in \mathscr{F}$, $\mathbb{E}_\mu[\mathbb{E}_f[u]] := \sum_{\omega \in \Omega} \mu(\omega) \sum_{x \in X} f(\omega)(x)u(x)$. Moreover, u is unique up to positive affine transformations and, if $\exists f,g \in \mathscr{F}$ such that $f \succ g$, μ is unique.

Proof. Again, we focus on the only if part. From Theorem 1, there is $u : X \times \Omega$ such that $f \succsim g$ if and only if $\sum_{\omega,x} f(\omega)(x)u(x,\omega) \geq \sum_{\omega,x} g(\omega)(x)u(x,\omega)$.

Let $U : \Delta(X) \times \Omega \to \mathbb{R}$ be defined as $U(p,\omega) := \sum_{x \in X} p(x)u(x,\omega)$ for all $\omega \in \Omega$, $p \in \Delta(X)$. Take any $p,q \in \Delta(X)$ and non-null $\{\omega\}$, $\omega \in \Omega$ such that $U(p,\omega) \geq U(q,\omega)$. From separability we have that for any non-null $\{\omega'\}$, $\omega' \in \Omega$, and any $h \in \mathscr{F}$,

$$\begin{aligned}
U(p,\omega) \geq U(q,\omega) &\Longleftrightarrow U(p,\omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''),\omega'') \geq U(q,\omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''),\omega'') \\
&\Longleftrightarrow V(\tilde{p}\{\omega\}h) \geq V(\tilde{q}\{\omega\}h) \\
&\Longleftrightarrow \tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h \Longleftrightarrow \tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h \\
&\Longleftrightarrow V(\tilde{p}\{\omega'\}h) \geq V(\tilde{q}\{\omega'\}h) \\
&\Longleftrightarrow U(p,\omega') + \sum_{\omega'' \in \Omega \setminus \{x\}} U(h(\omega''),\omega'') \geq U(q,\omega') + \sum_{\omega'' \in \Omega \setminus \{x\}} U(h(\omega''),\omega'') \\
&\Longleftrightarrow U(p,\omega') \geq U(q,\omega').
\end{aligned}$$

Let $\Omega^* \subseteq \Omega$ be the set of ω such that $\{\omega\}$ is non-null and fix $\omega^* \in \Omega^*$. As u is unique up to positive affine transformations, for all $\omega \in \Omega^*$, $\exists \alpha_\omega > 0, \beta_\omega \in \mathbb{R}$ such that $u(x,\omega) = \alpha_\omega u(x,\omega^*) + \beta_\omega$, for all $x \in X$.[1] Let $u : X \to \mathbb{R}$ be defined as $u(x) := u(x,\omega^*)$. Then, we have that

$$\begin{aligned}
f \succsim g &\Longleftrightarrow \sum_{\omega \in \Omega} U(f(\omega),x) = \sum_{\omega \in \Omega} \sum_{x \in X} f(\omega)(x) u(\omega,x) \geq \sum_{\omega \in \Omega} \sum_{x \in X} g(\omega)(x) u(\omega,x) \\
&\Longleftrightarrow \sum_{\omega \in \Omega^*} \sum_{x \in X} f(\omega)(x) \alpha_\omega u(x) \geq \sum_{\omega \in \Omega^*} \sum_{x \in X} g(\omega)(x) \alpha_\omega u(x) \\
&\Longleftrightarrow \sum_{\omega \in \Omega^*} \frac{\alpha_\omega}{\sum_{\omega' \in \Omega^*} \alpha_{\omega'}} \sum_{x \in X} f(\omega)(x) u(x) \geq \sum_{\omega \in \Omega^*} \frac{\alpha_\omega}{\sum_{\omega' \in \Omega^*} \alpha_{\omega'}} \sum_{x \in X} g(\omega)(x) u(x).
\end{aligned}$$

Now define $\mu : \Omega \to [0,1]$ such that $\mu(\omega) := \mathbf{1}_{\omega \in \Omega^*} \frac{\alpha_\omega}{\sum_{\omega' \in \Omega^*} \alpha_{\omega'}}$.[2]

Uniqueness of $u : X \to \mathbb{R}$ up to positive affine transformations follows by the fact that $u : X \times \Omega \to \mathbb{R}$ is too.

Now take any $f \succ g$; we will show that μ is uniquely defined. By separability, $\tilde{f}(\omega)\{\omega\}f \succ \tilde{g}(\omega)\{\omega\}f$ for every $\omega \in \Omega^*$. Then, there exists $z, y \in X$ such that $u(z) > u(y)$ (why?). By taking the acts $\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y$ for each $\omega' \in \Omega^*$, and the constant act $f = \mu(\omega')\tilde{\delta}_z + (1-\mu(\omega'))\tilde{\delta}_y$,[3] we have that $f \sim \tilde{\delta}_z\{\omega'\}\tilde{\delta}_y$, and therefore $\sum_{\omega \in \Omega} \mu(\omega) \sum_{x \in X} f(\omega)(x) u(x) = \sum_{x \in X} f(\omega')(x) u(x) =$

[1]Recall that for any $\omega \in \Omega$ such that $\{\omega\}$ is non-null, $V_\omega : \Delta(X) \to \mathbb{R}$ is an expected utility representation of preferences over $\Delta(X)$. Separability gives us that V_ω all represent the same preferences on $\Delta(X)$ *and* we know they are vNM representations; hence, the they much be affine transformations of each other.

[2]Technically speaking, μ is a probability measure and is therefore defined over a σ-algebra Σ on Ω. But as Ω is finite, we can redefine μ on 2^Ω (a σ-algebra) by doing $\mu(E) := \cup_{\omega \in E} \mu(\omega)$ for every $E \subseteq \Omega$, with the understanding that $\mu(\{\emptyset\}) = 0$.

[3]This is the constant act $f := \tilde{p}$, such that $p = \mu(\omega')\delta_z + (1-\mu(\omega'))\delta_y$, where $p \in \Delta(X)$.

$\mu(\omega')u(z)+(1-\mu(\omega'))u(y) = \sum_{\omega\in\Omega}\mu(\omega)\sum_{x\in X}(\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y)(\omega)(x)$, which shows uniqueness of μ (supposing existence of some other $\nu \in \Delta(\Omega)$ such that $\nu \neq \mu$ will give a contradiction, as f is a constant act and therefore the ν would not play a role in determining its expected utility). □

Exercise 1. We say that $\succsim$ on $\mathscr{F}$ satisfies **monotonicity** if $\forall f, g \in \mathscr{F}$, whenever $\tilde{f}(\omega) \succsim \tilde{g}(\omega)$ for all $\omega \in \Omega$ such that $\{\omega\}$ is non-null, then $f \succsim g$. Show that if $\succsim$ is preference relation that satisfies independence and continuity, then separability and monotonicity are equivalent.

While Anscombe and Aumann's (1963) is a more tractable approach to modeling subjective uncertainty, it has been pointed out as conceptually less appealing than Savage's (1954) exactly because of state-independence. We did want state-independence as this was the only way to derive from preferences the decision-maker's beliefs.

State-independence imposes some constraints on how we should think about states, acts, and consequences. A usual criticism to Anscombe and Aumann's (1963) framework goes as follows: Suppose we want to express that a decision-maker prefers to carry an umbrella when it is raining and not carrying it when it is not raining. It would be tempting to model this as Ω ={rain, no rain} and X ={carrying umbrella, not carrying umbrella}, but then we are ruling out that carrying umbrella $\succ$ not carrying umbrella when ω = rain and carrying umbrella $\prec$ not carrying umbrella when ω = no rain.

The issue with this example is that it is using Anscombe–Aumann and thinking Savage where we can have u(umbrella(rain))$>u$(umbrella(no rain)). We can see this example as how *not to* think about states, acts, and consequences in Anscombe and Aumann's (1963) framework. Carrying an umbrella or not is an *act*; we want to have it when it rains because we don't want to get wet; and we don't want to have it when it doesn't rain because we don't want to carry it around. So, instead, you need to rethink the set of consequences: X ={having to carry an umbrella, not having to carry an umbrella}$\times${getting wet, not getting wet}. This solves the apparent problem:

$\mathscr{F}$	Ω	
	rain	no rain
taking an umbrella	not wet but carrying the umbrella	not wet, carrying the umbrella
not taking an umbrella	wet, not carrying umbrella	not wet, not carrying the umbrella

In other words: once you specify the set of consequences appropriately, you come to realize that the reason for why the decision-maker wants the umbrella when it rains is to not get wet. An umbrella as an act then induces two state-dependent degenerate lotteries: getting wet if it rains for sure and not getting wet but having to carry the bloody thing for sure if it doesn't.

Exercise 2. Suppose that you want to learn what a person thinks is more likely, E or E^C (not E) and can't ask them directly. You decide to put what you learned about Anscombe–Aumann to work.

Assume that the person strictly prefers more money over less. Show how to define a decision problem where the person has to choose between two acts f, g and, from their choices alone, you can infer whether they think E is more like than E^C or not.

2.2. Savage's Framework

We now introduce Savage's (1954) framework. We discuss the main postulates of the model but we won't provide any proofs — you should see Kreps (1988, ch. 8) if you're interested.

The basic ingredients are as follows:

Ω: set of states of the world;

X: set of consequences or outcomes;

$f : \Omega \to X$: an act;

$\mathscr{F} := X^{\Omega}$: set of acts;

$\succsim \subseteq \mathscr{F}^2$: preference relation.

As you can see from the above, the crucial difference is that acts map directly to consequences and not to state-dependent lotteries on the set of consequences. To better appreciate the difference, let us put both representations next to each other:

- **Savage**: $\int_\Omega u(f(\omega))d\mu(\omega)$;
- **Anscombe–Aumann**: $\int_\Omega \int_X f(\omega)(x)u(x)d\mu(\omega)$.

The representation theorem goes as follows:

Theorem 3. (Savage 1954) $\succsim$ satisfies P1-P7 if and only if there exist

(i) a unique nonatomic and finitely additive probability measure μ on Ω, for which $\mu(E)=0$ if and only if E is a null event; and

(ii) a bounded function $u : X \to \mathbb{R}$, unique up to positive affine transformations

such that that for every $f, g \in \mathscr{F}$,

$$f \succsim g \quad \text{if and only if} \quad \mathbb{E}_\mu[u \circ f] := \int_\Omega u(f(\omega))d\mu(\omega) \geq \int_\Omega u(g(\omega))d\mu(\omega) = \mathbb{E}_\mu[u \circ g].$$

where P1-P7 are defined in Appendix A.

3. Bayesian Updating

Let's continue with the example What happens when our decision-maker, who is deciding whether or not to take an umbrella, looks out the window? They learn whether it is or it is not raining at that moment and this tells them something about whether it is going to rain later on.

Then, they should update their beliefs (by Bayes rule) and act accordingly: for any event $E \subseteq \Omega$, the posterior belief given $A \subseteq \Omega$ is given by

$$\mu \mid A(E) = \frac{\mu(A \cap E)}{\mu(A)}$$

for any A such that $\mu(A) > 0$. The issue is that their beliefs were *deduced* from their preferences, their behavior. Is it the case that these beliefs we deduced are updated according to Bayes rule? Yes — under some additional restrictions.

Let's continue with the Anscombe–Aumann setup[4] and enrich it in the following way. Events $A \subseteq \Omega$ are information, i.e. the decision-maker is learning that the state ω lies in A. Our new primitive is no longer just a single preference relation $\succsim$ but a collection of preferences $\{\succsim_A\}_{A\subseteq\Omega}$. Each $\succsim_A$ is a preference relation on acts $\mathscr{F}$ and the idea is that $\succsim_A$ describes their behavior upon obtaining information A. We write $\succsim \equiv \succsim_\Omega$ to denote the decision-maker's preference in absence of any information.

We will start by assuming that for each $A \subseteq \Omega$, $\succsim_A$ is a preference relation on the set of acts satisfying independence, continuity, and monotonicity (do exercise 1!).

We need to impose some consistency requirements.

Definition 2. We say that $\{\succsim_A\}_{A\subseteq\Omega}$ satisfy

(i) **constant-act consistency** if preferences over constant acts are consistent: for all lotteries $p, q \in \Delta(X)$ and events $A, B \subseteq \Omega$, $\tilde{p} \succsim_A \tilde{q}$ if and only if $\tilde{p} \succsim_B \tilde{q}$;

(ii) **dynamic consistency** if, for all non-null event (with respect to $\succsim$) $A \subseteq \Omega$ and all acts $f, g \in \mathscr{F}$, $fAg \succsim_\Omega g$ if and only if $f \succsim_A g$;

(iii) **consequentialism** if, for event $A \subseteq \Omega$, two acts $f, g \in \mathscr{F}$ deliver the same lottery $f(\omega) = g(\omega)$ for every $\omega \in A$, then $f \sim_A g$.

Some remarks on why these seem sensible assumptions to make. First, constant-act consistency says that if you take two acts whose (distribution over) consequences is completely orthogonal to the state, then whatever you learn should not change how you compare them. Together with independence, continuity, and monotonicity, we get that $\exists \alpha_A > 0, \beta_A \in \mathbb{R}$ such that $u_A = \alpha_A u + \beta_A$. In other words, constant-act consistency is doing the work in tying together the utility functions over consequences.

[4] You can also do this within the Savage framework and the requirements on preferences will be similar; see Ghirardato (2002).

If constant-act consistency deal makes preferences over consequences remain the same, then the other two assumptions are working on rendering the inferred beliefs consistent across the different information sets. Dynamic consistency is saying that if you take two acts that differ only when A occurs, then if you learn that A occurs, you compare them in the same way. This makes the decision-maker evaluate the 'relevant acts' under A the same way as they did before they had the information. We said that consistency of preferences over consequences was taken care of; then what is this assumption doing? It is making the decision-maker keep the same relative beliefs across any two non-null events.

Finally, consequentialism is taking care of rendering events B such that $B \cap A = \emptyset$ null events with respect to $\succsim_A$. So, in other words, one can interpret this assumption as 'the decision-maker believes the information received.'

The representation theorem follows:

Theorem 4. Let $\{\succsim_A\}_{A\subseteq\Omega}$ be a collection of preference relations on $\mathscr{F}$ and assume that there are $f, g \in \mathscr{F}$ such that $f \succ_\Omega g$.
$\{\succsim_A\}_{A\subseteq\Omega}$ satisfy constant-act consistency, dynamic consistency, and consequentialism and $\succsim_A$ satisfies continuity, independence, and monotonicity for every $A \subseteq \Omega$ if and only if there exist $u : X \to \mathbb{R}$ and a collection of probability measures $\{\mu_A\}_{A\subseteq\Omega}$ such that

(i) $\forall f, g \in \mathscr{F}$, $f \succsim_A g$ if and only if $\mathbb{E}_{\mu_A}[\mathbb{E}_f[u]] \geq \mathbb{E}_{\mu_A}[\mathbb{E}_g[u]]$, and

(ii) for all non-null events with respect to $\succsim$, $A \subseteq \Omega$, $\mu_A(B) = \frac{\mu_\Omega(A\cap B)}{\mu_\Omega(A)}$ for all $B \subseteq \Omega$.

Moreover, u is unique up to positive affine transformations and μ is unique.

Proof. By Theorem 2 (AA SEU theorem), for all non-null event $A \subseteq \Omega$, there is $u_A : X \to \mathbb{R}$ and a prior $\mu_A \in \Delta(X)$ such that $f \succsim_A g$ if and only if $\mathbb{E}_{\mu_A}[\mathbb{E}_f[u_A]] \geq \mathbb{E}_{\mu_A}[\mathbb{E}_g[u_A]]$.

We break the proof into steps that you should prove yourself:

Exercise 3.

Step 1. Show that constant-act consistency implies that for every non-null event $A \subseteq \Omega$, $\exists \alpha_A > 0, \beta_A \in \mathbb{R}$ such that $u_A = \alpha_A u_\Omega + \beta_A$.

Step 2. Show that there are $x, y \in X$ such that $\tilde{\delta}_x \succ_A \tilde{\delta}_y$ for all non-null events $A \subseteq \Omega$.

Step 3. Use the previous step to prove that for all non-null events $A \subseteq \Omega$, $\mu_\Omega(A) > 0$.

Step 4. Prove that, for any acts $f, g, h \in \mathscr{F}$ and any non-null event $A \subseteq \Omega$, $f \succsim_A g \Longleftrightarrow fAh \succsim gAh$.

Step 5. Conclude by showing that for any acts $f, g, h \in \mathscr{F}$ and any non-null event $A \subseteq \Omega$, $f \succsim_A g \Longleftrightarrow fAh \succsim gAh$ implies that $\mu_A(\omega) = \mu_\Omega(\omega)/\mu_\Omega(A)$.

Step 6. Argue the uniqueness claims based on the previous steps and on Theorem 2.

Step 7. Verify the 'if' part by showing that the representation implies the assumptions on $\{\succsim_A\}_{A\subseteq\Omega}$.

□

Theorem 4 imposes conditions on choices that agents must satisfy if they are *behaving like* Bayesian subjective expected utility maximizers. There are two points to make here. On the one hand, note that we cannot really observe people's beliefs; we can only infer them from their behavior. So, if someone's behavior is not consistent with these axioms, it does not mean that they are not updating beliefs in a Bayesian fashion (which is not falsifiable per se). On the other hand, it is interesting to point out that Savage's (1954) book is called *The Foundations of Statistics*. That is to say, there is a view that Bayesian statistics is *founded upon* statistical decision theory.

You don't have to take a stance on this: you may assume agents have and update beliefs in accordance to Bayes' rule and/or adhere (or not) to Bayesian statistics despite acknowledging that their behavior may not be in line with this model. There are other interesting models that provide rationales for Bayesian updating that are not subjective expected utility (e.g. Cripps 2019), and models that use Bayesian updating and are not rationalizable with subjective expected-utility maximizing behavior (e.g. Alós-Ferrer and Mihm 2021).

Bayesian updating and subjective expected utility remain the main framework in economic models: they are very appealing principles and their vices and virtues when it comes to behavioral implications are well-known. All-in-all, a model meant to be an approximation to reality that captures all the relevant aspects of the situation being represented. And only if people think something truly crucial is missing, that either Bayesian updating or subjective expected utility are making you unable to account for, do they seek alternatives.

This being said, let's see one well-known issue with subjective expected utility and some approaches that have been suggest to dealing with it.

4. Uncertainty Aversion

4.1. Ellsberg Paradox

This is an experiment by Ellsberg (1961):[5]

A box contains 60 balls: 20 are black and the rest are either red or green. Which would you prefer:

A £20 if a black ball is drawn;

B £20 if a red ball is drawn; or

C £20 if a green ball is drawn.

Most people choose A.

Now consider choosing among the following:

a £20 if a black or a green ball is drawn;

b £20 if a black or a red ball is drawn; or

c £20 if a red or a green ball is drawn.

And, to this question, most people choose c.

You should convince yourself that this is incompatible with SEU.

Exercise 4. Show that a decision-maker strictly preferring A over B and C, and c over a and b is incompatible with subjective expected utility.

4.2. A Set of Probability Measures: Maxmin Expected Utility

There have been multiple ways to approach this issue. One has been to relax the independence (always to blame) in a way such that μ no longer has to add up to one (i.e. is not a probability measure). This approach, followed in Schmeidler (1989), but many find it

[5] While I am not aware of such an interesting story surrounding this paradox as the one involving Allais and Savage, the Ellsberg paradox does feature Daniel Ellsberg, known world-wide because of the story of the pentagon papers: `https://en.wikipedia.org/wiki/Daniel_Ellsberg`.

	Ω	
	ω_1	ω_2
f	20	0
g	0	20
$\tilde{p}$	10	10

unappealing, as we know well the properties of probability measures and we want to keep them for modeling convenient.

Further, in the end what may be behind this is the intuitive idea that individuals may want to hedge their bets to avoid uncertainty. Suppose that the decision-maker is indifferent between f and g. While this implies that $\mu(\omega_1) = \mu(\omega_2) = 1/2$, it should not need to be the case that the decision-maker is also indifferent between f, g and $\tilde{p}$. To capture this idea we say that $\succsim$ on the set of (Anscombe–Aumann) acts $\mathscr{F}$ is **GS uncertainty averse** (neutral/seeking) if for any acts f, g such that $f \sim g$, we have that $\frac{1}{2}f + \frac{1}{2}g \succsim f$ $(\sim/\precsim)$.

This led to the following relaxation of independence (always taking the blame): we say that a preference relation $\succsim$ on $\mathscr{F}$ satisfies **C-independence** (C for certainty) if, for any two acts $f, g \in \mathscr{F}$ independence holds with respect to mixtures with a constant act $\tilde{p} \in \mathscr{F}$, with $p \in \Delta(X)$, that is, $\forall \alpha \in (0,1)$,

$$f \succsim g \Longleftrightarrow \alpha f + (1-\alpha)\tilde{p} \succsim \alpha g + (1-\alpha)\tilde{p}.$$

Clearly, C-independence is implied but does not imply independence. The reasoning is that we want to hedge, but hedging is only valuable when it can eliminate uncertainty, which is not the case if it uses a constant act (think about the example above).

This next exercise makes the point that independence is exactly C-independence plus with neutral attitudes toward uncertainty.

Exercise 5. Show that if $\succsim$ is a preference relation on $\mathscr{F}$ satisfying continuity and monotonicity, then independence is equivalent to C-independence and uncertainty neutrality.

These assumptions enable a representation of a different kind: **maxmin (subjective) expected utility** (Gilboa and Schmeidler 1989).

Theorem 5. (Gilboa–Schmeidler Maxmin Expected Utility) Let $\succsim$ be a preference relation on $\mathscr{F}$. Then $\succsim$ satisfies continuity, monotonicity, C-independence, and GS uncertainty aversion if and only if there is a utility function $u : X \to \mathbb{R}$ and a convex and compact set of probability measures $\mathscr{M} \subseteq \Delta(\Omega)$ such that

$$f \gtrsim g \quad \text{if and only if} \quad \min_{\mu \in \mathscr{M}} \mathbb{E}_\mu[\mathbb{E}_f[u]] \geq \min_{\mu \in \mathscr{M}} \mathbb{E}_\mu[\mathbb{E}_g[u]].$$

Note that GS uncertainty aversion is a statement regarding an agent's attitudes toward uncertainty. The decision-maker has not a prior, but a set of probability measures $\mathscr{M} \subseteq \Delta(\Omega)$ that is *endogenous* to the representation. Different preferences can induce representations with different sets of probability measures. The maxmin model implicitly assumes that agents are extremely uncertainty averse, behaving as if expecting the worst to happen among all the probability distributions over the state space that they entertain.

4.3. Beliefs over Unknown Probabilities: Smooth Uncertainty Aversion

We may then wonder whether we can't just get something like standard risk aversion but for uncertainty, where agents don't think about the worst possible outcome but instead trade-off uncertainty for better consequences. And the answer is yes, we can. Klibanoff et al. (2005) — and, more recently, Denti and Pomatto (2021) — prove conditions on preferences that allow us to have a utility representation that looks like this:

$$U(f) := \int_{\Delta(\Omega)} \phi\left(\int_\Omega u(f(\omega))d\mu(\omega)\right) d\pi(\mu)$$

where $f : \Omega \to X$ is a Savage act,

$u : X \to \mathbb{R}$ a von Neumann – Morgenstern utility,

$\phi : \mathbb{R} \to \mathbb{R}$ a strictly increasing and continuous function, $\mu \in \Delta(\Omega)$ is a probability measure on the state space, and

$\pi \in \Delta(\Delta(\Omega))$ is the decision-maker's *prior*, capturing their uncertainty about how the state is actually distributed.

So, we go one level up: the agent doesn't only have beliefs about what is the true state; they are uncertain about how it is distributed to start with! How can this be reasonable?

You can of this as a situation where you know that there is randomness — e.g. you know that machine impression produces biased coins (the probability of flipping heads and tails isn't exactly 1/2) — but you face uncertainty about this bias, i.e. don't really know how this bias is distributed.

It is called smooth uncertainty aversion because if the curvature of u captures risk attitudes, that of ϕ captures uncertainty attitudes in an analogous way.[6] That is, we separate preferences over risk and over uncertainty. Klibanoff et al. (2005) also show that we get equivalent characterizations of uncertainty aversion[7] in the same way as we saw equivalent characterizations of risk aversion. For instance, the decision-maker is uncertainty averse if and only if ϕ is concave, and we can make statements comparing uncertainty attitudes of different agents just like we did with risk aversion. Further, we recover the maxmin model as a limit case, in which the decision-maker is extremely uncertainty-averse.

[6]Note the analogy with compound lotteries and having different preferences over the compound lotteries and the reduced lotteries.

[7]This is a concept which we haven't really defined, as we made a point of distinguishing it from GS uncertainty aversion given the latter is defined on Anscombe–Aumann and not Savage acts. However, uncertainty aversion is defined for preferences over acts in an analogous manner as risk aversion is defined for preferences over lotteries; i.e. it means what you would expect.

5. References

Alós-Ferrer, Carlos, and Maximilian Mihm. 2021. "Updating Stochastic Choice." *Working Paper* 1–36. 10.2139/ssrn.3811298. 12

Anscombe, Frank J., and Robert J. Aumann. 1963. "A Definition of Subjective Probability." *Annals of Mathematical Statistics* 34 (1): 199–205. 10.1214/aoms/1177704255. 1, 2, 7

Cripps, Martin. 2019. "Divisible Updating." *Working Paper* 1–39, `https://drive.google.com/file/d/1YgfJI6p4E9sH8IoxmB1uL5eLBMJ_WZ7-/view`. 12

Denti, Tommaso, and Luciano Pomatto. 2021. "Model and Predictive Uncertainty: A Foundation for Smooth Ambiguity Preferences." *Working Paper* 1–50, `http://www.its.caltech.edu/~lpomatto/smooth_model.pdf`. 16

Ellsberg, Daniel. 1961. "Risk, Ambiguity, and the Savage Axioms." *Quarterly Journal of Economics* 75 (4): 643–669. 10.2307/1884324. 14

Ghirardato, Paolo. 2002. "Revisiting Savage in a conditional world." *Economic Theory* 20 83–92. 10.1007/s001990100188. 10

Gilboa, Itzhak, and David Schmeidler. 1989. "Maxmin expected utility with non-unique prior." *Journal of Mathematical Economics* 18 (2): 141–153. 10.1016/0304-4068(89)90018-9. 15

Klibanoff, Peter, Massimo Marinacci, and Sujoy Mukerji. 2005. "A Smooth Model of Decision Making under Ambiguity." *Econometrica* 73 (6): 1849–1892. 10.1111/j.1468-0262.2005.00640.x. 16, 17

Kreps, David M. 1988. *Notes on the Theory of Choice*. Westview Press. 8

Savage, Leornad J. 1954. *The Foundations of Statistics*. John Wiley and Sons. 1, 2, 7, 8, 9, 12, 19

Schmeidler, David. 1989. "Subjective Probability and Expected Utility without Additivity." *Econometrica* 57 (3): 571–587. 10.2307/1911053. 14

Appendix A. Details on Savage's Framework

To define Savage's (1954) postulates (the properties we require of $\succsim$ and, indirectly, of Ω) we restate some definitions in the Savage framework:

$E \subseteq \Omega$: an event;

fEg: a 'conditional act,' where for acts f,g and event E, $fEg \in \mathscr{F}$ is such that $(fEg)(\omega) = f(\omega)$ is $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if otherwise;

Null event E: an event such that for any $f,g,h \in \mathscr{F}$ for which $\succ g$, $fEh \sim gEh$;

$\tilde{x}$: a constant act, $\tilde{x}(\omega) = x, \forall \omega \in \Omega$.

Now the postulates:

P1 (Ordering): $\succsim$ is complete and transitive (a preference relation).

P2 (Sure-Thing Principle): For any acts f,g,h,h', and any event E, $fEh \succsim gEh$ if and only if $fEh' \succsim gEh'$.
(P2 gives a form of independence.)

P3 (Monotonicity): For every non-null event E and all constant acts, $\tilde{x}$ and $\tilde{y}$, $\tilde{x} \succsim \tilde{y}$ if and only if $\tilde{x}Eh \succsim \tilde{y}Eh$ for any act h.
(P3 allows us to rank acts based on the ranking of constant acts.)

P4 (Weak Comparative Probability): For all events A, B and constant acts $\tilde{x},\tilde{x}',\tilde{y},\tilde{y}'$, such that $\tilde{x} \succ \tilde{y}$ and $\tilde{x}' \succ \tilde{y}'$, then $\tilde{x}A\tilde{y} \succsim \tilde{x}B\tilde{y}$ if and only if $\tilde{x}'A\tilde{y}' \succsim \tilde{x}'B\tilde{y}'$.
(P4 is crucial to infer from preferences alone whether an event A is more likely than another event B.)

P5 (Nondegeneracy): There are constant acts $\tilde{x},\tilde{y}$ such that $\tilde{x} \succ \tilde{y}$.
(P5 just makes it a nontrivial preference relation.)

P6 (Small Event Continuity): For all acts f,g such that $f \succ g$ and all consequences degenerate acts $\tilde{x},\tilde{y}$, there is a finite partition $\{E_i\}_{i\in[n]}$ of Ω such that $\tilde{x}E_if \succ g$ and $f \succ \tilde{y}E_ig$ for every $i \in [n]$.
(P6 is a form of Archimedean property)

P7 (Uniform Monotonicity): For every event E and acts f, g, (i) if $fEh \succ \tilde{g}(\omega)Eh$ for any $\omega \in E$ — i.e. $\tilde{g}(\omega)$ is a constant act equal to x in every state, where $x = g(\omega)$ — and any act h, then $fEh \succsim gEh$; (ii) if $\tilde{f}(\omega)Eh \succ gEh$ for all $\omega \in E$, then $fEh \succsim gEh$.

ECON0106: **Microeconomics**

14. Monotone Comparative Statics in Games*

Duarte Gonçalves†

University College London

1. Overview

This lecture discusses monotone comparative statics of fixed points. This is a tool that you can use for research in a variety of topics and fields.

Here are some examples of questions that have been addressed using these tools:

- (Macro) Comparative statics on equilibrium prices and quantities when there is a demand shock induced by a change in consumers' preferences (e.g. Acemoglu and Jensen 2015).
- (Econometrics) Nonparametric partial identification of treatment response with social interactions (e.g. Lazzati 2015, with an application to studying the effect of police per capita on crime rates).
- (Health) Empirical antitrust implications of centralized matching systems on wages of medical residents (e.g. Agarwal 2015).[1]
- (Education) The empirical consequences of affirmative action in university admission (e.g. Dur et al. 2020; Aygün and Bó 2021).

Needless to say that theorems on fixed points and comparative statics are the bread and butter of theoretical research.

*Last updated: 6 December 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

[1]Fun fact: Nobel Prize winner, Alvin Roth not only made fundamental contributions to the theory of market design leveraging on lattice theory, these contributions then underlied crucial reforms in the US national residency matching program and organ donation programs, which he helped redesign. See `https://www.nobelprize.org/uploads/2018/06/advanced-economicsciences2012.pdf`.

2. Ordering Sets – Again

Recall the notion of **strong set order** $\geq_{ss}$ (Topkis 1979, 1998; Milgrom and Shannon 1994), where $\geq_{ss}$ is a binary relation on 2^X for some set X:

Definition 1. We say that S' **strong set dominates** S (writing $S' \geq_{ss} S$) if $\forall x' \in S', x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

As mentioned previously, this ordering of sets can be too demanding and therefore inapplicable to many situations.

One case where the strong set order is often unreasonable is when we want to compare sets of fixed points when some fundamental changes. For instance, suppose that you have a set of players I and each player i can choose strategies s_i in S_i. Their choices — characterized by some model, not necessarily best responses — are given by a mapping $B_i : S_{-i} \times \Theta_i \rightrightarrows S_i$ that depends on their opponents' choices and some parameter θ_i. You characterize the set of fixed points $\mathscr{F}(B,\theta)$, that is, the set of choices that are consistent, $s_i \in B_i(s_{-i},\theta_i)$ for every $i \in I$, and how they depend on θ. One major issue is that is hardly ever going to be the case that the sets of equilibria are strong-set ordered. This is because if s and s' are equilibrium strategy profiles in two different games, it is going to be extremely difficult to have that $s \vee s'$ and $s \wedge s'$ be equilibrium strategy profiles in either. So, to start with, we need a less stringent way of ordering sets. This is one possible motivation for the weak set order (Che et al. 2021).

Definition 2. We say that

(i) S' **upper weak set dominates** S (writing $S' \geq_{uws} S$) if $\forall x \in S$, $\exists x' \in S'$ such that $x' \geq x$;

(ii) S' **lower weak set dominates** S ($S' \geq_{lws} S$) if $\forall x' \in S'$, $\exists x \in S$ such that $x' \geq x$;

(iii) S' **weak set dominates** S ($S' \geq_{ws} S$) if S' both upper and weak set dominates S; that is, $\geq_{ws} = \geq_{uws} \cap \geq_{lws}$.

How do these two set orders compare?

Lemma 1. (i) The strong set order is transitive and antisymmetric on non-empty sets. It is not necessarily either reflexive or irreflexive.

(ii) The weak set order is transitive and reflexive,[2] but not necessarily antisymmetric.

(iii) For all nonempty subsets $S, T \subseteq X$, $S \geq_{ss} T \implies S \geq_{ws} T$.

(iv) The strong set order is closed under intersection, i.e. for all non-empty $S, S', T, T' \subseteq X$ such that $S' \geq_{ss} S$ and $T' \geq_{ss} T$, $S' \cap T' \geq_{ss} S \cap T$. It is not necessarily closed under union.

(v) The weak set order is closed under union, i.e. for all non-empty $S, S', T, T' \subseteq X$ such that $S' \geq_{ws} S$ and $T' \geq_{ws} T$, $S' \cup T' \geq_{ws} S \cup T$. It is not necessarily closed under intersection.

Exercise 1. Prove Lemma 1.

Exercise 2. Assume the conditions in Exercise 1 in Lecture Note 4. Provide necessary and sufficient conditions on $w', w \geq 0, p', p \in \mathbb{R}^k_{++}$ so that $B(p', w') \geq_{ws} B(p, w)$.

3. Fixed-Point Theorems

In this section we'll discuss two fixed-point theorems that are based on the strong and weak set orders. We will make use of them to show existence of equilibria later on when we look at comparative statics.

3.1. Tarski and Zhou Fixed-Point Theorems

Let X be a lattice.[3]

Definition 3. A function $f : X \to X$ is said to be **monotone** if it is order-preserving, i.e. $x \geq y \implies f(x) \geq f(y)$. A correspondence $F : X \rightrightarrows X$ is said to be **monotone** if $x \geq y \implies F(x) \geq_{ss} F(y)$.

Let $\mathscr{F}(F) := \{x \in X \mid x \in F(x)\}$ denote the set of **fixed points** of a self-correspondence F on X; we will abuse notation and write as well $\mathscr{F}(f) := \{x \in X \mid x = f(x)\}$ to denote the set of fixed points of a self-map f on X.

[2] Hence, a preorder.
[3] We omit the dependence on $\geq$ to not overburden the text.

Tarski's (1955) fixed-point theorem pertains to functions and was later generalized to correspondences by Zhou (1994). It can be stated as follows:

Theorem 1. (Tarski 1955) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is a non-empty complete lattice.

We will prove a more humble statement:

Lemma 2. Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is nonempty and has a largest element.

Proof. Let $S := \{x \in X : f(x) \geq x\}$. As X is a complete lattice $f(\inf_X X) \geq \inf_X X$, and therefore $S \neq \emptyset$. As $S \subseteq X$ and X is a complete lattice, $y := \sup_X S \in X$. Then, for any $x \in S$,

$$
\begin{aligned}
y \geq x &\Longrightarrow f(y) \geq f(x) \geq x && \text{as } f \text{ is monotone and } x \in S \\
&\Longrightarrow f(y) \geq y && \text{as } f(y) \text{ is an upper bound of } S \text{ and } y = \sup_X S \\
&\Longrightarrow f(f(y)) \geq f(y) && \text{as } f \text{ is monotone} \\
&\Longrightarrow f(y) \in S \\
&\Longrightarrow y := \sup_X S \geq f(y) \\
&\Longrightarrow y = f(y) && \text{by symmetry.}
\end{aligned}
$$

□

We provide a heuristic proof in the Appendix A to the lecture notes for completeness, but you don't need to know it; check it at your own risk.

Zhou (1994) then generalized the theorem to monotone correspondences:

Theorem 2. (Zhou 1994, Theorem 1) Let X be a complete lattice and $F : X \rightrightarrows X$ be nonempty-valued. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty complete lattice.

Again, we leave the details to the Appendix B, but we state a counterpart to Lemma 2 that requires only that very lemma (which we showed above) to prove it.

Lemma 3. Let X be a complete lattice and $F : X \rightrightarrows X$ be nonempty-valued. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty and has a largest element.

Exercise 3. Prove Lemma 3 by relying only on Lemma 2.

3.2. Li–Che–Kim–Kojima Fixed Point Theorem

Let $F : X \rightrightarrows Y$, where X, Y are partially ordered sets. For $S \in 2^X$, we write $F(S) := \cup_{x \in S} F(x)$.

Definition 4. F is said to be

(i) **upper weak set monotone** if $F(x') \geq_{uws} F(x)$ $\forall x' \geq x$;

(ii) **lower weak set monotone** if $F(x') \geq_{lws} F(x)$ $\forall x' \geq x$;

(iii) **weak set monotone** if $F(x') \geq_{ws} F(x)$ $\forall x' \geq x$;

(iv) **strong set monotone** if $F(x') \geq_{ss} F(x)$ $\forall x' \geq x$.

Lemma 4. (Che et al. 2021, Lemma 2) Let $F : X \rightrightarrows Y$, where X, Y are partially ordered sets. If F is weak set monotone, then for any subsets $S', S \subseteq X$ such that $S' \geq_{ws} S$, $F(S') \geq_{ws} F(S)$.

In the sequel we will say that a set X is a *compact partially ordered metric space*; you can read "compact subset a Euclidean space" ($\mathbb{R}^k$).[4] In fact, as in applications it will be often assumed that X is a subset of a Euclidean space, it may be convenient to know that in this case X being a compact (partially ordered set) is equivalent to it being a complete lattice.[5]

Theorem 3. (Li 2014; Che et al. 2021, Theorem 6) Let X be a compact partially ordered metric space. Let $F : X \rightrightarrows X$ be a nonempty- and closed-valued correspondence on X. If F is upper (resp. lower) weak set monotone and $\exists x, y \in X$ such that $x \leq y \in F(x)$ (resp. $x \geq y \in F(x)$), then it admits a maximal (resp. minimal) fixed point.

Proving the result goes far beyond the scope of this class; for those who are interested, a brief sketch of the steps is in the Appendix C.

[4] More rigorously, we use this to mean that it is endowed with the metric and natural topology induced by its partial order ($\geq$), and then we assume it is compact with respect to this topology.

[5] Note that we are saying a complete lattice, not a complete *sub*lattice of $\mathbb{R}^k$! An example: $\{(0,0),(1,0),(0,1),(2,2)\}$ is a complete lattice, a sublattice of $\mathbb{R}^2$, but not a complete *sub*lattice of $\mathbb{R}^2$.

4. Monotone Comparative Statics on Fixed Points

Villas-Boas (1997) proved weak monotone comparative statics results for functions that are very general.

Theorem 4. (Villas-Boas 1997, Theorems 4 and 5) Let X be a partially ordered set, and $f, g : X \to X$.

(i) If (i) $\forall x \in X$, $X_{\geq x}$ is a complete lattice, and (ii) f is weakly increasing, then for every fixed point of g, $x \in \mathscr{F}(g)$ such that $f(x) \geq (>) x$, there is a fixed point of f $y \in \mathscr{F}(f)$ for which $y \geq (>) x$. If in addition $f \geq (>) g$, then $y \geq (>) x$ for any $y \in \mathscr{F}(f), x \in \mathscr{F}(g)$.

(ii) If (i) $\forall x \in X$, $X_{x\leq}$ is a complete lattice, and (ii) g is weakly increasing, then for every fixed point of f, $y \in \mathscr{F}(f)$ such that $y \geq (>) g(y)$, there is a fixed point of g $x \in \mathscr{F}(g)$ for which $y \geq (>) x$. If in addition $f \geq (>) g$, then $y \geq (>) x$ for any $y \in \mathscr{F}(f), x \in \mathscr{F}(g)$.

Proof. We will prove (i); the proof for (ii) is symmetric. Let $x^* \in \mathscr{F}(g)$ such that $f(x^*) \geq (>) x^*$. By monotonicity, $f(x) \geq (>) x^*$ for any $x \in X_{\geq x^*}$. Let $\tilde{f} : X_{\geq x^*} \to X_{\geq x^*}$, where $\tilde{f}(x) = f(x)$ and apply Tarki's fixed point theorem (for (i), Lemma 2 is enough) to conclude that $\exists y \in X_{\geq x^*}$ such that $f(y) = y > x^*$. □

As we will see in the next section, Theorem 4 pairs very well with the strong set order to obtain comparative statics for equilibria. In addition, Villas-Boas (1997) provides extensions for Banach spaces and correspondences under very general conditions, which may prove useful if you are doing some complicated functional optimization (as is sometimes the case in macro).

Che et al. (2021, Theorem 7) provide results for correspondences that are adjusted to the weak set order.

Theorem 5. (Che et al. 2021, Theorem 7) Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$.

(i) If $\mathscr{F}(F) \neq \emptyset$, G is upper weak set monotone, nonempty- and closed-valued,
and $G(x)$ upper weak set dominates $F(x)$ for every $x \in X$,
then the set of fixed points of G upper weak set dominate those of F.

(ii) If $\mathscr{F}(G) \neq \emptyset$, F is lower weak set monotone, nonempty- and closed-valued, and $G(x)$ lower weak set dominates $F(x)$ for every $x \in X$, then the set of fixed points of G lower weak set dominate those of F.

Proof. Take any $x^* \in \mathscr{F}(F)$. For any $S \subseteq X$, define $S_{\geq x^*} := \{x \in S \mid x \geq x^*\}$ and $S_+(F) := \{x \in S \mid \exists y \geq x$ such that $y \in F(x)\}$ ($S_-(F)$ is analogously defined).
Let $\tilde{G}$ be a self-correspondence on $X_{\geq x^*}$ such that $\tilde{G}(x) := G(x)_{\geq x^*}$ for any $x \in X_{\geq x^*}$, that is, all the elements in $G(x)$ that dominate x^*.
We will prove that $\tilde{G}$ verifies the conditions for it to have a fixed point in $X_{\geq x^*}$, thereby showing upper weak set dominance.

(i) First, we want to show that $X_{\geq x^*}$ is a compact lattice. As any S closed, $S_{\geq x^*}$ is also closed, and as X is compact metric space, then $S_{\geq x^*}$ is compact. Hence, $X_{\geq x^*}$ is compact.
Furthermore, $X_{\geq x^*}$ is a lattice, as $\forall x, y \in X_{\geq x^*}$, $x \vee y, x \wedge y \in X$ (X is a lattice) and as $x^* \leq x, y \Longrightarrow x^* \leq x \wedge y \leq x \vee y \Longrightarrow x \wedge y, x \vee y \in X_{\geq x^*}$.

(ii) Note that $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$. This is because as $x^* \in F(x^*) \preceq_{uws} G(x^*)$ and therefore $\exists y \in G(x^*)$ such that $y \geq x^*$, i.e. $y \in \tilde{G}(x^*) = G(x^*) \cap X_{\geq x^*}$.
Closed-valuedness of $\tilde{G}$ follows from the facts that G is closed-valued and $X_{\geq x^*}$ is closed.

(iii) We show now that $\tilde{G}$ is nonempty-valued. As for all $x \in X_{\geq x^*}$, $G(x) \succeq_{uws} F(x^*) \ni x^*$, we have for all $x \in X_{\geq x^*}$, there is $y \in G(x)$ such that $y \geq x^*$; this in turn implies that any such $y \in \tilde{G}(x) = G(x) \cap X_{\geq x^*}$, and therefore $\tilde{G}$ is nonempty-valued.

(iv) Let us now show that $\tilde{G}$ is upper weak set monotone. $\forall x, x' \in X_{\geq x^*}$, such that $x \geq x'$, and any $y \in \tilde{G}(x) \subseteq G(x)$, $\exists y' \in G(x')$ such that $y' \geq y$, as G is uws monotone. As $y \geq x^*$, then $y' \geq x^* \Longrightarrow y' \in G(x') \cap X_{\geq x^*} = \tilde{G}(x')$.

(v) Therefore, the conditions for $\tilde{G}$ to have a fixed point as per Theorem 3 are satisfied, and $\exists y^* \in \mathscr{F}(\tilde{G}) \subseteq \mathscr{F}(G)$ such that $y^* \geq x^*$. We conclude $\mathscr{F}(G) \succeq_{uws} \mathscr{F}(F)$.

The proof for (ii) is symmetric. □

Corollary 1. Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$. If F and G are nonempty- and closed-valued, F is lower weak set monotone, G is upper weak set monotone, and $G(x) \geq_{ws} F(x)$ for every $x \in X$, then $\mathscr{F}(G) \geq_{ws} \mathscr{F}(F)$.

4.1. Games with Strategic Complementarities

In this section, we'll define games is a reduced-form manner, so as to obtain results that are applicable more generally. We call a **reduced-form game** G a tuple $G = \langle I, X, B\rangle$, where (i) I is a finite set of players, $X = \times_{i\in I} X_i$ with X_i denoting player i's strategy space, and $B = (B_i)_{i\in I}$ with $B_i : X_{-i} \rightrightarrows X_i$ being a correspondence that characterizes player i's behavior. We will also abuse notation and write $B(x) : X \rightrightarrows X$, such that $B(x) := \times_{i\in I} B_i(x_{-i})$, where the meaning is clear from the context. As B summarizes all the components of our reduced-form game[6] we will denote the set of fixed points of G by $\mathscr{F}(B)$, defined in the same way as before, $\mathscr{F}(B) := \{x \in X \mid x_i \in B_i(x_{-i})\}$.

Our goal is provide general results that say something like "if B_i increases, then the set of equilibria increases," in a well-defined sense. For instance, if payoffs for a player's action increase, when can we say that the player will choose it more often? Another example: suppose that the government decides to regulate prices for a specific good and imposes a price cap (e.g. EU price regulation on pharmaceutical products). How will that change pricing strategies of firms unaffected by the price cap? A similar question arises with quantity quotas — think about an oil cartel.

Note that B_i may or may not be given as the best-response correspondence in a game, i.e. $B_i(x_{-i}) = \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$. This way of defining things has the virtue of being applicable to equilibrium models and solution concepts other than Nash equilibrium. After discussing the results in terms of the properties of B_i that we need, we note how to get them from results we already saw when discussing monotone comparative statics of individual choices.

[6]Note that the definition of B depends on I and X

Strong Complementarities

First let's use Zhou's fixed point theorem to show an equilibrium exists:

Theorem 6. Let $X := \times_{i\in I}$ be such that, $\forall i \in I$, X_i is a complete lattice. If $B_i : X_{-i} \rightrightarrows X_i$ is strong set monotone and nonempty- and complete-sublattice-valued for every $i \in I$, then the set of fixed points of B, $\mathscr{F}(B)$, is a complete lattice.

Proof. As B_i is complete-sublattice-valued for every $i \in I$, $B(x)$ is a complete sublattice of X (in the product order). As B_i are monotone in the strong set order, then so is B. By Zhou's fixed point theorem (Theorem 2), the set of fixed points of B is a complete sublattice and therefore has a largest and smallest element. □

Now let's obtain the monotone comparative statics on the sets of equilibria:

Theorem 7. Let $X := \times_{i\in I} X_i, \tilde{X} := \times_{i\in I}\tilde{X}_i$ be such that, $\forall i \in I$, $X_i, \tilde{X}_i$ are complete lattices with respect to the same partial order, and $\tilde{X}_i \geq_{ss} X_i$. Let $B_i : X_{-i} \rightrightarrows X_i$ and $\tilde{B}_i : \tilde{X}_{-i} \rightrightarrows \tilde{X}_i$ be strong set monotone and nonempty- and complete-sublattice-valued for every $i \in I$. If $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$ for every $i \in I$ and $x_{-i} \in X_{-i}, \tilde{x}_{-i} \in \tilde{X}_{-i}$ such that $\tilde{x}_{-i} \geq x_{-i}$, then $\sup_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq \sup_{\mathscr{F}(B)} \mathscr{F}(B)$ and $\inf_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq \inf_{\mathscr{F}(B)} \mathscr{F}(B)$.

Note that if the above result implies that (but is not equivalent to) $\mathscr{F}(\tilde{B})$ *weak set dominates* $\mathscr{F}(B)$.

Proof. As $\mathscr{F}(\tilde{B}), \mathscr{F}(B)$ are lattices, their largest and smallest elements exist. We show that the largest fixed point of $\tilde{B}$ is greater than the largest fixed point of B; the proof for the smallest is symmetric.

Let $b_i^*(x_{-i}) := \sup_{X_i} B_i(x_{-i})$ and $b_{i*}(x_{-i}) := \inf_{X_i} B_i(x_{-i})$. As B_i is complete-sublattice-valued, $b_i^*(x_{-i}), b_{i*}(x_{-i}) \in B_i(x_{-i})$. Let $b^*(x) := \sup_X B(x) = (b_i^*(x_{-i}))_{i\in I}$ and symmetrically for $b_*(x)$. Define $\tilde{b}^*$ and $\tilde{b}_*$ analogously, on $\tilde{X}$.

Claim: The largest (smallest) fixed point of b^* (b_*) is the largest (smallest) fixed point of B.

The proof of this claim is left as an exercise.

Claim: $\tilde{b}^*(\tilde{x}) \geq b^*(x)$ for any $\tilde{x} \geq x$.

To see this, note that $\tilde{b}^*(\tilde{x}) \in \tilde{B}(\tilde{x})$ and $b^*(x) \in B(x)$. As $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$ for every $i \in I$ and $x_{-i} \in X_{-i}, \tilde{x}_{-i} \in \tilde{X}_{-i}$ such that $\tilde{x}_{-i} \geq x_{-i}$, then $\tilde{B}(\tilde{x}) \geq_{ss} B(x)$ for $\tilde{x} \geq x$.[7] Hence, $\tilde{x} \vee x \in \tilde{B}(\tilde{x})$ and then $x \leq \tilde{x} \vee x \leq \tilde{b}^*(\tilde{x}) = \sup_{\tilde{X}} \tilde{B}(\tilde{x})$.

Claim: $\tilde{b}^*$ is monotone.

As $\tilde{B}_i$ is strong set monotone and complete-sublattice-valued for each i, so is $\tilde{B}$. Completeness yields $\sup_{\tilde{X}} \tilde{B}(x) \in \tilde{B}(x)$ for each $x \in \tilde{X}$. Monotonicity implies that for any $x \geq y$, $\sup_{\tilde{X}} \tilde{B}(x) \vee \sup_{\tilde{X}} \tilde{B}(y) \in \tilde{B}(x)$. Therefore, $\tilde{b}^*(x) = \sup_{\tilde{X}} \tilde{B}(x) \geq \sup_{\tilde{X}} \tilde{B}(x) \vee \sup_{\tilde{X}} \tilde{B}(y) \geq \sup_{\tilde{X}} \tilde{B}(y) = \tilde{b}^*(y)$.

If $X = \tilde{X}$, noting that $X_{\geq x}$ is complete lattice for every x,[8] we can just use Theorem 4. As $X \neq \tilde{X}$, we will need an extra step:

Claim: Let x^* and $\tilde{x}^*$ be the largest fixed points of b^* and $\tilde{b}^*$. Then $\tilde{x}^* \geq x^*$.

Let $\tilde{X}_{\geq x^*} := \{x \in \tilde{X} \mid x \geq x^*\}$.

As $\tilde{X} \geq_{ss} X$, for any $x \in \tilde{X}$, $x^* \in X$, $x \vee x^* \in \tilde{X}$, and therefore $\tilde{X}_{\geq x^*}$ is nonempty. As $\tilde{X}$ is a complete lattice, so is $\tilde{X}_{\geq x^*}$.

Define $\tilde{g}^*$ on $\tilde{X}_{\geq x^*}$ as $\tilde{g}^*(x) = \tilde{b}^*(x)$. As $\forall x \in \tilde{X}_{\geq x^*}$, $x \geq x^*$, then $\tilde{g}^*(x) = \tilde{b}^*(x) \geq b^*(x^*)$, and therefore $\tilde{g}^*$ is a self-map on a complete lattice.

As $\tilde{b}^*$ is monotone, so is $\tilde{g}^*$.

By Tarski's fixed point theorem, $\tilde{g}^*$ has a fixed point $y^* \in \tilde{X}_{\geq x^*}$, which must be also a fixed point of $\tilde{b}^*$, by definition of $\tilde{g}^*$. Then, $\tilde{x}^* = \sup_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq y^* \geq x^*$. □

Now let's go back to normal-form games $\Gamma = \langle I, X, u \rangle$, where I and X are as above and $u = (u_i)_{i \in I}$, with $u_i : X \to \mathbb{R}$ denoting player i's payoff. We define B_i as player i's best-response correspondence: $B_i(x_{-i}) := \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$.

Given two normal-form games, Γ and $\tilde{\Gamma}$, what do we need in order to guarantee that (i) $\tilde{B}_i, B_i$ are strong set monotone, (ii) $\tilde{B}_i(\tilde{x}_{-i})$ strong set dominates $B_i(x_{-i})$ for every $\tilde{x} \geq x$, and (iii) $\tilde{B}_i, B_i$ are complete-sublattice-valued? The answer to the first two points we already know from our lecture on monotone comparative statics of individual choices. Let's restate Milgrom and Shannon's (1994) Monoticity theorem:

[7] This follows because the partial order on $X, \tilde{X}$ is the product order.

[8] The take any subset $S \subseteq X_{\geq x} \subseteq X$; we have $z := \inf_X S \in X$ (as X is a complete lattice). Noting that x is a lower bound for S and z is the greatest lower bound for S according to $\geq$, we have that $x \leq z \in X_{\geq x}$.

Theorem 8. (Monotonicity; (Milgrom and Shannon 1994, Theorem 4)) Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S';v) \geq_{ss} X(S;u)$.

That is, if $u_i, \tilde{u}_i$ are quasisupermodular, we get strong set monotonicity of $\tilde{B}_i$ and B_i. For single-crossing dominance, we need that $X_i, \tilde{X}_i$ be (i) subsets of a lattice Y_i, and (ii) $\tilde{X}_i \geq_{ss} X_i$. Note that if $u_i = \tilde{u}_i$, then single-crossing is trivially satisfied. Moreover, note that if S is a complete sublattice of X, then $X(S;u)$ is too, as $X(S;u) \geq_{ss} X(S;u)$.

For (iii) we use Berge's maximum theorem. Specifically, we need $\tilde{u}_i$ and u_i to be continuous and $\tilde{X}_i$ and X_i to be compact. This gets us that $\tilde{B}_i$ and B_i to be nonempty- and compact-valued, so we're not there yet. But if $X_i, \tilde{X}_i$ are compact subsets of an Euclidean space, compactness will buy us completeness. Combining everything we have the following result (Topkis 1979; Milgrom and Shannon 1994):

Corollary 2. Let $\Gamma = \langle I, X, u\rangle$ and $\tilde{\Gamma} = \langle I, \tilde{X}, \tilde{u}\rangle$ be two normal-form games such that, for each $i \in I$,

(i) $X_i, \tilde{X}_i$ are compact, complete sublattices of an Euclidean space, such that $\tilde{X}_i \geq_{ss} X_i$,

(ii) $u_i, \tilde{u}_i$ are continuous and quasisupermodular,[9] and

(iii) $\tilde{u}_i$ single-crossing dominates u_i,

then

(i) the set of Nash equilibria of each game is a nonempty complete lattice,

(ii) the largest (smallest) Nash equilibrium of $\tilde{\Gamma}$ is greater than the largest (smallest) Nash equilibrium of Γ.

The games satisfying the assumptions above are usually known as *supermodular* games.

Exercise 4. Consider an oligopolistic industry where each firm $i \in [n]$ simultaneously chooses a quantity $q_i \in [0, \overline{q}]$. Each firm has the same constant marginal cost $c > 0$, and the inverse demand function is given by $p = P(Q, \theta)$, where $Q = \sum_{i \in [n]} q_i$ is the total quantity of output produced by the firms and $\theta \in [0,1]$ is captures the demand level due to market

[9] To be more precise, we can relax qusisupermodularity in x — and the expense of being more verbose — to (i) qusisupermodularity in x_i, and (ii) satisfy the single-crossing property in $(x_i; x_{-i})$.

conditions. P is decreasing in Q, increasing in θ, and supermodular in (Q,θ), and assume that $P(n\overline{q},0) > 0$.

Suppose that $\overline{q}$ is a per firm production quota determined by the government every year in order to stability prices. It is expected that θ increases. What can we predict the government's reaction to be in setting the quota $\overline{q}$ for the following year? *(Hint: use the above framework to prove that the largest and smallest symmetric equilibria exist and describe how they relate to θ and $\overline{q}$.)*

Games with Weak Strategic Complementarities

We now provide weak set order counterparts to Theorems 6 and 7.

Theorem 9. (Che et al. 2021, Theorem 9(i)) Let $X := \times_{i\in I} X_i$ be such that, $\forall i \in I$, X_i is a compact partially ordered metric space, and $B_i : X_{-i} \rightrightarrows X_i$ is nonempty- and compact-valued. If

(i) $\exists x, y \in X$ such that, for every $i \in I$, $y_i \in B_i(x_{-i})$ and $y_i \geq x_i$ (resp. $\leq$), and

(ii) for every $i \in I$, B_i is upper (resp. lower) weak set monotone,

then the set of fixed points of B, $\mathscr{F}(B)$, is nonempty.

Proof. B_i is nonempty- and compact-valued by assumption. B is nonempty- and closed-valued, and upper (resp. lower) weak set monotone. Thus, by Theorem 3, $\mathscr{F}(B) \neq \emptyset$. □

Theorem 10. (Che et al. 2021, Theorem 9(ii)) Let $X := \times_{i\in I} X_i$ be such that, $\forall i \in I$, X_i is a compact partially ordered metric space, and $B_i, \tilde{B}_i : X_{-i} \rightrightarrows X_i$ are nonempty- and compact-valued. If

(i) $\exists x, y \in X$ such that, for every $i \in I$, $y_i \in \tilde{B}_i(x_{-i})$ and $y_i \geq x_i$ (resp. $\leq$),

(ii) for every $i \in I$, $\tilde{B}_i$ is upper (resp. lower) weak set monotone, and $\tilde{B}_i(x_{-i}) \geq_{uws} B_i(x_{-i})$ $\forall x_{-i} \in X_{-i}$ (resp. $\geq_{lws}$)

then $\mathscr{F}(\tilde{B}) \geq_{uws} \mathscr{F}(B)$ (resp. $\geq_{lws}$) whenever the latter is nonempty.

Proof. By Theorem 9, $\mathscr{F}(\tilde{B}) \neq \emptyset$. As $\tilde{B}$ upper (resp. lower) weak set dominates B, by Theorem 5, $\mathscr{F}(\tilde{B}) \geq_{uws} \mathscr{F}(B)$ (resp. $\geq_{lws}$). □

5. References

Acemoglu, Daron, and Martin Kaae Jensen. 2015. "Robust Comparative Statics in Large Dynamic Economies." *Journal of Political Economy* 123 (3): 587–640. 10.1086/680685. 1

Agarwal, Nikhil. 2015. "An Empirical Model of the Medical Match." *American Economic Review* 105 (7): 1939–78. 10.1257/aer.20131006. 1

Aygün, Orhan, and Inácio Bó. 2021. "College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case." *American Economic Journal: Microeconomics* 13 (3): 1–28. 10.1257/mic.20170364. 1

Che, Yeon-Koo, Jinwoo Kim, and Fuhito Kojima. 2021. "Weak Monotone Comparative Statics." *Working Paper* 1–65, `https://arxiv.org/pdf/1911.06442.pdf`. 2, 5, 6, 12

Dur, Umut, Parag A. Pathak, and Tayfun Sönmez. 2020. "Explicit vs. statistical targeting in affirmative action: Theory and evidence from Chicago's exam schools." *Journal of Economic Theory* 187 104996. 10.1016/j.jet.2020.104996. 1

Echenique, Federico. 2005. "A short and constructive proof of Tarski's fixed-point theorem." *International Journal of Game Theory* 33 215–218. 10.1007/s001820400192. 15, 16, 17

Lazzati, Natalia. 2015. "Treatment Response with Social Interactions: Partial Identification via Monotone Comparative Statics." *Quantitative Economics* 6 (1): 49–83. 10.3982/QE308. 1

Li, Jinlu. 2014. "Fixed Point Theorems on Partially Ordered Topological Vector Spaces and Their Applications to Equilibrium Problems with Incomplete Preferences." *Fixed Point Theory and Applications* 192 1—17. 10.1186/1687-1812-2014-192. 5

Milgrom, Paul, and Chris Shannon. 1994. "Monotone Comparative Statics." *Econometrica* 62 (1): 157—180. 10.1007/BF01215200. 2, 10, 11

Tarski, Alfred. 1955. "A lattice-theoretical fixpoint theorem and its applications." *Pacific Journal of Mathematics* 5 (2): 285–309. pjm/1103044538. 3, 4, 16

Topkis, Donald M. 1979. "Equilibrium Points in Nonzero-Sum n-Person Submodular Games." *SIAM Journal on Control and Optimization* 17 773—787. 10.1137/0317054. 2, 11

Topkis, Donald M. 1998. *Supermodularity and Complementarity*. Princeton, NJ: Princeton University Press. 2

Villas-Boas, J. Miguel. 1997. "Comparative Statics of Fixed Points." *Journal of Economic Theory* 73 (1): 183—198. 10.1006/jeth.1996.2224. 6

Zhou, Lin. 1994. "The Set of Nash Equilibria of a Supermodular Game Is a Complete Lattice." *Games and Economic Behavior* 7 (2): 295–300. 10.1006/game.1994.1051. 4, 17

Appendix A. Proof of Tarki's Fixed-Point Theorem (Theorem 1)

To show that $\mathscr{F}(f)$ has a smallest element, we need to take a small detour.

First, we need to introduce the concept of an **ordinal**, which generalizes the cardinal numbers. Think about the, as an extension of the natural numbers. An ordinal is the concept that translates the size or cardinality of a set even when this set is not countable. For instance, $\mathbb{R}$, while uncountable, has a cardinality.

Definition 5. A binary relation $>$ on X is said to be **well-founded** if for every nonempty subset $S \subseteq X$, there is a **minimal element** y of S in S, that is, $y \in S$ and $\nexists x \in S$ such that $y > x$.

The ordinal numbers are not only well-founded set, but also totally ordered. (Again think about the natural numbers; are they well-founded? are they totally ordered?)

One important property of well-founded binary relations on sets is that we can apply a particular form of definition by recursion, called *transfinite recursion*. This goes as follows:

Theorem 11. (Transfinite/Noetherian recursion) Let $(X, \geq)$ be a partially ordered set and $> \subseteq X \times X$ be well-founded. If, to each pair (x, g), where $x \in X$ and g is a function on $\{y \in X \mid y < x\}$, H assigns an object $H(x, g)$, then there is a unique function G on X such that, $\forall x \in X$, $G(x) = H(x, G\mid_{y \in X \mid y < x})$.

Theorem 12. (Transfinite/Noetherian induction) Let $(X, \geq)$ be a partially ordered set and $> \subseteq X \times X$ be well-founded. If $\forall x \in X$, property P being satisfied for all $y < x$ implies that P is satisfied for x, then P is satisfied for all $x \in X$.

We know that we can apply recursion and induction on countable sets. Transfinite recursion an induction extend these notions to uncountable sets.

Lemma 5. (Echenique 2005, Lemma 1) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $(\mathscr{F}(f), \geq)$ has a smallest element.

Proof. Let η be an ordinal number with cardinality greater than X and define $\xi := \eta + 1$. Let $g : \xi \to X$ be defined by transfinite recursion as follows: $g(0) = \inf_X X$ and $g(\beta) =$

$\sup_X\{f(g(\alpha)) \mid \beta > \alpha\}$, for $\beta > 0$. Now note that $f(g(0)) \geq g(0)(= \inf_X X)$, and therefore by transfinite induction $f(g(\beta)) \geq g(\beta)$ for all ordinals β, and we conclude that g is monotone. As ξ has cardinality larger than X, there is an ordinal $\gamma < \xi$ such that $g(\gamma) = g(\gamma+1)$ which further implies that there is a smallest $\underline{\gamma}$ (the set of ordinals is well-founded and totally ordered) such that $f(g(\underline{\gamma})) = f(g(\underline{\gamma})) = g(\underline{\gamma})$. Therefore, the set of fixed points of $\tilde{f}$ is nonempty: $\mathscr{F}(f) \neq \emptyset$.

We now show that $g(\underline{\gamma})$ is the smallest fixed point of f. If $\underline{\gamma} = 0$ we are done, as $g(0) = \inf_X X$. If not, then $g(0) < f(g(0))$. Take any $z \in \mathscr{F}(f)$. As, by monotononity of f, $z = f(z) \geq f(g(0)) \geq g(0)$, then, by transfinite induction $z \geq f(g(\alpha))$ for any α. Hence, $z \geq f(g(\underline{\gamma})) = g(\underline{\gamma})$. □

Theorem 13. (Tarski 1955; Echenique 2005, Theorem 2) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is a non-empty complete lattice.

Proof. By Lemma 5, $\mathscr{F}(f)$ is non-empty and has a smallest element. Let $S \subseteq \mathscr{F}(f)$ be non-empty. We want to show that $\sup_{\mathscr{F}(f)} S \in \mathscr{F}(f)$.

Let $y := \sup_X S$ and let $Y := \{x \in X \mid x \geq y\}$, be the set of upper bounds on S. If $y \in Y$, then $\forall x \in S$, $x \leq y \implies x = f(x) \leq f(y)$. Therefore, $f(Y) \subseteq Y$. Let $g = f\,|_Y$; g is then a monotone self-map on Y, a complete lattice. By Lemma 5, $(\mathscr{F}(g), \geq)$ has a smallest element, i.e. $\inf_X \mathscr{F}(g) \in \mathscr{F}(g) \subseteq \mathscr{F}(f) \cap Y$. By definition of g, $\min_X \mathscr{F}(g) = \sup_{\mathscr{F}(f)} S$. The proof for $\inf_{\mathscr{F}(f)} S$ is symmetric. □

Appendix B. Proof of Zhou's Fixed-Point Theorem (Theorem 2)

Lemma 6. (Echenique 2005, Lemma 3) Let X be a complete lattice and $F : X \rightrightarrows X$. If F is monotone and, $\forall x \in X$, $F(x)$ has a smallest element, then $(\mathscr{F}(F), \geq)$ has a smallest element.

Proof. Let $f : X \to X$ be given by $f(x) := \min_X F(x)$. By Lemma 5, $(\mathscr{F}(f), \geq)$ has a smallest element; denote it z. By construction, $z \in \mathscr{F}(F)$, as $z = \min_X F(z)$.

To see that z is the smallest element in $(\mathscr{F}(F), \geq)$, take any $e \in \mathscr{F}(F)$. Let g, and $\underline{\gamma}$ be defined as in the proof for Lemma 5 with respect to f. Take any $e \in \mathscr{F}(F)$. The steps are then identical: As $g(0) \leq f(g(0)) \leq e \implies f(g(\alpha)) \leq e$ for every ordinal α, by transfinite induction $z = g(\underline{\gamma}) = f(g(\underline{\gamma})) \leq e$. $\square$

Theorem 14. (Zhou 1994, Theorem 1; Echenique 2005, Theorem 4) Let X be a complete lattice and $F : X \rightrightarrows X$. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty complete lattice.

Proof. Take any subset of fixed points $E \subseteq \mathscr{F}(F)$, which exist, by Lemma 6.
We now show that $\sup_{\mathscr{F}(F)} E$ exists. Let $x := \sup_{X)} E$ and $Y := \{y \in X \mid x \leq y\}$. Note that Y is a complete lattice, as $\forall S \subseteq X$, $x \leq \sup_X S \in X \implies \sup_X S \in Y \implies \sup_X S = \sup_Y S$; an analogous argument holds for $\inf_Y S$.

Define $G : Y \rightrightarrows Y$ as $G(y) := Y \cap F(y)$; this is the set of elements in $F(y)$ that are weakly larger than x. The goal will be to show that G is (i) nonempty, (ii) monotone, and (iii) a complete sublattice of Y. Then we (iv) apply the previous lemma and show that it has a smallest point in Y, and (v) conclude by saying that this fixed is also a fixed point in X and that it must be in E. We now prove each of these claims:

(i) We want to show that $G(y) \neq \emptyset$. $e \leq x \leq y$, $\forall e \in E$ and $y \in Y$. Then, $\forall x_y \in F(y)$ and as $e \in F(e)$, $x_y \vee e \in F(y)$ by monotonicity of F. Fix an arbitrary $x_y \in F(y)$ and note that $\sup_X S =: x \leq \sup_X \{x_y \vee e \in X \mid e \in E\} \in F(y)$ as F is a complete sublatttice of X (by assumption). Hence, $G(y) := Y \cap F(y) \neq \emptyset$.

(ii) Now we show that G is monotone. Take any $y' \leq y \in Y$, $z \in G(y)$ and $z' \in G(y')$. Note that $y \wedge y' \in F(y)$ and $y \vee y' \in F(y')$ as F is monotone. Furthermore, $\forall e \in E$, $e \in F(e)$, $e \leq y \leq y'$, and thus, $y \wedge y' = (y \wedge y') \vee e \in F(y)$ and $y \vee y' = (y \vee y') \vee e \in F(y')$, again by monotonicity of F.

(iii) $\forall S \subseteq G(y)$, $\inf_Y S, \sup_Y S \in G(y)$. To see this note that as F is a complete sublattice, and as $S \subseteq G(y) = F(y) \cap Y$, $S \subseteq F(y)$, and then $\sup_Y S = \sup_X S \cap Y \in F(y)$, and analogously for $\inf_Y S$. As $\forall z \in S \subseteq Y$, $z \geq x$, $\sup_X S \cap Y \in Y \implies \sup_X S \cap Y \in F(y) \cap Y = G(y)$; and analogously for the $\inf_Y \in S$.

(iv) All the hypotheses of Lemma 6 are satisfied, and therefore let $y^* \in \mathscr{F}(G)$ be the smallest fixed point of G in Y. For any fixed point x^* of F in X that is an upper bound on E, we must have that $x \le x^* \implies x^* \in F(x^*) \cap Y = G(x^*)$, and therefore it is a fixed point of G in Y. Hence, $x^* \ge y^*$. As $y^* \in G(y^*) = F(y^*) \cap Y$, y^* is also a fixed point of F in X. We conclude that $y^* = \sup_{\mathscr{F}(F)} E$.

The proof that $\sup_{\mathscr{F}(F)} E$ exists is symmetric. □

Appendix C. Proof Sketch of LCKK's Fixed-Point Theorem (Theorem 3)

A very rough sketch of the main steps in the proof is as follows:

(1) X being a compact metric space implies that each nonempty chain has a least upper bound.

(2) Upper weak set monotonicity of F together with compactness allows for a proof for existence of a fixed point with the same spirit to finding the largest fixed point in Lemma 2 by doing a monotone selection of F.

(3) For existence of a maximal fixed point, take any chain of fixed points. Compactness implies that it admits a smallest upper bound and closed-valuedness and uws-monotocity of F will (eventually — the proof is complex) yield that the smallest upper bound is a fixed point.

(4) As (3) holds for any chain of fixed points, appeal to Zorn's Lemma to claim that there is a maximal fixed point.

ECON0106: **Microeconomics**

15. Dynamic Games and Refinements of Nash Equilibrium*

Duarte Gonçalves†

University College London

1. Backward Induction

Intuitive explanation of backward induction:

— start at terminal nodes and work your way backward toward the root of the game,

— at each information set, players choose the payoff-maximizing action assuming (i) the information set is reached, and (ii) taking as given what actions are taken at information sets corresponding to successor nodes

Whether or not agents *should* engage in this sort of reasoning and look ahead arbitrarily many "moves" is a matter for discussion; you'll learn more about limited foresight next term. For the moment we are going to make our lives easier and just assume perfect foresight.

Definition 1. (Informal) For any finite extensive-form game of perfect information Γ, a pure strategy profile s is said to be obtained by **backward induction** if it is given by an action profile, one for each node corresponding to an information set, such that the action profile is defined implicitly and recursively as follows:

(i) For all decision nodes at which any action leads to a terminal node, the player choosing at that node(= information set) takes the action that selects the terminal node that maximizes their payoff.

*Last updated: 10 December 2021.

† Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

(ii) Restrict the extensive-form game by making the decision nodes at (i) a terminal node of a restricted game tree, with payoffs being given by the payoff of the associated terminal node that was selected.

(iii) Take the restricted extensive-form game and repeat (i)-(ii) until the set of nodes is reduced to a singleton.

Theorem 1. (Zermelo) Every finite game of perfect information has a pure strategy Nash equilibrium that can be obtained by backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then backward induction results in a unique Nash equilibrium.

1.1. Subgame Perfection and Backward Induction

Definition 2. A **subgame** $\Gamma_\ell = \langle \mathscr{X}_\ell, \mathscr{A}, I, p_\ell, \alpha_\ell, \mathscr{H}_\ell, H_\ell, \iota_\ell, \rho_\ell, u^\ell \rangle$ of an extensive-form game $\Gamma = \langle \mathscr{X}, \mathscr{A}, I, p, \alpha, \mathscr{H}, H, \iota, \rho, u \rangle$ is another extensive-form game such that:

(i) The nodes in the subgame are a subset of the nodes of the original game: $\mathscr{X}_\ell \subseteq \mathscr{X}$.

(ii) There is a unique nonterminal node $x_0^\ell \in \mathscr{X}_\ell \setminus T$ such that its predecessor is not part of the subgame $p(x_0^\ell) \notin \mathscr{X}_\ell$; this is the root of the subgame.

(iii) The node corresponds to a singleton information set, $\{x_0^\ell\} = H(x_0^\ell)$.

(iv) The nodes of the subgame are the node and all of its successors: $S(x_0^\ell) \cup \{x_0^\ell\} = \mathscr{X}_\ell$.

(v) If a node is in the subgame, so are all the nodes in the same information set, i.e. $x \in \mathscr{X}_\ell \Longrightarrow H(x) \subseteq \mathscr{X}_\ell$.

(vi) Formally, the primitives of the original game are restricted to the subgame's nodes in the following manner:

— $p_\ell : \mathscr{X}_\ell \to \mathscr{X}_\ell \cup \{\emptyset\}$ such that $\forall x \in \mathscr{X}_\ell \setminus \{x_0^\ell\}$, $p_\ell(x) = p(x)$;

— $T_\ell = T \cap \mathscr{X}_\ell$;

— $\alpha_\ell : \mathscr{X}_\ell \setminus \{x_0^\ell\} \to \mathscr{A}$ such that $\forall x \in \mathscr{X}_\ell \setminus \{x_0^\ell\}$;

— $\alpha_\ell(x) = \alpha(x)$;

— $\mathscr{H}_\ell \subseteq \mathscr{H}$;

— $H_\ell : \mathscr{X}_\ell \setminus T_\ell \to \mathscr{H}_\ell$ such that $\forall x \in \mathscr{X}_\ell \setminus T_\ell$, $H_\ell(x) = H(x)$;

— $\iota_\ell : \mathscr{H}_\ell \to I$ such that $\iota_\ell(H') = \iota(H')$ for any $H' \in \mathscr{H}_\ell$;

— $u^\ell = (u_i\,|_{T_\ell})_{i \in I}$, i.e. the payoff functions are restricted to the subgame's terminal nodes;

— $\forall H' \in \mathscr{H}_\ell$ such that $\iota(H') = 0$, $\rho_\ell(H') = \rho(H')$.

Importantly:

(i) The root of the subgame is a singleton information set of the game;

(ii) The subgame includes all its successors and nothing more;

(iii) Subgames don't "cut across" information sets;

(iv) A subgame is an extensive-form game in its own right.

The rest is bookkeeping.

Proposition 1. Let $\Gamma = \langle \mathscr{X}, \mathscr{A}, I, p, \alpha, \mathscr{H}, H, \iota, \rho, u \rangle$ be an extensive-form game and $G = \{\Gamma_\ell\}_\ell$ be the set of all its subgames. Let $\trianglerighteq_{sg}$ be a binary relation on G such that $\Gamma_\ell \trianglerighteq_{sg} \Gamma_h$ if and only if Γ_ℓ is a subgame of Γ_h.

(i) $\trianglerighteq_{sg}$ is a partial order on G (reflexive, transitive, and antisymmetric binary relation).

(ii) G is nonempty.

(iii) If Γ is finite, then G is finite, that is, Γ has finitely many subgames.

Exercise 1. Prove Proposition 1.

A **proper subgame** Γ_ℓ of an extensive-form game Γ is such that $\Gamma \trianglerighteq_{sg} \Gamma_\ell$ but not the converse.

Definition 3. A strategy profile σ in extensive-form game Γ induces a strategy profile in its subgame Γ_ℓ given by $\sigma^\ell = (\sigma_i^\ell)_{i \in I}$ such that for every $i \in I, H' \in H_i{}^\ell$, $\sigma_i^\ell(H') = \sigma_i(H')$.

Definition 4. A Nash equilibrium σ of an extensive-form game Γ is a **subgame perfect Nash equilibrium** (SPNE) if it induces a Nash equilibrium in every subgame of Γ.

Definition 5. For any finite extensive-form game Γ, a mixed strategy profile σ is said to be obtained by **generalized backward induction** if it is outcome equivalent to a behavioral strategy profile λ^N, for $N = |G|$, where λ^N is defined recursively as follows:

$n = 1$ Let $G^1 := \arg\min_{\rhd_{sg}} G$. For every $\Gamma_\ell \in G^1$, pick a Nash equilibrium σ^ℓ and fix the associated behavioral strategies $\lambda^{1,\ell} = \{\lambda_i^\ell\}_{i \in \iota(\mathscr{H}_\ell)}$, and define $\lambda^1 := \cup_{\ell:\Gamma_\ell \in G^1} \lambda^{1,\ell}$.

$n \geq 2$ Let $G^n := \arg\min_{\rhd_{sg}} G \setminus \cup_{m=1}^{n-1} G^m$.
If $G^n \neq \emptyset$, for every $\Gamma_\ell \in G^n$, take the behavioral strategies fixed at every proper subgame Γ_h of Γ_ℓ, i.e. $\Gamma_h \in G^m, m < n : \Gamma_\ell \rhd_{sg} \Gamma_h$. Pick a Nash equilibrium of Γ_ℓ under the restriction that play at proper subgames is given by the fixed behavioral strategies. Fix the behavioral strategies associated with the Nash equilibrium $\lambda^{n,\ell}$ and define $\lambda^n := \lambda^{n-1} \cup \left(\cup_{\ell:\Gamma_\ell \in G^n} \lambda^{n,\ell}\right)$.
If $G^n = \emptyset$, then set $\lambda^N = \lambda^{n-1}$.

Proposition 2. Every finite extensive-form game has a subgame perfect Nash equilibrium. Moreover, a strategy profile is a subgame perfect Nash equilibrium of a finite extensive-form game if and only if it can be obtained by generalized backward induction.

Corollary 1. Every finite game of perfect information has a pure strategy subgame perfect Nash equilibrium that can be obtained by backward induction. If no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium coinciding with a the unique pure strategy subgame perfect Nash equilibrium. Moreover, a pure strategy profile is a subgame perfect Nash equilibrium of a finite extensive-form game of perfect information if and only if it can be obtained by backward induction.

2. Beliefs and Sequential Rationality

Definition 6. A **system of beliefs** is a mapping $\mu : \mathscr{H} \to \Delta(\mathscr{X})$ such that for all $H' \in \mathscr{H}$, $\text{supp}(\mu(H')) \subseteq H'$.[1]

A system of beliefs μ specifies the relative probabilies of being at each node of an information set, for every information set in the game. When the information set is a singleton, $H' = \{x\}$, then $\mu(x) = 1$.

[1] I am favoring this formulation to be able to potentially accommodate information sets with infinite nodes.

Let $T \mid H' := T \cap (\cup_{x \in H'} S(x))$ denote the terminal nodes that are successors of some note in information set H' and let $u_i \mid_{H'} : T \mid H' \to \mathbb{R}$ denote the payoff function of player i as restricted to be defined over such subset of terminal nodes. For a strategy profile σ let $\mathbb{E}_{\mu(H')}[u_i \mid_{H'} (\sigma)]$ denote the expected payoff for player i who is called upon to act at information set H' ($\iota(H') = i$), given that (i) player i's beliefs as to which node $x \in H'$ the player is in are given by $\mu(H')$ — i.e. the beliefs regarding the relative probability of being at any node $x \in H'$, (ii) players are playing according to the strategy profile σ at any information set *following* H'.[2]

Definition 7. A strategy profile σ is **sequentially rational** at information set H' given a system of beliefs μ if for $i = \iota(H')$

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i}) \mid H', \mu, \sigma] \geq \mathbb{E}[u_i(\tilde{\sigma}_i, \sigma_{-i}) \mid H', \mu, \sigma]$$

for all $\tilde{\sigma}_i \in \Sigma_i$.

A strategy profile is **sequentially rational** given a system of beliefs if it is sequentially rational at all information sets given that system of beliefs.

2.1. Weak Perfect Bayesian Equilibrium

Given an extensive-form game $\Gamma = \langle \mathscr{X}, \mathscr{A}, I, p, \alpha, \mathscr{H}, H, \iota, \rho, u \rangle$, and a strategy profile σ,[3] let $\lambda^\sigma : \mathscr{X} \to \Delta(\mathscr{X})$ be such that $\lambda^\sigma(x) \in \Delta(s(x))$ represents the probability distribution over immediate successors of x that given by outcome-equivalent induced behavioral strategy (including nature's move) at $H(x)$. We'll abuse notation and write $\lambda^\sigma(H')$ to denote the distribution over $\cup_{x \in H'} s(x)$ that is given by the outcome-equivalent induced behavioral strategy (including nature's move)

Definition 8. A decision node x is **reached given a strategy profile** σ if itself and all of its predecessor nodes $x' \in P(x) \cup \{x\}$ are in the support of the outcome-equivalent induced

[2] To be more precise, we ought to first define for each player j the information sets $\mathscr{H}_j \mid H' := \{H_j \in \mathscr{H}_j \mid S(x) \cap H_j \neq \emptyset$, for some $x \in H'\}$, i.e. the information sets that have nodes that are successors of some node in H'. Then, we associate with each strategy $s_j \in S_j$ the restricted strategy $s_j \mid H' : \mathscr{H}_j \mid H' \to \mathscr{A}$ such that $s_j \mid H'(H_j) = s_j(H_j)$. And finally, we associate each mixed strategy σ_j with a mixed strategy over the restricted strategies $\sigma_j \mid H'$, such that $\sigma_j \mid H'(\tilde{s}_j) = \sum_{s_j \in S_j \mid s_j \mid H' = \tilde{s}_j} \sigma_j(s_j)$. But this as the idea of what $\mathbb{E}_{\mu(H')}[u_i \mid_{H'} (\sigma)]$ means, we save on the notational burden.

[3] Some care is needed in defining mixed strategies when there a given node has uncountably many successors. See Aumman (1964).

behavioral strategy at the information set corresponding $p(x')$, i.e. $x' \in \text{supp}(\lambda^\sigma(p(x')))$.[4] An information set is **reached given a strategy profile** σ if any of its nodes are reached given σ.

Definition 9. A system of beliefs is said to be derived through Bayes rule whenever possible if, for any information set H' that is reached given σ, $\mu(H') = \lambda^\sigma(H')$.

In the case of a finite game, the definition reads:

Definition 10. A system of beliefs is said to be **derived through Bayes rule whenever possible given** σ if, for any information set H' such that $\mathbb{P}(H' \mid \sigma) > 0$, and any $x \in H'$, $\mu(x) = \frac{\mathbb{P}(x|\sigma)}{\mathbb{P}(H|\sigma)}$.

Proposition 3. A strategy profile σ is a Nash equilibrium of an extensive-form game if and only if there is a system of beliefs μ such that

(i) σ is sequential rational given the system of belief μ at all information sets that on reached given σ; and

(ii) the system of beliefs μ is derived through Bayes rule whenever possible given σ.

"On-path" beliefs — that is, beliefs implied by the system of beliefs at any information set that is reached according to the equilibrium — are *required* to be correct.

Exercise 2. Prove Proposition 3 for the case of finite extensive form games.

Definition 11. A strategy profile σ and a system of beliefs μ is a **weak perfect Bayesian Nash equilibrium** (wPBE) (σ, μ) of an extensive-form game Γ if

(i) σ is sequential rational given the sytem of beliefs μ; and

(ii) the system of beliefs μ is derived through Bayes rule whenever possible given σ.

The 'weak' in wPBE is because there are no restrictions on "off-path" beliefs.[5] Note that the definition of equilibrium includes now a strategy profile **and** a system of beliefs. Hence, when asked to provide all equilibria this means not only all equilibrium strategy profiles but also *all* systems of beliefs that render each equilbrium strategy profile an equilibrium.

[4] If the game is finite, this amounts to saying $\prod_{x' \in P(x) \cup \{x\}} \lambda^\sigma(x') > 0$.

[5] That is, the beliefs implied by the system of beliefs at any information set that is not reached according to the equilibrium strategy profile.

Another difference with respect to our extensive-form characterization of Nash equilibrium, using sequential rationality and a system of beliefs, is that now sequential rationality **is required** at all information sets. We have leeway in specifying "off-path" beliefs, but we do require "on-path" beliefs to be correct.

Corollary 2. Every wPBE is a Nash equilibrium; but the converse is not necessarily true.

How does wPBE relate to SPNE?

Proposition 4. In finite extensive-form games of perfect information, the set of strategy profiles that can be supported as a wPBE is identical to the set of SPNE.

Exercise 3. Prove Proposition 4.

However, in general:

Proposition 5. A strategy profile that is part of a wPBE need not be an SPNE and a SPNE need not be part of any wPBE.

A possible strengthening is to require wPBE to be subgame perfect. We follow Kartik in this next definition:

Definition 12. A strategy profile σ and a system of beliefs μ is a **perfect Bayesian Nash equilibrium** (PBE) (σ, μ) of an extensive-form game Γ if it induces a weak perfect Bayesian Nash equilibrium in every subgame.

2.2. Sequential Equilibrium

Definition 13. A strategy profile σ and a system of beliefs μ is a **sequential equilibrium** (SE) (σ, μ) of an extensive-form game Γ if

(i) σ is sequentially rational given μ;

(ii) there is a sequence of fully mixed strategy profiles $\{\sigma^n\}_n$ inducing a sequence of systems of beliefs μ^n derived through Bayes rule from σ^n such that $\sigma^n \to \sigma$ and $\mu^n \to \mu$.

Differently from wPBE, SE imposes restrictions on "off-path" beliefs. It requires that such beliefs be obtained as a limit of fully mixed beliefs in a way that these beliefs are in the limit consistent with equilibrium play.

Proposition 6. Any sequential equilibrium is a perfect Bayesian Nash equilibrium.

Sequential Equilibrium and Trembling-Hand Perfection

A related notion is that of **extensive-form trembling-hand perfect Nash equilibrium** (ETHPE).
This goes through interpreting the player choosing at any given information set as a different player, defining the normal-form game of such an auxillary game,[6] and solving for a trembling-hand perfect Nash equilibrium of the auxillary game. Note that

Proposition 7. Any extensive-form trembling-hand perfect Nash equilibrium can be supported as a sequential equilibrium by some system of beliefs.

However, in general, a (normal-form) THPE of an extensive-form game need not even be subgame perfect (in which case we need at least one player to choose at more than one information set).

$\{ETHPE\} \subseteq \{SE\} \subseteq \{PBE\} \subseteq \{wPBE\}, \{SPNE\} \subseteq \{NE\}$

2.3. Other Refinements of Nash Equilibria in Extensive-Form Games

If a player sees deviations from SPNE (off-path play), then perhaps they could question whether their opponent addheres to sequential rationality. In fact, you see yourself being called upon to make a choice at a node that is off-path according to SE, you would not think that your opponents' behavior is well-described by such model. This should give you pause to think what to do next. Think about it: if you find yourself at an off-path information set, one explanation would be that perhaps your opponent made a mistake. But, if this is the case, why keep on believing your opponent won't make further mistakes? This idea is the main motivation for Philip Reny's (1992 Ecta) weakening of sequential equilibrium and defining the concept of weakly sequentially rational (loosely speaking, we want to continue requiring sequential rationality given a system of beliefs, but only on-path).

[6]This could be called the agent-normal-form of an extensive-form game.

Lecture Notes for 1st Year Ph.D. Game Theory*

Navin Kartik†

Contents

*Last updated: March 11, 2020. These notes draw upon various published and unpublished sources, including notes by Doug Bernheim, Vince Crawford, George Mailath, David Miller, and Bill Sandholm. Thanks to David Miller and former and current students for comments.

†nkartik@gmail.com. Please feel free to send me suggestions or comments. I would particularly appreciate alerts to typos or other errors.

1. Introduction

Game theory is a formal methodology and a set of techniques to study the interaction of *rational* agents in *strategic* settings.[1] 'Rational' here means the standard thing in economics: maximizing over well-defined objectives; 'strategic' means that what decisions one agent wants to take depends on what she thinks other agents will be doing, which in turn may depend on what those other agents think this agent will be doing. Note that *decision theory*—which you should have elements of earlier in the Micro core—is the study of how an individual makes decisions in non-strategic settings; hence game theory is sometimes also referred to as *multi-person* or *interactive decision theory*. The common terminology for the field comes from its putative applications to games such as poker, chess, etc.[2] However, the applications we are usually interested in have little directly to do with such games. In particular, these are what we call "zero-sum" games in the sense that one player's loss is another player's gain; they are games of pure conflict. In most interesting economic applications, there is typically a mixture of conflict and cooperation motives.

1.1. A (Very!) Brief History

Modern game theory as a field owes much to the work of John von Neumann. In 1928, he wrote an important paper on two-person zero-sum games that contained the famous Minimax Theorem, which we'll see later on. In 1944, von Neumann and Oscar Morgenstern published their classic book, *Theory of Games and Strategic Behavior*, that extended the work on zero-sum games and also started cooperative game theory. In the early 1950s, John Nash made his seminal contributions to non-zero-sum games and started bargaining theory. In 1957, Robert Luce and Howard Raiffa published their book, *Games and Decisions: Introduction and Critical Survey*, popularizing game theory. In 1967–1968, John Harsanyi formalized methods to study games of incomplete information, which was crucial for widening the scope of applications. In the 1970s, there was an explosion of theoretical and applied work in game theory, and the methodology was well along its way to its current status as a preeminent tool in not only economics but other social sciences too.

1.2. Non-cooperative Game Theory

Throughout this course, we will focus on *noncooperative* game theory, as opposed to *cooperative* game theory. All of game theory describes strategic settings by starting with the set of *players*, i.e. the decision-makers. The difference between noncooperative and cooperative game theory is that the former takes each player's individual actions as primitives, whereas the latter takes joint actions as primitives. That is, cooperative game theory assumes that binding agreements can be made by players within various groups and players can communicate freely in order to do so. We will take the noncooperative viewpoint that each player acts as an individual, and the possibilities for agreements and communication must be explicitly modeled. Except for brief discussions in Appendix A of Chapter 18 and parts of Chapter 22, Mas-Colell, Whinston, and Green

[1] Naturally, any attempt like this to provide a succinct definition of a rich subject is bound to be incomplete; nevertheless, this working definition is suitable for the focus in this course.

[2] Ironically, game theory actually has limited prescriptive advice to offer on how to play either chess or poker. For example, we know that chess is "solvable" in a sense to be made precise later, but nobody actually knows what the solution is! This stems from the fact that chess is simply too complicated to "solve" (at present); this is of course why the best players are said to rely significantly on their intuition or feel in addition to logical computation.

(1995)—herafter, MWG—does not deal with cooperative game theory either. For an excellent introduction, see Chapters 13–15 in Osborne and Rubinstein (1994).

2. Strategic Settings

A game is a description of a strategic environment. Informally, the description must specify who is playing, what the rules are, what the outcomes are depending on any set of actions, and how players value the various outcomes.

Example 1. (Matching Pennies version A) Two players, *Anne* and *Bob*. Simultaneously, each picks *Heads* or *Tails*. If they pick the same, Bob pays Anne \$2; if they pick different, Anne pays Bob \$2.

Example 2. (Matching Pennies version B) Two players, *Anne* and *Bob*. First, Anne picks either *Heads* or *Tails*. Upon observing her choice, Bob then picks either *Heads* or *Tails*. If they pick the same, Bob pays Anne \$2; if they pick different, Anne pays Bob \$2.

Example 3. (Matching Pennies version N) Two players, *Anne* and *Bob*. Simultaneously, they pick either *Heads* or *Tails*. If they pick different, then they each receive \$0. If they pick the same, they wait for 15 minutes to see if it rains outside in that time. If it does, they each receive \$2 (from God); if it does not rain, they each receive \$0. Assume it rains with 50% chance.

In all the above examples, implicitly, players value money in the canonical way and are risk-neutral. Notice that Example 1 and Example 2 are zero-sum games — whatever Anne wins, Bob loses, and vice-versa. Example 3 is not zero-sum, since they could both win \$2. In fact, it is a particular kind of coordination game. Moreover, it is also a game that involves an action taken by "nature" (who decides whether it rains or not).

2.1. Extensive Form Representation

- Work through extensive form representation of the examples first.

Let us now be more precise about the description of a game.

Definition 1. An extensive form game is defined by a tuple $\Gamma_E = \{\mathcal{X}, \mathcal{A}, I, p, \alpha, \mathcal{H}, H, \iota, \rho, u\}$ as follows:

1. A finite set of I players. Denote the set of players as $I = \{0, 1, \ldots, I\}$. Players $1, \ldots, I$ are the "real" players; player 0 is used as an "auxiliary" player, nature.
2. A set of nodes, $\mathcal{X}$.[3]
3. A function $p : \mathcal{X} \to \mathcal{X} \cup \{\varnothing\}$ specifying a unique immediate predecessor of each node x such that $p(x)$ is the empty-set for exactly one node, called the *root node*, x_0.[4]
   (a) The immediate successors of node x are defined as $s(x) = \{y \in \mathcal{X} : p(y) = x\}$.

[3] Nodes are typically drawn as small solid circles, but note fn. 4.

[4] The root node is typically drawn as a small hollow circle.

(b) By iterating the functions p and s, we can find all predecessors and successors of any node, x, which we denote $P(x)$ and $S(x)$ respectively. We require that that $P(x) \cap S(x) = \varnothing$, i.e. no node is both a predecessor and a successor to any other node.

(c) The set of *terminal nodes* is $T = \{x \in \mathcal{X} : s(x) = \varnothing\}$. Any non-terminal node $x \in \mathcal{X} \backslash T$ is a *decision node*.

4. A set of actions, $\mathcal{A}$, and a function $\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$ that specifies for each node $x \neq x_0$, the action which leads to x from $p(x)$. We require that α be such that if distinct $x', x'' \in s(x)$, then $\alpha(x') \neq \alpha(x'')$. That is, from any node, each action leads to a unique successor. The set of available actions at any node, x, is denoted $c(x) = \{\alpha(x')\}_{x' \in s(x)}$.

5. A collection of information sets, $\mathcal{H}$, that forms a partition of $\mathcal{X} \backslash T$,[5] and a function $H : \mathcal{X} \backslash T \to \mathcal{H}$ that assigns each decision node into an information set. We require that $c(x) = c(x')$ if $H(x) = H(x')$; that is, two nodes in the same information set have the same set of available actions. It is therefore meaningful to write $C(H) = \{a \in \mathcal{A} : a \in c(x)\ \forall x \in H\}$ for any information set $H \in \mathcal{H}$ as the set of choices available at H.

6. A function $\iota : \mathcal{H} \to I$ assigning the player (possibly nature) to move at all the decision nodes in any information set. This defines a collection of information sets that any player i moves at, $\mathcal{H}_i \equiv \{H \in \mathcal{H} : i = \iota(H)\}$.

7. For each $H \in \mathcal{H}_0$, a probability distribution $\rho(H)$ on the set $C(H)$.[6] This dictates nature's moves at each of its information sets.

8. $u = (u_1, \ldots, u_I)$ is a vector of utility functions such that for each $i = 1, \ldots, I$, $u_i : T \to \mathbb{R}$ is a von-Neumann Morgenstern payoff function that represents (expected utility) preferences for i over terminal nodes.

Keep in mind that when drawing game trees, we use dotted lines between nodes (Kreps) or ellipses around nodes (MWG) to indicate nodes that fall into the same information set.

- Work through examples of what the definition of an extensive form game rules out.

To avoid technical complications, we restrict attention in the formal definition above to *finite* games:

Assumption 1. *The set of nodes, $\mathcal{X}$, is finite.*

Remark 1. If $\mathcal{X}$ is finite, then even if the set of actions, $\mathcal{A}$, is infinite, there are only a finite number of *relevant* actions; hence without loss of generality, we can take $\mathcal{A}$ as finite if $\mathcal{X}$ is finite.

At various points, we will study *infinite* games (where the number of nodes is infinite); the extension of the formal concept of a game to such cases will be intuitive and covered as needed.

It will often be convenient to talk about games without specifying payoffs for the players. Strictly speaking, this is called a *game form* rather than a game.

[5] Recall that a partition is a set of mutually exclusive and exhaustive subsets.

[6] One has to be a little careful in the definition if $C(H)$ is a continuum, which MWG ignore, and I will be casual about; cf. Assumption 1 below.

Definition 2. A game form is an otherwise complete description of a game, only lacking payoff specification.

A property Y is said to be *mutual knowledge* if all players know Y (but don't necessarily know that others know it). A property Y is *common knowledge* if everyone knows Y, everyone knows that everyone knows Y, everyone knows that everyone knows that everyone knows Y, ..., ad infinitum. Clearly, common knowledge implies mutual knowledge but not vice-versa.[7]

Definition 3. A complete information game is one where all players' payoff functions (and all other aspects of the game) are common knowledge.

You might worry that restricting attention to complete information games is pretty limited: what about a version of Matching Pennies where Anne does not know whether Bob wants to match or not match? We'll see there is a beautiful trick to analyze such situations within the framework of complete information.

Remark 2. We will always assume in this course that the game form is common knowledge. So the only source of informational asymmetry across players at the outset of a game can be about payoffs. More generally, however, the term "incomplete information" can refer to any game where at the outset, one player knows something about the game that another does not.

Remark 3. We will restrict our attention to games with *perfect recall*, as is typical. Loosely, this means that a player never forgets (i) a decision she took in the past, and (ii) any information that she possessed when making a past decision. More precisely: (i) if $y \in P(x)$ then $H(x) \neq H(y)$ — no node and any predecessor can be in the same information set; and (ii) if $H(x'') = H(x')$, $x \in P(x')$, and $\iota(H(x)) = \iota(H(x'))$, then $\exists \hat{x} \in H(x)$ s.t. $\hat{x} \in P(x'')$ and $\alpha\left(s(x) \cap P(x')\right) = \alpha\left(s(\hat{x}) \cap P(x'')\right)$. In words, (ii) says that if x'' and x' are in the same information set, x is a predecessor of x', and the same player moves at both x and x' (and hence also x''), then there must be an action at x's information set that can lead to both x' and x''.

Another piece of terminology to be aware of, but we don't want to impose in general, is *perfect information*.

Definition 4. A game has perfect information if all information sets are singletons. Otherwise, it has imperfect information.

Example 2 has perfect information, but Example 1 and Example 3 are of imperfect information. In terms of parlor games, chess has perfect information, whereas Mastermind has imperfect information.[8]

2.2. Strategies and Strategic Form of a Game

2.2.1. Strategies

A key concept in game theory is that of a player's *strategy*. A strategy, or a decision rule, is a *complete contingent plan* that specifies how a player will act at every information set that she is the decision-maker at, should it be reached during play of the game.

Definition 5. A [pure] strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

[7] Chapter 5 of Osborne and Rubinstein (1994) contains an accessible formal treatment of knowledge.

[8] In case you don't know it, Mastermind is the game where player 1 chooses an ordered sequence of four colored pegs (unobservable to player 2) and player 2 has to arrive at it through a sequence of guesses.

It is very important to be clear about what a strategy is. Here is point of clarification. Consider a "game" where you are walking North on Amsterdam Ave and trying to get to the department's entrance on 118th St.[9] Every cross street on Amsterdam is a decision node. The set of actions at each node is {Turn right, Continue on Amsterdam}. Consider a strategy that specifies *Continue* at all streets South of 118th, and *Turn right* at the 118th Street node. For a full specification, the strategy has to specify what to do if you get to the 119th St node, the 120th St, and so on — even though you won't actually get there if you follow the strategy! Remember: *complete contingent plan*. Moreover, do not confuse *actions* and *strategies*. An action is just a choice at a particular decision node. A strategy is a plan of action for every decision node that a player is the actor at. It may seem a little strange to define strategies in this way: why should a player have to plan for contingencies that his own actions ensure will never arise?! It turns out that what *would* happen at such "never-reached" nodes plays a crucial role in studying dynamic games, a topic we'll spend a lot of time on later.

The set of available strategies for a player i is denoted S_i. In finite games, this is a $|\mathcal{H}_i|$-dimensional space, where $|\mathcal{H}_i|$ is the number of information sets at which i acts. That is, $s_i \in S_i = \times_{H \in \mathcal{H}_i} C(H)$. Let $S = \prod_{i=1,\ldots,I} S_i$ be the product space of all players' strategy spaces, and $s = (s_1, \ldots, s_I) \in S$ be a *strategy profile* where s_i is the i^{th} player's strategy. We will sometimes write s_{-i} to refer to the $(I-1)$ vector of strategies of all players excluding i, and therefore $s = (s_i, s_{-i})$.

In Example 1, if we let Anne be player 1 and Bob be player 2, we can write $S_1 = S_2 = \{H, T\}$. Here, both players have 2 actions and also 2 strategies. In Example 2, we have $S_1 = \{H, T\}$ whereas $S_2 = \{(H,H), (H,T), (T,H), (T,T)\}$ where any $s_2 = (x, y)$ means that player 2 plays x *if player 1 plays* H and y *if player 1 plays* T. Thus, even though both players continue to have 2 actions each, observe that player 1 has 2 strategies (as before), but now player 2 has 4 strategies. (Question: how many strategies does each player have in Example 3?)

2.2.2. Strategic (Normal) Form

Every [pure] strategy profile induces a sequence of moves that are actually played, and a probability distribution over terminal nodes. (Probability distribution because nature may be involved; if there is no randomness due to nature, then there will be a unique final node induced.) Since all a player cares about is his opponents' actual play, we could instead just specify the game directly in terms of strategies and associated payoffs. This way of representing a game is known as the *Strategic* or *Normal* form of the game. To do this, first note that given a payoff function $u_i : T \to \mathbb{R}$, we can define an extended payoff function as the expected payoff for player i from a strategy profile s, where the expectation is taken with respect to the probability distribution induced on T by s. With some abuse of notation, I will denote this extended payoff function as $u_i : S \to \mathbb{R}$ again. Notice that the domain of u_i (S or T) makes it clear whether it is the primitive or extended payoff function we are talking about.

Definition 6. The normal form representation of a game, $\Gamma_N = \{I, S, u\}$, consists of the set of players, I, the strategy space, S, and the vector of extended payoff functions $u = (u_1, \ldots, u_I)$.

Often, the set of players will be clear from the strategy space, so we won't be explicit about the set I. For instance, the Normal Form for Example 2 can be written as

[9] This is really a decision-theory problem rather than a game, but it serves well to illustrate the point.

Bob

Anne

	(H,H)	(H,T)	(T,H)	(T,T)
H	$2,-2$	$2,-2$	$-2,2$	$-2,2$
T	$-2,2$	$2,-2$	$-2,2$	$2,-2$

where we follow the convention of writing payoffs as ordered pairs (x, y), with x being the payoff for the Row player (Anne) and y that of the Column player (Bob). This is a game where there is no role for Nature, so any strategy profile induces a unique terminal node. Consider on the other hand, Example 3, where this is not the case. The Normal Form is

Bob

Anne

	H	T
H	$1,1$	$0,0$
T	$0,0$	$1,1$

where the payoff of 1 if they match comes from the expected utility calculation with a 0.5 chance of rain (the expected payoff given the probability distribution induced over the terminal node).

***Normal Form Equivalence** There are various senses in which two strategic settings may be equivalent, even though they have different representations (in Normal or Extensive form). Indeed, as we already remarked, the same game can have different extensive form representations (think about MP-A and whether the game tree shows Bob moving first or Anne moving first). This is actually a deep question in Game Theory, but here is at least one simple case in which the equivalence should be obvious.

Definition 7 (Full Equivalence)**.** Two normal form games, $\Gamma_N = \{I, S, u\}$ and $\tilde{\Gamma}_N = \{I, S, \tilde{u}\}$, are fully equivalent if for each $i = 1, \ldots, I$, there exists $A_i > 0$ and B_i such that $\tilde{u}_i(s) = A_i u_i(s) + B_i$.

This definition is a consequence of the fact that the utility functions represent vNM expected-utility preferences, hence are only meaningful up to a linear transformation. For instance, this means that MP-A in Example 1 is fully equivalent to another version of Matching Pennies where we just multiply players' payoffs by the constant 2. Makes sense, right?

2.2.3. Randomized Choices

Mixed Strategies Thus far, we have taken it that when a player acts at any information set, he deterministically picks an action from the set of available actions. But there is no fundamental reason why this has to be case. For instance, in MP-A, perhaps Bob wants to flip a coin and make his choice based on the outcome of the coin flip. This is a way of making a *randomized choice*. Indeed, as we'll see, allowing for randomization in choices plays a very important role in game theory.

Definition 8 (Mixed Strategy)**.** A *mixed* strategy for player i is a function $\sigma_i : S_i \to [0, 1]$ which assigns a probability $\sigma_i(s_i) \geqslant 0$ to each pure strategy $s_i \in S_i$, satisfying $\sum\limits_{s_i \in S_i} \sigma_i(s_i) = 1$.

One way to think of this is that at the outset, i flips an $|S_i|$-sided die (with the right probabilities for each side), and based on its outcome, decides which pure strategy to play. Clearly, a pure strategy is a degenerate kind of mixed strategy, where $\sigma_i(s_i) = 1$ for some $s_i \in S_i$; in other words, mixed strategies subsume pure strategies. Sometimes, a mixed strategy that places positive probability on all pure strategies is called a *fully* or *totally mixed strategy.*[10]

Definition 9 (Fully Mixed Strategy)**.** A strategy, σ_i, for player i is fully (or completely, or totally) mixed if $\sigma_i(s_i) > 0$ for all $s_i \in S_i$.

As a piece of notation, we denote the set of probability distribution on S_i as $\Delta(S_i)$, which is the simplex on S_i. The space of mixed strategies then is $\Delta(S_i)$, which I will often denote as Σ_i.

Notice now that even if there is no role for nature in a game, when players use (non-degenerate) mixed strategies, this induces a probability distribution over terminal nodes of the game. But we can easily extend payoffs again to define payoffs over a profile of mixed strategies as follows:

$$u_i(\sigma_1, \ldots, \sigma_I) = \sum_{s \in S} [\sigma_1(s_1)\, \sigma_2(s_2) \cdots \sigma_I(s_I)]\, u_i(s)\,.$$

Remark 4. For the above formula to make sense, it is critical that each player is randomizing *independently.* That is, each player is independently tossing her own die to decide on which pure strategy to play. This rules out scenarios such as two players jointly observing the roll of a "public" die, and then correlating their choice of individual pure strategies based on the die's outcome. This independence assumption can be weakened in a more advanced treatment, but we maintain it throughout much of this course, except for brief remarks and the discussion in Subsection 3.5.

Return to Example 2. A mixed strategy for Anne can be specified by a single number $p_1 \in [0, 1]$ so that p_1 is the probability of playing the pure strategy H. This implicitly defines the probability of playing pure strategy T as $1 - p_1$. On the other hand, for Bob, a mixed strategy is a triple, $(q_1, q_2, q_3) \in [0, 1]^3$, where q_1 is the probability of playing (H, H), q_2 is the probability of playing (H, T), q_3 is the probability of (T, H), and $1 - q_1 - q_2 - q_3$ is the probability of playing (T, T).

Behavioral Strategies In the context of extensive form representations, there is an alternative way one can think about making randomized choices. Rather than randomizing over pure strategies, why not define a plan of action that specifies separately randomizing over the set of available actions at each information node? That is, in Example 2, why can't Bob simply specify how to randomize over Heads and Tails in each of two different scenarios: if Anne plays H, and if Anne plays T. Such a formulation is in fact feasible, and is called a *behavioral strategy.*

Definition 10 (Behavioral Strategy)**.** A *behavioral* strategy for player i is a function $\lambda_i : \mathcal{A} \times \mathcal{H}_i \to [0, 1]$ which assigns a probability $\lambda_i(a, H) \geqslant 0$ to each action $a \in \mathcal{A}$ at information set $H \in \mathcal{H}_i$, satisfying $\forall H \in \mathcal{H}_i$, $\lambda_i(a, H) = 0$ if $a \notin C(H)$ and $\sum\limits_{a \in C(H)} \lambda_i(a, H) = 1$.

To be clear, a behavioral strategy for Bob in Example 2 would be a pair $(q_1, q_2) \in [0, 1]^2$ such that q_1 is the probability of playing H *if Anne has played* H and q_2 is the probability of playing H *if Anne has played* T. Implicitly then, $1 - q_1$ is the probability of playing T if Anne has played H and $1 - q_2$ is the probability of playing T if Anne has played T. Compare this to a mixed strategy for Bob described earlier.

[10] There is no well established terminology for a strategy in $\Sigma_i \backslash S_i$; let's use *non-pure strategy.*

As you probably guessed, in games of perfect recall, behavioral strategies and mixed strategies are equivalent. That is, for any player, for any behavioral strategy there exists a mixed strategy that yields exactly the same distribution over terminal nodes given the strategies (behavioral or mixed) of other players, and vice-versa.[11] The formal theorem is this, where an outcome means a probability distribution over terminal nodes:

Theorem 1 (Kuhn's Theorem)**.** *For finite games with perfect recall, every mixed strategy of a player has an outcome-equivalent behavioral strategy, and conversely, every behavioral strategy has an outcome-equivalent mixed strategy.*

I won't prove this Theorem though the intuition is straightforward (you will work through a detailed example in a homework problem). Given the result, in this course, we will be a little casual and blur the distinction between mixed and behavioral strategies. Often, it is more convenient to use behavioral strategies in extensive form representations, and mixed strategies when a game is in strategic form. See Osborne and Rubinstein (1994, Section 11.4) for an excellent discussion.

2.3. An Economic Example

To illustrate a strategic setting with direct application to the study of markets, here is a classic model of imperfect competition. Conveniently, it also serves to introduce infinite action spaces. There are two firms, call them 1 and 2, producing an identical product. Market demand is given by $Q(P)$ with inverse demand $P(Q)$, both of which are decreasing functions mapping $\mathbb{R}_+$ to $\mathbb{R}_+$. Firm i produces a non-negative quantity q_i at cost $c_i(q_i)$, with $c_i(0) = 0$. Notice that since quantities and price are in a continuum here, none of the following games is finite. Nonetheless, we will just adapt our definitions from earlier in straightforward ways.

Simultaneous quantity-setting (Cournot) Suppose that each firm must simultaneously pick a quantity, and the market price gets determined as $P(q_1 + q_2)$. In Normal form, this game has $S_i = \mathbb{R}_+$, $s_i = q_i$, and $u_i(s_i, s_{-i}) = s_i P(s_1 + s_2) - c_i(s_i)$. We can draw it in extensive form too, using various game tree notations to represent the infinite number of available actions.

Simultaneous price-setting (Bertrand) Suppose that each firm must simultaneously pick a non-negative price, and the market quantity gets determined by $Q(\min\{p_1, p_2\})$, with all sales going to the firm with lower price and a 50-50 split in case of equal prices. In Normal form, this game has $S_i = \mathbb{R}_+$, $s_i = p_i$, and

$$u_i(s_i, s_{-i}) = \begin{cases} Q(s_i)\, s_i - c_i(Q(s_i)) & \text{if } s_i < s_{-i} \\ \frac{1}{2}Q(s_i)\, s_i - c_i\left(\frac{1}{2}Q(s_i)\right) & \text{if } s_i = s_{-i} \\ 0 & \text{if } s_i > s_{-i} \end{cases}$$

[11] The equivalence also breaks down when the set of actions available to a player — and hence nodes in the game — is infinite, and in particular a continuum; see Aumann (1964). A third way of defining randomization that works for a very general class of games (including many infinite games) is the *distributional strategy* approach of Milgrom and Weber (1985), but it comes at the cost of being unnecessarily cumbersome for finite games, so we don't use it typically.

Sequential quantity-setting (Stackelberg) Suppose now that firms sequentially pick quantities, where firm 2 observes firm 1's choice before acting. (Before reading further, see if you can represent the game in Normal form; it is an excellent check on whether you have fully grasped the difference between strategies and actions.) In Normal form, this game has $s_1 = q_1$ and $S_1 = \mathbb{R}_+$, $s_2 : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e. s_2 is function) and S_2 is a function space defined by $S_2 = \{\text{functions from } \mathbb{R}_+ \text{ to } \mathbb{R}_+\}$, and

$$\begin{aligned} u_1(s_1, s_2) &= s_1 P\left(s_1 + s_2\left(s_1\right)\right) - c_1\left(s_1\right) \\ u_2(s_1, s_2) &= s_2\left(s_1\right) P\left(s_1 + s_2\left(s_1\right)\right) - c_2\left(s_2(s_1)\right) \end{aligned}$$

Note that even though both firms' strategy spaces are infinite, firm 1's strategy space lies in a finite-dimensional space (it is 1-dimensional); whereas the dimensionality of firm 2's strategy space is (uncountably) infinite.[12]

3. Simultaneous-Move Games

In this Section, we are going to study behavior in Normal form games. One can either view this as taking a possibly-complicated extensive form game and collapsing it into its Normal form (in which case we may be losing relevant "dynamic" considerations that we will come back to later), or as simply considering "static" games where players only move once and move simultaneously. In any case, the primitive in this section will be a game in its Normal form, $\Gamma_N = \{I, S, u\}$.

3.1. Dominance

3.1.1. Strictly Dominant Strategies

You've probably heard of the *Prisoner's Dilemma* game (MWG Figure 8.B.1). I'm going to reinterpret it as a game of trust.

Example 4. (Trust Game) The Trust Game has the following Normal form.

Player 2

Player 1		$Trust$	$Cheat$
	$Trust$	$5, 5$	$0, 10$
	$Cheat$	$10, 0$	$2, 2$

Observe that regardless of what her opponent does, player i is strictly better off playing $Cheat$ rather than $Trust$. This is precisely what is meant by a strictly dominant strategy.

Definition 11 (Strictly Dominant strategy)**.** A strategy $s_i \in S_i$ is a strictly dominant strategy for player i if for all $\tilde{s}_i \neq s_i$ and all $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(\tilde{s}_i, s_{-i})$.

[12] Dealing with infinite dimensional spaces typically adds additional technical complications, but may be unavoidable as in this example.

That is, a strictly dominant strategy for i uniquely maximizes her payoff for any strategy profile of all other players. If such a strategy exists, it is highly reasonable to expect a player to play it. In a sense, this is a consequence of a player's "rationality".[13]

In the Trust Game, if both players play their strictly dominant strategies, the outcome of the game is *(Cheat, Cheat)*. But notice that this is a Pareto-dominated outcome. Another way to say this is that if the players could somehow write a binding contract that requires them to both play *Trust*, they would be better off doing that rather than playing this Trust Game. Lesson: self-interested behavior in games may not lead to socially optimal outcomes. This stems from the possibility that a player's actions can have a negative externality on another player's payoff. (Aside: think about the connection to the First Welfare Theorem.)

Exercise 1. *Prove that a player can have at most one strictly dominant strategy.*

Notice that we defined strictly dominant strategies by only considering alternative pure strategies for both player i and his opponents. Would it matter if we instead allowed mixed strategies for either i or his opponents? The answer is no.

Theorem 2. *If s_i is a strictly dominant strategy for player i, then for all $\sigma_i \in \Sigma_i \setminus \{s_i\}$ and $\sigma_{-i} \in \Sigma_{-i}$, $u_i(s_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$.*

Proof. For any σ_{-i} , σ_i, and s_i, we can write $u_i(s_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ as

$$\sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j (s_j) \right) \left[u_i (s_i, s_{-i}) - \sum_{\tilde{s}_i \in S_i} \sigma_i (\tilde{s}_i) \, u_i (\tilde{s}_i, s_{-i}) \right] > 0. \tag{1}$$

Since s_i is strictly dominant, $u_i (s_i, s_{-i}) - u_i (\tilde{s}_i, s_{-i}) > 0$ for all $\tilde{s}_i \neq s_i$ and all s_{-i}. Hence, $u_i (s_i, s_{-i}) - \sum_{\tilde{s}_i \in S_i} \sigma_i (\tilde{s}_i) \, u_i (\tilde{s}_i, s_{-i}) > 0$ for any $\sigma_i \in \Sigma_i \setminus \{s_i\}$. This implies inequality (1). □

Exercise 2. *Prove that there there can be no strategy $\sigma_i \in \Sigma_i$ such that for all $s_i \in S_i$ and $s_{-i} \in S_{-i}$, $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$.*

The preceding Theorem and Exercise show that there is absolutely no loss in restricting attention to pure strategies for all players when looking for strictly dominant strategies.

3.1.2. Strictly Dominated Strategies

What about if a strictly dominant strategy doesn't exist, such as in the following game?

Example 5. A game defined by the Normal form

Player 2

Player 1		a	b	c
	A	$5,5$	$0,10$	$3,4$
	B	$3,0$	$2,2$	$4,5$

[13] There is a better way to say this, but it is too much work at this point to be formal about it. Roughly though, if there is a strict dominant strategy, then no other strategy is optimal (in the sense of maximizing payoffs) regardless of what a player believes his opponents are playing. "Rationality" means that the player is playing optimally for *some* belief he holds about his opponents' play. Thus, he must play his strictly dominant strategy.

You can easily convince yourself that there are no strictly dominant strategies here for either player. However, notice that regardless of whether Player 1 plays A or B, Player 2 does strictly better by playing b rather than a. That is, a is "strictly dominated" by b. The motivates the next definition, where we allow for mixed strategies explicitly.

Definition 12 (Strict Dominance)**.** A (mixed) strategy $\sigma_i \in \Sigma_i$ is strictly dominated for player i if there exists a (mixed) strategy $\tilde{\sigma}_i \in \Sigma_i$ such that for all $s_{-i} \in S_{-i}$, $u_i(\tilde{\sigma}_i, s_{-i}) > u_i(\sigma_i, s_{-i})$. In this case, we say that $\tilde{\sigma}_i$ strictly dominates σ_i.

In words, $\tilde{\sigma}_i$ strictly dominates σ_i if it yields a strictly higher payoff regardless of what (pure) strategy rivals use. Note that the definition allows for both $\tilde{\sigma}_i$ and σ_i be pure or non-pure. Using this terminology, we can restate Definition 11: a strategy is strictly dominant for a player if it strictly dominates all other strategies for that player. Just as it is reasonable to expect a player to play a strictly dominant strategy if one exists; it is likewise reasonable that a player will *not* play a strictly dominated strategy — a consequence of rationality, again.

Why were we explicit about allowing for a strategy to be dominated by a mixed strategy in the definition? Here is a game where it does matter.

Example 6.

		Player 2		
		a	b	c
Player 1	A	$1, 5$	$0, 6$	$2, 9$
	B	$1, 9$	$2, 6$	$0, 5$

There is no pure strategy that strictly dominates any other pure strategy in this game. However, the mixed strategy σ_2 where $\sigma_2(a) = \sigma_2(c) = 0.5$ strictly dominates the strategy b for Player 2.

Remark 5. By the same argument as in Theorem 2, there is no loss in only comparing against all the pure strategies for all other players when evaluating whether there is a strictly dominated strategy for a particular player.

Remark 6. Convince yourself that a mixed strategy will be strictly dominated if it puts positive probability on any pure strategy that is strictly dominated. (This implies what we already noted: a strictly dominant strategy must be a pure strategy.) However, a mixed strategy may be strictly dominated even if none of the pure strategies it puts positive probability on are strictly dominated. Check this in a variant of Example 6 where b gives player 2 a payoff of 8 regardless of what player 1 does.

3.1.3. Iterated Deletion of Strictly Dominated Strategies

Return now to Example 5. We argued that a is strictly dominated (by b) for Player 2; hence rationality of Player 2 dictates she won't play it. But now, we can push the logic further: if Player 1 knows that Player 2 is rational, he should realize that Player 2 will not play strategy a. Notice that we are now moving from the rationality of each player to the *mutual knowledge* of each player's rationality. That is, not only are Player 1 and Player 2 rational, but moreover, Player 1 knows that Player 2 is rational (and vice-versa). Assuming this is the case, once Player 1 realizes that 2 will not play a and "deletes" this strategy from the strategy space, then strategy A becomes strictly dominated by strategy B for Player 1. So now, if we iterate the knowledge of rationality once again (that is: there is mutual knowledge of rationality, and moreover, Player 2 knows that Player 1 knows that Player 2 is rational), then Player 2 realizes that 1 will not play A, and hence "deletes" A, whereafter b is strictly dominated by c for Player 2. Thus, Player 2 should play c. We have arrived at a "solution" to the game through the *iterated deletion of strictly dominated strategies* (IDSDS): (B, c).

Here is a formal definition of IDSDS:

Definition 13. Given a game (I, S, u), the set $S^\infty \subseteq S$ survives IDSDS if $S^\infty = \times_i S_i^\infty$ and there is a collection $(S_i^k)_{k \geqslant 0}$ for each player i satisfying:

1. $S_i^0 = S_i$ and $S_i^\infty = \bigcap_{k \geqslant 0} S_i^k$.
2. For all $k \geqslant 1$, $S_i^k \subseteq S_i^{k-1}$.
3. For all $k \geqslant 1$, each $s_i \in S_i^{k-1} \backslash S_i^k$ is strictly dominated in the game $(I, \times_j S_j^{k-1}, u)$.
4. No $s_i \in S_i^\infty$ is strictly dominated in the game (I, S^∞, u).

Definition 14 (Strict-dominance solvable)**.** A game is strict-dominance solvable if iterated deletion of strictly dominated strategies results in a unique strategy profile, i.e. there is a S^∞ surviving IDSDS with $|S^\infty| = 1$.

Since in principle we might have to iterate numerous times in order to solve a strict-dominance solvable game, the process can effectively can only be justified by *common knowledge* of rationality.[14] As with strictly dominant strategies, it is also true that most games are not strict-dominance solvable. Consider for example MP-A (Example 1): no strategy is strictly dominated.

You might worry whether the order in which we delete strategies iteratively matters. In other words, can different deletion processes produce different S^∞ in Definition 13? Insofar as we are working with *strictly* dominated strategies so far (and finite games), this is not the case.

Remark 7. In finite games, the order of deletion does not affect the set of strategies that survive the process of iterated deletion of strictly dominated strategies. In particular, if a game is strict-dominance solvable, the outcome is independent of the order in which strategies are iteratively deleted according to strict dominance.

You will be asked to prove the above claim as a homework exercise. The following example demonstrates the potential power of iteratively deleting strictly dominated strategies.

Example 7 (Linear Cournot)**.** This is a specialized version of the Cournot competition game we introduced earlier. Suppose that inverse market demand is given by a linear function $p(Q) = a - bQ$, the cost functions for both firms are also linear, $c(q_i) = cq_i$, and the linear payoff functions are

$$u_i(q_i, q_{-i}) = q_i(a - b(q_i + q_{-i})) - cq_i,$$

which simplify to

$$u_i(q_i, q_{-i}) = (a - c)q_i - bq_i^2 - bq_i q_{-i}.$$

Assume that $a, b, c > 0$ and moreover $a > c$. To solve this game by iterated deletion of strictly dominated strategies, first define the "reaction" or "best response" functions $r_i : [0, \infty) \to [0, \infty)$ which specify firm i's

[14] This isn't quite right: in finite games, we would only need to iterate a finite number of times; hence in any given game, common knowledge of rationality isn't quite necessary. But an arbitrarily large order of iteration of knowledge may be needed.

optimal output for any given level of its opponent's output. These are computed through the first order conditions for profit maximization (assuming an interior solution),[15]

$$a - c - 2br(q_{-i}) - bq_{-i} = 0.$$

Hence,

$$r(q_{-i}) = \frac{a-c}{2b} - \frac{q_{-i}}{2},$$

where I am dropping the subscript on r since firms are symmetric.

Clearly, r is a decreasing function; that is, the more the opponent produces, the less a firm wants to produce. The strategy space we start with for each firm is $S_i^0 = [0, \infty)$. For each firm, playing anything above $r(0)$ is strictly dominated, since the opponent plays at least 0. Hence, deleting strictly dominated strategies once yields $S_i^1 = [0, r(0)]$. Now, since the opponent plays *no more than* $r(0)$, it is (iteratively) strictly dominated for firm i to play *less than* $r(r(0))$, which I'll denote $r^2(0)$. Thus, the second round of iterated deletion yields the strategy space $S_i^2 = [r^2(0), r(0)]$. In the third round, since the opponent is playing *at least* $r^2(0)$, it is (iteratively) dominated for a firm to play *more than* $r(r^2(0))$, which of course I denote $r^3(0)$. So eliminating iteratively dominated strategies yields the space $S_i^3 = [r^2(0), r^3(0)]$, ... and so on, ad infinitum. The lower bounds of these intervals form a sequence $r^{2n}(0)$, and the upper bounds form a sequence $r^{2n+1}(0)$. Define $\alpha \equiv \frac{a-c}{b}$. You can check by expanding out some of the $r^n(0)$ formulae that for all $n = 1, 2, \ldots$,

$$r^n(0) = -\alpha \sum_{k=1}^{n} \left(-\frac{1}{2}\right)^k.$$

This is a convergent series (by absolute convergence), and hence the intervals S_i^n converge to a single point. Thus the game is strict-dominance solvable. The solution can be found by evaluating the infinite series $-\alpha \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$, which turns out to be $\frac{a-c}{3b}$.[16] □

3.1.4. Weakly Dominated Strategies

Consider the following game, which has no strictly dominated strategies (hence is not strict-dominance solvable).

Example 8. The normal form for a game is

Player 2

Player 1		a	b
	A	$3,4$	$4,3$
	B	$5,3$	$3,5$
	C	$5,3$	$4,3$

[15] Check that the second order condition is satisfied.

[16] To see this, observe that we can write

$$-\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^k = -\sum_{k=1}^{\infty}\left[\left(-\frac{1}{2}\right)^{2k-1} + \left(-\frac{1}{2}\right)^{2k}\right] = \sum_{k=1}^{\infty}\left[\left(\frac{1}{2^{2k-1}}\right) - \left(\frac{1}{2^{2k}}\right)\right] = \sum_{k=1}^{\infty}\frac{1}{4^k} = \sum_{k=0}^{\infty}\frac{1}{4^k} - 1 = \frac{4}{3} - 1 = \frac{1}{3}.$$

Exercise 3. *Prove that there are no strictly dominated pure strategies in this game.*

Nonetheless, notice that unless Player 1 is *absolutely sure* that Player 2 is going to play a, he is strictly better off playing C rather than B. That is to say, for any (possibly mixed) strategy $\sigma_2 \neq a$, $u_1(C, \sigma_2) > u_1(B, \sigma_2)$. Moreover, $u_1(C, a) = u_2(B, a)$. Hence, C will do at least as well as B, and could do better. We say that B is weakly dominated by C. Generalizing,

Definition 15 (Weak dominance)**.** A strategy $\sigma_i \in \Sigma_i$ is weakly dominated for player i if there exists a mixed strategy $\tilde{\sigma}_i \in \Sigma_i$ such that for all $s_{-i} \in S_{-i}$, $u_i(\tilde{\sigma}_i, s_{-i}) \geqslant u_i(\sigma_i, s_{-i})$ and for some $s_{-i} \in S_{-i}$, $u_i(\tilde{\sigma}_i, s_{-i}) > u_i(\sigma_i, s_{-i})$. In this case, we say that $\tilde{\sigma}_i$ weakly dominates σ_i.

We say that a strategy s_i is weakly dominant if it weakly dominates all other strategies, $\tilde{s}_i \neq s_i$. As with the case of strict dominance, it is important to allow for mixed strategies for player i in the definition of weak dominance, but not for the other players. Notice that we cannot appeal to "rationality" to justify the deletion of weakly dominated strategies, since a player might optimally play a weakly dominated strategy if he were certain that his opponents were going to play a particular strategy profile. That said, it has a lot of plausibility and can be useful in simplifying a complicated game. Just as with strict dominance, we can iteratively delete weakly dominated strategies. However, there is a subtlety here, because the order of deletion matters. To see this, continue the Example above. Having deleted B through weak dominance (by C), we can then delete for Player 2 strategy b since it is weakly dominated by strategy a, and finally iterate once more and delete strategy A for player 1. The iterative process has yielded the outcome (C, a). On the other hand, starting from the outset all over, observe that A is also weakly dominated by C. If we delete A in the first round (rather than B as before), we can then delete a for Player 2 since it is weakly dominated by b; and in third round, now delete B as it is weakly dominated by C. This process has led to the outcome (C, b). This motivates two remarks.

Remark 8. The order of deletion can matter when iteratively deleting weakly dominated strategies. Arguably, this makes it less appealing as a solution concept. Oftentimes, in any round of deletion, we will delete *all* strategies that are weakly dominated for a player; this is referred to as *(iterated) admissibility.*[17]

Remark 9. There is no completely standard definition of what it means for a game to be weak-dominance solvable. Sometimes, it means that there is some order of iteratively deleting weakly dominated strategies that leads to a single strategy profile. Other times, it means that no matter what order in which we iteratively delete weakly dominated strategies, we end up at a unique (at least in terms of payoffs) strategy profile.[18]

One other terminological point: saying that a strategy dominates (or is dominated by) another is potentially ambiguous with regards to strict or weak dominance. Typically, dominance without a caveat means strict dominance, but the literature is not uniformly careful about this (I will try to remember to be!). Similarly, saying that a game is *dominance solvable* can mean either iterated deletion of strictly or weakly dominated strategies — typically, it means strict-dominance solvable.

[17] If you are interested in an advanced "epistemics" approach to solution concepts, see Brandenburger, Friedenberg, and Keisler (2008) on foundations for iterated admissibility.

[18] Osborne and Rubinstein (1994, p. 63) has a third (and substantively different) definition based on the idea of deleting all weakly dominated strategies in each round of deletion, i.e. iterated admissibility. Note that if we used this procedure on Example 8, iterative deletion would not get to a unique strategy profile (or unique payoff profile): after deleting both A and B in the first round, we can proceed no further.

Exercise 4. *In MP-B (Example 2), which strategies are strictly dominated for Player 2? Which are weakly dominated? Does the game have a unique prediction through iterated deletion of strictly-dominated strategies? What about through iterated deletion of weakly-dominated strategies?*

Note that lots of games have no weakly dominated strategies, such as MP-A. On the other hand, there are some interesting and useful examples of games that do.

Example 9 (Second-Price Auction)**.** A seller has one indivisible object. There are I bidders with respective valuations $0 \leqslant v_1 \leqslant \cdots \leqslant v_I$ for the object; these valuations are common knowledge. The bidders simultaneously submit bids $s_i \in [0, \infty)$. The highest bidder wins the object and pays the *second highest* bid. Given a profile of bids, s, let $W(s) \equiv \{k : \forall j, s_k \geqslant s_j\}$ be the set of highest bidders. Bidder i gets utility

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - \max_{j \neq i} s_j & \text{if } s_i > \max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = \max_{j \neq i} s_j \\ 0 & \text{if } s_i < \max_{j \neq i} s_j. \end{cases}$$

In this game, it is weakly dominant for each player to bid his true valuation, that is to play $s_i = v_i$. To see this, define $m(s_{-i}) \equiv \max_{j \neq i} s_j$.

Suppose first $s_i > v_i$. Then for any strategy profile, s_{-i}, if $m(s_{-i}) > s_i$, $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) = 0$. If $m(s_{-i}) \leqslant v_i$, then $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) \geqslant 0$. Finally, if $m(s_{-i}) \in (v_i, s_i]$, then $u_i(s_i, s_{-i}) < 0 = u_i(v_i, s_{-i})$. Hence, $s_i = v_i$ weakly dominates all $s_i > v_i$.

Consider next $s_i < v_i$. If $m(s_{-i}) \geqslant v_i$, then $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) = 0$. If $m(s_{-i}) < s_i$, then $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) > 0$. Finally, if $m(s_{-i}) \in [s_i, v_i)$, then $0 = u_i(s_i, s_{-i}) < u_i(v_i, s_{-i})$. Hence, $s_i = v_i$ weakly dominates all $s_i < v_i$.

Therefore, in a second price auction, it seems reasonable that rational bidders should bid their true valuation. The bidder with the highest valuation wins, and pays the second highest valuation, v_{I-1}. Note that since bidding one's valuation is a weakly dominant strategy, it does not matter even if player i does not know the other players' valuations — even if valuations are only known to each player privately (rather than being common knowledge), it still remains a weakly dominant strategy to bid truthfully. We'll come back to this last point later in the course. □

3.2. Rationalizability

Let's start by generalizing the notion of a best response we used in Example 7.

Definition 16 (Best Response)**.** A strategy $\sigma_i \in \Sigma_i$ is a best response to the strategy profile $\sigma_{-i} \in \Sigma_{-i}$ if $u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(\tilde{\sigma}_i, \sigma_{-i})$ for all $\tilde{\sigma}_i \in \Sigma_i$. A strategy $\sigma_i \in \Sigma_i$ is never a best response if there is no $\sigma_{-i} \in \Sigma_{-i}$ for which σ_i is a best response.

The idea is that a strategy, σ_i, is a best response if there is some strategy profile of the opponents for which σ_i does at least as well as any other strategy. Conversely, σ_i is never a best response if for every strategy profile of the opponents, there is some strategy that does strictly better than σ_i. Let me note here that we can also think of a best response in terms of optimal play against a *conjecture* about opponents' strategies. In particular, even if one knows (believes) that an opponent is using a pure strategy, one may be unsure *which* pure strategy, which is tantamount to the opponent mixing.

Clearly, in any game, a strategy that is strictly dominated is never a best response. In 2-player games, a strategy that is never a best response is strictly dominated. While this equivalence is true for all strategies in 2-player games, the exercise below asks you to prove it for pure strategies only (needless to say, because proving it for mixed strategies requires advanced tools).

Exercise 5. *Prove that in 2-player games, a pure strategy is never a best response if and only if it is strictly dominated.*[19]

In games with more than 2 players, there may be strategies that are *not* strictly dominated that are nonetheless never best responses.[20] As before, it is a consequence of "rationality" that a player should not play a strategy that is never a best response. That is, we can delete strategies that are never best responses. You can guess what comes next: by iterating on the knowledge of rationality, we iteratively delete strategies that are never best responses. The set of strategies for a player that survives this iterated deletion of never best responses is called her set of *rationalizable strategies* (Bernheim, 1984; Pearce, 1984).[21] A constructive way to define this is as follows.

Definition 17 (Rationalizability)**.**

1. $\sigma_i \in \Sigma_i$ is a 1-rationalizable strategy for player i if it is a best response to some strategy profile $\sigma_{-i} \in \Sigma_{-i}$.
2. $\sigma_i \in \Sigma_i$ is a k-rationalizable strategy ($k \geqslant 2$) for player i if it is a best response to some strategy profile $\sigma_{-i} \in \Sigma_{-i}$ such that each σ_j is in the convex hull of the set of $(k-1)$-rationalizable strategies for player $j \neq i$.
3. $\sigma_i \in \Sigma_i$ is rationalizable for player i if it is k-rationalizable for all $k \geqslant 1$.

Note a subtlety: at any round $k > 1$, we consider the *convex hull* of each opponent's $(k-1)$-rationalizable strategies.[22] The reason is that at any round, the set of "surviving strategies" for a player may not be convex (e.g., two pure strategies may each be best responses to some opponent strategy, but a mixture of them may not; recall Remark 6). Yet a player may be unsure which of her opponents' surviving strategies will be used.

Remark 10. You should convince yourself that any strategy that does not survive iterated deletion of strictly dominated strategies (Subsection 3.1.3) is not rationalizable. (This follows from the earlier comment that a strictly dominated strategy is never a best response.) Thus, the set of rationalizable strategies is no larger than the set of strategies that survives iterated deletion of strictly dominated strategies. In this sense,

[19] The "only if" part is a non-trivial problem, and I will give you a homework question that breaks down the steps. But as a hint now, you can use Kakutani's fixed point theorem (see Lemma 1 on page 20).

[20] This stems from the fact that we are assuming that each of player i's opponents is choosing his strategy independently. If we were to allow for *correlated* strategies, then the notions of being strictly dominated and never a best response coincide regardless of the number of players. This points to *why* the notions coincide in 2-player games even in our way of doing things — each player only has one opponent, hence trivially, the opponents are choosing strategies independently.

[21] Note that the order of deleting strategies that are never best responses doesn't matter, since we are deleting strategies that are not even *weakly* optimal for some strategy profile of the opponents. This is analogous to the case with deleting strictly dominated strategies.

[22] The convex hull of a set is the smallest convex set that contains that set. We interpret each element of the convex hull of a set of strategies as itself a mixed strategy.

rationalizability is *(weakly) more restrictive* than iterated deletion of strictly dominated strategies. It turns out that in 2-player games, the two concepts coincide. In n-player games ($n > 2$), they don't have to.[23]

One way to think of rationalizable strategies is through an *infinite or circular chain of justification*. This is best illustrated through examples. Let us return to Example 6. Is A rationalizable? Yes, by the following chain of justification: A is a best response to c, which is a best response to A. Here is another chain of justification that works: A is a best response to a, which is a best response to B, which is a best response to a. Is B rationalizable? Yes: B is a best response to a, which is a best response to B. Similarly, a and c are rationalizable, but b is not (and you know it is not rationalizable because we already saw that it is strictly dominated).

To see that how a chain of justification works when it involves a mixed strategy, consider a modification of Example 6 so that the payoff cell for (B, c) is $(0, -5)$, everything else staying the same. Then, b is rationalizable by the following chain: b is a best response to the mixed strategy $(0.75)A + (0.25)B$,[24] this mixed strategy for Row is a best response to a, which is a best response to B, which is a best response to a.

Rationalizability is still a weak solution concept in the sense that the set of rationalizable strategies is typically large in any complicated game. For example, even in something as simple as MP-A (Example 1), every strategy is rationalizable. This is also true in the richer example of the 2nd price auction (Example 9).

Exercise 6. *Prove that every strategy is rationalizable in the 2nd price auction.*[25]

I conclude this section by emphasizing that rationalizability is as far as we can go (in terms of refining our predictions for the outcome of a game) by using only common knowledge of rationality and the structure of the game. A little more precisely: *common knowledge of rationality and the structure of the game imply that players will play rationalizable strategies; conversely, any profile of rationalizable strategies is consistent with common knowledge of rationality and the structure of the game.*

3.3. Nash Equilibrium

Now we turn to the most well-known solution concept in game theory. We'll first discuss *pure strategy Nash equilibrium* (PSNE), and then later extend to mixed strategies.

3.3.1. Pure Strategy Nash Equilibrium

Definition 18 (PSNE)**.** A strategy profile $s = (s_1, \ldots, s_I) \in S$ is a pure strategy Nash equilibrium if for all i and $\tilde{s}_i \in S_i$, $u_i(s_i, s_{-i}) \geqslant u_i(\tilde{s}_i, s_{-i})$.

In a Nash equilibrium, each player's strategy must be a best response to *those strategies of his opponents that are components of the equilibrium.* Note that if s is a pure strategy Nash equilibrium, then it holds that for all i and $\sigma_i \in \Sigma_i$, $u_i(s_i, s_{-i}) \geqslant u_i(\sigma_i, s_{-i})$. In other words, s_i is a best response to s_{-i} if no pure strategy for i is strictly better against s_{-i}.

[23] Again, this is because we are not allowing for correlation in strategies across players. If we did, the concepts would coincide in general. See fn. 20, and Osborne and Rubinstein (1994, Sections 4.1 and 4.2) for more details.

[24] Given that Row is playing $(0.75)A + (0.25)B$, Column gets expected payoff of 6 by playing either a or b, and 5.5 from playing c.

[25] The result does *not* directly follow from the fact that no strategy is strictly dominated, since the equivalence between iterated deletion of strictly dominated strategies and rationalizability is only for 2-player games.

Remark 11 (Nash equilibrium). There are various conceptual points to make about Nash equilibrium:

- Unlike with our earlier solution concepts (dominance and rationalizability), Nash equilibrium applies to a profile of strategies rather than any individual's strategy. When people say "Nash equilibrium strategy", what they mean is "a strategy that is part of a Nash equilibrium profile."
- The term *equilibrium* is used because it connotes that *if a player knew* that his opponents were playing the prescribed strategies, then she is playing optimally by following her prescribed strategy. In a sense, this is like a "rational expectations" equilibrium, in that in a Nash equilibrium, a player's beliefs about what his opponents will do get confirmed (where the beliefs are precisely the opponents' prescribed strategies).
- Rationalizability only requires a player play optimally with respect to some "reasonable" conjecture about the opponents' play, where "reasonable" means that the conjectured play of the rivals can also be justified in this way. On the other hand, Nash requires that a player play optimally with respect to what his opponents *are actually* playing. That is to say, the conjecture she holds about her opponents' play is *correct.*
- The above point makes clear that Nash equilibrium is not simply a consequence of (common knowledge of) rationality and the structure of the game. Clearly, *each player's strategy in a Nash equilibrium profile is rationalizable, but lots of rationalizable profiles are not Nash equilibria.*

Let's look at some examples of how this works. In MP-B, the two PSNE are (H, TH) and (T, TH). MP-A has no PSNE (Why?). In Example 8, there are also two PSNE: (C, a) and (C, b). Similarly, in MP-N (Example 3), there are two Nash equilibria: (H, H) and (T, T). This last example really emphasizes the assumption of *correctly conjecturing* what your opponent is doing — even though it seems impossible to say which of these two Nash equilibria is "more reasonable", any one is an equilibrium only if each player can correctly forecast that his opponent is playing in the prescribed way.

Exercise 7. *Verify that in the Linear Cournot game (Example 7) and the 2nd price auction (Example 9), the solutions found via iterated dominance are pure strategy Nash equilibria. Prove that in the Linear Cournot game, there is a unique PSNE, whereas there are multiple PSNE in the 2nd price auction. (Why the difference?)*

Remark 12. Every finite game of perfect information has a *pure strategy* Nash equilibrium.[26] This holds true for "dynamic" games as well, so we'll prove it generally when we get there.

We next want to give some sufficient (but certainly not necessary) conditions for the existence of a PSNE, and prove it. To do so, we need the powerful fixed point theorem of Kakutani.

Lemma 1 (Kakutani's FPT)**.** *Suppose that* $X \subset \mathbb{R}^N$ *is a non-empty, compact, convex set, and that* $f : X \rightrightarrows X$ *is a non-empty and convex-valued correspondence with a closed graph.*[27] *Then there exists* $x^* \in X$ *such that* $x^* \in f(x^*)$.

[26] Recall that perfect information means all information sets are singletons.

[27] The correspondence f is convex-valued if $f(x)$ is a convex set for all $x \in X$; it has a closed graph if for all sequences $x^n \to x$ and $y^n \to y$ such that $y^n \in f(x^n)$ for all n, $y \in f(x)$. (Technical remark: in this context where X is a compact subset of $\mathbb{R}^N$, f having a closed graph is equivalent to it being upper hemi-continuous (uhc); but in more general settings, the closed graph property is necessary but not sufficient for uhc.)

(Question: why do we need the convex-valued assumption?)

It is also useful to define the idea of a best response correspondence (first recall the idea of a strategy being a best response, see Definition 16).

Definition 19 (BR Correspondence)**.** The best response correspondence for player i, $b_i : S_{-i} \rightrightarrows S_i$, is defined by $b_i(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geqslant u_i(\tilde{s}_i, s_{-i}) \ \forall \tilde{s}_i \in S_i\}$.

By this definition, it follows that $s \in S$ is a pure strategy Nash equilibrium if and only if $s_i \in b_i(s_{-i})$ for all i. We apply this observation in proving the following existence theorem.

Theorem 3 (Existence of PSNE)**.** *Suppose each $S_i \subset \mathbb{R}^N$ is compact and convex (and non-empty); and each $u_i : S \to \mathbb{R}$ is continuous in s and quasi-concave in s_i.*[28] *Then there exists a PSNE.*

Proof. Define the correspondence $b : S \rightrightarrows S$ by $b(s) = b_1(s_{-1}) \times \cdots \times b_I(s_{-I})$. A Nash equilibrium is a profile s^* such that $s^* \in b(s^*)$. Clearly, b is a correspondence from the non-empty, convex, and compact set S to itself.

Step 1: For all s, $b(s)$ is non-empty. This follows from the fact that each b_i is the set of maximizers of a continuous function u_i over a compact set S_i, which is non-empty by the Weierstrass Theorem of the Maximum.

Step 2: Each b_i (and hence b) is convex-valued. Pick any s_{-i}, and suppose that $s_i, \tilde{s}_i \in b_i(s_{-i})$. By definition of b_i, there is some $\bar{u}$ such that $\bar{u} = u_i(s_i, s_{-i}) = u_i(\tilde{s}_i, s_{-i})$. Applying quasi-concavity of u_i, $u_i(\lambda s_i + (1-\lambda)\tilde{s}_i, s_{-i}) \geqslant \bar{u}$ for all $\lambda \in [0,1]$. But this implies that $\lambda s_i + (1-\lambda)\tilde{s}_i \in b_i(s_{-i})$, proving the convexity of $b_i(s_{-i})$.

Step 3: Each b_i (and hence b) has a closed graph. Suppose $s_i^n \to s_i$ and $s_{-i}^n \to s_{-i}^n$ with $s_i^n \in b_i(s_{-i}^n)$ for all n. Then for all n, $u_i(s_i^n, s_{-i}^n) \geqslant u_i(\tilde{s}_i, s_{-i}^n)$ for all $\tilde{s}_i \in S_i$. Continuity of u_i implies that for all $\tilde{s}_i \in S_i$, $u_i(s_i, s_{-i}) \geqslant u_i(\tilde{s}_i, s_{-i})$; hence $s_i \in b_i(s_{-i})$.

Thus, all the requirements for Kakutani's FPT are satisfied. There exists a fixed point, $s^* \in b(s^*)$, which is a PSNE. $\square$

Remark 13. Note carefully that in Step 2 above, quasi-concavity a player's utility function plays a key role. It would be *wrong* (though tempting!) to say that $u_i(\lambda s_i + (1-\lambda)\tilde{s}_i, s_{-i}) = \bar{u}$ by vNM. This is wrong because $\lambda s_i + (1-\lambda)\tilde{s}_i$ is merely a point in the space S_i (by convexity of S_i), and *should not* be interpreted as a mixed strategy that places λ probability of s_i and $(1-\lambda)$ on $\tilde{s}_i$. Right now you should think of $\lambda s_i + (1-\lambda)\tilde{s}_i$ as just another pure strategy, which happens to be a convex combination of s_i of $\tilde{s}_i$. Indeed, this is the only point in the proof where quasi-concavity of u_i is used; come back and think about this point again after you see Theorem 4.

Remark 14. A finite strategy profile space, S, cannot be convex (why?), so this existence Theorem is only useful for infinite games.

3.3.2. Mixed Strategy Nash Equilibrium

The previous remark motivates the introduction of mixed strategies. It is straightforward to extend our definition of Nash equilibrium to this case, and this subsumes the earlier definition of PSNE.

[28] Recall that a function $g : \mathbb{R}^N \to \mathbb{R}$ is quasi-concave if, for all $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^N$ such that $g(x) \geqslant c$ and $g(y) \geqslant c$, $g(\lambda x + (1-\lambda)y) \geqslant c \ \forall \lambda \in [0,1]$.

Definition 20 (Nash Equilibrium)**.** A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I) \in \Sigma$ is a Nash equilibrium if for all i and $\tilde{\sigma}_i \in \Sigma_i$, $u_i(\sigma_i, \sigma_{-i}) \geqslant u_i(\tilde{\sigma}_i, \sigma_{-i})$.

To see why considering mixed strategies are important, observe that Matching Pennies version A (Example 1) has no PSNE, but does have a mixed strategy Nash equilibrium (MSNE): each player randomizes over H and T with equal probability. In fact, when player i behaves in this way, player $j \neq i$ is exactly indifferent between playing H or T! That is, in the MSNE, *each player who is playing a mixed strategy is indifferent amongst the set of pure strategies he is mixing over.* This remarkable property is very general and is essential in helping us solve for MSNE in many situations. Before tackling that, let's first give an existence Theorem for Nash equilibria in finite games using mixed strategies.

Theorem 4 (Existence of NE)**.** *Every finite game has a Nash equilibrium (possibly in mixed strategies).*

Proof. For each i, given the finite space of pure strategies, S_i, the space of mixed strategies, Σ_i, is a (non-empty) compact and convex subset of $\mathbb{R}^{|S_i|}$. The utility functions $u_i : \Sigma \to \mathbb{R}$ defined by

$$u_i(\sigma_1, \ldots, \sigma_I) = \sum_{s \in S} [\sigma_1(s_1)\, \sigma_2(s_2) \cdots \sigma_I(s_I)]\, u_i(s)$$

are continuous in σ and quasi-concave in σ_i (by linearity). Thus, Theorem 3 implies that there is a pure strategy Nash equilibrium of the infinite normal-form game $\langle I, \{\Sigma_i\}, \{u_i\} \rangle$; this profile is a (possibly degenerate) mixed strategy Nash equilibrium of the original finite game. □

Remark 15. The critical need to allow for mixed strategies is that in finite games, the pure strategy space is not convex, but allowing players to mix over their pure strategies "convexifies" the space.

This does *not* mean that mixed strategies are not important in infinite games when the pure strategy space is convex. I illustrate through the following examples showing that a convex infinite game which does not have a pure strategy Nash equilibrium can nonetheless have a mixed strategy equilibrium. The setting is the price-competition analogue to the Linear Cournot we considered earlier (in Example 7).

Example 10 (Symmetric Linear Bertrand)**.** Two firms compete in Bertrand price competition, each with identical linear costs given by $c(q_i) = cq_i$ ($c \geqslant 0$); market demand is given by a smooth decreasing function, $Q(p) > 0$. We will show that the unique PSNE is $s_1 = s_2 = c$. It is straightforward to verify that this is a PSNE, so let's argue that there cannot be any other PSNE. Wlog, suppose there is a PSNE with $s_1 \geqslant s_2$, with at least one $s_i \neq c$. There are three exhaustive cases to consider.

1. $s_2 > c$. Then firm 1 can do better by playing $\tilde{s}_1 = s_2 - \varepsilon$ (for some $\varepsilon > 0$), by the continuity of $Q(p)$.
2. $s_2 = c$. It must be that $s_1 > c$, but now firm 2 can do better by playing $\tilde{s}_2 = s_1$ (it makes positive rather than 0 profit).
3. $s_2 < c$. Then firm 2 is making losses, and can do better by playing $\tilde{s}_2 = c$. □

Example 11 (Asymmetric Linear Bertrand)**.** Continue with the setup of Example 10, but now suppose that costs are asymmetric, wlog $0 \leqslant c_1 < c_2$. Assume that there exists $\delta > 0$ such that $Q(p)[p - c_1]$ is strictly increasing on $p \in [c_1, c_2 + \delta]$.[29] You are asked to prove as an exercise that there is no PSNE in this game.

[29] The economics of this assumption is that it guarantees that the price firm 1 would charge if it were a monopolist in the market is strictly larger than c_2.

However, we can construct a MSNE as follows: firm 1 plays $s_1 = c_2$, and firm 2 plays a mixed strategy, σ_2 which randomizes uniformly over all pure strategies in $[c_2, c_2 + \varepsilon]$, for some $\varepsilon > 0$. Denote the cdf of firm 2's price choice by $F(p; \varepsilon)$, with density $f(p; \varepsilon) = 1/\varepsilon$. Clearly, firm 2 is playing a best response to s_1, since it earns 0 profits in equilibrium and cannot do any better. To show that firm 1 is playing optimally, we only need to show that it does not prefer to deviate to any $\tilde{s}_1 \in (c_2, c_2 + \varepsilon]$, since clearly any $\tilde{s}_1 < c_2$ or $\tilde{s}_1 > c_2 + \varepsilon$ does strictly worse than $s_1 = c_2$. Consider firm 1's profit function by picking any price in $[c_2, c_2 + \varepsilon]$, given firm 2's strategy:

$$\pi(p) = (1 - F(p; \varepsilon))(p - c_1)Q(p).$$

Differentiating gives

$$\pi'(p) = (1 - F(p; \varepsilon))[(p - c_1)Q'(p) + Q(p)] - f(p; \varepsilon)(p - c_1)Q(p).$$

By picking $\varepsilon > 0$ small enough we can make $\min_{p \in [c_2, c_2+\varepsilon]} f(p; \varepsilon)$ arbitrarily large; it follows that for small enough $\varepsilon > 0$, $\pi'(p) < 0$ for all $p \in [c_2, c_2 + \varepsilon]$, which implies $s_1 = c_2$ is optimal for firm 1. $\square$

Exercise 8. *Prove that there is no PSNE in Example 11. Why does Theorem 3 not apply to this infinite game?*

This construction of a mixed strategy equilibrium to the asymmetric Bertrand game is due to Blume (2003). It obviates having to resort to "tricks" such as discretizing the price space, resolving price ties in favor of the lower cost firm, etc. — such assumptions would be needed to get existence in pure strategies.

***Infinite Pure Strategy Spaces** Let us record a mixed-strategy existence theorem for infinite games.

Theorem 5. *Suppose each S_i is a compact subset of a metric space and each $u_i : S \to \mathbb{R}$ is continuous. Then the game has a (possibly mixed) Nash equilibrium.*

The theorem relies on the following generalization of the Kakutani FPT to infinite-dimensional spaces; consult Aliprantis and Border (2006) for more on these definitions and results.

Lemma 2 (Fan-Glicksberg FPT)**.** *Suppose that X is a non-empty, compact, convex subset of a locally convex Hausdorff (topological vector) space, and that $f : X \rightrightarrows X$ is a non-empty and convex-valued correspondence with a closed graph.*[30] *Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.*

A proof of Theorem 5 proceeds similarly to that of Theorem 4 (and hence uses arguments similar to the proof of Theorem 3), but invoking the Fan-Glicksberg FPT instead of Kakutani. One invokes two facts about a non-empty compact metric space X: (i) the space of (Borel) probability measures on X is a non-empty, compact, and convex subset of a locally convex Hausdorff space;[31] and (ii) if $g : X \to \mathbb{R}$ is a continuous function then $\int g \mathrm{d}\mu$ is continuous in the probability measure μ. Point (i) implies that the set of mixed strategies (which, it bears emphasis, is infinite dimensional here) satisfies the domain hypotheses

[30] A topological vector space X is Hausdorff if for any $x \in X$ and $y \in X \backslash \{x\}$, there are neighborhoods N_x of x and N_y of y such that $N_x \cap N_y = \varnothing$. The space is locally convex if every neighborhood of zero includes a convex neighborhood of zero. In particular, every normed vector space is a locally convex Hausdorff space.

[31] Actually, the set of these probability measures can be normed so that it is a (non-empty, compact, and convex) normed vector space.

of Fan-Glicksberg, while point (ii) gives continuity of payoffs in mixed strategies, which, along with the linearity of expected utility, implies that the best response correspondences satisfy the self-map's hypotheses in Fan-Glicksberg.

To sum up, if we allow for mixed strategies, we can always find Nash equilibria in finite games. In infinite games, Nash equilibria need not exist (construct an example!), but Theorem 3 and Theorem 5 gives sufficient conditions to assure existence. Example 11 shows that these conditions are certainly not necessary.[32]

3.3.3. Finding Mixed Strategy Equilibria

As noted earlier, the MSNE we computed for MP-A had the property that each player is indifferent, in equilibrium, between the pure strategies that he is randomizing over. The next result show that this is a general property.

Proposition 1 (MSNE Indifference Condition)**.** *Fix a strategy profile, σ^*. Define $S_i^* \equiv \{s_i \in S_i | \sigma_i^*(s_i) > 0\}$ as the set of pure strategies that player i plays with positive probability according to σ^*. Then, σ^* is a Nash equilibrium if and only if*

1. $u_i(s_i, \sigma_{-i}^*) = u_i(s_i', \sigma_{-i}^*)$ *for all* $s_i, s_i' \in S_i^*$;
2. $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*)$ *for all* $s_i \in S_i^*$ *and* $s_i' \in S_i$.

Proof. (Necessity.) If either condition fails, there are strategies $s_i' \in S_i$ and $s_i \in S_i^*$ such that $u_i(s_i', \sigma_{-i}^*) > u_i(s_i, \sigma_{-i}^*)$. Construct the mixed strategy σ_i by setting $\sigma_i(\tilde{s}_i) = \sigma_i^*(\tilde{s}_i)$ for all $\tilde{s}_i \notin \{s_i', s_i\}$ and $\sigma_i(s_i') = \sigma_i^*(s_i) + \sigma_i^*(s_i')$. Clearly, $u_i(\sigma_i, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$; hence σ_i^* is not a best response to σ_{-i}^*, and consequently, σ^* is not a Nash equilibrium.

(Sufficiency.) If both conditions hold, there is no strategy for player i that does strictly better against σ_{-i}^* than σ_i^*; hence σ^* is Nash. □

Corollary 1. *No strictly dominated strategy can be played with positive probability in a Nash equilibrium.*

Proof. Let σ^* be a Nash equilibrium in which $\sigma_i^*(s_i) > 0$ for some s_i that is strictly dominated. By strict domination of s_i, there exists a s_i' such that $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$. (Why, even though the strict domination need not be by a pure strategy?) This contradicts condition (2) of Proposition 1. □

The necessity part of Proposition 1 says that in any Nash equilibrium, a player must be indifferent over the pure strategies she is randomizing over.[33] This places a restriction on the mixed strategy of *her opponents*. That is, in general, player i will *not* be indifferent between playing s_i and $s_i'(\neq s_i)$, unless σ_{-i} is carefully chosen. Let's see how this operates.

Example 12 (Mixed Strategies in MP-A)**.** Consider MP-A from Example 1. We can solve for *all* mixed strategy equilibria as follows. At least one player must be non-degenerately randomizing over H and T.

---

[32] See, for example, Athey (2001), Jackson, Simon, Swinkels, and Zame (2002), and Reny (1999) for even more on existence theorems.

[33] In fact, the logic establishes something stronger: if σ_i is a best response to some σ_{-i} (not necessarily part of a Nash equilibrium), then each $s_i \in \textit{support}[\sigma_i]$ is a best response to σ_{-i}.

Wlog, suppose it is player 1 (Anne). For her to be indifferent between the two actions means that σ_2 (Bob's mixed strategy) must be such that $u_1(H,\sigma_2) = u_1(T,\sigma_2)$. Observe that

$$\begin{aligned} u_1(H,\sigma_2) &= 2\sigma_2(H), \\ u_1(T,\sigma_2) &= 2\sigma_2(T). \end{aligned}$$

Indifference thus requires that $\sigma_2(H) = \sigma_2(T) = \frac{1}{2}$. That is, Bob must randomize uniformly over his pure strategies for Anne to be willing to mix in equilibrium. By a symmetric argument, Anne must also be mixing uniformly (for Bob to be willing to mix at all, and in particular uniformly). Hence, the unique Nash equilibrium in MP-A is both players randomizing equally over H and T. □

The next example applies the same idea to an important economic application, also showing that it works for infinite action spaces.[34]

Example 13 (Common Value All-Pay Auction)**.** There are $I > 1$ bidders for an object, each of whom values the object at $v > 0$. They all simultaneously submit bids, $s_i \geqslant 0$. The object goes to the highest bidder (randomly chosen amongst highest-bidders if there is a tie); everyone pays their bid to the auctioneer regardless of whether they win or not. Hence, payoffs for bidder i are $v - s_i$ is he wins, and $-s_i$ if he does not.

You can (and should!) verify that there are no PSNE in this game. To find a MSNE, we look for a *symmetric* MSNE, i.e. one where all players use the same mixed strategy. Let $F(x)$ denote the cdf over bids of a single player that is induced by the strategy; assume it has no atoms. For a player to be indifferent over all the bids that he is mixing over, it must be that for all $x \in Supp(F)$,

$$[F(x)]^{I-1}v - x = \bar{u}$$

for some constant utility level $\bar{u}$. Rearranging gives

$$F(x) = \left(\frac{\bar{u} + x}{v}\right)^{\frac{1}{I-1}}.$$

We know that $F(v) = 1$ (i.e., a player never bids more than his value, v) because any bid strictly above v is strictly dominated. Plugging in $F(v) = 1$ above yields $\bar{u} = 0$, and hence we get the solution

$$F(x) = \left(\frac{x}{v}\right)^{\frac{1}{I-1}}.$$

To complete the argument, we must show that no $x \notin Supp(F)$ yields a strictly higher utility than $\bar{u} = 0$, but this is immediate since $supp(F) = [0, v]$ and bidding any $x > v$ yields an expected payoff of $v - x < 0$ (because it wins for sure).

Notice that in the equilibrium, each player has an expected payoff of $\bar{u} = 0$ — competition amongst the buyers has left them with 0 surplus from the auction. You can check that the expected payment of any bidder is v/I, so that the auctioneer's revenue in expectation is v. □

[34] Actually, if you were paying attention, the indifference property already came up in an infinite action space: the asymmetric Bertrand game of Example 11.

Remark 16. All-pay auctions are useful models in various contexts. For example, in Industrial Organization and related fields, one can think of R&D as being an all-pay auction. That is, there are many firms competing against each other to develop a new product. Each firm independently and simultaneously decides how much money to sink into R&D. The "winner" is the one who invests the most money, but all players bear the R&D costs regardless of whether they win or not. All-pay auctions are also widely used in political economy to model lobbying or political campaigns.

3.3.4. Interpreting Mixed Strategy Equilibria

When we introduced mixed strategies in Subsection 2.2.3, I suggested that the easiest way to think about them was as though players were rolling dice to determine which pure strategy to use. While this is pedagogically true, some people find it a little uncomfortable to think that agents are actually choosing their (pure) strategies though an act of explicit randomization. This may be particularly discomforting given that we have already seen that in any Nash equilibrium, each player is *indifferent* over the set of pure strategies that he is mixing over! So why would a player then randomize, instead of just picking one with certainty? Of course, if he does pick one with certainty, this would in general destroy the indifference of the *other* players over the strategies they are randomizing over, and break the equilibrium altogether.

One response is to say that the player is indifferent, hence is happy to randomize. But this won't sit well with you if you don't like the idea that players randomize in practice. Fortunately, it turns out that we do not need players to be actually randomizing in a MSNE. All that matters is that as far as *other* players are concerned, player i's choice *seem* like a randomized choice. That is, what matters is the uncertainty that other players have about i's strategy. To give an example, consider an NBA basketball game where team A has possession, is down by 2 points, and there is only time for one more play. Team A is in a time-out, and has to decide whether to go for a 2-pointer to tie the game, or a 3-pointer to win.[35] Team B is obviously deciding whether to focus its defense against a 2-point shot or a 3-point shot. This is basically a generalized game of Matching Pennies, version A: team A wants to mismatch; team B wants to match. It may be that Team A's coach has a deterministic way of deciding whether to go for the win or the tie — for example, he uses his star shooter's morning practice 3-point accuracy as the critical factor. So long as Team B did not observe the shooter's morning practice accuracy, it is a randomized choice as far as they are concerned. Hence, B's belief about A's play is a non-degenerate one, even though A may actually be playing a pure strategy based on on some private information not available to B. That is, when we talk about A's mixed strategy, we are really talking about B's beliefs about A's strategy.

This way of justifying mixed strategy equilibria is known as *purification* (Harsanyi, 1973). We'll come back to it somewhat more precisely a bit later when we have studied incomplete information.

3.4. Normal Form Refinements of Nash Equilibrium

Many games have lots of Nash equilibria, and we've seen examples already. It's natural therefore to ask whether there are systematic ways in which we can refine our predictions within the set of Nash equilibria. The idea we pursue here is related to weak dominance.

[35] Incidentally, NBA wisdom has it that the "road" team should go (more often) for a win, whereas a "home" team should go (more often) for the tie.

Example 14 (Voting Game)**.** Suppose there are an odd number, $I > 2$, members of a committee, each of whom must simultaneously vote for one of two alternatives: Q (for status quo) or A (for alternative). The result of the vote is determined by majority rule. Every member strictly prefers the alternative passing over the status quo.

There are many Nash equilibria in this game. Probably the most implausible is this: every member plays $s_i = Q$; and this results in the status quo remaining. There is a more natural PSNE: every member plays $s_i = A$; and this results in the alternative passing. □

Why is it Nash for everyone to vote Q in this game? Precisely because if all other players do so, then no individual player's vote can change the outcome. That is, no player is *pivotal*. However, it is reasonable to think that a player would vote conditioning on the event that he *is* pivotal. In such an event, he should vote A. One way to say this formally is that $s_i = Q$ is weakly dominated by $s_i = A$ (recall Definition 15). We suggested earlier that players should not play weakly dominated strategies if they believe that there is the slightest possibility that opponents will play a strategy profile for which the weakly dominated strategy is not a best response. One way to justify this belief is that a player assumes that even though his opponents might intend to play their Nash equilibrium strategies, they might make a mistake in executing them. This motivates the notion of *trembling-hand perfect Nash equilibrium*: Nash equilibria that are robust to a small possibility that players may make mistakes.

Given pure strategy spaces, $\{S_i\}_i$, and a function $\varepsilon : \bigcup S_i \to (0, 1)$, define

$$\Delta_\varepsilon(S_i) = \{\sigma_i \in \Delta(S_i) \mid \sigma_i(s_i) \geqslant \varepsilon(s_i) \text{ for all } s_i \in S_i\}$$

as the space of ε-constrained mixed strategies for player i. This is the set of mixed strategies for i that place at least $\varepsilon(s_i) > 0$ probability on each of pure strategy s_i. If you recall our initial discussion of randomization, every such strategy is a *fully mixed strategy* (see Definition 9). The idea here is that the non-zero probabilities on each pure strategy capture the notion of "unavoidable mistakes". We can now define an ε-constrained equilibrium as a Nash equilibrium in which players play ε-constrained mixed strategies.

Definition 21. An ε-constrained equilibrium of a normal form game, $\Gamma_N \equiv \{I, \{S_i\}_{i=1}^I, \{u_i\}_{i=1}^I\}$, is a pure strategy Nash equilibrium of the perturbed game, $\Gamma_\varepsilon \equiv \{I, \{\Delta_\varepsilon(S_i)\}_{i=1}^I, \{u_i\}_{i=1}^I\}$.

Note that in the original game, $u_i : S \to \mathbb{R}$ (with mixed strategy profiles evaluated according to expected utility), while in the perturbed game we have extended to $u_i : \Delta_\varepsilon(S_1) \times \cdots \times \Delta_\varepsilon(S_I) \to \mathbb{R}$ using expected utility (with mixed strategy profiles—which are now compound lotteries over the original pure strategy profiles—again evaluated using expected utility).

A trembling-hand perfect equilibrium is any limit of a sequence of ε-constrained equilibria.

Definition 22 (Trembling-Hand Perfect Equilibrium)**.** A Nash equilibrium, σ^*, is a trembling-hand perfect equilibrium (THPE) if there is a sequence, $\{\varepsilon^k\}_{k=1}^\infty$, with each $\varepsilon^k : \bigcup S_i \to (0, 1)$, such that $\varepsilon^k(s_i) \to 0$ for all s_i as $k \to \infty$, and an associated sequence of ε^k-constrained equilibria, $\{\sigma^k\}_{k=1}^\infty$, such that $\sigma^k \to \sigma^*$ as $k \to \infty$.

Selten (1975) proved the following existence theorem, paralleling that of Nash.

Theorem 6 (THPE Existence)**.** *Every finite game has a THPE.*

Proof Sketch. Given a finite game, Γ_N, for any $\varepsilon : \bigcup S_i \to (0,1)$, we can define the perturbed game Γ_ε as above. It is straightforward to verify that Γ_ε satisfies all the assumptions of Theorem 3 when each $\varepsilon(s_i)$ is small enough,[36] and hence has a PSNE. Consider a sequence of perturbed games as $\varepsilon(s_i) \to 0$ for all s_i, and any sequence of equilibria. By the compactness of $\Delta(S)$, the equilibrium sequence has a convergent subsequence. Since $u_i : \Delta(S) \to \mathbb{R}$ is continuous, this subsequence converges to a Nash equilibrium of the original game. □

Notice that the definition of THPE only requires that we be able to find *some* sequence of ε-constrained equilibria that converges to a THPE. A stronger requirement would be that *every* sequence of ε-constrained equilibria converge to it. Unfortunately, this requirement would be too strong in the sense that many (finite) games do not have equilibria that satisfy this property.[37]

The definition of THPE given above using ε-constrained equilibria is conceptually elegant because it nicely captures the idea of mistakes that we started out with. However, it can be difficult to work with in practice. A useful result is the following.

Proposition 2. *A Nash equilibrium, σ^*, is a THPE if and only if there exists a sequence of fully mixed strategy profiles, $\sigma^k \to \sigma^*$, such that for all i and k, σ_i^* is a best response to σ_{-i}^k.*

Proof. (Only If.) Let σ^* be a THPE. Take the sequence of ε^k-constrained equilibria, $\sigma^k \to \sigma^*$, from the definition of THPE. We claim that for all i and s_i, if $\sigma_i^*(s_i) > 0$ then s_i is a best response to σ_{-i}^k for all k large enough (and hence σ_i^* is a best response to σ_i^k for all k large enough). Suppose not. Then there is some player i and a sub-sequence of $\{\sigma^k\}$, call it $\{\sigma^n\}$, such that for all n, there is s_i^n with $u_i(s_i^n, \sigma_{-i}^n) > u_i(s_i, \sigma_{-i}^n)$. But then, as σ^n is an ε^n-constrained equilibrium, it must hold that $\sigma_i^n(s_i) \to 0 < \sigma_i^*(s_i)$, contradicting $\sigma_i^n \to \sigma_i^*$.

(If.) Fix $\sigma^k \to \sigma^*$ such that each σ_i^k is fully mixed and each σ_i^* is a best response to each σ_{-i}^k. Define $\varepsilon^k(s_i) \equiv \sigma_i^k(s_i)$ if $\sigma_i^*(s_i) = 0$ and $\varepsilon^k(s_i) \equiv 1/k$ if $\sigma_i^*(s_i) > 0$. Note that $\varepsilon^k(s_i) \to 0$ for all s_i as $k \to \infty$. We claim that σ^k is an ε^k-constrained equilibrium for all large enough k. This follows from the observation that by construction, if $\sigma_i^k(s_i) > \varepsilon^k(s_i)$ then $\sigma_i(s_i) > 0$, which, using the hypothesis that σ_i is a best response to σ_{-i}^k, implies that s_i is a best response to σ_{-i}^k (recall Proposition 1). In words: each σ_i^k only puts greater than minimum probability (minimum according to ε^k) on pure strategies that are best responses to σ_{-i}^k, and hence σ_i^k is a best response to σ_{-i}^k in the ε^k-perturbed game. □

The Proposition says that in a THPE, σ^*, each σ_i^* is a best response to not only σ_{-i}^* (as required by Nash), but moreover, to each element of some sequence of fully mixed strategy profiles, $\sigma_{-i}^k \to \sigma_{-i}^*$. Proposition 2 immediately implies:

Corollary 2. *If σ^* is a Nash equilibrium in which σ_i^* is fully mixed for each player i, then σ^* is a THPE.*

Furthermore, we can use the Proposition to show that no weakly dominated strategy can be part of a THPE:

---

[36] Without the qualification we could have $\Delta_\varepsilon(S_i) = \varnothing$.

[37] This is tied to the fact that there are games with multiple THPE, which cannot be justified by the same sequence of ε-constrained equilibria. One could alternatively consider a set-valued concept of equilibria, where rather than referring to a particular strategy profile as an equilibrium, one instead targets a collection of strategy profiles that jointly satisfy certain desiderata. For instance, can we find a (non-trivial) *minimal* collection of Nash equilibria with the property that every sequence of ε-constrained equilibria converges to some Nash equilibrium in the set? The set may not be a singleton, but perhaps it is still typically "small". For a solution concept along these lines, see the notion of *stability* in Kohlberg and Mertens (1986).

Proposition 3 (THPE and Weak Dominance)**.** *If σ^* is a THPE, then for all i: (1) σ_i^* is not weakly dominated; (2) for all $s_i \in S_i$ such that $\sigma_i^*(s_i) > 0$, s_i is not weakly dominated.*[38]

Proof. The first claim is an immediate implication of Proposition 2. For the second claim, take a profile σ^* such that for some i and $s_i \in S_i$ that is weakly dominated, $\sigma_i^*(s_i) > 0$. Then, by the definition of weak dominance, there exists $s'_{-i} \in S_{-i}$ and $\sigma_i \in \Sigma_i$ such that $u_i(\sigma_i, s'_{-i}) > u_i(s_i, s'_{-i})$, and for all $s_{-i} \in S_{-i}$, $u_i(\sigma_i, s_{-i}) \geqslant u_i(s_i, s_{-i})$. Now consider a strategy $\tilde{\sigma}_i$ constructed as follows: for any $\tilde{s}_i \in S_i \backslash \{s_i\}$, $\tilde{\sigma}_i(\tilde{s}_i) = \sigma_i^*(\tilde{s}_i) + \sigma_i^*(s_i)\sigma_i(\tilde{s}_i)$; and $\tilde{\sigma}_i(s_i) = \sigma_i^*(s_i)\sigma_i(s_i)$. (Exercise: verify that $\tilde{\sigma}_i$ defines a valid strategy.) It follows that for any σ_{-i} in which $\sigma_{-i}(s'_{-i}) > 0$, $u_i(\tilde{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i^*, \sigma_{-i})$. Hence, σ_i^* is not a best response to any fully mixed strategy profile, σ_{-i}. By Proposition 2, σ^* is not a THPE. $\square$

Combined with the existence result for THPE, the fact that THPE rules out weakly dominated strategies makes it an appealing refinement of Nash equilibrium. You might wonder if we can say more: is any Nash equilibrium in which players use strategies that are not weakly dominated a THPE? Unfortunately, this is not true in general.[39]

Example 15. Consider the following three player game:

	L	R
T	1,1,1	1,0,1
B	1,1,1	0,0,1

l

	L	R
T	1,1,0	0,0,0
B	0,1,0	1,0,0

r

You can verify that (B, L, l) is a Nash equilibrium where no strategy is weakly dominated. However, it is not a THPE, because for any fully mixed strategies for players 2 and 3 that assign sufficiently low probabilities to R and r respectively, player 1 strictly prefers to play T rather than B due to (L, r) being an order of magnitude more likely to occur than (R, r). As an exercise, check that (T, L, l) is the *only* THPE in this game (hint: think first about the set of Nash equilibria). $\square$

The result is true in the case of two-player games, however.

Proposition 4. *In a two-player game, if σ^* is a Nash equilibrium where both σ_1^* and σ_2^* are not weakly dominated, then σ^* is a THPE.*

Proof Sketch. Assume that σ^* is a Nash equilibrium where both σ_1^* and σ_2^* are not weakly dominated. It can be shown that that for each player, i, there exists some fully mixed strategy of the opponent, σ_j ($j \neq i$), such that σ_i^* is a best response to σ_j.[40] For any positive integer n, define for each player the mixed strategy $\sigma_i^n = \frac{1}{n}\sigma_i + \left(1 - \frac{1}{n}\right)\sigma_i^*$. The sequence of fully mixed strategies profiles (σ_1^n, σ_2^n) converges to σ^*, and σ_i^* is a best response to each σ_j^n. By Proposition 2, σ^* is a THPE. $\square$

[38] Strictly speaking, it is unnecessary to state the two parts separately, because the first implies the second. You can prove this along the lines of the proof below.

[39] As the example below and the logic of the proof of the following Proposition demonstrate, this is due to the independence we assume in the players' randomization when playing mixed strategies. If we allowed for correlation, it would indeed be true.

[40] See Osborne and Rubinstein (1994, p. 64, Exercise 64.2).

(Do you see where in the above proof we used the assumption of two players? Hint: think about where it fails to apply to Example 15.)

Finally, we show that THPE is consistent with a particular class of reasonable Nash equilibria.

Definition 23 (Strict Nash Equilibrium)**.** A strategy profile, s^*, is a strict Nash equilibrium if for all i and $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_i^*)$.

Remark 17. There is no loss of generality in only considering PSNE in the definition of strict Nash equilibrium, since in any MSNE, a player is indifferent over the pure strategies he is mixing over.

In a strict Nash equilibrium, each player has a *unique* best response to his opponents' strategy profile. This implies that every strict Nash equilibrium is trembling-hand perfect.

Proposition 5 (Strict NE $\implies$ THPE)**.** *Every strict Nash equilibrium is a THPE.*

Proof. Exercise. □

Remark 18. The converse is certainly not true: not every THPE is a strict Nash equilibrium. (We've already seen multiple counter-examples.)

Sometimes a Nash equilibrium that is not strict is called a *weak* Nash equilibrium, but this is not very common terminology. Keep also in mind that there is a distinct concept from strict Nash called *strong* Nash equilibrium, which has to do with (lack of) "coalitional deviations"—outside our scope.

3.5. Correlated Equilibrium

As a final section on static games of complete information, we are going to briefly discuss the concept of *correlated equilibrium*. At a few different points earlier, I mentioned that requiring players to randomize independently when playing mixed strategies can be restrictive. Relaxing this is important for a fuller treatment for at least two reasons: (1) certain results depend on it (such as the equivalence of iterative deletion of strictly dominated strategies and rationalizability with more than 2 players); (2) it can be practically relevant in thinking about behavior in some strategic situations. To illustrate the latter point, we start with an example.

Example 16 (Battle of the Sexes)**.** Spouses, Anne and Bob, must decide whether to go to the concert or the baseball game. Both want to coordinate with the other, but Anne prefers to coordinate on the concert whereas Bob prefers baseball. In normal form,

	b	c
B	2,6	0,0
C	0,0	6,2

You can verify that there are exactly three Nash equilibria to this game: (B, b), (C, c) and a MSNE $(0.25B + 0.75C, 0.75b + 0.25c)$. The corresponding (expected) payoffs are: $(2, 6)$, $(6, 2)$, and $(1.5, 1.5)$. However, suppose that they can jointly observe a coin toss (or whether it is raining or sunny outside, or any other publicly observable random variable) before acting. Then they can attain a new payoff outcome, one that is more equitable than either PSNE, and Pareto-dominates the MSNE. For example, they toss a fair coin, and if the outcome is Heads, they both play B; if the outcome is Tails, they both play C.[41] Clearly, given that the opponent is following the prescribed strategy, it is optimal for each player to follow it. In expectation, this achieves a convex combination of two PSNE payoffs, giving the payoff profile $(4,4)$. More generally, by using an appropriately weighted randomization device, *any* convex combination of the the NE payoffs (or action-profiles) can be achieved. □

The next example demonstrates that by using correlated but private signals, players may be able to do even better than by public randomization.

Example 17. Consider a modification of the Battles of the Sexes, as follows:[42]

	b	c
B	5,1	0,0
C	4,4	1,5

Now, the three Nash equilibria are (B,b), (C,c) and $(0.5B+0.5C, 0.5b+0.5c)$, with corresponding payoffs $(5,1)$, $(1,5)$, and $(2.5,2.5)$. By using public randomization as before, any convex combination of these can be attained. But the players can do even more by using a device that sends each player correlated but privately observed signals. For example, suppose they hire an independent third party who tosses a three-sided fair die and acts as follows: she reveals whether the roll is 1 or in the set $\{2,3\}$ to Anne; but to Bob, she reveals whether the roll is 3 or in the set $\{1,2\}$. Consider a strategy for Anne where she plays B if told 1, and C if told $\{2,3\}$; a strategy for Bob where he plays c if told 3, and b if told $\{1,2\}$. Let us check that it is optimal for Anne to follow her strategy, given that Bob is following his: (i) when Anne is told 1, she knows that Bob will play b (since he will be told $\{1,2\}$), hence it is optimal for her to play B; (ii) when Anne is told $\{2,3\}$, she knows that with probability 0.5, Bob was told $\{1,2\}$ and will play b, and with probability 0.5, Bob was told 3 and will play c, hence she is indifferent between her two actions. A similar analysis shows that it is optimal for Bob to follow his strategy given that Anne is following hers. Hence, the prescribed behavior is self-enforcing, and attains a payoff of $\left(3\frac{1}{3}, 3\frac{1}{3}\right)$, which is outside the convex hull of the original Nash equilibrium payoffs. □

Generalizing from the examples, we now define a correlated equilibrium.

Definition 24 (Correlated Equilibrium)**.** A probability distribution p on the product pure strategy space, $S = S_1 \times \cdots \times S_I$, is a correlated equilibrium if for all i and all s_i chosen with positive probability under p,

$$\sum_{s_{-i}\in S_{-i}} p(s_{-i}|s_i)u_i(s_i,s_{-i}) \geqslant \sum_{s_{-i}\in S_{-i}} p(s_{-i}|s_i)u_i(s_i',s_{-i}) \qquad \forall s_i' \in S_i.$$

The way to think of this definition of correlated equilibrium is that everyone knows ex-ante that a "device" (or outside party) will choose the pure strategy profile s with probability $p(s)$, but each player only learns his component of the profile that was selected, s_i. We have a correlated equilibrium if all players want to follow their recommendation to play s_i, given that all other players are following their recommendations.

[41] Indeed, your own personal experience in such situations may suggest that this is precisely how couples operate!

[42] Unintendedly, I wrote this as Anne (row player) preferring baseball whereas Bob (column player) prefers the concert.

Observe that the key difference with the definition of Nash equilibrium is that the distribution p may have correlation between s_i and s_{-i}, and accordingly the optimality conditions account for the conditional probability of s_{-i} given s_i. This suggests that the set of correlated equilibria generalizes the set of Nash equilibria.

Proposition 6 (NE $\implies$ Correlated Equilibrium)**.** *Every Nash equilibrium is a correlated equilibrium.*

Proof. Given a Nash equilibrium, σ, simply define $p(s) = \prod_{i=1}^{I} \sigma_i(s_i)$. In this case, $p(s_{-i}|s_i) = p(s_{-i})$, and the optimality conditions for correlated equilibrium reduce to that of Nash. $\square$

Corollary 3. *Every finite game has a correlated equilibrium.*

Correlated equilibrium is a useful solution concept when thinking about pre-play communication (possibly through a mediator) and other contexts. However, we will stick to Nash equilibrium for the rest of this course.

3.6. Bayesian Nash Equilibrium and Incomplete Information

The assumption heretofore maintained that all players know each other's preferences over terminal nodes (or strategy profiles) is clearly a very restrictive one. In fact, it is reasonable to think that in many games, one doesn't really know what the opponents' payoffs are. For example, in the Bertrand or Cournot competition games, each firm may not know the other firm's cost function or cost parameter in the linear case. Alternatively, in the auction examples, players won't generally know their opponents' valuations for the object. Moreover, one may not even know one's own payoff from some strategy profiles, since the payoff could depend upon something that is only to known to another player (Example 18 below illustrates).

Definition 25 (Incomplete Information Game)**.** A game has (or is of) incomplete information when at least one player does not know the payoff that some player receives from some strategy profile (or terminal node).

Dealing with incomplete information would seem to require consideration of a player's beliefs about other players' preferences, her beliefs about their beliefs about her preferences, and so on, ad infinitum. This can quickly get very complicated. Fortunately, we have a now standard and beautiful way to approach this problem, due to Harsanyi. His idea is to *transform any game of incomplete information into a game of complete but imperfect information* as follows: we imagine that a player's payoffs are determined at the outset of the game by Nature, which chooses realizations of a random variable for each player. The vector of realizations of random variables determines the payoffs for all players from each strategy profile. A player observes the realization of his own random variable, but not necessarily that of others. As with other moves of Nature, the probability distribution that Nature uses for each random variable is common knowledge. This extended game is known as a *Bayesian game (of incomplete information)*, and we call a Nash equilibrium of this extended game a *Bayesian Nash equilibrium*. The realization of a player's random variable is often called his *type*.

Example 18 (For Love or Money)**.** Suppose a wealthy Anne is considering whether to marry a pauper Bob, but is not completely sure whether he loves her (probability α), or just wants her money (probability $1-\alpha$). This is a game where each player must choose to marry or not. If either one chooses not to marry, then both players gets a payoff of 0. If both choose to marry, then Bob gets a payoff of 5 if he is a lover, and 3 if he is a scoundrel; whereas Anne gets 5 if Bob is a lover and -3 if he is a scoundrel. Clearly, if it were known that Bob is a scoundrel, then every Nash equilibrium involves Anne choosing not to marry. Conversely, if Bob were known to be a lover, then there is a Nash equilibrium where both Anne and Bob choose marriage.

The question is, what happens if Bob privately knows his *type* (lover or scoundrel), but Anne does not? We can represent this game of incomplete information in extensive form using Nature's move at the root node. In this extended game, a pure strategy for Anne is simply a choice of marry or not marry; so she has 2 pure strategies. However, for Bob, a pure strategy is a *contingent* plan that specifies whether to marry or not in each of two cases: if his type is lover, and if his type is scoundrel. Hence, Bob has 4 pure strategies in the extended game: (marry if lover, marry if scoundrel), (marry if lover, don't if scoundrel), (don't if lover, marry if scoundrel), and (don't if lover, don't if scoundrel). $\square$

We now define a Bayesian game formally.

Definition 26 (Bayesian Game)**.** A Bayesian game is defined by a tuple $\langle I, \{A_i\}, \{u_i\}, \{\Theta_i\}, p\rangle$, where I is the set of players, A_i is the action space for player i, Θ_i is the set of types for player i, $u_i : A \times \Theta \to \mathbb{R}$ is the payoff function for player i, and $p : \Theta \to [0, 1]$ is the prior probability distribution over type profiles.[43] p is assumed to be common knowledge, but each player i only knows her own θ_i.

Note that in the Bayesian game, players privately learn their types and then choose their actions simultaneously. You should think of the action space for each player as possibly representing normal-form strategies of some underlying game. I use the terminology "action" rather than "strategy" because in the context of a Bayesian game, there is something else we will want to call a strategy.

Remark 19. We allow each utility function, u_i, to depend on the entire vector of types — not just player i's type. This is more general than MWG (p. 255), and is useful in many applications. We say that the game has *private values* when each u_i only depends on i's type. The more general situation has *interdependent values*, sometimes also referred to as *correlated values*.

If the prior distribution p is a product distribution, so that for each i the conditional distributional $p(\theta_{-i}|\theta_i)$ is independent of θ_i, we say the game has *independent types*.

Remark 20. The above formulation of a Bayesian game is *ex-ante*: there is a stage at which players have identical information/no private information, and they subsequently learn their private information. But in many applications—including Example 18—it is more natural to take an *interim* perspective: players simply begin with private information, and there is no literal ex-ante stage. From this perspective, we would define a Bayesian game as a tuple $\langle I, \{A_i\}, \{u_i\}, \{\Theta_i\}, \{q_i\}\rangle$, where the objects are as in Definition 26, except that each $q_i : \Theta_i \to \Delta(\Theta_{-i})$. Here $q_i(\theta_i)$ captures type θ_i's (subjective) belief about the opponents' type profile. Up to technical details, which don't arise when Θ is finite, it is straightforward that for each i we can find a "prior" $p_i \in \Delta(\Theta)$ such that q_i obtains from p_i via Bayesian updating. For example, set $p_i(\theta) = (1/|\Theta_i|)q_i(\theta_{-i}; \theta_i)$; although there are infinitely many others, as we could use any full-support marginal over Θ_i instead of the uniform. But there is no guarantee that we can find single prior $p \in \Delta(\Theta)$ that is simultaneously valid for all players. That requires some degree of consistency in the subjective beliefs $\{q_i\}$, which is referred to as the *common prior assumption* (CPA). Under the CPA, the ex-ante interpretation is justified in an "as if" sense even if there is no real ex-ante stage. The CPA is typically assumed in applications and we will maintain it here—although I want to stress that there is no logical reason it must be required. One can study Bayesian

[43] Here, $A \equiv A_1 \times \cdots \times A_I$ and $\Theta \equiv \Theta_1 \times \cdots \times \Theta_I$.

games without the CPA, contrary to some misconceptions. Indeed the CPA can be controversial; see Gul (1998).

Remark 21. It is far from obvious that Definition 26 of a Bayesian game is sufficient to represent all games of incomplete information. In particular, one key issue is, following the discussion after Definition 25, whether Bayesian games (as defined above) are sufficiently rich to capture an infinite hierarchy of beliefs of every player about other players' beliefs. This was resolved by Mertens and Zamir (1985) and Brandenburger and Dekel (1993) with constructions of the so-called *universal type space*. This is material suited for a more advanced course, but see Zamir (2008) for a nice discussion. In a nutshell, they established that Bayesian games are indeed sufficient under some reasonable "coherence" conditions on belief heirarchies. Caveat: whether the CPA can be satisfied is another question.

A pure strategy for player i in a Bayesian game specifies what action to take for each of her possible type realizations. That is, a (pure) strategy is a mapping $s_i : \Theta_i \to A_i$. Given a profile of straegies for all players, we compute i's expected payoffs as

$$\tilde{u}_i(s_1, \ldots, s_I) = \mathbb{E}_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i, \theta_{-i})].$$

With these preliminaries, we can define a (pure strategy) Bayesian Nash equilibrium in the natural way.

Definition 27 (Bayesian Nash Equilibrium)**.** A (pure strategy) Bayesian Nash equilibrium of the Bayesian game, $\langle I, \{A_i\}, \{u_i\}, \{\Theta_i\}, p\rangle$, is a profile of (pure) strategies $(s_1, \ldots, s_I)$ such that for all i and all $s_i' \in S_i$, $\tilde{u}_i(s_i, s_{-i}(\cdot)) \geqslant \tilde{u}_i(s_i', s_{-i})$.

In the natural way, we can allow for *mixed* strategies, which are just probability distributions over the pure strategies, and then define mixed strategy Bayesian Nash equilibrium. An important observation is that by the Nash existence theorem, a (possibly mixed) Bayesian Nash equilibrium (BNE) exists if for all i, A_i and Θ_i are finite.

The approach taken above is ex ante in the sense that players are choosing their strategies prior to knowing their types. On the other hand, one can imagine that a player picks an action once he knows his type, but not that of others; this is at the interim stage.[44] It is straightforward that a strategy can be part of a BNE if and only if it maximizes a player's expected utility conditional on each θ_i that occurs with positive probability.[45]

Proposition 7. *A profile of strategies, s, is a (pure strategy) BNE if and only if, for all i and all $\theta_i \in \Theta_i$ occurring with positive probability and all $a_i \in A_i$,*

$$\mathbb{E}_{\theta_{-i}}[u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i] \geqslant \mathbb{E}_{\theta_{-i}}[u_i(a_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i].$$

Proof. Exercise (or see MWG p. 256). □

The Proposition implies that we can essentially think of each type of a player as maximizing its own payoffs independent of what other types of the player are doing. This is a very useful result since it permits

[44] *Ex post* would refer to a third stage where the types of all players become known, for example at the end of the game when payoffs are realized.

[45] This is for finite Θ_i; in the infinite case, it applies to "almost all" θ_i.

us to “decompose”, type by type, the process of finding an optimal response for a player to any strategy profile of his opponents.

Remark 22. While we are focussing on BNE, one can also parallel other complete-information solution concepts. For example, a strategy s_i is *strictly dominant* if for all $s_i' \neq s_i$ and s_{-i}, $\tilde{u}_i(s_i, s_{-i}) > \tilde{u}_i(s_i', s_{-i})$. In the private-value case, we can write this alternatively as for all θ_i (that occur with positive probability), $a_i \neq s_i(\theta_i)$, and a_{-i}, $u_i(s_i(\theta_i), a_{-i}, \theta_i) > u_i(a_i, a_{-i}, \theta_i)$.

In Bayesian games, a weakening of weak dominance is of interest: a strategy s_i is “very weakly dominant” if for all s_i', and s_{-i}, $\tilde{u}_i(s_i, s_{-i}) \geqslant \tilde{u}_i(s_i', s_{-i})$. (This terminology is not common. Another way to say it is that s_i is a best response to any strategy of the opponents. Do you see why it is weaker than weak dominance?) In the context of mechanism/market design, where the action space is frequently $A_i = \Theta_i$ and the strategy of interest is $s_i(\theta_i) = \theta_i$, this requirement is called *strategyproofness*.

Another concept of interest is that of *ex-post (Bayesian Nash) equilibrium*. A pure BNE s is an ex-post equilibrium if for all θ, i, and a_i, $u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta) \geqslant u_i(a_i, s_{-i}(\theta_{-i}), \theta)$. Note that we are using equilibrium information here; what we are dropping is information about other players’ types. Ex-post equilibrium is appealing because it means we can be agnostic about players’ beliefs about others’ information; it thus comes up in design problems.

3.6.1. Examples

Example 19 (BNE in For Love or Money)**.** Continuing with Example 18, let us now find the (pure strategy) BNE of this game. Denote Anne as player 1 and Bob as player 2; Bob’s types as l (lover) and c (scoundrel); and the actions are M (marry) and N (not marry). We could proceed in two ways, corresponding to the *ex-interim* and *ex-ante* definitions.

1. For the ex-interim procedure, we can treat a BNE as a triple $(s_1, s_2(l), s_2(c)) \in \{M, N\}^3$. Then (N, N, N) and (N, N, M) are always BNE. In addition, there is at least one other BNE, though what it is depends on the parameter α. If $\alpha \geqslant \frac{3}{8}$, (M, M, M) is a BNE. If $\alpha \leqslant \frac{3}{8}$, (N, M, M) is a BNE. Notice that (M, N, N) is never a BNE, and moreover, if $\alpha \in (0, 1)$, (M, M, N), (M, N, M), and (N, M, N) are not BNE.
2. For the ex-ante formulation, a BNE is a pair $(s_1, s_2) \in \{M, N\} \times \{MM, MN, NM, NN\}$, where for player 2, (MN) is the strategy of playing M if l and N if c, for example. You can write out the BNE in this formulation that are equivalent to the ex-interim formulation. □

Example 20 (Incomplete Information 2^{nd} Price Auction)**.** Recall the 2^{nd} price auction we studied in Example 9. Now, each player’s valuation v_i is assumed to be *private* information. It is straightforward that it remains a BNE for each player to bid the truth, i.e. play $s_i(v_i) = v_i$. This follows from the fact that it was weakly dominant to bid the truth in the complete information case. □

Example 21 (1^{st} Price Auction)**.** Now we consider a 1^{st} price Auction, which is an auction where the highest bidder wins (bids are non-negative), the winner pays his own bid, and losers pay nothing. As before, if there is a tie, the winner is randomly selected from the highest bidders. Assume that each v_i is drawn independently from the distribution F that is continuous on $\mathbb{R}$ and strictly increasing on the support $[\underline{v}, \overline{v}]$, where $\underline{v} \geqslant 0$. This is a setting of *independent private values.*

To find a BNE, we look for a symmetric pure-strategy equilibrium in which each player is playing the same strictly increasing and differentiable strategy, denoted $s^*(v)$. Suppose that all other bidders are playing $s^*(v)$. Player i's expected payoff from a bid b when he has valuation v is[46]

$$\pi(b,v) = [F((s^*)^{-1}(b))]^{I-1}[v-b].$$

The expected payoff in equilibrium for a bidder with valuation v is then

$$\Pi(v) = \pi(s^*(v),v) = [F(v)]^{I-1}[v-s^*(v)].$$

Note that $\Pi(\underline{v}) = 0$ because our hypothesis that $s^*(v)$ is strictly increasing (and that F is continuous) implies that a bidder with the lowest possible valuation wins with probability zero. Differentiating,

$$\Pi'(v) = \pi_1(s^*(v),v)\frac{\mathrm{d}s^*}{\mathrm{d}v}(v) + \pi_2(s^*(v),v) = \pi_2(s^*(v),v) = [F(v)]^{I-1},$$

where the second equality uses the envelope theorem, i.e., that $\pi_1(s^*(v),v) = 0$ because $s^*(v)$ is an optimal bid with valuation v.[47]

Recall that by the Fundamental Theorem of Calculus,

$$\Pi(v) - \Pi(\underline{v}) = \int_{\underline{v}}^{v} \Pi'(x)\mathrm{d}x.$$

Substituting in to the above from the previous derivations gives

$$[F(v)]^{I-1}[v-s^*(v)] = \int_{\underline{v}}^{v} [F(x)]^{I-1}\mathrm{d}x,$$

which rearranges as

$$s^*(v) = v - \int_{\underline{v}}^{v} \left[\frac{F(x)}{F(v)}\right]^{I-1} \mathrm{d}x.$$

It is clear that $s^*(v)$ is strictly increasing and differentiable, as required. Notice that $s^*(v) < v$ if $v > \underline{v}$, which makes sense: bidders should "shade down" their bids because winning is only beneficial if one pays less than one's valuation. (Indeed, verify that the strategy $s(v) = v$ is weakly dominated.)

To complete the argument, one must verify that $s^*(v)$ is in fact an optimal bid for type v—we only derived necessary conditions above—which I omit. $\square$

Remark 23. Above, we constructed a symmetric Bayes Nash equilibrium by assuming that bidders are using a differentiable pure strategy. In fact, one can prove that in *any* symmetric equilibrium, each bidder must use a continuous and strictly increasing (hence a.e. differentiable) pure strategy. This is proved by establishing that there cannot be a "gap" in the range of bids played in equilibrium (because then it would be suboptimal

[46] Below, let $(s^*)^{-1}(b) = \underline{v}$ for $b < s^*(\underline{v})$ and $(s^*)^{-1}(b) = \overline{v}$ for $b > \overline{v}$.

[47] You may worry that this is not assured for type $\underline{v}$, since $s^*(\underline{v})$ may be zero, and as such one is not assured the first order condition $\pi_1(s^*(\underline{v}),\underline{v}) = 0$. But this won't matter because it will be sufficient that the formula for $\Pi'(v)$ hold for $v > \underline{v}$.

to bid just above the gap; one should instead bid at or just above the bottom of the gap) and there cannot be any "atoms" in the equilibrium distribution of bids (because then no bidder would make an offer at or just below the atom as it is better to deviate to just above the atom, leading to a gap). Thus, the symmetric equilibrium is essentially unique.

Remark 24 (Revenue Equivalence). Let's figure out the seller's expected revenue in the 1st price auction if the symmetric equilibrium is played. It is simply the expected winning bid, or the expected bid of the bidder with the highest valuation. We may denote this $\mathbb{E}[s^*(V^{1:I})]$, where, for any integer $k \in [1, I]$, $V^{k:I}$ denotes the k-th highest valuation out of I bidders. It is common to refer to $V^{k:I}$ as the $(I+1-k)$-th order statistic of F (with sample size I), so that $V^{1:I}$ is the I-th order statistic. To compute the expected revenue more explicitly, define $F^{1:I}(v) \equiv [F(v)]^I$, so that $F^{1:I}$ is the cdf of $V^{1:I}$. We have

$$
\begin{aligned}
s^*(v) &= v - \int_{\underline{v}}^{v} \left[\frac{F(x)}{F(v)}\right]^{I-1} \mathrm{d}x \\
&= \frac{1}{F^{1:I-1}(v)} \left[F^{1:I-1}(v)v - \int_{\underline{v}}^{v} F^{1:I-1}(x)\mathrm{d}x\right] \\
&= \frac{1}{F^{1:I-1}(v)} \int_{\underline{v}}^{v} x\mathrm{d}F^{1:I-1}(x) \\
&= \mathbb{E}[V^{1:I-1}|V^{1:I-1} < v], \qquad (2)
\end{aligned}
$$

where the third equality is derived through an integration by parts. This says that if a bidder has valuation v, he sets his bid equal to the expectation of the highest of the other $I-1$ valuations, conditional on him having the highest overall valuation. So, the expected revenue for the seller is

$$
\mathbb{E}[s^*(V^{1:I})] = \mathbb{E}[V^{1:I-1}|V^{1:I-1} < V^{1:I}] = \mathbb{E}[V^{2:I}], \qquad (3)
$$

where the first equality is from (2) and the second equality owes to iterated expectations.[48] But observe that $\mathbb{E}[V^{2:I}]$, which is the expectation of the 2nd highest valuation among I bidders, is also the expected revenue from a 2nd price auction! (Why?) We have thus proved that, given the relevant equilibria, the 1st price auction and the 2nd price auction generate exactly the same expected revenue for the seller, even though the payment rules and bidding behavior are very different.[49] This is a particular case of the *Revenue Equivalence Theorem*: the key points are that (i) the setting is one of independent private values (each bidder's type

[48] More detail on the iterated expectations: for any vector of realized valuations, the second-highest valuation can also be viewed as the highest remaining valuation after excluding the highest valuation. Thus, $\mathbb{E}[V^{2:I}] = \mathbb{E}[\mathbb{E}[V^{1:I-1}|V^{1:I-1} < v]]$, where the outer expectation is with respect to $V^{I:I}$ having realization v. But this is just the middle expression of (3).

[49] Another way to see the revenue equivalence is to note that in the 2nd price auction, the expected payment for a bidder i with valuation v is

$$
\begin{aligned}
&\text{Prob}[\text{i wins}|v_i = v] \times \mathbb{E}[\text{2nd highest bid}|v \text{ is the highest bid}] \\
=&\text{Prob}[\text{i wins}|v_i = v] \times \mathbb{E}[\text{2nd highest value}|v \text{ is the highest value}] \\
=&F^{1:I}(v) \times \mathbb{E}[V^{1:I-1}|V^{1:I-1} < v],
\end{aligned}
$$

which, using our derivation (2), is the same as in the 1st price auction (because (2) gives the payment of bidder i with valuation v when he wins, which occurs with probability $F^{1:I}(v)$). If the expected payment is the same for every bidder (conditional on his type, and hence on expectation too), then the expected revenue is the same.

is his own valuation and types are drawn independently); (ii) both auction formats allocate the object to the bidder with the highest valuation; and (iii) a bidder with the lowest possible valuation (i.e., $\underline{v}$) gets 0 expected payoff in equilibrium. More generally, the issue of how to design an auction to maximize the seller's revenue lies in the field of mechanism design.

As a third example of Bayes Nash equilibrium, we look at Cournot competition game with incomplete information. It serves to illustrate a case where players' types are not independent.

Example 22 (Incomplete Information Cournot)**.** The setting is the same as the Linear Cournot case considered in Example 7, but modified so that each firm now has one of two potential cost parameters: c_H or c_L, where $c_H > c_L \geqslant 0$. Each firm's parameter is privately known to it alone, and the prior distribution is given by $p(c_H, c_H) = p(c_L, c_L) = \frac{1}{2}\alpha$ and $p(c_H, c_L) = p(c_L, c_H) = \frac{1}{2}(1-\alpha)$, with $\alpha \in (0,1)$ commonly known. Recall that inverse market demand is given by $p(Q) = a - bQ$; firm i's cost function is $c_i q_i$, where $c_i \in \{c_H, c_L\}$.

Let's look for a symmetric pure-strategy BNE where each firm plays $s^*(c_H)$ and $s^*(c_L)$ for each of its two types. To solve for these, we observe that if a firm is of type c_H, then its maximization problem, taking as given that the other firm is using $s^*(\cdot)$, is:

$$\max_q \left[\alpha(a - b(s^*(c_H) + q) - c_H)q + (1-\alpha)(a - b(s^*(c_L) + q) - c_H)q\right].$$

By hypothesis that $s^*(\cdot)$ is a symmetric equilibrium, a solution the above problem must be $s^*(c_H)$. Taking the FOC (presuming an interior solution) thus gives an equilibrium condition:

$$\alpha\left(a - c_H - bs^*(c_H) - 2bs^*(c_H)\right) + (1-\alpha)\left(a - c_H - bs^*(c_L) - 2bs^*(c_H)\right) = 0. \tag{4}$$

Similarly, the maximization problem when type is c_L is

$$\max_q \left[(1-\alpha)(a - b(s^*(c_H) + q) - c_L)q + \alpha(a - b(s^*(c_L) + q) - c_L)q\right],$$

for which $s^*(c_L)$ must be a solution. This requires

$$(1-\alpha)\left(a - c_L - bs^*(c_H) - 2bs^*(c_L)\right) + \alpha\left(a - c_L - bs^*(c_L) - 2bs^*(c_L)\right) = 0. \tag{5}$$

The equilibrium is found by solving Eqns. (4) and (5), which involves tedious but straightforward algebra. □

3.6.2. Purification Theorem

In Subsection 3.3.4, we introduced the notion of *purification* (Harsanyi, 1973) as a justification for MSNE. Using incomplete information, we can now state the idea more precisely. Start with a (finite) normal-form game, $\Gamma_N = \{I, \{S_i\}, \{u_i\}\}$. Let θ_i^s be a random variable with range $[-1, 1]$, and $\varepsilon > 0$ a constant. Player i's perturbed payoff function depends on the collection $\theta_i \equiv \{\theta_i^s\}_{s \in S}$ and is defined as

$$\tilde{u}_i(s, \theta_i) = u_i(s) + \varepsilon\theta_i^s$$

We assume that the θ_i's are independent across players, and each θ_i is drawn from a distribution P_i with density p_i.

Theorem 7 (Purification)**.** *Fix a set of players, I, and strategy spaces $\{S_i\}$. For almost all vectors of payoffs, $u = (u_1, \ldots, u_I)$,*[50] *for all independent and twice-differentiable densities, p_i on $[-1,1]^{|S|}$, any MSNE of the payoffs $\{u_i\}$ is the limit as $\varepsilon \to 0$ of a sequence of pure-strategy BNE of the perturbed payoffs $\{\tilde{u}_i\}$. Moreover, the sequence of pure-strategy BNE are "essentially" strict.*

The Theorem, whose proof is quite complicated, is the precise justification for thinking of MSNE as PSNE of a "nearby" game with some added private information. Note that according to the Theorem, a single sequence of perturbed games can be used to purify all the MSNE of the underlying game. In his proof, Harsanyi showed that equilibria of the perturbed games exist (note that these are infinite games), and moreover, that in any equilibrium of a perturbed game, almost all $s_i(\theta_i)$ must be pure strategies of the underlying game, and that for almost all θ_i, $s_i(\theta_i)$ is the unique best response (this is the "essentially strict" portion of the result).

3.6.3. The Importance of Higher-Order Beliefs

In the examples/applications so far, higher order beliefs have not played a major role. But consider the following symmetric two-player game, where $x \in \{0, 2\}$:

	A	B
A	x,x	$-\varepsilon$,1
B	1,$-\varepsilon$	1,1

In what follows, $\varepsilon > 0$ is common knowledge; you can think of it as small. If it is commonly known that $x = 2$ then we have a coordination game: both (A, A) and (B, B) are strict Nash equilibria; (A, A) is Pareto dominant while (B, B) is risk dominant.[51] If, on the other hand, $x = 0$ is common knowledge, then B is a strictly dominant strategy, so (B, B) is the unique Nash equilibrium.

Suppose there isn't common knowledge about which game is being played, i.e., there is incomplete information about x. Assume the prior $\Pr(x = 2) > 1/2$ (and $\Pr(x = 0) = 1 - \Pr(x = 2)$). If players don't receive any other information about x, then the corresponding Bayesian game has two Bayesian Nash equilibria: (A, A) and (B, B); both are strict and (A, A) remains Pareto dominant.

Let us add private information as follows. There is a "state" $\omega \in \Omega \equiv \{\omega_1, \omega_2, \ldots, \omega_K\}$, where $K > 1$ is an odd integer. If $\omega < \omega_K$ then $x = 2$; if $\omega = \omega_K$ then $x = 0$. The distribution of ω is uniform. Player 1's type is depicted by the partition

$$\{\{\omega_1\}, \{\omega_2, \omega_3\}, \ldots, \{\omega_{K-1}, \omega_K\}\},$$

which represents what he learns about ω. Player's 2 type is depicted by the partition

$$\{\{\omega_1, \omega_2\}, \ldots \{\omega_{K-2}, \omega_{K-1}\}, \{\omega_K\}\}.$$

(Exercise: write down the Bayesian game in our standard notation. Note that the distribution of types is correlated.)

[50] That is, for all but a set of payoffs of Lebesgue measure 0.

[51] In a 2×2 two-player game, a strategy is risk dominant if it is the unique best response to the other player playing a 50-50 mixture.

The key property here is that while there can be arbitrarily high finite order of knowledge of x (as $K \to \infty$), there is never common knowledge of x.[52] For example, when $\omega = \omega_{K-2}$, there is mutual knowledge that $x = 2$, but player 2 does not know that player 1 knows x; rather player 2 puts probability 1/2 on player 1 thinking $x = 2$ and probability 1/2 on player holding a uniform distribution over $x \in \{0, 2\}$. When $\omega = \omega_{K-3}$, not only do both players know $x = 2$, but there is mutual knowledge of this (i.e., both players know that both players know $x = 2$); however, player 1 does not know that player 2 knows that player 1 knows x. Notice that for any integer M and any probability $p < 1$, as $K \to \infty$, there is Mth-order knowledge of x with ex-ante probability larger than p. It may appear, then, that for large K, it is "almost common knowledge" that $x = 2$.[53] But:

Claim 1. *In the above Bayesian game, IDSDS yields $s_i(t_i) = B$ for $i = 1, 2$ and all t_i.*

Proof. Deleting dominated strategies rules out any strategy that does not have $s_2(\{\omega_K\}) = B$. But then, $s_1(\{\omega_{K-1}, \omega_K\}) = A$ is iteratively dominated because B is risk dominant. The result follows by induction. $\square$

One take-away is that we have to be careful with what we mean by "almost common knowledge": the topology with respect to which continuity with common knowledge is viewed is crucial (cf. fn. 53.)

The argument in Claim 1 is due to ideas in Rubinstein (1989); it starkly illustrates the logic by which outcomes in strategic settings can be sensitive to "tails" of higher-order beliefs. Notice that the argument is based on "contagion" — dominance for some types feeds into (iterative) dominance for other types. Carlsson and Van Damme (1993) introduced the terminology of "global games" to model situations in which each player observes the true payoffs of the game with a very small amount of noise; dominance plus contagion arguments prove powerful. I will give you a homework problem illustrating this. Morris and Shin (1998) popularized global games for applications.[54] More commonly, higher-order uncertainty is usually downplayed in applied settings by modeling them with simple type spaces.

[52] It is common knowledge that player 2 knows x. (Why?) What is not common knowledge is whether player 1 knows x.

[53] However, for any probability p close enough to 1, no matter the value of $K > 1$, there is no state ω in which the following statement is true: both players ascribe probability (at least) p to $x = 2$, both players ascribe probability (at least) p to both players ascribing probability (at least) p to $x = 2$, ... ad infinitum. In other words, for all high p, there is a failure of "common p-belief" (Monderer and Samet, 1989), which is what drives the ensuing argument.

[54] While applied scholars often take the message of the global games literature to be that there is a "correct" unique equilibrium in settings that *prima facie* have multiple equilibria, this is not the right interpretation. See me for more discussion if you are interested, or have a look at Weinstein and Yildiz (2007).

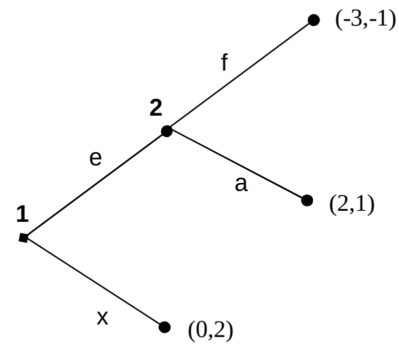


Figure 1 – Extensive Form for Predation Game

4. Dynamic Games and Extensive Form Refinements of Nash Equilibrium

We now turn to a study of dynamic games, i.e. games where players are not just moving simultaneously. The underlying theme will be to refine the set of Nash equilibria in these games.

4.1. The Problem of Credibility

One view towards studying dynamic games is to simply write down their normal form, and then proceed as we did when studying simultaneous games. The problem however is that certain Nash equilibria in dynamic games can be very implausible predictions. Let's illustrate this through the following example.

Example 23 (Predation)**.** Firm 1 (the entrant) can choose whether to enter a market against against a single incumbent, Firm 2, or exit. If Firm 1 enters, Firm 2 can either respond by fighting or accommodating. The extensive form and payoffs are drawn in Figure 1.

To find Nash equilibria of this game, we can write out the normal form as follows.

	f	a
e	-3,-1	2,1
x	0,2	0,2

Clearly, (x, f) is a Nash equilibrium, as is (e, a).[55] However, (x, f) does not seem like a plausible prediction: conditional upon Firm 1 having entered, Firm 2 is strictly better off accommodating rather than fighting. Hence, if Firm 1 enters, Firm 2 should accommodate. But then, Firm 1 should foresee this and enter, since it prefers the outcome (e, a) to what it what it gets by playing x. $\square$

The problem in the Example is that the "threat" of playing f, that is fighting upon entry, is not credible. The outcome (x, f) is Nash because *if* Firm 2 would fight upon entry, then Firm 1 is better off exiting. However, in the dynamic game, Firm 1 should not believe such an "empty threat". The crux of the matter is that the Nash equilibrium concept places no restrictions on players' behavior at nodes that are

[55] There are also some MSNE involving x.

never reached on the equilibrium path. In this example, given that Firm 1 is playing x, any action for Firm 2 is a best response, since all its actions are at a node that is never reached when Firm 1 places x. Thus, by choosing an action (f) that it certainly wouldn't want to play if it were actually forced to act, it can ensure that Firm 1's [unique, in this case] best response is to play x, guaranteeing that it in fact won't have to act.

4.2. Backward Induction and Subgame Perfection

The natural way to solve the problem above is to require that a player's strategy specify optimal actions at *every node of the game tree*. That is, when contemplating an action, a player takes as given that the relevant node has been reached, and thus should playing something that is optimal here on out (given her opponents' strategies). This is the principle of *sequential rationality* (which we define formally later). In Example 23, action f is not optimal conditional on the relevant node being reached; only a is. Thus, sequential rationality requires that 2's strategy be a, to which the unique best response for 1 to play e, resulting in the more reasonable outcome (e, a).

4.2.1. Backward Induction

In general, we can try to apply this logic to any extensive game in the following way: start at the "end" of the game tree, and work "back" up the tree by solving for optimal behavior at each node, determining optimal behavior earlier in the game by anticipating the later optimal behavior. This procedure is known as *backward induction*. In the class of finite games with perfect information (i.e. finite number of nodes and singleton information sets), this is a powerful procedure.

Theorem 8 (Zermelo)**.** *Every finite game of perfect information has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then backward induction results in a unique Nash equilibrium.*

Proof. The uniqueness part is straightforward. The rest is in MWG pp. 272–273. □

Zermelo's Theorem says that backward induction can be powerful in various *finite* games. For example it implies that even a game as complicated as chess is solvable through backward induction; in chess, one of the following mutually exclusive statements is true:

- White has a strategy that results in a win for him regardless of what Black does;
- Black has a strategy that results in a win for him regardless of what White does;
- Each player has a strategy that results in either a draw or a win for him regardless of what the other player does.

In this sense, chess is "solvable"; alas, no-one knows what the solution is![56,57]

[56] Note well what the application to chess is: it doesn't just say that every chess game must end in one of the players winning or in a draw (this is trivial, modulo the game ending—see the next footnote); it says that one of the players has a strategy that *guarantees* a win for the player (regardless of what strategy the other player uses), or both player have strategies that *guarantee* at least a draw (regardless of what the strategy the other player uses).

[57] There is actually a subtle caveat here with regards to real chess. Although game theorists often think of a chess as a finite game, and this is needed to apply the Theorem above directly, in fact the official rules of chess make it an infinite game. Nevertheless, the result above concerning chess (that either one player has a winning strategy, or both players have strategies that individually ensure at least a draw) is true. See Ewerhart (2002).

Corollary 4. *Every finite game of perfect information has a PSNE.*

As in illustration of how backward induction operates, here is a famous game.

Example 24 (Centipede Game)**.** Two players, 1 and 2, take turns choosing one of two actions each time, *continue* or *stop*. They both start with \$1 in their respective piles, and each time i says continue, \$1 is taken away from his pile, and \$2 are added to the other player's pile. The game automatically stops when both players have \$1000 in their respective piles. Backward induction implies that a player should say *stop* whenever it is his turn to move. In particular, Player 1 should say *stop* at the very first node, and both players leave with just the \$1 they start out with.[58] □

4.2.2. Subgame Perfect Nash Equilibrium

We are now going to define a refinement of Nash equilibrium that captures the notion of backward induction. To do so, we need some preliminaries. Recall from Definition 1 that an extensive form game, Γ_E, specifies a host of objects, including a set of nodes, $\mathcal{X}$, an immediate predecessor mapping $p(x)$ that induces a successor nodes mapping $S(x)$, and a mapping $H(x)$ from nodes to information sets.

Definition 28 (Subgame)**.** A subgame of an extensive form game, Γ_E, is a subset of the game such that

1. (a) there is a unique node in the subgame, x^*, such that $p(x^*)$ is not in the subgame. Moreover, (b) x^* is not a terminal node, and (c) $H(x^*) = \{x^*\}$;
2. a node, x, is in the subgame if and only if $x \in \{x^*\} \cup S(x^*)$;
3. if node x is in the subgame, then so is any $\tilde{x} \in H(x)$.[59]

Remark 25. Notice that any extensive form game as whole is always a subgame (of itself). Thus, we use the term *proper subgame* to refer to a subgame where $x^* \neq x_0$ (recall that x_0 is the root of the game).

Exercise 9. *Draw an extensive form game and indicate three different parts of it that respectively each fail the three components of the definition of a subgame.*

The key feature of a subgame is that it is a game in its own right, and hence, we can apply the concept of Nash equilibrium to it. We say that a strategy profile, σ, in the game Γ_E *induces* a Nash equilibrium in a particular subgame of Γ_E if the [probability distribution over] moves specified by σ for information sets in the subgame constitute a Nash equilibrium when the subgame is considered as a game by itself.

Definition 29 (Subgame Perfect NE)**.** A Nash equilibrium, σ^*, in the extensive form game, Γ_E, is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in *every* subgame of Γ_E.

In finite extensive form games with possibly imperfect information, we conduct *generalized backward induction* as follows:

1. Consider the maximal subgames (that is, subgames that have no further proper subgames) and pick a Nash equilibrium in each maximal subgame (one exists! why?).

[58] Experiments with this game show that most players tend to continue until there is a substantial sum of money in both their piles, and then one will say *stop*, almost always before the game automatically stops.

[59] Actually, this condition together with condition 1(a) implies 1(c). Can you prove it?

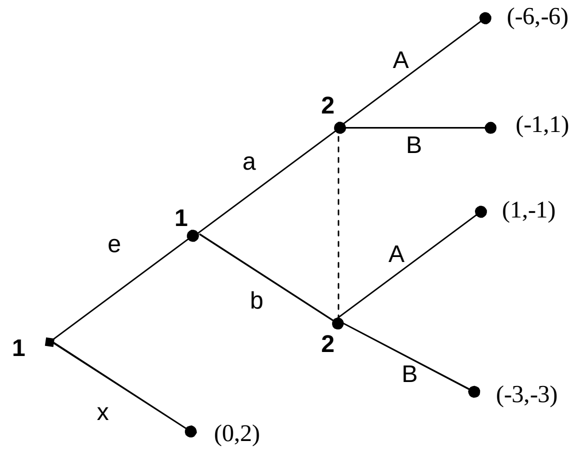


Figure 2 – Extensive Form for Predation with Niches Game

2. Replace each maximal subgame with a "terminal" node that has the payoffs of the Nash equilibrium we picked in the subgame. After replacement, call this a "reduced" game.
3. Iterate the process on a successive sequence of "reduced" games until the whole tree has been replaced with a single terminal node.

It should be intuitive that the process of generalized backward induction yields a SPNE of a game of imperfect information, and conversely every SPNE survives generalized backward induction.[60] The following example illustrates the idea.

Example 25 (Predation with Niches)**.** This is an extended version of the Predation game from Example 23. Now, Firm 1 (the entrant) first chooses to enter or not. If it enters, then the two firms simultaneously choose a niche of the market to compete in: a (A) or b (B). Niche b is the "larger" niche. The extensive form and payoffs are drawn in Figure 2.

To find the pure strategy SPNE equilibria of this game, we employ generalized backward induction as follows. Notice that there is only one proper subgame here. We can write out the normal form of this proper subgame as follows.

	A	B
a	-6,-6	-1,1
b	1,-1	-3,-3

There are two PSNE in the subgame: (a, B) and (b, A). If we replace the subgame with a terminal node corresponding to (a, B) payoffs, then it follows that Firm 1 prefers to play x at its first move. If on the other hand we replace the subgame with a terminal node corresponding to (b, A) payoffs, then it follows that Firm 1 prefers to play e at its first move. Therefore, the two pure strategy SPNE of this game are (xa, B) and (eb, A). □

[60] See MWG Proposition 9.B.3 (p. 277) for a precise statement.

Since every subgame of a finite game has a Nash equilibrium, and a SPNE of the whole game is derived by "folding" together Nash equilibria of each subgame, we have the following existence result.

Theorem 9 (SPNE Existence)**.** *Every finite game has a SPNE.*

Remark 26. MWG — and other textbooks — tend not to mention the above result because existence of SPNE can also be derived as a corollary to the existence of *Sequential Equilibrium*, which we discuss later on.

Obviously, in games of perfect information, every SPNE can be derived through the basic backward induction process, since generalized backward induction reduces to this with perfect information. Using Zermello's Theorem, it follows that

Corollary 5 (Pure Strategy SPNE)**.** *Every finite game of perfect information has a pure strategy SPNE. Moreover, if no player has the same payoffs at any two terminal nodes, there is a unique SPNE.*

Exercise 10. *Recall the centipede game from Example 24. Prove that the unique SPNE is where each player plays "*stop*" at every node he is the actor at (this is simply an application of the results above!). Show that there are a plethora of Nash equilibria. However, prove that in every Nash equilibrium, Player 1 must play "*stop*" at the first node with probability 1 (hint: you have to consider mixed strategies).*

Example 26 (Finite Horizon Bilateral Bargaining)**.** Players 1 and 2 are bargaining over the split of $v > 0$ dollars. The game lasts a finite odd number of $T \in \{1, 3, \ldots\}$ periods. In period 1, player 1 offers player 2 some amount $b_1 \in [0, v]$, which player 2 can either accept or reject. If player 2 accepts, the game ends, with the proposed split enforced. If player 2 rejects, then we move onto period 2, where player 2 offers player 1 some amount $b_2 \in [0, v]$, and player 1 can either accept or reject. The game proceeds in this way for up to T periods; in the $T + 1$ period, the game necessarily ends and both players get nothing if agreement has not been reached. Players discount the future by a factor $\delta \in (0, 1)$, so that a dollar received in period t gives a payoff of δ^{t-1}. What is the set of SPNE?

Remarkably, there is a unique SPNE. (This is not a consequence of Zermello's Theorem.) We find the SPNE through backward induction. Start at the end: in the final period, T, player 2 weakly prefers to accept any offer $b_T \geqslant 0$, and strictly so if $b_T > 0$. Given this, the *only* offer for player 1 in period T that can be part of a SPNE is $b_T = 0$, which player 2 must respond to by accepting in a SPNE.[61] The corresponding payoffs in the subgame starting at period T are $(\delta^{T-1}v, 0)$. Now consider the subgame starting in period $T - 1$, where player 2 is the proposer. Player 1 weakly prefers to accept any offer that gives him a payoff of at least $\delta^{T-1}v$, and strictly so if it provides a payoff of strictly more than $\delta^{T-1}v$. Thus, the only offer for player 2 that can be part of a SPNE is $b_{T-1} = \delta v$, which player 1 must respond to by accepting in a SPNE. The corresponding payoffs in the subgame starting at period $T - 1$ are thus $(\delta^{T-1}v, \delta^{T-2}(1 - \delta)v)$. Now consider period $T - 2$. The above logic implies that the only offer that is part of a SPNE is such that $\delta^{T-3}b_{T-2} = \delta^{T-2}(1 - \delta)v$, i.e. $b_{T-2} = \delta(1 - \delta)v$, which must be accepted in a SPNE.

[61] Any $b_T > 0$ is strictly worse for player 1 than some $b_T - \varepsilon > 0$, which player 2 necessarily accepts in a SPNE. If player 2 rejects $b_T = 0$ with positive probability, then player 1 would strictly prefer to offer some $b_T = \varepsilon > 0$. Note that player 2 is indifferent when offered 0, but it is critical that we resolve his indifference in favor of accepting in order to have a SPNE.

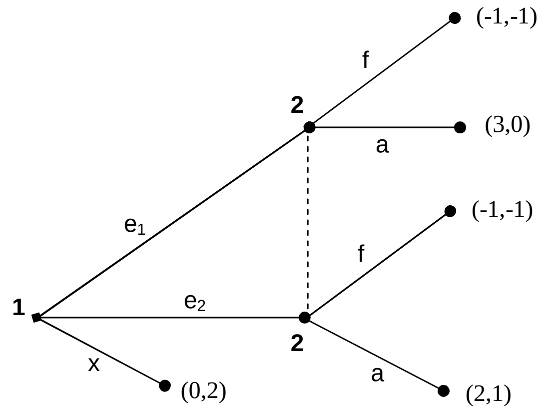


Figure 3 – Extensive Form for Predation version 2 Game

Continuing this process all the way back to period 1, by induction we see that there is a unique SPNE that involves the sequence of offers, $b_T^* = 0$ and for all $t \in \{1, \ldots, T-1\}$,

$$b_{T-t}^* = -v \sum_{\tau=1}^{t} (-\delta)^\tau = \frac{\delta(1-(-\delta)^t)}{1+\delta} v.$$

In any period t, the responder's accepts an offer if and only if it is at least b_t^*. Player 1's equilibrium payoff is therefore

$$\pi_1^* = v - b_1 = v \left[1 - \frac{\delta(1-(-\delta)^{T-1})}{1+\delta}\right] = v\frac{1+\delta^T}{1+\delta},$$

and player 2's equilibrium payoff is $\pi_2^* = b_1 = v\frac{\delta - \delta^T}{1+\delta}$.

Observe that player 1's equilibrium payoff is larger than player 2's; this because player 1 is both the first and last proposer—each aspect confers some benefit.[62] As $T \to \infty$, the equilibrium payoff vector $\to (v/(1+\delta), v\delta/(1+\delta))$, which in turn is $\approx (v/2, v/2)$ when $\delta \approx 1$. In other words, when the bargaining horizon is long and both players are patient, each gets close to half the pie. Impatience pushes in favor of the first proposer. $\square$

Exercise 11. *Construct a Nash equilibrium in the finite horizon bargaining game in which player 1 gets a payoff of* 0.

4.3. Systems of Beliefs and Sequential Rationality

A limitation of the preceding analysis is subgame perfection is powerless in dynamic games where there are no proper subgames.

Example 27 (Predation version 2)**.** Modify the Predation game in Example 23 so that Firm 1 now has two ways in which it can enter the market. The extensive form is in Figure 3. Now, the set of SPNE is exactly

[62] Notice that in a 2-period version of this problem, for large δ, the second proposer gets most of the pie in equilibrium; on the other hand, for small δ, the first proposer gets most of the pie in equilibrium.

the same as the set of NE, because there are no proper subgames to this game. Hence, the NE (x, f) is a SPNE. But (x, f) is no more plausible here than it was in the original Predation example; regardless of whether Firm 1 entered using e_1 or e_2, given that it has actually entered, Firm 2 is strictly better off playing a rather than f. □

4.3.1. Weak Perfect Bayesian Equilibrium

Accordingly, we need a theory of "reasonable" choices by players at all nodes, and not just at those nodes that are parts of proper subgames. One way to approach this problem in the above example is to ask: could f be optimal for Firm 2 when it must actually act for *any* belief that it holds about whether Firm 1 played e_1 or e_2? Clearly, no. Regardless of what Firm 2 thinks about the likelihood of e_1 versus e_2, it is optimal for it to play a. This motivates a formal development of beliefs in extensive form games. Recall the notation we use in such games: the set of nodes is $\mathcal{X}$, the set of successor nodes to any x is denoted $S(x)$, $\mathcal{H}$ is the set of information sets, $H(x)$ is the information set that a node x belongs to, and $\iota(H)$ is the player who acts at an information set H.

Definition 30 (System of Beliefs)**.** A system of beliefs is a mapping $\mu : \mathcal{X} \to [0, 1]$ such that for all $H \in \mathcal{H}$, $\sum_{x \in H} \mu(x) = 1$.

In words, a system of beliefs, μ, specifies the relative probabilities of being at each node of an information set, for every information set in the game. Obviously, $\mu(x) = 1$ for all x such that $H(x) = \{x\}$. That is, at singleton information sets, beliefs are degenerate.

Using this notion of beliefs, we can state formally what it means for strategies to be sequentially rational given some beliefs. Let $\mathbb{E}[u_i \mid H, \mu, \sigma_i, \sigma_{-i}]$ denote player i's expected utility starting at her information set H if her beliefs regarding the relative probabilities of being at any node, $x \in H$ is given by $\mu(x)$, and she follows strategy σ_i while the other plays follow the profile of strategies σ_{-i}. Think about this via behavioral strategies: conditional on any node in the information set H, the behavioral strategy σ induces a probability distribution over terminal nodes, and so some expected utility for player i; we integrate these expected utilities over all the nodes in H using weights given by μ.

Definition 31 (Sequential Rationality)**.** A strategy profile, σ, is sequentially rational at information set H given a system of beliefs, μ, if

$$\mathbb{E}[u_{\iota(H)} \mid H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geqslant \mathbb{E}[u_{\iota(H)} \mid H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Sigma_{\iota(H)}$.

A strategy profile is sequentially rational given a system of beliefs if it is sequentially rational at all information sets given that system of beliefs.

In words, a strategy profile is sequentially rational given a system of beliefs if there is no information set such that once it is reached, the actor would strictly prefer to deviate from his prescribed play, given his beliefs about the relative probabilities of nodes in the information set and opponents' strategies.

With these concepts in hand, we now define a *weak perfect Bayesian equilibrium.* The idea is straightforward: strategies must be sequentially rational, and beliefs must be derived from strategies whenever possible via Bayes rule. Recall that the statistical version of Bayes Rule (discrete case) says that given any events A, B, C, where B occurs with positive probability given that C does,

$$Prob(A|B,C) = \frac{Prob(B|A,C)Prob(A|C)}{Prob(B|C)}. \tag{6}$$

Definition 32 (Weak PBE)**.** A profile of strategies, σ, and a system of beliefs, μ, is a weak perfect Bayesian equilibrium (weak PBE), (σ, μ), if:

1. σ is sequentially rational given μ; and
2. μ is derived from σ through Bayes rule whenever possible. That is, for any information set H such that $Prob(H|\sigma) > 0$, and any $x \in H$,

$$\mu(x) = \frac{Prob(x|\sigma)}{Prob(H|\sigma)}. \tag{7}$$

The first part of the definition is the requirement of sequential rationality, and the 2nd part is the consistency of beliefs as embodied by Bayes Rule.[63] Keep in mind that strictly speaking, a weak PBE is a strategy profile-belief pair. However, we will sometimes be casual and refer to just a strategy profile as a weak PBE. This implicitly means that there is at least one system of beliefs such the pair forms a weak PBE.

Remark 27. The moniker "weak" in weak PBE is because absolutely no restrictions are being placed on beliefs at information sets that do not occur with positive probability in equilibrium, i.e. on *out-of-equilibrium* information sets. To be more precise, no consistency restriction is being placed; we do require that they be well-defined in the sense that beliefs are probability distributions. As we will see, in many games, there are natural consistency restrictions one would want to impose on out of equilibrium information sets as well.

To see the power of weak PBE, we return to the motivating example of Predation version 2.

Example 28 (Weak PBE in Predation version 2)**.** Recall that Example 27 had (x, f) as a SPNE. I claim that it is not a weak PBE equilibrium (or more precisely, there is no weak PBE involving the strategy profile (x, f)). This is proved by showing something stronger: that there is a unique weak PBE, and it does not involve (x, f). To see this, observe that any system of beliefs in this game, μ, can be described by a single number, $\lambda \in [0, 1]$, which is the probability that μ places on the node following e_1. But for any λ, the uniquely optimal strategy for Firm 2 is a. Hence, only a is sequentially rational for any beliefs. It follows that *only* (e_1, a) can be part of a weak PBE. Given this, we see that beliefs are also pinned down as $\lambda = 1$ by Bayes Rule. There is thus a unique strategy-belief pair that forms a weak PBE in this game. □

Proposition 8 (Nash and weak PBE)**.** *A strategy profile, σ, is a Nash equilibrium of an extensive form game if and only if there exists a system of beliefs, μ, such that*

1. *σ is sequentially rational given μ at all H such that* $\mathrm{Prob}(H|\sigma) > 0$ *(not necessarily at those H such that* $\mathrm{Prob}(H|\sigma) = 0$*); and*

[63] To see that the 2nd part of the definition is just Bayes Rule, think of the the left-hand side of (7) as $Prob(x|H, \sigma)$ and the right-hand side as $\frac{Prob(H|x,\sigma)Prob(x|\sigma)}{Prob(H|\sigma)}$ and then compare to (6). We are able to simplify because $Prob(H|x, \sigma) = 1$ by the hypothesis that $x \in H$.

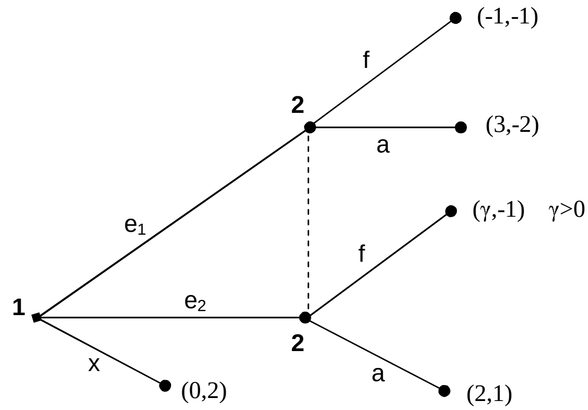


Figure 4 – Extensive Form for Predation version 3 Game

2. μ is derived from σ through Bayes rule at every information set H such that $\text{Prob}(H|\sigma) > 0$.

Proof. Exercise. □

Since we have already seen that not every NE is a weak PBE, and since the only difference in the above Proposition with the definition of a weak PBE is that sequential rationality is only required at a subset of information sets (those that occur with positive probability, rather than all of them), we immediately have the following corollary.

Corollary 6. *Every weak PBE is a Nash equilibrium; but not every Nash equilibrium is a weak PBE.*

(You might be wondering what we can say about SPNE and weak PBE equilibrium: Example 28 showed that a SPNE need not be a not be a weak PBE ... for the converse, hold off for just a little bit!)

The next example shows how to solve for weak PBE in a more complicated case than before.

Example 29 (Predation version 3)**.** This is yet another variant of a predation game. The extensive form is drawn in Figure 4. The key difference with before is that Firm 2's optimal action if it must act depends on whether Firm 1 played e_1 or e_2. To solve for weak PBE, let μ_1 be the probability the system of beliefs assigns to the node following e_1, let σ_f be Firm 2's probability of playing f, and σ_x, σ_{e_1} and σ_{e_2} denote the respective probabilities in Firm 1's strategy.

First, observe that x can never part of a weak PBE because e_2 strictly dominates x for Firm 1, hence it is never sequentially rational for Firm 1 to play x. Next, observe that it is sequentially rational for Firm 2 to put positive probability on f if and only if $-1(\mu_1) + -1(1-\mu_1) \geqslant -2\mu_1 + (1-\mu_1)$, i.e. if and only if $\mu_1 \geqslant \frac{2}{3}$. Now we consider two cases.

1. Suppose that $\mu_1 > \frac{2}{3}$ in a weak PBE. Then Firm 2 must be playing f (this is uniquely sequentially rational given the beliefs). But then, Firm 1 must be playing e_2 (since $\gamma > 0 > -1$), and Bayes rule requires $\mu_1 = 0$, a contradiction.
2. Suppose that $\mu_1 < \frac{2}{3}$ in a weak PBE. Then Firm 2 must be playing a (this is uniquely sequentially rational given the beliefs). But then, Firm 1 must be playing e_1 (since $3 > 2 > 0$), and Bayes rule beliefs requires $\mu_1 = 1$, a contradiction.

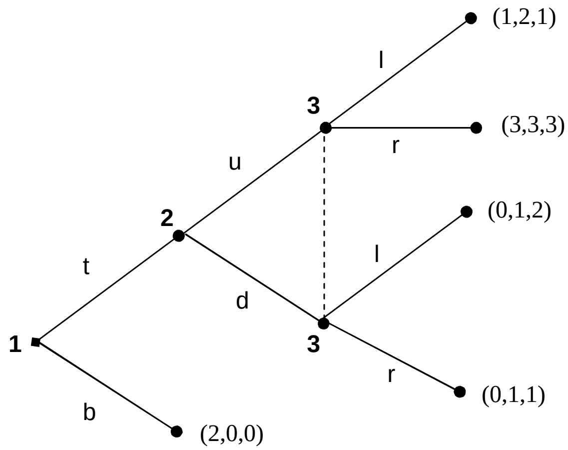


Figure 5 – Extensive Form for Example 30

Therefore, any weak PBE has $\mu_1 = \frac{2}{3}$. Given the earlier observation that $\sigma_x = 0$ in any weak PBE, Bayes rule holds if and only if $\sigma_{e_1} = \frac{2}{3}$ and $\sigma_{e_2} = \frac{1}{3}$. For this to be sequentially rational for Firm 1 requires it to be indifferent between e_1 and e_2, which is the case if and only if $-1\sigma_f + 3(1-\sigma_f) = \gamma\sigma_f + 2(1-\sigma_f)$, i.e. if and only if $\sigma_f = \frac{1}{\gamma+2}$.

We conclude that there is a unique weak PBE in this game. □

Exercise 12. *Solve for the weak PBE in the above game when $\gamma \in (-1, 0)$.*

Now we return to the issue of relating weak PBE to Nash equilibria, in particular, to SPNE. As we saw earlier in Example 28, not every SPNE is a weak PBE. The following example demonstrates that not every weak PBE is a SPNE.

Example 30 (wPBE $\nsubseteq$ SPNE)**.** In the game given by Figure 5, (b, u, l) is a weak PBE (with what beliefs?) but the unique SPNE is (t, u, r). [Question: what can you say about player 2's strategy in any wPBE?]

The intuition behind the example is simple: weak PBE places minimal restrictions on behavior at a subgame that is off the equilibrium path (i.e., not reached under the equilibrium strategies), while SPNE requires more. Loosely speaking, within an off-path subgame, weak PBE just rules out dominated actions, while SPNE requires mutual best responses. We have thus proved the following.

Proposition 9. *A weak PBE need not be a SPNE; a SPNE need not be a weak PBE.*

Exercise 13. *Prove that in any extensive form game of perfect information, the set of weak PBE is the same as the set of SPNE.*

The fact that a weak PBE may not be subgame perfect (in imperfect information games) motivates a strengthening of the solution concept. One strengthening is that of a *perfect Bayesian equilibrium* (PBE, without the "weak" moniker). This requires that the strategy profile-belief pair be not only a weak PBE,

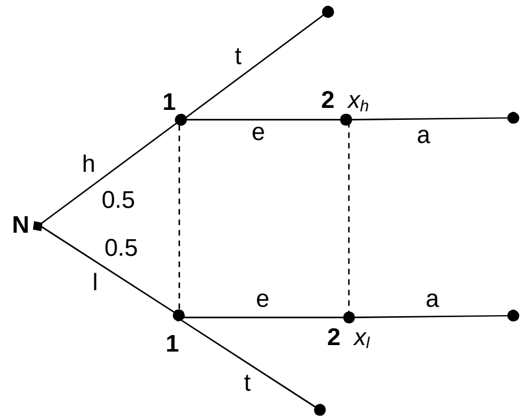


Figure 6 – Extensive Form for Example 31

but moreover induce a weak PBE in every proper subgame.[64] Clearly, this would assure subgame perfection. However, it is still too weak in its restrictions on beliefs off the equilibrium path, as the following example demonstrates.

Example 31 (Implausible Beliefs in PBE)**.** Figure 6 shows a game form.[65] Suppose payoffs were specified so that it were strictly dominant for player 1 to choose t. Then, there are a continuum of PBE, all with the same strategy profile, (t, a), but supported by different beliefs: any $\mu(x_h) \in [0, 1]$ and $\mu(x_l) = 1 - \mu(x_h)$. The reason we can do this is that there are no proper subgames, hence the set of PBE is the same as set of weak PBE, and no restrictions are placed on beliefs at information sets that are off the equilibrium path. However, it should be clear that the only reasonable belief for player 2 is $\mu(x_h) = \mu(x_l) = \frac{1}{2}$, since player 1's move cannot depend on N's choice, and N picks h and l with equal probabilities.[66] □

4.3.2. Sequential Equilibrium

A stronger equilibrium concept that we now introduce is that of *sequential equilibrium*, due to Kreps and Wilson (1982).

Definition 33 (Sequential Equilibrium)**.** A strategy profile, σ, and a system of beliefs, μ, is a sequential equilibrium (SE), (σ, μ), if:

1. σ is sequentially rational given μ;
2. There exists a sequence of *fully mixed* strategy profiles $\{\sigma^k\}_{k=1}^{\infty} \to \sigma$, such that $\{\mu^k\}_{k=1}^{\infty} \to \mu$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes rule.

[64] In textbook treatments, PBE is often defined as something that is a bit stronger than the definition I have given; that is, not only is it a weak PBE that is subgame perfect, but other subtle conditions are typically imposed. See for example Fudenberg and Tirole (1991a, pp. 331-333) or MWG (p. 452).

[65] Recall that this is called a game form rather than a game because payoffs are not specified.

[66] To keep it simple, this example is such that even the implausible PBE are outcome-equivalent (i.e. have the same stategy profile) to plausible PBE. But it is easy to construct variants where this need not be the case: see MWG Example 9.C.5 (p. 289) for one.

Part 1 of the definition is the same as that of weak PBE. The 2nd part is a consistency of beliefs notion that is more demanding than that of weak PBE.[67] To interpret it, first note that given a fully mixed strategy profile, every information set is reached with positive probability, and hence Bayes rule completely determines a belief system. Thus, the definition roughly requires that the equilibrium beliefs be "close to" beliefs that are fully determined via Bayes rule from a fully mixed strategy profile that is "near by" the equilibrium strategy profile. There is obviously a connection here with trembling hand perfect equilibrium,[68] since these "near by" fully mixed strategy profiles can be thought of as arising from mistakes in playing the equilibrium strategies. Since sequential equilibrium places more restrictions on belief consistency than weak PBE, it follows that *every sequential equilibrium is a weak PBE*. The converse is not true, as we now show by example.

Example 32. Recall Example 30 that had (b, u, l) as weak PBE strategies. Let us argue that the unique SE strategy profile is the unique SPNE: (t, u, r). Let σ_u and σ_d denote the probabilities used by player 2; H_3 denote the only non-singleton information set; and let the four decision nodes be denoted as x_1, x_2, x_{3u} and x_{3d} respectively. In any fully mixed strategy profile, σ, Bayes rule implies

$$\mu_\sigma(x_{3u}) = \frac{Prob(x_{3u}|\sigma)}{Prob(H_3|\sigma)} = \frac{Prob(x_{3u}|x_2,\sigma)Prob(x_2|\sigma)}{Prob(H_3|x_2,\sigma)Prob(x_2|\sigma)}.$$

[Read $Prob(x_{3u}|x_2, \sigma)$ as the probability that x_{3u} is reached given that x_2 has been reached and σ is the profile being played, and so forth.]

Canceling terms and noting that $Prob(H_3|x_2, \sigma) = 1$, we have

$$\mu_\sigma(x_{3u}) = Prob(x_{3u}|x_2, \sigma) = \sigma_u.$$

Thus, for any sequence of fully mixed profiles $\{\sigma^k\}_{k=1}^\infty$ that converges to a profile σ^*, the limit of the sequence of beliefs derived from Bayes rule necessarily has $\mu_{\sigma^*}(x_{3u}) = \sigma^*_u$. Since player 2 being sequentially rational imposes that $\sigma^*_u = 1$ in any SE, σ^*, it follows that $\mu_{\sigma^*}(x_{3u}) = 1$ and hence player 3 must play r in any sequential equilibrium. It is then uniquely sequentially rational for player 1 to play t in any SE. □

Remark 28 (Lower hemicontinuity of Beliefs in a SE). To get a better intuition for how sequential equilibrium works, it is useful to think about the mapping from mixed strategies to beliefs for the relevant player(s) in a game. Consider Example 30. Holding fixed a σ_u, we can think of the allowable beliefs in a weak PBE by defining the maximal belief correspondence $B(\sigma_t)$ which satisfies the property that $B(\sigma_t)$ is derived from Bayes Rule whenever possible, i.e. so long as $\sigma_t > 0$. Graphically, this is illustrated in Figure 7. The key

[67] It is possible to show that for any strategy profile σ, there is a system of beliefs satisfying this consistency requirement.

[68] Note, however, that we only defined THPE for normal form games. It turns out that a normal form THPE need not even be a SPNE. There is a definition of THPE for extensive form games that I won't pursue in detail (cf. MWG pp. 299-300). Briefly, an extensive form THPE is defined using normal form THPE but applied to the "agent normal form" of the extensive form, in which each information set of a player is treated as belonging to a separate copy of that player—so that trembles are independent across a player's information sets. The upshot is that extensive form THPE is slightly more demanding than sequential equilibrium (i.e., every extensive form THPE is a sequential equilibrium strategy profile, but not vice-versa); nonetheless, generically they coincide (i.e., if we fix a finite extensive game form, the two concepts produce different sets of equilibria for a set of payoffs of Lebesgue measure 0).

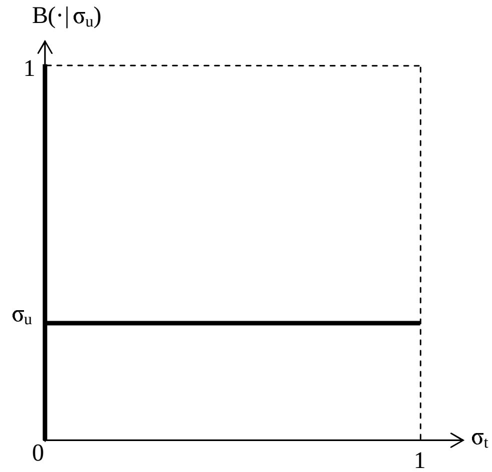


Figure 7 – SE and weak PBE Beliefs

point of course is that when $\sigma_t = 0$, Bayes Rule does not apply, and any $B(0) \in [0,1]$ is permitted in a weak PBE. However, SE requires that $B(0)$ be the limit of $B(\sigma_t)$ as $\sigma_t \to 0$. In other words, SE requires that we take a selection from correspondence B that satisfies *lower hemicontinuity*. Indeed, this was one of motivations behind how Kreps and Wilson (1982) arrived at their definition of SE.

Exercise 14. *Verify that in any SE in Example 31, $\mu(x_h) = \mu(x_l) = \frac{1}{2}$, and work through the lower hemicontinuity of beliefs idea for this case.*

Earlier, we noted that it is clear that every SE is a weak PBE. In fact, one can show that any SE is subgame perfect (MWG Proposition 9.C.2), hence it follows that every SE is a PBE (not just a weak PBE). We can summarize with the following partial ordering.

Theorem 10 (Solution Concepts Ordering)**.** *In any extensive form game, the following partial ordering holds amongst solution concepts:*

$$\{SE\} \subseteq \{PBE\} \subseteq \begin{array}{c}\{weak\ PBE\}\\ \bigcap \\ \{SPNE\}\end{array} \subseteq \begin{array}{c}\{weak\ PBE\}\\ \bigcup \\ \{SPNE\}\end{array} \subseteq \{NE\}\,.$$

It is fair to say that in extensive form games, sequential equilibrium is a fairly uncontroversial solution concept to apply, at least amongst those who are willing to accept any type of "equilibrium" concept (rather than sticking to just rationalizability, for example). However, it turns out that even sequential equilibrium may not be completely satisfying in some games.

Example 33 (Predation version 4)**.** This is another version of the Predation game, whose extensive form is drawn in Figure 8. I leave it as an exercise to show that (x, f) and (e_1, a) are both SE strategy profiles (the latter is straightforward; showing the former requires you to find a sequence of fully mixed strategies and implied beliefs that make the required equilibrium beliefs consistent). However, the equilibrium play (x, f) can be argued to be implausible. Observe that by playing e_2, player 1 *will get a payoff of no more than -1,*

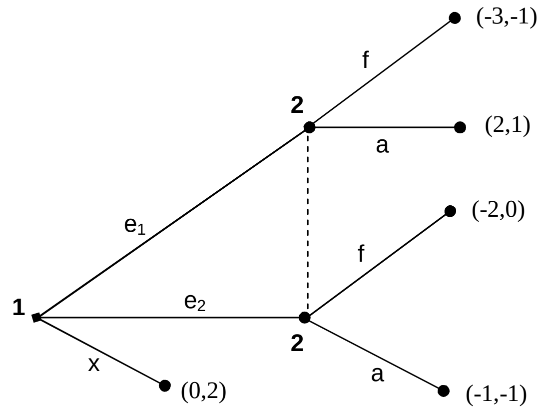


Figure 8 – Extensive Form for Predation version 4

with certainty. Hence, that strategy is strictly dominated by the strategy x. Accordingly, if player 2 must actually move in the game, he should realize that player 1 must have played e_1 (with the anticipation that player 2 will play a), to which his best response is indeed a. Given this logic, player 1 should play e_1 rather than x; hence (x, f) is not plausible. □

This is an example of *forward induction*, because we are requiring player 2 to reason about what he should play at his information set not just on the basis of what will happen *after* his move (that is backward induction, which is degenerate in this game for player 2 since he is the last mover), but moreover on the basis of what player 1 could have rationally done *prior* to 2's move.

Exercise 15. *Prove that in the Example 33, there is a SE with strategy profile* (x, f).

Applying forward induction to refine predictions in a game can be somewhat controversial.[69] However, as you will see when studying signaling games, without it, the set of (sequential) equilibria can be unbearably large; whereas applying it yields a narrow set that accords with intuition.[70]

5. Market Power

In this section, we are going look at a few static models of markets with a small number of firms. We've already seen some examples before: two-firm Bertrand and Cournot competition. The idea now is to

[69] This is partly because there is some tension between the motivating idea behind forward induction and the idea of trembles, which is used to intuitively justify SE. In particular, forward induction operates on the notion that an out-of-equilibrium action should be interpreted as a "rational" decision by a player, and subsequent players should ask themselves what to play based on this interpretation (think of the logic we used in Example 33). On the other hand, trembles are based on the idea that out-of-equilibrium actions result from "mistakes" rather than intended deviations. This is a subtle issue that is more suitable for an advanced course, but do come to talk to me if you are interested in discussing it more.

[70] For those of you more cynical: this is of course the usual rhetoric that means "accords with the dominant thinking of many people who have thought about this."

generalize those examples into some broad ideas of oligopoly markets. Throughout, we take a partial-equilibrium approach, focussing on only one market.

5.1. Monopoly

You've seen competitive market analysis earlier in the year; that is a benchmark against which we can view the effect of market power. We begin this section by looking at the other extreme where a single firm produces the good, known as *a monopoly*.

Market demand for the good at price $p \geqslant 0$ is given by a function $x(p)$, where $x : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. Implicit in this formulation is the assumption that consumers are not strategic players, and they are anonymous to the firm in the sense that the firm cannot charge separate prices to different consumers. We assume that $x(p) = 0$ for all $p \geqslant \bar{p} \in (0, \infty)$, and that $x(\cdot)$ is strictly decreasing on $[0, \bar{p}]$ (thus $x(q) = +\infty \implies q = 0$). The monopolist knows the demand function, and can produce quantity $q \in \mathbb{R}_+$ at a cost $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$. Define the *inverse demand function*, $p : [0, x(0)] \to \mathbb{R}_+$ by $p(q) = \min\{p : x(p) = q\}$. Observe that $p(0) = \bar{p}$, and for all $q \in (0, x(0)]$, $p(q) = x^{-1}(q)$, so that $p(\cdot)$ is strictly decreasing on $[0, x(0)]$.

Rather than writing the monopolist's objective as choosing price, it is convenient to take the equivalent approach of maximizing quantity, so that the objective is

$$\max_{q \in \mathbb{R}_+} p(q)q - c(q). \tag{8}$$

Under some assumptions, this problem will have a unique solution that can be found by analyzing the first order condition. The assumptions are:

i) $p(\cdot)$ and $c(\cdot)$ are twice differentiable on their domains. [So we can take 1st and 2nd derivatives.]

ii) $p(0) > c'(0)$. [So choosing $q = 0$ will not be optimal for the monopolist.]

iii) There is a unique $q^e \in (0, \infty)$ such that $p(q^e) = c'(q^e)$; and $p'(q^e) < c''(q^e)$. [This will be the socially optimal quantity, and also ensure a solution to the monopolist's problem.]

iv) $p''(q)q + 2p'(q) - c''(q) < 0$ for all $q \in (0, q^e]$. [Will ensure that FOC is sufficient.]

Assumption (iv) seems convoluted, but is satisfied under various more primitive assumptions. For example, it holds under linear demand ($p(q) = a - bq$) and convex costs.

Recall that in a competitive market, price = marginal cost, so that q^e defined above is the unique competitive market quantity, and also that the socially optimal quantity.[71] In contrast:

Proposition 10. *Under the stated assumptions, the monopolist's problem has a unique solution, $q^m \in (0, q^e)$, given by*

$$p'(q^m)q^m + p(q^m) = c'(q^m). \tag{9}$$

The monopolist produces less than the socially optimal quantity. Moreover, $p(q^m) > c(q^m)$, so that price under monopoly exceeds marginal cost.

[71] The latter part of Assumption (iii) is needed to guarantee that q^e is indeed a social optimum. Can you see why?

Proof. If a solution exists to (8), call it q^m, Assumption (i) implies that the objective is differentiable and must satisfy the FOC:

$$p'(q^m)q^m + p(q^m) \leqslant c'(q^m), \quad \text{with equality if } q^m > 0.$$

The LHS above is marginal revenue while the RHS is marginal cost. By Assumption (iii), $p(q) < c'(q)$ for all $q > q^e$, and since $p'(\cdot) < 0$, it follows that $q^m \in [0, q^e]$. Since a continuous function attains a maximum on a compact set, there is a solution to (8).

Assumption (ii) implies that $q^m > 0$ and the FOC must hold with equality. Assumption (iv) implies that the objective is strictly concave on $(0, q^e]$, hence there is a unique solution to the FOC. □

The key observation here is that a monopolist recognizes that by reducing quantity, it increases revenue on *all the units sold* because the price goes up on all units, captured by the term $p'(q^m)q^m$ in (9). On the other hand, the direct effect of reducing quantity only decreases revenue on the marginal unit, captured by the term $p(q^m)$ in (9). When the quantity is q^e, the marginal reduction in revenue is compensated equally by the cost savings, since $p(q^e) = c'(q^e)$. Thus, the *inframarginal* effect of raising revenue on all other units makes it is optimal for the monopolist to produce quantity below q^e. Note that the inframarginal effect is absent for firms in a competitive market.

The *deadweight welfare loss from a monopoly* can be quantified as

$$\int_{q^m}^{q^e} [p(q) - c'(q)]dq.$$

Graphically, this would be the region between the inverse demand curve and the marginal cost curve that is foregone under a monopoly relative to a competitive market.

Remark 29. As is suggested by the above discussion of inframarginal effects, the social inefficiency arising from a monopoly is crucially linked to the (often plausible) assumption that the monopolist must charge the same price to all consumers. If the monopolist could instead perfectly *price discriminate* in the sense of charges a distinct price to each consumer (knowing individual demand functions), then the inefficiency would disappear — although all the surplus would be extracted by the monopolist. See MWG (p. 387) for a formal treatment.

5.2. Basic Oligopoly Models

Let us now turn to oligopoly markets with at least 2 firms. We already studied price competition (Bertrand) and quantity competition (Cournot) with exactly two firms (a duopoly). The first task is to generalize to an arbitrary number of firms.

5.2.1. Bertrand oligopoly

The general Bertrand case is straightforward extension of the duopoly analysis and left as an exercise.

Exercise 16. *Consider the Bertrand model of Example 10, except that there are now an arbitrary number of $n \geqslant 2$ firms, each with symmetric linear costs. Assume that if multiple firms all charge the lowest price, they each get an equal share of the market demand at that price. Show that when $n > 2$, (i) there are pure* *strategy Nash equilibria where not all firms charge marginal cost; (ii) but in any Nash equilibrium, all sales take place at price = marginal cost (do it first for PSNE; then extend it to MSNE).*[72]

Thus, under Bertrand competition, we see that just two firms is sufficient to make the market perfectly competitive. Although this is striking, it doesn't seem realistic in some applications. As we will see next, the Cournot model is more satisfying in this regard (although arguably not as appealing in the sense that we often think about firms as choosing prices rather than quantities—though in some cases, quantity choice is not unreasonable).

5.2.2. Cournot oligopoly

We generalize the linear Cournot setting of Example 7 as follows. There are $n \geqslant 2$ identical firms. Each simultaneously chooses quantity $q_i \in \mathbb{R}_+$. Each has a twice differentiable, strictly increasing, and weakly convex cost function $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$. The inverse market demand given the *total quantity*, $Q = \sum_i q_i$, is given by $p(Q)$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is twice differentiable, strictly decreasing, and weakly concave for all $Q \geqslant 0$.[73] Assume that $p(0) > c'(0)$. Also assume that there is a unique socially efficient total quantity $Q^e \in (0, \infty)$. Social efficiency requires that the aggregate output be distributed efficiently across firms; since costs are weakly convex, one way to do this is to have each firm produce $\frac{1}{n}Q^e$ (this will the unique way if costs are strictly convex). Thus, we have $p(Q^e) = c'(\frac{1}{n}Q^e)$, i.e. the marginal cost for each firm is the price at the socially efficient aggregate quantity.

Now let us look for a symmetric pure strategy Nash equilibrium for the profit-maximizing firms in this market. Taking as given other firms' quantity choices, firm i's objective is to choose q_i to maximize

$$\pi_i(q_i, q_{-i}) = p(\sum_{j \neq i} q_j + q_i)q_i - c(q_i).$$

In a symmetric equilibrium, we cannot have $q_i = 0$, since $p(0) > c'(0)$. So a symmetric equilibrium must have $q_i > 0$ for all i. This means that the FOC must be satisfied with equality for each firm:

$$p'(\sum_{j \neq i} q_j + q_i)q_i + p(\sum_{j \neq i} q_j + q_i) - c'(q_i) = 0. \tag{10}$$

The assumptions we made guarantee that the problem is strictly concave when $q_i > 0$ (for any q_{-i}), so the FOC is also sufficient for a maximum. In a symmetric equilibrium, each firm produces the same quantity, $q^* = \frac{1}{n}Q^*$, where Q^* solves

$$p'(Q^*)\frac{1}{n}Q^* + p(Q^*) = c'(\frac{1}{n}Q^*). \tag{11}$$

Our assumptions guarantee that there is a unique solution $Q^* > 0$ to the above equation. We therefore conclude that:

Proposition 11. *$Q^e > Q^* > q^m$, where q^m is the monopoly quantity. Moreover, the market price is strictly above each firm's marginal cost.*

[72] For the MSNE argument, you can assume that that there is a well-defined monopoly price (i.e., $(p - c)Q(p)$ has a unique, finite maximum).

[73] The assumption that $p(\cdot)$ is weakly concave is not a conventional property of demand functions, although it is satisfied for linear demand. It is assumed to ensure that the payoff function for each firm is quasiconcave in its own quantity, which recall is an assumption to guarantee existence of PSNE in infinite games (Theorem 3).

Proof. If $Q^* = q^m$, the LHS of (11) is strictly greater than the LHS of (9); while the RHS is weakly smaller than that of (9). Thus equation (11) is not satisfied when $Q^* = q^m$. By concavity of LHS and convexity of RHS of (11), we must have $Q^* > q^m$.

If $Q^* = Q^e$, the RHS of (11) is equal to second term of the LHS, so is strictly smaller than the LHS, since $p' < 0$. Again, by concavity/convexity, we must have $Q^* < Q^e$.

That price is above marginal cost follows immediately from (11) and that $p' < 0$. □

Thus, the total quantity under Cournot oligopoly with 2 or more firms lies strictly between the socially optimal quantity and the monopoly quantity. The intuition is as follows: following the same logic as in the monopoly case, each firm has an incentive to reduce its individual quantity when the aggregate output is the social optimum, because of the inframarginal effect. However, as compared to the monopoly case, reducing quantity slightly only has an individual inframarginal benefit of $p'(Q)\frac{1}{n}Q$, since each firm has $\frac{1}{n}$ market share. This leads to a smaller reduction in quantity than would occur under a monopolist. To put it differently: when reducing quantity, each oligopolist has a positive externality on all the other firms (since it leads to an increase in price on all the units they sell) that it does not internalize.

This line of reasoning suggests that $n \to \infty$, the incentive to reduce quantity starting at the social optimum would vanish, since each individual firm would only be producing a very small quantity and thus the individual inframarginal gain is small. On the other hand, one might worry that because there are more and more firms, even if any individual firm's quantity reduction (relative to the social optimum) vanishes, the total quantity reduction does not. This turns out not to be the case under so long as the socially optimal quantity remains bounded.[74] In the proposition below, we introduce a subscript index for the number of firms.

Proposition 12. *Assume that* $\{Q^e_n\}_{n=1}^{\infty}$ *is bounded. Then* $\lim_{n\to\infty} Q^*_n = \lim_{n\to\infty} Q^e_n$.

Proof. Rewrite equation (11) as

$$-p'(Q^*_n)\frac{1}{n}Q^*_n = p(Q^*_n) - c'(\frac{1}{n}Q^*_n).$$

Let a finite upper bound on Q^e_n be $\bar{Q}^e$. Proposition 11 implies that for any n, $Q^*_n \in [q^m, \bar{Q}^e]$, hence Q^*_n is uniformly bounded. Since p is twice differentiable, $p'(Q^*_n)$ is also uniformly bounded. Thus the LHS above converges to 0 as $n \to \infty$. So the RHS must, as well. This implies the result. □

Exercise 17. *Using the linear Cournot model of Example 7, but now allowing for $n \geqslant 2$ identical firms, solve explicitly for the symmetric PSNE quantities and market price. Verify directly the statements of Proposition 11 and Proposition 12 for this example.*

5.3. Stackelberg Duopoly

In a Stackelberg duopoly, both firms choose quantities, but one firm moves first and the second observes the first mover's choice before making its choice. The important point is that the first mover is able to make a commitment to its quantity that the second mover must incorporate into its choice. The standard

[74] The socially optimal quantity could, in principle, go to ∞ even though demand does not change, because by adding firms we are changing the aggregate production technology. For instance, when $c(\cdot)$ is strictly convex with $c(0) = c'(0) = 0$, then with a very large number of firms, society can produce large quantities at close to zero total cost by making each firm produce a very small quantity.

application is to a market where there is an incumbent who is facing a potential entrant, with the incumbent being able to pre-commit before the entrant arrives.

To illustrate the main idea, consider a linear market competition model, where inverse demand is given by $p(Q) = \max\{0, a - bQ\}$, and two identical firms have linear costs $c(q_i) = cq_i$, where $a, b, c > 0$ and $a > c$. Firms choose non-negative quantities, but firm 1 chooses its quantity before firm 2 (the game is one of perfect information). We call this the linear Stackelberg Duopoly. Note that a strategy for firm 2 is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$.

We begin by observing that there there are plenty of pure strategy Nash equilibria.

Exercise 18. *Show that in the linear Stackelberg Duopoly, for any* $(q_1, q_2) \in \mathbb{R}^2_+$ *such that* $q_1 \leqslant \frac{a-c}{b}$ *and* $q_2 = \frac{a-c-bq_1}{2b}$*, there is a pure strategy Nash equilibrium where firm 1 chooses* q_1 *and firm 2 chooses* q_2 *on the equilibrium path.*

However, most of these Nash equilibria rely upon incredible threats by firm 2: they are not subgame perfect.

Proposition 13. *The linear Stackelberg Duopoly has a unique SPNE: firm 2 plays* $s_2(q_1) = \max\{0, \frac{a-c-bq_1}{2b}\}$ *and firm 1 plays* $s_1 = \frac{a-c}{2b}$*. The equilibrium path quantity choices are therefore* $q_1 = \frac{a-c}{2b}$ *and* $q_2 = \frac{a-c}{4b}$*, so that total equilibrium quantity is* $\frac{3(a-c)}{4b}$*.*

Proof. Exercise. □

Thus, in the linear Stackelberg duopoly, we see that firm 1 is more aggressive (produced more) and firm 2 is less aggressive (produces less) than in the Cournot duopoly. Total market quantity is between the total of a Cournot duopoly ($\frac{2(a-c)}{3b}$) and the efficient quantity ($\frac{a-c}{b}$). You can verify that firm 1's profit is higher here than in the Cournot game, and also higher than firm 2's. On the other hand, firm 2 does worse here than in the Cournot game.

The themes apply to general Stackelberg duopoly settings (not just this linear case). The main point is that there is a *first mover advantage* to firm 1. Since quantity choices are *strategic substitutes* (each firm's best response is a decreasing function of the other firm's quantity), the first mover is able to exploit its ability to commit by increasing its quantity relative to the Cournot individual quantity, ensuring that the second mover will reduce its quantity in response.

Remark 30. Notice that we have restricted attention to quantity competition above in treating the Stackelberg setting. This is the standard practice. But, one could study sequential moves by the two firms with price competition. I will assign a homework problem on this.

5.4. Price Competition with Endogenous Capacity Constraints

In studying Bertrand competition to this point, we have assumed that a firm can produce any quantity that is demanded at the price it charges. From a short run perspective, this is unrealistic: given that various factors of production are fixed in the short run (after all, that is the definition of short run), there may be an upper bound on how much a firm can supply, i.e. there may be a *capacity constraint*. In this section, I mention how capacity constraints can be used to "bridge" the Bertrand and Cournot models: intuitively, the quantity choices in Cournot can be viewed as long-run choices of capacity, so that firms choose prices in the short-run a la Bertrand given these capacity choices.

The following example illustrates how capacity choices can fundamentally change the nature of Bertrand competition.

Example 34. Consider the linear Bertrand duopoly model of Example 10. Suppose now that each firm is exogenously capacity constrained so that it cannot produce more than $\bar{q} > 0$ units. Without specifying all the details of how the market works, let us assume only that if $p_i > p_j$, then firm j will sell at price p_j a quantity that does not exceed its capacity, and if this is not sufficient to cover the market demand at price p_j, then firm i will sell to some strictly positive mass of consumers at price p_i.

The main observation is that if $\bar{q} \geqslant Q(c)$, then there is a PSNE where both firms price at marginal cost. But if $\bar{q} < Q(c)$, then it is no longer a Nash equilibrium for both firms to price at marginal cost. Why?

As you will see in a homework problem, for an appropriate specification of how the market works and a wide range of capacity constraints, $\bar{q}_1$ and $\bar{q}_2$ (this allows the constraints to differ across firms), the unique equilibrium in the pricing game involves both firms setting an identical price equal to $p(\bar{q}_1 + \bar{q}_2)$, where $p(\cdot)$ is the inverse demand function. In other words, the outcome of Bertrand with exogenous capacity constraints is just like what happens in a Cournot model if firms happen to choose quantities equal to these capacity constraints! But now, consider a prior stage where firms get to endogenously choose their capacity constraints, anticipating that after these are mutually observed, they will play a subsequent Bertrand-with-capacity-constraints pricing game. Intuitively, SPNE of this two-stage game have to yield capacity choices equal to the Nash quantity choices of the Cournot model, since in any subgame following capacity choices (q_1, q_2), the resulting payoffs are the same as in the Cournot model with quantity choices (q_1, q_2). See Kreps and Scheinkman (1983) for a thorough analysis. This formally justifies the interpretation of Cournot quantity competition as capturing long-run capacity choices followed by short-run Bertrand price competition.

Remark 31. In the above two-stage model, it is essential that the capacity choices are mutually observed before the price competition. If firms cannot observe each other's capacities when choosing prices, then it is as though they choose capacities and prices simultaneously. Convince yourself that in such a variation, Nash equilibria must yield zero profit (marginal cost pricing), just like unconstrained Bertrand competition.

6. Repeated Games

An important class of dynamic games are so-called *repeated games*. They are used to study strategic interactions that are ongoing over time. For instance, a typical application of the Prisoner's Dilemma or Trust Game (Example 4) is to represent a situation where two parties have to exert some individually costly effort in order to achieve a socially beneficial outcome, but there is an incentive to free ride on the other party. Often, such interactions do not occur once and for all; rather they occur repeatedly, and each player is able to observe something about the past history of play and condition her own behavior on that information.[75] This implies that actions are strategically linked over time. Going back to the Trust Game, one intuition is that under repeated interaction, a player may not Cheat because she fears that her partner will retaliate by in turn Cheating in the future. (We will see under what conditions this reasoning can be justified.) As a general principle, the inter-temporal strategic linkage can result in a much richer set of possible behavior than mere repetition of the static game prediction.

[75] For example, you can think about your own experience exerting effort in a study group.

6.1. Description of a Repeated Game

Stage Game. Formally, a repeated game consists of repetitions of a *stage game*. Although the stage game itself could be a dynamic game, we will focus on the normal-form representation of the stage game, so that a stage game is a one-period simultaneous move game of complete information given by $\langle I, \{A_i\}, \{\pi_i\}\rangle$, where I is the set of players, each A_i is the set of actions (or pure strategies of the stage game) for player i, and $\pi_i : A \to \mathbb{R}$ is the stage-game von Neumann- Morgenstern utility function for player i (where $A := A_1 \times \cdots \times A_I$). Assume the stage game is finite in the sense that I and A are finite. As usual, this can be relaxed at the cost of technical complications.[76] In the standard way, we can extend each $\pi_i : \Delta(A) \to \mathbb{R}$. I will use the notation α_i to denote an element of $\Delta(A_i)$ and refer to this as a *mixed action*.

Repeated Game. A repeated game (sometimes also referred to as a *supergame*) is formed by a repetition of the stage game for $T \in \{1, \dots, \infty\}$ periods. If T is finite, we call it a *finitely repeated game*; if $T = \infty$, we call it an *infinitely repeated game*. We will only consider repeated games of *perfect monitoring*: this means that at the end of every period, every player observes the actions chosen by all players in that period.[77]

Strategies. It is convenient to represent strategies in the repeated game as behavioral strategies. Denote the history of actions at period t as $h^t = \{a^1, \dots, a^{t-1}\}$, where for any t, $a^t = (a_1^t, \dots, a_I^t)$. Thus, a_i^t refers to player i's action at time t. Let H^t denote the space of all possible histories at time t, and let $H := \bigcup_{t=1}^T H^t$ denote the space of all possible histories in the repeated game. A (pure) strategy for player i in the repeated game can be represented as a function $s_i : H \to A_i$. In words, it specifies what action to take in any period following any history at that period. A (behavioral) mixed strategy can be represented as $\sigma_i : H \to \Delta(A_i)$.

Payoffs. If T is finite, then we could represent payoffs as usual via terminal nodes. But in an infinitely repeated game, there are no terminal nodes. So we need to take a different approach. Notice that any pure strategy profile in the repeated game maps into a unique profile of actions in each period (i.e., a unique path through the game tree), and a mixed strategy maps into a distribution over actions in each period. Thus, any (possibly mixed) strategy profile generates a sequence of expected payoffs for each player in each period. It suffices therefore to specify how a player aggregates a sequence of period-payoffs into an overall utility index. The standard approach is to assume exponential discounting, so that a sequence of expected payoffs $(v_i^1, v_i^2, \dots)$ yields player i an aggregate utility of

$$\tilde{u}_i(\{v_i^t\}) = \sum_{t=1}^{T} \delta^{t-1} v_i^t,$$

[76] For example, if A is not finite, one needs to assume that a Nash equilibrium exists in the stage game, and also make a boundedness assumption on payoffs.

[77] Repeated games of *imperfect monitoring*, or *unobservable actions*, are important but beyond the current scope. An excellent textbook reference is Mailath and Samuelson (2006).

where $\delta \in [0, 1)$ is the discount factor, assumed to be the same across players for simplicity.[78] When dealing with infinitely repeated games, it is useful to normalize this into *average discounted payoffs*,

$$u_i(\{v_i^t\}) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_i^t. \tag{12}$$

To understand the normalization, let U_i be the value of (12) for some sequence of (possibly time-varying) payoffs in each period. Now if we insert the constant U_i in place of v_i^t for all t into (12), we would get $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}U_i$, which evaluates precisely as U_i. Thus, in the infinite horizon, we are justified in interpreting (12) as the average discounted payoff: it gives a utility level such that if the player received that utility level in every period, he would have the same average discounted utility as the discounted average of the original stream of period-utilities. Notable then is that by using the average discounted payoff we put the entire repeated game payoff on the "same scale" as any one period.

Remark 32. Above, I have assumed that $\delta < 1$. This means that periods are not treated symmetrically, since a player has a preference for the same period-payoff sooner than later. For finitely repeated games there is no difficulty whatsoever with allowing $\delta = 1$. For infinitely repeated games, the difficulty is that when $\delta = 1$, total payoffs can be unbounded (even though stage-game payoffs are bounded), which can create problems. Nevertheless, if one wants to treat periods symmetrically in an infinitely repeated, there are criteria that can be used instead of exponential discounting, but I will not pursue those here.[79]

Remark 33. The previous remark also points to why the normalization from $\tilde{u}_i$ to u_i (by multiplying by $(1-\delta)$) is useful for infinitely repeated games: it allows us to take limits as $\delta \to 1$ while keeping discounted payoffs bounded.

Remark 34. The discount factor can be interpreted in the literal way of time preference. But there is an another interpretation: it can capture uncertainty about when the game will end. That is, an infinitely repeated game can be interpreted as situation where players know that the game will end in finite time with probability one, but are unsure of exactly when it will end. To be specific, suppose that players have a discount factor of $\rho \in [0, 1]$ and in addition, also think that conditional on the game having reached period t, it will continue to period $t+1$ with probability $\lambda \in [0, 1]$ (if the game does not continue, we normalize all payoffs thereafter to zero). Under the condition that either λ or ρ is strictly less than one 1, one can show that this induces the same expected payoff as a setting where the game continues with probability 1 in each period, but players have a discount factor $\delta = \rho\lambda$.

Given this setup for repeated games, we will look at Nash and SPNE equilibria as usual. Observe that the beginning of every period $t > 1$ marks a proper subgame, and these are the only proper subgames because since we have assumed simultaneous moves in the stage game.

[78] To write payoffs directly as a function of mixed strategies, we would therefore write

$$\tilde{u}_i(\sigma) = \sum_{t=1}^{T}\delta^{t-1}\sum_{h^t \in H^t}\Pr(h^t|\sigma)\pi_i(\sigma(h^t)).$$

[79] Probably the most common alternative is known as *limit of means criterion*, which evaluates a sequence of period-payoffs by the formula $\lim_{T\to\infty}\inf\frac{1}{T}\sum_{t=1}^{\infty}v_i^t$ (note: lim inf because limit may not exist); another alternative is known as the *overtaking criterion* which just uses $\lim_{T\to\infty}\inf\sum_{t=1}^{\infty}v_i^t$.

6.2. The One-Shot Deviation Principle

An essential result in the study of repeated games—indeed, dynamic games in general—is the *one-shot deviation principle*. It says that when considering profitable deviations for any player from a SPNE, it suffices to consider strategies where he plays as he is supposed to at all information sets except one, i.e. he only behaves differently at a single history. To state the idea precisely, let $\sigma_i|_{h^t}$ be the restriction of strategy σ_i to the subgame following history h^t.

Definition 34 (Profitable One-Shot Deviation)**.** Fix a strategy profile σ. A profitable one-shot deviation for player i is a strategy $\sigma_i' \neq \sigma_i$ s.t.

1. there is a unique history $h^{t'}$ such that for all $\tilde{h}^t \neq h^{t'}$, $\sigma_i'(\tilde{h}^t) = \sigma_i(\tilde{h}^t)$.
2. $u_i(\sigma_i'|_{h^{t'}}, \sigma_{-i}|_{h^{t'}}) > u_i(\sigma|_{h^{t'}})$.

In the Definition above, the first part is what it means to be a one-shot deviation: there is exactly one history at which the strategies differ. Note, however, that even if a deviation is one-shot, it can have a significant effect on the path of play, since the behavior at histories subsequent to $h^{t'}$ can depend significantly on exactly what what was played at $h^{t'}$ (we'll see examples later). The second part of the definition is what it means to for the deviation to be profitable. Note here that the profitability is defined *conditional* on the history $h^{t'}$ being reached, even though it may not actually be reached given the profile σ. This means that a Nash equilibrium can have profitable one-shot deviations off the equilibrium path. Yet:

Proposition 14 (One-Shot Deviation Principle)**.** *A strategy σ is a SPNE if and only if there are no profitable one-shot deviations.*

Proof. The "only if" is immediate: if there is a profitable one-shot deviation from σ, it cannot be a SPNE. What needs to be shown shown is the "if" part. We prove the contrapositive: if σ is not a SPNE then there is a profitable one-shot deviation.

So pick any strategy profile σ and suppose it is not a SPNE. Then there exists a deviation $\tilde{\sigma}_i$ that is profitable for some player i in a subgame following some history $\hat{h}^t$. Without loss of generality, assume that $\hat{h}^t = \varnothing$, so that the deviation is profitable in the entire game (this is wlog because we can just "zoom in" on the subgame and treat it as an entire game restricting everything that follows to the subgame). Let $\varepsilon > 0$ be the discounted utility gain from the deviation in absolute (rather than average) terms, i.e. $\varepsilon := \tilde{u}_i(\tilde{\sigma}_i, \sigma_{-i}) - \tilde{u}_i(\sigma)$. Since the stage game is finite,[80] there exists some (potentially large) $T^* < \infty$ s.t.

$$\frac{\delta^{T^*}}{1-\delta}\left(\max_{a\in A}\pi_i(a) - \min_{a\in A}\pi_i(a)\right) < \frac{\varepsilon}{2}.$$

This implies that at least half the discounted utility gain from deviating to $\tilde{\sigma}_i$ must accrue to player i by period T^* — for no matter how big the gains in every period after T^*, the discounted value is less than $\frac{\varepsilon}{2}$.[81] Hence, there must exist a profitable deviation $\hat{\sigma}_i$ that differs from σ_i at only a finite number of histories.

[80] or, more generally, by an assumption that payoffs in the stage game are bounded

[81] There is nothing important about choosing a half. The point is that the gains from the deviation in terms of period-payoffs cannot accrue only too far out in the future, because of discounting. Note that this holds if the game is finitely repeated even if there is no discounting.

Now we use induction to complete the proof. Look at any history, $h^{t'}$, such that there is no possible later history at which $\hat{\sigma}_i$ differs from σ_i. (This is well-defined because $\hat{\sigma}_i$ and σ_i only differ at a finite number of histories.) If we modify $\hat{\sigma}_i$ by only replacing $\hat{\sigma}_i(h^{t'})$ with $\sigma_i(h^{t'})$, is this new strategy still a profitable deviation from σ_i? There are only two possibilities:

- If the answer is NO, we are done, since we could take σ_i, switch it at only history $h^{t'}$ with $\hat{\sigma}_i(h^{t'})$, and we have created a profitable one-shot deviation.
- If the answer is YES, redefine $\hat{\sigma}_i$ by just replacing the behavior at $h^{t'}$ with $\sigma_i(h^{t'})$, and go back to the inductive step. Eventually, we must hit the previous bullet and would have found a profitable one-shot deviation. $\square$

Remark 35. That the game is a repeated game is not important for the one-shot deviation principle. The argument applies to various classes of extensive form games that are either finite, or more generally, satisfy a "*continuity-at-infinity*" condition that is ensured by discounting.[82] The canonical class is referred to as multi-stage games with observable actions: roughly, the only nonsingleton information sets capture simultaneous moves. See Fudenberg and Tirole (1991a).

The one-shot deviation principle is very useful because it means that we can always focus on "simple" deviations when thinking about SPNE. Let me stress that the one-shot deviation principle does *not* apply to Nash equilibria; we'll see an example later.

6.3. A Basic Result

The one-shot deviation principle allows us to make a simple but useful observation: repetition of stage-game Nash equilibria is a SPNE of a repeated game. To state this precisely, say that a strategy profile σ is *history-independent* if for all h^t and $\tilde{h}^t$ (i.e., any two histories at the same time period), $\sigma(h^t) = \sigma(\tilde{h}^t)$.

Proposition 15. *A history-independent profile, σ, is a SPNE if and only if for each t, $\sigma(h^t)$ (for any h^t) is a Nash equilibrium of the stage game.*

Proof. For the if part, observe that because each $\sigma(h^t)$ is a Nash equilibrium of the stage game, there is no profitable one-shot deviation, and hence no profitable deviation by the one-shot deviation principle. For the only if part, suppose that some $\sigma(h^t)$ is not a Nash equilibrium of the stage game. Then some player i has a profitable deviation where he deviates at h^t but otherwise plays just as σ_i; since all others are playing history-independent strategies, this is indeed a profitable deviation in the subgame starting at h^t. $\square$

It is important to emphasize two things about the Proposition: first, it applies to both finitely- and infinitely-repeated games; second, it does not require that the *same* Nash equilibrium be played at every history; only that in any given period, the Nash equilibrium being played not vary across possible histories at that period. So, for example, if we consider a finitely- or infinitely-repeated version of Battle of the Sexes (Example 16), the following strategies form a SPNE: in every odd period, regardless of history, row plays B and column plays b; in every even period, regardless of history, they play C and c respectively.

[82] Here is a (somewhat contrived, but perfectly valid) counter-example illustrating why some condition is needed in infinite games: suppose a single player makes an infinite sequence of decisions of Left or Right. If he chooses Left an infinite number of times, he gets a payoff of 1, otherwise he gets 0. Then there is a profitable deviation from the strategy of playing Right always (e.g., deviate to playing Left always), but there is no profitable one-shot deviation.

Consequently, there are an infinite number of pure strategy SPNE in the infinitely repeated Battle of Sexes, or a large number in the finitely repeated case when there are many periods.

Although Proposition 15 is not terribly exciting, one reason to state it is that it implies existence of SPNE in infinitely repeated games (under our maintained assumptions).

Corollary 7. *If the stage game has a Nash equilibrium,*[83] *then the repeated game has a SPNE.*

6.4. Finitely Repeated Games

Proposition 15 doesn't say anything about strategies that are not history independent. Moreover, in games with a unique stage game Nash equilibrium (such as Matching Pennies or the Trust Game), it doesn't offer existence beyond repetition of the same (possibly mixed) action profile. As it turns out, there are no other SPNE in such cases for finitely repeated games!

Proposition 16. *Suppose $T < \infty$ and the stage game has a unique (possibly mixed) Nash equilibrium, α^*. Then the unique SPNE of the repeated game is the history-independent strategy profile σ^* s.t. for all t and h^t, $\sigma^*(h^t) = \alpha^*$.*

Proof. The argument is by generalized backward induction. In any SPNE, we must have $\sigma^*(h^T) = \alpha^*$ for all h^T. Now consider period $T-1$ with some history h^{T-1}. There are no dynamic incentive considerations, since no matter what players do at this period, they will each get $\pi_i(\alpha^*)$ in the last period. Thus each player must be playing a stage-game best response to others' play in this period. Since α^* is the unique stage-game Nash equilibrium, we must have $\sigma^*(h^{T-1}) = \alpha^*$; otherwise there is a profitable deviation for some player. Induction yields the same conclusion for all periods. □

The Proposition implies, for instance, that no matter how many periods it is repeated, so long as there are only a finite number of repetitions, the Trust Game has a unique SPNE where each player plays Cheat in every period.

On the other hand, we can have interesting history-*dependent* play in finitely repeated games if there are multiple Nash equilibria of the stage game. In such cases, there can be SPNE in the repeated game where some periods involve action profiles being played that are not Nash equilibria of the stage game. The following example illustrates some general principles about sustaining "good" outcomes in repeated games.

Example 35 (Modified Prisoner's Dilemma)**.** Let $\delta = 1$ (no discounting) and $T = 2$ and consider the following stage game:

	L	C	R
T	10,10	2,8	-5,13
M	8,2	5,5	0,0
B	13,-5	0,0	1,1

There are two PSNE in the stage game: MC and BR. This has a Prisoner's Dilemma flavor because in the stage game, both players would be better off if they could somehow manage to play TL, but the problem is that each has a unilateral deviation from that profile that is profitable (B and R respectively). So the stage game PSNE are pareto-inefficient. From Proposition 15 we know there are at least four history-independent

[83] To be clear: this is automatic if the stage game is finite, but the statement is intended to emphasize that the result also applies to infinite stage games that have Nash equilibria.

SPNE of the repeated game, where the outcomes are playing MC in each period, BR in each period, or MC in one period and BR in the other.

But we can actually do better: there is a SPNE of the repeated game where TL is played in the first period. Consider the following strategy profile: TL is played in the first period, and in the 2nd period, $s^2(TL) = MC$ and $s^2(h^2) = BR$ for all $h^2 \neq TL$. Clearly, there are no profitable deviations in the 2nd period alone, since MC and BR are both Nash equilibria of the stage game (why would it not suffice to only note that MC is?). In the first period, a player's expected payoff from following his prescribed strategy (given that the opponent is playing his prescribed strategy) is $10 + 5 = 15$. By deviating from T or L respectively, either player can get at most $13 + 1 = 14$. Hence, there is no profitable deviation.

Now suppose we relax the assumption that $\delta = 1$. Plainly, if δ is close enough to 1, the same logic would apply. How high must δ be in order to sustain a pure-strategy SPNE (PSSPNE) where TL is played in the first period? As already noted, in period 2, following any history, either MC or BR must be played. Intuitively, the best hope of sustaining TL in period 1 is to play the above-specified s^2 in period 2 — this provides the "highest reward" for playing TL in period 1 and the "worst punishment" for playing anything else, subject to the requirement of subgame perfection. So we can sustain TL in the first period of a PSSPNE if and only if $10 + 5\delta \geqslant 13 + \delta$, or $\delta \geqslant \frac{3}{4}$.

A related point is that if we added periods, then the requirement on the discount factor becomes less demanding. As an illustration, consider now $T = 3$. If $\delta \geqslant \frac{3}{4}$, we can sustain outcomes of TL in the first two periods. If $\delta < \frac{3}{4}$, we know from the above analysis that TL cannot be sustained in the 2nd period of a PSSPNE. But how about having it played in just the first period? This can be done in a PSSPNE if and only if $10 + 5\delta + 5\delta^2 \geqslant 13 + \delta + \delta^2$, or $\delta \geqslant \frac{1}{2}$.[84]

As a final variation, suppose again $T = 2$, but we now modify the game so that each player has an additional action in the stage game, D (for Destruction) such that the payoffs from DD are $(-x, -x)$, the payoffs from aD for any $a \neq D$ is $(0, -x)$ and symmetrically the payoffs from Da for any $a \neq D$ are $(-x, 0)$. Let $x > 0$. Since D is strictly dominated for both players, it does not change any of the prior analysis for PSSPNE of the repeated game. However, if we turn to Nash equilibria of the repeated game, things are quite different.[85] TL can be sustained in the first period in a Nash equilibrium so long as $10 + 5\delta \geqslant 13$, or $\delta \geqslant \frac{3}{5}$. A NE strategy profile would be as follows: play TL in the first period, and $s^2(TL) = MC$ and $s^2(h^2) = DD$ for all $h^2 \neq TL$. This is a Nash equilibrium when $\delta \geqslant 3/5$ because given column's play, a best response for row is either to follow the prescribed behavior or to play B in both periods (but not to play B followed by D), and the former is at least as good as the latter if and only if $\delta \geqslant 3/5$. Notice that when $\delta < 3/5$, the row player, say, would deviate with a *two-stage deviation* — this reiterates that the one-shot deviation principle does not apply to Nash equilibria. Of course, even when $\delta \geqslant 3/5$, the prescribed strategy profile is not a SPNE because at any $h^2 \neq TL$ (which are off the equilibrium path), players are not playing optimally. Thus there are profitable one-shot deviations according to Definition 34; they just occur at histories that are never reached. □

Remark 36. Note well that actions which are dominated in the stage-game *can* be played in a SPNE of the repeated game, even on the equilibrium path. They just cannot be played in the last period of the (finitely) repeated game. Indeed, we have demonstrated this point in the example above: why so?

[84] Showing the "only if" part requires considering various pure strategy profiles and ruling them out as SPNE.

[85] You can check that in the original game without D, one cannot sustain TL in the first period in a pure strategy Nash equilibrium unless $\delta \geqslant 3/4$, just as with PSSPNE.

General principles illustrated by the example:

1. What is needed to sustain stage-game outcomes that are not sustainable in a one-shot game is that the ability to "reward" current-period behavior with future behavior, or the flip side of the coin, to "punish" deviations by switching to less-desirable future behavior.
2. The importance of the future — either via high discount factor or the length of the future horizon — is key: the future losses from deviations must outweigh current gains.
3. Looking only at Nash equilibria instead of SPNE allows a greater scope to deter current deviations, since more threats about future play are possible. Although this can helpful in sustaining cooperative outcomes, it is not satisfactory when these threats are incredible (just as NE that are not subgame perfect are not compelling in simple dynamic games).

Exercise 19. *Does Proposition 16 apply to Nash equilibria? That is, does a finitely repeated game have a unique Nash equilibrium if the stage-game has a unique Nash equilibrium? Prove or give a counter-example.*

6.5. Infinitely Repeated Games

Now we turn to infinitely repeated games where $T = \infty$. (Keep in mind that that $\delta < 1$ throughout now, and when we refer to "payoff" of the infinitely repeated game, we mean the normalized or average discounted payoff.) Our goal is to develop simple versions of a classic result known as the *Folk Theorem*. Let us begin with an example.

Example 36 (Infinitely Repeated PD)**.** Consider an infinitely repeated game where the stage game is the following Prisoner's Dilemma:[86]

Player 2

Player 1		C	D
	C	$5,5$	$0,6$
	D	$6,0$	$2,2$

As we already noted, Proposition 16 implies that with only a finite number of repetitions, there is a unique SPNE (no matter the discount factor), with the outcome of DD in every period. In the current infinitely repeated case, we also know that history-independent repetition of DD is a SPNE. But are there other SPNE?

Grim-Trigger: Consider the following pure strategy for each player: in the first period, play C; in any subsequent period, play C if the history is such that neither player has ever played D before, and play D otherwise. This strategy is known as *grim-trigger*. Under what conditions, if any, is it a SPNE for both players to play grim-trigger? By the one-shot deviation principle, we only need to check that there is no profitable one-shot deviation. In principle, this could be very complicated to check, since there are an infinite number of histories. But given the simple structure of the strategy profile, it turns out to be straightforward. First, observe that there is no profitable one-shot deviation at any history where D has already been played by some player in the past, since such a one-shot deviation would only lower the current period payoff and

[86] The payoffs are slightly modified from the original Trust Game, but preserving the same qualitative structure — the reason will become clear.

not affect future payoffs.[87] Thus, we can focus on one-shot deviations at histories where CC has always been played in the past. Wlog, we can now consider a one-shot deviation for player 1 of playing D at period 1 (why?). In so doing, player 1 triggers a switch from perpetual CC to $(DC, DD, DD, ...)$. So the deviation is not profitable if and only if:

$$5 \geqslant (1-\delta)\left[6 + 2\frac{\delta}{1-\delta}\right],$$

or $\delta \geqslant \frac{1}{4}$. Remarkably, so long as the discount factor is not too low, we have a SPNE where on the equilibrium path there is mutual cooperation in all periods! *The reason we get this stark difference compared to the finitely repeated game is that now there is no specter of the last period hanging over the players.*[88,89] Mathematically, we have a failure of continuity—specifically, lower hemi-continuity—of the SPNE payoff set at $T = \infty$.

Tit-for-Tat SPNE: Now consider another strategy: play C in the first period; in any subsequent period, play whatever the opponent played in the previous period. This is known as *tit-for-tat*. For mutual play of tit-for-tat to be a SPNE, we must again show that there is no profitable one-shot deviation. Wlog, focus on player 1's deviations. Given the structure of tit-for-tat, whether a one-shot deviation is profitable at some history h^t only depends upon the action profile played in the previous period, a^{t-1}, since this is what determines how the opponent plays in the current period (i.e., what happened in periods $t-2, t-3, ...$ is irrelevant). So we consider the four possibilities for a^{t-1}. First, suppose $a^{t-1} = CC$ or $t = 1$. In the subgame starting at h^t, not deviating leads to CC in every period on the path of play; a (one-shot) deviation to D leads to DC,CD,DC,CD,... So the constraint is

$$\begin{aligned} 5 &\geqslant (1-\delta)\left[6 + \delta(0) + \delta^2(6) + \delta^3(0) + \delta^4(6) + \cdots\right] \\ &= (1-\delta)\frac{6}{1-\delta^2}, \end{aligned}$$

or $\delta \geqslant \frac{1}{5}$. Next, suppose $a^{t-1} = CD$. Not deviating at h^t leads to DC, CD, DC,... whereas deviating to C leads to CC, CC, ... So the constraint is

$$(1-\delta)\frac{6}{1-\delta^2} \geqslant 5,$$

or $\delta \leqslant \frac{1}{5}$. Third, suppose $a^{t-1} = DC$. Not deviating at h^t leads to CD, DC, CD, ... whereas deviating leads to DD, DD, ... So the constraint is

$$(1-\delta)\frac{6\delta}{1-\delta^2} \geqslant 2,$$

[87] You might wonder why we even had to consider such a deviation, since if players follow the prescribed strategies, there never would be such a history. But it is crucial that we rule out such deviations because we are looking at SPNE (and the one-shot deviation principle relies on subgame perfection).

[88] It is worth emphasizing at this point that what is important is not that the game literally be infinitely repeated, but rather that there always be a (non-vanishing) possibility that there will be another period, i.e. that "today is not the end".

[89] The difference between finitely- and infinitely-repeated games is less stark when there are multiple Nash equilibria of the stage game. See Benoit and Krishna (1985).

or $\delta \geqslant \frac{1}{2}$. Finally, suppose $a^{t-1} = DD$. Not deviating at h^t leads to DD, DD, ... whereas deviating leads to CD, DC, CD, ... So the constraint is

$$2 \geqslant (1-\delta)\frac{6\delta}{1-\delta^2}$$

or $\delta \leqslant \frac{1}{2}$. Plainly, there is no δ that can satisfy all the four requirements. Thus, mutual tit-for-tat is not a SPNE in this game, no matter the discount factor. (An exercise below will clarify how generally this point holds.)

Tit-for-Tat NE: On the other hand, is mutual tit-for-tat a Nash Equilibrium? Answering this requires some care, because we cannot appeal to the one-shot deviation principle any longer, so we have to consider all possible deviations. You asked to show in an exercise below that if the opponent is playing tit-for-tat, then one of the three following strategies is a (not necessarily unique) best response for a player: either (i) play tit-for-tat; (ii) play D in every period; or (iii) alternate between D and C, beginning with D in period 1. From this, it follows that tit-for-tat is a best response if and only if

$$5 \quad \geqslant (1-\delta)\max\left\{\frac{6}{1-\delta^2}, 6+2\frac{\delta}{1-\delta}\right\},$$

or $\delta \geqslant \frac{1}{4}$. Thus, mutual tit-for-tat is a NE for all sufficiently high discount factors. (This is not a general property of Prisoner Dilemma games; the exact specification of the payoff matrix matters, and for some specifications mutual tit-for-tat is not a NE for any δ.[90])

Other SPNE: So far we have seen that for sufficiently high discount factors, we can achieve a payoff profile in SPNE of the repeated game that is equal to the efficient payoff profile $(5,5)$ of the stage game, and also one equal to the stage game Nash equilibrium payoff profile $(2,2)$. But we can also achieve various other payoff profiles. For example, consider a strategy that modifies grim-trigger as follows: play C in the first period; in any even period, play D; in any odd period > 1, play D if either player ever played D in a prior odd period, otherwise play C. One can show that mutual play of this strategy is a SPNE if δ is sufficiently large. In this SPNE, each player gets a payoff of $(1-\delta)\left(5+2\delta+5\delta^2+2\delta^3+...\right) = (1-\delta)\left(\frac{5}{1-\delta^2}+\frac{2\delta}{1-\delta^2}\right) = \frac{5+2\delta}{1+\delta}$. As you would expect, this converges to 3.5 as $\delta \to 1$, which is the simple average of the stage-game payoffs from CC and DD. $\square$

Exercise 20. *Suppose the stage game prisoner's dilemma has different payoffs:* $u(C,C) = (3,3)$, $u(D,D) = (1,1)$, $u(C,D) = (0,4)$ *and* $u(D,C) = (4,0)$. *For what discount factors (if any) is tit-for-tat a SPNE in the infinitely repeated game?*

Exercise 21. *In Example 36, show that the modified grim-trigger profile described at the end is a SPNE for all discount factors high enough, and identify the minimum discount factor needed.*

Exercise 22. *Show that in Example 36, if player 2 plays tit-for-tat, a best response for player 1 is either (i) play tit-for-tat; (ii) play D in every period; or (iii) alternate between D and C, beginning with D in period 1. [Hint: first consider the payoff to player 1 from using any strategy such that the path of play is CC in every period. Then argue that if 1's best response is a strategy such that he takes action D in some period, then either strategy (ii) or (iii) is a best response.]*

[90] For example, if you replace the DC and CD payoffs by 10,0 and 0,10 respectively, then you can check that the incentive constraint we just derived cannot be satisfied for any δ; hence mutual tit-for-tat would not be a NE. (This is the reason I changed payoffs from the original Trust game.)

6.5.1. Folk Theorems

The natural question raised by the discussion in Example 36 is: what are all the payoff profiles that can be achieved by a SPNE of an infinitely repeated game? It is this question that various *folk theorems* provide answers to. We will state two such theorems, starting with the more straightforward one. We need two definitions.

Definition 35 (Feasible Payoffs)**.** Let $\tilde{V} := \{(\pi_1(a), \ldots, \pi_I(a))\}_{a \in A}$ be the set of all payoff vectors that are attained by some action profile of the stage game. The set of *feasible payoffs*, V, is defined as $V := \text{co}(\tilde{V})$, i.e., the convex hull of $\tilde{V}$.

To interpret this, draw the set of feasible payoffs for Example 36. The reason we are interested in the convex hull of the payoffs from action profiles is that as $\delta \to 1$, any payoff that is in the convex hull can be obtained (ignoring any equilibrium or incentive issues for now, just as a matter of "technology") by having players play an appropriate sequence of (possibly time-varying) action profiles. Indeed, recall the last construction in Example 36. The general point was made by Sorin (1986).

It is intuitive that no payoff vector *outside* the set of feasible payoffs can be achieved as the average discounted payoffs in a SPNE (or Nash Equilibrium, for that matter) of an infinitely repeated game. But is any feasible payoff vector supportable in SPNE? It is not hard to see that the answer is no (can you provide an example?). But it turns out that "almost everything of interest" in V — in the sense of economic interest, not mathematical — with sufficiently patient players.

Definition 36 (Nash-threat Payoff)**.** Any player i's Nash-threat payoff is

$$\underline{v}_i := \inf\{v_i : \exists \text{ stage-game (possibly mixed) Nash equilibrium } \alpha \text{ s.t. } \pi_i(\alpha) = v_i.\}$$

Theorem 11 (Nash-threats Folk Theorem)**.** *Fix a stage game and pick any $v \in \mathbb{R}^I$ such that $v \in V$ and for all i, $v_i > \underline{v}_i$. There is $\underline{\delta} \in [0, 1)$ such that for all $\delta > \underline{\delta}$, there is a SPNE of the infinitely repeated game with average discounted payoff profile v.*

Proof. To simplify the proof, we will make two assumptions that can be dispensed with:

- at the start of each period, players observe the realization of a (sufficiently rich) public randomization device that allows them to correlate their strategies.[91] This means that in any period, they can play any mixed action profile in $\Delta(A)$, as opposed to just mixed action profiles in $\Delta(A_1) \times \cdots \times \Delta(A_I)$.
- at the end of each period, players observe not just the realization of the mixed action profile being played that period, but the actual mixture itself. This means that if a player is supposed to mix over actions in a particular way (potentially degenerately) at some history but deviates to some other mixture, this will observed by everyone.

Since $v \in V$ is a feasible payoff vector, there is some $\alpha^* \in \Delta(A)$ such that $\pi(\alpha^*) = v$. Now consider the following strategy profile:

[91] Formally, let $\{\omega^1, \ldots, \omega^t, \ldots\}$ be a sequence of independent draws from a uniform distribution on $[0, 1]$. We expand the notion of a history as follows: for each $t \geqslant 1$, $h^t = (a^1, \ldots, a^{t-1}, \omega^1, \ldots, \omega^t)$. Thus, players are able to condition their behavior at t on not only the history of actions but also the history of (and current) realizations of the randomization device.

1. In the first period, play α^*.
2. At any history where α^* was played in every prior period, play α^*.
3. At any history h^t where some $\alpha \neq \alpha^*$ was played in some prior period, let $t' := \min\{\tilde{t} : \alpha^{\tilde{t}} \neq \alpha^*\}$ and let $j := \min\{i : \alpha_i^{t'} \neq \alpha_i^*\}$. That is, j is a "first deviator". Let α be the stage-game Nash equilibrium (possibly mixed) such that $\pi_j(\alpha) = \underline{v}_j$. (If there are multiple such stage-game Nash equilibria, any can be used, but pick the same one every time.) At h^t, players play α.

Let's argue that this strategy profile is a SPNE. On the path of play, α^* is played every period. Observe that once there is a deviation in some period, call it t, players are just repeating the same stage-game Nash equilibrium profile regardless of what happens in any period following t. Thus, by the logic of Proposition 15, there are no profitable deviations in any subgame following the first deviation. So it suffices to argue that a unilateral first deviation is not profitable. The first deviator, call him j, can gain at most some finite amount of period-utility in the period he deviates. But in all future periods, he foregoes $v_j - \underline{v}_j > 0$. Let $d := \min_i\{v_i - \underline{v}_i\} > 0$. Since $\lim_{\delta \to 1} \frac{\delta}{1-\delta} d = \infty$, no deviation is profitable for large enough δ. $\square$

The strategy profile used in the proof is known as *Nash Reversion*, since the key idea is to punish a deviator by just playing a stage-game Nash equilibrium in every period thereafter that gives him the lowest payoff amongst all stage-game Nash equilibria. Note the parallel with grim-trigger in the repeated Prisoner's Dilemma. In a homework problem, you will apply this idea to a simple model of repeated market power.

Nash reversion is a very intuitive way to punish a player for deviating from the desired action profile. It turns out, however, that there may be more "severe" punishment schemes than Nash reversion. The following example illustrates.

Example 37 (Minmax Punishments)**.** Consider the following stage-game:

		Player 2	
		C	D
Player 1	C	$3,3$	$0,4$
	D	$4,1$	$1,0$

Observe that player 1 has a strictly dominant strategy of D, whereas player 2's best response is to "mismatch" with player 1's action. So the unique Nash equilibrium of the stage game is DC.

In the infinitely repeated game, can we find a SPNE where CC is played every period with high enough discount factor? Nash reversion is of no help, since player 1 prefers the the unique Nash equilibrium of the stage game to CC. Nevertheless, one can do it (and you are asked to, below!). Note that it is easy to sustain play of CC in a Nash Equilibrium (with patient players); what makes the question interesting is the requirement of subgame perfection. $\square$

Exercise 23. *For the above example, for high enough discount factors, construct a SPNE where CC is played in every period on the equilibrium path.*

To generalize the example, make the following definition.

Definition 37 (Individually Rational Payoffs)**.** A player i's *minmax* payoff of the stage game is given by:

$$\underline{\underline{v}}_i := \min_{\alpha_{-i}} \max_{\alpha_i} \pi_i(\alpha_i, \alpha_{-i}).$$

A vector of payoffs, $v = (v_1, \dots, v_I)$ is *strictly individual rational* if for all i, $v_i > \underline{\underline{v}}_i$.

You've seen the idea of minmax in a homework problem a while back. A player's minmax payoff is the lowest payoff his opponents can "force" him down to in the stage game so long as he plays an optimal response to his opponents' play. Thus, a player will obtain at least his minmax payoff in a Nash equilibrium of the stage game.[92] It is in this sense that no payoff below the minmax payoff is "individually rational." But then, in any Nash equilibrium—indeed, any rationalizable strategy profile—of the infinitely repeated game, a player's average discounted payoff cannot be lower than his minmax payoff either, for he can assure at least his minmax payoff by just playing a strategy where at each history he myopically best responds to what his opponents are doing at that history. Our final result of this section is that essentially any vector of feasible and strictly individually rational payoffs can be obtained as a SPNE of an infinitely repeated game.

Theorem 12 (Minmax Folk Theorem)**.** *Fix a (finite) stage game and let $V^* \subseteq V$ be the set of feasible and strictly individually rational payoffs. Assume the interior of V^* is nonempty.*[93] *For any $v \in V^*$, there is $\underline{\delta} \in [0, 1)$ such that for all $\delta > \underline{\delta}$ there is a SPNE of the infinitely repeated game with average discounted payoff profile v.*

A full proof is somewhat involved, but we can illustrate the key idea how minimax rather than Nash reversion can be used as a credible threat.

Proof (Partial). Let us assume there are only two players, that the payoff vector we seek, v, is the payoff vector from some pure action profile, a^* (i.e., $v = \pi(a^*)$), and that $v_i > \underline{\underline{v}}_i^P := \min_{a_{-i}} \max_{a_i} \pi_i(a_i, a_{-i})$. The last assumption means that each player is getting strictly more than his *pure strategy minmax*, not just his his individually rational payoff (cf. fn. 92). Let a_i^m be some action of i that that holds his opponent j to j's pure strategy minmax payoff, and consider the following *stick-and-carrot strategy*:

1. Play a_i^* initially or if a^* was played in previous period.
2. If there was a deviation from (1), play a_i^m K times and then restart (1).
3. If there is a deviation from (2), begin (2) again.

The length K of phase 2 (the "punishment phase") will be determined momentarily. Let $\bar{v}_i := \max_a \pi_i(a)$ be the maximum stage payoff for i, and let $v_i^m := \pi_i(a_i^m, a_{-i}^m)$ be i's stage payoff in phase (2). Note that

$$v_i^m \leqslant \underline{\underline{v}}_i^P < v_i \leqslant \bar{v}_i.$$

Pick any integer $K > 0$ such that for both players i, $K(v_i - v_i^m) > \bar{v}_i - v_i$.

All we need to check is that there is no profitable one-shot deviation from phase 1 or phase 2.

[92] Note that the definition above explicitly considers mixed strategies for both player i and his opponents. This is not innocuous. For example, if you consider simultaneous Matching Pennies where the utility from matching is 1 and mismatching is 0 for player 1, say, then the minmax payoff for player 1 is 0.5 according to our definition; whereas if we only consider pure strategies, the "pure strategy minmax" payoff would be 1.

[93] The interior is relative to I-dimensional Euclidean space.

There is no profitable one-shot deviation from phase 1 of the strategy profile if

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t v_i \geqslant (1-\delta)\left(\bar{v}_i + \sum_{t=1}^{K}\delta^t v_i^m + \sum_{t=K+1}^{\infty}\delta^t v_i\right)$$
$$\iff \sum_{t=1}^{K}(v_i - v_i^m) \geqslant \bar{v}_i - v_i.$$

Since the LHS $\to K(v_i - v_i^m)$ as $\delta \to 1$, this inequality holds for all $\delta < 1$ large enough because of our choice of K above. Intuitively, a deviation from phase 1 triggers phase 2, in which a player will get a per-period payoff no higher than his pure strategy minmax, which is strictly less than his payoff in phase 1; if phase 2 is long enough then a patient player will not find this profitable.

In phase 2, deviation is most tempting in the initial period, as that is when there are the most remaining punishment periods. A one-shot deviation in the initial period is not profitable if

$$(1-\delta)\left(\sum_{t=0}^{K-1}\delta^t v_i^m + \sum_{t=K}^{\infty}\delta^t v_i\right) \geqslant (1-\delta)\left(\underline{\underline{v}}_i^P + \sum_{t=1}^{K}\delta^t v_i^m + \sum_{t=K+1}^{\infty}\delta^t v_i\right)$$
$$\iff v_i^m + \delta^K v_i \geqslant \underline{\underline{v}}_i^P + \delta^K v_i^m$$
$$\iff \delta^K v_i + \left(1-\delta^K\right) v_i^m \geqslant \underline{\underline{v}}_i^P.$$

The last inequality holds when δ is large enough because $v_i > \underline{\underline{v}}_i^P$. Intuitively, given the specification of phase 3, the effect of a deviation in the initial period of phase 2 is to swap a future period's payoff of v_i with a payoff of $\underline{\underline{v}}_i^P$ today; when a player is patient, this is not profitable.[94]

The above logic shows why phase 2 is the "stick" that enforces phase 1, while at the same time phase 1 acts as a "carrot" for carrying out phase 2. This carrot is used to ensure that a player carrying out the punishment of his opponent even when it is not in her myopic interest to do so. □

See Mailath and Samuelson (2006, p. 101) for a full proof. The general idea remains that a player who deviates from the equilibrium path is minmaxed by his opponents. To induce the opponents to carry out the minmaxing, they must be given a small reward once the punishment phase is completed. The assumption that the interior of V^* is nonempty ensures that rewards/punishment incentives can be provided to each player independently of the others. In particular, the condition rule out any pair of players having identical payoff functions.[95]

We can apply the Theorem to Example 37 by deducing what the set V^* is there. (Draw a picture.) We see immediately that the payoff profile from CC $(3,3)$ is supportable in SPNE as $\delta \to 1$, which you were asked to prove earlier with an explicit equilibrium construction.

Remark 37 (Comments on Folk Theorems). Theorem 11 and Theorem 12 show that standard equilibrium requirements do very little to narrow down predictions in infinitely repeated games:[96] in terms of payoffs,

[94] Note that if we fix $\delta < 1$, then the last inequality places an upper bound on K. But that upper bound diverges to ∞ as $\delta \to 1$. We have fixed an arbitrary large enough K, and we are then taking δ large.

[95] In fact, instead of assuming V^* has a nonempty interior, the following weaker "nonequivalent utilities" condition suffices (Abreu, Dutta, and Smith, 1994): either there are two players, or no two players have identical preferences in the sense that there are no $i, j \in I$ $(i \neq j)$ and $a > 0$, $b \in \mathbb{R}$ with $\pi_i(\cdot) = a\pi_j(\cdot) + b$.

[96] There are also folk theorems for finitely repeated games when there are multiple Nash equilibria of the stage

more or less "anything can happen" as $\delta \to 1$ (subject to being feasible and strictly individually rational). Some comments:

1. This can make comparative statics difficult.
2. With repeated interactions, players are able to get around inabilities to write binding contracts (at least in the perfect monitoring environment we have been considering). Anything achievable through a binding contact can also be attained as a non-binding equilibrium.
3. Which equilibrium gets played could be thought of determined by some kind of pre-play negotiation amongst players. It is natural to then think that the payoff profile will be efficient, and which efficient profile is selected may depend on bargaining powers. (There is not much formal work on this yet.)
4. Although the Theorems only discuss SPNE payoffs, one can say a fair bit about what strategies can be used to attain these payoffs. An important technique is known as *self-generation*; see Abreu, Pearce, and Stacchetti (1990).
5. Although they yield multiplicity of equilibria, folk theorems are important because of what they tell us about *how* payoffs can be achieved, what the qualitative structure of reward and punishment is (more relevant under imperfect monitoring, which we haven't studied here), and what the limitations are.
6. In applications, people usually use various refinements to narrow down payoffs/equilibria, such as efficiency, stationarity, Markov strategies, symmetric strategies, etc.
7. It is also important to understand what payoffs are supportable for a given δ (not close to 1). There is some work on this.

---

game, see for example Benoit and Krishna (1985).

7. Signaling and Cheap Talk

Dynamic games of incomplete information are particularly rich in application. In this section, we study a class of such games that involve *asymmetric information*, viz. situations where some players hold private information that is valuable to other players.

7.1. Costly Signaling

We first consider the classic idea of Spence (1973). We'll frame it in the context of labor markets, as he did, but it is important to stress that the same principle has been applied in a host of other contexts.

7.1.1. The Setting

There is a worker and two firms.[97] For simplicity, we study here the canonical case where there are only two types of workers, θ_H and θ_L, with $\theta_H > \theta_L > 0$. Let $\lambda = \text{Prob}(\theta = \theta_H) \in (0, 1)$ be the fraction of high type workers. A worker's type is his private information. Prior to entering the labor market, workers can obtain education at some cost. In particular, a worker of type $\theta \in \{\theta_L, \theta_H\}$ can obtain education level $e \geqslant 0$ at a cost $c(e, \theta)$. We assume that $c_1(e, \theta) > 0$ and $c_{12}(e, \theta) < 0$. In words, the first condition says that acquiring more education is always costly on the margin; the second condition says that the marginal cost is lower for the higher type worker. It is critical that firms observe a worker's education level after he has acquired it (but they do not observe his productivity type). We set the reservation wage for both types to be 0; the important assumption here is that it is the same for both types, but otherwise 0 is a normalization. The game that the worker then plays with the firms is as follows:[98]

1. Nature chooses worker's type, $\theta \in \{\theta_L, \theta_H\}$, according to $\lambda = \text{Prob}(\theta = \theta_H)$.
2. Having privately observed θ, worker chooses e.
3. Having observed e but not θ, each firm $i \in \{1, 2\}$ simultaneously offers a wage, w_i.
4. Worker accepts one or neither job.

Payoffs are as follows: the worker gets $u(w, e, \theta) = w - c(e, \theta)$ if she accepts an offer and $-c(e, \theta)$ if she does not; the firm that employs the worker gets $\theta - w$; the other firm gets 0. Note that as specified, education is absolutely worthless in terms of increasing productivity — it is solely an instrument to potentially signal some private information.[99] This is known as *purely dissipative* signaling.

[97] We could also do this more workers and more firms. Two firms is sufficient because when firms compete for workers through their wage offerings, Bertrand competition between any two firms is sufficient to drive wages up to the marginal product of labor.

[98] You can think of the multiple workers case as simply each worker playing this game simultaneously with the firms, with types drawn independently across workers.

[99] Of course, in practice, education also has a productivity-enhancing purpose . . . we hope. Even in that case, it can serve as a signaling instrument. As you'll see, what is important is the difference in marginal costs of acquiring education for the different worker types.

7.1.2. Basic Properties

The goal is to study whether and how education can be used by workers to signal their type to the firms. In what follows, we are going to analyze *pure strategy weak PBE* that satisfy the following additional property: for any level of education chosen by the worker, e, both firms have the same belief over the worker's type. That is, if $\mu_i(e)$ represents the belief of firm $i \in \{1, 2\}$ that the worker is type θ_H given that he has chosen education e, then we require that $\mu_1(e) = \mu_2(e)$ for all e.[100] It turns out that the set of such weak PBE is identical to the set of sequential equilibria in this model (strictly speaking, in a discretized version of the model, since sequential equilibrium is only defined for finite games); see Fudenberg and Tirole (1991b) for a more general equivalence of (a version of) PBE and sequential equilibria in (a class of) signaling games. For short, I will just refer to any of these weak PBE as an equilibrium in the ensuing discussion.

Using generalized backward induction, we start analyzing play at the end of the game.

Stage 4: In the last stage, sequential rationality requires that a worker accepts a job offer from the firm that offers the higher wage, so long as it is non-negative.[101] If they both offer the same wage, he randomly accepts one of them.

Stage 3: Given Stage 4 behavior of the worker, we can see that for any belief $\mu(e)$, both firms must offer the wage

$$w(e) = \mathbb{E}[\theta|\mu(e)] = \mu(e)\theta_H + (1 - \mu(e))\theta_L. \tag{13}$$

To see this, note that a firm expects strictly negative payoffs if it hires the worker at a wage larger than $\mathbb{E}[\theta|\mu(e)]$, and strictly positive payoffs if it hires him at less than $\mathbb{E}[\theta|\mu(e)]$. Since firms are competing in Bertrand price competition, the unique sequentially rational best responses are $w(e)$ as defined above.

What more can be said about $w(e)$? At this stage, not much, except that in any equilibrium, for all e, $w(e) \in [\theta_L, \theta_H]$ because $\mu(e) \in [0, 1]$. Note that in particular, we cannot say that that $w(\cdot)$ even need be increasing. Another point to note is that there is a 1-to-1 mapping from $\mu(e)$ to $w(e)$. Remember that equilibrium requires $\mu(e)$ to be derived in accordance with Bayes rule applied to the worker's on-path education choices .

Stage 2: To study the worker's choice of education, we must consider her preferences over wage-education pairs. To this end, consider the utility from acquiring education e and then receiving a wage w, for type θ: $u(w, e, \theta) = w - c(e, \theta)$. To find indifference curves, we set $u(w, e, \theta) = \bar{u}$ (for any constant $\bar{u}$), and implicitly differentiate, obtaining

$$\left.\frac{\mathrm{d}w}{\mathrm{d}e}\right|_{u=\bar{u}} = c_1(e, \theta) > 0.$$

Thus indifference curves are upward sloping in e-w space, and moreover, at any particular (e, w), they are steeper for θ_L than for θ_H by the assumption that $c_{12}(e, \theta) < 0$. Thus, an indifference curve for type θ_H crosses an indifference curve for type θ_L only once (so long as they cross at all). This is known as the *(Spence-Mirlees) single crossing property*, and it plays a key role in the analysis of many signaling models. Figure 9 shows a graphical representation.[102]

---

[100] MWG call this a PBE, and indeed, in accordance with our terminology, any weak PBE equilibrium with commonality of beliefs will be subgame perfect. However, subgame perfection does not require the commonality of beliefs; this is an added restriction, albeit a natural one.

[101] Strictly speaking, he can reject if the wage is exactly 0, but we resolve indifference in favor of acceptance, for simplicity. This is not important.

[102] Here is another way to state the requisite single-crossing property in a manner that is portable to many types:

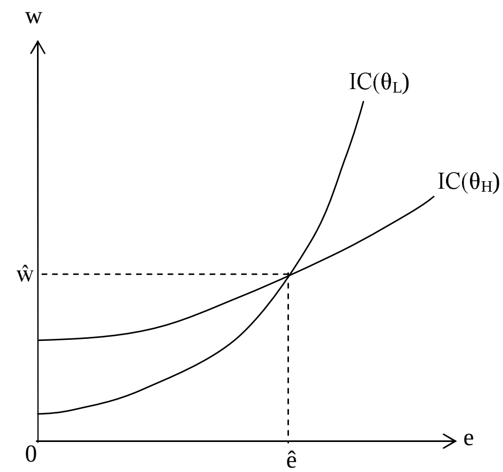


Figure 9 – The Single-Crossing Property

Obviously, the choice of e for a worker of either type will depend on the wage function $w(e)$ from Stage 3. But in turn, the function $w(e)$ (or equivalently, $\mu(e)$) must be derived from Bayes rule for any e that is chosen by the worker (of either type). For any e not chosen by either type, any $w(e) \in [\theta_L, \theta_H]$ is permissible since Bayes rule doesn't apply. This flexibility in specifying "off-the-equilibrium-path" wages yields a multiplicity of equilibria. Equilibria can be divided into two classes:

1. Separating equilibria. Here, the two types of worker choose different education levels, thus "separating" themselves.

2. Pooling equilibria. Here, the two types choose the same education level, thereby "pooling" together.

We study each of them in turn.[103]

7.1.3. Separating Equilibria

Let $e^*(\theta)$ denote a worker's equilibrium education choice, and $w^*(e)$ denote an equilibrium wage offer given the equilibrium beliefs $\mu^*(e)$.

Claim 2. *In any separating equilibrium, $w^*(e^*(\theta_i)) = \theta_i$ for $i \in \{L, H\}$, i.e., a worker is paid her marginal product.*

Proof. By definition, in a separating equilibrium, the two types choose different education levels, call them $e^*(\theta_L) \neq e^*(\theta_H)$. Bayes rule applies on the equilibrium path, and implies that $\mu(e^*(\theta_L)) = 0$ and $\mu(e^*(\theta_H)) = 1$. The resulting wages by substituting into equation (13) are therefore θ_L and θ_H respectively. □

$\forall (w, e) > (w', e')$, where $>$ is in the vector sense, and $\forall \theta > \theta'$, $u(w, e, \theta') \geqslant u(w', e', \theta') \implies u(w, e, \theta) > u(w', e', \theta)$. Given the separability $u(w, e, \theta) = w - c(e, \theta)$, it is sufficient that c satisfy decreasing differences, which under smoothness of c can be stated as $c_{e\theta}(e, \theta) < 0$.

[103] The two classes are exhaustive given the restriction to pure strategies. If we were to consider mixed strategies, there would be a third class of *partial pooling* or *hybrid* equilibria where types could be separating with some probability and pooling with some probability.

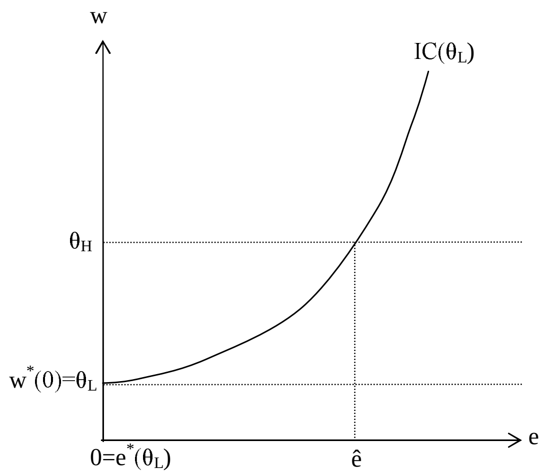


Figure 10 – Separating Equilibrium Low Type's Allocation

Claim 3. *In any separating equilibrium, $e^*(\theta_L) = 0$, i.e., the low type worker chooses 0 education.*

Proof. Suppose towards contradiction that $e^*(\theta_L) > 0$. By Claim 2, type θ_L's equilibrium utility is $\theta_L - c(e^*(\theta_L), \theta_L)$. If she instead chose education level 0, she would receive a utility of at least $\theta_L - c(0, \theta_L)$, because $w^*(0) \geqslant \theta_L$. Since $c_1(\cdot, \theta_L) > 0$, it follows that the worker gets a strictly higher utility by deviating to an education level of 0, a contradiction with equilibrium play. □

Claim 2 and Claim 3 combined imply that the equilibrium utility for a low type is $u(\theta_L, 0, \theta_L)$. This puts the low type on the indifference curve passing through the point $(0, \theta_L)$ in e-w space, as drawn in Figure 10.

We can use this picture to construct a separating equilibrium. By Claim 2, the θ_H worker must receive a wage of θ_H, hence an allocation somewhere on the horizontal dotted line at θ_H. If the allocation were to the left of where $IC(\theta_L)$ crosses that dotted line, then type θ_L would prefer to mimic θ_H worker rather than separate (i.e. it would prefer to choose the education level that θ_H is supposed to, rather than 0); this follows from the fact that allocations to the "left" of a given indifference curve are more desirable. So, a candidate allocation for the high type is education level $\hat{e}$ with wage θ_H in Figure 10. That is, we set $e^*(\theta_H) = \hat{e}$ and $w^*(\hat{e}) = \theta_H$, where $\hat{e}$ is formally the solution to

$$u(\theta_H, \hat{e}, \theta_L) = u(\theta_L, 0, \theta_L).$$

It remains only to specify the wage schedule $w^*(e)$ at all points $e \notin \{0, \hat{e}\}$. Since we are free to specify any $w^*(e) \in [\theta_L, \theta_H]$, consider the one that is drawn in Figure 11.

Given this wage schedule, it is clear that both types are playing optimally by choosing 0 and $\hat{e}$ respectively, i.e., neither type strictly prefers choosing any other education level and receiving the associated wage over its prescribed education and associated wage. Beliefs (or wages) are correct on the equilibrium path, and thus firms are playing optimally. Thus, e^* and w^* as defined is in fact a separating equilibrium. It is obvious that there are various wage schedules that can support the same equilibrium education choices: an alternate schedule that works, for example, is $w(e) = \theta_L$ for all $e \in [0, \hat{e})$ and $w(e) = \theta_H$ for all $e \geqslant \hat{e}$.

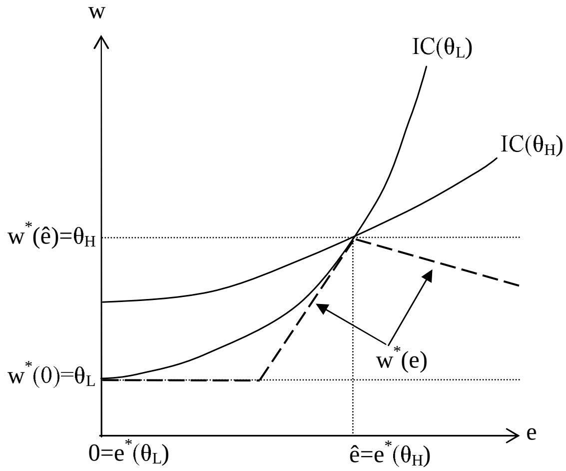


Figure 11 – A Separating Equilibrium

The more interesting question is whether there are other education levels that can be sustained in a separating equilibrium. Claim 3 says that the low type must always play $e^{(}\theta_L) = 0$, but could we vary $e^*(\theta_H)$? Yes. Let $\bar{e}$ be the education level that solves

$$u(\theta_H, \bar{e}, \theta_H) = u(\theta_L, 0, \theta_H).$$

In words, $\bar{e}$ is the education level that makes type θ_H indifferent between acquiring $\bar{e}$ with pay θ_H and acquiring 0 with pay θ_L. The single-crossing property stemming from $c_{12}(\cdot,\cdot) < 0$ ensures that $\bar{e} > \hat{e}$. A separating equilibrium where $e^*(\theta_H) = \bar{e}$ is illustrated in Figure 12.

It follows from the construction's logic that for every $e \in [\hat{e}, \bar{e}]$, there is a separating equilibrium with $e^*(\theta_H) = e$; and there is no separating equilibrium with $e^*(\theta_H) \notin [\hat{e}, \bar{e}]$. It is easy to see that we can Pareto-rank these separating equilibrium allocations.

Proposition 17. *A separating equilibrium with $e^*(\theta_H) = e_1$ Pareto-dominates a separating equilibrium with $e^*(\theta_H) = e_2$ if and only if $e_1 < e_2$.*

Proof. Straightforward, since firms are making 0 profit in any separating equilibrium, the low type of worker receives the same allocation in any separating equilibrium, and the high type of worker prefers acquiring less education to more at the same wage. □

So the first separating equilibrium we considered with $e^*(\theta_H) = \hat{e}$ Pareto-dominates all others with different high-type allocations; it has the least inefficiency via the costly signaling. This outcome—note that because there the flexibility with off-path beliefs means that there are many equilibria leading to the same outcome or on-path behavior—is referred to as the *least-cost separating equilibrium outcome* or the *Riley outcome* after Riley (1979).

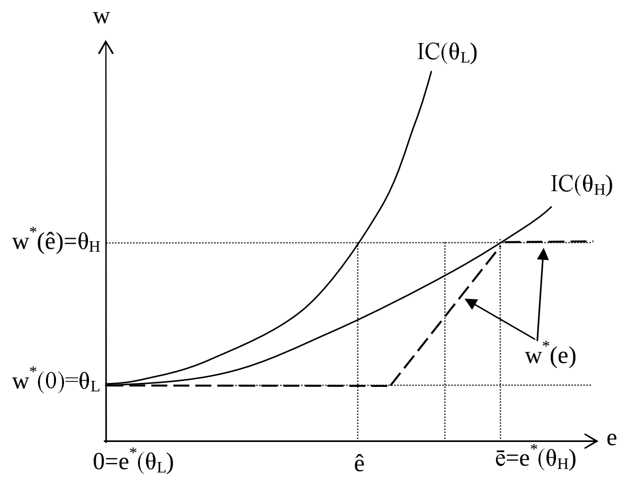


Figure 12 – A Separating Equilibrium with $e^*(\theta_H) = \bar{e}$

7.1.4. Pooling Equilibria

Now we turn to the other class of equilibria, where both types of worker choose the same education level, thus pooling together. That is, we are looking for equilibria where $e^*(\theta_H) = e^*(\theta_L) = e^P$ for some e^P. In any such equilibrium, Bayes rule implies that $\mu^*(e^P) = \lambda$, hence from equation (13), $w^*(e^P) = \mathbb{E}(\theta)$. To determine which e^P are feasible in a pooling equilibrium, define $\underline{e}$ as the education level that makes type θ_L indifferent between acquiring education $\underline{e}$ with wage $\mathbb{E}(\theta)$ and acquiring education 0 with wage θ_L. Formally, $\underline{e}$ is the solution to

$$u(\mathbb{E}(\theta), \underline{e}, \theta_L) = u(\theta_L, 0, \theta_L)$$

Since $\mathbb{E}(\theta) \in (\theta_L, \theta_H)$, it follows that $\underline{e} \in (0, \hat{e})$. Figure 13 shows the construction of a pooling equilibrium with $e^P = \underline{e}$. Of course, there are multiple wage schedules that can support this pooling choice of $e^P = \underline{e}$.

Moreover, the logic of this construction implies that there is a pooling equilibrium for any $e^P \in [0, \underline{e}]$; but not for any $e^P > \underline{e}$. The reason for the latter is that a worker of type θ_L would strictly prefer to choose education 0 and get and $w^*(0) \geqslant \theta_L$ rather than choose $e^P > \underline{e}$ and get wage $\mathbb{E}(\theta)$. We can also Pareto-rank the pooling equilibria.

Proposition 18. *A pooling equilibrium with education level* e^P *Pareto-dominates a pooling equilibrium with education level* $\tilde{e}^P$ *if and only if* $e^P < \tilde{e}^P$.

Proof. Straightforward, since firms are making 0 profit (in expectation) in any pooling equilibrium, both types of of worker receive the same wage in any pooling equilibrium, and both type of workers strictly prefer acquiring lower education levels for a given wage. □

Note also that any pooling equilibrium is completely wasteful in the sense both types of worker would be better off if the ability to signal had been absent altogether, and the market just functioned with no education acquisition and a wage rate of $\mathbb{E}(\theta)$. Contrast this with separating equilibria, where at least the high type is able to reap some benefit from signaling in terms of a higher wage (though it may not compensate him for the cost of signaling relative to the absence of the signaling altogether).

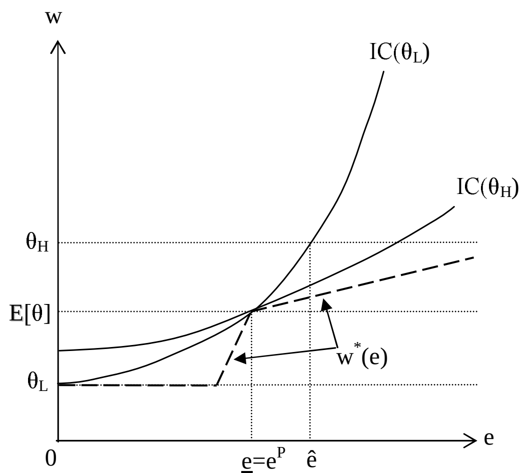


Figure 13 – A Pooling Equilibrium

7.1.5. Equilibrium Refinement

The fact that we have been able to construct a continuum of equilibrium allocations in both separating and pooling classes is somewhat troublesome. To put it another way, is there a reason why some of these equilibria should be thought of as more "reasonable" than others? The latitude in selecting $w(e)$ for all e that are not chosen in equilibrium is what leads to the multiplicity of equilibria. Based on the culminating discussion in Section 4, you might guess that we have to turn to forward induction arguments. No time like the present, so let's do so without further ado!

The refinement we analyze is called *equilibrium dominance* or the *intuitive criterion* and the ideas are due to Cho and Kreps (1987) and Banks and Sobel (1987).[104]

Definition 38 (Equilibrium Dominance)**.** A signaling equilibrium, $(e^*(\theta), w^*(e))$ and beliefs $\mu^*(e)$, satisfies the equilibrium dominance condition if $\mu^*(\tilde{e}) = 1$ for any $\tilde{e}$ such that

1. $\tilde{e}$ is not chosen by either type in the equilibrium;
2. for all $w \in [\theta_L, \theta_H]$, $u(w^*(e^*(\theta_L)), e^*(\theta_L), \theta_L) > u(w, \tilde{e}, \theta_L)$;
3. for some $w \in [\theta_L, \theta_H]$, $u(w^*(e^*(\theta_H)), e^*(\theta_H), \theta_H) < u(w, \tilde{e}, \theta_H)$.

What does the equilibrium dominance condition require? Condition 2 is the key. It says that type θ_L gets strictly higher utility in the equilibrium than any $w \in [\theta_L, \theta_H]$ it could get in return for choosing the out-of-equilibrium education $\tilde{e}$. Condition 3 says that there is some $w \in [\theta_L, \theta_H]$ that would make type θ_H prefer acquiring $\tilde{e}$ if it received w in return, relative to what it gets in equilibrium. Note that it is sufficient to check this condition using the most attractive wage, i.e., $w = \theta_H$.

[104] Strictly speaking, there is a difference between the equilibrium dominance criterion and the intuitive criterion (Cho and Kreps, 1987, Section 3). In general, the former is stronger (i.e., more restrictive) because, vaguely speaking, they turn on the same principle but the former entails iterated applications. The criteria are, however, equivalent in our environment with two types. I use the term "equilibrium dominance" because it is more informative than "intuitive criterion".

The intuition behind this condition is quite simple: since type θ_L can only do worse by playing $\tilde{e}$ relative to what it gets in equilibrium (so long as it gets in response some $w \in [\theta_L, \theta_H]$ — sequential rationality is important), if we do observe the choice of $\tilde{e}$, we ought to rule out the possibility that it was made by θ_L. Moreover, there is some response $w \in [\theta_L, \theta_H]$ that justifies θ_H choosing $\tilde{e}$. Thus, we must infer that the deviation to $\tilde{e}$ was made by θ_H, hence we must put $\mu^*(\tilde{e}) = 1$, or equivalently $w^*(\tilde{e}) = \theta_H$. (This is a good time to go back and look at Example 33 and the logic we discussed there, it's very similar.)

It turns out that this equilibrium dominance condition is very powerful in signaling games with 2 types of the privately informed player. Let's apply it to the current model.

Proposition 19. *The only signaling equilibria (amongst both pooling and separating) that satisfy the equilibrium dominance condition are the separating equilibria with* $e^*(\theta_H) = \hat{e}$.

Proof. For any separating equilibrium with $e^*(\theta_H) > \hat{e}$, consider any out-of-equilibrium $\tilde{e} \in (\hat{e}, e^*(\theta_H))$. It can be seen (graphical argument) that Conditions 2 and 3 of Definition 38 are met. So we must have $\mu^*(\tilde{e}) = 1$, or equivalently, $w^*(\tilde{e}) = \theta_H$. But then type θ_H has a profitable deviation to $\tilde{e}$.

For any pooling equilibrium with education level e^P (recall $e^P \leqslant \underline{e} < \hat{e}$), define $\check{e}$ by $u(\mathbb{E}(\theta), e^P, \theta_L) = u(\theta_H, \check{e}, \theta_L)$. Note that $\check{e} \in (e^P, \hat{e})$, and because of the single-crossing property, $u(\mathbb{E}(\theta), e^P, \theta_H) < u(\theta_H, \check{e}, \theta_H)$. Consider the out-of-equilibrium $\tilde{e} = \check{e} + \varepsilon$ for a small $\varepsilon > 0$ such that $u(\mathbb{E}(\theta), e^P, \theta_H) < u(\theta_H, \tilde{e}, \theta_H)$. It can be seen (graphical argument) that Conditions 2 and 3 of Definition 38 are met. So we must have $\mu^*(\tilde{e}) = 1$, or equivalently, $w^*(\tilde{e}) = \theta_H$. But then type θ_H has a profitable deviation to $\tilde{e}$. □

Thus, application of the equilibrium dominance condition yields a unique equilibrium outcome (i.e., the equilibria that survive can only differ in off-the-equilibrium path wages), which is the least-cost separating outcome. In fact, hybrid or partially-pooling equilibria (see fn. 103) are also eliminated by the equilibrium dominance condition.

7.2. Cheap Talk

We'll also cover the classic cheap-talk model of Crawford and Sobel (1982). **TBAdded**.

8. Mechanism Design

We only have time to peek into mechanism design.

We consider a framework with $I \equiv \{1, \ldots, I\}$ agents. Each agent has a type $\theta_i \in \Theta_i$, which is the agent's private information. There is prior distribution of the vector $\theta \in \Theta \equiv \Theta_1 \times \cdots \times \Theta_I$.[105] There is a set Z of allocations/alternatives/outcomes. Each agent has a vNM utility function $u_i(z, \theta)$. The environment has private values if we can reduce to $u_i(z, \theta_i)$; otherwise there is interdependent values. (Recall the terminology from our discussion of Bayesian games.)

A canonical special case is to take $Z \equiv X \times T$, where $T \equiv \mathbb{R}^I$, with the interpretation that $t \in \mathbb{R}^I$ is the vector of transfers paid by the agents. In this case, we refer to just the element of X as the allocation. We will assume *quasi-linearity* here: utility for each i is given simply by $u_i(x, \theta) - t_i$. Note that this imposes risk neutrality with respect to the transfer.

We will be interested in achieving or implementing (in a sense to be made precise below) an objective given by a social choice function, SCF, or allocation rule $\psi : \Theta \to Z$. In the quasi-linear environment, typical objectives of interest include efficiency (maximize $\sum_I u_i(x, \theta)$),[106] revenue maximization (maximize $\sum_I t_i$), or profit maximization (maximize $u_0(x, \theta) + \sum_I t_i$, with the interpretation that $u_0(x, \theta)$ is the "cost" for some additional player, usually called the principal).

A *mechanism* is a game form, given by $(A_1 \ldots A_I, g)$, or just (A, g) for short, where $A \equiv A_1 \times \cdots \times A_I$ and $g : A \to Z$. The interpretation is that for each i, A_i is the pure action set in some game set up by the designer; note that the game need not be a simultaneous-move game, just like in our discussion for Bayesian games. A mechanism, together with the other primitives (utilities and type distribution) defines a Bayesian game; a (pure) strategy for an agent in this game is $s_i : \Theta_i \to A_i$.

We say that a mechanism *implements SCF* ψ in a given solution concept if the Bayesian game has an "equilibrium" of that concept, $(s_1^*, \ldots, s_I^*)$, such that for all θ, $\psi(\theta) = g(s_1^*(\theta_1), \ldots s_I^*(\theta_I))$. "Equilibrium" should be understood broadly; it is just some solution concept. Note that we are restricting to pure strategy equilibria. This can be relaxed. More importantly, we are focussing on what is called *weak/partial* implementation — i.e., we are not requiring that every equilibrium must lead to the desired outcomes.[107] Three common solutions concepts are Bayesian Nash equilibrium (BNE), ex post BNE, and dominant strategies. It is important to note that dominant strategies here refers to the "very weakly dominant" notion we mentioned in Remark 22, i.e., for each agent i, s_i^* should be a best response to any strategy profile of the opponents in the Bayesian game.

8.1. Revelation Principle

A *direct mechanism* has $A_i = \Theta_i$ for all i. A direct mechanism is said to be *incentive compatible* if it has a truth-telling equilibrium, i.e, all agents playing $s_i^*(\theta_i) = \theta_i$ is an equilibrium. An incentive compatible direct

[105] As usual, for general results below we treat Θ_i as finite. Furthermore, to simplify, I'll assume every θ_i has positive prior probability. These can be relaxed.

[106] The interpretation of this as (Pareto) efficiency requires some clarification. The best way to think of it is efficiency subject to holding constant the sum of transfers. That is, given any θ, if we don't choose x to maximize $\sum_I u_i(x, \theta)$, we can always find a redistribution of transfers that will lead to a Pareto improvement while not changing the sum of transfers.

[107] That requirement is called *full implementation*. Classic references on this include Maskin (1999) (circulated since 1977) and Jackson (1991). There is a nice survey by Jackson (2001).

mechanism implements a SCF if the implementation is achieved in the truth-telling equilibrium.

The following result is the *revelation principle*.

Proposition 20. *If some mechanism implements a SCF in dominant strategies (BNE), then there is an incentive compatible direct mechanism that also implements the SCF in dominant strategies (BNE).*

Proof. We prove it for dominant strategies and the case of private values. Suppose (A, g) implements ψ. Write $s^*(\theta)$ as shorthand for $(s_i^*(\theta_i))_{i=1}^I$ and similarly $s_{-i}(\theta_{-i})$. Consider the direct mechanism (Θ, γ), where $\gamma(\theta) \equiv g(s^*(\theta))$.

For any i, θ_i, $\hat{\theta}_i$, and θ_{-i}, the dominance of s_i^* implies that

$$u_i\left(g\left(s_i^*\left(\theta_i\right), s_{-i}^*(\theta_{-i})\right), \theta_i\right) \geqslant u_i(g(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})), \theta_i),$$

and hence that

$$u_i\left(\gamma\left(\theta_i, \theta_{-i}\right), \theta_i\right) \geqslant u_i(\gamma(\hat{\theta}_i, \theta_{-i}), \theta_i),$$

But this means that in the direct mechanism (Θ, γ), truth-telling is a [very weakly!] dominant strategy for each player. Moreover, it is clear that the truth-telling equilibrium of the direct mechanism implements the SCF.

The argument for BNE and private values is analogous, and a good exercise. The argument is also similar absent private values. □

The revelation principle is a crucial result because it affords a tremendous simplification when we search for what SCFs can be implemented: not only can we restrict attention to a relatively small space of mechanisms, but moreover, the constraints we need consider are just those of incentive compatibility.

8.2. Vickrey-Clarke-Groves Mechanisms

Now consider an environment with private values and quasi-linearity. Let $x^* : \Theta \to X$ be some (ex post) efficient allocation rule: $\forall\theta,\ x^*(\theta) \in \arg\max_{x \in X} \sum_i u_i(x, \theta_i)$. (Assume maximizers exist.)

Consider the following transfer rule:

$$t_i^G(\theta) = -\sum_{j \neq i} u_j(x^*(\theta), \theta_j) + h_i(\theta_{-i}),$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is arbitrary. A direct mechanism with allocation and transfer functions $(x^*, (t_i^G)_{i=1}^I)$ is called a *Groves mechanism*. (Hence the superscript G on t_i.)

Proposition 21. *Any Groves mechanism is dominant strategy incentive compatible.*

Proof. Type θ_i of agent i's payoff from reporting $\hat{\theta}_i$ when others report $\hat{\theta}_{-i}$ is

$$\begin{aligned} & u_i(x^*(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) - t_i^G(\hat{\theta}_i, \hat{\theta}_{-i}) \\ = & u_i(x^*(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) + \sum_{j \neq i} u_j(x^*(\hat{\theta}_i, \hat{\theta}_{-i}), \hat{\theta}_j) - h_i(\hat{\theta}_{-i}). \end{aligned} \tag{14}$$

Since x^* is an efficient allocation rule, it follows that the function above is maximized over $\hat{\theta}_i$ when $\hat{\theta}_i = \theta_i$. Dominant strategy incentive compatibility follows. □

The key intuition here is that the specification of the transfer rule, along with an efficient allocation rule, ensures that each agent is maximizing efficiency given his type and that his opponents' types are their reports. To make that clear, consider the first two terms in (14); we can ignore the third term as it is independent of $\hat{\theta}_i$. You see that by announcing $\hat{\theta}_i = \theta_i$, agent i maximizes the sum of those two terms, producing an efficient allocation for the type profile $(\theta_i, \hat{\theta}_{-i})$. To say this yet another way, ignoring the h_i term, a Groves mechanism pays each agent the total utility of the other agents; hence, when including his own utility, each agent ends up maximizing the total utility of all agents — he internalizes any externalities caused by his report.

Let $x^*_{-i} : \Theta_{-i} \to X$ denote an efficient allocation rule if we ignore agent i's payoff: $\forall \theta$, $x^*_{-i}(\theta_{-i}) \in \arg\max_{x \in X} \sum_{j \neq i} u_j(x, \theta_j)$. A special case of Groves mechanisms, called the *Vickrey-Clarke-Groves* (VCG) mechanism, obtains when we set

$$h_i(\theta_{-i}) = \sum_{j \neq i} u_j(x^*_{-i}(\theta_{-i}), \theta_j),$$

so that the transfer rule becomes

$$t_i^V(\theta) = \sum_{j \neq i} u_j(x^*_{-i}(\theta_{-i}), \theta_j) - \sum_{j \neq i} u_j(x^*(\theta), \theta_j). \tag{15}$$

The direct mechanism here is defined so as to make each agent precisely pay for the externality his presence (or his announcement) creates. To see this, note that the first term in t_i^V above is the total utility for all agents but i from an efficient allocation that ignores i's presence, while the second term is their total utility from an efficient allocation when i is present. In particular, $t_i^V(\theta) = 0$ if i is not "pivotal", i.e., if $x^*_{-i}(\theta_{-i}) = x^*(\theta)$.

The VCG mechanism (or Groves mechanisms, more broadly) is important because it says that we can achieve efficiency despite private information in various problems. Here is one example.

Example 38. Consider allocating an indivisible good. Let $X = I$, denoting which agent gets the good. Let $u_i(i, \theta_i) = \theta_i$ and $u_i(j, \theta_i) = 0$ for $j \neq i$. So the agent who gets the good gets payoff θ_i, and all other agents get 0. Consequently, ignoring the issue of ties (i.e., assuming $\theta_j \neq \theta_i$ for all $i, j \in I$), there is a unique efficient allocation rule:

$$x^*(\theta) = \arg\max_{j \in I} \theta_j, \text{ and } x^*_{-i}(\theta_{-i}) = \arg\max_{j \neq i} \theta_j.$$

Plugging into (15) yields

$$t_i^V(\theta) = \begin{cases} \max_{j \neq i} \theta_j & \text{if } x^*(\theta) = i \\ 0 & \text{otherwise.} \end{cases}$$

That is, the VCG mechanism in this context is simply a second-price auction!

References

Abreu, D., P. K. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A Neu Condition," *Econometrica*, 62(4), 939–948.

Abreu, D., D. Pearce, and E. Stacchetti (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, 58(5), 1041–1063.

Aliprantis, C. D., and K. C. Border (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide.* Springer.

Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69(4), 861–889.

Aumann, R. J. (1964): "Mixed and Extensive Strategies in Infinite Extensive Games," in *Advances in Game Theory*, ed. by M. Dresher, L. Shapley, and A. Tucker, vol. 52 of *Annals of Mathematics Studies*, pp. 627–650. Princeton University Press.

Banks, J. S., and J. Sobel (1987): "Equilibrium Selection in Signaling Games," *Econometrica*, 55(3), 647–661.

Benoit, J.-P., and V. Krishna (1985): "Finitely Repeated Games," *Econometrica*, 53(4), 905–922.

Bernheim, B. D. (1984): "Rationalizable Strategic Behavior," *Econometrica*, 52(4), 1007–28.

Blume, A. (2003): "Bertrand Without Fudge," *Economics Letters*, 78(2), 167–168.

Brandenburger, A., and E. Dekel (1993): "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59(1), 189–198.

Brandenburger, A., A. Friedenberg, and H. J. Keisler (2008): "Admissibility in Games," *Econometrica*, 76(2), 307–352.

Carlsson, H., and E. Van Damme (1993): "Global games and equilibrium selection," *Econometrica: Journal of the Econometric Society*, pp. 989–1018.

Cho, I.-K., and D. Kreps (1987): "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 102(2), 179–221.

Crawford, V., and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50(6), 1431–1451.

Ewerhart, C. (2002): "Backward Induction and the Game-Theoretic Analysis of Chess," *Games and Economic Behavior*, 39, 206–214.

Fudenberg, D., and J. Tirole (1991a): *Game Theory.* MIT Press, Cambridge, MA.

——— (1991b): "Perfect Bayesian Equilibrium and Sequential Equilibrium," *Journal of Economic Theory*, 53(2), 236–260.

Gul, F. (1998): "A Comment on Aumann's Bayesian View," *Econometrica*, 66(4), 923–927.

Harsanyi, J. C. (1973): "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points," *International Journal of Game Theory*, 2, 1–23.

Jackson, M., L. Simon, J. Swinkels, and W. Zame (2002): "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica*, 70(5), 1711–1740.

Jackson, M. O. (1991): "Bayesian Implementation," *Econometrica*, 59(2), 461–77.

Jackson, M. O. (2001): "A Crash Course in Implementation Theory," *Social Choice and Welfare*, 18(4), 655–708.

Kohlberg, E., and J.-F. Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54(5), 1003–1037.

Kreps, D., and R. Wilson (1982): "Sequential Equilibria," *Econometrica*, 50(4), 863–894.

Kreps, D. M., and J. A. Scheinkman (1983): "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes," *Bell Journal of Economics*, 14(2), 326–337.

Mailath, G. J., and L. Samuelson (2006): *Repeated Games and Reputations*. Oxford University Press.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press.

Maskin, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66(1), 23–38.

Mertens, J.-F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14(1), 1–29.

Milgrom, P. R., and R. J. Weber (1985): "Distributional Strategies for Games of Incomplete Information," *Mathematics of Operations Research*, 10(4), 619–632.

Monderer, D., and D. Samet (1989): "Approximating common knowledge with common beliefs," *Games and Economic Behavior*, 1(2), 170–190.

Morris, S., and H. S. Shin (1998): "Unique equilibrium in a model of self-fulfilling currency attacks," *American Economic Review*, pp. 587–597.

Osborne, M. J., and A. Rubinstein (1994): *A Course in Game Theory*. MIT Press.

Pearce, D. G. (1984): "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica*, 52(4), 1029–50.

Reny, P. J. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

Riley, J. G. (1979): "Informational Equilibrium," *Econometrica*, 47(2), 331–359.

Rubinstein, A. (1989): "The Electronic Mail Game: Strategic Behavior under "Almost Common Knowledge."," *American Economic Review*, 79(3), 385–91.

Selten, R. (1975): "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25–55.

Sorin, S. (1986): "On Repeated Games with Complete Information," *Mathematics of Operations Research*, 11(1), 147–160.

Spence, M. (1973): "Job Market Signaling," *Quarterly Journal of Economics*, 87(3), 355–374.

Weinstein, J., and M. Yildiz (2007): "A structure theorem for rationalizability with application to robust predictions of refinements," *Econometrica*, 75(2), 365–400.

Zamir, S. (2008): "Bayesian games: Games with incomplete information," Discussion Paper Series dp486, Center for Rationality and Interactive Decision Theory, Hebrew University, Jerusalem.

Lectures on Stochastic Choice

Tomasz Strzalecki

this version: March 8, 2019

Table of Contents

Disclaimer

- I won't get too deeply into any one area
- The monograph (in preparation) fills in more details
  - Theorem† means there are some terms I did not define
  - Theorem‡ means that additional technical conditions are needed
- I cover mostly work in decision theory. I am not an expert on neighboring fields, such as discrete choice econometrics, structural IO and labor, experimental economics, psychology and economics, cognitive science. Happy to talk if you are one.
- All comments welcome at tomasz_strzalecki@harvard.edu

Notation

X $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ set of alternatives

$x, y, z \in X$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ typical alternatives

$A, B, C \subseteq X$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ finite choice problems (menus)

$\rho(x, A)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ probability of x being chosen from A

ρ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ stochastic choice function (rule)

Stochastic Choice

- **Idea**: The analyst/econometrician observes an agent/group of agents
- **Examples**:
  - Population-level field data: McFadden (1973)
  - Individual-level field data: Rust (1987)
  - Between-subjects experiments: Kahneman and Tversky (1979)
  - Within-subject experiments: Tversky (1969)

Is individual choice random?

Stylized Fact: Choice can change, even if repeated shortly after

- Tversky (1969)
- Hey (1995)
- Ballinger and Wilcox (1997)
- Hey (2001)
- Agranov and Ortoleva (2017)

Why is individual choice random?

- Randomly fluctuating tastes
- Noisy signals
- Attention is random
- Experimentation (experience goods)

} Agent does not see his choice as random. Stochastic choice is a result of informational asymmetry between agent and analyst.

- People just like to randomize
- Trembling hands

} Analyst and agent on the same footing

Questions

1. What are the properties of ρ (axioms)?

   - Example: "*Adding an item to a menu reduces the choice probability of all other items*"

2. How can we "explain" ρ (representation)?

   - Example: "*The agent is maximizing utility, which is privately known*"

Goals

1. Better understand the properties of a model. What kind of predictions does it make? What axioms does it satisfy?
   - Ideally, prove a *representation theorem* (ρ satisfies Axioms A and B if and only if it has a representation R)

2. Identification: Are the parameters pinned down uniquely?

3. Determine whether these axioms are reasonable, either normatively, or descriptively (testing the axioms)

4. Compare properties of different models (axioms can be helpful here, even without testing them on data). Outline the modeling tradeoffs

5. Estimate the model, make a counterfactual prediction, evaluate a policy (I won't talk about those here)

Testing the axioms

- Axioms expressed in terms of ρ, which is the limiting frequency
- How to test such axioms when observed data is finite?
- Hausman and McFadden (1984) developed a test of Luce's IIA axiom that characterizes the logit model
- Kitamura and Stoye (2018) develop tests of the static random utility model based on axioms of McFadden and Richter (1990)
- I will mention many other axioms here, without corresponding "tests"

Richness

- The work in decision theory often assumes a "rich" menu structure
  - Menu variation can be generated in experiments
  - But harder in field data
  - But don't need a full domain to *reject* the axioms
- The work in discrete choice econometrics often assumes richness in "observable attributes"
  - I will mostly abstract from this here
- The two approaches lead to somewhat different identification results
  - Comparison?

Random Utility (RU)

Idea: Choice is random because:

- There is a population of heterogenous individuals
- Or there is one individual with varying preferences

Story (Data Generating Process):

- The menu A is drawn at random
  - maybe by the analyst or the experimenter
- A utility function is drawn at random (with probability distribution $\mathbb{P}$)
  - independently of the menu
- Agent chooses $x \in A$ whenever x maximizes the utility on A

Questions

- The most we can obtain from the data (with infinite sample) is
  - The distribution over menus
  - The conditional choice probability of choosing x from A: $\rho(x, A)$
- So you can think of the likelihood function $\mathbb{P} \mapsto \rho(x, A)$
  - Is this mapping one-to-one? (Identification/ Partial Identification)
  - What is the image of this mapping? (Axiomatization)

Random Utility (RU)

How to model a random utility function on X?

Depending on the context it will be either:

- a probability distribution μ over utility functions living in $\mathbb{R}^X$
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a $\mathcal{F}$-measurable random utility function $\tilde{U} : \Omega \to \mathbb{R}^X$

Notes:

- Given μ we can always take the canonical state space where $\Omega = \mathbb{R}^X$, $\tilde{U}$ the identity mapping, and $\mathbb{P} = \mu$.
- Or even $\Omega =$ all strict preferences
- Ω is a subjective state space, related to Kreps (1979) and Dekel, Lipman, and Rustichini (2001) $\rightsquigarrow$ Lecture 3

Random Utility (RU)

$C(x, A)$ is the event in which the agent chooses x from A

$$C(x, A) := \{\omega \in \Omega : \tilde{U}_\omega(x) \geq \tilde{U}_\omega(y) \text{ for all } y \in A\}$$

This is the event in which the utility is maximized at $x \in A$

Definition: ρ has a *random utility representation* if there exists $\tilde{U}$ and

$$\rho(x, A) = \mathbb{P}\big(C(x, A)\big)$$

Key assumption:

- $\mathbb{P}$ is independent of the menu; it's the structural invariant of the model
- Menu-dependent $\mathbb{P}$ can trivially explain any ρ

Discrete Choice (DC)

- Let $v \in \mathbb{R}^X$ be a deterministic utility function
- Let $\tilde{\epsilon} : \Omega \to \mathbb{R}^X$ be a random *unobserved utility shock* or *error*
  - the distribution of $\tilde{\epsilon}$ has a density and full support

Definition: ρ has a *discrete choice* representation if it has a RU representation with $\tilde{U}(x) = v(x) + \tilde{\epsilon}(x)$

Remark: This is sometimes called the *additive random utility* model

Ties

- $T^{\tilde{U}}$ is the event in which there is a tie

$$T^{\tilde{U}} := \{\omega \in \Omega : \tilde{U}_\omega(x) = \tilde{U}_\omega(y) \text{ for some } x \neq y\}$$

- Notice that RU implies that $\mathbb{P}(T^{\tilde{U}}) = 0$
  - this is because $\sum_{x \in A} \rho(x, A) = 1$
  - in DC guaranteed by assuming that $\tilde{\epsilon}$ has a density

- So not every $\tilde{U}$ leads to a legitimate ρ

Ways to deal with ties

- Sometimes convenient to allow ties (esp. when X is infinite)
- For example, randomize uniformly over $\text{argmax}_{x \in A} \tilde{U}_\omega(x)$
- A more general idea of tie-breaking was introduced by Gul and Pesendorfer (2006)
- A different approach is to change the primitive (stochastic choice correspondence: Barberá and Pattanaik, 1986; Lu, 2016; Gul and Pesendorfer, 2013)

Random Utility (with a tiebreaker)

- A *tie-breaker* is a random utility function $\tilde{W} : \Omega \to \mathbb{R}^X$, (which is always a strict preference)
- The agent first maximizes $\tilde{U}$ and if there is a tie, it gets resolved using $\tilde{W}$

Definition: ρ has a *random utility representation with a tie-breaker* if there exists $(\Omega, \mathcal{F}, \mathbb{P})$, $\tilde{U}, \tilde{W} : \Omega \to \mathbb{R}^X$ such that $\mathbb{P}(T^{\tilde{W}}) = 0$, and

$$\rho(x, A) = \mathbb{P}\left(\{\omega \in \Omega : \tilde{W}_\omega(x) \geq \tilde{W}_\omega(y) \text{ for all } y \in \operatorname{argmax}_{x \in A} \tilde{U}_\omega(x)\}\right).$$

Equivalence

Theorem: The following are equivalent when X is finite:

- ρ has a RU representation
- ρ has a RU representation with uniform tie breaking
- ρ has a RU representation with a tiebreaker

Thus, even though the representation is more general, the primitive is not.

When X is infinite (for example lotteries) these things are different.

Positivity

The full support assumption on $\tilde{\epsilon}$ ensures that all items are chosen with positive probability

Axiom (Positivity). $\rho(x, A) > 0$ for all $x \in A$

- This leads to a non-degenerate likelihood function—good for estimation
- Positivity cannot be rejected by any finite data set

Equivalence

Theorem: If X is finite and ρ satisfies Positivity, then the following are equivalent:

(i) ρ has a random utility representation

(ii) ρ has a discrete choice representation

Questions:

- What do these models assume about ρ?
- Are their parameters identified?
- Are there any differences between the two approaches?

i.i.d. DC

- It is often assumed that $\tilde{\epsilon}_x$ are i.i.d. across $x \in X$
  - logit, where ϵ has a mean zero extreme value distribution
  - probit, where ϵ has a mean zero Normal distribution

- In i.i.d. DC the binary choice probabilities are given by

$$\begin{aligned}\rho(x, \{x, y\}) &= \mathbb{P}(v(x) + \tilde{\epsilon}_x \geq v(y) + \tilde{\epsilon}_y) \\ &= \mathbb{P}(\tilde{\epsilon}_y - \tilde{\epsilon}_x \leq v(x) - v(y)) = F(v(x) - v(y)),\end{aligned}$$

where F is the cdf of $\tilde{\epsilon}_y - \tilde{\epsilon}_x$ (such models are called Fechnerian)

Fechnerian Models

Definition: ρ has a *Fechnerian* representation if there exist a utility function $v : X \to \mathbb{R}$ and a strictly increasing transformation function F such that

$$\rho(x, \{x, y\}) = F(v(x) - v(y))$$

Comments:

- This property of ρ depends only on its restriction to binary menus
- RU in general is not Fechnerian because it violates Weak Stochastic Transitivity (Marschak, 1959)
- Some models outside of RU are Fechnerian, e.g., APU $\rightsquigarrow$ Lecture 2

References: Davidson and Marschak (1959); Block and Marschak (1960); Debreu (1958); Scott (1964); Fishburn (1998)

The Luce Model

Definition: ρ has a *Luce representation* iff there exists $w : X \to \mathbb{R}_{++}$ such that

$$\rho(x, A) = \frac{w(x)}{\sum_{y \in A} w(y)}.$$

Intuition 1: The Luce representation is like a conditional probability: the probability distribution on A, is the conditional of the probability distribution on the grand set X.

Intuition 2: $w(x)$ is the "response strength" associated with x. Choice probability is proportional to the response strength.

Theorem (McFadden, 1973): The following are equivalent

(i) ρ has a logit representation with v

(ii) ρ has a Luce representation with $w = e^v$

Proof: This is a calculation you all did in 1^{st} year metrics $\square$

DC with characteristics

- Typically menu is fixed, $A = X$
- $\xi \in \mathbb{R}^n$ vector of observable characteristics
- $\tilde{U}(x;\xi) = v(x;\xi) + \tilde{\epsilon}(x)$
- Observed choices $\rho(x, X; \xi)$

Note: Allen and Rehbeck (2019) also study $\rho(x, X; \xi)$, but with a perturbed utility representation instead of discrete choice representation $\rightsquigarrow$ Lecture 2

Generalizations

- Removing Positivity (Echenique and Saito, 2015; Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini, 2018; Ahumada and Ülkü, 2018)
- Nested logit (Train, 2009, for axioms see Kovach and Tserenjigmid, 2019)
- GEV (generalized extreme value; Train, 2009)
- Multivariate probit (Train, 2009)
- Mixed logit (McFadden and Train, 2000; Gul, Natenzon, and Pesendorfer, 2014; Saito, 2018)

Generalizations

- Elimination by aspects (Tversky, 1972)
- Random Attention (Manzini and Mariotti, 2014)
- Attribute rule (Gul, Natenzon, and Pesendorfer, 2014)
- Additive Perturbed Utility (Fudenberg, Iijima, and Strzalecki, 2015)
- Perception adjusted Luce (Echenique, Saito, and Tserenjigmid, 2018)
- Imbalanced Luce (Kovach and Tserenjigmid, 2018)
- Threshold Luce (Horan, 2018)

Regularity

Axiom (Regularity). If $x \in A \subseteq B$, then $\rho(x, A) \geq \rho(x, B)$

Intuition When we add an item to a menu, existing items have to "make room" for it.

Theorem (Block and Marschak, 1960). If ρ has a random utility representation, then it satisfies Regularity.

Proof:

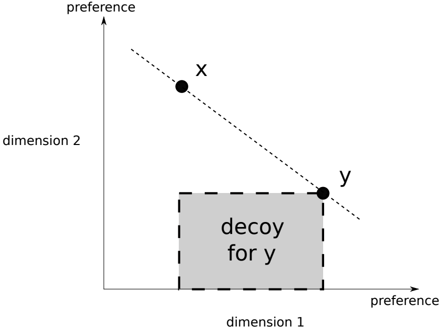
Step 1: If x maximizes u on B, then x maximizes u on A (because A is smaller). Thus, For any $x \in A \subseteq B$ we have $C(x, A) \supseteq C(x, B)$.

Step 2: $\mathbb{P}$ is set-monotone, so $\mathbb{P}(C(x, A)) \geq \mathbb{P}(C(x, B))$ $\square$

Violations of Regularity

1. Iyengar and Lepper (2000): tasting booth in a supermarket
   - 6 varieties of jam — 70% people purchased no jam
   - 24 varieties of jam — 97% people purchased no jam

2. Huber, Payne, and Puto (1982): adding a "decoy" option raises demand for the targeted option



Theorem (Block and Marschak, 1960). Suppose that $|X| \leq 3$. If ρ satisfies Regularity, then ρ has a random utility representation.

Proof Idea:

- For each menu A sets $C(x, A)$ form a partition of Ω
- ρ defines a probability distribution over each partition
- Need to ensure that they are consistent with a single $\mathbb{P}$

Proof

Wlog $\Omega = \{xyz, xzy, yxz, yzx, zxy, zyx\}$. To define $\mathbb{P}(xyz)$ note that $C(y, \{y, z\}) = \{xyz, yxz, yzx\}$ and $C(y, X) = \{yxz, yzx\}$, so define

$$\mathbb{P}(xyz) := \rho(y, \{y, z\}) - \rho(y, X).$$

Likewise,

$$\begin{aligned}
\mathbb{P}(xzy) &:= \rho(z, \{y, z\}) - \rho(z, X) \\
\mathbb{P}(yxz) &:= \rho(x, \{x, z\}) - \rho(x, X) \\
\mathbb{P}(yzx) &:= \rho(z, \{x, z\}) - \rho(z, X) \\
\mathbb{P}(zxy) &:= \rho(x, \{x, y\}) - \rho(x, X) \\
\mathbb{P}(zyx) &:= \rho(y, \{x, y\}) - \rho(y, X)
\end{aligned}$$

By Regularity, they are nonnegative. They sum up to $3 - 2 = 1$. Finally, $\rho(x, A) = \mathbb{P}(C(x, A))$ follows from the above definitions as well. $\square$

Comments:

- Unfortunately, when $|X| > 3$, Regularity alone is not enough
- More axioms are needed, but they are hard to interpret
- More elegant axioms if X consists of lotteries (Gul and Pesendorfer, 2006) $\rightsquigarrow$ later in this lecture

Block and Marschak

Axiom (Block and Marschak, 1960): For all $x \in A$

$$\sum_{B \supseteq A} (-1)^{|B \setminus A|} \rho(x, B) \geq 0.$$

Comments:

- Generalizes the idea that we get information from looking at difference between $\rho(x, A)$ and $\rho(x, B)$
- Inclusion-Exclusion formula (Möbus transform)

Other Axioms

Axiom (McFadden and Richter, 1990): For any n, for any sequence $(x_1, A_1), \ldots, (x_n, A_n)$ such that $x_i \in A_i$

$$\sum_{i=1}^{n} \rho(x_i, A_i) \leq \max_{\omega \in \Omega} \sum_{i=1}^{n} \mathbb{1}_{C^{\succsim}(x_i, A_i)}(\succsim).$$

Axiom (Clark, 1996): For any n, for any sequence $(x_1, A_1), \ldots, (x_n, A_n)$ such that $x_i \in A_i$, and for any sequence of real numbers $\lambda_1, \ldots, \lambda_n$

$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{C^{\succsim}(x_i, A_i)} \geq 0 \Longrightarrow \sum_{i=1}^{n} \lambda_i \rho(x_i, A_i) \geq 0.$$

Remark: These axioms refer to the canonical random preference representation where Ω is the set of all strict preference relations and the mapping $\succsim$ is the identity

Axiomatic Characterizations

Theorem: The following are equivalent for a finite X

(i) ρ has a random utility representation

(ii) ρ satisfies the Block–Marschak axiom

(iii) ρ satisfies the McFadden–Richter axiom

(iv) ρ satisfies the Clark axiom.

Comments:

- The equivalence *(i)–(ii)* was proved by Falmagne (1978) and Barberá and Pattanaik (1986).
- The equivalences *(i)–(iii)* and *(i)–(iv)* were proved by McFadden and Richter (1990, 1971) and Clark (1996) respectively. They hold also when X is infinite (Clark, 1996; McFadden, 2005; Chambers and Echenique, 2016).

Axioms for Luce/Logit

Axiom (Luce's IIA). For all $x, y \in A \cap B$ whenever the probabilities are positive

$$\frac{\rho(x, A)}{\rho(y, A)} = \frac{\rho(x, B)}{\rho(y, B)}.$$

Axiom (Luce's Choice Axiom). For all $x \in A \subseteq B$

$$\rho(x, B) = \rho(x, A)\rho(A, B).$$

Theorem (Luce, 1959; McFadden, 1973): The following are equivalent

(i) ρ satisfies Positivity and Luce's IIA

(ii) ρ satisfies Positivity and Luce's Choice Axiom

(iii) ρ has a Luce representation

(iv) ρ has a logit representation

Proof

Luce $\Rightarrow$ **IIA** is straightforward:

$$\frac{\rho(x, A)}{\rho(y, A)} = \frac{w(x)}{w(y)} = \frac{\rho(x, B)}{\rho(y, B)}$$

Luce $\Rightarrow$ **Positivity** is also straightforward since $w(x) > 0$ for all $x \in X$

Proof

To show **Luce's IIA+Positivity** $\Rightarrow$ **Luce** for X finite, define $w(x) := \rho(x, X)$.

Fix A and $x^* \in A$. By IIA,

$$\rho(x, A) = \rho(x, X)\frac{\rho(x^*, A)}{\rho(x^*, X)} = w(x)\frac{\rho(x^*, A)}{w(x^*)}.$$

Summing up over $y \in A$ and rearranging we get $\frac{\rho(x^*, A)}{w(x^*)} = \frac{1}{\sum_{y \in A} w(y)}$.
When X is infinite, need to modify the proof slightly.

Proof

To show **IIA+Positivity** $\Rightarrow$ **Luce** for X finite, define $w(x) := \rho(x, X)$.

Fix A. By Luce's Choice Axiom,

$$\begin{aligned} \rho(x, X) &= \rho(x, A)\rho(A, X) \\ &= \rho(x, A) \sum_{y \in A} \rho(y, X) \end{aligned}$$

so $w(x) = \rho(x, A) \sum_{y \in A} w(y)$. When X is infinite, need to modify the proof slightly. $\square$

Remark: Luce's IIA is equivalent to Luce's Choice Axiom even without Positivity, see Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2018).

Other forms of IIA

Remark: IIA has a cardinal feel to it (we require ratios of probabilities to be equal to each other). Consider the following ordinal axiom.

Axiom (GNP IIA). If $A \cup B$ and $C \cup D$ are disjoint, then

$$\rho(A, A \cup C) \geq \rho(B, B \cup C) \Longrightarrow \rho(A, A \cup D) \geq \rho(B, B \cup D)$$

Theorem† (Gul, Natenzon, and Pesendorfer, 2014). In the presence of Richness†, ρ satisfies GNP IIA iff it has a Luce representation.

Blue bus/red bus paradox for i.i.d. DC

Example: Transportation choices are: train, or bus. There are two kinds of buses: blue bus and red bus. So $X = \{t, bb, rb\}$. Suppose that we observed that

$$\rho(t, \{t, bb\}) = \rho(t, \{t, rb\}) = \rho(bb, \{bb, rb\}) = \frac{1}{2}.$$

If ρ is i.i.d. DC, then $\rho(t, X) = \frac{1}{3}$. But this doesn't make much sense if you think that the main choice is between the modes of communication (train or bus) and the bus color is just a tie breaker. In that case we would like to have $\rho(t, X) = \frac{1}{2}$.

If you are still not convinced, imagine that there n colors of buses. Would you insist on $\rho(t, X) \to 0$ as $n \to \infty$?

Solution to the blue bus/red bus paradox

- Don't use i.i.d. DC, but some RU model
  - for example put equal probability on orders
    $bb \succ rb \succ t,\ rb \succ bb \succ t,\ t \succ bb \succ rb,\ t \succ rb \succ bb$
  - or use the attribute rule of Gul, Natenzon, and Pesendorfer (2014)
  - or use parametric DC families (nested logit, GEV)
- But no need to go outside of the RU class. This is not a paradox for RU but for i.i.d. DC

Weak Stochastic Transitivity

Definition: $x \succsim^s y$ iff $\rho(x, A) \geq \rho(y, A)$ for $A = \{x, y\}$

Definition: ρ satisfies *Weak Stochastic Transitivity* iff $\succsim^s$ is transitive

Satisfied by: Fechnerian models because $x \succsim^s y$ iff $v(x) \geq v(y)$

Can be violated by:

- RU (Marschak, 1959)
- random attention (Manzini and Mariotti, 2014)
- deliberate randomization (Machina, 1985)

Stylized Fact: Weak Stochastic Transitivity is typically satisfied in lab experiments (Rieskamp, Busemeyer, and Mellers, 2006)

Forms of Stochastic Transitivity

Let $p = \rho(x, \{x, y\})$, $q = \rho(y, \{y, z\})$, $r = \rho(x, \{x, z\})$.

Definition: Suppose that $p, q \geq 0.5$. Then ρ satisfies

- *Weak Stochastic Transitivity* if $r \geq 0.5$
- *Moderate Stochastic Transitivity* if $r \geq \min\{p, q\}$
- *Strong Stochastic Transitivity* if $r \geq \max\{p, q\}$

Forms of Stochastic Transitivity

Tversky and Russo (1969) characterize the class of binary choice models that satisfy (a slightly stronger version of) Strong Stochastic Transitivity. Under Positivity, they are the models that have a *simple scalability* representation.

Definition: ρ has a *simple scalability* representation if $\rho(x, \{x, y\}) = F(v(x), v(y))$ for some $v : X \to \mathbb{R}$ and $F : \mathbb{R}^2 \to [0, 1]$, defined on an appropriate domain, is strictly increasing in the first argument and strictly decreasing in the second argument.

Note: Fechnerian is a special case where $F(v(x), v(y)) = F(v(x) - v(y))$.

Forms of Stochastic Transitivity

He and Natenzon (2018) characterize the class of models that satisfy (a slightly stronger version of) Moderate Stochastic Transitivity. These are the models that are represented by *moderate utility*.

Definition: ρ has a *moderate utility* representation if $\rho(x, \{x, y\}) = F\left(\frac{u(x)-u(y)}{d(x,y)}\right)$ for some $u : X \to \mathbb{R}$, distance metric $d : X \times X \to \mathbb{R}_+$, and $F : \mathbb{R} \to [0, 1]$ strictly increasing transformation, defined on an appropriate domain, that satisfies $F(t) = 1 - F(1 - t)$

Introduction

Random Utility/Discrete Choice

Representations

Special Cases

Axioms

Identification/Uniqueness

Random Expected Utility (REU)

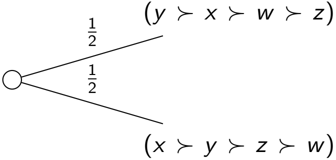
Representation and Axioms

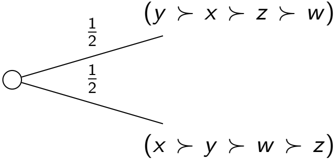
Measuring Risk Preferences

Identification of Utilities

- Since utility is ordinal, we cannot identify its distribution—at best we can hope to pin down the distribution of ordinal preferences
- But it turns out we can't even do that

Example (Fishburn, 1998). Suppose that $X = \{x, y, z, w\}$. The following two distributions over preferences lead to the same ρ.





Note that these two distributions have disjoint supports!

Identification of "Marginal" Preferences

Theorem (Falmagne, 1978). If $\mathbb{P}_1$ and $\mathbb{P}_2$ are RU representations of the same ρ, then for any $x \in X$

$$\mathbb{P}_1(x \text{ is } k\text{-th best in } X) = \mathbb{P}_2(x \text{ is } k\text{-th best in } X)$$

for all $k = 1, \ldots, |X|$.

Identification in DC

Theorem: If $(v_1, \tilde{\epsilon}_1)$ is a DC representation of ρ, then for any $v_2 \in \mathbb{R}^X$ there exists $\tilde{\epsilon}_2$ such that $(v_2, \tilde{\epsilon}_2)$ is another representation of ρ

Comments:

- So can't identify v (even ordinally) unless make assumptions on unobservables
- If assume a given distribution of $\tilde{\epsilon}$, then can pin down more
- Also, stronger identification results are obtained in the presence of "observable attributes"

i.i.d. DC with known distribution of $\tilde{\epsilon}$

Theorem: If $(v_1, \tilde{\epsilon})$ and $(v_2, \tilde{\epsilon})$ are i.i.d DC representations of ρ that share the distribution of $\tilde{\epsilon}$, then there exists $k \in \mathbb{R}$ such that $v_2(x) = v_1(x) + k$ for all $x \in X$.

Proof: Fix $x^* \in X$ and normalize $v(x^*) = 0$. Let F be the cdf of the ϵ difference. By Fechnerianity

$$\rho(x, \{x, x^*\}) = F(v(x)),$$

so $v(x) = F^{-1}(\rho(\{x, \{x, x^*\}))$. $\square$

Remark: If we know F, this gives us a recipe for identifying v from data.

Unknown distribution of $\tilde{\epsilon}$; observable attributes

- Under appropriate assumptions can identify $v(x;\xi)$ and the distribution of $\tilde{\epsilon}$
- Matzkin (1992) and the literature that follows

Random Expected Utility (REU)

- Gul and Pesendorfer (2006) study choice between lotteries
- Specify the RU model to $X = \Delta(Z)$, where Z is a finite set of prizes
- Typical items are now $p, q, r \in X$

Definition: ρ has a REU representation if has a RU representation where with probability one $\tilde{U}$ has vNM form:

$$\tilde{U}(p) := \mathbb{E}_p \tilde{u} := \sum_{z \in Z} \tilde{u}(z) p(z)$$

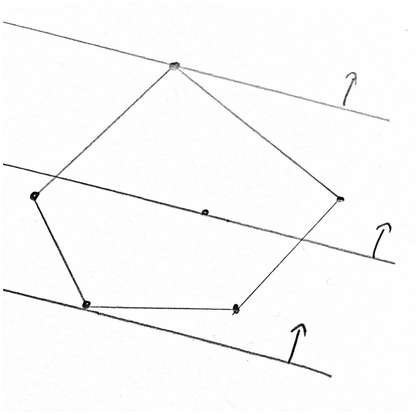
for some random Bernoulli utility function $\tilde{u} \in \mathbb{R}^Z$

REU—Axioms

Notation: $Ext(A)$ is the set of extreme points of A

Axiom (Extremeness). $\rho(Ext(A), A) = 1$

Idea: The indifference curves are linear, so maximized at an extreme point of the choice set (modulo ties)



REU—Axioms

Definition: $\alpha A + (1-\alpha)q := \{\alpha p' + (1-\alpha)q : p' \in A\}$

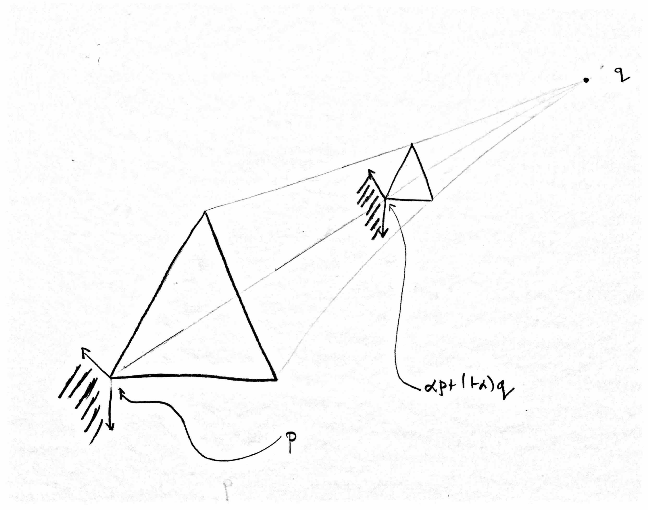
Axiom (Linearity). For any $\alpha \in (0,1)$ and $p \in A$ and $q \in X$

$$\rho(p, A) = \rho(\alpha p + (1-\alpha)q, \alpha A + (1-\alpha)q)$$

Idea: The vNM Independence axiom applied utility by utility

$$\tilde{u}_w \in C(p, A) \Longleftrightarrow u_w \in C\left(\alpha p + (1-\alpha)q, \alpha A + (1-\alpha)q\right)$$

Linearity



$C(p, A)$ is the normal cone of A at p

same for the mixture with q

they are equal because they are "corresponding angles"

REU—Gul and Pesendorfer (2006) Results

Theorem† (Characterization). ρ has a REU representation if and only if it satisfies

- Regularity
- Extremeness
- Linearity
- Continuity†

Theorem† (Uniqueness). In a REU representation the distribution over ordinal preferences is identified.

REU—Comments

- Simpler axioms
- Better identification results
- Stronger assumptions: vNM relaxed $\rightsquigarrow$ Lecture 2
  - Allais (1953) paradox is a rejection of Linearity
  - Agranov and Ortoleva (2017) is a rejection of Extremeness
- Model used as a building block for a lot to come
- This is only one possible specification of risk preferences ...

REU—Comments

- Gul and Pesendorfer (2006) introduce tiebreakers
  - weakening of Continuity, tiebreaker are finitely additive
  - Extremeness hinges on tiebreakers being EU themselves (uniform tiebreaking violates Extremenes)
- finite support: Ahn and Sarver (2013)
  - Add a Finiteness axiom to get finitely many $\tilde{U}$
  - Useful in dynamic model to avoid conditioning on zero-probability events

Measuring Risk Preferences

- Let U_θ be a family of vNM forms with CARA or CRRA indexes
- Higher θ is more risk-aversion (allow for risk-aversion and risk-loving)

Model 1 (à la REU): There is a probability distribution $\mathbb{P}$ over error shocks $\tilde{\epsilon}$ to the preference parameter θ

$$\rho_\theta^{REU}(p, A) = \mathbb{P}\{U_{\theta+\tilde{\epsilon}}(p) \geq U_{\theta+\tilde{\epsilon}}(q) \text{ for all } q \in A\}$$

Model 2 (à la DC): There is a probability distribution $\mathbb{P}$ over error shocks $\tilde{\epsilon}$ to the expected value

$$\rho_\theta^{DC}(p, A) = \mathbb{P}\{U_\theta(p) + \tilde{\epsilon}(p) \geq U_\theta(q) + \tilde{\epsilon}(q) \text{ for all } q \in A\}$$

Comment: In Model 2, preferences over lotteries are not vNM!

Measuring Risk Preferences

Notation:

- FOSD—First Order Stochastic Dominance
- SOSD—Second Order Stochastic Dominance

Observation 1: Model 1 has intuitive properties:

- If p FOSD q, then $\rho_\theta^{REU}(p, \{p, q\}) = 1$
- If p SOSD q, then $\rho_\theta^{REU}(p, \{p, q\})$ is increasing in θ

Observation 2: Model 2 not so much:

- If p FOSD q, then $\rho_\theta^{DC}(p, \{p, q\}) < 1$
- If p SOSD q, then $\rho_\theta^{DC}(p, \{p, q\})$ is not monotone in θ

Theorem‡: (Wilcox, 2008, 2011; Apesteguia and Ballester, 2017) If p SOSD q, then $\rho_\theta^{DC}(p, \{p, q\})$ is strictly decreasing for large enough θ.

Comments:

- This biases parameter estimates
- Subjects may well violate FOSD and SOSD. Better to model these violations explicitly rather than as artifacts of the error specification?
- A similar lack of monotonicity for discounted utility time-preferences
- Apesteguia, Ballester, and Lu (2017) study a general notion of single-crossing for random utility models

Lecture 2 on Stochastic Choice

Tomasz Strzalecki

Recap of Random Utility

- $(\Omega, \mathcal{F}, \mathbb{P})$ $\cdots\cdots\cdots\cdots\cdots\cdots$ probability space

- $\tilde{U} : \Omega \to \mathbb{R}^X$ $\cdots\cdots\cdots\cdots\cdots$ random utility

- $C(x, A) := \{\omega \in \Omega : \tilde{U}_\omega(x) \geq \tilde{U}_\omega(y)$ for all $y \in A\}$
  - agent learns the state (his utility) and chooses optimally

- $\rho(x, A) = \mathbb{P}(C(x, A))$
  - analyst does not see the state; the observed choice frequency of x from A is the probability that x is the argmax of the agent's utility on A

Learning

- In RU choice is stochastic because preferences are fluctuating
- Another possible reason: choices are driven by agent's noisy signals
  - Agent does not learn the state perfectly but gets a signal of it
- What kinds of ρ does this lead to?
  - If information is independent of the menu, this is a special case of RU
  - Strict subset of RU if model is rich enough
  - What if information can depend on the menu? $\rightsquigarrow$ later today

Learning—probabilistic model

- Let $\mathcal{G}$ represent *menu-independent* information the agent is learning.
- Conditional on the signal the agent maximizes $\mathbb{E}[\tilde{U}(x)|\mathcal{G}]$

Example: Hiring an applicant based on an interview

- Interview is a noisy signal
- Interview goes well $\Longrightarrow \mathbb{E}[\tilde{U}(\mathit{hire})|\mathcal{G}] > \mathbb{E}[\tilde{U}(\mathit{not})|\mathcal{G}]$
- Interview goes badly $\Longrightarrow \mathbb{E}[\tilde{U}(\mathit{hire})|\mathcal{G}] < \mathbb{E}[\tilde{U}(\mathit{not})|\mathcal{G}]$

Learning—probabilistic model

Comment: Choices are random because they depend on the signal realization

- No information ($\mathcal{G}$ trivial) $\Rightarrow$ choices are deterministic (agent maximizes ex ante expected utility)
- Full information ($\mathcal{G} = \mathcal{F}$) $\Rightarrow$ this is just a RU model
- In general, the finer the $\mathcal{G}$, the more random the choices, keeping $(\Omega, \mathcal{F}, \mathbb{P})$ constant

Learning—probabilistic model

Proposition: ρ has a probabilistic learning representation iff it has a RU representation

Proof:

- For any $\mathcal{G}$ the induced ρ has a RU representation $(\Omega, \mathcal{G}, \mathbb{P}, \tilde{V})$ with $\tilde{V} := \mathbb{E}[\tilde{U}|\mathcal{G}]$

- Any RU has a learning representation where $\mathcal{G} = \mathcal{F}$ (signal is fully revealing) $\square$

Comment: Need to enrich the model to get a strictly special case

- Separation of tastes and beliefs $\rightsquigarrow$ next slide

- Strictly special case of RU in a dynamic setting (Frick, Iijima, and Strzalecki, 2019) $\rightsquigarrow$ Lecture 3

Learning—statistical model

S $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ set of unknown states

$p \in \Delta(S)$ $\cdots\cdots\cdots\cdots\cdots$ prior belief

$u : S \to \mathbb{R}^X$ $\cdots\cdots\cdots$ deterministic state-dependent utility function

$\mathbb{E}_p u(x)$ $\cdots\cdots\cdots\cdots\cdots\cdots$ (ex ante) expected utility of x

- Signal structure: in each state s there is a distribution over signals
- For each signal realization, posterior beliefs are given by the Bayes rule
- The prior p and the signal structure $\Rightarrow$ random posterior $\tilde{q}$
  - For each posterior $\tilde{q}$ the agent maximizes $\max_{x \in A} \mathbb{E}_{\tilde{q}} u(x)$

Learning—statistical model

For each s, the model generates a choice distribution $\rho^s(x, A)$

- In some lab experiments the analyst can control/observe s
- This is a special case of observable attributes ξ from Lecture 1

An average of ρ^s according to the prior p generates $\rho(x, A)$

- That's when the analyst does not observe s

Learning—statistical model

Example (Classical experimental design in perception literature):

- $s\%$ of dots on the screen are moving left, $100 - s\%$ are moving right
- subject has to guess where most dots are moving
- imperfect perception, so noisy guesses
- experimenter controls s, observes ρ^s

Comments:

- The class of ρ generated this way equals the RU class
- For each s conditional choices ρ^s also belong to the RU class
  - Consistency conditions of ρ^s across $s \rightsquigarrow$ Caplin and Martin (2015)
- The (statistical) learning model becomes a strictly special case of RU when specified to Anscombe–Aumann acts (Lu, 2016)

Learning—the Lu (2016) model

- Random Utility model of choice between Anscombe–Aumann acts
- This means $X = \Delta(Z)^S$
  - In each state the agent gets a lottery over prizes in a finite set Z
  - Typical acts are denoted $f, g, h \in X$
- Random Utility $\tilde{U}(f) = \sum_{s \in S} u(f(s))\tilde{q}(s)$, where
  - u is a (deterministic) linear utility over $\Delta(Z)$
  - $\tilde{q}$ is the (random) posterior over S
- The distribution over $\tilde{q}$ is given by μ

Learning—the Lu (2016) model

Let $A(s) := \{f(s) : f \in A\}$.

Axiom (S-monotonicity): If $\rho(f(s), A(s)) = 1$ for all $s \in S$ then $\rho(f, A) = 1$.

Axiom (C-determinism): If A is a menu of constant acts, then $\rho(f, A) = 1$ for some $f \in A$.

Axiom (Non-degeneracy): $\rho(f, A) \in (0, 1)$ for some $f \in A$.

Learning—the Lu (2016) model

Theorem‡ (Characterization). ρ has a (statistical) learning representation iff it satisfies the Gul and Pesendorfer (2006) axioms *plus* S-monotonicity, C-determinisim.

‡ (Ties dealt with by stochastic choice correspondence)

Theorem‡ (Uniqueness). Under Non-degeneracy the the information structure μ is unique and the utility function *u* is cardinally-unique.

- In fact, the parameters can be identified on binary menus
- *Test functions*: calibration through constant acts

Theorem‡ (Comparative Statics). Fix *u* and *p* and consider two information structures μ and μ'. ρ is "more random" than ρ' if and only if μ is Blackwell-more informative than μ'.

Menu-dependent Learning

- Models of learning so far:
  - the probabilistic model (information is $\mathcal{G}$)
  - the statistical model (information is μ)
    - the Lu (2016) model
- In all of them information is independent of the menu
- But it could depend on the menu (so we would have $\mathcal{G}^A$ or μ^A):
  - if new items provide more information
  - or if there is limited attention $\rightsquigarrow$ later today

Example

	$\tilde{U}_\omega(steak\ tartare)$	$\tilde{U}_\omega(chicken)$	$\tilde{U}_\omega(fish)$
$\omega = good\ chef$	10	7	3
$\omega = bad\ chef$	0	5	0

- *fish* provides an informative signal about the quality of the chef
  - $\mathcal{G}^{\{s,c,f\}}$ gives full information:
    - if the whole restaurant smells like fish $\rightarrow$ chef is bad
    - if the whole restaurant doesn't smell like fish $\rightarrow$ chef is good
  - $\rho(s, \{s, c, f\}) = \rho(c, \{s, c, f\}) = \frac{1}{2}$ and $\rho(f, \{s, c, f\}) = 0$
- in absence of f get no signal
  - $\mathcal{G}^{\{s,c\}}$ gives no information
  - $\rho(s, \{s, c\}) = 0$, $\rho(c, \{s, c\}) = 1$ (if prior uniform)
- violation of the Regularity axiom!
  - menu-dependent information is like menu-dependent (expected) utility

Bayesian Probit

- Natenzon (2018) develops a Bayesian Probit model of this, where the agent observes noisy signal of the utility of each item in the menu
  - signals are jointly normal and correlated
  - model explains decoy effect, compromise effect, and similarity effects
  - correlation $\Rightarrow$ new items shed light on relative utilities of existing items

- Note: adding an item gives Blackwell-more information about the state, the state is *uncorrelated* with the menu

- **Question**: What is the family of ρ that has a general menu-dependent learning representation? What is the additional bite of Blackwell monotonicity? What if Blackwell is violated (information overload)?

Learning so far

- Information independent of the menu (RU or special case of RU)

- Information dependent on the menu (more general than RU)

In both cases, the true state was uncorrelated with the menu. What if there is such a correlation? $\rightsquigarrow$ in general can explain any ρ

Example (Luce and Raiffa, 1957)

	$\tilde{U}_\omega(\textit{steak tartare})$	$\tilde{U}_\omega(\textit{chicken})$	$\tilde{U}_\omega(\textit{frog legs})$
$\omega = \textit{good chef}$	10	7	3
$\omega = \textit{bad chef}$	0	5	0

- *frog legs* provides an informative signal about the quality of the chef
  - only good chefs will attempt to make *frog legs*
  - so $\{s, c, f\}$ signals $\omega = \textit{good chef}$
  - so $\{s, c\}$ signals $\omega = \textit{bad chef}$

- this implies
  - $\rho(s, \{s, c, f\}) = 1$, $\rho(c, \{s, c, f\}) = \rho(f, \{s, c, f\}) = 0$
  - $\rho(s, \{s, c\}) = 0$, $\rho(c, \{s, c\}) = 1$ (if prior uniform)

- so here the menu is directly *correlated* with the state
  - unlike in the *fish* example where there is no correlation
  - Kamenica (2008)–model where consumers make inferences from menus (model explains choice overload and compromise effect)

Learning—recap

- Information independent of menu
  - Special case of RU (or equivalent to RU depending on the formulation)
  - More informative signals $\Rightarrow$ more randomness in choice

- Information depends on the menu
  - More general than RU (can violate Regularity)
  - Two flavors of the model:
    - more items $\Rightarrow$ more information (Natenzon, 2018)
    - correlation between menu and state (Kamenica, 2008)
  - General analysis? Axioms?

Optimal Attention

- Imagine now that the signal structure is chosen by the agent
  - instead of being fixed
- The agent may want to choose to focus on some aspect
  - depending on the menu
- One way to model this margin of choice is to let the agent choose attention optimally:
  - *Costly Information Acquisition* (Raiffa and Schlaifer, 1961)
  - *Rational Inattention* (Sims, 2003)
  - *Costly Contemplation* (Ergin, 2003; Ergin and Sarver, 2010)

Value of Information

For each information structure μ its value to the agent is

$$V^A(\mu) = \sum_{\tilde{q} \in \Delta(S)} [\max_{x \in A} \mathbb{E}_{\tilde{q}} v(x)] \mu(\tilde{q})$$

Comment: Blackwell's theorem says the value of information is always positive: more information is better

Optimal Attention

- For every menu A, the agent chooses μ to maximize:

$$\max_{\mu} \; V^A(\mu) - C(\mu)$$

- where $C(\mu)$ is the cost of choosing the signal structure μ
  - could be a physical cost
  - or mental/cognitive

- this is another case where information depends on the menu A
  - this time endogenously

Optimal Attention

Example (Matejka and McKay, 2014): $\rho(x, \{x, y, z\}) > \rho(x, \{x, y\})$ because adding z adds incentive to learn about the state

	s_1	s_2
x	0	2
y	1	1
z	2	0

- Prior is $(\frac{1}{2}, \frac{1}{2})$
- Cost of learning the state perfectly is 0.75
- No other learning possible (cost infinity)
- $\rho(x, \{x, y\}) = 0$, $\rho(x, \{x, y, z\}) = \frac{1}{2}$

Optimal Attention

Special cases of the cost function:

- Separable cost functions $C(\mu) = \int \phi(\tilde{q})\mu(d\tilde{q})$
  - for some function $\phi : \Delta(S) \to \mathbb{R}$
- Mutual information: separable where $\phi(q)$ is the relative entropy (Kullback-Leibler divergence) of q with respect to the prior $\int \tilde{q}\mu(d\tilde{q})$
- General cost functions: C is just Blackwell-monotone and convex

Optimal Attention

Question: Is it harder to distinguish "nearby" states than "far away" states?

- In the dots example, is it harder to distinguish $s = 49\%$ from $s = 51\%$ or $s = 1\%$ from $s = 99\%$?
- Caplin and Dean (2013), Morris and Yang (2016), Hébert and Woodford (2017)

Optimal Attention

- Matejka and McKay (2014) analyze the mutual information cost function used in Sims (2003)
  - show the optimal solution leads to weighted-Luce choice probabilities ρ^s
  - can be characterized by two Luce IIA-like axioms on ρ^s

Optimal Attention

- Caplin and Dean (2015) characterize general cost C
  - assume choice is between Savage acts
  - assume the analyst knows the agent's utility function and the prior
  - can be characterized by two acyclicity-like axioms on ρ^s
  - partial uniqueness: bounds on the cost function

- Denti (2018) and Caplin, Dean, and Leahy (2018) characterize separable cost functions (and mutual information)
  - additional axioms beyond the two acyclicity-like axioms on ρ^s

- Chambers, Liu, and Rehbeck (2018) characterize a more general model without the $V^A(\mu) - C(\mu)$ separability
  - like Caplin and Dean (2015), they assume that the analyst knows the agent's utility function and the prior

Optimal Attention

- Lin (2017) characterizes general cost C
  - building on Lu (2016) and De Oliveira, Denti, Mihm, and Ozbek (2016)
  - the utility and prior are recovered from the data
  - can be characterized by a relaxation of REU axioms plus the De Oliveira, Denti, Mihm, and Ozbek (2016) axioms
  - essential uniqueness of parameters: minimal cost function unique
- Duraj and Lin (2019a) the agent can buy a fixed signal
  - either at a cost, or experimentation takes time
  - axiomatic characterization and uniqueness results

Random Attention

- In the Optimal Attention model, paying attention meant optimally choosing an informative signal about its utility (at a cost)
- In the Random Attention model, attention is exogenous (and random)
  - $\tilde{\Gamma}(A) \subseteq A$ is a random *Consideration Set*
  - $v \in \mathbb{R}^X$ is a deterministic utility function
  - for each possible realization $\tilde{\Gamma}(A)$ the agent maximizes v on $\tilde{\Gamma}(A)$
  - so for each menu we get a probability distribution over choices
- So this could be called Random Consideration

Random Attention

- Manzini and Mariotti (2014)
  - each $x \in A$ belongs to $\tilde{\Gamma}(A)$ with prob $\gamma(x)$, independently over x
  - if $\tilde{\Gamma}(A) = \emptyset$, the agent chooses a default option
  - axiomatic characterization, uniqueness result
    - turns out this is a special case of RU

- Imagine that $\tilde{\Gamma}(A) = \tilde{\Gamma} \cap A$ for some random set $\tilde{\Gamma}$. Items outside of $\tilde{\Gamma}$ have their utility set to $-\infty$; inside of $\tilde{\Gamma}$ utility is unchanged. If $\tilde{\Gamma}$ is independent of the menu, then this is a special case of RU.

Random Attention

- Brady and Rehbeck (2016): allow for correlation
  - axiomatic characterization, uniqueness result
  - now can violate Regularity
- Cattaneo, Ma, Masatlioglu, and Suleymanov (2018): even more general
  - *attention filters*, following Masatlioglu, Nakajima, and Ozbay (2011)
  - axiomatic characterization, uniqueness result

Random Attention

- Suleymanov (2018) provides a clean axiomatic classification of these models
- Aguiar, Boccardi, Kashaev, and Kim (2018)
  - theoretical and statistical framework to test limited and random consideration at the population level
  - experiment designed to tease them apart

Random Attention

- Abaluck and Adams (2017): model with characteristics ξ
  - a version of Manzini and Mariotti (2014) where $\gamma(x)$ depend only on ξ^x
  - a version where the probability of being asleep (only looking at status quo) depends only on $\xi^{\text{status quo}}$
  - identification results and experimental proof of concept

Satisficing

- Aguiar, Boccardi, and Dean (2016): agent is Satisficing
  - draws a random order
  - goes through items till an item is "good enough"
  - randomness in orders generates randomness in choice

Learning

Attention

Optimal Attention

Random Attention

Controlled Randomization

Controlled Randomization

Idea: The agent directly chooses a probability distribution on actions $\rho \in \Delta(A)$ to maximize some non-linear value function $V(\rho)$

Examples:

- Trembling hands with implementation costs
- Allais-style lottery preferences
- Hedging against ambiguity
- Regret minimization

Trembling Hands

Idea: The agent implements her choices with an error (trembling hands)

- can reduce error at a cost that depends on the tremble probabilities

- When presented with a menu A choose $\rho \in \Delta(A)$ to maximize

$$V(\rho) = \sum_x v(x)\rho(x) - C(\rho)$$

- $v \in \mathbb{R}^X$ is a deterministic utility function
- C is the cost of implementing ρ
  - zero for the uniform distribution
  - higher as ρ focuses on a particular outcome
- This is called the Perturbed Utility model, used in game theory

Additive Perturbed Utility

Typically used specification: Additive Perturbed Utility

$$C(\rho) = \eta \sum_{x \in A} c(\rho(x)) + k$$

- log cost: $c(t) = -\log(t)$ (Harsanyi, 1973)
- quadratic cost: $c(t) = t^2$ (Rosenthal, 1989)
- entropy cost: $c(t) = t \log t$ (Fudenberg and Levine, 1995),

General C function used in

- Mattsson and Weibull (2002), Hofbauer and Sandholm (2002), van Damme and Weibull (2002)

a recent decision theoretic study is Allen and Rehbeck (2019)

The Quadruple Equivalence

Theorem (Anderson, de Palma, and Thisse, 1992): The following are equivalent

(i) ρ satisfies Positivity and IIA

(ii) ρ has a Luce representation

(iii) ρ has a logit representation

(iv) ρ has an entropy APU representation

Comments:

- Another application to game theory: Quantal Response Equilibrium (McKelvey and Palfrey, 1995, 1998) uses logit

Additive Perturbed Utility

Axiom (Acyclicity): For any n and bijections $f, g : \{1, ..., n\} \to \{1, ..., n\}$,

$$\begin{aligned} \rho(x_1, A_1) &> \rho(x_{f(1)}, A_{g(1)}) \\ \rho(x_k, A_k) &\geq \rho(x_{f(k)}, A_{g(k)}) \quad \text{for } 1 < k < n \end{aligned}$$

implies

$$\rho(x_n, A_n) < \rho(x_{f(n)}, A_{g(n)}).$$

Condition (Ordinal IIA): For some continuous and monotone $\phi : (0, 1) \to \mathbb{R}_+$

$$\frac{\phi(\rho(x, A))}{\phi(\rho(y, A))} = \frac{\phi(\rho(x, B))}{\phi(\rho(y, B))}$$

for each menu $A, B \in A$ and $x, y \in A \cap B$.

Additive Perturbed Utility

Theorem† (Fudenberg, Iijima, and Strzalecki, 2015): The following are equivalent under Positivity:

(i) ρ has an APU representation with steep cost†

(ii) ρ satisfies Acyclicity

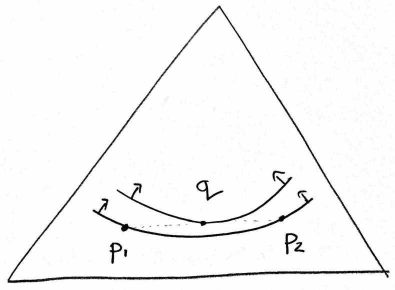
(iii) ρ satisfies Ordinal IIA

Comments:

- Weaker forms of Acyclicity if c is allowed to depend on A or on z (Clark, 1990; Fudenberg, Iijima, and Strzalecki, 2014)
- The model explains any ρ if c is allowed to depend on both A and z
- Hedging against ambiguity interpretation (Fudenberg, Iijima, and Strzalecki, 2015)

Allais-style lottery preferences

- Agent is choosing between lotteries, $X = \Delta(Z)$
- She has a deterministic nonlinear lottery preference $\succsim^\ell$ over $\Delta(Z)$
- If $\succsim^\ell$ is quasiconcave, then the agent likes to toss a "mental coin"



- Example: $p_1 \sim^\ell p_2$
- Strictly prefer q
- To implement this, choice from $A = \{p_1, p_2\}$ is $\rho(p_1, A) = \rho(p_2, A) = \frac{1}{2}$
- what if $B = \{p_1, p_2, q\}$? (Is the "mental coin" better or worse than actual coin?)

Allais-style lottery preferences

- Machina (1985): derives some necessary axioms that follow from maximizing any general $\succsim^{\ell}$

- Cerreia-Vioglio, Dillenberger, Ortoleva, and Riella (2017):
  - characterize maximization of a general $\succsim^{\ell}$ $\longrightarrow$ Rational Mixing axiom
  - show that violations of Regularity obtain iff $\succsim^{\ell}$ has a point of strict convexity
  - characterize maximization of a specific $\succsim^{\ell}$ that belongs to the Cautious Expected Utility class $\longrightarrow$ Rational Mixing + additional axioms

- Lin (2019) shows lack of uniqueness for other classes of risk preferences
  - betweenness
  - also can rationalize REU as betweenness

Evidence

- In experiments (Agranov and Ortoleva, 2017; Dwenger, Kubler, and Weizsacker, 2013) subjects are willing to pay money for an "objective" coin toss
- So "objective" coin better than "mental" coin
- No room in above models for this distinction...

Lecture 3 on Stochastic Choice

Tomasz Strzalecki

Recap of Static Random Utility

- $(\Omega, \mathcal{F}, \mathbb{P})$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ probability space
- $\tilde{U} : \Omega \to \mathbb{R}^X$ $\cdots\cdots\cdots\cdots\cdots\cdots$ random utility
- $C(x, A) := \{\omega \in \Omega : \tilde{U}_\omega(x) = \max_{y \in A} \tilde{U}_\omega(y)\}$
  - agent learns the state (his utility) and chooses optimally
- $\rho(x, A) = \mathbb{P}(C(x, A))$
  - analyst does not see the state; the observed choice frequency of x from A is the probability that x is the argmax of the agent's utility on A

Dynamic Random Utility (DRU)

In every period ρ_t has a RU representation with utility $\tilde{U}_t(x_0)$

Conditional choice probability (given a history of choices h^t):

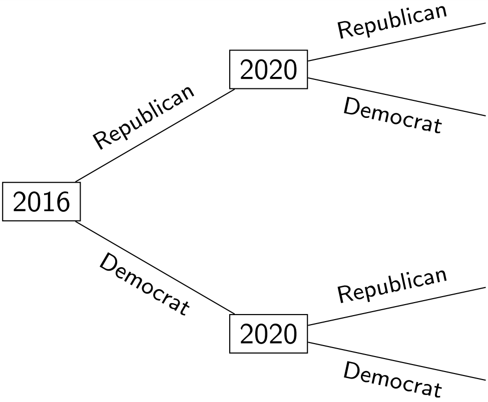
$$\rho_t(x_t, A_t|h^t) = \mathbb{P}\Big[C(x_t, A_t)\Big|h^t\Big]$$

Two main dynamic effects that connect ρ_t and ρ_{t+1}

- **Backward Looking**: (if $\tilde{U}_t$ and $\tilde{U}_{t+1}$ are correlated)
  - History-Dependence, Choice-Persistence

- **Forward Looking**: (if $\tilde{U}_t$ satisfies the Bellman Equation)
  - Agent is Bayesian, has rational expectations, and correctly calculates option value

History Dependence and Selection on Unobservables



If political preferences persistent over time, expect history dependence:

$$\rho(R_{2020}|R_{2016}) > \rho(R_{2020}|D_{2016})$$

History independence only if preferences completely independent over time.

History Dependence is a result of informational asymmetry between agent

Types of History Dependence (Heckman, 1981)

1. **Choice Dependence**: A consequence of the informational asymmetry between the analyst and the agent
   - Selection on unobservables
   - Utility is serially correlated (past choices partially reveal it)

2. **Consumption Dependence**: Past consumption changes the state of the agent
   - Habit formation or preference for variety (preferences change)
   - Experimentation (beliefs change)

Questions:

- How to distinguish between the two?
- How much history-dependence can there be?
- What are the axioms that link ρ_t and ρ_{t+1}?

Decision Trees: $x_t = (z_t, A_{t+1})$

– Choice today leads to an immediate payoff and a menu for tomorrow

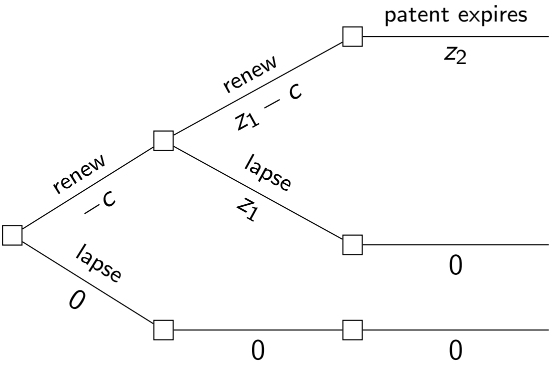
Examples:

– fertility and schooling choices (Todd and Wolpin, 2006)

– engine replacement (Rust, 1987)

– patent renewal (Pakes, 1986)

– occupational choices (Miller, 1984)



Primitive

- The analyst observes the conditional choice probabilities $\rho_t(\cdot|h_{t-1})$
  - at each node of a decision tree
- Dynamic Discrete Choice literature
  - typically for a fixed tree, but have covariates ξ
- Decision Theory literature
  - typically across decision trees

Bellman Equation

In addition, it is often assumed that:

- In period 0 the agent's utility is

$$\tilde{U}_0(z_0, A_1) = \tilde{u}_0(z_0) + \delta \mathbb{E}_0 \left[\max_{z_1 \in A_1} \tilde{u}_1(z_1) \right]$$

- $\tilde{u}_0$ is private information in $t = 0$
- $\tilde{u}_1$ is private information in $t = 1$ (so may be unknown in $t = 0$)

Question: What do these additional assumptions mean?

Decision Trees

Time: $t = 0, 1$

Per-period outcomes: Z

Decision Nodes: $\mathcal{A}_t$ defined recursively:

- period 1: menu A_1 is a subset of $X_1 := Z$
- period 0: menu A_0 is a subset of $X_0 := Z \times \mathcal{A}_1$

  pairs $x_0 = (z_0, A_1)$ of current outcome and continuation menu

Comment: Everything extends to finite horizon by backward induction; infinite horizon—need more technical conditions (a construction similar to universal type spaces)

Conditional Choice Probabilities

ρ is a sequence of **history-dependent** choice distributions:

period 0: for each menu A_0, observe choice distribution

$$\rho_0(\cdot, A_0) \in \Delta(A_0)$$

period 1: for each menu A_1 and history h^0 that leads to menu A_1, observe choice distribution conditional on h^0

$$\rho_1(\cdot, A_1 | h^0) \in \Delta(A_1)$$

$\mathcal{H}_0 \cdots\cdots\cdots\cdots$ period-0 histories

$$\mathcal{H}_0 := \{h^0 = (A_0, x_0) : \rho_0(x_0, A_0) > 0\}$$

$\mathcal{H}_0(A_1) \cdots\cdots\cdots$ is set of histories that lead to menu A_1

$$\mathcal{H}_0(A_1) := \{h^0 = (A_0, x_0) \in \mathcal{H}_0 : x_0 = (z_0, A_1) \text{ for some } z_0 \in Z\}$$

Dynamic Random Utility

Definition: A *DRU* representation of ρ consists of

- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- a stochastic process of utilities $\tilde{U}_t : \Omega \to \mathbb{R}^{X_t}$

such that for all $x_0 \in A_0$

$$\rho_0(x_0, A_0) = \mathbb{P}\left[C(x_0, A_0)\right]$$

and for all $x_1 \in A_1$ and histories $(A_0, x_0) \in \mathcal{H}_0(A_1)$,

$$\rho_1(x_1, A_1 | h^0) = \mathbb{P}\left[C(x_1, A_1) | C(x_0, A_0)\right]$$

where $C(x_t, A_t) := \{\omega \in \Omega : \tilde{U}_{t,\omega}(x_t) = \max_{y_t \in A_t} \tilde{U}_{t,\omega}(y_t)\}$

- for technical reasons allow for ties and use tie-breaking

History Independence

General idea:

- Agent's choice history $h^0 = (A_0, x_0)$ reveals something about his period-0 private information, so expect $\rho_1(\cdot|h^0)$ to depend on h^0

- But dependence cannot be arbitrary: some histories are *equivalent* as far as the private information they reveal

- The axioms of Frick, Iijima, and Strzalecki (2019)
  - Identify two types of equivalence classes of histories
  - Impose history *independence* of ρ_1 within these classes

Contraction History Independence

Axiom (Contraction History Independence): If

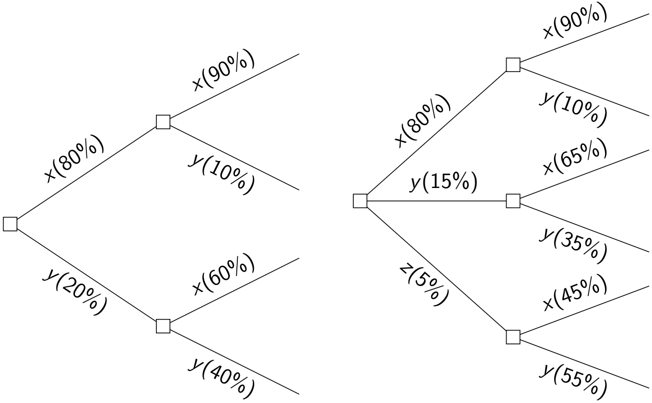
(i) $A_0 \subseteq B_0$

(ii) $\rho_0(x_0, A_0) = \rho_0(x_0, B_0)$,

then

$$\rho_1(\cdot, \cdot | A_0, x_0) = \rho_1(\cdot, \cdot | B_0, x_0)$$

Example



- z does not steal any customers from x in period $t = 0$
- so what people do in $t = 1$ after choosing x should be the same
- (note that z steals from y, so we have a mixture)

Adding Lotteries

Add lotteries: $X_t = \Delta(Z \times \mathcal{A}_{t+1})$, assume each utility function is vNM

- Denote lotteries by $p_t \in X_t$
- Helps formulate the second kind of history-independence
- Makes it easy to build on the REU axiomatization
- Helps overcome the limited observability problem
  - not all menus observed after a given history; how to impose axioms?
- Helps distinguish choice-dependence from consumption-dependence

$$h^0 = (A_0, x_0) \;\text{ vs }\; h^0 = (A_0, p_0, z_0)$$

Consumption History Independence

For now, assume away consumption dependence and allow only for choice dependence

Axiom (Consumption Independence): For any $p_0 \in A_0$ with $p_0(z_0), p_0(z_0') > 0$

$$\rho_1(\cdot|(A_0, p_0, z_0)) = \rho_1(\cdot|(A_0, p_0, z_0'))$$

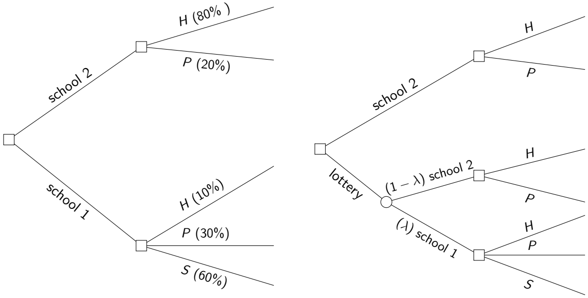
Weak Linear History Independence

Idea: Under EU-maximization, choosing p_0 from A_0 reveals the same information as choosing option $\lambda p_0 + (1-\lambda) q_0$ from menu $\lambda A_0 + (1-\lambda)\{q_0\}$.

Axiom (Weak Linear History Independence)

$$\rho_1(\cdot,\cdot|A_0, p_0) = \rho_1\big(\cdot,\cdot|\lambda A_0 + (1-\lambda) q_0, \lambda p_0 + (1-\lambda) q_0\big).$$

Example



- school 2 offers two after-school programs, school 1 offers three
- different parents self-select to different schools
- how would school-1 parents choose between $\{H, P\}$?
- lottery to get in to the school
- Axiom says choice between $\{H, P\}$ independent of λ

Linear History Independence

Axiom (Weak Linear History Independence)

$$\rho_1(\cdot,\cdot|A_0,p_0)=\rho_1\big(\cdot,\cdot|\lambda A_0+(1-\lambda)q_0,\lambda p_0+(1-\lambda)q_0\big).$$

Idea was to mix-in a lottery q_0. Next we mix-in a set of lotteries B_0

Axiom (Linear History Independence)

$$\rho_1(\cdot,\cdot|A_0,p_0)\rho_0(p_0,A_0)$$
$$=\sum_{q_0\in B_0}\rho_1\Big(\cdot,\cdot|\lambda A_0+(1-\lambda)B_0,\lambda p_0+(1-\lambda)q_0\Big)\cdot\rho_0\Big(\lambda p_0+(1-\lambda)q_0,\lambda A_0+(1-\lambda)B_0\Big)$$

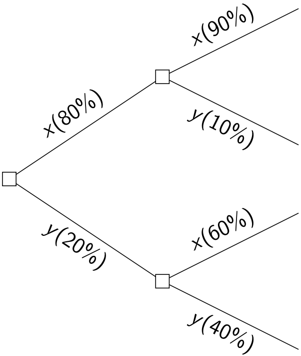
Dynamic Random Expected Utility

Theorem 1: ρ has a DREU representation if and only it satisfies

- Contraction History Independence
- Consumption History Independence
- Linear History Independence
- REU axioms in each period†
- History-Continuity†

Remark: For REU axioms we use the approach of Gul and Pesendorfer (2006); Ahn and Sarver (2013). We need to extend their result to infinite spaces because X_1 is infinite (our Theorem 0).

Consumption Persistence



- $\rho_1(x|x) > \rho_1(x|y)$
- again, there is no habit here
- but serially correlated utility
- widely studied in marketing literature
- Frick, Iijima, and Strzalecki (2019) formulate behavioral notions of persistence and relate them to the serial correlation of utility

How to incorporate Dynamic Optimality?

- In the definition above, no structure on the family $(\tilde{U}_t)$
- But typically $\tilde{U}_t$ satisfies the Bellman equation

Definition: ρ has an *Evolving Utility* representation if it has a DREU representation where the process $(\tilde{U}_t)$ satisfies the Bellman equation

$$\tilde{U}_t(z_t, A_{t+1}) = \tilde{u}_t(z_t) + \delta \mathbb{E}\left[\max_{p_{t+1} \in A_{t+1}} \tilde{U}_{t+1}(p_{t+1}) | \mathcal{F}_t\right]$$

for $\delta > 0$ and a $\mathcal{F}_t$-adapted process of vNM utilities $\tilde{u}_t : \Omega \to \mathbb{R}^Z$

Question: What are the additional assumptions?

Answer:

- Option value calculation (Preference for Flexibility)
- Rational Expectations (Sophistication)

Simplifying assumption: No selection

Simplifying Assumption:

1. The payoff in $t = 0$ is fixed
2. There is no private information in $t = 0$

What this means:

- Choices in $t = 0$:
  - are deterministic
  - can be represented by a preference $A_1 \succsim_0 B_1$
- Choices in $t = 1$:
  - are random, represented by ρ_1
  - are history-independent
    - $t = 0$ choices do not reveal any information

Preference for Flexibility

Suppose that there are no lotteries, so $X_1 = Z_1$ is finite and $X_0 = M(X_1)$.

Definition: $\succsim_0$ has an *option-value representation* if there exists a random $u_1 : \Omega \to \mathbb{R}^Z$ such that

$$U_0(A_1) = \mathbb{E}_0 \left[\max_{z_1 \in A_1} \tilde{u}_1(z_1) \right]$$

Axiom (Preference for Flexibility): If $A \supseteq B$, then $A \succsim_0 B$

Theorem† (Kreps, 1979): Preference $\succsim_0$ has an option-value representation iff it satisfies Completeness, Transitivity, Preference for Flexibility, and Modularity†

Comments:

- This representation has very weak uniqueness properties
- To improve uniqueness, Dekel, Lipman, and Rustichini (2001); Dekel, Lipman, Rustichini, and Sarver (2007) specialize to choice between lotteries, $X_1 = \Delta(Z_1)$
- In econometrics U_0 is called the *consumer surplus* or *inclusive value*

Rational Expectations

Specify to $X_1 = \Delta(Z_1)$ and suppose that

- $\succsim_0$ has an option-value representation $(\Omega, \mathcal{F}, \mathbb{P}_0, u_1)$
- ρ_1 has a REU representation with $(\Omega, \mathcal{F}, \mathbb{P}_1, u_1)$

Definition: $(\succsim_0, \rho_1)$ has *Rational Expectations* iff $\mathbb{P}_0 = \mathbb{P}_1$

Axiom (Sophistication)[†]: For any menu without ties[†] $A \cup \{p\}$

$$A \cup \{p\} \succ_0 A \Longleftrightarrow \rho_1(x, A \cup \{p\}) > 0$$

Theorem[‡] (Ahn and Sarver, 2013): $(\succsim_0, \rho_1)$ has Rational Expectations iff it satisfies Sophistication.

Comment: Relaxed Sophistication ($\Rightarrow$ or $\Leftarrow$) pins down either an *unforeseen contingencies* model or a *pure freedom of choice* model

Identification of Beliefs

Theorem[‡] (Ahn and Sarver, 2013): If $(\succsim_0, \rho_1)$ has Rational Expectations, then the distribution over cardinal utilities u_1 is uniquely identified.

Comments:

- Just looking at ρ_1 only identifies the distribution over ordinal risk preferences (Gul and Pesendorfer, 2006)
- Just looking at $\succsim_0$ identifies even less (Dekel, Lipman, and Rustichini, 2001)
- But jointly looking at the evaluation of a menu and the choice from the menu helps with the identification

Analogues in econometrics

- Analogue of Sophistication is the Williams-Daly-Zachary theorem
  - ρ_1 is the gradient of U_0 (in the space of utilities)
  - see, e.g., Koning and Ridder (2003); Chiong, Galichon, and Shum (2016)
- Hotz and Miller (1993) and the literature that follows exploits this relationship
- Sophistication is in a sense a "test" of this property

Putting Selection Back In

- In general, want to relax the simplifying assumption
  - in reality there are intermediate payoffs
  - and informational asymmetry in each period
  - choice is stochastic in each period
  - and there is history dependence

- To characterize BEU need to add Preference for Flexibility and Sophistication
  - but those are expressed in terms of $\succsim_0$
  - when the simplifying assumption is violated we only have access to ρ_0
  - Frick, Iijima, and Strzalecki (2019) find a way to extract $\succsim_0$ from ρ_0
  - Impose stochastic versions of time-separability, DLR, and Sophistication

Passive and Active Learning

- BEU: randomness in choice comes from changing tastes

- Passive Learning: randomness in choice comes from random signals
  - tastes are time-invariant, but unknown $\tilde{u}_t = \mathbb{E}[\tilde{u}|\mathcal{G}_t]$ for some time-invariant vNM utility $\tilde{u} : \Omega \to \mathbb{R}^Z$

- To characterize the passive learning model, need to add a "martingale" axiom

  - Uniqueness of the utility process, discount factor, and information

- Frick, Iijima, and Strzalecki (2019) relax consumption-independence and characterize habit-formation and active learning (experimentation) models

  - parametric models of active learning used by, e.g., Erdem and Keane (1996), Crawford and Shum (2005)

Related Work

- The Bayesian probit model of Natenzon (2018) can be viewed as a model of a sequence of static choice problems where choice probabilities are time dependent

- Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2017) also study a sequence of static choice problems using a Luce-like model

- Gabaix and Laibson (2017) use a model of gradual learning to microfound "as-if" discounting and present bias

- Lu and Saito (2018) study $t = 0$ choices between consumption stream

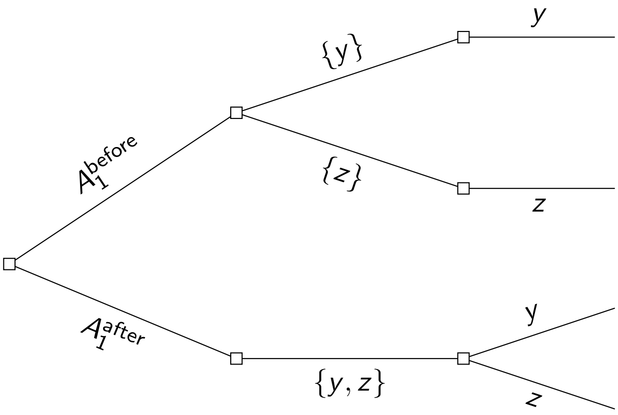
- Krishna and Sadowski (2012, 2016) characterize a class of models similar to Evolving Utility by looking at menu-preferences

Related Work

- Lu and Saito (2019)
  - study a random utility model where separability is violated, as in Epstein and Zin (1989)
  - show that even on simple domains where the continuation menu is fixed the analysts estimates of the function u are biased because they are contaminated by the nonlinear continuation utility.
- Duraj (2018) adds an objective state space to DREU and studies dynamic stochastic choice between Anscombe–Aumann acts
  - new primitive: augmented stochastic choice function
  - direct test of whether the agent's beliefs reflect the true data generating process

Preference for making choices late

- Suppose you got admitted to PhD programs at Harvard and MIT
- Do you make your decision before the visit days or after?



Preference for making choices late

Theorem[†]: If ρ has a BEU representation, then absent ties[†]

$$\rho_0(A_1^{\text{after}}, \{A_1^{\text{before}}, A_1^{\text{after}}\}) = 1$$

Comments:

- BEU has positive value of information: desire to delay the choice as late as possible to capitalize on incoming information (unless there is a cost)
- Here delaying decision does not delay consumption. The situation is different in optimal stopping models $\rightsquigarrow$ later today

DDC model

Definition: The *DDC model* is a restriction of DREU to deterministic decision trees that additionally satisfies the Bellman equation

$$\tilde{U}_t(z_t, A_{t+1}) = v_t(z_t) + \delta\mathbb{E}\left[\max_{y_{t+1}\in A_{t+1}} \tilde{U}_{t+1}(y_{t+1})|\mathcal{F}_t\right] + \tilde{\epsilon}_t^{(z_t,A_{t+1})},$$

with deterministic utility functions $v_t : \Omega \to \mathbb{R}^Z$; discount factor $\delta \in (0,1)$; and $\mathcal{F}_t$-adapted zero-mean *payoff shocks* $\tilde{\epsilon}_t : \Omega \to \mathbb{R}^{Y_t}$.

Special cases of DDC

- **BEU** is a special case, which can be written by setting $\tilde{\epsilon}_t^{(z_t,A_{t+1})} = \tilde{\epsilon}_t^{(z_t,B_{t+1})}$
  - **shocks to consumption**

- **i.i.d. DDC** where $\tilde{\epsilon}_t^{(z_t,A_{t+1})}$ and $\tilde{\epsilon}_\tau^{(y_t,B_{t+1})}$ are i.i.d.
  - **shocks to actions**

Remarks:

- i.i.d. DDC displays history-independence because $\tilde{U}_t$ are independent
- BEU can also be history-independent
- but these two models are different

Other special cases of DDC

- **permanent unobserved heterogeneity**: $\tilde{\epsilon}_t^{(z_t, A_{t+1})} = \tilde{\pi}_t^{z_t} + \tilde{\theta}_t^{(z_t, A_{t+1})}$, where
  - $\tilde{\pi}_t^{z_t}$ is a "permanent" shock that is measurable with respect to $\mathcal{F}_0$
  - $\tilde{\theta}_t^{(z_t, A_{t+1})}$ is a "transitory" shock, i.i.d. conditional on $\mathcal{F}_0$
  - Kasahara and Shimotsu (2009)

- **transitory but correlated shocks to actions**: $\tilde{\epsilon}_t^{(z_t, A_{t+1})}$ and $\tilde{\epsilon}_\tau^{(x_\tau, B_{\tau+1})}$ are i.i.d. whenever $t \neq \tau$, but might be correlated within any fixed period $t = \tau$

- **unobservable serially correlated state variables**: almost no structure on ϵ
  - Norets (2009); Hu and Shum (2012)

Dynamic logit

- A special case of i.i.d. DDC where $\tilde{\epsilon}_t$ are distributed extreme value
- Dynamic logit is a workhorse for estimation
  - e.g., Miller (1984), Rust (1989), Hendel and Nevo (2006), Gowrisankaran and Rysman (2012)
- Very tractable due to the "log-sum" expression for "consumer surplus"

$$V_t(A_{t+1}) = \log\left(\sum_{x_{t+1}\in A_{t+1}} e^{v_{t+1}(x_{t+1})}\right)$$

(This formula is also the reason why nested logit is so tractable)

Axiomatization (Fudenberg and Strzalecki, 2015)

Notation: $x \succsim_t^s y$ iff $\rho_t(x, A) \geq \rho_t(y, A)$ for $A = \{x, y\}$

Axiom (Recursivity):

$$(z_t, A_{t+1}) \succsim_t^s (z_t, B_{t+1})$$
$$\Updownarrow$$
$$\rho_{t+1}(A_{t+1}, A_{t+1} \cup B_{t+1}) \geq \rho_{t+1}(B_{t+1}, A_{t+1} \cup B_{t+1})$$

Axiom (Weak Preference for Flexibility): If $A_{t+1} \supseteq B_{t+1}$, then

$$(z_t, A_{t+1}) \succsim_t^s (z_t, B_{t+1})$$

Comments:

- Recursivity leverages the "log-sum" expression
- Preference for flexibility is weak because support of $\tilde{\epsilon}_t$ is unbounded
- Also, identification results, including uniqueness of δ

Models that build on Dynamic Logit

- View $\tilde{\epsilon}_t$ as errors, not utility shocks
  - Fudenberg and Strzalecki (2015): errors lead to "choice aversion" (each menu is penalized by a function of its size)
  - Ke (2016): a dynamic model of mistakes (agent evaluates each menu by the expectation of the utility under her own stochastic choice function)
- Dynamic attribute rule
  - Gul, Natenzon, and Pesendorfer (2014)

Questions about DDC

- Characterization of the general i.i.d. DDC model? General DDC?
  - In general, no formula for the "consumer surplus", but the Williams-Daly-Zachary theorem may be useful here?

- There is a vast DDC literature on identification (Manski, 1993; Rust, 1994; Magnac and Thesmar, 2002; Norets and Tang, 2013)
  - δ not identified unless make assumptions about "observable attributes"
  - How does this compare to the "menu variation" approach

Understanding the role of i.i.d. ϵ

Key Assumption: Shocks to actions, $\epsilon_t^{(z_t, A_{t+1})}$ and $\epsilon_t^{(z_t, B_{t+1})}$ are i.i.d. regardless of the nature of the menus A_{t+1} and B_{t+1}

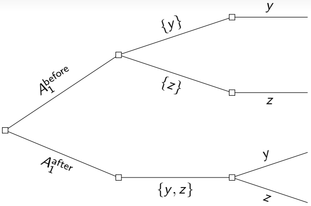
Let $A_0 := \{(z_0, A_1^{\text{small}}), (z_0, A_1^{\text{big}})\}$ where $A_1^{\text{small}} = \{z_1\}$ and $A_1^{\text{big}} = \{z_1, z_1'\}$.

Proposition (Frick, Iijima, and Strzalecki, 2019): If ρ has a i.i.d. DDC representation, then

$$0 < \rho_0 \left((z_0, A_1^{\text{small}}), A_0 \right) < 0.5.$$

Moreover, if the ε shocks are scaled by $\lambda > 0$, then this probability is strictly increasing in λ whenever $v_1(z_1') > v_1(z_1)$.

Understanding the role of i.i.d. ϵ



Proposition (Fudenberg and Strzalecki, 2015; Frick, Iijima, and Strzalecki, 2019): If ρ has a i.i.d. DDC representation with $\delta < 1$, then

$$0.5 < \rho_0\left((x, A_1^{\text{early}}), A_0\right) < 1.$$

Moreover, if ε is scaled by $\lambda > 0$, then $\rho_0((x, A_1^{\text{early}}), A_0)$ is strictly increasing in λ (modulo ties).

Intuition:

- The agent gets the ϵ not at the time of consumption but at the time of decision (even if the decision has only delayed consequences)
- So making decisions early allows him to get the $\max \epsilon$ earlier

Beyond i.i.d. DDC

- This result extends in a straightforward way to DDC with permanent unobserved heterogeneity
  - this is just a mixture of i.i.d DDC models, so inherits this property

- Also to DDC with transitory but correlated shocks to actions

- serially correlated unobservable heterogeneity:
  - the result may not hold in general
  - example: in the mixture model of i.i.d. DDC with BEU there is a horse race between the two effects

Other Decision Problems

- So far, looked at pure manifestations of option value
  - direct choice between nested menus
  - costless option to defer choice
- DDC models typically not applied to those
- But these forces exist in "nearby" choice problems
- So specification of shocks matters more generally

Modeling Choices

- BEU: so far few convenient parametrization (Pakes, 1986) but
  - bigger menus w/prob. 1
  - late decisions w/prob. 1

- i.i.d. DDC: widely used because of statistical tractability, but
  - smaller menus w/prob. $\in (0, \frac{1}{2})$
  - early decisions w/prob. $\in (\frac{1}{2}, 1)$

Comments:

- i.i.d. DDC violates a key feature of Bayesian rationality: positive option value

- Biased Parameter Estimates

- Model Misspecification
  - Maybe a fine model of (behavioral) consumers
  - But what about profit maximizing firms?

Modeling Choices

Comments:

- Note that in the static setting i.i.d. DC is a special case of RU
  - though it has its own problems (blue bus/red bus)

- But in the dynamic setting, i.i.d. DDC is outside of BEU!

- Negative option value is not a consequence of history independence
  - e.g., no such problem in the i.i.d. BEU

- It is a consequence of shocks to actions vs shocks to payoffs

Decision Times

New Variable: How long does the agent take to decide?

Time: $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = \{0, 1, 2, \ldots\}$

Observe: Joint distribution $\rho \in \Delta(A \times \mathcal{T})$

Question:

- Are fast decisions "better" or "worse" than slow ones?

Are quick decisions better than slow ones?

Informational Effect:

- More time $\Rightarrow$ more information $\Rightarrow$ better decisions
  - if forced to stop at time t, make better choices for higher t
  - seeing more signals leads to more informed choices

Selection Effect:

- Time is costly, so you decide to stop depending on how much you expect to learn (option value of waiting)
  - Want to stop early if get an informative signal
  - Want to continue if get a noisy signal

- This creates dynamic selection
  - stop early after informative signals
  - informative signals more likely when the problem is easy

Decreasing accuracy

The two effects push in opposite directions. Which one wins?

Stylized fact: Decreasing accuracy: fast decisions are “better”

- Well established in perceptual tasks (dots moving on the screen), where “better” is objective
- Also in experiments where subjects choose between consumption items

When are decisions "more accurate?"

In cognitive tasks, **accurate = correct**

In choice tasks, **accurate = preferred**

$\boldsymbol{p(t)}$:= probability of making the correct/preferred choicce conditional conditional on deciding at t

Definition:

$$P \text{ displays } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix} \text{ accuracy iff } p(t) \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix}$$

Experiment of Krajbich, Armel, and Rangel (2010)

- X: 70 different food items

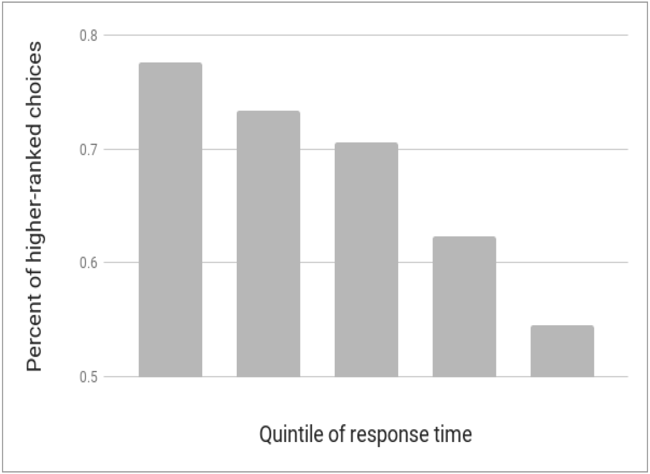
- Step 1: Rate each $x \in X$ on the scale -10, . . . , 10

- Step 2: Choose from $A = \{\ell, r\}$ (100 different pairs)
  - record choice and decision time

- Step 3: Draw a random pair and get your choice

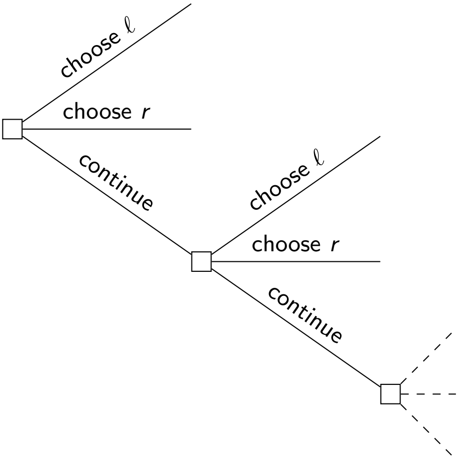
Remark: Here we are using the ratings as a proxy for true preferences. Of course, this is imperfect, as the ratings themselves are probably stochastic as well, so this approach should be treated only as the first step.

Decreasing Accuracy



(based on data from Krajbich, Armel, and Rangel, 2010)

Domain



Models

There are two kinds of models:

1. **Optimal stopping models**: The agent is optimally choosing when to stop (and what to choose). The benefit of waiting is that it gives more information. But there is a cost of waiting too, so the optimal decision balances the two.

   – Wald's model (with a binary prior)

   – Chernoff's model (with a Gaussian prior)

2. **Hitting-time models**: The agent stops when some accumulation process hits a certain boundary. This is a heuristic model, there is no optimization here.

   – Drift-Diffusion models

Under certain conditions, 2 is a solution to 1.

Optimal Stopping Model

- $S \cdots$ set of unknown states
- $p \in \Delta(S) \cdots$ prior belief
- $v : S \to \mathbb{R}^X \cdots$ state-dependent utility function
- $(\mathcal{G}_t) \cdots$ information of the agent (filtration)
- $\tau \cdots$ stopping time, $\{\tau \leq t\} \in \mathcal{G}_t$
- Conditional on stopping, the agent maximizes expected utility

$$\text{choice}_\tau = \text{argmax}_{x \in A} \mathbb{E}[v(x)|\mathcal{G}_\tau]$$

- So the only problem is to choose the stopping time

Interpretation of the Signal Process

- recognition of the objects on the screen
- retrieving pleasant or unpleasant memories
- coming up with reasons pro and con
- introspection
- signal strength depends on the utility difference or on the ease of the perceptual task

In animal experiments, some neuroscientists record neural firing and relate it to these signals

We don't do this, treat signals as unobserved by the analyst

Exogenous vs Endogenous Stopping

Example: If stopping is exogenous (τ is independent of signal G_t), and prior is symmetric, there is **increasing** accuracy: waiting longer gives better information so generates better decisions

- Key assumption above: stopping independent of signal
- If stopping is conditional on the signal, this could get reversed
- Intuition: with endogenous stopping you
  - *#1* stop early after informative signals (and make the right choice); wait longer after noisy signals (and possibly make a mistake)
  - *#2* probably faced an easier problem if you decided quickly

Optimal Stopping Problem

The agent chooses the stopping time optimally

$$\max_{\tau} \mathbb{E}[v(\text{choice}_\tau)] - C(\tau)$$

Comments:

- Assume first that cost is linear $C(t) = ct$
- $(\mathcal{G}_t)$ and τ generate a joint distribution of choices and times
  - conditional on the state $\rho^s \in \Delta(A \times \mathcal{T})$
  - unconditional (averaged out according to p) $\rho \in \Delta(A \times \mathcal{T})$
- Even though $(\mathcal{G}_t)$ is fixed, there is an element of optimal attention
  - Waiting longer gives more information at a cost
  - Choosing τ is like choosing the distribution over posteriors μ
  - How close is this to the static model of optimal attention? $\rightsquigarrow$ later

Further Assumptions

- Binary choice $A = \{x, y\}$
- $s = \left(u^{\ell}, u^{r}\right) \in \mathbb{R}^2$; utility function is identity
- Continuous time
- Signal: $\mathcal{G}_t$ is generated by (G_t^{ℓ}, G_t^{r}) where

$$G_t^i = t \cdot u^i + B_t^i$$

  and B_t^{ℓ}, B_t^{r} are Brownian motions; often look at $G_t := G_t^{\ell} - G_t^{r}$

Examples of Prior/Posterior Families

- "certain difference" (Wald's model)
  - binomial prior: either $s = (1, 0)$ or $s = (0, 1)$
  - binomial posterior: either $s = (1, 0)$ or $s = (0, 1)$
- "uncertain difference" (Chernoff's model)
  - Gaussian prior: $u^i \sim N(X_0^i, \sigma_0^2)$, independent
  - Gaussian posterior: $u^i \sim N(X_t^i, \sigma_t^2)$, independent

The "certain difference" model

* Assumptions:
  - binomial prior: either $s = (1, 0)$ or $s = (0, 1)$
  - binomial posterior: either $s = (1, 0)$ or $s = (0, 1)$

* Key intuition: **stationarity**
  - suppose that you observe $G_t^\ell \approx G_t^r$ after a long t
  - you think to yourself: "the signal must have been noisy"
  - so you don't learn anything $\Rightarrow$ you continue

* Formally, the option value is constant in time

The "certain difference" model

Theorem (Wald, 1945): With binary prior the optimal strategy in the stopping model takes a boundary-hitting form: there exists $b \geq 0$ such that

$$\tau := \inf\{t \geq 0 : |G_t| \geq b\} \qquad \text{choice}_\tau := \begin{cases} \ell & \text{if} \quad G_\tau = b \\ r & \text{if} \quad G_\tau = -b \end{cases}$$

The "certain difference" model

Theorem (Wald, 1945): With binary prior the optimal strategy in the stopping model takes a boundary-hitting form: there exists $b \geq 0$ such that

$$\tau := \inf\{t \geq 0 : |G_t| \geq b\} \qquad \text{choice}_\tau := \begin{cases} \ell & \text{if} \quad G_\tau = b \\ r & \text{if} \quad G_\tau = -b \end{cases}$$

The "certain difference" model

Theorem (Wald, 1945): With binary prior the optimal strategy in the stopping model takes a boundary-hitting form: there exists $b \geq 0$ such that

$$\tau := \inf\{t \geq 0 : |G_t| \geq b\} \qquad \text{choice}_\tau := \begin{cases} \ell & \text{if} \quad G_\tau = b \\ r & \text{if} \quad G_\tau = -b \end{cases}$$

Comments

- The solution to the optimal stopping problem is a hitting-time model
- Can use this as a reduced-form model to generate $\rho \in \Delta(A \times \mathcal{T})$
  - No optimization problem, just a boundary-hitting exercise
- Brought to the psychology literature by Stone (1960) and Edwards (1965) to study perception; memory retrieval (Ratcliff, 1978)
- Used extensively for perception tasks since the 70's; pretty well established in psych and neuroscience
- Closed-form solutions for choice probabilities (logit) and expected decision time

Comments

- More recently used to study choice tasks by a number of teams of authors including Colin Camerer and Antonio Rangel

- Many versions of the model
  - ad-hoc tweaks (not worrying about optimality)
    - assumptions about the process G_t
    - functional forms for the time-dependent boundary
  - much less often, optimization used:
    - time-varying costs (Drugowitsch, Moreno-Bote, Churchland, Shadlen, and Pouget, 2012)
    - endogenous attention (Woodford, 2014)

Hitting Time Models

Definition:

- Stochastic "stimulus process" G_t starts at 0
- Time-dependent boundary $b : \mathbb{R}_+ \to \mathbb{R}_+$
- Hitting time $\tau = \inf\{t \geq 0 : |G_t| \geq b(t)\}$
- Choice $= \begin{cases} \ell & \text{if} \quad G_\tau = +b(\tau) \\ r & \text{if} \quad G_\tau = -b(\tau) \end{cases}$

Anything Goes

Proposition (Fudenberg, Strack, and Strzalecki, 2018): Any Borel $\rho \in \Delta(A \times T)$ has a hitting time representation where the stochastic process G_t is a time-inhomogeneous Markov process and the barrier is constant

Remarks:

- This means that the general model is without loss of generality
  - for a fixed binary menu (but maybe not across menus?)
- In particular, it is without loss of content to assume that b is independent of time
- However, in the general model the process G_t may have jumps
- From now on we focus on the DDM special cases

Drift Diffusion Model (DDM)

Special case where the stimulus process G_t is a diffusion with constant drift and volatility

$$G_t = \delta t + B_t$$

Definition: ρ has a **DDM representation** if it can be represented by a stimulus process $G_t = \delta t + B_t$ and a time-dependent boundary b. We write this as $\rho = DDM(\delta, b)$

Remarks:

- The optimal solution to the certain difference model is a DDM with $\delta = u^\ell - u^r$ and constant b.
- The Brownian assumption has bite. A partial axiomatization was obtained by Baldassi, Cerreia-Vioglio, Maccheroni, and Marinacci (2018) but they only look at expected decision times, so ignore the issue of correlation of times and decisions

Average DDM

Definition: ρ has an **average DDM representation** $DDM(\mu, b)$ with $\mu \in \Delta(\mathbb{R})$ if $\rho = \int \rho(\delta, b) d\mu(\delta)$.

- In an average DDM model the analyst does not know δ, but has a correct prior
- Intuitively, it is unknown to the analyst how hard the problem is for the agent
- This is the unconditional choice function ρ (the average of ρ^s)

Accuracy

Definition: *Accuracy* is the probability of making the correct choice

$$\alpha(t) := \mathbb{P}\left[\text{choice}(\tau) = \text{argmax}_{x \in A}\, v(x) | \tau = t\right]$$

Problem: In DDM $\alpha(t)$ is constant in t, so the model does not explain the stylized fact

Intuition:

- Unconditional on stopping:
  - higher $t \Rightarrow$ more information $\Rightarrow$ better accuracy
- But t is not chosen at random: it depends on information
  - stop early after informative signals
- The two effects balance each other out perfectly!

Accuracy in DDM

Theorem (Fudenberg, Strack, and Strzalecki, 2018): Suppose that $\rho = DDM(\delta, b)$.

$$\text{accuracy } \alpha \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix} \text{ iff boundary } b \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix}$$

Intuition for decreasing accuracy: this is our selection effect #1

- higher bar to clear for small t, so if the agent stopped early, G must have been very high, so higher likelihood of making the correct choice

Accuracy in DDM models

Theorem (Fudenberg, Strack, and Strzalecki, 2018): Suppose that $\rho = DDM(\mu, b)$, with $\mu = \mathcal{N}(0, \sigma_0)$

$$\text{accuracy } \alpha \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix} \text{ iff } b(t) \cdot \sigma_t \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix}$$

where $\sigma_t^2 := \frac{1}{\sigma_0^{-2} + t}$

Intuition for decreasing accuracy: this is our selection effect #2

- σ_t is a decreasing function; this makes it an easier bar to pass

selection effect #2

Proposition (Fudenberg, Strack, and Strzalecki, 2018): Suppose that $\mu = \mathcal{N}(0, \sigma_0)$, and $b(t) \cdot \sigma_t$ non-increasing. Then $|\delta|$ **decreases in** τ in the sense of FOSD, i.e. for all $d > 0$ and $0 < t < t'$

$$\mathbb{P}\left[\,|\delta| \geq d \mid \tau = t\right] > \mathbb{P}\left[\,|\delta| \geq d \mid \tau = t'\right] .$$

- larger values of $|\delta|$ more likely when the agent decides quicker
- problem more likely to be "easy" when a quick decision is observed
- this is a selection coming from the analyst not knowing how hard the problem is

Extended DDM

- Constant DDM cannot explain decreasing accuracy

- **Extended DDM** adds more parameters to match the data better:

  – random starting point of Z_t

  – random drift δ

  – random initial latency T

- Sometimes this is also called **full DDM**

- See, e.g., Ratcliff and McKoon (2008); Bogacz, Brown, Moehlis, Holmes, and Cohen (2006); Ratcliff and Smith (2004)

Example: random starting point

- random starting point may seem ad-hoc, but it sometimes makes sense:
  - example: there is a window of time in which the agent gathers information but cannot act yet
- Let T_0 be the length of this window and the drift be $\lambda(u^\ell - u^r)$
- Then the starting point of Z will be distributed $N(T_0\lambda(u^\ell - u^r), T_0)$
- Chiong, Shum, Webb, and Chen (2018) study non-skippable video ads in apps (estimate the model and simulate skippable counterfactual)

Microfounding the Boundary

* So far, only the constant boundary b was microfounded

* Do any other boundaries come from optimization?

* Which boundaries should we use?

* We now derive the optimal boundary

The "uncertain difference" model

* Assumptions:
  – Gaussian prior: $u^i \sim N(X_0^i, \sigma_0^2)$
  – Gaussian posterior: $u^i \sim N(X_t^i, \sigma_t^2)$

* Key intuition: **nonstationarity**
  – suppose that you observe $G_t^l \approx G_t^r$ after a long t
  – you think to yourself: "I must be indifferent"
  – so you have learned a lot $\Rightarrow$ you stop

* Formally $\sigma_t^2 = \frac{1}{\sigma_0^{-2} + t}$ so option value is decreasing in time

* Intuition for the difference between the two models:
  – interpretation of signal depends on the prior

The "uncertain difference" model

Theorem (Fudenberg, Strack, and Strzalecki, 2018): In the uncertain difference model the optimal behavior has a DDM representation. Moreover, unconditional on the state, accuracy is decreasing.

Other Boundaries

Question: How to microfound other non-constant boundaries? Do they correspond to any particular optimization problem?

Theorem‡ (Fudenberg, Strack, and Strzalecki, 2018): For any b there exists a (nonlinear) cost function C such that b is the optimal solution to the stopping problem

Optimal Attention

- Pure optimal stopping problem (given a fixed $(\mathcal{G}_t)$, choose τ):

$$\max_{\tau} \mathbb{E}\left[\max_{x \in A} \mathbb{E}[\tilde{u}(x)|\mathcal{G}_\tau]\right] - C(\tau)$$

- Pure optimal attention (given a fixed τ, choose $(\mathcal{G}_t)$)

$$\max_{(\mathcal{G}_t)} \mathbb{E}\left[\max_{x \in A} \mathbb{E}[\tilde{u}(x)|\mathcal{G}_\tau]\right] - C(\mathcal{G}_t)$$

- Joint optimization

$$\max_{\tau,(\mathcal{G}_t)} \mathbb{E}\left[\max_{x \in A} \mathbb{E}[\tilde{u}(x)|\mathcal{G}_\tau]\right] - C(\tau, \mathcal{G}_t)$$

Optimal Attention

- In the pure optimal attention problem information choice is more flexible than in the pure stopping problem
  - The agent can focus on one item, depending on what she learned so far
- Woodford (2014) solves a pure optimal attention problem
  - with a constant boundary
  - shows that optimal behavior leads to a decreasing choice accuracy
- Joint optimization puts the two effects together
- In experiments eye movements are often recorded (Krajbich, Armel, and Rangel, 2010; Krajbich and Rangel, 2011; Krajbich, Lu, Camerer, and Rangel, 2012)
  - Do the optimal attention models predict them?

Optimal Attention

- Fudenberg, Strack, and Strzalecki (2018) show that in their model it is always optimal to pay equal attention to both alternatives
- Liang, Mu, and Syrgkanis (2017) study the pure attention as well as joint optimization models
  - Find conditions under which the dynamically optimal strategy is close to the myopic strategy
- Che and Mierendorff (2016) study the joint optimization problem in a Poisson environment with two states; find that coexistence of two strategies is optimal:
  - Contradictory strategy that seeks to challenge the prior
  - Confirmatory strategy that seeks to confirm the prior
- Zhong (2018) shows that in a broad class of models Poisson signals are optimal.

Other Models

- Ke and Villas-Boas (2016) joint optimization with two states per alternative in the diffusion environment
- Steiner, Stewart, and Matějka (2017) optimal attention with the mutual information cost and evolving (finite) state
- Branco, Sun, and Villas-Boas (2012); Ke, Shen, and Villas-Boas (2016) application to consumers searching for products
- Epstein and Ji (2017): ambiguity averse agents may never learn
- Gabaix and Laibson (2005): a model of bounded rationality
- Duraj and Lin (2019b): decision-theoretic foundations for optimal sampling

Optimal Stopping vs Optimal Attention

- In the pure optimal stopping problem $(\mathcal{G}_t)$ is fixed like in the passive learning model
- But there is an element of optimal attention
  - Waiting longer gives more information at a cost
  - Choosing τ is like choosing the distribution over posteriors μ
  - Morris and Strack (2017) show all μ can be obtained this way if $|S| = 2$
- So in a sense this boils down to a static optimal attention problem
  - With a specific cost function: Morris and Strack (2017) show that the class of such cost functions is equal to separable cost functions as long as the flow cost depends only on the current posterior
- Hébert and Woodford (2017) show a similar reduction to a static separable problem in the joint optimization problem
  - Converse to their theorem?

Other Questions

Question:

- Are "close" decisions faster or slower?

Intuitions:

- People "overthink" decision problems which don't matter, "underthink" those with big consequences
- But it is optimal to think more when options are closer
  - The option value of thinking is higher
  - Would you like to spend more time thinking about the choice "Harvard vs MIT" or "Harvard vs Alabama State"?

Experiment: Oud, Krajbich, Miller, Cheong, Botvinick, and Fehr (2014)

Other questions

Question: Are fast decisions impulsive/instinctive and slow deliberate/cognitive?

- Kahneman (2011); Rubinstein (2007); Rand, Greene, and Nowak (2012); Krajbich, Bartling, Hare, and Fehr (2015); Caplin and Martin (2016)

Question: How does the decision time depend on the menu size?

- "Hick–Hyman Law:" the decision time increases logarithmically in the menu size (at least for perceptual tasks Luce, 1986)

Question: Use reaction times to understand how people play games?

- Costa-Gomes, Crawford, and Broseta (2001); Johnson, Camerer, Sen, and Rymon (2002); Brocas, Carrillo, Wang, and Camerer (2014)

Thank you!

References I

Abaluck, J., and A. Adams (2017): "What Do Consumers Consider Before They Choose? Identification from Asymmetric Demand Responses," .

Agranov, M., and P. Ortoleva (2017): "Stochastic choice and preferences for randomization," *Journal of Political Economy*, 125(1), 40–68.

Aguiar, V. H., M. J. Boccardi, and M. Dean (2016): "Satisficing and stochastic choice," *Journal of Economic Theory*, 166, 445–482.

Aguiar, V. H., M. J. Boccardi, N. Kashaev, and J. Kim (2018): "Does Random Consideration Explain Behavior when Choice is Hard? Evidence from a Large-scale Experiment," *arXiv preprint arXiv:1812.09619*.

Ahn, D. S., and T. Sarver (2013): "Preference for flexibility and random choice," *Econometrica*, 81(1), 341–361.

Ahumada, A., and L. Ülkü (2018): "Luce rule with limited consideration," *Mathematical Social Sciences*, 93, 52–56.

Allais, M. (1953): "Le Comportment de 1'Homme Rational devant 1e Risque, Critique des Postulates et Axiomes de 1'Eco1e Americaine," *Econometrica*, 21, 803–815.

Allen, R., and J. Rehbeck (2019): "Revealed Stochastic Choice with Attributes," Discussion paper.

References II

Anderson, S., A. de Palma, and J. Thisse (1992): *Discrete choice theory of product differentiation*. MIT Press.

Apesteguia, J., M. Ballester, and J. Lu (2017): "Single-Crossing Random Utility Models," *Econometrica*.

Apesteguia, J., and M. A. Ballester (2017): "Monotone Stochastic Choice Models: The Case of Risk and Time Preferences," *Journal of Political Economy*.

Baldassi, C., S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci (2018): "Simulated Decision Processes An axiomatization of the Drift Diffusion Model and its MCMC extension to multi-alternative choice," .

Ballinger, T. P., and N. T. Wilcox (1997): "Decisions, error and heterogeneity," *The Economic Journal*, 107(443), 1090–1105.

Barberá, S., and P. Pattanaik (1986): "Falmagne and the rationalizability of stochastic choices in terms of random orderings," *Econometrica*, pp. 707–715.

Block, D., and J. Marschak (1960): "Random Orderings And Stochastic Theories of Responses," in *Contributions To Probability And Statistics*, ed. by I. O. et al. Stanford: Stanford University Press.

Bogacz, R., E. Brown, J. Moehlis, P. Holmes, and J. D. Cohen (2006): "The physics of optimal decision making: a formal analysis of models of performance in two-alternative forced-choice tasks.," *Psychological review*, 113(4), 700.

References III

Brady, R. L., and J. Rehbeck (2016): "Menu-Dependent Stochastic Feasibility," *Econometrica*, 84(3), 1203–1223.

Branco, F., M. Sun, and J. M. Villas-Boas (2012): "Optimal search for product information," *Management Science*, 58(11), 2037–2056.

Brocas, I., J. D. Carrillo, S. W. Wang, and C. F. Camerer (2014): "Imperfect choice or imperfect attention? Understanding strategic thinking in private information games," *The Review of Economic Studies*, p. rdu001.

Caplin, A., and M. Dean (2013): "Behavioral implications of rational inattention with shannon entropy," Discussion paper, National Bureau of Economic Research.

——— (2015): "Revealed preference, rational inattention, and costly information acquisition," *The American Economic Review*, 105(7), 2183–2203.

Caplin, A., M. Dean, and J. Leahy (2018): "Rationally Inattentive Behavior: Characterizing and Generalizing Shannon Entropy," Discussion paper.

Caplin, A., and D. Martin (2015): "A testable theory of imperfect perception," *The Economic Journal*, 125(582), 184–202.

——— (2016): "The Dual-Process Drift Diffusion Model: Evidence from Response Times," *Economic Inquiry*, 54(2), 1274–1282.

Cattaneo, M., X. Ma, Y. Masatlioglu, and E. Suleymanov (2018): "A Random Attention Model: Identification, Estimation and Testing," *mimeo*.

References IV

Cerreia-Vioglio, S., D. Dillenberger, P. Ortoleva, and G. Riella (2017): "Deliberately Stochastic," *mimeo*.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and A. Rustichini (2017): "Multinomial logit processes and preference discovery," .

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and A. Rustichini (2018): "Multinomial logit processes and preference discovery: inside and outside the black box," Discussion paper.

Chambers, C. P., and F. Echenique (2016): *Revealed preference theory*, vol. 56. Cambridge University Press.

Chambers, C. P., C. Liu, and J. Rehbeck (2018): "Costly Information Acquisition," Discussion paper.

Che, Y.-K., and K. Mierendorff (2016): "Optimal Sequential Decision with Limited Attention," *in prepraration*.

Chiong, K., M. Shum, R. Webb, and R. Chen (2018): "Split-second Decision-Making in the Field: Response Times in Mobile Advertising," *Available at SSRN*.

Chiong, K. X., A. Galichon, and M. Shum (2016): "Duality in dynamic discrete-choice models," *Quantitative Economics*, 7(1), 83–115.

References V

Clark, S. (1996): "The random utility model with an infinite choice space," *Economic Theory*, 7(1), 179–189.

Clark, S. A. (1990): "A concept of stochastic transitivity for the random utility model," *Journal of Mathematical Psychology*, 34(1), 95–108.

Costa-Gomes, M., V. P. Crawford, and B. Broseta (2001): "Cognition and behavior in normal-form games: An experimental study," *Econometrica*, 69(5), 1193–1235.

Crawford, G. S., and M. Shum (2005): "Uncertainty and learning in pharmaceutical demand," *Econometrica*, 73(4), 1137–1173.

van Damme, E., and J. Weibull (2002): "Evolution in games with endogenous mistake probabilities," *Journal of Economic Theory*, 106(2), 296–315.

Davidson, D., and J. Marschak (1959): "Experimental Tests of Stochastic Decision Theory," in *Measurement Definitions and Theories*, ed. by C. W. Churchman. John Wiley and Sons.

De Oliveira, H., T. Denti, M. Mihm, and M. K. Ozbek (2016): "Rationally inattentive preferences and hidden information costs," *Theoretical Economics*, pp. 2–14.

Debreu, G. (1958): "Stochastic Choice and Cardinal Utility," *Econometrica*, 26(3), 440–444.

References VI

Dekel, E., B. Lipman, and A. Rustichini (2001): "Representing preferences with a unique subjective state space," *Econometrica*, 69(4), 891–934.

Dekel, E., B. L. Lipman, A. Rustichini, and T. Sarver (2007): "Representing Preferences with a Unique Subjective State Space: A Corrigendum1," *Econometrica*, 75(2), 591–600.

Denti, T. (2018): "Posterior-Separable Cost of Information," Discussion paper.

Drugowitsch, J., R. Moreno-Bote, A. K. Churchland, M. N. Shadlen, and A. Pouget (2012): "The cost of accumulating evidence in perceptual decision making," *The Journal of Neuroscience*, 32(11), 3612–3628.

Duraj, J. (2018): "Dynamic Random Subjective Expected Utility," Discussion paper.

Duraj, J., and Y.-H. Lin (2019a): "Costly Information and Random Choice," Discussion paper.

——— (2019b): "Identification and Welfare Analysis in Sequential Sampling Models," Discussion paper.

Dwenger, N., D. Kubler, and G. Weizsacker (2013): "Flipping a Coin: Theory and Evidence," Discussion paper.

Echenique, F., and K. Saito (2015): "General Luce Model," *Economic Theory*, pp. 1–16.

References VII

Echenique, F., K. Saito, and G. Tserenjigmid (2018): "The Perception Adjusted Luce Model," Discussion paper.

Edwards, W. (1965): "Optimal strategies for seeking information: Models for statistics, choice reaction times, and human information processing," *Journal of Mathematical Psychology*, 2(2), 312–329.

Epstein, L. G., and S. Ji (2017): "Optimal Learning and Ellsberg's Urns," .

Epstein, L. G., and S. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

Erdem, T., and M. P. Keane (1996): "Decision-making under uncertainty: Capturing dynamic brand choice processes in turbulent consumer goods markets," *Marketing science*, 15(1), 1–20.

Ergin, H. (2003): "Costly contemplation," *Unpublished paper, Department of Economics, Duke University.[22] Ergin, Haluk and Todd Sarver (2010),A unique costly contemplation representation. Econometrica*, 78, 1285–1339.

Ergin, H., and T. Sarver (2010): "A unique costly contemplation representation," *Econometrica*, 78(4), 1285–1339.

Falmagne, J. (1978): "A representation theorem for finite random scale systems," *Journal of Mathematical Psychology*, 18(1), 52–72.

References VIII

Fishburn, P. C. (1998): "Stochastic Utility," *Handbook of Utility Theory: Volume 1: Principles*, p. 273.

Frick, M., R. Iijima, and T. Strzalecki (2019): "Dynamic Random Utility," *mimeo*.

Fudenberg, D., R. Iijima, and T. Strzalecki (2014): "Stochastic choice and revealed perturbed utility," *working paper version*.

——— (2015): "Stochastic choice and revealed perturbed utility," *Econometrica*, 83(6), 2371–2409.

Fudenberg, D., and D. K. Levine (1995): "Consistency and Cautious Fictitious Play," *Journal of Economic Dynamics and Control*, 19, 1065–1089.

Fudenberg, D., P. Strack, and T. Strzalecki (2018): "Speed, Accuracy, and the Optimal Timing of Choices," *American Economic Review*, 108, 3651–3684.

Fudenberg, D., and T. Strzalecki (2015): "Dynamic logit with choice aversion," *Econometrica*, 83(2), 651–691.

Gabaix, X., and D. Laibson (2005): "Bounded rationality and directed cognition," *Harvard University*.

——— (2017): "Myopia and Discounting," .

Gowrisankaran, G., and M. Rysman (2012): "Dynamics of Consumer Demand for New Durable Goods," *mimeo*.

References IX

Gul, F., P. Natenzon, and W. Pesendorfer (2014): "Random Choice as Behavioral Optimization," *Econometrica*, 82(5), 1873–1912.

Gul, F., and W. Pesendorfer (2006): "Random expected utility," *Econometrica*, 74(1), 121–146.

Gul, F., and W. Pesendorfer (2013): "Random Utility Maximization with Indifference," *mimeo*.

Harsanyi, J. (1973): "Oddness of the number of equilibrium points: A new proof," *International Journal of Game Theory*, 2(1), 235–250.

Hausman, J., and D. McFadden (1984): "Specification tests for the multinomial logit model," *Econometrica: Journal of the Econometric Society*, pp. 1219–1240.

He, J., and P. Natenzon (2018): "Moderate Expected Utility," Discussion paper, Working Paper, Washington University in Saint Louis.

Hébert, B., and M. Woodford (2017): "Rational Inattention with Sequential Information Sampling," *mimeo*.

Heckman, J. J. (1981): "Heterogeneity and state dependence," in *Studies in labor markets*, pp. 91–140. University of Chicago Press.

Hendel, I., and A. Nevo (2006): "Measuring the implications of sales and consumer inventory behavior," *Econometrica*, 74(6), 1637–1673.

HEY, J. D. (1995): "Experimental investigations of errors in decision making under risk," *European Economic Review*, 39(3), 633–640.

——— (2001): "Does repetition improve consistency?," *Experimental economics*, 4(1), 5–54.

HOFBAUER, J., AND W. SANDHOLM (2002): "On the global convergence of stochastic fictitious play," *Econometrica*, 70(6), 2265–2294.

HORAN, S. (2018): "Threshold Luce Rules," Discussion paper.

HOTZ, V. J., AND R. A. MILLER (1993): "Conditional choice probabilities and the estimation of dynamic models," *The Review of Economic Studies*, 60(3), 497–529.

HU, Y., AND M. SHUM (2012): "Nonparametric identification of dynamic models with unobserved state variables," *Journal of Econometrics*, 171(1), 32–44.

HUBER, J., J. W. PAYNE, AND C. PUTO (1982): "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis," *Journal of Consumer Research*, 9.

IYENGAR, S. S., AND M. R. LEPPER (2000): "When choice is demotivating: Can one desire too much of a good thing?," *Journal of Personality and Social Psychology*, 79(6), 995–1006.

References XI

Johnson, E. J., C. Camerer, S. Sen, and T. Rymon (2002): "Detecting failures of backward induction: Monitoring information search in sequential bargaining," *Journal of Economic Theory*, 104(1), 16–47.

Kahneman, D. (2011): *Thinking, fast and slow*. Macmillan.

Kahneman, D., and A. Tversky (1979): "Prospect theory: An analysis of decision under risk," *Econometrica*, pp. 263–291.

Kamenica, E. (2008): "Contextual Inference in Markets: On the Informational Content of Product Lines," *American Economic Review*, 98, 2127–2149.

Kasahara, H., and K. Shimotsu (2009): "Nonparametric identification of finite mixture models of dynamic discrete choices," *Econometrica*, 77(1), 135–175.

Ke, S. (2016): "A Dynamic Model of Mistakes," working paper.

Ke, T., and M. Villas-Boas (2016): "Optimal Learning before Choice," *mimeo*.

Ke, T. T., Z.-J. M. Shen, and J. M. Villas-Boas (2016): "Search for information on multiple products," *Management Science*, 62(12), 3576–3603.

Kitamura, Y., and J. Stoye (2018): "Nonparametric analysis of random utility models," *Econometrica*, 86(6), 1883–1909.

Koning, R. H., and G. Ridder (2003): "Discrete choice and stochastic utility maximization," *The Econometrics Journal*, 6(1), 1–27.

Kovach, M., and G. Tserenjigmid (2018): "The Imbalanced Luce Model," Discussion paper.

——— (2019): "Behavioral Foundations of Nested Stochastic Choice and Nested Logit," Discussion paper.

Krajbich, I., C. Armel, and A. Rangel (2010): "Visual fixations and the computation and comparison of value in simple choice," *Nature neuroscience*, 13(10), 1292–1298.

Krajbich, I., B. Bartling, T. Hare, and E. Fehr (2015): "Rethinking fast and slow based on a critique of reaction-time reverse inference.," *Nature Communications*, 6(7455), 700.

Krajbich, I., D. Lu, C. Camerer, and A. Rangel (2012): "The attentional drift-diffusion model extends to simple purchasing decisions," *Frontiers in psychology*, 3, 193.

Krajbich, I., and A. Rangel (2011): "Multialternative drift-diffusion model predicts the relationship between visual fixations and choice in value-based decisions," *Proceedings of the National Academy of Sciences*, 108(33), 13852–13857.

Kreps, D. (1979): "A representation theorem for" preference for flexibility"," *Econometrica*, pp. 565–577.

Krishna, V., and P. Sadowski (2012): "Dynamic Preference for Flexibility," *mimeo*.

References XIII

——— (2016): "Randomly Evolving Tastes and Delayed Commitment," *mimeo*.

Liang, A., X. Mu, and V. Syrgkanis (2017): "Optimal Learning from Multiple Information Sources," .

Lin, Y.-H. (2017): "Stochastic Choice and Rational Inattention," *mimeo*.

——— (2019): "Random Non-Expected Utility: Non-Uniqueness," *mimeo*.

Lu, J. (2016): "Random choice and private information," *Econometrica*, 84(6), 1983–2027.

Lu, J., and K. Saito (2018): "Random intertemporal choice," *Journal of Economic Theory*, 177, 780–815.

Lu, J., and K. Saito (2019): "Repeated Choice," *mimeo*.

Luce, D. (1959): *Individual choice behavior*. John Wiley.

Luce, R. D. (1986): *Response times*. Oxford University Press.

Luce, R. D., and H. Raiffa (1957): *Games and decisions: Introduction and critical survey*. New York: Wiley.

Machina, M. (1985): "Stochastic choice functions generated from deterministic preferences over lotteries," *The Economic Journal*, 95(379), 575–594.

Magnac, T., and D. Thesmar (2002): "Identifying dynamic discrete decision processes," *Econometrica*, 70(2), 801–816.

References XIV

Manski, C. F. (1993): "Dynamic choice in social settings: Learning from the experiences of others," *Journal of Econometrics*, 58(1-2), 121–136.

Manzini, P., and M. Mariotti (2014): "Stochastic choice and consideration sets," *Econometrica*, 82(3), 1153–1176.

Marschak, J. (1959): "Binary Choice Constraints on Random Utility Indicators," Cowles Foundation Discussion Papers 74, Cowles Foundation for Research in Economics, Yale University.

Masatlioglu, Y., D. Nakajima, and E. Y. Ozbay (2011): "Revealed attention," 102, 2183–2205.

Matejka, F., and A. McKay (2014): "Rational inattention to discrete choices: A new foundation for the multinomial logit model," *The American Economic Review*, 105(1), 272–298.

Mattsson, L.-G., and J. W. Weibull (2002): "Probabilistic choice and procedurally bounded rationality," *Games and Economic Behavior*, 41, 61–78.

Matzkin, R. L. (1992): "Nonparametric and distribution-free estimation of the binary threshold crossing and the binary choice models," *Econometrica*, pp. 239–270.

McFadden, D. (1973): "Conditional logit analysis of qualitative choice behavior," in *Frontiers in Econometrics*, ed. by P. Zarembka. Institute of Urban and Regional Development, University of California.

McFadden, D., and M. Richter (1971): "On the Extension of a Set Function on a Set of Events to a Probability on the Generated Boolean σ-algebra," *University of California, Berkeley, working paper*.

McFadden, D., and M. Richter (1990): "Stochastic rationality and revealed stochastic preference," *Preferences, Uncertainty, and Optimality, Essays in Honor of Leo Hurwicz*, pp. 161–186.

McFadden, D., and K. Train (2000): "Mixed MNL models for discrete response," *Journal of applied Econometrics*, pp. 447–470.

McFadden, D. L. (2005): "Revealed stochastic preference: a synthesis," *Economic Theory*, 26(2), 245–264.

McKelvey, R., and T. Palfrey (1995): "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior*, 10, 6–38.

McKelvey, R. D., and T. R. Palfrey (1998): "Quantal response equilibria for extensive form games," *Experimental economics*, 1(1), 9–41.

Miller, R. (1984): "Job matching and occupational choice," *The Journal of Political Economy*, 92, 1086–1120.

Morris, S., and P. Strack (2017): "The Wald Problem and the Equivalence of Sequential Sampling and Static Information Costs," *mimeo*.

Morris, S., and M. Yang (2016): "Coordination and Continuous Choice," .

References XVI

Natenzon, P. (2018): "Random choice and learning," *Journal of Political Economy*.

Norets, A. (2009): "Inference in dynamic discrete choice models with serially orrelated unobserved state variables," *Econometrica*, 77(5), 1665–1682.

Norets, A., and X. Tang (2013): "Semiparametric Inference in dynamic binary choice models," *The Review of Economic Studies*, p. rdt050.

Oud, B., I. Krajbich, K. Miller, J. H. Cheong, M. Botvinick, and E. Fehr (2014): "Irrational Deliberation in Decision Making," *mimeo*.

Pakes, A. (1986): "Patents as options: Some estimates of the value of holding European patent stocks," *Econometrica*, 54, 755–784.

Raiffa, H., and R. Schlaifer (1961): *Applied statistical decision theory*. Boston: Division of Research, Harvard Business School.

Rand, D. G., J. D. Greene, and M. A. Nowak (2012): "Spontaneous giving and calculated greed," *Nature*, 489(7416), 427–430.

Ratcliff, R. (1978): "A theory of memory retrieval.," *Psychological review*, 85(2), 59.

Ratcliff, R., and G. McKoon (2008): "The diffusion decision model: Theory and data for two-choice decision tasks," *Neural computation*, 20(4), 873–922.

Ratcliff, R., and P. L. Smith (2004): "A comparison of sequential sampling models for two-choice reaction time.," *Psychological review*, 111(2), 333.

References XVII

RIESKAMP, J., J. R. BUSEMEYER, AND B. A. MELLERS (2006): "Extending the Bounds of Rationality: Evidence and Theories of Preferential Choice," *Journal of Economic Literature*, 44(3), 631–661.

ROSENTHAL, A. (1989): "A bounded-rationality approach to the study of noncooperative games," *International Journal of Game Theory*, 18.

RUBINSTEIN, A. (2007): "Instinctive and cognitive reasoning: A study of response times," *The Economic Journal*, 117(523), 1243–1259.

RUST, J. (1987): "Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher," *Econometrica*, pp. 999–1033.

——— (1989): "A Dynamic Programming Model of Retirement Behavior," in *The Economics of Aging*, ed. by D. Wise, pp. 359–398. University of Chicago Press: Chicago.

——— (1994): "Structural estimation of Markov decision processes," *Handbook of econometrics*, 4, 3081–3143.

SAITO, K. (2018): "Axiomatizations of the Mixed Logit Model," .

SCOTT, D. (1964): "Measurement structures and linear inequalities," *Journal of mathematical psychology*, 1(2), 233–247.

SIMS, C. A. (2003): "Implications of rational inattention," *Journal of monetary Economics*, 50(3), 665–690.

References XVIII

Steiner, J., C. Stewart, and F. Matějka (2017): "Rational Inattention Dynamics: Inertia and Delay in Decision-Making," *Econometrica*, 85(2), 521–553.

Stone, M. (1960): "Models for choice-reaction time," *Psychometrika*, 25(3), 251–260.

Suleymanov, E. (2018): "Stochastic Attention and Search," *mimeo*.

Todd, P. E., and K. I. Wolpin (2006): "Assessing the Impact of a School Subsidy Program in Mexico: Using a Social Experiment to Validate a Dynamic Behavioral Model of Child Schooling and Fertility," *American economic review*, 96(5), 1384–1417.

Train, K. (2009): *Discrete choice methods with simulation*. Cambridge university press, 2nd edn.

Tversky, A. (1969): "Intransitivity of Preferences," *Psychological Review*, 76, 31–48.

——— (1972): "Choice by Elimination," *Journal of Mathematical Psychology*, 9, 341–367.

Tversky, A., and J. E. Russo (1969): "Substitutability and similarity in binary choices," *Journal of Mathematical psychology*, 6(1), 1–12.

Wald, A. (1945): "Sequential tests of statistical hypotheses," *The Annals of Mathematical Statistics*, 16(2), 117–186.

Wilcox, N. T. (2008): "Stochastic models for binary discrete choice under risk: A critical primer and econometric comparison," in *Risk aversion in experiments*, pp. 197–292. Emerald Group Publishing Limited.

——— (2011): "Stochastically more risk averse:A contextual theory of stochastic discrete choice under risk," *Journal of Econometrics*, 162(1), 89–104.

Woodford, M. (2014): "An Optimizing Neuroeconomic Model of Discrete Choice," *Columbia University working paper*.

Zhong, W. (2018): "Optimal Dynamic Information Acquisition," Discussion paper.