Advanced Microeconomic Theory

Lecture 4: Applications of Bayesian Games I

Strategic Inferences from “Pitoval Events”

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Plan of the Lecture

- Simple illustrations of economic applications of the model of games with incomplete information:
  - Bilateral trade in the presence of adverse selection
  - Second-price auctions
  - Strategic voting under common interests

Recurring Theme

- The game’s outcome depends on a critical “pivotal event”
  - Trade takes place
  - Winning an auction
  - One’s vote makes a difference
- Equilibrium reasoning involves statistical inferences from the pivotal event.
  - Statistical inferences from hypothetical contingencies

Bilateral Trade

- A game between a seller (s) and a buyer (b)
- A seller owns an object of uncertain value.
  - The value to the seller is $v \sim U[0,1]$.
  - The value to the buyer is $1.5v$.
- The seller knows v, the buyer is entirely uninformed.
  - $\Omega = [0,1]$; $\tau_s(v) \equiv v$; $\tau_b(v) = t^*$ for all v

Bilateral Trade

- The two players simultaneously submit price offers, a_s and a_b.
- If $a_s > a_b$, there is no trade and players earn **zero** payoffs.
- If $a_s \leq a_b$, there is trade at the price a_b:
  - The seller's payoff is $a_b - v$.
  - The buyer's payoff is $1.5v - a_b$.

Bilateral Trade

- Trade is efficient for every v.
- In a complete-information game in which v is certain:
  - $a_s = a_b = v$ would be a Nash equilibrium resulting in trade.
- Let us look for a Nash equilibrium in the Bayesian game
  - Restricting attention to equilibrium in which the seller always plays a **weakly dominant action**

Weakly Dominant Actions

- An action $a_i \in A_i$ is **weakly dominant** for player i given the signal t_i if for all $a_i' \neq a_i$, $U_i(a_i, s_{-i}|t_i) \geq U_i(a_i', s_{-i}|t_i)$ for every s_{-i}, with a strict inequality for at least one s_{-i}.
  - The other actions are **weakly dominated** in this case.
- Unlike the case of strict domination, playing a weakly dominated action is consistent with Nash equilibrium.
  - Classic example: Bertrand competition

Weakly Dominant Actions

- Nevertheless, eliminating weakly dominated actions is a popular criterion for selecting among Nash equilibria.
- In the current buyer-seller example, the seller has a weakly dominant action $a_s = v$, for every v.
  - E.g., deviation from $a_s > v$ to v:
    - Increases payoff from zero to $a_b - v$ when $a_b \in (v, a_s)$.
    - Makes no difference otherwise.

Bilateral Trade

- Suppose, then, that the seller plays $a_s = v$ for every v.
- The buyer chooses a_b to maximize

$$\sum_v p(v) u_b(a_b, a_s(v), v)$$

- Because $v \sim U[0,1]$ (a continuous variable), we need to write

$$\int u_b(a_b, a_s(v), v) dv$$

Bilateral Trade

- $u_b(a_b, a_s, v) = 0$ whenever $a_b < a_s$, independently of v.

- This enables us to rewrite the buyer's objective function:

$$\Pr(a_b \geq a_s) \cdot E(1.5v - a_b | a_b \geq a_s)$$

- Since $a_s = v$ for every v, this is equivalent to

$$\Pr(v \leq a_b) \cdot [1.5E(v | v \leq a_b) - a_b]$$

Bilateral Trade

- Because $v \sim U[0,1]$, the following holds for every $a \in [0,1]$:
  - $\Pr(v \leq a) = a$ for every $a \in [0,1]$
  - $\mathrm{E}(v|v \leq a) = 0.5a$ for every $a \in [0,1]$
- The buyer would never want to play $a_b > 1$.

$$\Pr(v \leq a_b) \cdot [1.5E(v|v \leq a_b) - a_b]$$

$$= a_b \cdot [1.5 \cdot 0.5a_b - a_b] = -0.25(a_b)^2$$

- The optimal action is therefore $a_b = 0$!

Bilateral Trade

- Zero probability of trade in equilibrium, despite the efficiency of trade for all v
- Game-theoretic formulation of Akerlof's insight regarding market failure due to adverse selection
- Key argument: The buyer draws a statistical inference about v from the "pitoval event" that trade occurs, taking into account his equilibrium knowledge of the seller's strategy.

Second-Price Auction

- Two bidders, denoted 1 and 2, compete for an object.
- The bidders simultaneously submit bids $b_1, b_2 \geq 0$.
- The object is allocated to the player who submitted the highest bid.
- The winner pays the **loser**'s bid.

Second-Price Auction

- The state of the world is the pair $(t_1, t_2) \sim U[0,1]^2$.
  - t_i is player i's signal.
  - $\tau_i(t_1, t_2) = t_i$
- When player i wins the auction, his payoff is $t_i + \alpha t_j - b_j$.
  - $\alpha \in [0,1]$
  - j is i's opponent.
- When player i loses the auction, his payoff is zero.

Interpretations of the Payoff Function

$$t_i + \alpha t_j - b_j$$

- $\alpha = 0$: Private values
  - Purely idiosyncratic tastes
- $\alpha = 1$: Common values
  - Example: Bidding for an oil tract
- $\alpha \in (0,1)$: Intermediate case
  - Example: Partial technological spillovers

Nash Equilibrium

- Look for a symmetric Nash equilibrium with linear bidding strategies: $b_i = kt_i$
  - $k > 0$ is a constant that needs to be derived.
- A "pivotal event": Winning the auction
- The bidder's payoff is zero with certainty outside the pivotal event.

Nash Equilibrium

- Take player 2's bidding strategy as given.
- Consider player 1's maximization problem given t_1. He chooses b_1 to maximize

$$\Pr(b_1 > b_2) \cdot E[t_1 + \alpha t_2 - b_2 | b_1 > b_2]$$

  - We can ignore ties ($b_1 = b_2$) because this is a zero-probability event, given that player 2's bid distribution is continuous.

Parsing The Player's Objective Function

$$\Pr(b_2 < b_1) \cdot \ [t_1 + \alpha E(t_2|b_2 < b_1) - E(b_2|b_2 < b_1)]$$

- By assumption, $b_2 = kt_2$

- $t_2 \sim U[0,1]$, independently of t_1.

  $\Longrightarrow$ $b_2 \sim U[0, k]$, independently of t_1.

  $\Longrightarrow$ $\Pr(b_2 < b_1) = \frac{b_1}{k}$ for every $b_1 \leq k$.

- Player 2 has no reason to play $b_1 > k$.

Parsing The Player's Objective Function

$$\frac{b_1}{k} \cdot [t_1 + \alpha E(t_2 | b_2 < b_1) - E(b_2 | b_2 < b_1)]$$

- $b_2 \sim U[0, k]$, independently of t_1.

$$\Longrightarrow E(b_2 | b_2 < b_1) = \frac{b_1}{2} \text{ for every } b_1 \leq k.$$

Parsing The Player's Objective Function

$$\downarrow$$

$$\frac{b_1}{k} \cdot [t_1 + \alpha E(t_2 | b_2 < b_1) - \frac{b_1}{2}]$$

- $b_2 \sim U[0, k]$, independently of t_1.

$$\Longrightarrow E(b_2 | b_2 < b_1) = \frac{b_1}{2} \text{ for every } b_1 \leq k.$$

- Because $t_2 = \frac{b_2}{k}$, $E(t_2 | b_2 < b_1) = \frac{b_1}{2k}$ for every $b_1 \leq k$.

Parsing The Player's Objective Function

$$\frac{b_1}{k} \cdot [t_1 + \alpha \frac{b_1}{2k} - \frac{b_1}{2}]$$

- The meaning of $E(t_2 | b_2 < b_1) = \frac{b_1}{2k}$:
  - Below the unconditional expectation $E(t_2) = 0.5$
  - Often referred to as "the winner's curse"

Parsing The Player's Objective Function

$$\frac{b_1}{k} \cdot \left[t_1 + \alpha \frac{b_1}{2k} - \frac{b_1}{2}\right]$$

- The meaning of $E(t_2 | b_2 < b_1) = \frac{b_1}{2k}$:
  - Despite the suggestive term, **no** clear-cut incentive to shade one's bid (intensive vs. extensive margins)!
  - Statistical inference from a hypothetical event

Deriving the Equilibrium Strategy

$$\frac{b_1}{k} \cdot [t_1 + \alpha \frac{b_1}{2k} - \frac{b_1}{2}]$$

- First-order condition w.r.t b_1 gives $b_1 = \frac{k}{k-\alpha} t_1$.

- But by assumption, $b_1 = kt_1 \implies k = \frac{k}{k-\alpha}$

- We obtain:

$$b_1 = (1+\alpha)t_1$$

The Private Values Case

- All the player cares about is the opponent's bid distribution
- No need to draw inferences from the pivotal event
- Bidding one's value is a weakly dominant action in the second-price auction with private values
  - Similar logic to the bilateral trade example: Bidding affects the prospect of winning, not the payment conditional on winning.

Interim Summary

- In both applications so far, a player’s expected payoff is the probability he wins an object times his expected net payoff conditional on winning the object.
- To calculate the object’s conditional expected value, the player relies on his knowledge of opponents’ strategies.
- This is the source of the phenomena: market failure due to adverse selection, the winner’s curse in auctions.

Strategic Voting

- Voting is a non-monetary mechanism for aggregating:
  - Preferences
  - Information
- We will consider a common-interest environment that focuses purely on the information aggregation function
- Surprisingly, strategic considerations will matter despite the lack of a conflict of interests!

The Jury Model

- A group of n voters submit simultaneous recommendations between two social alternatives, denoted 1 and -1.
  - n is an odd number.
  - $a_i \in \{-1,1\}$ denotes voter i's recommendation.
- The implemented outcome is $z \in \{-1,1\}$.

The Jury Model

- The method of aggregating recommendations is non-weighted majority voting.
- The implemented outcome is $z = 1$ if and only if $\sum_i a_i \geq k$.
  - $k = 0$ is a simple-majority rule.
  - $k = n$ is a unanimity rule.

The Jury Model

- The state of the world is $(\theta, t_1, \ldots, t_n)$:
  - $\theta \in \{-1, 1\}$ is the objectively desirable alternative.
  - $t_i \in \{-1, 1\}$ is voter i's signal.
  - $\tau_i(\theta, t_1, \ldots, t_n) = t_i$
- Each voter's payoff is $z\theta$.
  - Common interests: The outcome should match the state.
  - Voters care about actions only insofar as they affect z.

The Jury Model

- The prior p over $(\theta, t_1, \ldots, t_n)$ is:
  - $p(\theta = 1) = 0.5$
  - $p(t_i = \theta | \theta) = q \in (0.5, 1)$ for all θ, independently of t_{-i}.
  - q measures the signal's accuracy
- Interpreting voters' conditionally independent signals:
  - Different expertise
  - Differential attention

Interpretational Difficulties

- Small n (juries, committees):
  - Lack of communication despite common interests?
- Large n (large elections):
  - Do voters think strategically?

Simple Majority

- Let $k = 0$
- Non-strategic benchmark: Voters report their signals ($a_i = t_i$)
- Condorcet's Jury Theorem: As n grows larger, the probability that the majority decision is correct approaches 1.
  - "Wisdom of the crowd"
  - A precursor of the law of large numbers

The Single-Voter Case

- Let $n = 1$ (a dictatorial decision maker: $z = a_i$)

$$p(\theta = 1 | t_1 = 1) = \frac{0.5 \cdot q}{0.5 \cdot q + 0.5 \cdot (1 - q)} = q$$

- Likewise, $p(\theta = -1 | t_1 = -1) = q$.
- Since $q > 0.5$, the voter prefers to play $a_i \equiv t_i$.
  - Optimal individual decision coincides with truthful reporting
  - Follows from the symmetric prior and payoff function.

Nash Equilibrium for (Odd) $n > 1$

- A "bad" equilibrium in which all voters vote for the same alternative, independently of their signal:
  - A "weak" equilibrium: No individual voter can change the outcome by unilateral deviation.
  - It would become strict if we introduced small information acquisition costs.
- Is truthful reporting consistent with Nash equilibrium?

Nash Equilibrium for (Odd) $n > 1$

- Assume every voter i plays $a_i \equiv t_i$.
- W.l.o.g, consider voter 1 and suppose $t_1 = 1$.
- When calculating the expected utility from an action, the voter sums over all payoff-relevant contingencies $(\theta, a_2, \dots, a_n)$.
- Voter 1's action affects z if and only if $\sum_{i>1} a_i = 0$.

Nash Equilibrium for (Odd) $n > 1$

$a_1 \backslash \Sigma_{i>1} a_i$	-(n-1)	$\cdots$	-2	0	2	$\cdots$	n-1
1	-1	$\cdots$	-1	**1**	1	$\cdots$	1
-1	-1	$\cdots$	-1	**-1**	1	$\cdots$	1

The outcome as a function of a_1 & $\sum_{i>1} a_i$

- We can ignore all the contingencies in which the voter's action doesn't make a difference.
- Manifestation of the independence property of EU Theory

Nash Equilibrium for (Odd) $n > 1$

- Voter 1 effectively calculates $Pr(\theta = 1 | t_1 \,;\, \sum_{i>1} a_i = 0)$
  - He plays $a_1 = 1$ (-1) whenever this posterior is above (below) 0.5.
- The calculation takes into account other voters' strategies.
- Yet another instance of the "statistical inferences from hypothetical events" theme

Nash Equilibrium for (Odd) $n > 1$

- We have guessed that $a_i \equiv t_i$ for every i. Then:

$$Pr(\theta = 1 | t_1\, ; \sum_{i>1} a_i = 0) = p(\theta = 1 | t_1\, ; \sum_{i>1} t_i = 0)$$

- The R.H.S is expressed entirely in terms of the prior p.

- The equal numbers of 1 and -1 signals among voters

  $2, \ldots, n$ mean that these signal cancel each other out:

$$p(\theta = 1 | t_1\, ; \sum_{i>1} t_i = 0) = p(\theta = 1 | t_1)$$

Nash Equilibrium for (Odd) $n > 1$

$$p(\theta = 1 | t_1 ; \sum_{i>1} t_i = 0) = p(\theta = 1 | t_1)$$

- Now the R.H.S is just as in the single-voter case.
- We've established that in this case playing $a_1 \equiv t_1$ is optimal.
- Therefore, a strategy profile in which every voter reports his signal constitutes a Nash equilibrium.

The Simple-Majority Case: Discussion

- The voter imagines being pivotal, because that is the only scenario in which his vote matters.
- In that event, the other votes have no informational content because the 1 and -1 signals cancel each other out.
- It would be intuitive to draw that inference ex-post; the idea that voters do it **in anticipation of this event** is somewhat less intuitive.

Unanimity

- Let $k = n$
- The outcome -1 is the default social outcome; switching to the other outcome requires unanimous agreement.
- The norm in criminal jury trials
- A "bad" Nash equilibrium: All voters always recommend -1.
- Is truthful reporting consistent with Nash equilibrium?

Truthful Nash Equilibrium?

- Assume every voter i plays $a_i \equiv t_i$.
- W.l.o.g, consider voter 1 and suppose $t_1 = -1$.
- When calculating the expected utility from an action, the voter sums over all payoff-relevant contingencies $(\theta, a_2, \dots, a_n)$.
- Voter 1's action affects z if and only if $\sum_{i>1} a_i = n - 1$.

Truthful Nash Equilibrium?

$a_1 \backslash \Sigma_{i>1} a_i$	$-(n-1)$	$\cdots$	$n-3$	$n-1$
1	-1	$\cdots$	-1	**1**
-1	-1	$\cdots$	-1	**-1**

The outcome as a function of a_1 & $\sum_{i>1} a_i$

- Voter 1 effectively best-replies to the distribution

$$Pr(\theta|t_1\,;\,\textstyle\sum_{i>1} a_i = n-1) = Pr(\theta|t_1\,;\textstyle\sum_{i>1} t_i = n-1)$$

Plugging Equilibrium strategies

Explicit Posterior Calculation

$$Pr(\theta = 1 | t_1 = -1\,; t_i = 1 \; for \; all \; i > 1)$$

$$= \frac{\overbrace{0.5 \cdot (1-q) \cdot q^{n-1}}^{I'm \; wrong}}{0.5 \cdot (1-q) \cdot q^{n-1} + \underbrace{0.5 \cdot q \cdot (1-q)^{n-1}}_{I'm \; right}} = \frac{1}{1 + \left(\frac{1-q}{q}\right)^{n-2}} > \frac{1}{2}$$

- Voter 1's best-reply is to vote against his signal,

  contradicting the equilibrium assumption.

Unanimity: Discussion

- Ex-post, seeing my signal differs from everybody else's, it would be sensible for me to adopt the majority view.
- What's counter-intuitive is that this inference is made in the **interim** stage, before that (rare) pivotal event happen.
- Equilibrium with partial truthful reporting requires mixing.
- Do people actually reason along these lines? Mixed experimental evidence

Summary

- Many examples of games in which payoffs depend on a “pivotal event” (auctions, voting, trade)
- Recurring theme: Nash equilibrium analysis involves statistical inferences from the pivotal event, taking the opponents’ strategies into account
- Non-trivial effects: The winner’s curse, swing voter’s curse