Advanced Microeconomic Theory

Lecture 3: Games with Incomplete Information

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Games with Incomplete Information (a.k.a Bayesian Games)

- Strategic interactions in which players do not know everything about the game
  - Variables that affect my own payoffs
  - My opponents’ preferences
  - My opponents’ knowledge (including knowledge of my knowledge, etc…)

Relevant Environments

- Auctions: Incomplete information about bidders' preferences or the value of the sold object
- Adverse selection in bilateral trade
- Speculative trade in financial markets
- Strategic voting
- Bank runs, currency crises

Plan of the Following Lectures

- Enriching the model of strategic games to express players' uncertainty
- Problematic treatment in available textbooks; supplementary lecture notes
- Lots of examples and applications
- Exercises are super-important.

The Formal Model

- We retain the following components of the basic model:
  - A set of players $N = \{1, \dots, n\}$
  - For each player $i \in N$, a set of feasible actions A_i
  - $A = \times_{i \in N} A_i$ is the set of action profiles.
- For simplicity, we rule out mixed strategies.
- No uncertainty about the set of feasible actions (w.l.o.g)

The New Ingredients: State Space

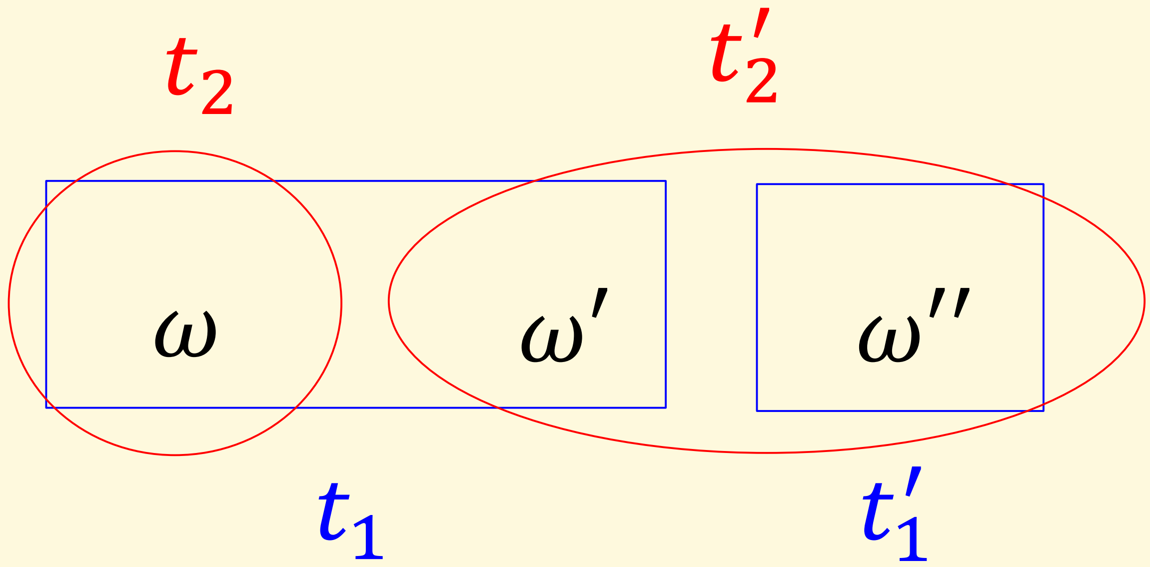
- A set of states of the world Ω
  - A state resolves all exogenous uncertainty that is relevant to the model.
- A prior probability distribution $p \in \Delta(\Omega)$ over the state space
  - Controversial: What does it mean? Why is it common?
- For each player $i \in N$, a vNM utility function $u_i: A \times \Omega \to \mathbb{R}$

The New Ingredients: Signals

- For each player $i \in N$:
  - A set of possible signals T_i
  - A signal function $\tau_i : \Omega \to T_i$ (deterministic because recall that the state resolves **all** uncertainty)
- A player's signal $t_i \in T_i$ represents his information, or state of knowledge, regarding the state of the world.
  - It is often referred to as the player's type.

The Information Structure

- The new components $\langle \Omega, p, (T_i)_{i \in N}, (\tau_i)_{i \in N} \rangle$ define the game's information structure.
- Useful diagram: **Information partitions**



Examples of Information Structures

- A seller knows the value of an object; the buyer is uninformed.
- The state of the world is the object's value v.
  - $\tau_{seller}(v) = v$ for all v
  - $\tau_{buyer}(v) = t^*$ for all v
- The prior p describes the distribution of v.

Examples of Information Structures

- Two firms, 1 and 2, receive noisy information about uncertain market demand.
- A state of the world is a triple (θ, t_1, t_2).
  - The size of market demand and the firms' signals
  - $\tau_i(\theta, t_1, t_2) = t_i$
- The prior p describes the distribution of market demand and the conditional distribution of the firms' signals.

Posterior Beliefs

Assumption: Player i's belief over the state space given his signal t_i is governed by Bayesian updating (hence the nickname "Bayesian Games"):

– If $\tau_i(\omega) = t_i$, then

$$p(\omega|t_i) = \frac{p(\omega)}{p(t_i)} = \frac{p(\omega)}{\sum_{\omega' \in \tau_i^{-1}(t_i)} p(\omega')}$$

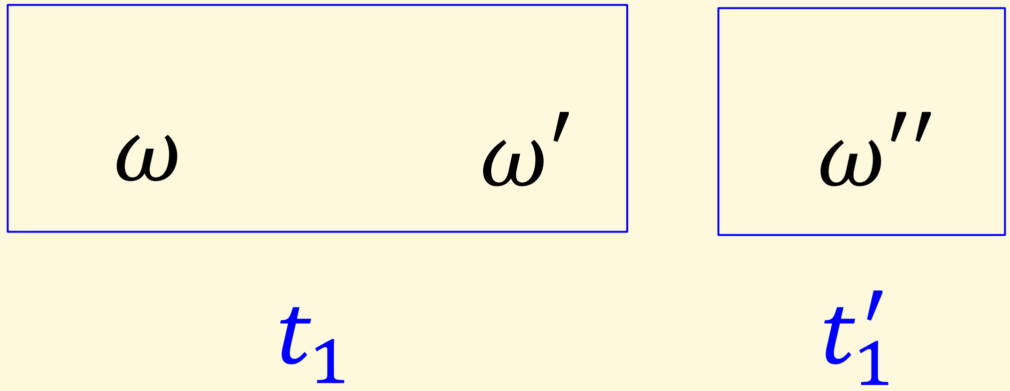
– If $\tau_i(\omega) \neq t_i$, then $p(\omega|t_i) = 0$

Posterior Beliefs: Illustration

- $p(\omega) = p(\omega'') = 0.25$, $p(\omega') = 0.5$

- $p(\omega|t_1) = \frac{0.25}{0.25+0.5} = \frac{1}{3}$ $\quad$ $p(\omega'|t_1) = \frac{0.5}{0.25+0.5} = \frac{2}{3}$

- $p(\omega''|t_1') = 1$



Strategies

- A player can only condition his action on his information.
- A pure strategy for player i is a function $s_i\colon T_i \to A_i$.
  - $s_i(t_i) \in A_i$ is the action that player i takes when his signal is t_i.
- We will rule out mixed strategies, for simplicity.

Strategies

- Ex-ante interpretation: The player **plans** the strategy before receiving the information
- Interim interpretation: s_i describes other players' belief regarding player i's contingent behavior; optimality of player i's action is evaluated **given** his information.
- As in the case of mixed-strategy Nash equilibrium, we will mostly work with the interim version.

Nash Equilibrium

Definition: A strategy profile $(s_1, \dots, s_n)$ is a Nash equilibrium if for every player i and every $t_i \in T_i$:

$$s_i(t_i) \in argmax_{a_i} \sum_{\omega \in \Omega} p(\omega | t_i) u_i\big(a_i, (s_j(\tau_j(\omega)))_{j \neq i}, \omega\big)$$

- Each player chooses an action that maximizes his expected utility, given his belief over the state space and regarding the opponents’ strategies.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum\nolimits_{\omega \in \Omega} p(\omega|t_i) u_i\big(a_i, (s_j(\tau_j(\omega)))_{j \neq i}, \omega\big)$$

- The player has double uncertainty, regarding the state of the world **and** the opponents' actions.
- Opponents' actions are uncertain because their information is uncertain.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega \in \Omega} p(\omega|t_i) u_i\big(a_i, (s_j(\tau_j(\omega)))_{j \neq i}, \omega\big)$$

- But in Nash equilibrium, the player correctly perceives the mapping state → opponent’s signal → opponent’s action
- This reduces his uncertainty de facto to uncertainty about the state of the world.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega\in\Omega} p(\omega|t_i) u_i\left(a_i, (s_j(\tau_j(\omega)))_{j\neq i}, \omega\right)$$

- The player's residual uncertainty regarding the state is given by his Bayesian posterior belief.
- He sums over the states and weighs them according to his posterior belief.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega\in\Omega} p(\omega|t_i)u_i\big(a_i, (s_j(\tau_j(\omega)))_{j\neq i}, \omega\big)$$

- For each possible state, the player correctly predicts the opponents' action profile.
- Important motto: Statistical inferences from contingent events

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Two equally likely states of Nature:
  - Bad (unprofitable investment)
  - Good (profitable investment, provided both players invest)
- NI is a safe action; I is a risky but potentially profitable one.

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- If the state of Nature is common knowledge, players think about each payoff matrix as an isolated game.
- In bad state, NI is a strictly dominant action.
- In good state, two pure Nash equilibria: (I,I) and (NI,NI)

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- We’ll consider two alternative information structures; each induces a different Bayesian game.
- “Never invest” is a Nash equilibrium for **every** information structure. Are there other equilibria?

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Player 1 knows the state of Nature.
- Player 2 is uninformed.
- There is no additional uncertainty.

Information Structure No. 1

Bad state

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Good state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

- $\Omega = \{g, b\}$
- $\tau_1(\omega) = t^{\omega}$ for all ω
- $\tau_2(\omega) = t^{*}$ for all ω

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Player 2 (the uninformed party) plays a constant action in any pure-strategy Nash equilibrium.
- If he plays NI, player 1's best-reply is always NI, and we're back with the "never invest" Nash equilibrium.

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Let's guess an equilibrium in which player 2 plays I.
- In the bad state, player 1 learns this and plays the strictly dominant action NI $\Longrightarrow$ $s_1\left(t^b\right) = NI$
- In the good state, his best-reply is I $\Longrightarrow$ $s_1(t^g) = I$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$p(\omega = g) \cdot u_2(a_2 = I, s_1(t^g), \omega = g)$$

$$+ p(\omega = b) \cdot u_2\left(a_2 = I, s_1\left(t^b\right), \omega = b\right) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$0.5 \cdot u_2(a_2 = I, s_1(t^g), \omega = g)$$

$$+\, 0.5 \cdot u_2\big(a_2 = I, s_1\big(t^b\big), \omega = b\big) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.

- Player 2's action I is a best-reply if and only if

$$0.5 \cdot u_2(a_2 = I, a_1 = I, \omega = g)$$

$$+ 0.5 \cdot u_2(a_2 = I, a_1 = NI, \omega = b) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$0.5 \cdot 1 + \ 0.5 \cdot (-2) \ \geq 0$$

- This inequality does not hold.

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Therefore, we are unable to sustain any pure Nash equilibrium apart from both “never invest”.
- An example of asymmetric information as a friction that prevents an efficient outcome

Information Structure No. 2

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

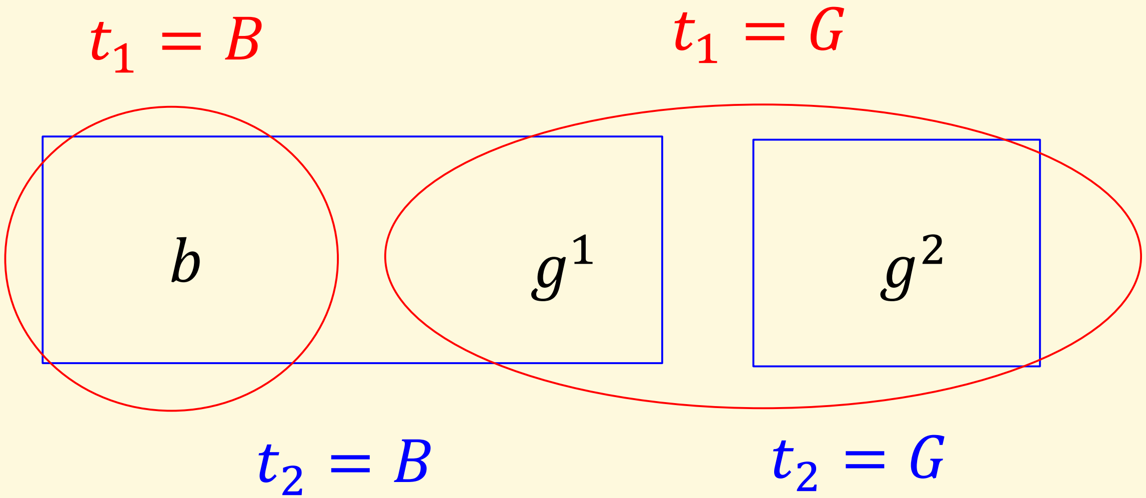
Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

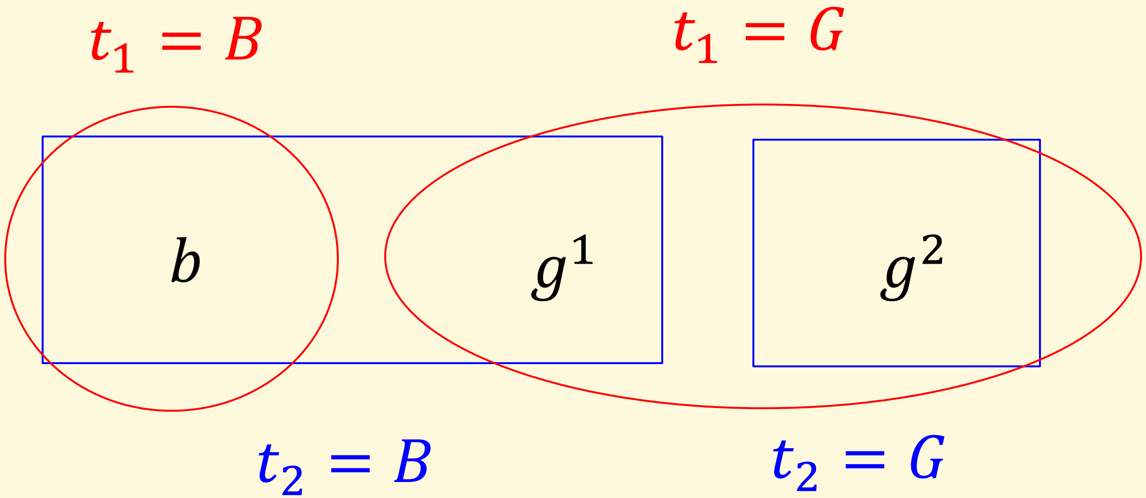
- Player 1 knows the state of Nature.
- When the state is good (but only then), player 2 gets tipped about this with probability $1 - \varepsilon$.
- Player 1 does not know whether good news have leaked.

Information Structure No. 2



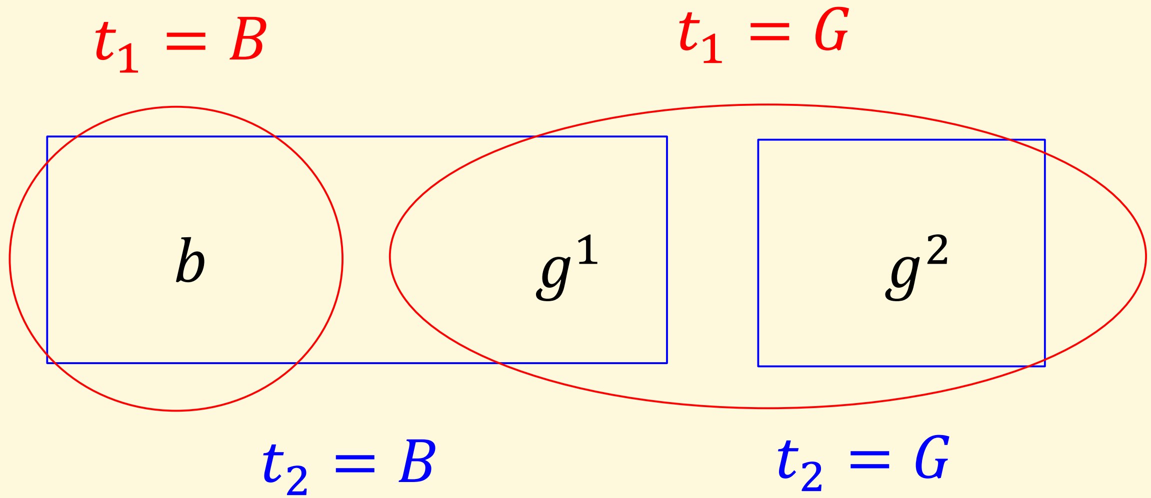
- $\Omega = \{b, g^1, g^2\}$ (bad state, good state with/without leak)
- $\tau_1(b) = B$, $\tau_1(g^1) = \tau_1(g^2) = G$
- $\tau_2(b) = \tau_2(g^1) = B$, $\tau_2(g^2) = G$

Information Structure No. 2



- A strategy for each player specifies how he acts when he gets the signals B and G.
- Is there a Nash equilibrium in which players sometimes play I?

Information Structure No. 2

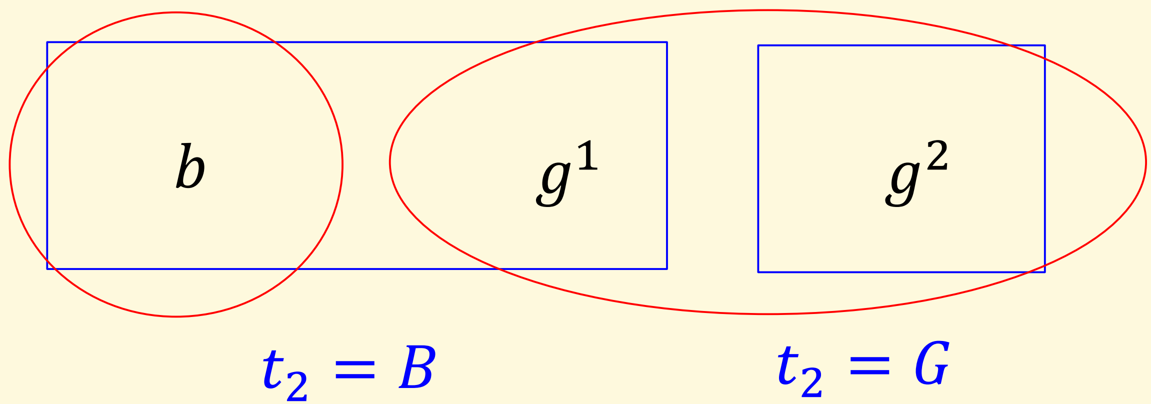


- When $t_1 = B$, player 1 knows that the state is b for sure.

- Therefore, $a_1 = NI$ is strictly dominant when $t_1 = B$.
  - This is what player 1 will play in any Nash equilibrium.

Information Structure No. 2

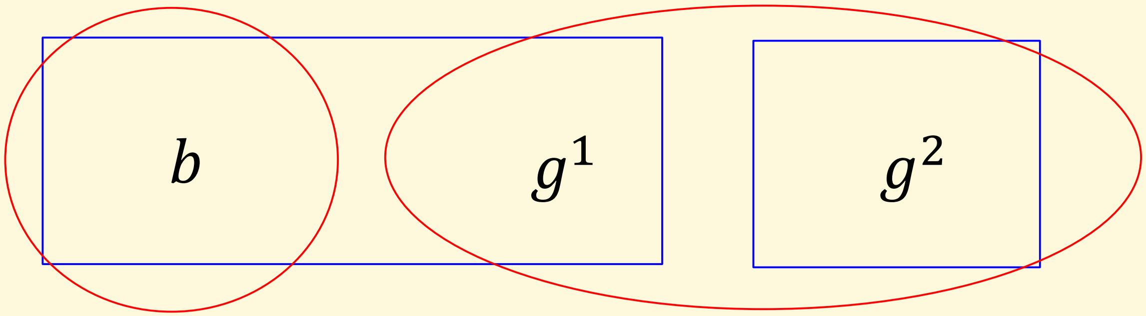
$\boldsymbol{a_1(t_1 = B) = NI}$ $t_1 = G$



- To break away from the "never invest" Nash equilibrium, we must guess that player 1 plays I when $t_1 = G$.
- Let us check whether the guess is consistent.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



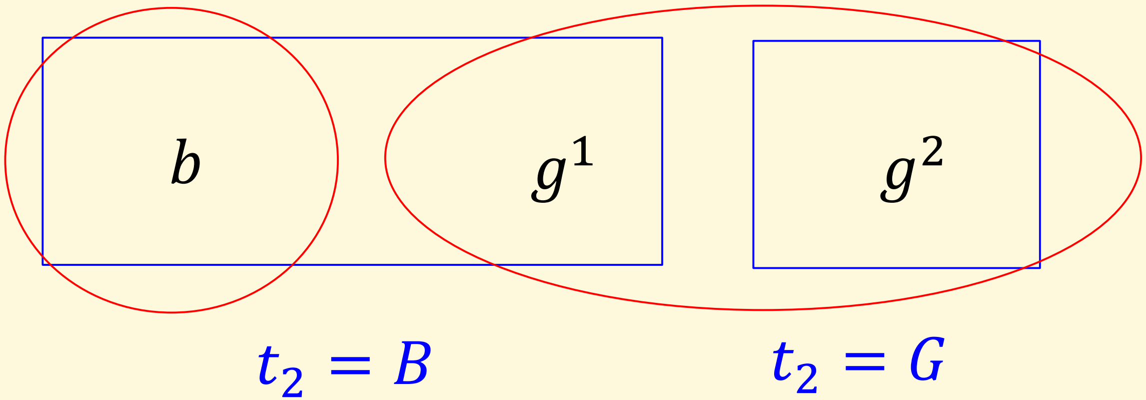
$t_2 = B$ $t_2 = G$

- When $t_2 = G$, player 2 knows that the state is g^2 for sure.

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\qquad$ $\boldsymbol{a_1(t_1 = G) = I}$

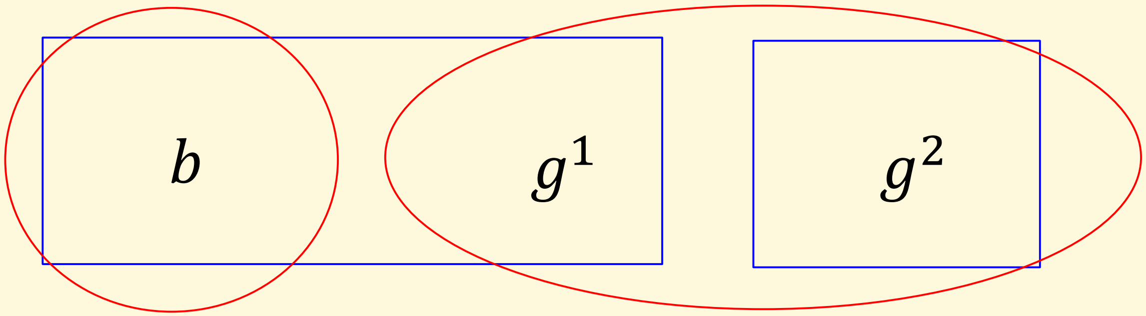


- Player 2 infers that player 1 plays I (according to our guessed equilibrium strategy).

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



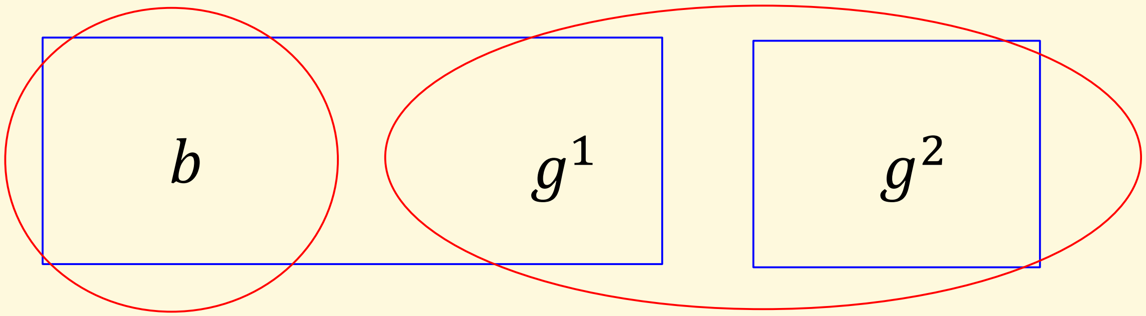
$t_2 = B$ $t_2 = G$

- Player 2's best-reply is I.

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



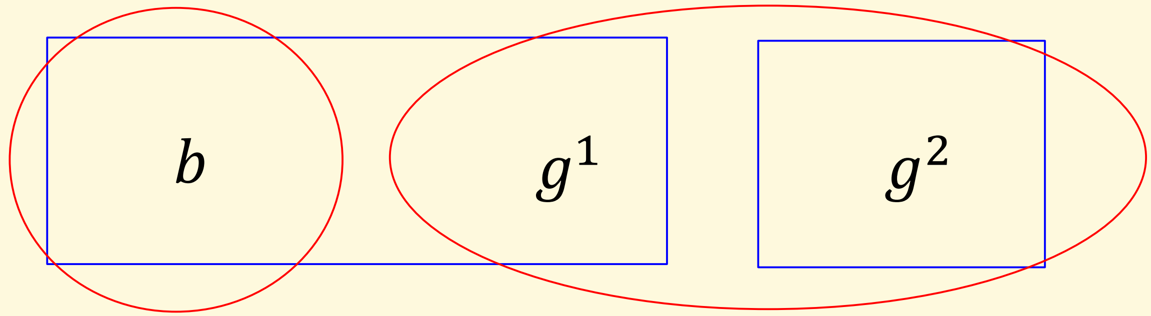
$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

- Player 2's posterior belief given $t_2 = B$:

$$p(\omega = b | t_2 = B) = \frac{0.5}{0.5 + 0.5 \cdot \varepsilon} = \frac{1}{1 + \varepsilon}$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



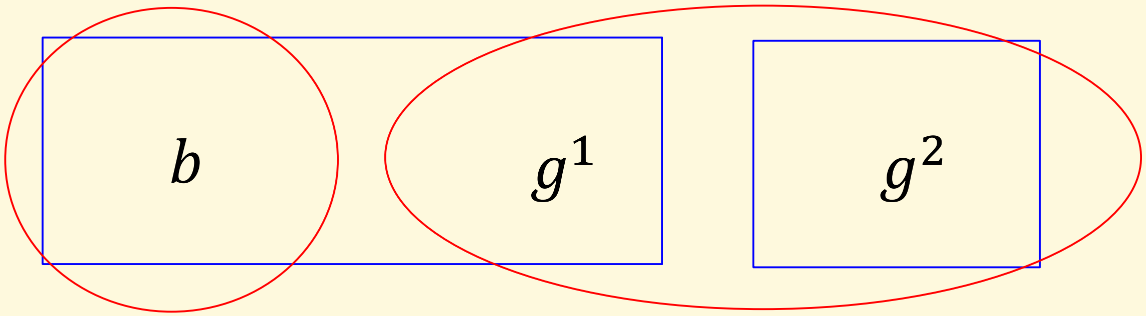
$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

$$p(\omega = b|t_2 = B) = \frac{0.5}{0.5 + 0.5 \cdot \varepsilon} = \frac{1}{1 + \varepsilon}$$

$$p(\omega = g^1|t_2 = B) = \frac{0.5 \cdot \varepsilon}{0.5 + 0.5 \cdot \varepsilon} = \frac{\varepsilon}{1 + \varepsilon}$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

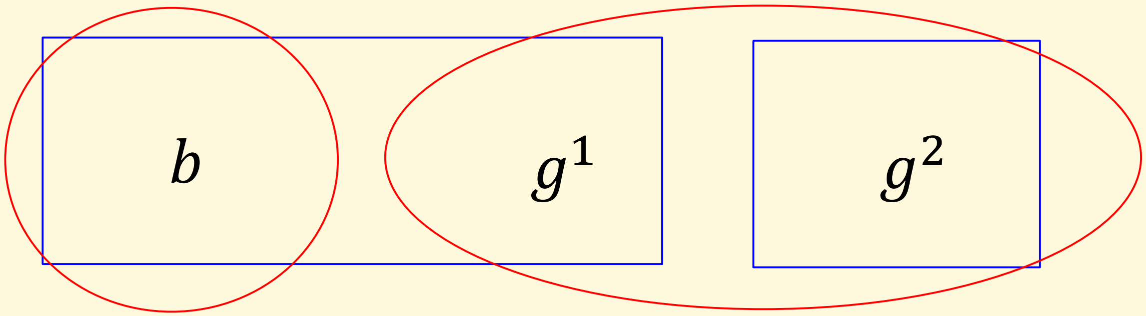
- Player 2's expected payoff from I is

$$p(\omega = b|t_2 = B) \cdot u_2(a_2 = I, s_1(\tau_1(b)), \omega = b)$$

$$+\, p(\omega = g^1|t_2 = B) \cdot u_2(a_2 = I, s_1(\tau_1(g^1)), \omega = g^1)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



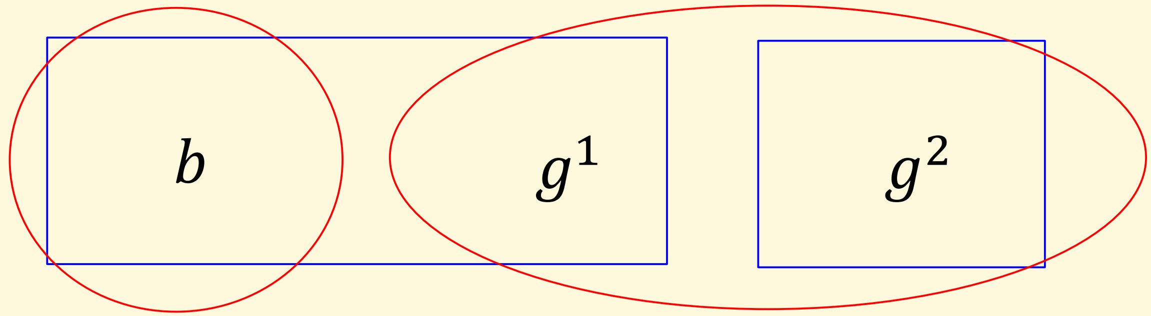
$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\frac{1}{1+\varepsilon} \cdot u_2(a_2 = I, s_1(\tau_1(b)), \omega = b)$$

$$+ \frac{\varepsilon}{1+\varepsilon} \cdot u_2(a_2 = I, s_1(\tau_1(g^1)), \omega = g^1)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\frac{1}{1+\varepsilon} \cdot u_2(a_2 = I, a_1 = NI, \omega = b)$$

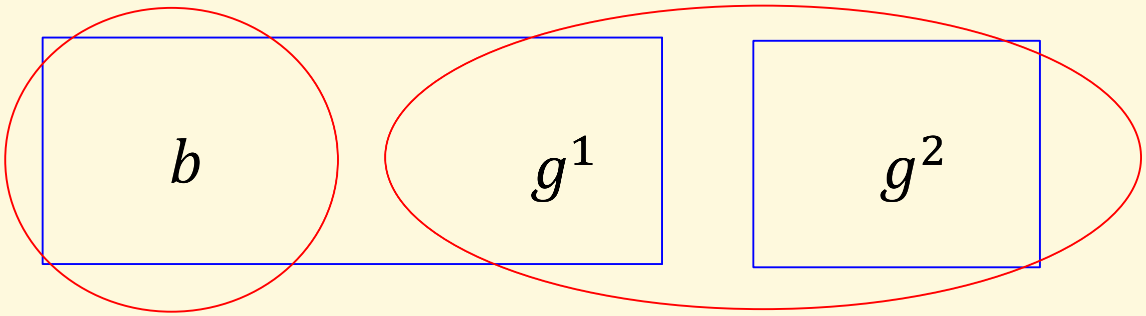
	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

$$+\frac{\varepsilon}{1+\varepsilon} \cdot u_2(a_2 = I, a_1 = I, \omega = g^1)$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$a_1(t_1 = B) = NI$ $\qquad$ $a_1(t_1 = G) = I$



$t_2 = B$ $\qquad$ $a_2(t_2 = G) = I$

$$\frac{1}{1+\varepsilon} \cdot (-2) + \frac{\varepsilon}{1+\varepsilon} \cdot 1 < 0$$

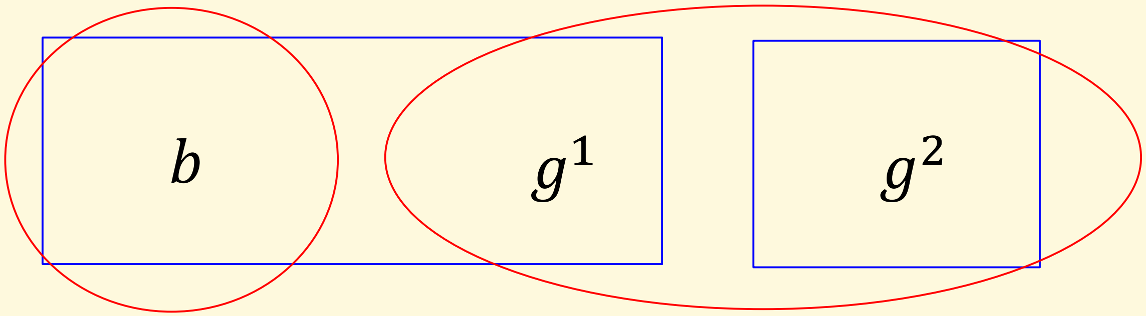
	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

- Player 2's best-reply at $t_2 = B$ is NI.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$

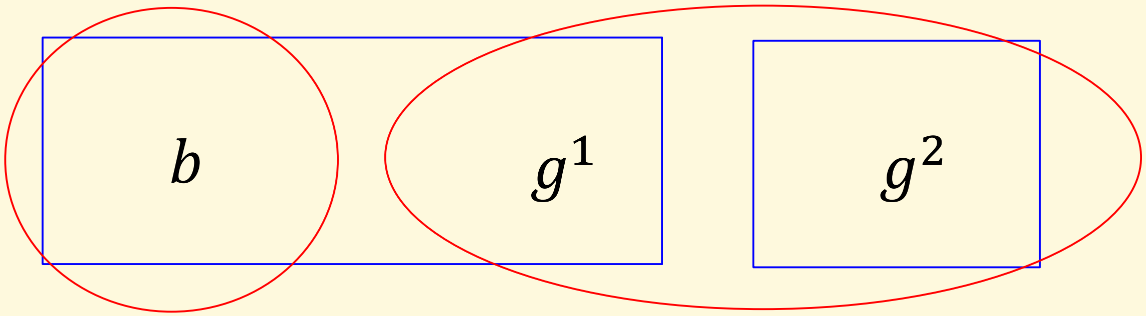


$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- It remains to check whether player 1 indeed wants to play I when $t_1 = G$.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



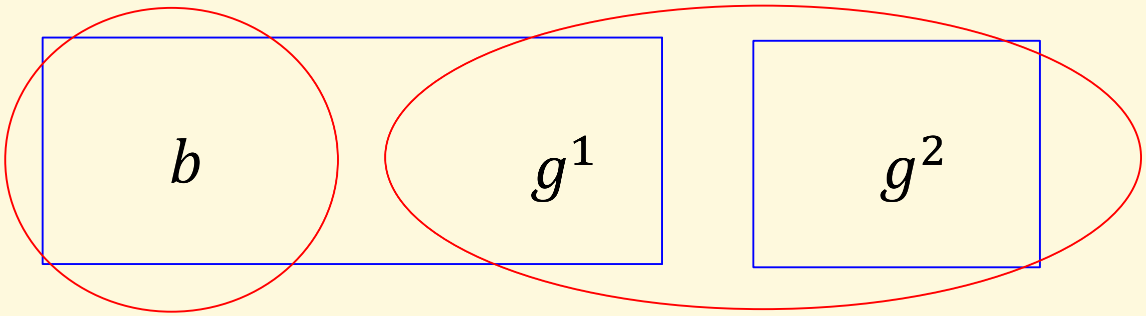
$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- Player 1's posterior belief given $t_1 = G$:

$$p(\omega = g^2 | t_1 = G) = \frac{0.5 \cdot (1-\varepsilon)}{0.5 \cdot \varepsilon + 0.5 \cdot (1-\varepsilon)} = 1 - \varepsilon$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

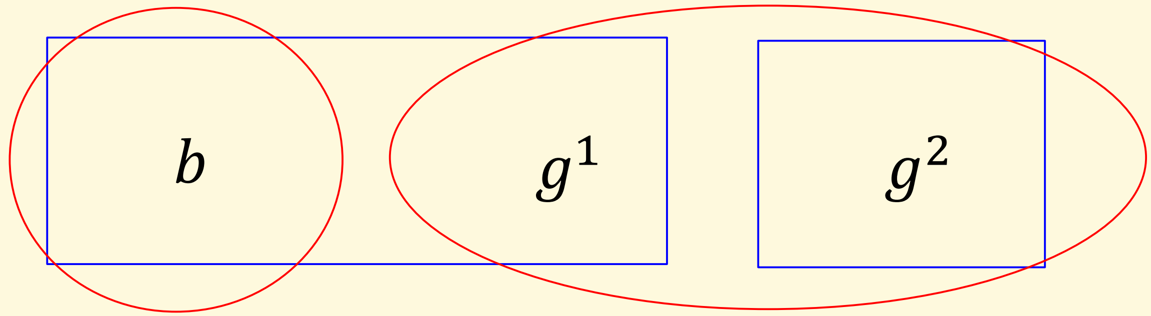
- Player 1's expected payoff from I is

$$p(\omega = g^1 | t_1 = G) \cdot u_1(a_1 = I, s_2(\tau_2(g^1)), \omega = g^1)$$

$$+ p(\omega = g^2 | t_1 = G) \cdot u_1(a_1 = I, s_2(\tau_2(g^2)), \omega = g^2)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



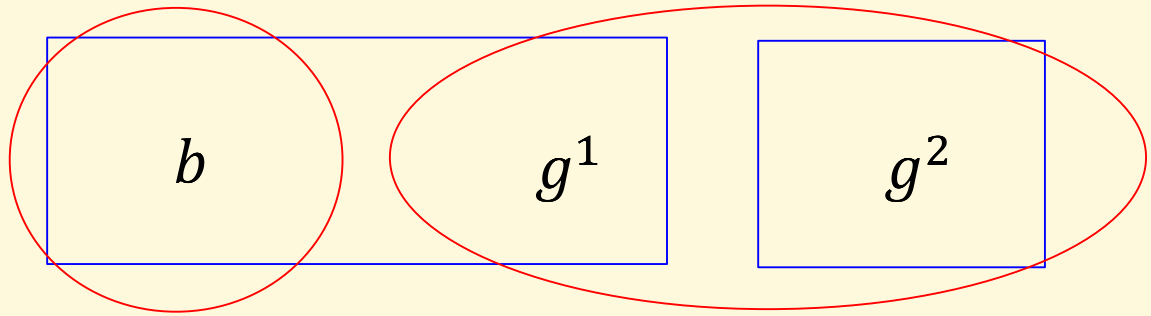
$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\varepsilon \cdot u_1(a_1 = I, s_2(\tau_2(g^1)), \omega = g^1)$$

$$+ (1 - \varepsilon) \cdot u_1(a_1 = I, s_2(\tau_2(g^2)), \omega = g^2)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\qquad$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\qquad$ $\boldsymbol{a_2(t_2 = G) = I}$

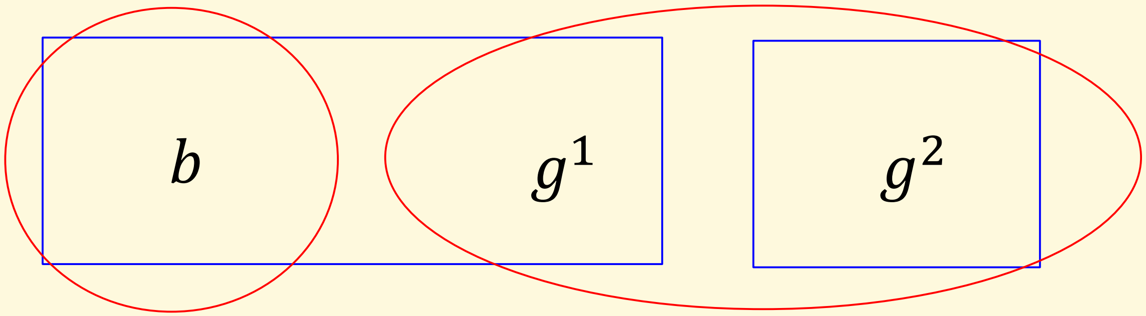
$$\varepsilon \cdot u_1(a_1 = I, a_2 = NI, \omega = g^1)$$

$$+ (1 - \varepsilon) \cdot u_1(a_1 = I, a_2 = I, \omega = g^2)$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

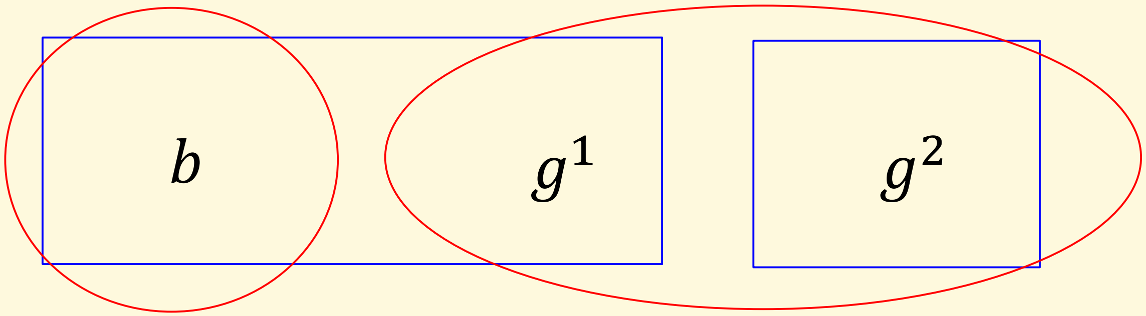
$$\varepsilon \cdot (-2) + (1-\varepsilon) \cdot 1$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

- Weakly above zero if and only if $\varepsilon \leq \frac{1}{3}$.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$

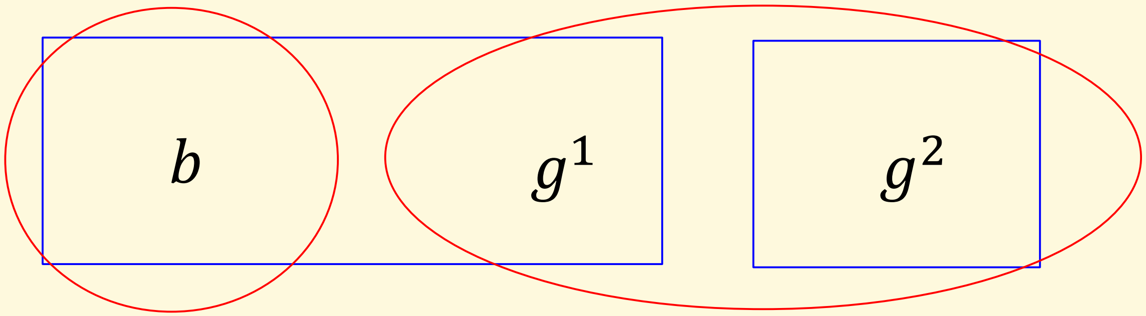


$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- Conclusion: There is a Nash equilibrium in addition to "never invest" if and only if $\varepsilon \leq \frac{1}{3}$.
  - Each players invests if and only if his signal is good.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- An intuitive prediction that lower informational frictions facilitate good coordination