Advanced Micro Theory: Part 2

Lecture 2

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3-10 March 2022

Overview

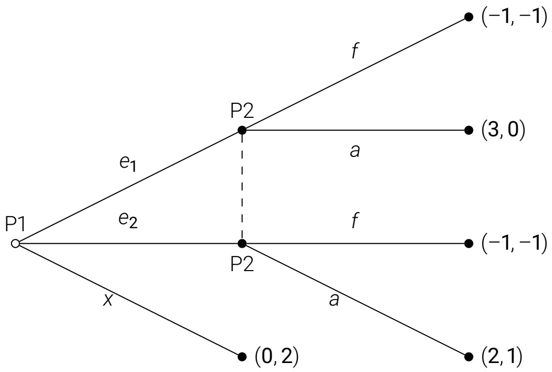
1. Beliefs and Sequential Rationality
   - Credibility, Part 2
   - Belief System
   - Sequential Rationality
   - Restrictions on Beliefs

2. weak Perfect Bayesian Equilibrium
   - Comparison to Nash Equilibrium
   - Examples
   - Comparing wPBE and SPNE
   - Perfect Bayesian Equilibrium

3. Sequential Equilibrium

4. Applications
   - Job Market Signaling
   - Reputation

Credibility, Part 2



Only one subgame: the whole game! SPNE = NE

(x, f) is SPNE, but, in a sense, it's a non-credible threat:

If the incumbent is called to move, it would not be payoff maximizing to choose to fight

Want to have a way to rule out such equilibria

Beliefs

Definition

A **belief system** (or system of beliefs) in an extensive-form game $\Gamma = \langle I, \mathcal{A}, H, \mathcal{I}, \rho, u \rangle$ specifies, for each information set of each player, $I_i \in \mathcal{I}_i$, a probability distribution over the histories in that information set, $\mu(I_i) \in \Delta(I_i)$.

E.g. for $h \in I_i$, $\mu(I_i)(h)$ determines the belief player i holds upon being called to play at information set I_i that history h has occurred, conditional on information set I_i having been reached (i.e. player i having been called to play at information set I_i)

When the $I_i = \{h\}$ contains only one history, $\mu(I_i)(h) = \mathbf{1}$ (no uncertainty on what happened before)

Sequential Rationality

Sequential rationality is a simple concept: each player, when called upon to play, chooses the best behavioral strategy given their beliefs about what has happened and given what opponents are doing in the sequel

$\mu(I_i)$: distribution over histories h in I_i

σ: induce distribution over terminal histories T

$T\,|_{I_i}$: terminal histories $h \in T$ such that there is some history $h' \in I_i$ that is a proper subhistory of h (i.e. terminal histories that follow from some history in I_i)

$\sigma\,|_{I_i}$: distribution over terminal histories $T\,|_{I_i}$

$u_i\,|_{I_i}$: payoff function of player i restricted to $T\,|_{I_i}$

$\mathbb{E}[u_i(\sigma_i, \sigma_{-i}) \mid I_i, \mu]$: player i's expected payoff at information set I_i given belief system μ and strategy profile σ

(more properly, $\mathbb{E}[u_i\,|_{I_i}\,(\sigma\,|_{I_i}) \mid \mu(I_i)]$, but it is too cumbersome to carry I_i around)

Sequential Rationality

Definition

A strategy profile σ is **sequentially rational at information set** I_i *given a belief system* μ if

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i}) \mid I_i, \mu] \geq \mathbb{E}[u_i(\sigma_i', \sigma_{-i}) \mid I_i, \mu]$$

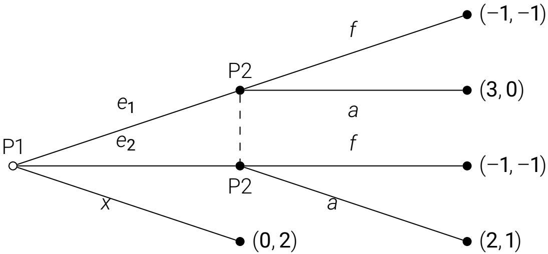
for all $\sigma_i' \in \Sigma_i$.

A strategy profile is **sequentially rational** *given a belief system* if it is sequentially rational at all information sets given that belief system.

Beliefs matter! Different μ can lead to different strategy sequentially rational strategies

It is sequentially rational *given* a belief system

Credibility, Part 2

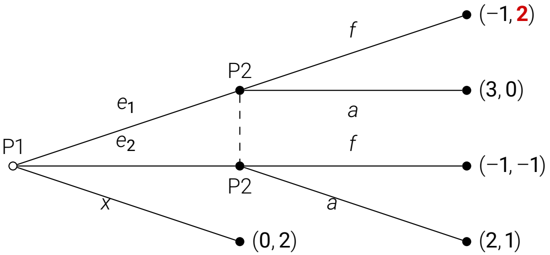


(x, f) is SPNE, but, in a sense, it's a non-credible threat:

- No belief that the incumbent (P2) may hold at $I_2 = \{e_1, e_2\}$ would justify choosing to fight if called upon to move Given they are at I_2, a is strictly better than f *for any beliefs* about whether the entrant (P1) chose e_1 ($\mu(I_2)(e_1)$) or e_2 ($\mu(I_2)(e_2) = 1 - \mu(I_2)(e_1)$)

Never sequentially rational at I_2 to choose f with positive probability

Credibility, Part 2



Now (x, f) may be sequentially rational at I_2 given μ, but only if

$$\mathbb{E}[u_2(f,x) \mid I_2, \mu] \geq \mathbb{E}[u_2(a,x) \mid I_2, \mu] \iff p2 + (1-p)(-1) \geq p0 + (1-p)1 \iff p \in [1/2, 1]$$

where $p = \mu(I_2)(e_1) = 1 - \mu(I_2)(e_2)$

Restrictions on Beliefs

For simplicity, let's focus on the case where H is finite

A strategy profile $\boldsymbol{\sigma}$ determines the probability that history h (a sequence of actions) is played

Definition

An information set I_i is **reached given** $\boldsymbol{\sigma}$ if there is a positive probability that some history $h \in I_i$ is played with positive probability, $\mathbb{P}(I_i \mid \boldsymbol{\sigma}) > 0$

Restrictions on Beliefs

Definition

A belief system is **derived through Bayes rule whenever possible given** σ if, for any information set I_i that is reached given σ, beliefs $\mu(I_i)$ over histories in I_i equal the distribution over histories in I_i conditional on I_i, as induced by σ.

If H is finite, a belief system is derived through Bayes rule whenever possible given σ if, whenever $\mathbb{P}(I_i \mid \sigma) > 0$,

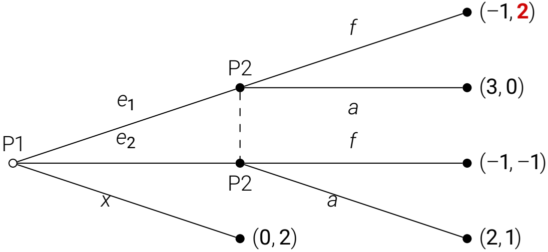
$$\mu(I_i)(h) = \frac{\mathbb{P}(h \mid \sigma)}{\mathbb{P}(I_i \mid \sigma)}$$

To be able to use Bayes rule, we need that, according to σ, [the probability of history h being played given some history in I_i was played] is well-defined

If there are finitely many histories, this amounts to histories in I_i being played with positive probability according to σ ($\mathbb{P}(I_i \mid \sigma) > 0$)

as otherwise, the denominator on RHS = 0 and Bayes rule is not well-defined

Credibility, Part 2



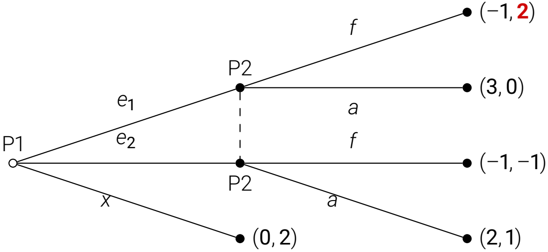
Suppose $\sigma_1(e_1) = 1/6$, $\sigma_1(e_2) = 1/3$, $\sigma_1(x) = 1/2$

For $\mu(I_2)$ to be derived by Bayes' rule:

$$\mu(I_2)(e_1) = \frac{P(e_1 \mid \sigma)}{P(I_2 \mid \sigma)} = \frac{\sigma_1(e_1)}{\sigma_1(e_1) + \sigma_1(e_2)} = \frac{1/6}{1/6 + 1/3} = 1/3$$

and then $\mu(I_2)(e_2) = 1 - \mu(I_2)(e_1) = 2/3$

Credibility, Part 2



Suppose $\sigma_1(e_1) = 0$, $\sigma_1(e_2) = 0$, $\sigma_1(x) = 1$

Then, $\mu(I_2)$ cannot be derived by Bayes' rule from σ as $P(I_2 \mid \sigma) = 0$: I_2 is never reached given σ

Refining Nash Equilibrium by Sequential Rationality

Definition

A strategy profile σ and a belief system μ is a **weak perfect Bayesian Nash equilibrium** (wPBE) (σ, μ) of an extensive-form game Γ if

(i) σ is sequential rational given the belief sytem μ at all information sets that are reached given σ; and

(ii) the belief system μ is derived through Bayes rule whenever possible given σ

wPBE: need to define both the strategy profile **and the belief system**

Reinterpreting Nash Equilibrium

Proposition

A strategy profile σ is a Nash equilibrium of extensive-form game Γ if and only if there is a belief system μ such that

(i) σ is sequential rational given the system of belief μ at all information sets that are reached given σ; and

(ii) the belief system μ is derived through Bayes rule whenever possible given σ

Note: (i) only require sequential rationality at information sets that are reached, and (ii) beliefs at information sets that are reached are correct (coincide with probability of history being played given σ)

To rule out non-credible threats, we strengthened (i): in wPBE sequential rationality is required at *all* information sets

Reinterpreting Nash Equilibrium

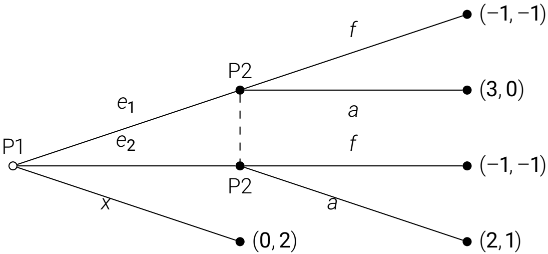
Corollary

If (σ, μ) is a wPBE of an extensive-form game Γ, then σ is a NE of that same game

wPBE's strategy profile is a NE

Not all NE can be supported (with some belief system) as a wPBE

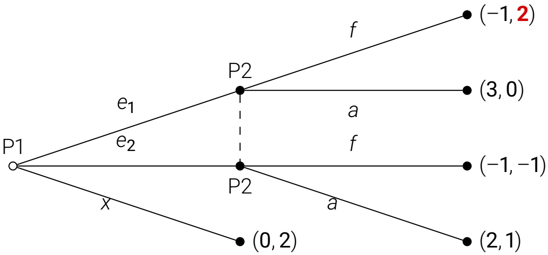
Credibility, Part 2



(x, f) is SPNE, but, in a sense, it's a non-credible threat:

- Sequential rationality requires P2 to choose a with probability 1 at $I_2 = \{e_1, e_2\}$ for any belief system μ
- Then, sequential rationality requires P1 to choose e_1 with probability 1
- Finally, as $P(I_2 \mid (e_1, a)) = 1 > 0$, we have $\mu(I_2)(e_1) = \frac{P(e_1|(e_1,a))}{P(I_2|(e_1,a))} = 1$
- The unique wPBE is $((e_1, a), \mu)$ where $\mu(I_2)(e_1) = 1$

Credibility, Part 2



If $\mu(I_2)(e_1) \geq 1/2$

f is sequentially rational $\implies$ P1 chooses x

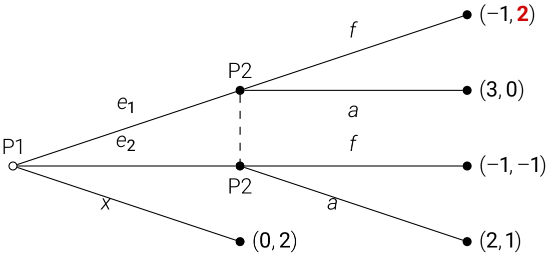
$((x, f), \mu)$ is wPBE for $\mu : \mu(I_2)(e_1) \geq 1/2$

If $\mu(I_2)(e_1) \leq 1/2$

a is sequentially rational $\implies$ P1 chooses e_1

By Bayes' rule, $\mu(I_2)(e_1) = 1 > 1/2$, contradiction!

Credibility, Part 2



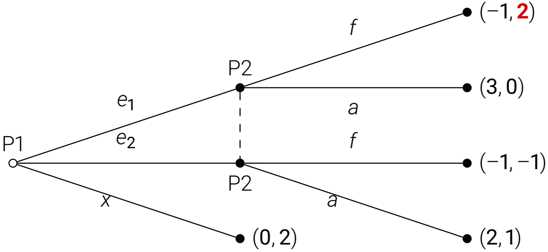
If $\mu(I_2)(e_1) = 1/2$: P2 indifferent between f and a

P1 chooses x if $0 \geq \max\{\sigma_2(a)3 + (1 - \sigma_2(a))(-1), \sigma_2(a)2 + (1 - \sigma_2(a))(-1)\}$

$\implies \sigma_2(a) \in [0, 1/4]$

For any $\sigma_2 : \sigma_2(a) \in [0, 1/4]$, $((x, \sigma_2), \mu)$ is wPBE for $\mu : \mu(I_2)(e_1) = 1/2$

Credibility, Part 2



If $\mu(I_2)(e_1) = 1/2$: P2 indifferent between f and a

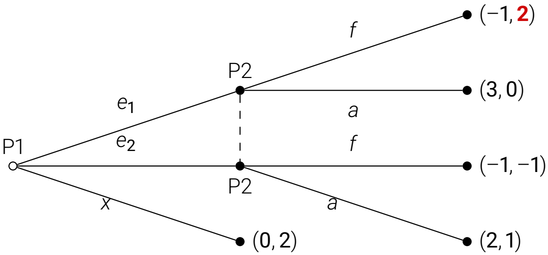
P1 never chooses e_2 as if e_2 is preferred to x: $\sigma_2(a)2 + (1 - \sigma_2(a))(-1) \geq 0$,

then e_1 is strictly preferred to both e_2 and x:

$\sigma_2(a)3 + (1 - \sigma_2(a))(-1) > \sigma_2(a)2 + (1 - \sigma_2(a))(-1) \geq 0$

If P1 chooses e_1 with positive probability, then by Bayes' rule $\mu(I_2)(e_1) = 1$ a contradiction!

Credibility, Part 2



Conclusion: wPBE are $((\sigma_1, \sigma_2), \mu)$ such that

(i) $\sigma_1(x) = 1$, $\sigma_2(a) = 0$, and $\mu(I_2)(e_1) \in [1/2, 1]$

or (ii) $\sigma_1(x) = 1$, $\sigma_2(a) \in [0, 1/4]$, and $\mu(I_2)(e_1) = 1/2$

Comparing wPBE and SPNE

We already saw a case such that σ is SPNE but there is no μ such that (σ, μ) is wPBE

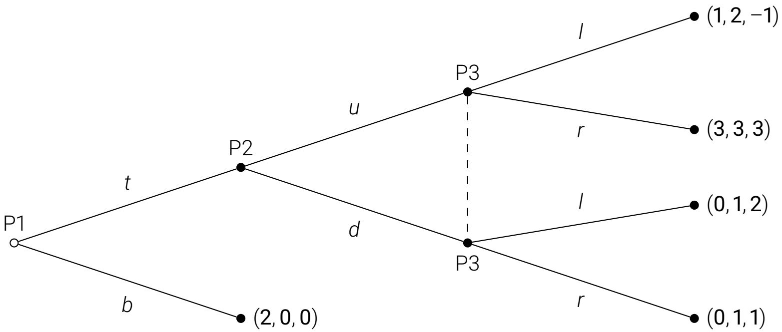
It is also the case that there wPBE (σ, μ) such that σ is not SPNE

In general:

Remark

A strategy profile that is part of a wPBE need not be an SPNE and a SPNE need be part of any wPBE.

Comparing wPBE and SPNE



(t, u, r) is unique SPNE but... (b, u, l) can be supported as wPBE

$p := \mu(\{tu, td\})(tu)$; P3 would choose l over r given p only if

$p(-1) + (1-p)2 \geq p3 + (1-p)1 \iff 1/5 \geq p$

Sequential rationality: P3 chooses l given p; P2 chooses u (by sequential rationality, never chooses d); P1 chooses b

So for any $\mu(\{tu, td\})(tu) \in [0, 1/5]$, $((b, u, l), \mu)$ is wPBE

Comparing wPBE and SPNE

We already saw a case such that σ is SPNE but there is no μ such that (σ, μ) is wPBE

It is also the case that there wPBE (σ, μ) such that σ is not SPNE

In general:

Remark

A strategy profile that is part of a wPBE need not be an SPNE and a SPNE need be part of any wPBE.

Proposition

In finite extensive-form games of perfect information, the set of strategy profiles that can be supported as a wPBE (with some belief system) is identical to the set of SPNE.

- wPBE strategy profiles and SPNE strategy profiles coincide on games of perfect information, but not necessarily on games of imperfect information

Perfect Bayesian Equilibrium

Why the weak in wPBE? Because wPBE places no restrictions on beliefs in subgames that are not reached given the equilibrium strategy profile

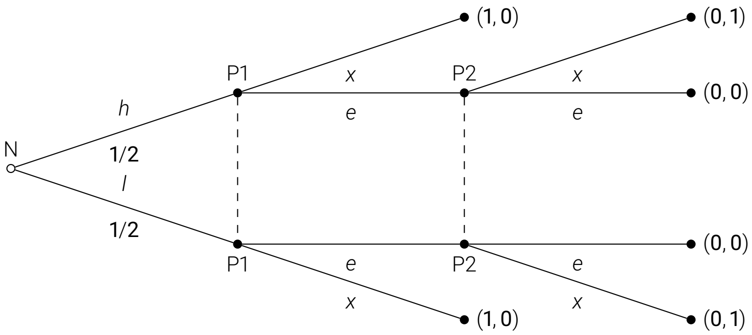
We here diverge from OR in favor of a more natural definition of 'subgame perfection'

Definition

A strategy profile σ and a belief system μ is a **perfect Bayesian Nash equilibrium** (PBE) (σ, μ) of an extensive-form game Γ if it induces a wPBE in every subgame.

As a wPBE induces a NE, a PBE induces a SPNE

Not so Perfect PBE



$\forall \mu : \mu(\{he, le\})(he) \in [0, 1]$, $((x, x), \mu)$ is PBE

However, any reasonable belief would have $\mu(\{he, le\})(he) = 1/2$

Spurious multiplicity

Sequential Equilibrium

Definition

A strategy profile σ and a belief system μ is a **sequential equilibrium** (SE) (σ, μ) of an extensive-form game Γ if

(i) σ is sequentially rational given μ;

(ii) there is a sequence of fully mixed strategy profiles $\{\sigma^n\}_n$ inducing a sequence of belief systems μ^n derived through Bayes rule from σ^n such that $\sigma^n \to \sigma$ and $\mu^n \to \mu$.

Differently from wPBE, SE imposes restrictions on "off-path" beliefs

It requires that such beliefs be obtained as a limit of fully mixed beliefs in a way that these beliefs are in the limit consistent with equilibrium play

Sequential Equilibrium

Theorem

Let Γ be an extensive-form game where H is finite. Then a sequential equilibrium exists.

Proposition

If (σ, μ) is a sequential equilibrium of an extensive-form game Γ, then it is also a PBE.

(σ, μ) SE $\Longrightarrow$ (σ, μ) PBE $\Longrightarrow$ (σ, μ) wPBE and σ SPNE $\Longrightarrow$ σ NE

Job Market Signaling

Setup

Nature chooses the worker's type (ability) $\theta \in \{1, 2\}$, where $\theta = 2$ with probability q

Worker learns type and chooses education investment $e \in \mathbb{R}_+$

Firms F1 and F2 observe worker's choice of e, but not type θ

Firms compete to hire the worker by offering wage $w_i \in \mathbb{R}_+$

- F1 offers w_1, followed by F2 offering w_2 (who does not observe w_1)
- The worker observes both w_1, w_2 and chooses $f \in \{1, 2\}$ which to accept

Payoff functions:

- Worker $u(e, f, w_1, w_2, \theta) = w_f - e/\theta$
- Firm i: $\pi_i(e, f, w_1, w_2, \theta) = \theta - w_i$ if $f = i$ and 0 if otherwise

Job Market Signaling

Setup

Assumptions:

- θ reflects higher productivity and lower cost of education
- Education does not have any direct effect on productivity (very debatable assumption! think about physicians, lawyers, professors, architects, etc)
- Education can still serve to signal high ability

Job Market Signaling

Analysis. Solve for PBE (σ, μ) where $e(\theta)$ is the investment in education given the type

Given $w_1(e), w_2(e)$, the worker chooses the highest; call it $w(e)$

Given e, firms then compete on salary

$w(e) \leq \mathbb{E}[\theta \mid \sigma, \mu]$ (otherwise they'd make a loss)

$w(e) \geq \mathbb{E}[\theta \mid \sigma, \mu]$ (otherwise competitor would offer more and get profit>0)

$\implies w(e) = \mathbb{E}[\theta \mid \sigma, \mu]$

Letting $p(e) := \mu(\{(\theta = 1, e), (\theta = 2, e)\})((\theta = 2, e))$ we have $w(e) = p(e) + 1$

Job Market Signaling

Analysis

Incentive compatibility: Worker of type θ prefers to invest $e(\theta)$ and not e

$w(e(1)) - e(1) \geq w(e) - e \iff e + p(e(1)) - p(e) \geq e(1)$ and

$w(e(2)) - e(2)/2 \geq w(e) - e/2 \iff 2p(e(2)) + e - 2p(e) \geq e(2)$

Separating PBE: different types invest different amounts in education

$p(e(1)) = 0$ and $p(e(2)) = 1 \implies e(1) + 2 \leq e(2) \geq e(1) + 1$

As $p(e(1)) = 0 \leq p(e)$ for any $e \geq 0$, $e + p(e(1)) - p(e) = e - p(e) \geq e \geq e(1)$ for any $e \geq 0$

$\implies e(1) = 0$ and $e \geq p(e) \forall e \geq 0$

As $p(e(2)) = 1$, $e(2) \leq 2 - 2p(e) + e \leq 2 - e$ for any $e \in [0, 1]$ and thus $e(2) = 1$

Finally, $\min\{e, 1/2 + e/2\} \geq p(e)$

$((w_1(e), w_2(e), f(w_1, w_2, e), e(\theta)), \mu)$ is a separating PBE iff

$\max\{w_1(e), w_2(e)\} = 1 + p(e)$, $e(2) = 1$, $e(1) = 0$,

$\min\{e, 1/2 + e/2\} \geq p(e) := \mu(\{(\theta = 1, e), (\theta = 2, e)\})((\theta = 2, e))$

Analysis

Incentive compatibility: Worker of type θ prefers to invest $e(\theta)$ and not e

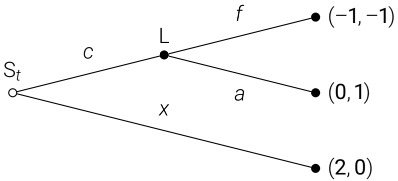
$w(e(1)) - e(1) \geq w(e) - e \iff e + p(e(1)) - p(e) \geq e(1)$ and

$w(e(2)) - e(2)/2 \geq w(e) - e/2 \iff 2p(e(2)) + e - 2p(e) \geq e(2)$

Pooling PBE: $e(1) = e(2)$, $p(e(1)) = p(e(2)) = q$, $\max\{w_1(e), w_2(e)\} = 1 + p(e)$

and, from ICs, $\min\{q - p(e) + e, 2(q - p(e)) + e\} \geq e(1)$

Reputation



Setup

Every period t, a small firm t develops a new product

A large firm copies the new product

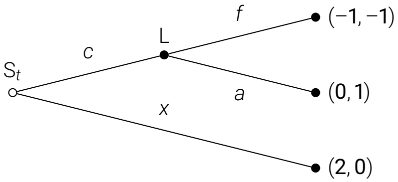
The small firm decides on whether to challenge the large firm in court (c) or not (x)

The large firm then either fights the challenge (f) or accommodates the challenge and settles (a)

The small firm t is short-lived and enters and exits the market at period t

The large firm is long-lived and stays on the market for T (finite) periods

Reputation



Setup

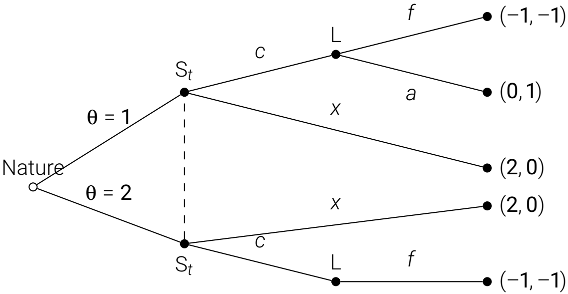
If this were the whole story, SPNE gives a unique prediction:

by backward induction, we find that all small firms challenge the large firm, and the large firm always accommodates the challenge

We model reputation by assuming that the large firm can simply not even consider accommodating and always fight

We do this by introducing types $\theta = 1, 2$

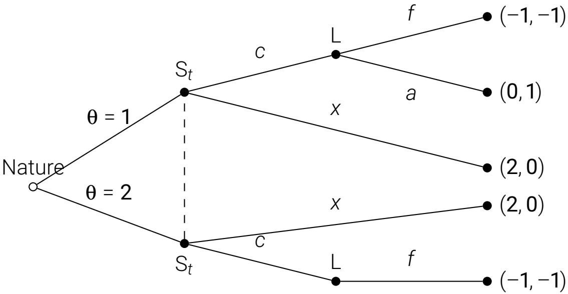
Reputation



Setup

- Types are drawn at the beginning of the game, at $t = 1$, and persist throughout the game up to the last period $t = T$
- With prob. q, $\theta = 1$, and L is as before; With prob. $1 - q$, $\theta = 2$, and L always fights
- The idea is that $\theta = 1$ L can build a reputation of being a hawkish firm by hiding under the idea that it could simply not even consider accommodating the challenge

Reputation



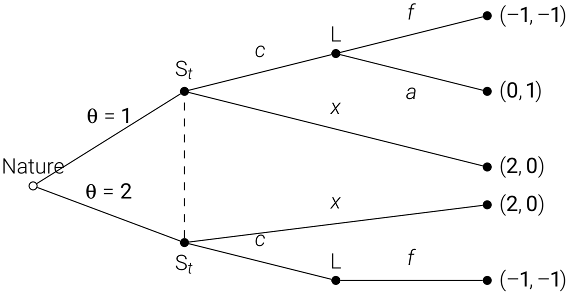
Analysis: $T = 1$

At any SPNE, L chooses a when $\theta = 1$

If $q < 1/2$, the unique SPNE is $(x, (a, f))$; If $q > 1/2$, the unique SPNE is $(c, (a, f))$

For $T = 2$, solve for PBE

Reputation



Analysis: $T = 2$, $q < 1/2$; L refers to $(\theta = 1)$

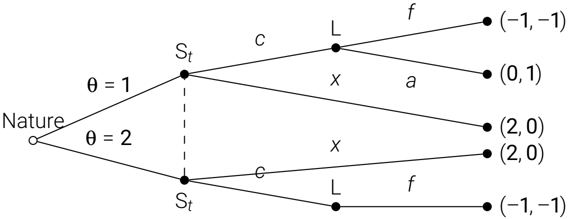
$t = 2$: L always accommodates; S_2 challenges if L accommodated in period 1 and does not challenge if otherwise

This is because having accommodated would unambiguously reveal $\theta = 1$, whereas observing L fighting in period $t = 1$ leads to S_2 beliefs about $\theta = 2$

$$\frac{1-q}{(1-q)+q\mathbb{P}(L \text{ fights at } t{=}1)} \geq 1 - q$$

$t = 1$: L should fight: accommodating would lead to S_2 challenging, resulting in a

Reputation



Analysis: $T = 2$, $q > 1/2$; L refers to $(\theta = 1)$

$t = 2$: L always accommodates;

If S_1 didn't challenge, then $\mathbb{P}(\theta = 2) = 1 - q < 1/2$ and S_2 does challenge

If S_1 did challenge, then S_2 does not challenge when L fights in $t = 1$ if

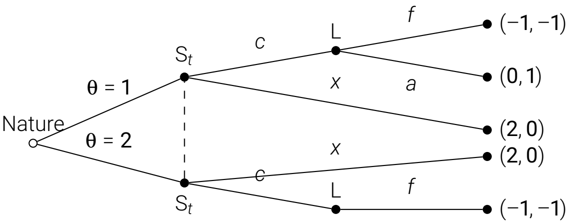
$\mathbb{P}(L \text{ fights at } t = 1) > \frac{1-q}{q}$, challenges if $\mathbb{P}(L \text{ fights at } t = 1) < \frac{1-q}{q}$, is indifferent if ow

Observing L fighting in period $t = 1$ leads to S_2 beliefs about $\theta = 2$

$$\frac{1-q}{(1-q)+q\mathbb{P}(L \text{ fights at } t{=}1)} \geq 1/2 \iff \mathbb{P}(L \text{ fights at } t = 1) \leq \frac{1-q}{q}$$

S_2 belief that $\theta = 2 \geq 1/2 \implies$ prefer x to c

Reputation



Analysis: $T = 2$, $q > 1/2$; L refers to $(\theta = 1)$

$t = 1$: L playing a (or f) with prob. 1 would lead to S_2 challenging, with payoff 0 (-2);

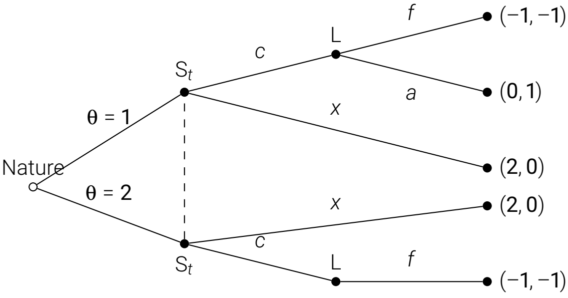
fighting with prob. $p > \frac{1-q}{q}$ leads to payoff $p(-1-1) + (1-p)0 = -2p < 0$

fighting with prob. $0 < p < \frac{1-q}{q}$ leads to payoff $p(-1+2) + (1-p)0 = p > 0$

at $p = \frac{1-q}{q}$, we have payoff $pk(-1+0) + p(1-k)(-1+2) + (1-p)(0+0) = p(1-2k)$,

where $k = \mathbb{P}(S_2 \text{ fights})$

Reputation



Analysis: $T = 2$, $q > 1/2$; L refers to $(\theta = 1)$

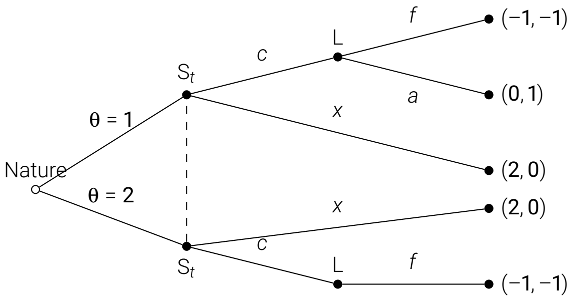
$t = 1$: L mixing $\Longrightarrow$ L indifferent between accommodating and fighting, i.e.

$p(1 - 2k) = 0 \Longrightarrow k = 1/2$

L fights with prob. $p = \frac{1-q}{q}$; S_1 challenges if

$q(p(-1) + (1-p)1) + (1-q)(-1) \geq 0 \Longleftrightarrow q > 3/4$

Reputation



Analysis: $T = 2$, $q > 1/2$; L refers to $(\theta = 1)$; in short

$t = 2$: L always accommodates; If S_1 didn't challenge, then $\mathbb{P}(\theta = 2) = 1 - q < 1/2$ and S_2 does challenge

If S_1 did challenge, then S_2 challenges with prob. $k = 1/2$

$t = 1$: L fights with prob. $p = \frac{1-q}{q}$; S_1 challenges if $q > 3/4$, does not challenge if $q < 3/4$, and is indifferent if ow