Advanced Microeconomic Theory

Lecture 1: Nash Equilibrium in Strategic Games

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About Myself

- Ran Spiegler, Professor of Economics at UCL since 2006
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  - E-mail: r.spiegler@ucl.ac.uk
  - “Office” hour: Flexible, coordinate by e-mail
- Research areas: Microeconomic theory
  - Bounded rationality in markets
  - Behavioral implications of causal misperceptions
  - Incentive issues in modern platforms

General Introduction

- **Game theory**: The theoretical study of strategic decision making
- Decision making is strategic if the consequences of one's actions depend on other agents' actions.
  - This requires the decision maker (DM) to form beliefs about opponents' actions.
  - This in turn may require the DM to think about opponents' preferences, knowledge and mode of reasoning.

General Introduction

- Strategic decision making is at the core of microeconomics
  - Oligopoly, auctions, voting, bargaining, electoral competition, trade in markets
- Game theory is historically and intrinsically inter-disciplinary
  - Connections with evolutionary biology
  - Distributed computing in computer science

General Introduction

- This course:
  - Going more deeply into the game-theoretic concepts you tasted in the core micro theory course, and new concepts
  - Economic applications
- Restricted focus:
  - Theoretical orientation
  - Rational DMs

Plan of the Course

- Weeks 1-5 (Rani) :
  - Static games with complete information (Nash equilibrium, mixed strategies, zero-sum games and max-minimization, rationalizability)
  - Static games with incomplete information
- Weeks 7-11 (Duarte):
  - Extensive games w/o complete information
  - Applications: Repeated games, bargaining, signaling

Weekly Lecture Plan (weeks 1-5)

- Roughly 75 minutes of (recorded) live lecture
  - Starting Thursday 9:00am London time
  - The basic material for the week, discussion
- Roughly 35 minutes of pre-recorded videos
  - Additional material: examples, worked out proofs, etc.
- Class sessions (weeks 2-5, 6-11)

Written Material

- Textbooks:
  - Osborne-Rubinstein (terse, graduate level)
  - Osborne (precise yet more verbal, more examples, advanced undergrad level)
- Supplementary lecture notes, slides

Assessment

- The final grade is determined as a sum of:
  - Final exam during Term 3: 70%
  - Short midterm exam after sixth lecture: 20%
  - Online multiple-choice questions attached to four problem sets (two on RS's material, two on DG's material): 10% total

Game Theory: The Basic Insight

- Two game theories: “Non-cooperative” and “coalitional”
- This course focuses on the former.
- The basic idea: Model a situation of conflict as if it were a parlor game with rigid rules.
- Some real-life interactions (auctions, voting) literally conform to this description; in others (bargaining) the game-theoretic description is a caricature / an abstraction.

Strategic Games

- The most basic model of strategic interaction
- Agents act independently, once and for all
- No explicit time element
- No explicit model of information

Elements of the Model

- A set of **players** N (typically enumerated $1, \dots, n$)
- For each player $i \in N$, a set of **actions** A_i
  - The set of game outcomes (also referred to as **action profiles**) is $A = \times_{i \in N} A_i$.
  - Convenient notation: a_{-i} is the profile of actions by player i's opponents.
  - An action profile $a \in A$ can be decomposed as (a_i, a_{-i}).

Elements of the Model

- For each player $i \in N$, a **preference relation** $\succsim_i$ over A
- $\succsim_i$ is represented by a utility function $u_i : A \to \mathbb{R}$.
  - For now, u_i only has ordinal meaning.
- A strategic game is defined by $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$.
- Interpretation: Actions are taken independently (simultaneously?), once and for all.

Payoff Matrices

	Don't Advertise	Advertise
Don't Advertise	5,5	2,6
Advertise	6,2	3,3

- Two-player games with small action sets are conveniently represented by a payoff matrix.
- The game in this matrix is an example of the "Prisoner's Dilemma".

Payoff Matrices

	Cafe	Bar
Cafe	2,1	0,0
Bar	0,0	1,2

- The game in this matrix is an example of the “Battle of the Sexes”.

Payoff Matrices

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- This game is “matching pennies”.
- Also interpreted as “hide and seek”
- Basic illustrations of the “artistic” aspect of game-theoretic modeling

Solution Concepts

- **Solution concept**: A function that selects (predicts?) a subset of outcomes in each game in some class
- Traditionally, a solution concept embodies a specific theory of how players reason about games.
- Two salient approaches:
  - Equilibrium / steady-state
  - One-shot / introspective

Nash Equilibrium

Definition: An action profile $a^* \in A$ is a Nash equilibrium if for every player $i \in N$, $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$ for every $a_i \in A_i$.

- In words: Every player's action is a best-reply to the actions taken by his opponents.

Interpretations of Nash Equilibrium

- Axiomatic: Correct beliefs and individual rationality
- No individual regret
- The most common interpretation: Convention, steady state of an unspecified process of social learning
  - Close in spirit to competitive equilibrium
- Pre-play communication: A self-enforcing agreement

Nash Equilibrium: Examples

	Cafe	Bar
Cafe	2,1	0,0
Bar	0,0	1,2

	Don't Advertise	Advertise
Don't Advertise	5,5	2,6
Advertise	6,2	3,3

An Example: The Hotelling-Downs Model

- A model of electoral competition
- Two parties commit to a policy that lies on a left-right axis.
- Each party wants to maximize its vote share.
- Each voter votes for the party that commits to a policy that is closest to his idiosyncratic “ideal policy”.
- Voters’ ideal policies are distributed according to a continuous and strictly increasing *cdf* F over $[0,1]$.

An Example: The Hotelling-Downs Model

- $N = \{1,2\}$

- $A_1 = A_2 = [0,1]$

- $u_1 + u_2 \equiv 1$

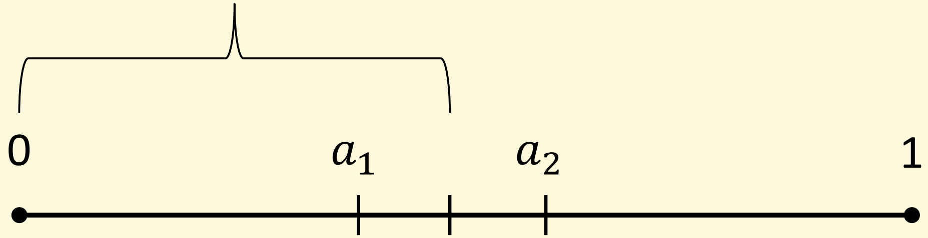
- x^* is the median voter's ideal policy: $F(x^*) = 0.5$

- Let $a_i < a_j$. Then, $u_i(a_1, a_2) = F\left(\frac{a_1+a_2}{2}\right)$

- Let $a_i = a_j$. Then, $u_i(a_1, a_2) = 0.5$.

An Example: The Hotelling-Downs Model

$$u_1(a_1, a_2) = F\left(\frac{a_1 + a_2}{2}\right)$$



An Example: The Hotelling-Downs Model

Claim: There is a unique Nash equilibrium in this game. Both parties choose x^*.

- A “median voter” result: Parties cater to the preferences of the median voter.
- Other interpretations: locating stores on a high street; strategic location in “product characteristic space”

Proof

Step 1: Show that $a_1 = a_2 = x^*$ is a Nash equilibrium.

Proof: When $a_1 = a_2 = x^*$, both parties get 50% of the votes.

- If party 1 deviates to $a_1 < x^*$, say, its vote share will be

  $F\left(\frac{a_1+x^*}{2}\right) < 0.5$.

- Rightward deviation is similarly unprofitable.

Proof

Step 2: Show that $a_1 = a_2 \neq x^*$ is not a Nash equilibrium.

Proof: Suppose $a_1 = a_2 > x^*$. Both parties get 50%.

- Suppose party 1 deviates to $a_1 - \varepsilon$, where $\varepsilon > 0$ is small.

- Its vote share will be $F\left(\frac{a_1+a_1-\varepsilon}{2}\right)$, which is greater than 0.5

  because $\frac{a_1+a_1-\varepsilon}{2} > x^*$.

- Intuition: Moving toward the median voter

Proof

Step 3: Show that $a_1 \neq a_2$ is not a Nash equilibrium.

Proof: Suppose $a_1 < a_2$. Party 1's vote share is $F\left(\frac{a_1+a_2}{2}\right)$.

- Suppose the party deviates to $a_1 + \varepsilon$, where $\varepsilon > 0$ is small.
- Its vote share will be $F\left(\frac{a_1+\varepsilon+a_2}{2}\right)$.
- Intuition: Moving toward the opponent gets swing voters

Best-Reply Correspondence

Definition: For each player $i \in N$ and every a_{-i},

$$BR_i(a_{-i}) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \ \forall a'_i \in A_i\}$$

– Player i's optimal actions against a_{-i}

- Suppose BR_i is well-defined for all players. Then, $a^* \in A$ is a Nash equilibrium if and only if $a_i^* \in BR_i(a_{-i}^*) \ \forall\, i \in N$.
  – Computationally easier; sometimes reduces the task of finding Nash equilibria to solving a system of equations.

Best-Reply Correspondence

Definition: For each player $i \in N$ and every a_{-i},

$$BR_i(a_{-i}) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \;\; \forall a'_i \in A_i\}$$

– Player i's optimal actions against a_{-i}

- Suppose BR_i is well-defined for all players. Then, $a^* \in A$ is a Nash equilibrium if and only if $a_i^* \in BR_i(a_{-i}^*) \;\; \forall\, i \in N$.
  – But sometimes (as in the Hotelling-Downs model) the BR correspondence is **not** well-defined.

Existence of Nash Equilibrium

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- This game has no Nash equilibria.
- Existence theorems:
  - Restrictions on action spaces and preferences
  - Mathematical tool: Fixed-point theorems applied to the best-reply correspondence

Incorporating Strategic Uncertainty

- In Nash equilibrium, players harbor no uncertainty regarding their opponents' behavior.
- However, in many contexts stable behavior in games can co-exist with strategic uncertainty.

Sources of Endogenous Uncertainty

- Players may deliberately try to be unexpected.
  - Changing behavior over time with no discernable pattern
- Players are drawn from a population with heterogeneous norms.
- Players may condition on random variables that are outside the model.

Mixed Strategies

- We capture uncertainty about a player's behavior by lotteries over his actions.
- $\Delta(A_i)$ is the set of all probability distributions over A_i.
- We refer to $\sigma_i \in \Delta(A_i)$ as a mixed strategy.
- $\sigma_i(a_i)$ is the probability that $\sigma_i \in \Delta(A_i)$ assigns to the action a_i.
- When σ_i is degenerate, it is a pure strategy.

Mixed Strategies

- Assumption: Players' actions are statistically independent
  - Rules out conditioning on a common random variable
- σ_{-i} is the Cartesian product of the mixed strategies of player i's opponents.

$$\sigma_{-i}(a_{-i}) = \prod_{j \in N-\{i\}} \sigma_j(a_j)$$

Expected Utility

- Uncertainty regarding opponents' actions means that the player is uncertain about the consequences of his actions.
- We need to extend the player's preferences to $\Delta(A)$.
- Assumption: Players are expected-utility maximizers
  - u_i is a vNM utility function.
  - u_i has cardinal meaning (unique up to affine transformations).

Extending Nash Equilibrium

- Notation: $U_i(a_i, \sigma_{-i}) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}) u_i(a_i, a_{-i})$

Definition: A profile of mixed strategies $(\sigma_1, \dots, \sigma_n)$ is a Nash equilibrium if, whenever $\sigma_i(a'_i) > 0$ for some $i \in N$ and $a'_i \in A_i$, it is the case that $a'_i \in argmax_{a_i} U_i(a_i, \sigma_{-i})$.

Mixed-Strategy Nash Equilibrium: Discussion

- Players have correct beliefs.
- Each player is indifferent among all actions in the support of his equilibrium mixed strategy; they are all best-replies.
- Pure-strategy Nash equilibrium is a special case.
- The concept is agnostic regarding the source of players' random behavior.

Illustration: Matching Pennies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- If player 1 plays a pure strategy, player 2 has a unique best-reply and therefore must also play a pure strategy.
- But we know there is no pure-strategy Nash equilibrium.
- Both players must randomize. Therefore, each player must be indifferent between the two actions.

Illustration: Matching Pennies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- Denote $q = \sigma_2(even)$. Then:

$$U_1(even, \sigma_2) = q \cdot 1 + (1 - q) \cdot (-1)$$

$$U_1(odd, \sigma_2) = q \cdot (-1) + (1 - q) \cdot 1$$

- Indifference requires $q = 0.5$.

Illustration: Matching Pennies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- Player 1's indifference pinned down player 2's strategy!
- By the same logic, player 2's indifference implies

$$\sigma_1(even) = \sigma_1(odd) = 0.5$$

- Unique Nash equilibrium: Each player mixes uniformly between the two actions.

Illustration: Matching Pennies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- A win/lose game: Effectively, only two outcomes
- Therefore, utilities have no cardinal meaning.

Example: Reporting a Crime

- A group of $n \geq 2$ agents witness a crime.
- Each agent chooses independently between reporting (r) and remaining silent (s).
- A player's payoff is:
  - 0 when no one reports the crime.
  - $1 - c$ when he reports the crime, $1 > c > 0$.
  - 1 when someone else reports the crime.

Pure-Strategy Nash Equilibria

- Full characterization: All profiles in which exactly one player chooses r.
  - If exactly one player reports, no one wants to deviate.
  - If no one reports, any player wants to deviate.
  - If at least two players report, any one of them wants to deviate.

Symmetric Nash Equilibria

- An asymmetric norm; implausible in some contexts.
- Suppose we insist on symmetric Nash equilibria.
- Then, we must allow for **mixed strategies**.
- Let $p \in (0,1)$ denote the probability that any player plays s (remaining silent) in a candidate equilibrium.
- The player must be indifferent between the two actions.

Symmetric Nash Equilibria

$$\underbrace{1-c}_{reporting} = \underbrace{p^{n-1}\cdot 0 + (1-p^{n-1})\cdot 1}_{remaining\ silent}$$

- Unique solution: $p = \sqrt[n-1]{c}$
  - Unsurprisingly, p (the probability that an individual player remains silent in equilibrium) increases with n.

Comparative Statics

- Equilibrium probability that **no one reports**: $p^n = p \cdot p^{n-1}$
- We saw that p increases with n.

$$\underbrace{1-c}_{reporting} = \underbrace{(1-p^{n-1})\cdot 1}_{remaining\ silent}$$

- $p^{n-1} = c$; it is invariant to n.
- Therefore, a higher n leads to a lower equilibrium probability that the crime is reported!

Ex-Ante Formulation

- An alternative, more conventional formulation of mixed-strategy Nash equilibrium
- Ex-ante approach: Players choose a mixed strategy
  - Fits the “deliberate randomization” interpretation
- A strategic game with an extended strategy space and expected-utility preferences over (extended) strategy profiles

Ex-Ante Formulation

- A strategic game in which player i's strategy space is $\Delta(A_i)$ and his preferences over $\Delta(A_1) \times \cdots \times \Delta(A_n)$ are represented by

$$V_i(\sigma_1, \ldots, \sigma_n) = \sum_{a_i} \sigma_i(a_i) U_i(a_i, \sigma_{-i})$$

- A profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium if for every $i \in N$, $V_i(\sigma_i^*, \sigma_{-i}^*) \geq V_i(\sigma_i, \sigma_{-i}^*)$ for every $\sigma_i \in \Delta(A_i)$.

Ex-Ante Formulation

- The two formulations of mixed-strategy Nash equilibrium are equivalent in finite games.
  - The expected-utility assumption is crucial.
- Players' indifference between their equilibrium actions casts doubt on the deliberate randomization interpretation.
- **Theorem** (Nash 1950): Every finite game has a mixed-strategy Nash equilibrium.

Strategic Games: Summary

- The simplest class of game-theoretic models, which abstracts from time and exogenous uncertainty.
- Nash equilibrium: A “steady state” solution concept
- Mixed strategies: An extension that allows for steady states with strategic uncertainty
- The indifference property and its strange implications

Advanced Microeconomic Theory

Lecture 2: “One-Shot” Solution Concepts for Strategic Games

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Plan of this Lecture

- Two solution concepts that fit the “one shot” interpretation
  - Max-minimization
  - Rationalizability
- Connections with Nash equilibrium in some games

Max-Minimization

- A purely decision-theoretic concept
- Worst-case reasoning:
  - Find the worst possible outcome for every action.
  - Then choose the best among these worst cases.
- A player's max-min strategy guarantees that he gets at least his max-min payoff.

Max-Minimization

- **Formally**: An action $a_i \in A_i$ is a max-minimizer for player i if it solves the problem

$$max_{a_i} min_{a_{-i}} u_i(a_i, a_{-i})$$

- A player's max-min strategy guarantees that he gets at least his max-min payoff.

General Introduction

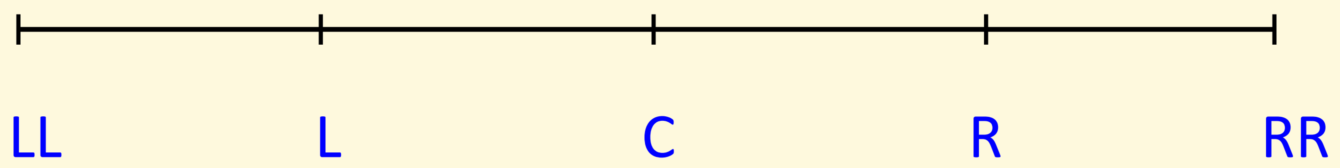
	L	R
T	3	0
B	1	2

General Introduction

	L	R
T	3	<u>0</u>
B	<u>1</u>	2

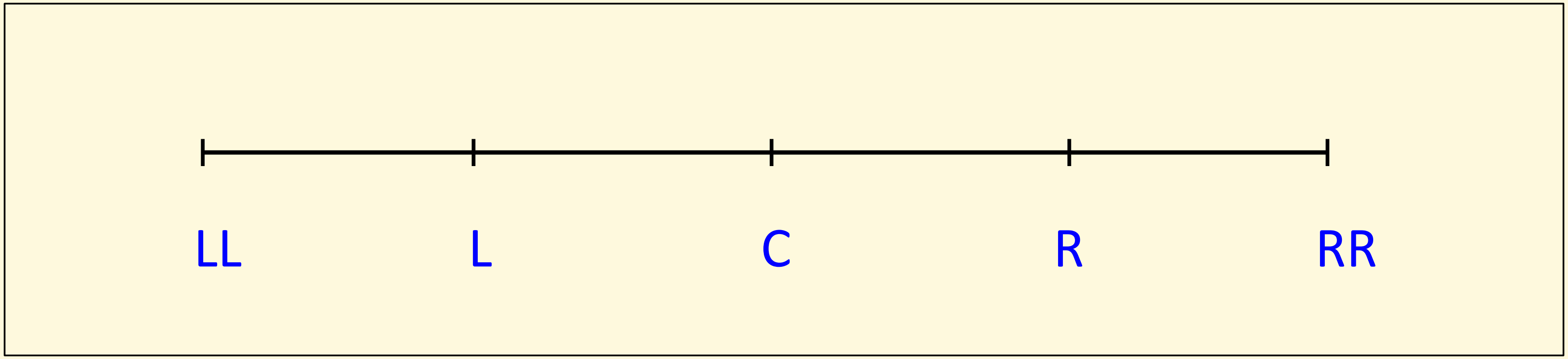
- Psychology of max-minimization:
  - Believing that the opponents can condition their actions on mine
  - Believing that they act collectively to hurt me
- Pessimism, “paranoia”

A Discrete Hotelling-Downs Example



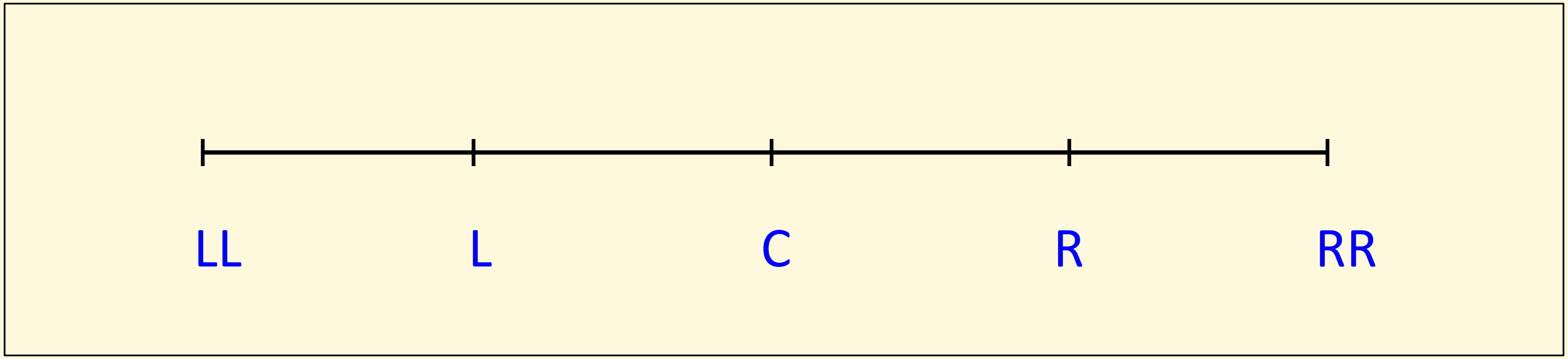
- The two parties can only choose five locations:
  - Extreme left, moderate left, center, moderate right, extreme right
  - Voters' ideal points are uniformly distributed over these five locations.
- Unique Nash equilibrium: (C, C)

A Discrete Hotelling-Downs Example



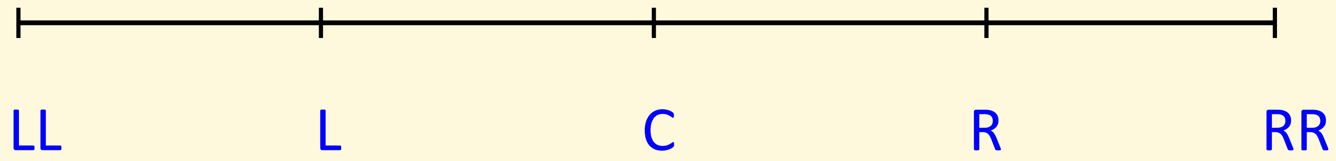
a_1	Worst-case a_2	u_1
LL	L	0.2
L	C	0.4
C	C	0.5
R	C	0.4
RR	R	0.2

A Discrete Hotelling-Downs Example



a_1	Worst-case a_2	u_1
LL	L	0.2
L	C	0.4
C	C	0.5
R	C	0.4
RR	R	0.2

A Discrete Hotelling-Downs Example



- Max-minimization predicts that each party selects C.
- This coincides with the Nash equilibrium prediction.
- Coincidence?

Max-Minimization with Randomization

- Suppose the player can randomize – i.e., play a **mixed** strategy.
- The player assumes that his opponents can condition their behavior on his mixed strategy, but **not** on its realization.
- Formally, a mixed strategy $\sigma_i \in \Delta_i$ is a max-minimizer for player i if it solves the problem $max_{\sigma_i} min_{a_{-i}} V_i(\sigma_i, a_{-i})$.

Worst-Case Scenario: Profiles of Actions or Profiles of Mixed Strategies?

- Because V_i is **linear in probabilities**, we can replace

$$max_{\sigma_i} min_{a_{-i}} V_i(\sigma_i, a_{-i})$$

with

$$max_{\sigma_i} min_{\sigma_{-i}} U_i(\sigma_i, \sigma_{-i})$$

where

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}) V_i(\sigma_i, a_{-i})$$

Max-Minimization with Mixed Strategies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

- Both actions generate the same worst-case payoff -1.
- Now consider max-minimization with mixed strategies.
- **Assumption: The opponent conditions his action on the player's mixed strategy, not on its realization!**

Max-Minimization with Mixed Strategies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

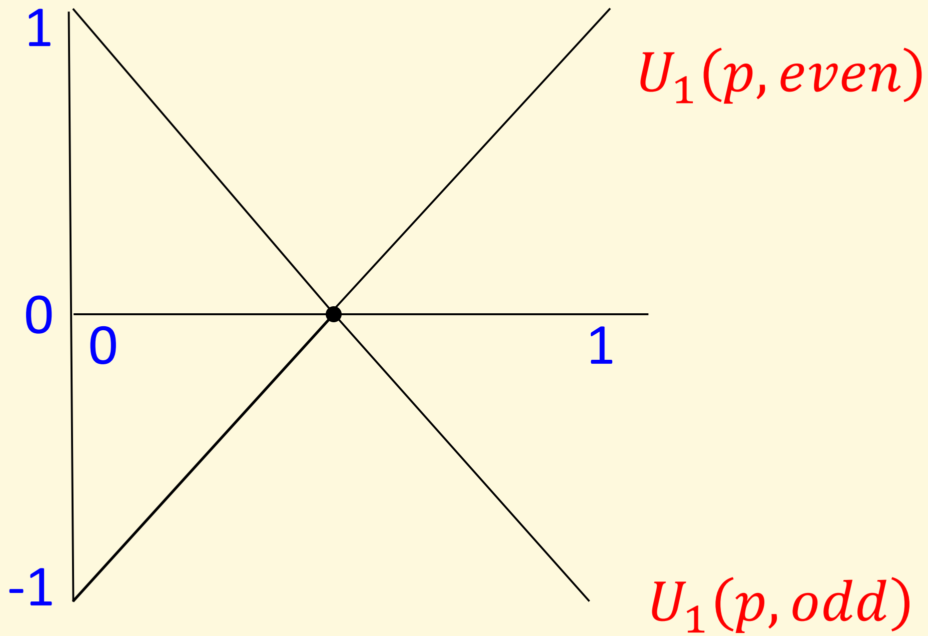
- Notation:
  - $p = \sigma_1(even)$
  - $U_1(p, a_2)$ is player 1's expected payoff from the mixed strategy given by p against a_2

Max-Minimization with Mixed Strategies

	Even	Odd
Even	1,-1	-1,1
Odd	-1,1	1,-1

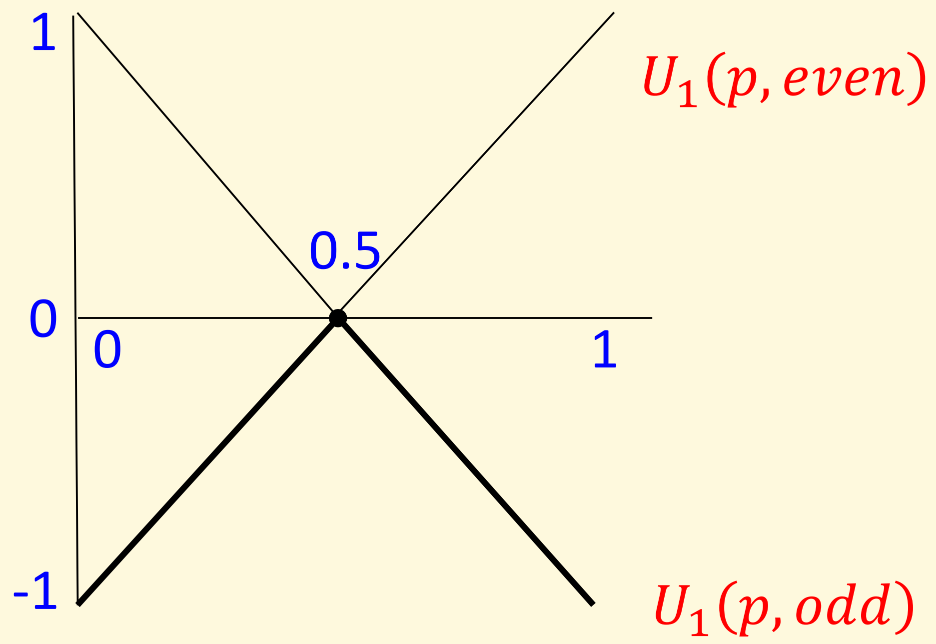
- $U_1(p, even) = p \cdot 1 + (1-p) \cdot (-1)$
- $U_1(p, odd) = p \cdot (-1) + (1-p) \cdot 1$
- For every p we check which of player 2's two actions generates the lower U_1, and that's the worst case.

Max-Minimization with Mixed Strategies



- $U_1(p, even) = p \cdot 1 + (1 - p) \cdot (-1) = 2p - 1$
- $U_1(p, odd) = p \cdot (-1) + (1 - p) \cdot 1 = 1 - 2p$

Max-Minimization with Mixed Strategies



$$max_p \min\{2p-1\,, 1-2p\} = 0.5$$

- Again, max-min prediction coincides with Nash equilibrium!

Strictly Competitive Games

- A strategic game is **strictly competitive** if it has exactly two players whose preferences are diametrically opposed.
  - Without randomization: $a \succ_1 a'$ if and only if $a' \succ_2 a$ for every $a, a' \in A$.
  - With randomization: $\sigma \succ_1 \sigma'$ if and only if $\sigma' \succ_2 \sigma$ for every $\sigma, \sigma' \in \Delta(A)$.
- We'll focus on the "with randomization" case.

Strictly Competitive Games

- Such diametrically opposed preferences can be represented by vNM utility functions that satisfy $u_1(a) \equiv -u_2(a)$.
- Hence, strictly competitive games are famously known as "**zero-sum games**".
- The zero-sum representation is not unique because we can apply affine transformations to u_1 and u_2.
  - E.g., in the Hotelling-Downs game, $u_1 + u_2 \equiv 1$.

The Minimax Theorem (von Neumann 1928)

1. In a finite strictly competitive game, a mixed-strategy profile is a Nash equilibrium if and only if each player's strategy is a max-minimizer.

2. Furthermore, player 1 earns an expected payoff of v^* in all Nash equilibria, where

$$v^* = max_{\sigma_1} min_{\sigma_2} U_1(\sigma_1, \sigma_2) = min_{\sigma_2} max_{\sigma_1} U_1(\sigma_1, \sigma_2)$$

is known as the value of the game.

The Minimax Theorem

- Surprising connection between equilibrium behavior and individual worst-case reasoning
- The "pessimism" and "paranoia" implicit in max-minimization are more intuitive in a zero-sum game.

$$v^* = max_{\sigma_1} min_{\sigma_2} U_1(\sigma_1, \sigma_2) = min_{\sigma_2} max_{\sigma_1} U_1(\sigma_1, \sigma_2)$$

- This equality implies an "irrelevance of order of moves result".

Partial Proof: Equivalence between Nash and Max-min **Payoffs**

Claim: In any game, a player's payoff in any mixed-strategy Nash equilibrium is weakly above his max-min payoff.

Proof: Let (σ_1^*, σ_2^*) be a Nash Equilibrium. Then:

$$U_1(\sigma_1^*, \sigma_2^*) = max_{\sigma_1} U_1(\sigma_1, \sigma_2^*)$$

But $U_1(\sigma_1, \sigma_2^*) \geq min_{\sigma_2} U_1(\sigma_1, \sigma_2)$ for any σ_1. Then,

$$U_1(\sigma_1^*, \sigma_2^*) \geq max_{\sigma_1} min_{\sigma_2} U_1(\sigma_1, \sigma_2)$$

Partial Proof: Equivalence between Nash and Max-min **Payoffs**

Claim: In a zero-sum game, a player's payoff in any mixed-strategy Nash equilibrium is weakly below his max-min payoff.

Proof: Let (σ_1^*, σ_2^*) be a Nash Equilibrium. Then:

$U_2(\sigma_1^*, \sigma_2^*) \geq U_2(\sigma_1^*, \sigma_2)$ for any σ_2. $\Longrightarrow$

$U_1(\sigma_1^*, \sigma_2^*) \leq U_1(\sigma_1^*, \sigma_2)$ for any σ_2. $\Longrightarrow$

$U_1(\sigma_1^*, \sigma_2^*) = min_{\sigma_2} U_1(\sigma_1^*, \sigma_2) \leq max_{\sigma_1} min_{\sigma_2} U_1(\sigma_1, \sigma_2)$

The Max-min – Min-Max Equivalence

- $U_1(\sigma_1^*, \sigma_2^*) = max_{\sigma_1} min_{\sigma_2} U_1(\sigma_1, \sigma_2)$
- Likewise, $U_2(\sigma_1^*, \sigma_2^*) = max_{\sigma_2} min_{\sigma_1} U_2(\sigma_1, \sigma_2)$
- But $U_2 = -U_1$.
- $-U_1(\sigma_1^*, \sigma_2^*) = max_{\sigma_2} min_{\sigma_1} [-U_1(\sigma_1, \sigma_2)]$
- $U_1(\sigma_1^*, \sigma_2^*) = min_{\sigma_2} max_{\sigma_1} U_1(\sigma_1, \sigma_2)$

Minimax Theorem: Comments

- Reducing Nash equilibrium to a maximization problem often facilitates finding equilibria.
- Interchangeability: if (x, y) and (x', y') are Nash equilibria, then so are (x, y') and (x', y).
  - Atypical in general games (e.g. coordination games)

Rationalizability

- A solution concept that draws logical implications from players' common belief in each other's rationality:

1) Which actions are impossible if players are rational?
2) Which actions are impossible if players know (1)?
3) Which actions are impossible if players know (1) & (2)?

⋮

Rationality

- Assume players are expected utility maximizers.

<u>Definition</u>: An action $a_i \in A_i$ is rational if there exists a belief

$\sigma_{-i} \in \Delta(A_{-i})$ such that $U_i(a_i, \sigma_{-i}) \geq U_i(a_i', \sigma_{-i})$ for all $a_i' \in A_i$.

- σ_{-i} may involve correlation between opponents.

<u>Definition</u>: An action is a never-best-reply if it is not rational.

Strictly Dominated Actions

	Don't Advertise	Advertise
Don't Advertise	5,5	2,6
Advertise	6,2	3,3

- "Don't advertise" yields strictly lower payoff for a player than the action "Advertise" against **any** action of the opponent.
- "Don't advertise" is a never-best-reply.
- Individual rationality predicts both players play "Advertise".

Strictly Dominated Actions

Definition: An action $a_i \in A_i$ is strictly dominated if there is

$a_i' \in A_i$ such that $u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

- If an action is strictly dominated, it is a never-best-reply.
  - A rational player will never play a strictly dominated action.
- Is the converse true?
  - We'll revisit this question later.

Iterated Elimination of Strictly Dominated Actions

	L	R
T	4,2	0,3
M	1,1	1,0
B	3,0	2,2

- Player 2 doesn’t have a strictly dominated action.
- As to player 1, M is strictly dominated by B.
- If player 1 is rational, he will never play M.

Iterated Elimination of Strictly Dominated Actions

	L	R
T	4,2	0,3
M	1,1	1,0
B	3,0	2,2

- If Player 2 believes that player 1 is rational, he will regard M as an impossible scenario.
- Effectively, we delete M from the game.

Iterated Elimination of Strictly Dominated Actions

	L	R
T	4,2	0,3
B	3,0	2,2

- In the reduced game, L is a strictly dominated action.
- Thus, if player 2 is rational and believes that player 1 is rational, he will never play L.
- If player 1 believes this, we can effectively delete L from the reduced game.

Iterated Elimination of Strictly Dominated Actions

	R
T	0,3
B	2,2

- In the reduced game, T is trivially a strictly dominated action.
- Thus, if player 1 rational & believes that player 2 is rational & believes that player 2 believes that player 1 is rational, he will play B.

Iterated Elimination of Strictly Dominated Actions

	L	R
T	4,2	0,3
M	1,1	1,0
B	3,0	2,2

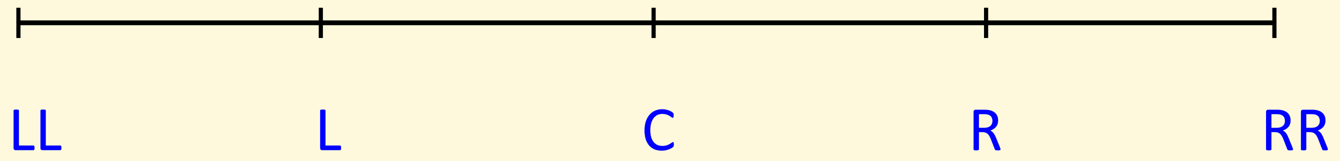
- We deduced (B, R) as the unique outcome that is consistent with common knowledge of rationality.
- It also happens to coincide with Nash equilibrium.
- Is this a coincidence…?

The Iteration Procedure

1. Look for a strictly dominated action for any player.
2. If you can’t find one, terminate the procedure.
3. If you find one, delete it and obtain a reduced game. Iterate!

- The set of outcomes that survive this procedure is insensitive to the order of elimination.
- This set contains all Nash equilibria in the game.

Discrete Hotelling-Downs Revisited

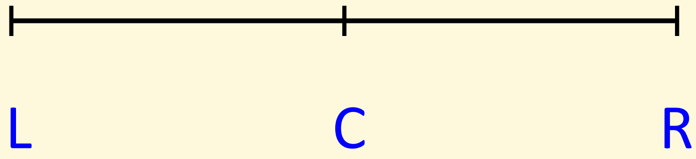


- Party i's vote share from L and LL as a function of party j's action:

$a_i \backslash a_j$	LL	L	C	R	RR
L	0.8	0.5	0.4	0.5	0.6
LL	0.5	0.2	0.3	0.4	0.5

- LL is strictly dominated by L.

Discrete Hotelling-Downs Revisited



- By the same logic, RR is strictly dominated by R.
- In the reduced game, L and R are strictly dominated by C, by a similar calculation.
- (C, C) is the only outcome that survives iterated elimination of strictly dominated actions.

Never-Best-Reply vs. Strict Domination

	L	R
T	3	0
M	0	3
B	1	1

- The action B is a never-best-reply:

$U_1(B, \sigma_2) = 1$ against any σ_2.

$max\{U_1(T, \sigma_2), U_1(M, \sigma_2)\} = max\{3\sigma_2(L), 3\sigma_2(R)\} \geq 1.5$

- But B is not a strictly dominated action.

Never-Best-Reply vs. Strict Domination

	L	R
T	3	0
M	0	3
B	1	1

- However, B **is** strictly dominated by a **mixed strategy**.
- E.g., if player 1 plays T (M) with probability 0.6 (0.4), his expected payoff is $0.6 \cdot 3 + \ 0.4 \cdot 0 = 1.8 > 1$ against L, and $0.6 \cdot 0 + \ 0.4 \cdot 3 = 1.2 > 1$ against R.

Never-Best-Reply vs. Strict Domination

<u>Result</u>: An action is a never-best-reply if and only if it is strictly dominated by some mixed strategy.

- The proof makes crucial use of the Minimax Theorem!
- This is a recurring theme in Game Theory: The Minimax Theorem sometimes proves technically useful outside the context of zero-sum games.

Never-Best-Reply vs. Strict Domination

- Perfect equivalence between rationality and the extended property of not playing a strictly dominated action
- Extend the iterative procedure accordingly: At each round, we look for actions that are strictly dominated by a mixed strategy.
- The outcomes that survive the procedure are the ones that are consistent with common knowledge of rationality!

Rationalizability: Equivalent Definition

- The set of outcomes that survive the extended procedure is also known as the set of rationalizable outcomes.
- It is a product set: The product of the set of rationalizable actions for each player.
- It contains all Nash equilibria of the game.

Rationalizability: Equivalent Definition

<u>**Equivalent definition**</u>: The set of rationalizable outcomes in the game is the largest product set $A^* = A_1^* \times \cdots \times A_n^*$ such that for every player i and every $a_i \in A_i^*$, a_i maximizes player i's expected utility against some $\sigma_{-i} \in \Delta(A_{-i}^*)$.

- A "fixed point" definition, analogous to Nash equilibrium
- Unlike Nash equilibrium, players' beliefs are not necessarily correct or coordinated.

An Example: Cournot Competition

- Two firms choose a quantity in $[0,1]$.
- $u_i(q_1, q_2) = q_i \cdot [1 - q_i - q_j]$
- Interpretation: Firms incur zero costs; market price is consistent with linear demand $P = 1 - Q$.
- **Claim**: The game has a unique rationalizable outcome, $\left(\frac{1}{3}, \frac{1}{3}\right)$.
  - Therefore, it is also the unique Nash equilibrium.

An Example: Cournot Competition

- Let's try the iterative procedure.

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 1 - 2q_1 - q_2$$

- This derivative is strictly negative whenever $q_1 > 0.5$, independently of q_2.
- Therefore, any $q_1 > 0.5$ is strictly dominated by $q_1 = 0.5$.
- Eliminate actions above 0.5 for both players in 1st round.

An Example: Cournot Competition

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 1 - 2q_1 - q_2$$

- This derivative is strictly positive for every $q_1 < 0.25$, as long as $q_2 \leq 0.5$.
- Therefore, any $q_1 < 0.25$ is strictly dominated by $q_1 = 0.25$, once we ruled out $q_2 > 0.5$.
- Eliminate actions below 0.25 for both players in 2nd round.

An Example: Cournot Competition

- Continuing like this, each round will shave off an interval from the set of possible quantities.
  - An upper interval in odd rounds, a lower interval in even rounds.
- As the number of rounds tends to infinity, the remaining interval converges to the point $\frac{1}{3}$.
- This is tedious; the alternative definition offers a more efficient method.

An Example: Cournot Competition

- Suppose player 1 has a belief σ_2 over q_2 with expectation μ_2.

$$U_1(q_1, \sigma_2) = \sum_{q_2} \sigma_2(q_2) \cdot q_1 \cdot [1 - q_1 - q_2]$$

$$= q_1 \cdot \left[1 - q_1 - \sum_{q_2} \sigma_2(q_2) q_2\right] = q_1 \cdot [1 - q_1 - \mu_2]$$

- Player 1's best-reply is given by the first-order condition, which gives $q_1 = (1 - \mu_2)/2$.

An Example: Cournot Competition

- Because the game is symmetric, the set of rationalizable actions is the same for both players.
- Let h (l) denote the highest (lowest) rationalizable action.
- By definition, $l \leq \mu_2 \leq h$.
- Because player 1's best-reply is decreasing in μ_2:

  $l = (1 - h)/2$ and $h = (1 - l)/2$

- Solving the equations, we get $l = h = 1/3$.

Rationalizability: Summary

- Two equivalent definitions of a concept that draws implications of common knowledge of rationality
  - An algorithmic, iterative elimination procedure
  - A “fixed point” definition that relaxes the aspect of correct, coordinated beliefs in Nash equilibrium
- Which definition is easier to implement varies with the context.

Advanced Microeconomic Theory

Lecture 3: Games with Incomplete Information

Ran Spiegler, UCL

January 2022

Games with Incomplete Information (a.k.a Bayesian Games)

- Strategic interactions in which players do not know everything about the game
  - Variables that affect my own payoffs
  - My opponents' preferences
  - My opponents' knowledge (including knowledge of my knowledge, etc…)

Relevant Environments

- Auctions: Incomplete information about bidders' preferences or the value of the sold object
- Adverse selection in bilateral trade
- Speculative trade in financial markets
- Strategic voting
- Bank runs, currency crises

Plan of the Following Lectures

- Enriching the model of strategic games to express players' uncertainty
- Problematic treatment in available textbooks; supplementary lecture notes
- Lots of examples and applications
- Exercises are super-important.

The Formal Model

- We retain the following components of the basic model:
  - A set of players $N = \{1, \dots, n\}$
  - For each player $i \in N$, a set of feasible actions A_i
  - $A = \times_{i \in N} A_i$ is the set of action profiles.
- For simplicity, we rule out mixed strategies.
- No uncertainty about the set of feasible actions (w.l.o.g)

The New Ingredients: State Space

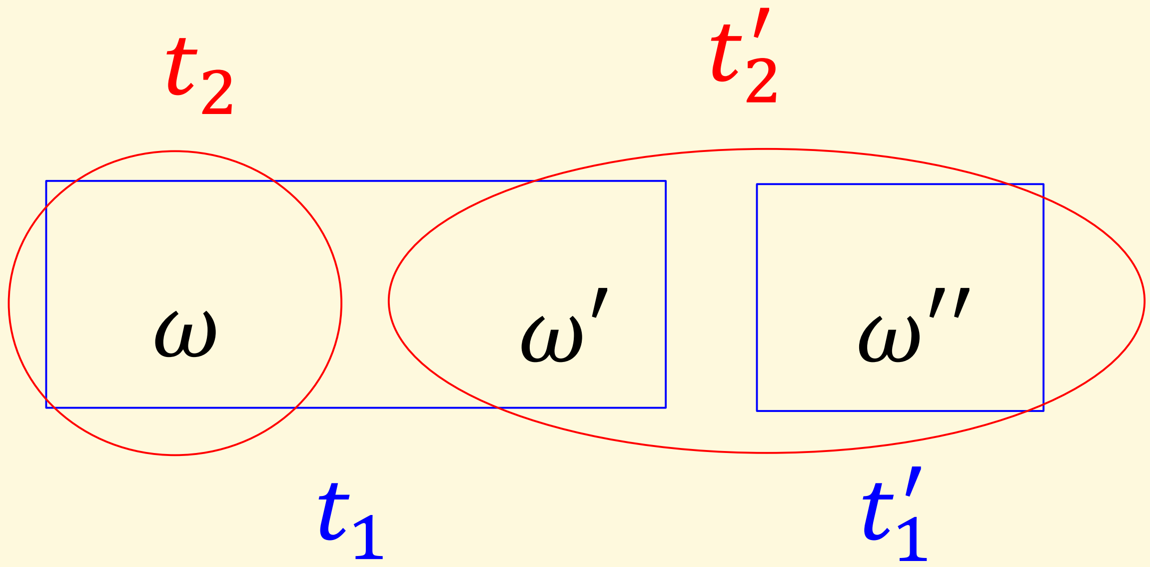
- A set of states of the world Ω
  - A state resolves all exogenous uncertainty that is relevant to the model.
- A prior probability distribution $p \in \Delta(\Omega)$ over the state space
  - Controversial: What does it mean? Why is it common?
- For each player $i \in N$, a vNM utility function $u_i: A \times \Omega \to \mathbb{R}$

The New Ingredients: Signals

- For each player $i \in N$:
  - A set of possible signals T_i
  - A signal function $\tau_i : \Omega \to T_i$ (deterministic because recall that the state resolves **all** uncertainty)
- A player's signal $t_i \in T_i$ represents his information, or state of knowledge, regarding the state of the world.
  - It is often referred to as the player's type.

The Information Structure

- The new components $\langle \Omega, p, (T_i)_{i \in N}, (\tau_i)_{i \in N} \rangle$ define the game's information structure.
- Useful diagram: **Information partitions**



Examples of Information Structures

- A seller knows the value of an object; the buyer is uninformed.
- The state of the world is the object's value v.
  - $\tau_{seller}(v) = v$ for all v
  - $\tau_{buyer}(v) = t^*$ for all v
- The prior p describes the distribution of v.

Examples of Information Structures

- Two firms, 1 and 2, receive noisy information about uncertain market demand.
- A state of the world is a triple (θ, t_1, t_2).
  - The size of market demand and the firms' signals
  - $\tau_i(\theta, t_1, t_2) = t_i$
- The prior p describes the distribution of market demand and the conditional distribution of the firms' signals.

Posterior Beliefs

Assumption: Player i's belief over the state space given his signal t_i is governed by Bayesian updating (hence the nickname "Bayesian Games"):

- If $\tau_i(\omega) = t_i$, then

$$p(\omega | t_i) = \frac{p(\omega)}{p(t_i)} = \frac{p(\omega)}{\sum_{\omega' \in \tau_i^{-1}(t_i)} p(\omega')}$$

- If $\tau_i(\omega) \neq t_i$, then $p(\omega | t_i) = 0$

Posterior Beliefs: Illustration

- $p(\omega) = p(\omega'') = 0.25$, $p(\omega') = 0.5$
- $p(\omega|t_1) = \frac{0.25}{0.25+0.5} = \frac{1}{3}$ $\quad p(\omega'|t_1) = \frac{0.5}{0.25+0.5} = \frac{2}{3}$
- $p(\omega''|t_1') = 1$

$$\underset{t_1}{\boxed{\quad \omega \qquad\qquad \omega' \quad}} \qquad \underset{t_1'}{\boxed{\quad \omega'' \quad}}$$

Strategies

- A player can only condition his action on his information.
- A pure strategy for player i is a function $s_i: T_i \to A_i$.
  - $s_i(t_i) \in A_i$ is the action that player i takes when his signal is t_i.
- We will rule out mixed strategies, for simplicity.

Strategies

- Ex-ante interpretation: The player **plans** the strategy before receiving the information

- Interim interpretation: s_i describes other players' belief regarding player i's contingent behavior; optimality of player i's action is evaluated **given** his information.

- As in the case of mixed-strategy Nash equilibrium, we will mostly work with the interim version.

Nash Equilibrium

Definition: A strategy profile $(s_1, \dots, s_n)$ is a Nash equilibrium if

for every player i and every $t_i \in T_i$:

$$s_i(t_i) \in argmax_{a_i} \sum_{\omega \in \Omega} p(\omega | t_i) u_i\left(a_i, (s_j(\tau_j(\omega)))_{j \neq i}, \omega\right)$$

- Each player chooses an action that maximizes his expected utility, given his belief over the state space and regarding the opponents' strategies.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega \in \Omega} p(\omega|t_i) u_i\big(a_i, (s_j(\tau_j(\omega)))_{j \neq i}, \omega\big)$$

- The player has double uncertainty, regarding the state of the world **and** the opponents' actions.
- Opponents' actions are uncertain because their information is uncertain.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega\in\Omega} p(\omega|t_i)u_i\big(a_i, (s_j(\tau_j(\omega)))_{j\neq i}, \omega\big)$$

- But in Nash equilibrium, the player correctly perceives the mapping state → opponent’s signal → opponent’s action
- This reduces his uncertainty de facto to uncertainty about the state of the world.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega\in\Omega} p(\omega|t_i) u_i\big(a_i, (s_j(\tau_j(\omega)))_{j\neq i}, \omega\big)$$

- The player's residual uncertainty regarding the state is given by his Bayesian posterior belief.
- He sums over the states and weighs them according to his posterior belief.

Parsing this Complicated Expression

$$U_i(a_i, s_{-i}|t_i) = \sum_{\omega\in\Omega} p(\omega|t_i)u_i\left(a_i, (s_j(\tau_j(\omega)))_{j\neq i}, \omega\right)$$

- For each possible state, the player correctly predicts the opponents’ action profile.
- Important motto: Statistical inferences from contingent events

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Two equally likely states of Nature:
  - Bad (unprofitable investment)
  - Good (profitable investment, provided both players invest)
- NI is a safe action; I is a risky but potentially profitable one.

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- If the state of Nature is common knowledge, players think about each payoff matrix as an isolated game.
- In bad state, NI is a strictly dominant action.
- In good state, two pure Nash equilibria: (I,I) and (NI,NI)

Example: An Investment Game

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- We’ll consider two alternative information structures; each induces a different Bayesian game.
- “Never invest” is a Nash equilibrium for **every** information structure. Are there other equilibria?

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Player 1 knows the state of Nature.
- Player 2 is uninformed.
- There is no additional uncertainty.

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- $\Omega = \{g, b\}$
- $\tau_1(\omega) = t^{\omega}$ for all ω
- $\tau_2(\omega) = t^{*}$ for all ω

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Player 2 (the uninformed party) plays a constant action in any pure-strategy Nash equilibrium.
- If he plays NI, player 1's best-reply is always NI, and we're back with the "never invest" Nash equilibrium.

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Let's guess an equilibrium in which player 2 plays I.
- In the bad state, player 1 learns this and plays the strictly dominant action NI $\Longrightarrow$ $s_1\left(t^b\right) = NI$
- In the good state, his best-reply is I $\Longrightarrow$ $s_1(t^g) = I$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$p(\omega = g) \cdot u_2(a_2 = I, s_1(t^g), \omega = g)$$

$$+ p(\omega = b) \cdot u_2\left(a_2 = I, s_1\left(t^b\right), \omega = b\right) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$0.5 \cdot u_2(a_2 = I, s_1(t^g), \omega = g)$$

$$+\, 0.5 \cdot u_2\big(a_2 = I, s_1\big(t^b\big), \omega = b\big) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$0.5 \cdot u_2(a_2 = I, a_1 = I, \omega = g)$$

$$+ 0.5 \cdot u_2(a_2 = I, a_1 = NI, \omega = b) \geq 0$$

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Recall that NI generates a payoff of 0 **for sure**.
- Player 2's action I is a best-reply if and only if

$$0.5 \cdot 1 + 0.5 \cdot (-2) \geq 0$$

- This inequality does not hold.

Information Structure No. 1

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

- Therefore, we are unable to sustain any pure Nash equilibrium apart from both “never invest”.
- An example of asymmetric information as a friction that prevents an efficient outcome

Information Structure No. 2

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

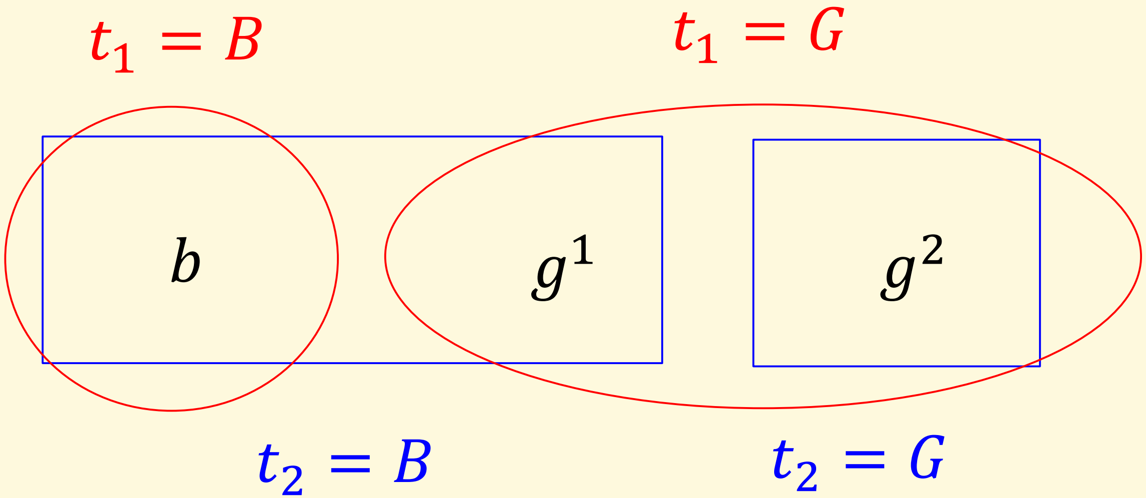
Bad state

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Good state

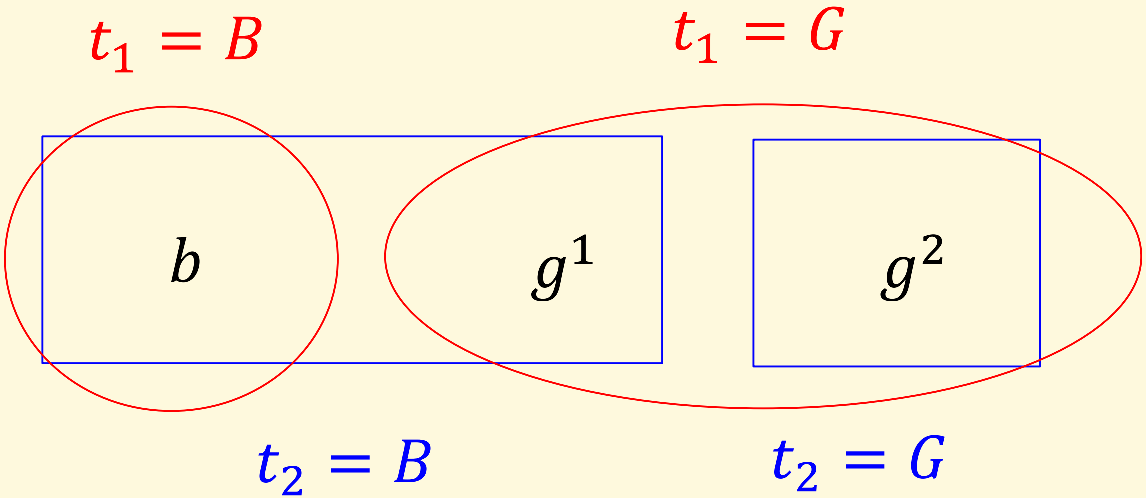
- Player 1 knows the state of Nature.
- When the state is good (but only then), player 2 gets tipped about this with probability $1-\varepsilon$.
- Player 1 does not know whether good news have leaked.

Information Structure No. 2



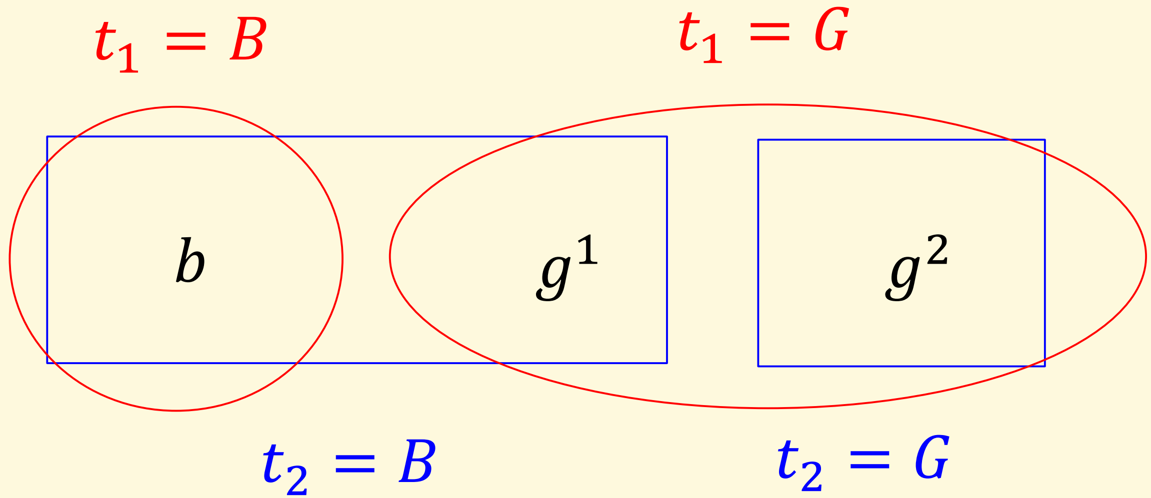
- $\Omega = \{b, g^1, g^2\}$ (bad state, good state with/without leak)
- $\tau_1(b) = B$, $\tau_1(g^1) = \tau_1(g^2) = G$
- $\tau_2(b) = \tau_2(g^1) = B$, $\tau_2(g^2) = G$

Information Structure No. 2



- A strategy for each player specifies how he acts when he gets the signals B and G.
- Is there a Nash equilibrium in which players sometimes play I?

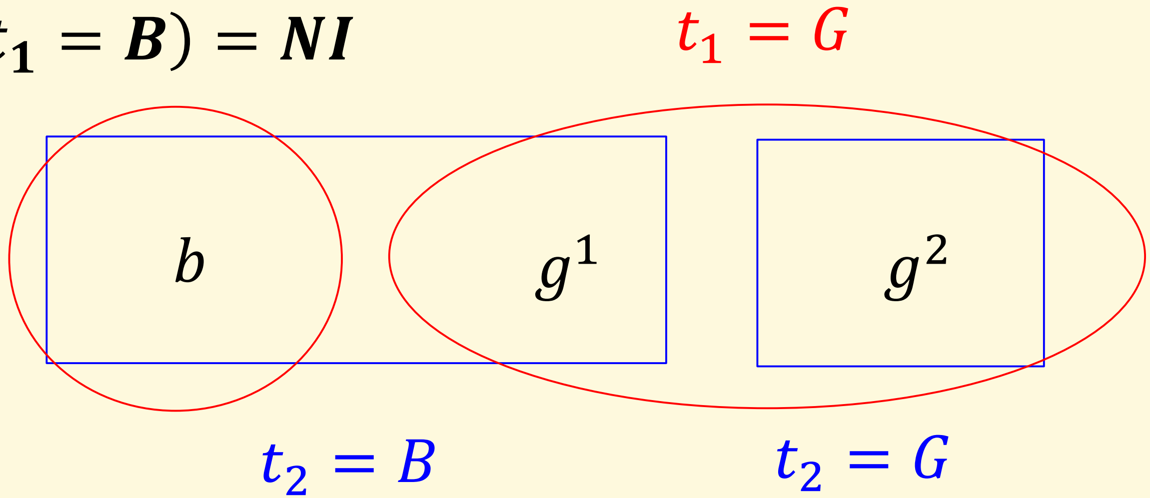
Information Structure No. 2



- When $t_1 = B$, player 1 knows that the state is b for sure.
- Therefore, $a_1 = NI$ is strictly dominant when $t_1 = B$.
    - This is what player 1 will play in any Nash equilibrium.

Information Structure No. 2

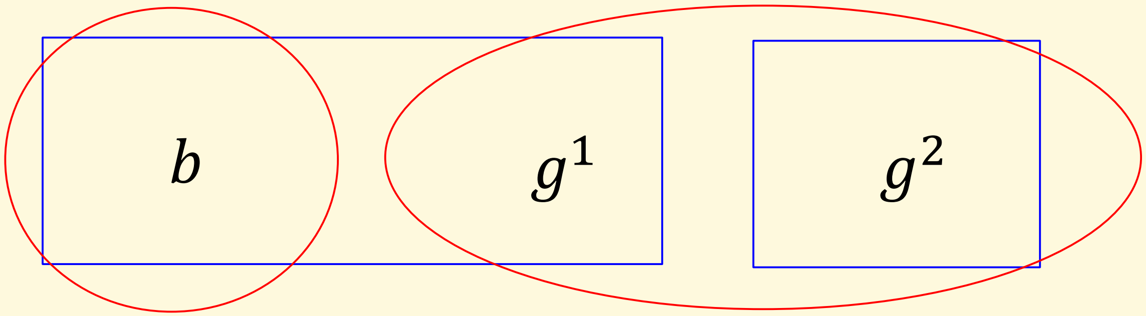
$\boldsymbol{a_1(t_1 = B) = NI}$



- To break away from the "never invest" Nash equilibrium, we must guess that player 1 plays I when $t_1 = G$.
- Let us check whether the guess is consistent.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



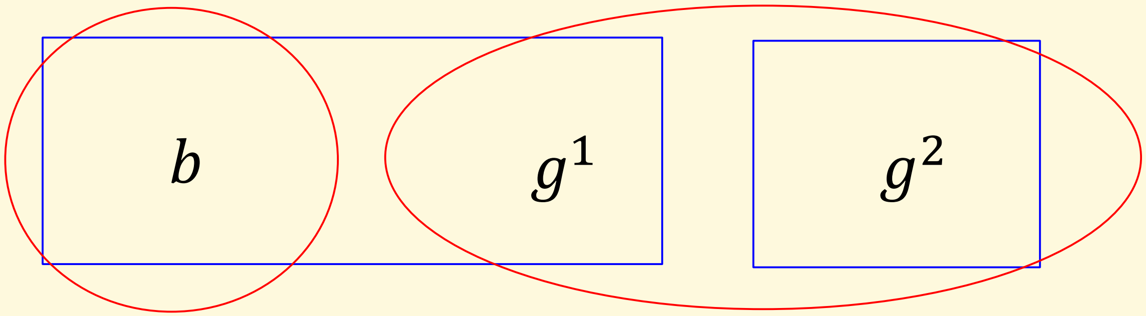
$t_2 = B$ $t_2 = G$

- When $t_2 = G$, player 2 knows that the state is g^2 for sure.

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\qquad$ $\boldsymbol{a_1(t_1 = G) = I}$



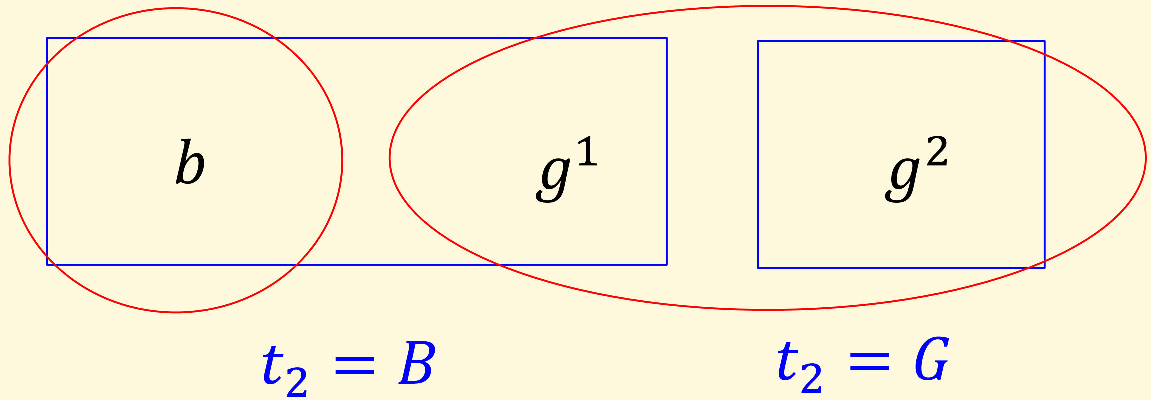
$t_2 = B$ $\qquad$ $t_2 = G$

- Player 2 infers that player 1 plays I (according to our guessed equilibrium strategy).

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

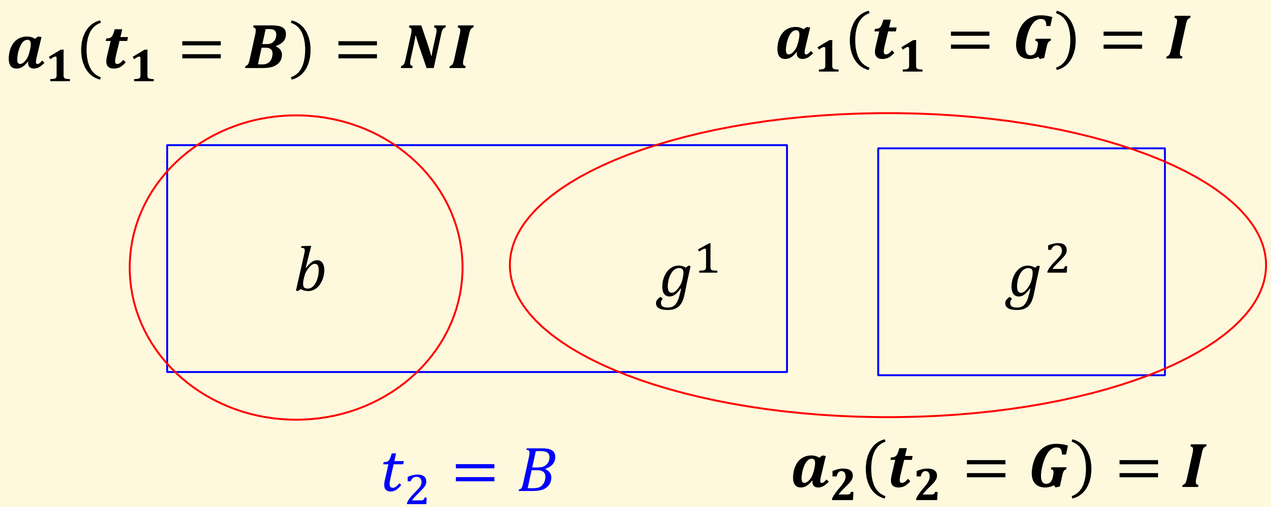
$a_1(t_1 = B) = NI$ $\qquad$ $a_1(t_1 = G) = I$



- Player 2’s best-reply is I.

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

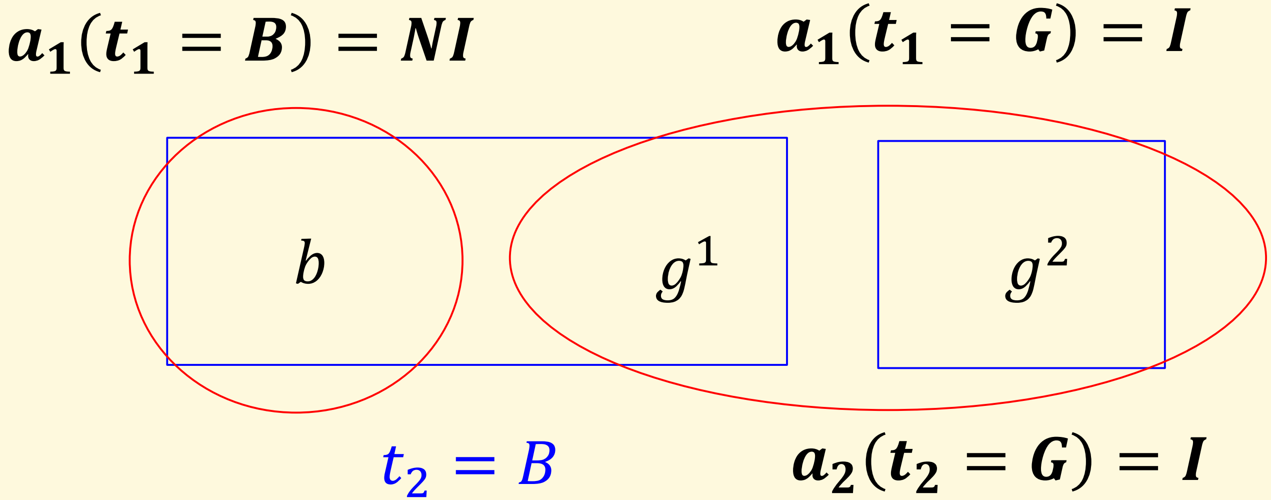
Information Structure No. 2



- Player 2's posterior belief given $t_2 = B$:

$$p(\omega = b | t_2 = B) = \frac{0.5}{0.5 + 0.5 \cdot \varepsilon} = \frac{1}{1 + \varepsilon}$$

Information Structure No. 2

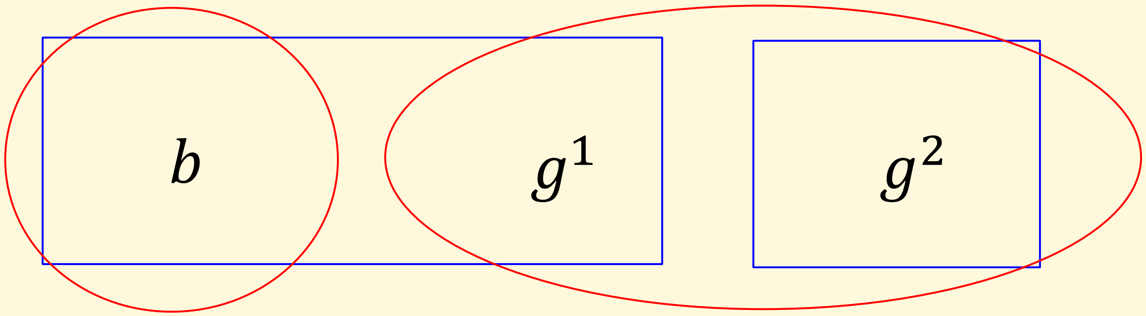


$$p(\omega = b | t_2 = B) = \frac{0.5}{0.5 + 0.5 \cdot \varepsilon} = \frac{1}{1 + \varepsilon}$$

$$p(\omega = g^1 | t_2 = B) = \frac{0.5 \cdot \varepsilon}{0.5 + 0.5 \cdot \varepsilon} = \frac{\varepsilon}{1 + \varepsilon}$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

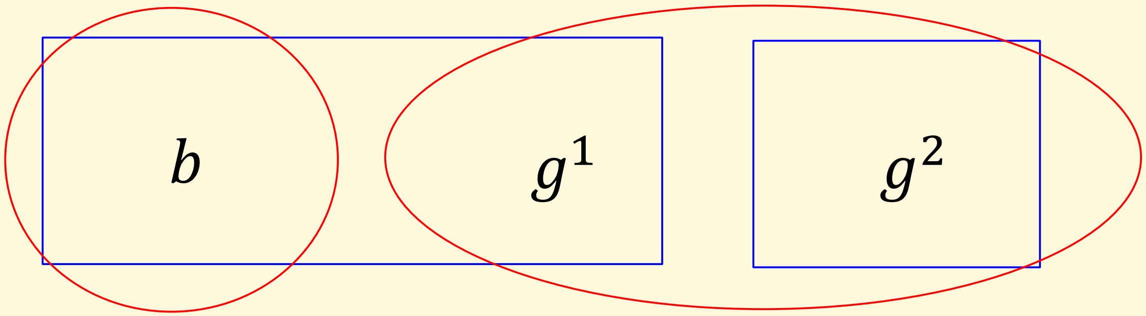
- Player 2's expected payoff from I is

$$p(\omega = b | t_2 = B) \cdot u_2(a_2 = I, s_1(\tau_1(b)), \omega = b)$$

$$+ p(\omega = g^1 | t_2 = B) \cdot u_2(a_2 = I, s_1(\tau_1(g^1)), \omega = g^1)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



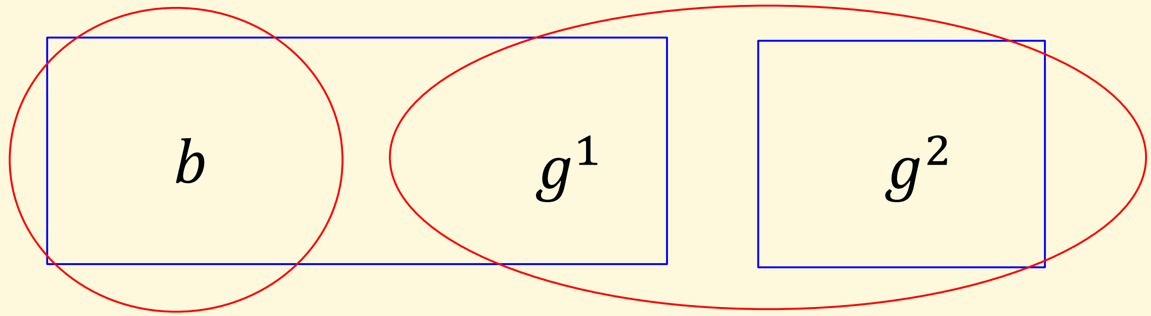
$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\frac{1}{1+\varepsilon} \cdot u_2(a_2 = I, s_1(\tau_1(b)), \omega = b)$$

$$+ \frac{\varepsilon}{1+\varepsilon} \cdot u_2(a_2 = I, s_1(\tau_1(g^1)), \omega = g^1)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

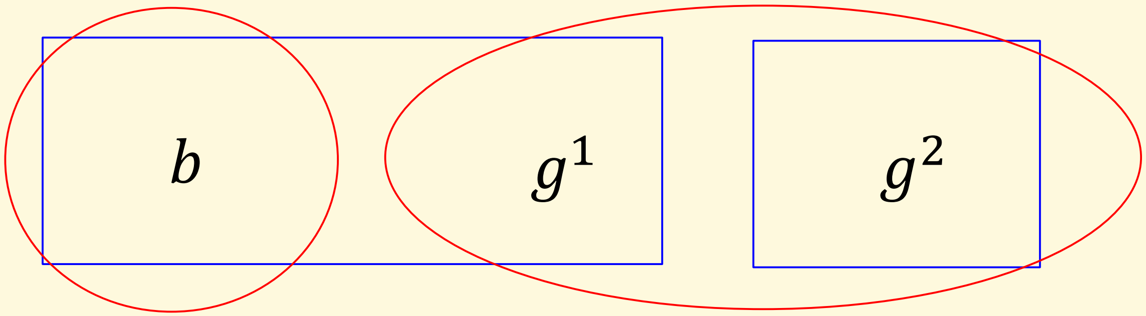
$$\frac{1}{1+\varepsilon} \cdot u_2(a_2 = I, a_1 = NI, \omega = b)$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

$$+ \frac{\varepsilon}{1+\varepsilon} \cdot u_2(a_2 = I, a_1 = I, \omega = g^1)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$t_2 = B$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\frac{1}{1+\varepsilon}\cdot(-2)+\frac{\varepsilon}{1+\varepsilon}\cdot 1 < 0$$

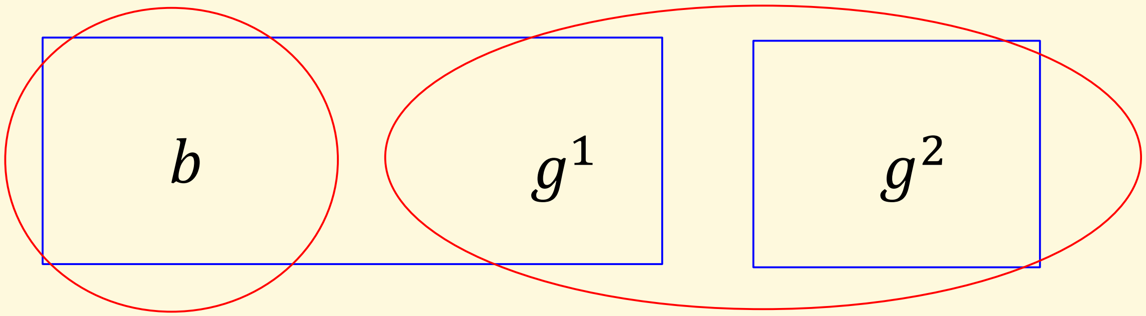
	I	NI
I	-2,-2	-2,0
NI	0,-2	0,0

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

- Player 2's best-reply at $t_2 = B$ is NI.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\qquad$ $\boldsymbol{a_1(t_1 = G) = I}$

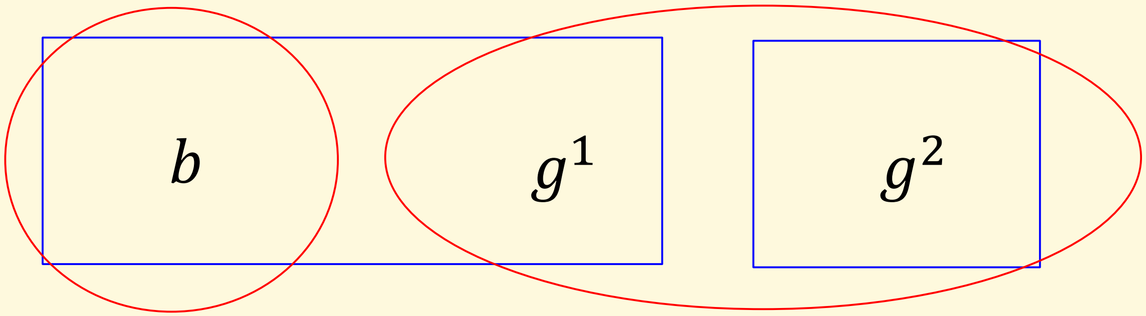


$\boldsymbol{a_2(t_2 = B) = NI}$ $\qquad$ $\boldsymbol{a_2(t_2 = G) = I}$

- It remains to check whether player 1 indeed wants to play I when $t_1 = G$.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



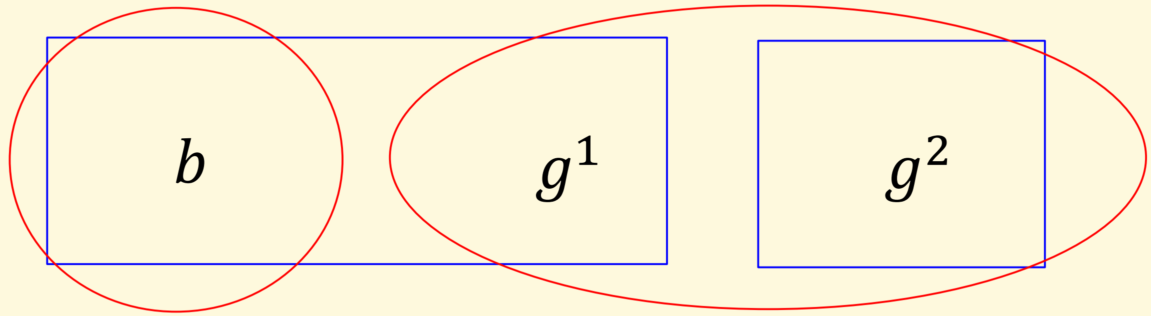
$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- Player 1's posterior belief given $t_1 = G$:

$$p(\omega = g^2 | t_1 = G) = \frac{0.5 \cdot (1 - \varepsilon)}{0.5 \cdot \varepsilon + 0.5 \cdot (1 - \varepsilon)} = 1 - \varepsilon$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

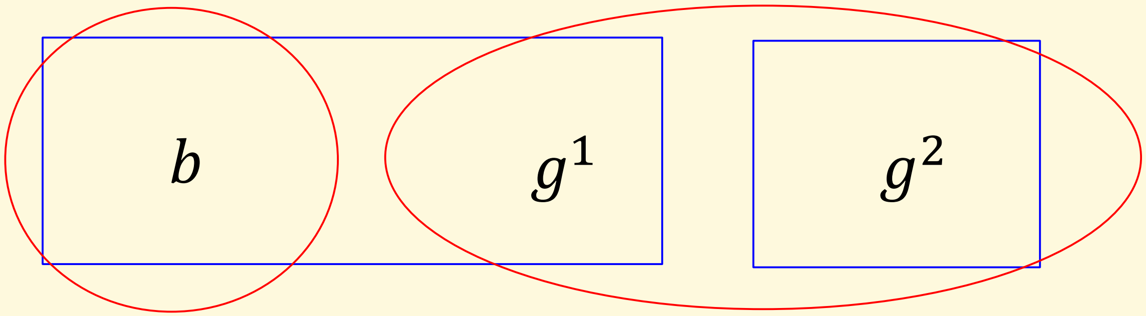
- Player 1's expected payoff from I is

$$p(\omega = g^1 | t_1 = G) \cdot u_1(a_1 = I, s_2(\tau_2(g^1)), \omega = g^1)$$

$$+\, p(\omega = g^2 | t_1 = G) \cdot u_1(a_1 = I, s_2(\tau_2(g^2)), \omega = g^2)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



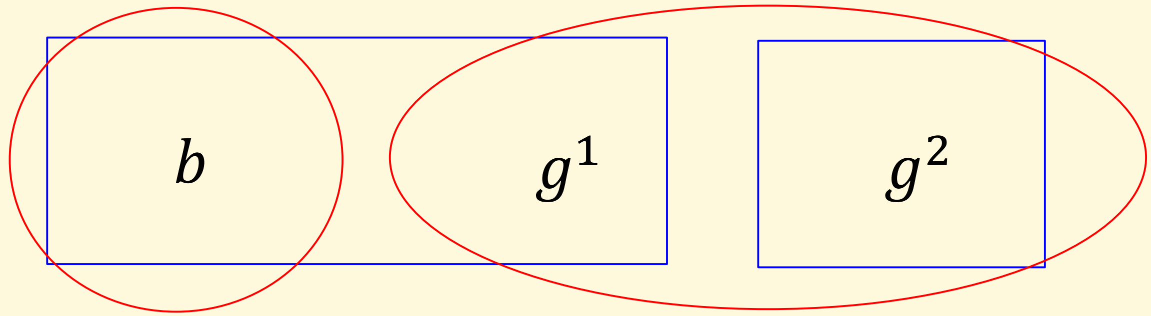
$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

$$\varepsilon \cdot u_1(a_1 = I, s_2(\tau_2(g^1)), \omega = g^1)$$

$$+ (1 - \varepsilon) \cdot u_1(a_1 = I, s_2(\tau_2(g^2)), \omega = g^2)$$

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

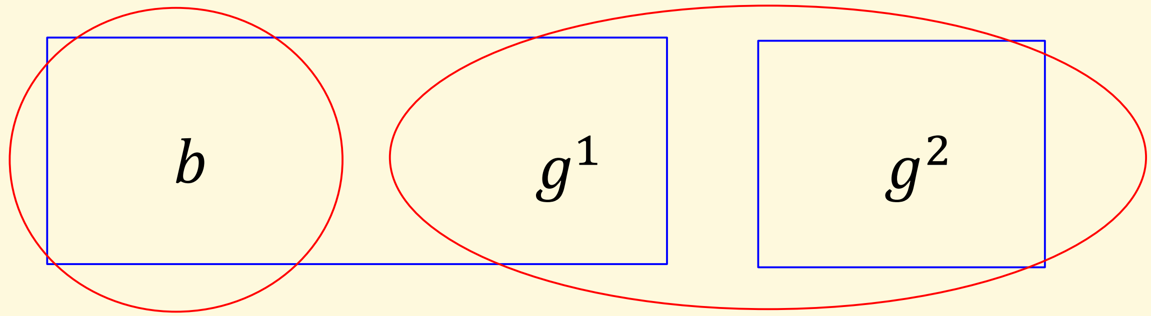
$$\varepsilon \cdot u_1(a_1 = I, a_2 = NI, \omega = g^1)$$

$$+ (1 - \varepsilon) \cdot u_1(a_1 = I, a_2 = I, \omega = g^2)$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

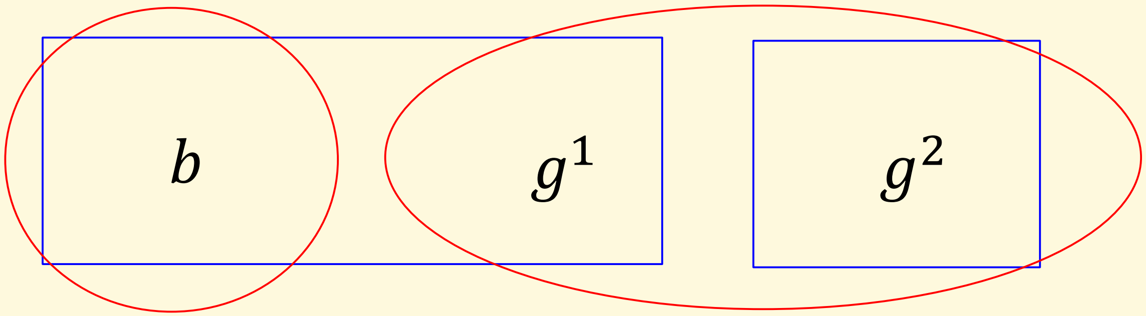
$$\varepsilon \cdot (-2) + (1 - \varepsilon) \cdot 1$$

	I	NI
I	1,1	-2,0
NI	0,-2	0,0

- Weakly above zero if and only if $\varepsilon \leq \frac{1}{3}$.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\qquad$ $\boldsymbol{a_1(t_1 = G) = I}$

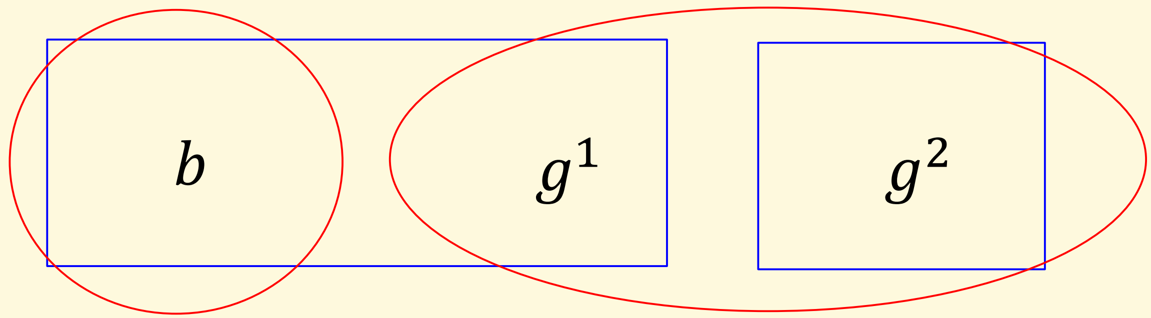


$\boldsymbol{a_2(t_2 = B) = NI}$ $\qquad$ $\boldsymbol{a_2(t_2 = G) = I}$

- Conclusion: There is a Nash equilibrium in addition to "never invest" if and only if $\varepsilon \leq \frac{1}{3}$.
  - Each players invests if and only if his signal is good.

Information Structure No. 2

$\boldsymbol{a_1(t_1 = B) = NI}$ $\boldsymbol{a_1(t_1 = G) = I}$



$\boldsymbol{a_2(t_2 = B) = NI}$ $\boldsymbol{a_2(t_2 = G) = I}$

- An intuitive prediction that lower informational frictions facilitate good coordination

Advanced Microeconomic Theory

Lecture 4: Applications of Bayesian Games I

Strategic Inferences from “Pitoval Events”

Ran Spiegler, UCL

February 2022

Plan of the Lecture

- Simple illustrations of economic applications of the model of games with incomplete information:
  - Bilateral trade in the presence of adverse selection
  - Second-price auctions
  - Strategic voting under common interests

Recurring Theme

- The game's outcome depends on a critical "pivotal event"
  - Trade takes place
  - Winning an auction
  - One's vote makes a difference
- Equilibrium reasoning involves statistical inferences from the pivotal event.
  - Statistical inferences from hypothetical contingencies

Bilateral Trade

- A game between a seller (s) and a buyer (b)
- A seller owns an object of uncertain value.
  - The value to the seller is $v \sim U[0,1]$.
  - The value to the buyer is $1.5v$.
- The seller knows v, the buyer is entirely uninformed.
  - $\Omega = [0,1]$; $\tau_s(v) \equiv v$; $\tau_b(v) = t^*$ for all v

Bilateral Trade

- The two players simultaneously submit price offers, a_s and a_b.
- If $a_s > a_b$, there is no trade and players earn **zero** payoffs.
- If $a_s \leq a_b$, there is trade at the price a_b:
  - The seller's payoff is $a_b - v$.
  - The buyer's payoff is $1.5v - a_b$.

Bilateral Trade

- Trade is efficient for every v.
- In a complete-information game in which v is certain:
  - $a_s = a_b = v$ would be a Nash equilibrium resulting in trade.
- Let us look for a Nash equilibrium in the Bayesian game
  - Restricting attention to equilibrium in which the seller always plays a **weakly dominant action**

Weakly Dominant Actions

- An action $a_i \in A_i$ is **weakly dominant** for player i given the signal t_i if for all $a_i' \neq a_i$, $U_i(a_i, s_{-i}|t_i) \geq U_i(a_i', s_{-i}|t_i)$ for every s_{-i}, with a strict inequality for at least one s_{-i}.
  - The other actions are **weakly dominated** in this case.
- Unlike the case of strict domination, playing a weakly dominated action is consistent with Nash equilibrium.
  - Classic example: Bertrand competition

Weakly Dominant Actions

- Nevertheless, eliminating weakly dominated actions is a popular criterion for selecting among Nash equilibria.
- In the current buyer-seller example, the seller has a weakly dominant action $a_s = v$, for every v.
  - E.g., deviation from $a_s > v$ to v:
    - Increases payoff from zero to $a_b - v$ when $a_b \in (v, a_s)$.
    - Makes no difference otherwise.

Bilateral Trade

- Suppose, then, that the seller plays $a_s = v$ for every v.
- The buyer chooses a_b to maximize

$$\sum_v p(v) u_b(a_b, a_s(v), v)$$

- Because $v \sim U[0,1]$ (a continuous variable), we need to write

$$\int u_b(a_b, a_s(v), v) dv$$

Bilateral Trade

- $u_b(a_b, a_s, v) = 0$ whenever $a_b < a_s$, independently of v.
- This enables us to rewrite the buyer's objective function:

$$\Pr(a_b \geq a_s) \cdot E(1.5v - a_b | a_b \geq a_s)$$

- Since $a_s = v$ for every v, this is equivalent to

$$\Pr(v \leq a_b) \cdot [1.5E(v | v \leq a_b) - a_b]$$

Bilateral Trade

- Because $v \sim U[0,1]$, the following holds for every $a \in [0,1]$:
  - $\Pr(v \leq a) = a$ for every $a \in [0,1]$
  - $\mathrm{E}(v|v \leq a) = 0.5a$ for every $a \in [0,1]$

- The buyer would never want to play $a_b > 1$.

$$\Pr(v \leq a_b) \cdot [1.5E(v|v \leq a_b) - a_b]$$

$$= a_b \cdot [1.5 \cdot 0.5a_b - a_b] = -0.25(a_b)^2$$

- The optimal action is therefore $a_b = 0$!

Bilateral Trade

- Zero probability of trade in equilibrium, despite the efficiency of trade for all v
- Game-theoretic formulation of Akerlof's insight regarding market failure due to adverse selection
- Key argument: The buyer draws a statistical inference about v from the "pitoval event" that trade occurs, taking into account his equilibrium knowledge of the seller's strategy.

Second-Price Auction

- Two bidders, denoted 1 and 2, compete for an object.
- The bidders simultaneously submit bids $b_1, b_2 \geq 0$.
- The object is allocated to the player who submitted the highest bid.
- The winner pays the **loser**'s bid.

Second-Price Auction

- The state of the world is the pair $(t_1, t_2) \sim U[0,1]^2$.
  - t_i is player i's signal.
  - $\tau_i(t_1, t_2) = t_i$
- When player i wins the auction, his payoff is $t_i + \alpha t_j - b_j$.
  - $\alpha \in [0,1]$
  - j is i's opponent.
- When player i loses the auction, his payoff is zero.

Interpretations of the Payoff Function

$$t_i + \alpha t_j - b_j$$

- $\alpha = 0$: Private values
  - Purely idiosyncratic tastes
- $\alpha = 1$: Common values
  - Example: Bidding for an oil tract
- $\alpha \in (0,1)$: Intermediate case
  - Example: Partial technological spillovers

Nash Equilibrium

- Look for a symmetric Nash equilibrium with linear bidding strategies: $b_i = kt_i$
  - $k > 0$ is a constant that needs to be derived.
- A "pivotal event": Winning the auction
- The bidder's payoff is zero with certainty outside the pivotal event.

Nash Equilibrium

- Take player 2's bidding strategy as given.
- Consider player 1's maximization problem given t_1. He chooses b_1 to maximize

$$\Pr(b_1 > b_2) \cdot E[t_1 + \alpha t_2 - b_2 | b_1 > b_2]$$

  - We can ignore ties ($b_1 = b_2$) because this is a zero-probability event, given that player 2's bid distribution is continuous.

Parsing The Player's Objective Function

$$\Pr(b_2 < b_1) \cdot \ [t_1 + \alpha E(t_2|b_2 < b_1) - E(b_2|b_2 < b_1)]$$

- By assumption, $b_2 = kt_2$

- $t_2 \sim U[0,1]$, independently of t_1.

  $\Longrightarrow \ b_2 \sim U[0,k]$, independently of t_1.

  $\Longrightarrow \ \Pr(b_2 < b_1) = \frac{b_1}{k}$ for every $b_1 \leq k$.

- Player 2 has no reason to play $b_1 > k$.

Parsing The Player's Objective Function

$$\frac{b_1}{k} \cdot [t_1 + \alpha E(t_2 | b_2 < b_1) - E(b_2 | b_2 < b_1)]$$

- $b_2 \sim U[0, k]$, independently of t_1.

$\Longrightarrow E(b_2 | b_2 < b_1) = \frac{b_1}{2}$ for every $b_1 \leq k$.

Parsing The Player's Objective Function

$$\downarrow$$

$$\frac{b_1}{k} \cdot [t_1 + \alpha E(t_2|b_2 < b_1) - \frac{b_1}{2}]$$

- $b_2 \sim U[0, k]$, independently of t_1.

$$\Longrightarrow E(b_2|b_2 < b_1) = \frac{b_1}{2} \text{ for every } b_1 \leq k.$$

- Because $t_2 = \frac{b_2}{k}$, $E(t_2|b_2 < b_1) = \frac{b_1}{2k}$ for every $b_1 \leq k$.

Parsing The Player's Objective Function

$$\frac{b_1}{k} \cdot \left[t_1 + \alpha \frac{b_1}{2k} - \frac{b_1}{2}\right]$$

- The meaning of $E(t_2 | b_2 < b_1) = \frac{b_1}{2k}$:
  - Below the unconditional expectation $E(t_2) = 0.5$
  - Often referred to as "the winner's curse"

Parsing The Player's Objective Function

$$\frac{b_1}{k}\cdot\left[t_1+\alpha\frac{b_1}{2k}-\frac{b_1}{2}\right]$$

- The meaning of $E(t_2|b_2 < b_1) = \frac{b_1}{2k}$:
  - Despite the suggestive term, **no** clear-cut incentive to shade one's bid (intensive vs. extensive margins)!
  - Statistical inference from a hypothetical event

Deriving the Equilibrium Strategy

$$\frac{b_1}{k} \cdot [t_1 + \alpha \frac{b_1}{2k} - \frac{b_1}{2}]$$

- First-order condition w.r.t b_1 gives $b_1 = \frac{k}{k-\alpha} t_1$.
- But by assumption, $b_1 = k t_1 \implies k = \frac{k}{k-\alpha}$
- We obtain:

$$b_1 = (1 + \alpha) t_1$$

The Private Values Case

- All the player cares about is the opponent's bid distribution
- No need to draw inferences from the pivotal event
- Bidding one's value is a weakly dominant action in the second-price auction with private values
  - Similar logic to the bilateral trade example: Bidding affects the prospect of winning, not the payment conditional on winning.

Interim Summary

- In both applications so far, a player's expected payoff is the probability he wins an object times his expected net payoff conditional on winning the object.
- To calculate the object's conditional expected value, the player relies on his knowledge of opponents' strategies.
- This is the source of the phenomena: market failure due to adverse selection, the winner's curse in auctions.

Strategic Voting

- Voting is a non-monetary mechanism for aggregating:
  - Preferences
  - Information
- We will consider a common-interest environment that focuses purely on the information aggregation function
- Surprisingly, strategic considerations will matter despite the lack of a conflict of interests!

The Jury Model

- A group of n voters submit simultaneous recommendations between two social alternatives, denoted 1 and -1.
  - n is an odd number.
  - $a_i \in \{-1,1\}$ denotes voter i's recommendation.
- The implemented outcome is $z \in \{-1,1\}$.

The Jury Model

- The method of aggregating recommendations is non-weighted majority voting.
- The implemented outcome is $z = 1$ if and only if $\sum_i a_i \geq k$.
  - $k = 0$ is a simple-majority rule.
  - $k = n$ is a unanimity rule.

The Jury Model

- The state of the world is $(\theta, t_1, \ldots, t_n)$:
  - $\theta \in \{-1, 1\}$ is the objectively desirable alternative.
  - $t_i \in \{-1, 1\}$ is voter i's signal.
  - $\tau_i(\theta, t_1, \ldots, t_n) = t_i$
- Each voter's payoff is $z\theta$.
  - Common interests: The outcome should match the state.
  - Voters care about actions only insofar as they affect z.

The Jury Model

- The prior p over $(\theta, t_1, \dots, t_n)$ is:
  - $p(\theta = 1) = 0.5$
  - $p(t_i = \theta | \theta) = q \in (0.5,1)$ for all θ, independently of t_{-i}.
  - q measures the signal's accuracy
- Interpreting voters' conditionally independent signals:
  - Different expertise
  - Differential attention

Interpretational Difficulties

- Small n (juries, committees):
  - Lack of communication despite common interests?
- Large n (large elections):
  - Do voters think strategically?

Simple Majority

- Let $k = 0$

- Non-strategic benchmark: Voters report their signals ($a_i = t_i$)

- Condorcet's Jury Theorem: As n grows larger, the probability that the majority decision is correct approaches 1.
  - "Wisdom of the crowd"
  - A precursor of the law of large numbers

The Single-Voter Case

- Let $n = 1$ (a dictatorial decision maker: $z = a_i$)

$$p(\theta = 1 | t_1 = 1) = \frac{0.5 \cdot q}{0.5 \cdot q + 0.5 \cdot (1 - q)} = q$$

- Likewise, $p(\theta = -1 | t_1 = -1) = q$.
- Since $q > 0.5$, the voter prefers to play $a_i \equiv t_i$.
  - Optimal individual decision coincides with truthful reporting
  - Follows from the symmetric prior and payoff function.

Nash Equilibrium for (Odd) $n > 1$

- A "bad" equilibrium in which all voters vote for the same alternative, independently of their signal:
  - A "weak" equilibrium: No individual voter can change the outcome by unilateral deviation.
  - It would become strict if we introduced small information acquisition costs.
- Is truthful reporting consistent with Nash equilibrium?

Nash Equilibrium for (Odd) $n > 1$

- Assume every voter i plays $a_i \equiv t_i$.
- W.l.o.g, consider voter 1 and suppose $t_1 = 1$.
- When calculating the expected utility from an action, the voter sums over all payoff-relevant contingencies $(\theta, a_2, \ldots, a_n)$.
- Voter 1's action affects z if and only if $\sum_{i>1} a_i = 0$.

Nash Equilibrium for (Odd) $n > 1$

$a_1 \backslash \Sigma_{i>1} a_i$	-(n-1)	$\cdots$	-2	0	2	$\cdots$	n-1
1	-1	$\cdots$	-1	**1**	1	$\cdots$	1
-1	-1	$\cdots$	-1	**-1**	1	$\cdots$	1

The outcome as a function of a_1 & $\sum_{i>1} a_i$

- We can ignore all the contingencies in which the voter's action doesn't make a difference.
- Manifestation of the independence property of EU Theory

Nash Equilibrium for (Odd) $n > 1$

- Voter 1 effectively calculates $Pr(\theta = 1 | t_1 \, ; \sum_{i>1} a_i = 0)$
  - He plays $a_1 = 1$ (-1) whenever this posterior is above (below) 0.5.
- The calculation takes into account other voters' strategies.
- Yet another instance of the "statistical inferences from hypothetical events" theme

Nash Equilibrium for (Odd) $n > 1$

- We have guessed that $a_i \equiv t_i$ for every i. Then:

$$Pr(\theta = 1|t_1\,;\textstyle\sum_{i>1} a_i = 0) = p(\theta = 1|t_1\,;\textstyle\sum_{i>1} t_i = 0)$$

- The R.H.S is expressed entirely in terms of the prior p.

- The equal numbers of 1 and -1 signals among voters

  $2, \dots, n$ mean that these signal cancel each other out:

$$p(\theta = 1|t_1\,;\textstyle\sum_{i>1} t_i = 0) = p(\theta = 1|t_1)$$

Nash Equilibrium for (Odd) $n > 1$

$$p(\theta = 1 | t_1 ; \sum_{i>1} t_i = 0) = p(\theta = 1 | t_1)$$

- Now the R.H.S is just as in the single-voter case.
- We've established that in this case playing $a_1 \equiv t_1$ is optimal.
- Therefore, a strategy profile in which every voter reports his signal constitutes a Nash equilibrium.

The Simple-Majority Case: Discussion

- The voter imagines being pivotal, because that is the only scenario in which his vote matters.
- In that event, the other votes have no informational content because the 1 and -1 signals cancel each other out.
- It would be intuitive to draw that inference ex-post; the idea that voters do it **in anticipation of this event** is somewhat less intuitive.

Unanimity

- Let $k = n$
- The outcome -1 is the default social outcome; switching to the other outcome requires unanimous agreement.
- The norm in criminal jury trials
- A "bad" Nash equilibrium: All voters always recommend -1.
- Is truthful reporting consistent with Nash equilibrium?

Truthful Nash Equilibrium?

- Assume every voter i plays $a_i \equiv t_i$.
- W.l.o.g, consider voter 1 and suppose $t_1 = -1$.
- When calculating the expected utility from an action, the voter sums over all payoff-relevant contingencies $(\theta, a_2, \dots, a_n)$.
- Voter 1's action affects z if and only if $\sum_{i>1} a_i = n - 1$.

Truthful Nash Equilibrium?

$a_1 \backslash \Sigma_{i>1} a_i$	-(n-1)	⋯	n-3	n-1
1	-1	⋯	-1	**1**
-1	-1	⋯	-1	**-1**

The outcome as a function of a_1 & $\sum_{i>1} a_i$

- Voter 1 effectively best-replies to the distribution

$$Pr(\theta | t_1 \,;\, \textstyle\sum_{i>1} a_i = n-1) = Pr(\theta | t_1 \,; \textstyle\sum_{i>1} t_i = n-1)$$

Plugging Equilibrium strategies

Explicit Posterior Calculation

$$Pr(\theta = 1 | t_1 = -1\,; t_i = 1 \; for\; all\; i > 1)$$

$$= \frac{\overbrace{0.5\cdot(1-q)\cdot q^{n-1}}^{I'm\;wrong}}{0.5\cdot(1-q)\cdot q^{n-1} + \underbrace{0.5\cdot q\cdot(1-q)^{n-1}}_{I'm\;right}} = \frac{1}{1+\left(\frac{1-q}{q}\right)^{n-2}} > \frac{1}{2}$$

- Voter 1's best-reply is to vote against his signal,

  contradicting the equilibrium assumption.

Unanimity: Discussion

- Ex-post, seeing my signal differs from everybody else's, it would be sensible for me to adopt the majority view.
- What's counter-intuitive is that this inference is made in the **interim** stage, before that (rare) pivotal event happen.
- Equilibrium with partial truthful reporting requires mixing.
- Do people actually reason along these lines? Mixed experimental evidence

Summary

- Many examples of games in which payoffs depend on a "pivotal event" (auctions, voting, trade)
- Recurring theme: Nash equilibrium analysis involves statistical inferences from the pivotal event, taking the opponents' strategies into account
- Non-trivial effects: The winner's curse, swing voter's curse

Advanced Microeconomic Theory

Lecture 5: Applications of Bayesian Games II

Elaborate Information Structures and High-Order Beliefs

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Plan of the Lecture

- The ex-ante formulation of Bayesian games
  - Application: Speculative trade
- The role of high-order beliefs: The investment game revisited with two information structures:
  - One humorous
  - The other (supposedly) serious

Ex-Ante Formulation of Bayesian Games

- Ex-ante perspective: Players commit ex-ante to a strategy
- At the ex-ante stage, there are no informational asymmetries.
- Reducing the model to a strategic game with complete information

Ex-Ante Formulation of Bayesian Games

- The set of player is $N = \{1, \dots, n\}$
- For each player $i \in N$, the set of strategies is the set of all functions $s_i: T_i \to A_i$.
- Player i's utility from the strategy profile $(s_1, \dots, s_n)$ is

$$U_i(s_1, \dots, s_n) = \sum_{\omega \in \Omega} p(\omega) u_i\big(s_1(\tau_1(\omega)), \dots, s_n(\tau_n(\omega)), \omega\big)$$

Ex-Ante Formulation of Bayesian Games

- A profile of strategies $(s_1, \ldots, s_n)$ is a Nash equilibrium if for every player i, $U_i(s_i, s_{-i}) \geq U_i(s_i', s_{-i})$ for every $s_i' \in S_i$.
  - Tests deviations at a hypothetical planning stage
  - Equivalent to the original, interim definition – thanks to the expected-utility assumption
  - Usually hard to work with because of large strategy space
  - Useful for general results about classes of games

Speculative Trade

- Trade motivated purely by differences in beliefs
- Arguably the predominant motive for trade in financial markets
- Can differences in beliefs that give rise to trade be entirely due to informational asymmetries?
- Traders' strategic inferences from their counterparts' willingness to trade is an impediment to trade

Speculative Trade: A Two-Player Example

- Consider a bet $f: \Omega \to \{-1,1\}$.
  - $f(\omega)$ is the amount that player 1 receives from player 2 in state ω.
- A trading game: Each player i chooses an action $a_i \in \{0,1\}$.

$$u_1(a_1, a_2, \omega) = a_1[a_2 f(\omega) - \varepsilon]$$

$$u_2(a_1, a_2, \omega) = a_2[-a_1 f(\omega) - \varepsilon]$$

  - $\varepsilon > 0$ is an arbitrarily small transaction cost.

Speculative Trade: A Two-Player Example

- $a_i = 1$ means agreeing to trade.
- The role of the transaction cost is to break ties.
- An arbitrary information structure $(\Omega, p, T_1, T_2, \tau_1, \tau_2)$
  - The prior p has full support.

Speculative Trade: A Two-Player Example

- A trivial example: $T_1 = T_2 = \{t\}$: Both traders are uninformed.
- Player i will play $a_i = 1$ only if $a_j = 1$, due to transaction cost.
- player 1 will play $a_1 = 1$ only if $\sum_{\omega} p(\omega) f(\omega) > 0$.
- But then player 2 doesn't want to trade!
- The only Nash equilibrium is $a_1 = a_2 = 0$ (no trade).

Speculative Trade: A Two-Player Example

- Another example: Player 1 knows ω , player 2 is uninformed.
- For trade to take place, we need $a_2 = 1$.
- player 1 will play $a_1 = 1$ if and only if $f(\omega) = 1$.
- But then player 2 earns a negative payoff. He can profitably deviate to $a_2 = 0$.
- The only Nash equilibrium involves no trade.

A “No-Trade Theorem”

Proposition: For any information structure, the unique Nash equilibrium in the induced Bayesian game is for each player i to play $s_i(t_i) = 0$ for every t_i.

- Speculative trade cannot be due to differential information, under the assumption that traders play Nash equilibrium.
- An example of a rich literature on “no-trade theorems”.

Proof (Using the Ex-ante Formulation)

- When $a_i = 0$ with certainty, $a_j = 0$ is a best-reply for player j regardless of his information, because trade doesn't occur anyway and playing 0 saves the transaction cost.
- Now consider a candidate Nash equilibrium in which each player sometimes plays 1.

Proof

- Each player can ensure an ex-ante payoff of 0 by always refusing to trade. This is a lower bound on his equilibrium payoff.
- By assumption, players incur the transaction cost with positive probability in the candidate equilibrium.
- Therefore, each player’s ex-ante expected monetary transfer is strictly positive.

Proof

- Player 1's ex-ante monetary payoff:

$$\sum_{\omega} p(\omega) s_1\big(\tau_1(\omega)\big) s_2\big(\tau_2(\omega)\big) f(\omega) > 0$$

- Player 2's ex-ante monetary payoff:

$$-\sum_{\omega} p(\omega) s_1\big(\tau_1(\omega)\big) s_2\big(\tau_2(\omega)\big) f(\omega) > 0$$

- A contradiction!

Discussion

- Many culprits:
  - The common prior belief
  - Partitional information structures
  - Expected utility maximization
  - Rational expectations
- A common trick in the finance literature: “Noise traders”

High-Order Beliefs

- Information structures can express rich patterns of high-order beliefs ("my information about your information about my information…")
- The state space can have a dimensionality far beyond the payoff-relevant states.
- This richness can be strategically relevant.

The E-Mail Game

	I	NI
I	-1,-1	-1,0
NI	0,-1	0,0

Bad state

	I	NI
I	1,1	-1,0
NI	0,-1	0,0

Good state

- The investment game revisited
- A slight change in the payoff structure
- The probability of the bad state of Nature is $q > 0.5$.

The Information Structure

- Each player sits in front of a computer screen.
- When the state is good (and **only** then), player 1's computer sends an automatic message to player 2's computer.
- When player 2's computer receives the message, it sends a confirmation to player 1's computer, which sends a re-confirmation, and so forth…
- Each message gets lost with independent probability $\varepsilon > 0$.

The Information Structure

- The process terminates with probability one after finitely many rounds. Each player's computer screen displays the number of messages that the computer sent.
  - This number is the player's signal.
  - Players simultaneously take actions after receiving it.
- Ω is the set of all pairs of non-negative integers (t_1, t_2) for which $t_2 \in \{t_1 - 1, t_1\}$.

The Information Structure

$$p(0,0) = q$$

$$p(1,0) = (1 - q)\varepsilon$$

$$p(1,1) = (1 - q)(1 - \varepsilon)\varepsilon$$

$$p(2,1) = (1 - q)(1 - \varepsilon)^2\varepsilon$$

$$p(2,2) = (1 - q)(1 - \varepsilon)^3\varepsilon$$

$$p(3,2) = (1 - q)(1 - \varepsilon)^4\varepsilon$$

$$\vdots$$

The Information Structure

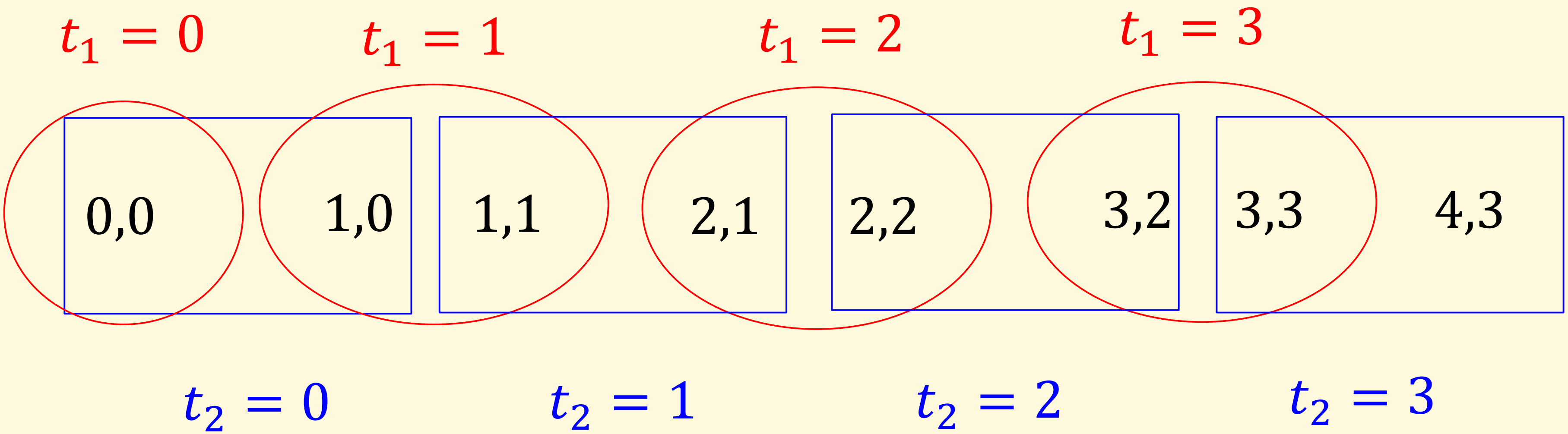
- $\tau_i(t_1, t_2) = t_i$ encodes player i's high-order knowledge regarding the state of Nature:
  - $t_1 = 1$: Player 1 knows it is good but doesn't know whether player 2 knows it is good.
  - $t_1 = 2$: He knows the state is good and that player 2 knows it is good, but doesn't know whether 2 knows all this.

  $\vdots$

Almost Common Knowledge

- Small ε ensures that when the state of Nature is good, players are very likely to have a high degree of mutual knowledge of this event.
- However, common knowledge is never attained.
- What would **you** do if you saw a large number on your computer screen?

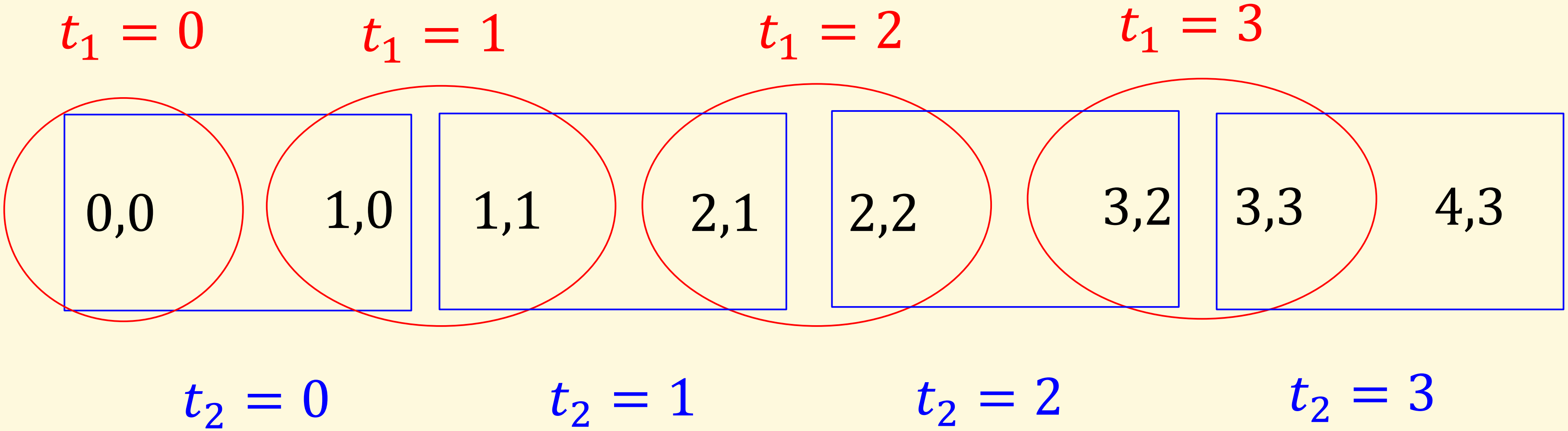
Diagrammatic Representation



- Interlocking information sets

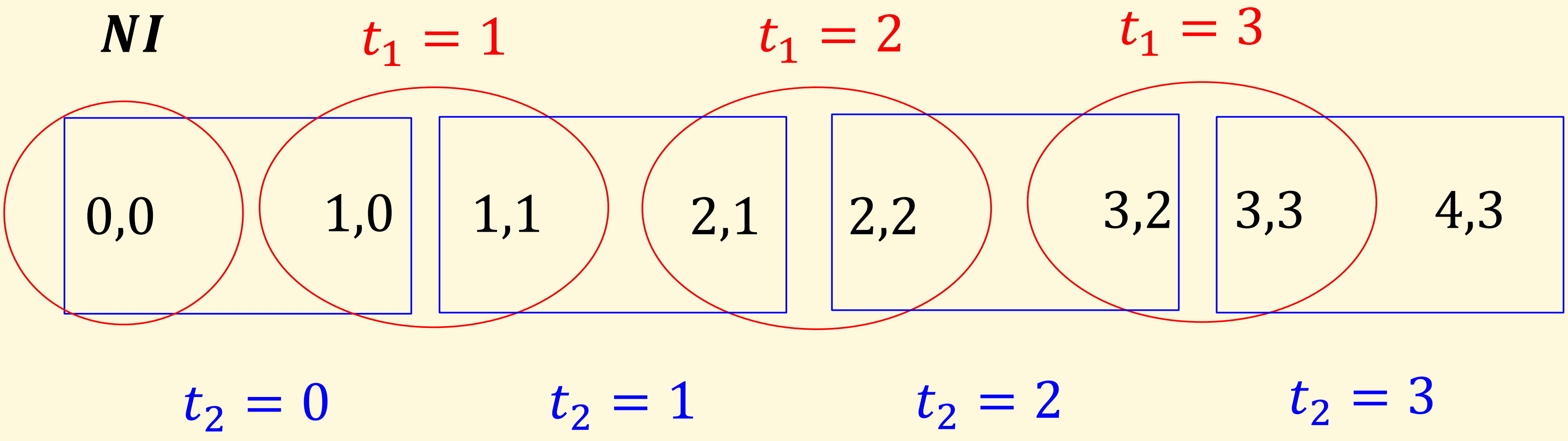
Proposition: The game has a unique Nash equilibrium. For every player $i = 1,2$, $s_i(t_i) = NI$ for every $t_i = 0,1,2, \ldots$

Proof



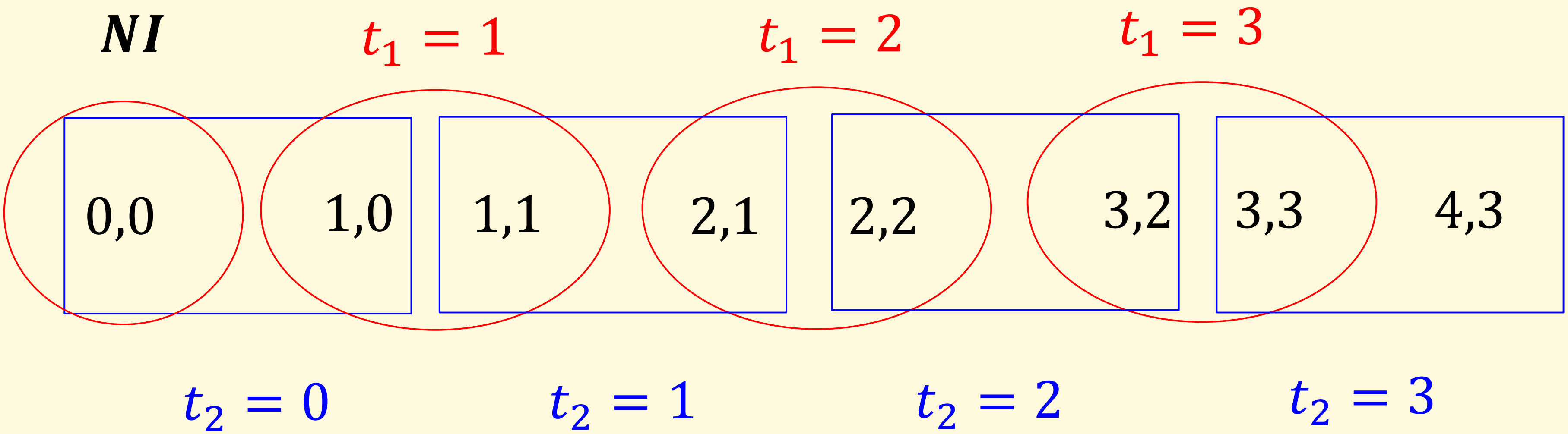
- The proof is by induction on the players' interlocking information sets: $t_1 = 0$, $t_2 = 0$, $t_1 = 1$, $t_2 = 1, \ldots$
- At $t_1 = 0$, NI is strictly dominant for player 1.

Proof



- At $t_2 = 0$, player 2 assigns probability $\frac{q}{q+(1-q)\varepsilon} > 0.5$ to the bad state of Nature.

Proof

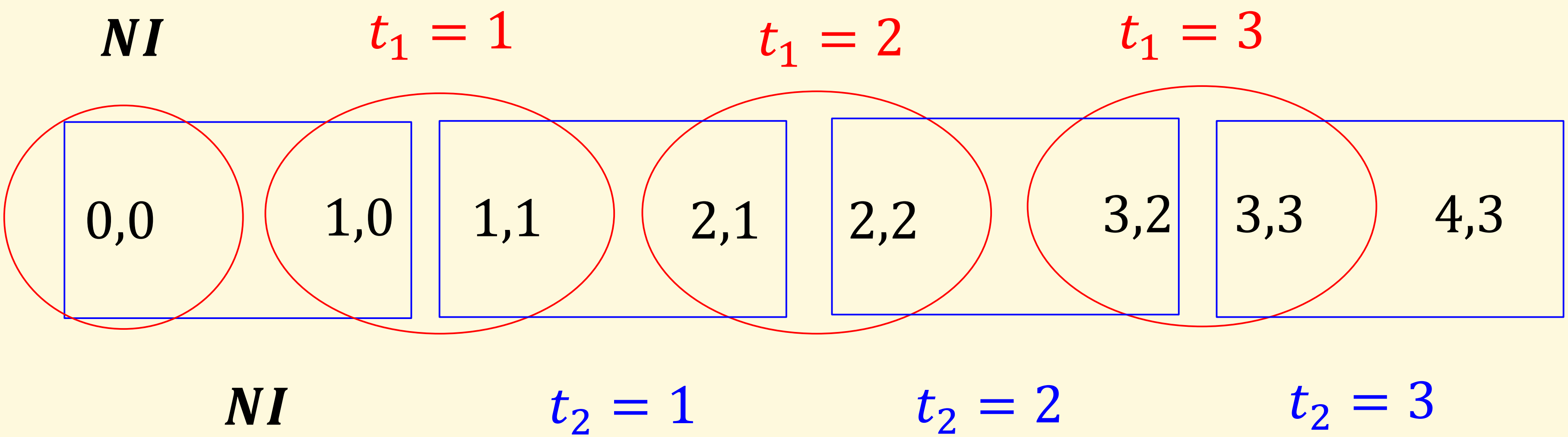


- The player's expected payoff from I is therefore at most

$$\frac{q}{q+(1-q)\varepsilon}\cdot(-1)+\frac{(1-q)\varepsilon}{q+(1-q)\varepsilon}\cdot 1<0.$$

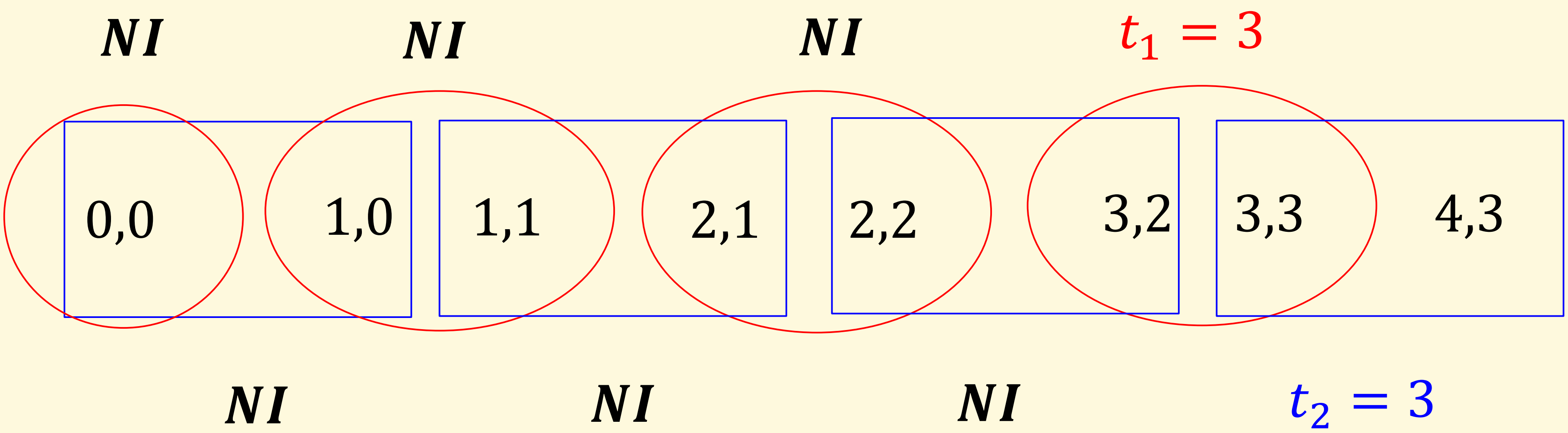
- Therefore, player 2 plays NI at $t_2=0$ in any Nash equilibrium.

Proof



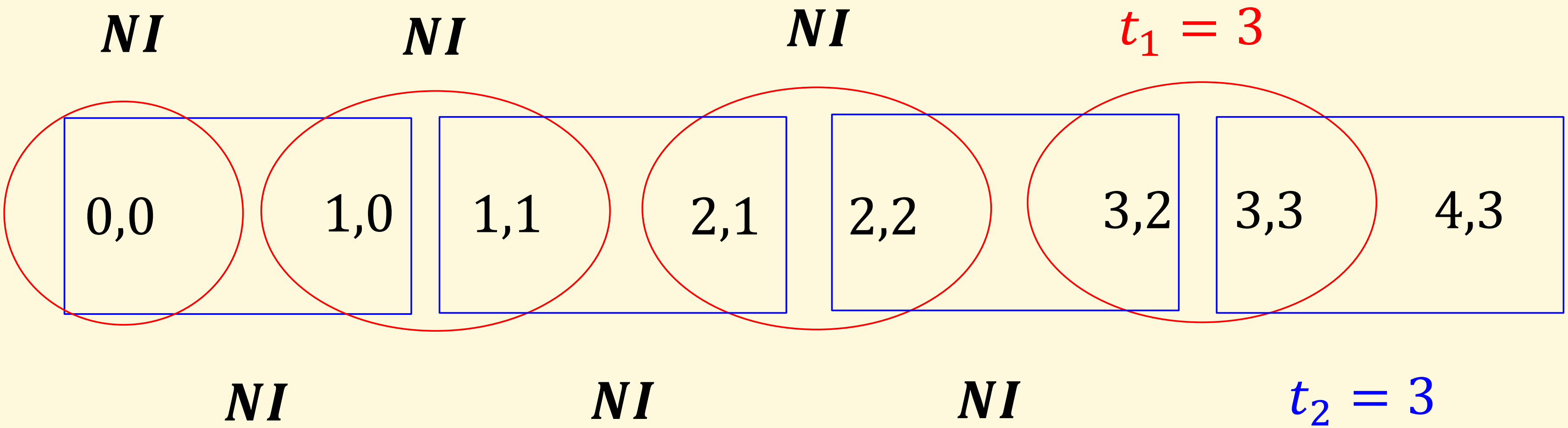
- We have established that $s_i(t_i = 0) = NI$ for both $i = 1,2$ in any Nash equilibrium.
- Now we'll put the inductive argument to work.

Proof



- Suppose we proved that players play NI in all information sets in the sequence up to some information set.
- In the diagram, that information set is $t_2 = 2$.

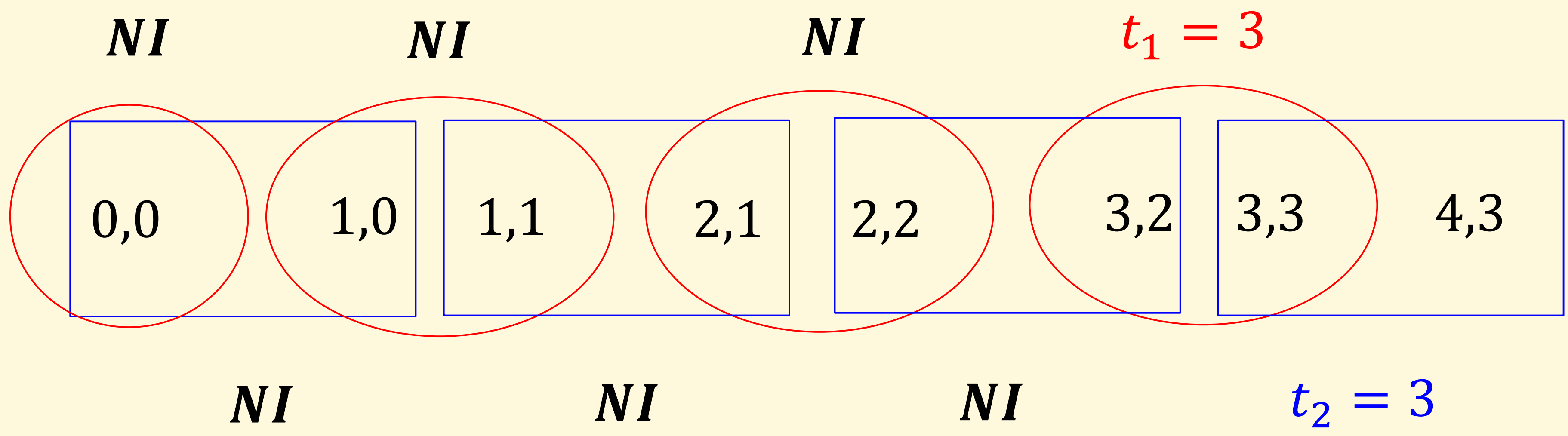
Proof



- At $t_1 = 3$, player 1's posterior probability the state $(3,2)$ is

$$\frac{(1-q)(1-\varepsilon)^4\varepsilon}{(1-q)(1-\varepsilon)^4\varepsilon + (1-q)(1-\varepsilon)^5\varepsilon} = \frac{1}{2-\varepsilon} > 0.5$$

Proof

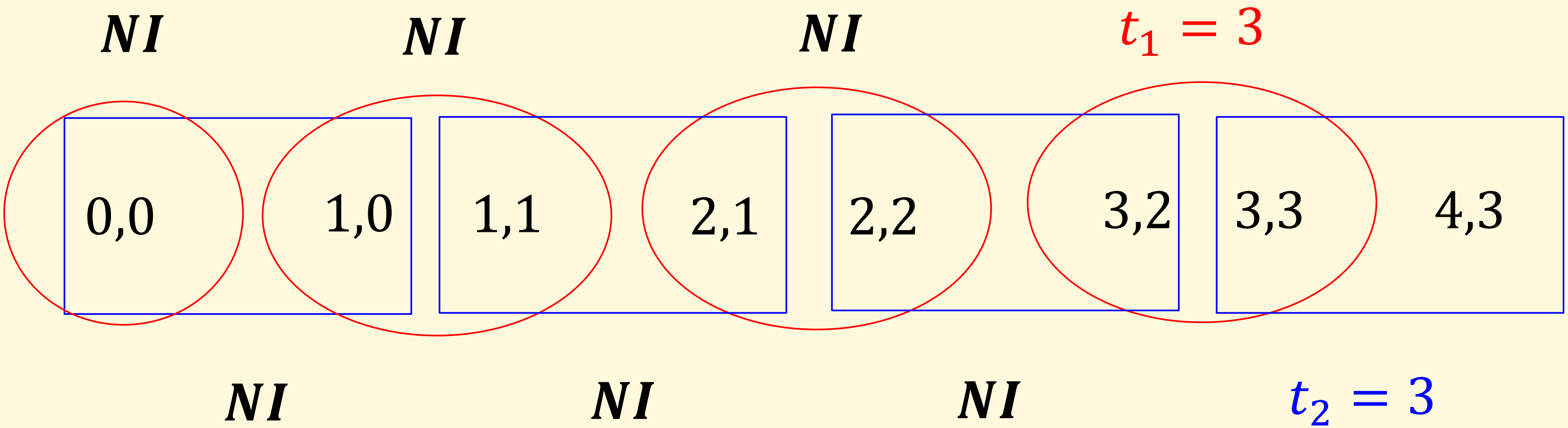


Intuition: Given that my last message hasn't been confirmed, which scenario is more likely?

– My original message got lost (probability ε).

– The confirmation got lost (probability $\varepsilon(1-\varepsilon)$).

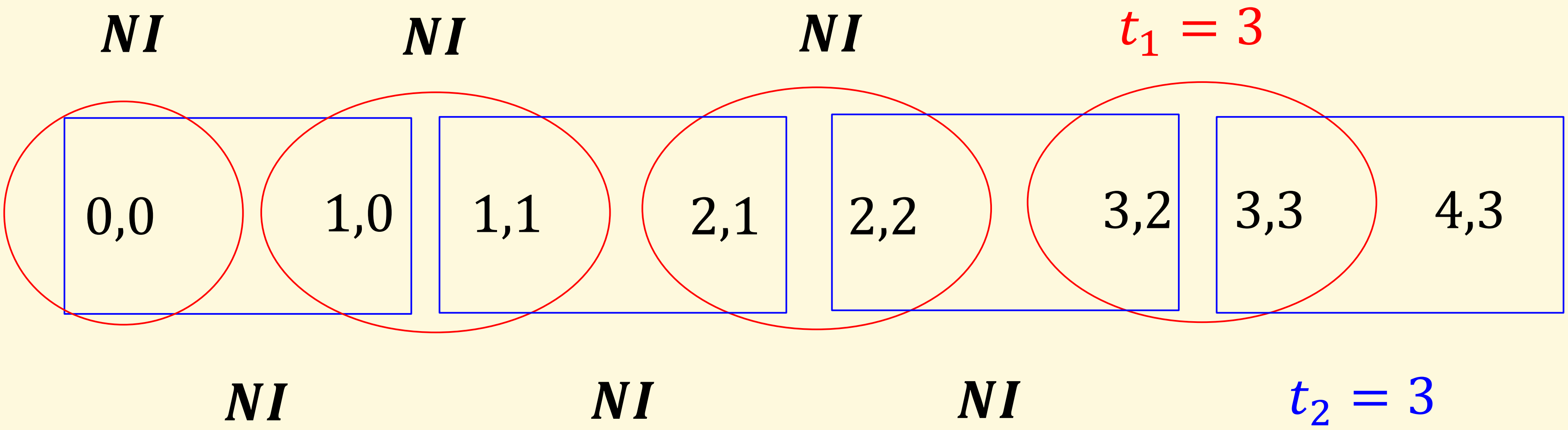
Proof



- Bayes' rule says: The first scenario is strictly more likely.
- Player 1's expected utility from I is therefore at most

$$\frac{1}{2-\varepsilon}\cdot u_1(I, a_2(t_2=2), good) + \frac{1-\varepsilon}{2-\varepsilon}\cdot u_1(I, a_2(t_2=3), good)$$

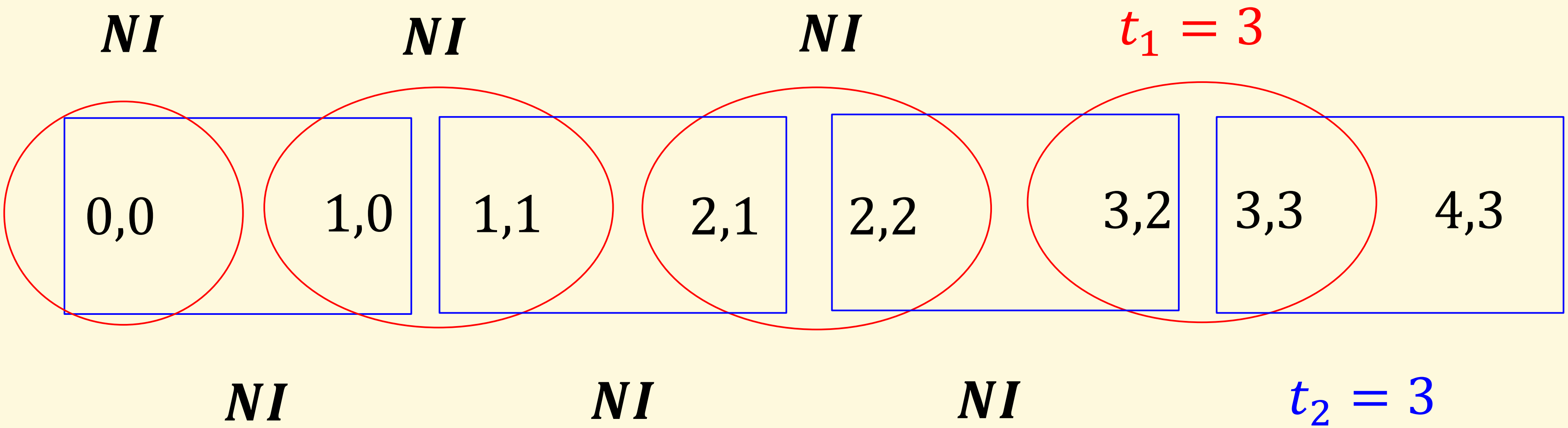
Proof



- Bayes' rule says: The first scenario is strictly more likely.
- Player 1's expected utility from I is therefore at most

$$\frac{1}{2-\varepsilon} \cdot u_1(I, NI, good) + \frac{1-\varepsilon}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 3), good)$$

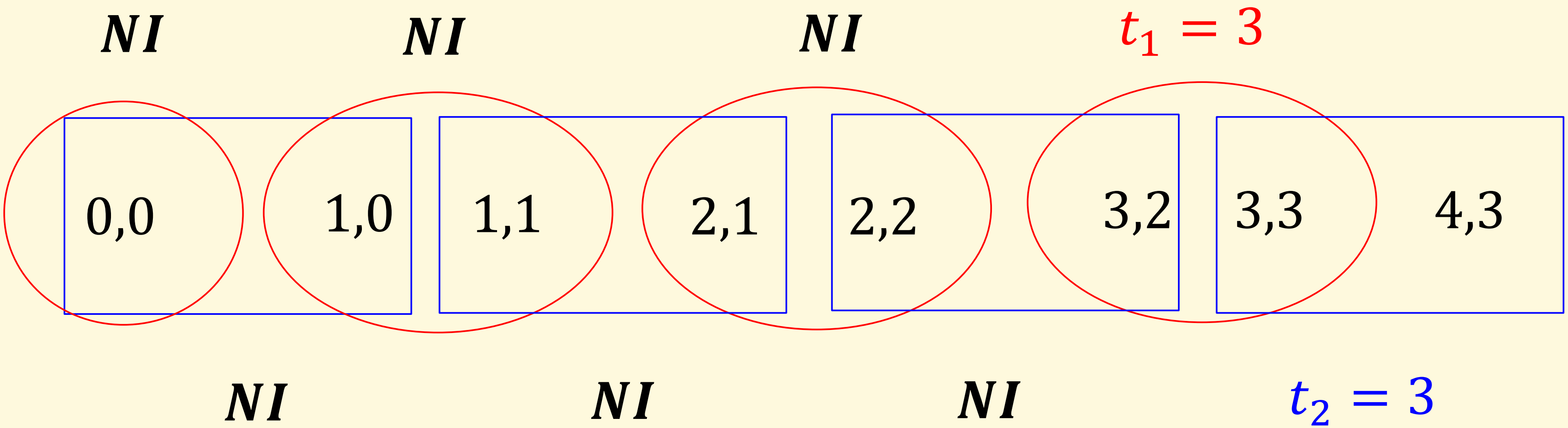
Proof



$$\frac{1}{2-\varepsilon} \cdot (-1) + \frac{1-\varepsilon}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 3), good)$$

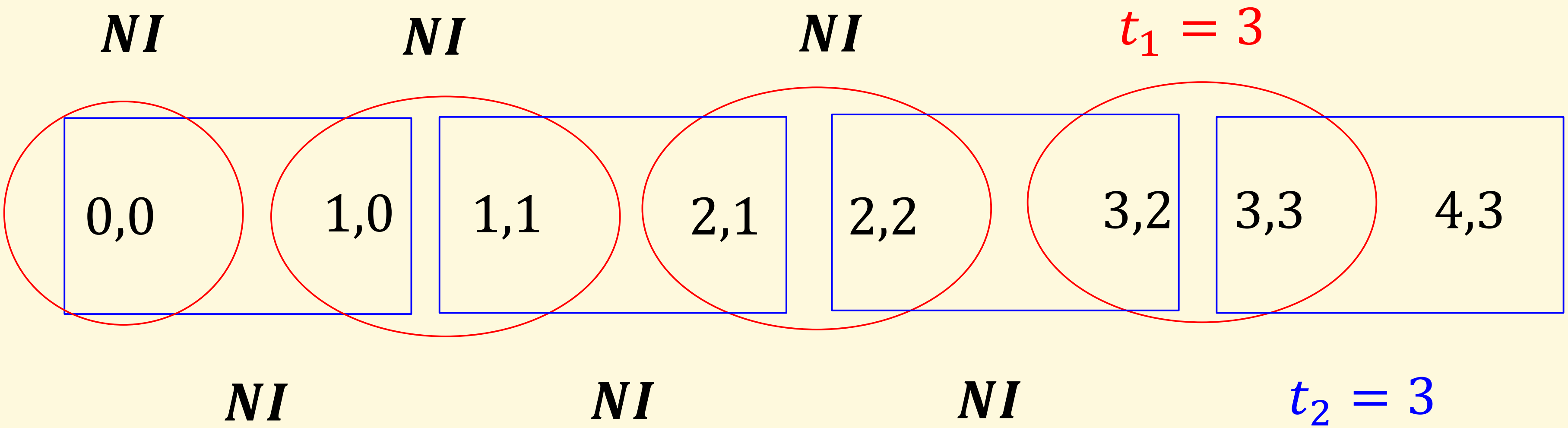
$$\leq \frac{1}{2-\varepsilon} \cdot (-1) + \frac{1-\varepsilon}{2-\varepsilon} \cdot 1 < 0$$

Proof



- Therefore, player 1's best-reply at $t_1 = 3$ in any Nash equilibrium is NI.
- Recall that we used $t_1 = 3$ for purely illustrative purposes.

Proof



- The same argument works for any information set $t_i > 0$.
- It follows that both players always play NI in Nash equilibrium.

This completes the proof.

Discussion

- An example of "finite layers of strategic reasoning" paradoxes
- The difference between the states $(1,1), (2,1), (2,2), \ldots$ is in the players' high-order beliefs.
- The inductive proof is fundamentally iterative elimination of strictly dominated strategies.
  - The Nash equilibrium is the unique **rationalizable** outcome.

Global Games

- The information structure of the E-mail game makes the players' high-order beliefs explicit.
- However, it is "artificial" – not something that a self-respecting applied economist would assume...
- But very similar effects arise under more "respectable" information structures!

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- An example by Carlsson & van Demme (1993)
- θ is the state of Nature that indicates investment quality.
- Customary assumption: θ is distributed according to an "improper" uniform prior over $(-\infty, \infty)$.

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Player i's signal is $t_i = \theta + \varepsilon_i$, where ε_i is independently drawn according to $N(0,\sigma^2)$.
- $\omega = (\theta, \varepsilon_1, \varepsilon_2)$; $\tau_i(\theta, \varepsilon_1, \varepsilon_2) = \theta + \varepsilon_i$

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- When $\sigma^2 = 0$, θ is common knowledge.
  - $\theta > 1$ $\Longrightarrow$ I is a strictly dominant action.
  - $\theta < 0$ $\Longrightarrow$ NI is a strictly dominant action.
  - $\theta \in [0,1]$ $\Longrightarrow$ (I, I) and (NI, NI) are Nash equilibria.

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

Proposition: When $\sigma^2 > 0$, there is an essentially unique Nash equilibrium. Each player i plays I whenever $t_i > 0.5$, and he plays NI whenever $t_i < 0.5$.

Discussion

- Slight incomplete-information perturbation of the complete information game leads to equilibrium selection.
  - Efficient coordination with near certainty when $\theta > 0.5$
  - Inefficient coordination when near certainty when $\theta < 0.5$
- Striking difference between the common knowledge and "almost common knowledge" environments

Why is it an Equilibrium?

- Suppose σ^2 is vanishing.
- At $t_i = 0.5$, player i believes $\theta \approx 0.5$.
- He also assigns probability 0.5 to $a_j = I$ because of his knowledge of player j's cut-off strategy.
- Therefore, he is indifferent between the two actions.
- When we raise (lower) t_i, the incentive to play I becomes stronger (weaker).

Why is it the Unique Equilibrium?

- Recall $\theta = t_i - \varepsilon_i$, $t_j = t_i - \varepsilon_i + \varepsilon_j$.

- Therefore, conditional on observing t_i, player i's posterior implies $\theta \sim N(t_i, \sigma^2)$ and $t_j \sim N(t_i, 2\sigma^2)$.

- When $t_i < 0$, $E(\theta | t_i) < 0$, and therefore NI is strictly dominant for player i.

$\Longrightarrow$ In any Nash equilibrium, $a_i = NI$ when $t_i < 0$.

Why is it the Unique Equilibrium?

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Now suppose that $t_i > 0$ but close to zero.
- By the previous argument, player i's posterior probability that $t_j < 0$ – and hence $a_j = NI$ – is close to 0.5.

Why is it the Unique Equilibrium?

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Given that $E(\theta|t_i)$ is close to zero, NI is a best reply for i.
- And so in any Nash equilibrium, $a_i = NI$ also when t_i is positive but close to zero.

Why is it the Unique Equilibrium?

- The last argument was based on player i's second-order belief – i.e., his belief regarding player j's signal.
- We continue in this iterative manner, further expanding the range of signal realizations for which NI is a best reply for i.
- This iterative argument mirrors the inductive proof in the E-mail game.

Idea of the Proof

- The limit of this iterative argument is that in any Nash equilibrium, player i plays NI whenever $t_i < 0.5$.
- An analogous argument applies to the other side (starting with I being dominant when $t_i > 1$).
- When σ is small, players can be almost certain that investment is efficient and nevertheless they coordinate on the bad outcome because of lack of common knowledge.

Summary

- Despite the apparent gap between the whimsical E-mail game and the “applied look” of the last example, the game-theoretic analysis is very similar.
- While the equilibrium in the E-mail game looks paradoxical, the cutoff strategies in the last example look natural…
- …And has been extended to “applied” models of phenomena like currency attacks and bank runs.

Summary

- In Game Theory, there is a fine line between the applied and the paradoxical.
- An appropriate motto to conclude with…

THANK YOU!