# Lecture Notes on Bayesian Games

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## 1 The Formalism of Games with Incomplete Information

A strategic game with incomplete information consists of the following components:

- A set of players $N$
- A set of actions $A_i$ for each player $i \in N$
- A set of states $\Omega$ (assumed to be finite for the sake of exposition)
- A prior probability distribution $p$ over $\Omega$, representing the players' common prior belief regarding the state of the world.
- A set of signals $T_i$ for each player $i \in N$
- A signal function $\tau_i : \Omega \to T_i$ for each player $i \in N$; $\tau_i(\omega)$ is the signal that player $i$ receives in state $\omega$.
- A vNM utility function $u_i : (\times_{j \in N} A_j) \times \Omega \to \mathbb{R}$; $u_i(a, \omega)$ is the payoff that player $i$ receives when the action profile is $a$ and the state is $\omega$.

When player $i$ receives a signal $t_i \in T_i$, she updates her belief as follows. If $\tau_i(\omega) = t_i$ - i.e., if the state is consistent with the signal the player receives - the posterior probability she assigns to the state $\omega$ is computed according to Bayes' rule:

$$p(\omega \mid t_i) = \frac{p(\omega)}{\displaystyle\sum_{\omega' \mid \tau_i(\omega') = t_i} p(\omega')}$$

If $\tau_i(\omega) \neq t_i$ - i.e., if the state is inconsistent with the signal the player receives - the posterior probability she assigns to the state $\omega$ is $p(\omega \mid t_i) = 0$. Because of the reliance on Bayesian updating, games with incomplete information are also called Bayesian games.

## 2 Nash Equilibrium

When player $i$ receives the signal $t_i$, she updates her belief as described above, and chooses an action $a_i \in A_i$. In order to make this choice, the player needs to form a conjecture regarding the actions that her opponents are taking, given any information they might have. For every player $j \neq i$, let us fix the action she takes when she receives the signal $t_j$ and denote it by $a_{j,t_j}$. Then, player $i$'s expected utility from the action $a_i$, given that she receives the signal $t_i$ and expects player $j$ to play the action $a_{j,t_j}$ when receiving the signal $t_j$ is

$$U_i(a_i, (a_{j,t_j})_{j \neq i, t_j \in T_j} \mid t_i) = \sum_{\omega \in \Omega} p(\omega \mid t_i) \cdot u_i[a_i, (a_{j,\tau_j(\omega)})_{j \neq i}, \omega]$$

Let us interpret this expression, because it captures a complicated thought process. Player $i$ understands the structure of the game. In particular, she understands the information that each player $j \neq i$ receives in each state. While player $i$ may not know the state, she knows the signal that each of her opponents receives in each state. Furthermore, she holds an expectation of the action that each of her opponents $j$ takes as a function of the signal $t_j$ that she receives. Therefore, player $i$ expects player $j$ to take the action $a_{j,\tau_j(\omega)}$ in the state $\omega$. She can therefore calculate her vNM utility from $a_i$ in each possible state, because she knows what to expect from her opponents in each state. Player $i$ can then calculate her expected utility from $a_i$, by summing her utility over all possible states, where each state is weighted according to its posterior probability, given the signal $t_i$ that player $i$ receives.

Consider a collection $(a_{i,t_i})$ of signal-dependent actions for each player. That is, we go over all players $i \in N$ and all the possible signals $t_i \in T_i$ that player $i$ can receive. This collection constitutes a Nash equilibrium if for very $i \in N$ and every $t_i \in T_i$, the following holds:

$$a_{i,t_i} \in \arg\max_{a'_i} U_i(a_i, (a_{j,t_j})_{j \neq i, t_j \in T_j} \mid t_i)$$

In other words, each player's action given her signal maximizes her expected utility, given her posterior beliefs over the state space and her expectation of how her opponents behave for every signal they may receive.

## 3 Examples

Throughout this section, we will assume the following structure. The set of players is $\{1, 2\}$. The set of actions for each player is $\{I, NI\}$, where $I$ represents investing in a project while $NI$ represents not investing in it. With prior probability $\frac{1}{2}$, the project is profitable, leading to the following payoff matrix:

| | $I$ | $NI$ |
|---|---|---|
| $I$ | $1, 1$ | $-2, 0$ |
| $NI$ | $0, -2$ | $0, 0$ |

(Table 1)

With the remaining probability, the project is unprofitable, leading to the following payoff matrix:

|  | $I$ | $NI$ |
|---|---|---|
| $I$ | $-2, -2$ | $-2, 0$ |
| $NI$ | $0, -2$ | $0, 0$ |

(Table 2)

We will examine a number of informational structures imposed on top of this set-up.

### 3.1 Scenario I: Complete Information

Assume that whether the project is profitable is common knowledge. We can use the familiar framework of strategic games with complete information to analyze this situation. We simply treat the two cases (the project is profitable / the project is unprofitable) separately and independently. When the project is profitable, we can see from the payoff matrix that we have a coordination game, with two pure-strategy Nash equilibria, $(I, I)$ and $(NI, NI)$. When the project is unprofitable, each player has a strictly dominant action, $NI$. Therefore, in this case the only Nash equilibrium is $(NI, NI)$.

### 3.2 Scenario II: Player $1$ ($2$) is Fully Informed (Uninformed)

Now suppose that player 1 knows whether the project is profitable or not, while player 2 is uninformed. There is no additional uncertainty. This situation can be represented by the following information structure:

- $\Omega = \{g, b\}$
- $p(g) = p(b) = \frac{1}{2}$
- $T_1 = \{G, B\}$, $T_2 = \{t^*\}$
- $\tau_1(g) = G$, $\tau_1(b) = B$, $\tau_2(g) = \tau_2(b) = t^*$

The state $g$ ($b$) represents a profitable (unprofitable) project, and the players' payoffs in this state are given by Table 1 (2). Player 1's posterior belief always assigns probability one to the true state. Player 2's posterior belief is equal to her prior belief because she receives no information.

Let us find Nash equilibria in this Bayesian game. First, note that when both players play $NI$ regardless of their information, we have a Nash equilibrium. The reason is as follows. For player 1, $NI$ is a dominant action in state $b$ and a best-reply to $NI$ in state $g$. As to player 2, since she expects player 1 to play $NI$ in both states, playing $I$ yields a sure payoff of $-2$, whereas playing $NI$ yields a sure payoff of 0.

Is there an additional Nash equilibrium? In any putative equilibrium, player 1 must choose $NI$ in state $b$. If player 1 chooses $NI$ in state $g$, we already saw that player 2's best-reply is $N\not{I}$. Let us assume that player 1 chooses $I$ in state $g$. If player 2 chooses $NI$, she gets a sure payoff of 0. If, on the other hand, if she plays $I$, she gets an expected payoff of

$$\frac{1}{2} \cdot u_2(I, NI, b) + \frac{1}{2} u_2(I, I, g) = \frac{1}{2} \cdot (-2) + \frac{1}{2} \cdot 1 < 0$$

It follows that $I$ is not a best-reply for player 2 against player 1's supposed strategy of playing $I$ if and only if the state is $g$. Therefore, we have eliminated the possibility of an additional equilibrium.

This is an example of an inefficient outcome due to asymmetric information. If it were commonly known that the state is $g$, we would have an equilibrium in which both players invest, and this outcome Pareto-dominates the Nash equilibrium in the asymmetric information environment.

### 3.3 Scenario III: Player 1 is Fully Informed; Good Information is Leaked to Player 2 with Positive Probability

Now suppose that player 1 knows whether the project is profitable or not. When player 1 receives good news (but only then), these news leak to player 2 with probability $1-\varepsilon$, where $\varepsilon > \frac{1}{3}$. There is no additional uncertainty. This situation can be represented by the following information structure:

- $\Omega = \{g_1, g_2, b\}$
- $p(b) = \frac{1}{2}$, $p(g_1) = \frac{1}{2}\varepsilon$, $p(g_2) = \frac{1}{2}(1-\varepsilon)$
- $T_1 = \{G, B\}$, $T_2 = \{S, L\}$
- $\tau_1(g_1) = \tau_1(g_2) = G$, $\tau_1(b) = B$
- $\tau_2(b) = \tau_2(g_1) = S$, $\tau_2(g_2) = L$

The state $g_1$ represents a profitable project without a news leak, the state $g_2$ represents a profitable project with a news leak, and the state $b$ represents an unprofitable project. The players' payoffs are given by Table 1 in the states $g_1$ and $g_2$, and by Table 2 in the state $b$. Note that the distinction between $g_1$ and $g_2$ is thus payoff-irrelevant. The two states differ only in the information that the players possess.

Player 1's posterior beliefs given the possible signals that she may receive are as follows:

$$\begin{aligned}
p(g_1 \mid B) &= p(g_2 \mid B) = 0 \\
p(g_1 \mid G) &= \frac{\frac{1}{2}\varepsilon}{\frac{1}{2}\varepsilon + \frac{1}{2}(1-\varepsilon)} = \varepsilon \\
p(g_2 \mid G) &= \frac{\frac{1}{2}(1-\varepsilon)}{\frac{1}{2}\varepsilon + \frac{1}{2}(1-\varepsilon)} = 1-\varepsilon
\end{aligned}$$

The reason is as follows. When player 1 receives the signal $B$, she knows for sure that the state is $b$. When player 1 receives the signal $G$, she knows that the state cannot be $b$, but she cannot tell between $g_1$ and $g_2$ - in other words, she knows that the project is profitable but does not know whether the good news leaked to player 2. She uses Bayes' rule to update the probability of each of these two states.

Similarly, player 2's posterior beliefs given the possible signals that she may receive are as follows:

$$\begin{aligned} p(g_1 \quad &| \quad S) = \frac{\frac{1}{2}\varepsilon}{\frac{1}{2}+\frac{1}{2}\varepsilon} = \frac{\varepsilon}{1+\varepsilon} \\ p(b \quad &| \quad S) = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}\varepsilon} = \frac{1}{1+\varepsilon} \\ p(g_2 \quad &| \quad L) = 1 \end{aligned}$$

The reason is as follows. When player 2 receives the signal $L$, she knows for sure that the state is $g_2$. When player 2 receives the signal $S$ - i.e., when she does not hear good news - she does not know whether this is because there are no good news to begin with (the project is unprofitable) or the good news did not leak. She uses Bayes' rule to update the probability of each of these two states.

Let us turn to Nash equilibrium analysis. As before, when both players play $NI$ regardless of their information, we have a Nash equilibrium. The reason is as follows. For player 1, $NI$ is a dominant action in state $b$ and a best-reply to $NI$ in states $g_1$ and $g_2$. As to player 2, since she expects player 1 to play $NI$ in all states, playing $I$ yields a sure payoff of $-2$, whereas playing $NI$ yields a sure payoff of 0.

Are there additional equilibria? In any equilibrium, $a_{1,B} = NI$ - i.e., player 1 chooses $NI$ when she receives the signal $B$. This is because given this signal she assigns probability one to the state $b$, and $NI$ is a dominant action in that state. From now on, let us guess that $a_{1,G} = I$ - i.e., player 1 chooses $I$ when she receives the signal $G$.

If $a_{2,S} = NI$ - i.e., player 2 chooses $NI$ when she receives the signal $S$ - then player 2 gets a sure payoff of 0. In contrast, if $a_{2,S} = I$ - i.e., player 2 chooses $I$ when she receives the signal $S$ - her expected payoff (given her signal and her expectation that $a_{1,B} = NI$ and $a_{1,G} = I$) is

$$\begin{aligned} p(b \quad &| \quad S) \cdot u_2(a_{1,B}, I, b) + p(g_1 \mid S) \cdot u_2(a_{1,G}, I, g_1) \\ &= \quad p(b \mid S) \cdot u_2(NI, I, b) + p(g_1 \mid S) \cdot u_2(I, I, g_1) \\ &= \quad \frac{1}{1+\varepsilon} \cdot (-2) + \frac{\varepsilon}{1+\varepsilon} \cdot 1 < 0 \end{aligned}$$

Therefore, player 2 necessarily plays $NI$ when she receives the signal $S$ - i.e., $a_{2,S} = NI$. If player 2 also plays $NI$ when she receives the signal $L$, then player 1's best-reply is necessarily $NI$ regardless of her information. Suppose that $a_{2,L} = I$ - i.e., player 2 plays $I$ when she receives the signal $L$. This is a best-reply if and only if player 2 expects player 1 to play $I$ when she receives the signal $G$ - i.e., $a_{1,G} = I$. Let us check whether this is consistent with best-replying. If player 1 chooses $NI$, she gets a sure payoff of 0. If, on the other hand, $a_{1,G} = I$, player 1's expected payoff (given her signal and her expectation that $a_{2,S} = NI$ and $a_{2,L} = I$) is

$$\begin{aligned} p(g_1 &\mid G) \cdot u_1(I, a_{2,S}, g_1) + p(g_2 \mid G) \cdot u_1(I, a_{2,L}, g_2) \\ &= p(g_1 \mid G) \cdot u_1(I, NI, g_1) + p(g_2 \mid G) \cdot u_1(I, I, g_2) \\ &= \varepsilon \cdot (-2) + (1 - \varepsilon) \cdot 1 \end{aligned}$$

And since we assumed $\varepsilon > \frac{1}{3}$, this expression is negative. Therefore, playing $I$ given the signal $G$ is not a best-reply for player 1. We have thus eliminated the possibility of other equilibria apart from the equilibrium in which players choose $NI$ regardless of their information.

# 4 Examples

In this section, I apply the formalism of Bayesian games and the notion of Nash equilibrium as applied to such games to a number of examples that address a variety of economic interactions: bilateral trade with adverse selection, speculative trade, auctions, voting and investment.

## 4.1 Bilateral Trade with Adverse Selection

Consider the following market situation with adverse selection. A seller enters a market aiming to sell a single object. A buyer enters with the intent to buy the object. The seller's valuation of the object, denoted $\omega$, is drawn uniformly from $\Omega = [0, 1]$ and represents the object's quality. The value of $\omega$ is the seller's private information. The buyer's valuation of the object is $v = \omega + b$, where $b \in (0, 1)$ is a parameter that measures the gains from trade. Since $b > 0$, trade is always efficient, independently of the state. The market mechanism is a simple two-sided auction. The buyer and seller submit a bid $p$ and an ask price $a$, respectively. If $p \geq a$, trade takes place at the price $p$. If $p < a$, trade does not occur.

The seller has a weakly dominant strategy - namely, to submit an ask price equal to her valuation. Let us construct a Nash equilibrium in which the seller follows this strategy - formally, $f(\omega) = \omega$ - and focus on the buyer's considerations. In order to best-reply against the seller's strategy, the buyer needs to choose her bid $p$ to maximize the following expression:

$$\Pr(\omega \leq p) \cdot [E(v \mid \omega \leq p) - p]$$

where $\Pr(\omega \leq p)$ is the probability that trade takes place because it is the probability that the seller's ask price does not exceed the buyer's bid, and $E(v \mid \omega \leq p)$ is the buyer's expected valuation of the object conditional on trade taking place.

Since $v = \omega + b$ and $\omega$ is uniformly distributed over $[0,1]$, $\Pr(\omega \leq p) = p$ and $E(v \mid \omega \leq p) = \frac{1}{2}p + b$. Therefore, the buyer chooses $p$ to maximize $p(b - \frac{1}{2}p)$, yielding a solution $p_r^* = b$. We can see that asymmetric information in this case causes market failure. While the efficient outcome is to have trade in every state, in this market equilibrium trade occurs only when the seller's valuation is below $b$. As $b$ tends to zero, the probability of trade converges to zero.

## 4.2 Speculative Trade: A "No Trade" Theorem

Consider the following situation. Two players decide simultaneously whether to accept the following bet. If company X earns a positive net profit in 2010, player 1 pays \$1 to player 2. If company X does not earn a positive net profit in 2010, player 2 pays \$1 to player 1. The bet is realized if and only if both players accept it. Each player incurs a small transaction fee $\varepsilon > 0$ when she accepts the bet.

Let us impose an arbitrary information structure on this platform. Thus, we have the following Bayesian game. The set of players is $\{1, 2\}$. The set of actions for each player is $\{Y, N\}$, where $Y$ denotes accepting the bet and $N$ denotes rejecting it. The (finite) state space is $\Omega$. Let $p \in \Delta(\Omega)$ be the players' common prior belief. The common-prior assumption will be crucial for our analysis. For each player $i$, $T_i$ denotes the player's set of signals and $\tau_i : \Omega \to T_i$ denotes her signal function.

In order to capture the above betting scenario, we impose the following restrictions on the players' payoff function, for every state $\omega \in \Omega$. First, $u_i(N, a_j, \omega) = 0$ for every $a_j \in \{Y, N\}$. That is, if player $i$ rejects the bet, trade does not take place and the player incurs no transaction fee. Second, $u_i(Y, N, \omega) = -\varepsilon$. That is, if player $j$ rejects the bet while player $i$ accepts it, no trade takes place but player $i$ pays the transaction fee. Finally, $u_i(Y, Y, \omega) \in \{1 - \varepsilon, -1 - \varepsilon\}$, and $u_1(Y, Y, \omega) + u_2(Y, Y, \omega) = -2\varepsilon$. That is, when both players accept the bet, either player 1 wins or player 2 wins, and both players pay the transaction fee.

**Proposition 1** *There is a unique Nash equilibrium in this Bayesian game, in which each player chooses $N$ independently of her information.*

To prove this result, let us first observe that when player $i$ plays $N$ independently of her signal, it is indeed a best-reply for player $j$ to play $N$ regardless of her signal, for the following reason. Player $i$'s action vetoes the bet, and therefore player $j$'s payoff is 0 if she plays $N$ and $-\varepsilon$ if she plays $Y$, regardless of the state of the world. Thus, when both players reject the bet independently of their signal, we have a Nash equilibrium. The question is whether other equilibria exist.

Assume that such an equilibrium exists. Let us fix a strategy profile which constitutes a Nash equilibrium, in which at least one player $i$ plays $Y$ for some signal $t_i \in T_i$. Note that it must be the case that player $j$ also plays $Y$ for some signal $t_j \in T_j$ - otherwise, as we saw above, the only best-reply for player $i$ would be to play $N$ regardless of her signal, in violation of what we assumed.

Define $x_i(\omega)$ as the gross monetary payoff of player $i$ in state $\omega$ (by "gross" I mean that the transaction fee is not taken into account) *given the players' equilibrium strategy*. Crucially, by the definition of the players' utility functions, $x_1(\omega) + x_2(\omega) = 0$ for every $\omega$.

For every signal $t_1$ at which player 1 plays $Y$, it must be the case that the following inequality holds:

$$\sum_{\omega \in \Omega} p(\omega \mid t_1) \cdot x_1(\omega) > 0 \tag{1}$$

That is, the player must receive a strictly positive expected monetary payout conditional on her signal (and given the equilibrium strategies). Otherwise, it wouldn't be optimal for him to play $Y$ given $t_1$.

When player 1 plays $N$ at a signal $t_1$, the following equation holds:

$$\sum_{\omega \in \Omega} p(\omega \mid t_1) \cdot x_1(\omega) = 0 \tag{2}$$

The reason is simple: by playing $N$, player 1 ensures a payoff of 0 in each of the states that are consistent with $t_1$At a signal $t_1$.

By the definition of posterior probabilities, for every $\omega$:

$$\begin{aligned} p(\omega \quad &| \quad t_1) = 0 \text{ if } \tau_1(\omega) \neq t_1 \\ p(\omega \quad &| \quad t_1) = \frac{p(\omega)}{p(t_1)} \text{ if } \tau_1(\omega) = t_1 \end{aligned}$$

where

$$p(t_1) = \sum_{\omega' \in \Omega | \tau_1(\omega') = t_1} p(\omega')$$

is simply the prior probability of the event in which player 1 receives the signal $t_1$. Therefore, we can rewrite

$$\sum_{\omega \in \Omega} p(\omega \mid t_1) \cdot x_1(\omega) = \frac{1}{p(t_1)} \cdot \sum_{\omega \in \Omega | \tau_1(\omega) = t_1} p(\omega) \cdot x_1(\omega)$$

Summing over all signals $t_1 \in T_1$ and using (1) and (2), we obtain:

$$\sum_{t_1 \in T_1} p(t_1) \cdot \sum_{\omega \in \Omega} p(\omega \mid t_1) \cdot x_1(\omega) = \sum_{\omega \in \Omega} p(\omega) \cdot x_1(\omega) > 0$$

That is, player 1's ex-ante gross monetary payout is strictly positive in the putative equilibrium. But by the same analysis, it must be the case that

$$\sum_{\omega \in \Omega} p(\omega) \cdot x_2(\omega) > 0$$

Since $x_2(\omega) = -x_1(\omega)$ for every state $\omega$, these two inequalities are mutually contradictory.

We have thus shown that there exists no Nash equilibrium in which players sometimes accept the bet. This is an example of how speculative trade - i.e., trade motivated purely by differences in beliefs - is inconsistent with Nash equilibrium in Bayesian games with common priors. If we relaxed the common-prior assumption, we could sustain equilibria with speculative trade.

## 4.3 A Common-Value Second-Price Auction

Two players bid for an object, the value of which is $t_1 + t_2$ to *both* players - hence the term "common-value auction". Each player $i$ is informed of $t_i$ only, and believes that $t_j$ is uniformly distributed over $[0, 1]$. The auction format is a second-price auction: that is, the highest-bidding player receives the object and pays the loser's bid. We will not need to worry about ties.

Let us formulate the interaction as a Bayesian game. The set of players is $\{1, 2\}$, and the set of actions for each player is $[0, \infty)$. Let $b_i$ denote an action for player $i$. The set of states is $\Omega = [0, 1]^2$. The common prior is the uniform distribution over $\Omega$. The set of signals for each player $i$ is $T_i = [0, 1]$ and the signal function is $\tau_i(t_1, t_2) = t_i$. The payoff function is as follows. When $b_i > b_j$, player $i$'s payoff in state $(t_1, t_2)$ is $t_1 + t_2 - b_j$ and player $j$'s payoff is 0.

Let us guess a symmetric Nash equilibrium in which each player plays the bidding function $b(t_i) = 2t_i$. That is, each player bids twice her signal. Assuming that player 2 follows this strategy, let us verify that it is indeed optimal for player 1 to follow the same strategy. Suppose that player 1's signal is $t_1$. If player 1 submits the bid $b \leq 2$, she wins when $b > 2t_2$. Since $t_2$ is uniformly distributed over $[0, 1]$, the probability that player 1 wins is $\frac{b}{2}$, the expected value of the object conditional on player 1 winning is $t_1 + E[t_2 \mid b > 2t_2] = t_1 + \frac{b}{4}$, and player 2's expected bid conditional on player 1 winning is $E[2t_2 \mid b > 2t_2] = \frac{b}{2}$. It follows that player 1's expected payoff from the bid $b$ is

$$\frac{b}{2}[(t_1 + \frac{b}{4}) - \frac{b}{2}]$$

The bid $b$ that maximizes this expression is $b = 2t_1$, which is consistent with our guess.

The equilibrium strategy has an interesting feature. If player 1 tried to evaluate the object only on the basis of her private information, she would put it at $t_1 + \frac{1}{2}$, because she knows $t_1$ and uninformed about $t_2$ and believes that in expectation it is $\frac{1}{2}$. However, in equilibrium the player uses her knowledge of player 2's strategy and evaluates the object at $t_1 + \frac{b(t_1)}{4} = t_1 + \frac{t_1}{2}$, which is strictly below $t_1 + \frac{1}{2}$ with probability one. That is, the expected value of the object conditional on winning (and given the player's private information) is lower than its expected value given the player's private information alone. This is an instance of the phenomenon called "the winner's curse": winning a common-value auction is bad news in, because it means that the other bidders received worse signals about the value of the object. Therefore, bidders who play a Nash equilibrium shade their bid to take this effect into account.

Incorporating the winner's curse into equilibrium behavior in common-value auctions reflects great strategic sophistication. What players do, when calculating their best-reply, is consider a hypothetical "pivotal event" - namely, winning the auctions - and draw statistical inferences regarding the unknown variable (the opponent's private information) from this event, taking into account their knowledge of the opponent's equilibrium behavior.

## 4.4 Strategic Voting

The following model shares the ultra-sophisticated type of strategic thinking we highlighted in the previous sub-section, despite the superficially different environment.

Consider a jury consisting of $n$ members. The jury needs to decide whether to convict a defendant. A priori, the probability that the defendant is guilty is $\frac{1}{2}$. Each jury member receives a signal regarding the defendant's innocence. The signal's accuracy is $q > \frac{1}{2}$. That is, if the defendant is guilty, an individual jury member receives a "guilty" signal with probability $q$, independently of the other jurors' signals. Similarly, if the defendant is innocent, the juror receives an "innocent" signal with probability $q$, independently of the other jurors' signals.

After receiving their signals, jurors simultaneously submit a recommendation whether to convict or acquit the defendant. The voting mechanism follows a unanimity rule. That is, the defendant is convicted if and only if all jurors recommend to convict.

Jurors have common preferences: they all want the jury to reach the right decision. Specifically, a juror receives a vNM utility 0 if the jury's decision is correct and a vNM utility of $-1$ if the jury's decision is incorrect. The symmetry in preferences and priors is imposed merely for expositional simplicity, and the student is encouraged to tamper with it as an exercise.

Let us formulate the situation as a Bayesian game. The set of players is $N = \{1, ..., n\}$. The set of actions for each player is $\{f, c\}$, where $f$ and $c$ denote a recommendation to acquit and convict, respectively. The set of states is the set of all sequences $(x, t_1, ..., t_n)$, where $x \in \{G, B\}$ represents the defendant's true culpability ($G$ and $B$ denote guilt and innocence, respectively), and $t_i \in \{G, B\}$ represents the signal that juror $i$'s receives ($G$ and $B$ denote a "guilty" signal and an "innocent" signal, respectively). The prior distribution is as follows. For every $x \in \{G, B\}$ and every subset $A \subseteq N$, the probability that $x$ is the true state of guilt and $A$ is the jurors who receive a signal $t_i = x$ is $\frac{1}{2}q^k(1-q)^{n-k}$, where $k = |A|$. The set of signals for each player $i \in N$ is $\{G, B\}$, and her signal function is $\tau_i(x, t_1, ..., t_n) = t_i$. Player $i$'s payoff function is as follows: $u_i((a_1, ..., a_n), (x, t_1, ..., t_n)) = 0$ if $a_i = f$ for at least one player $i$ and $x = B$, or if $a_i = c$ for all players $i$ and $x = G$. Otherwise, $u_i((a_1, ..., a_n), (x, t_1, ..., t_n)) = -1$.

Consider the case of a single juror: $n = 1$. In this case, we are facing an individual decision problem, rather than a proper game. When the juror receives a "guilty" signal, the posterior probability that the defendant is truly guilty is given by Bayes' rule:

$$\frac{\frac{1}{2}q}{\frac{1}{2}q + \frac{1}{2}(1-q)} = q$$

if the juror chooses to convict the defendant, her expected payoff is $-(1-q)$, while if she chooses to acquit, her expected payoff is $-q$. Since $q > \frac{1}{2}$, the juror will decide to convict. Similarly, when she receives an "innocent" signal, she will choose to acquit. In other words, because the juror's signal is informative and both her prior and her preferences are symmetric, she will always act on her signal.

We will now show that if $n > 2$, the strategy of following one's signal is inconsistent with Nash equilibrium. Assume the contrary - i.e., that in equilibrium, each juror votes to convict if and only if she receives a "guilty" signal. Suppose that juror 1 receives an "innocent" signal. If she recommends to acquit, the defendant will be acquitted because of the unanimity rule. If, on the other hand, she recommends to convict, the defendant will be convicted only if all other jurors vote to convict. To put it differently, juror 1's decision matters for him *only in the event that all other jury members vote to convict*. This is a "pivotal" event, akin to the event of winning the common-value auction in the previous sub-section. Thus, we should only compare the juror's expected utility conditional on this pivotal event.

By assumption, all jurors act on their signal in equilibrium. Thus, conditional on juror 1 receiving an "innocent" signal and jurors $2, ..., n$ recommending to convict the defendant, the posterior probability that the defendant is truly innocent is

$$\frac{\frac{1}{2}q(1-q)^{n-1}}{\frac{1}{2}q(1-q)^{n-1} + \frac{1}{2}q^{n-1}(1-q)} \tag{3}$$

Thus, conditional on the pivotal event, all jurors except juror 1 have received an "innocent" signal. To put it crudely, all jury members except juror 1 think that the defendant is guilty. Either juror 1 is wrong and all the others are right, or juror 1 is right and all the others are wrong. Expression (3) gives the probability that the latter possibility is true. This expression simplifies into

$$\frac{(1-q)^{n-2}}{(1-q)^{n-2} + q^{n-2}}$$

When $n > 2$, this expression falls below $\frac{1}{2}$. Thus, conditional on the pivotal event, the defendant is more likely to be guilty. Therefore, juror 1 prefers to recommend to convict! That is, she doesn't act on her signal. Thus, in equilibrium, it is impossible for all players to act on their signal as they would if they were the sole decision maker.

Note that in this game there is an equilibrium in which all jurors vote to acquit independently of the signal. By the unanimity rule, if a juror expects all other jury members to recommend acquittal, she is indifferent between her two actions regardless of her signal, and thus doesn't mind voting to acquit, too.

### 4.5 The E-mail Game

Consider a variant on the investment game we introduced to illustrate the model of Bayesian games. The set of players is $\{1, 2\}$. The set of actions for each player is $\{I, NI\}$, where $I$ represents investing in a project while $NI$ represents not investing in it. With prior probability $p < \frac{1}{2}$, the project is profitable, leading to the following payoff matrix:

| | $I$ | $NI$ |
|---|---|---|
| $I$ | $1, 1$ | $-1, 0$ |
| $NI$ | $0, -1$ | $0, 0$ |

(Table 3)

With probability $1-p$, the project is unprofitable, leading to the following payoff matrix:

| | $I$ | $NI$ |
|---|---|---|
| $I$ | $-1, -1$ | $-1, 0$ |
| $NI$ | $0, -1$ | $0, 0$ |

(Table 4)

If the project's true profitability is common knowledge, the players have a strict dominant action $NI$ when the project is unprofitable and there are two Nash equilibria when the project is profitable: $(I, I)$ and $(NI, NI)$. Note that when the project is profitable and one player believes that her opponent plays $NI$ with probability greater than $\frac{1}{2}$, she prefers to play $NI$ - in other words, the action $NI$ is "risk dominant". We will rely on this property in the sequel.

Impose the following information structure on this platform. Player 1 knows whether the project is profitable. When she receives information that the project is profitable, an automatic e-mail message is sent to player 2's computer, which sends an automatic confirmation e-mail to player 1's computer, which sends an automatic confirmation of player 2's confirmation back to player 2's computer, which sends an automatic confirmation of player 1's confirmation of player 2's confirmation back to player 1's computer, and so forth. Each e-mail gets lost with an independent probability $\varepsilon > 0$, where $\varepsilon$ is arbitrarily small. Once an e-mail gets lost, the communication is terminated and there are no further confirmations. This technology implies that with probability one, the communication will be terminated after a finite number of rounds. Assume that at the end of the communication, each player sees the number of messages sent from her computer on her computer screen.

Let us write down the information structure formally. A state is a profile of numbers that appear on the screens of the two players' computers, denoted $(K_1, K_2)$. Because of the communication technology, $K_1 = K_2$ or $K_1 = K_2 + 1$.

Thus, the set of states is the set of all pairs $(K_1, K_2)$ for which $K_1, K_2 = 0, 1, 2, ...$ and $K_2 \in \{K_1, K_1 - 1\}$. The pair $(0, 0)$ means that the project is unprofitable. The pair $(1, 0)$ means that the project is profitable but only player 1 knows it, because the e-mail her computer sent to player 2 got lost. The pair $(1, 1)$ means that both players know that the project is profitable, but player 1 does not know that player 2 knows, because player 2's confirmation e-mail got lost. The number $(2, 1)$ means that both players know that the project is profitable, player 1 knows that player 2 knows, but player 2 does not know that player 1 knows that she knows, because player 1's second e-mail got lost.

The prior probability of $(0,0)$ is $p$, whereas the prior probability of any other legitimate $(K_1, K_2)$ is $(1-p)(1-\varepsilon)^{K_1+K_2-1}\varepsilon$. For each player $i$, the set of signals $T_i$ is simply the set of non-negative integers. Given a state $(K_1, K_2)$, $\tau_i(K_1, K_2) = K_i$.

The players' payoff function is as given by Tables 3 and 4. Thus, while the state space is enormous, payoffs only depend on whether the project is profitable. We have the same payoff matrix for all states $(K_1, K_2) \neq (0,0)$. This is an excellent demonstration that in the formalism of Bayesian games, the state space enables us to distinguish between situations that are identical in terms of players' payoffs, but different in terms of the players' high-order beliefs.

**Proposition 2** *The game has a unique Nash equilibrium, in which both players choose $NI$ regardless of their information.*

The proof is by induction on the players' signal. When player 1 receives the signal $K_1 = 0$, she has a dominant action $NI$.

When player 2 receives the signal $K_2 = 0$, she does not know whether $K_1 = 0$ or $K_1 = 1$. The posterior probability of the former possibility is

$$\frac{p}{p+(1-p)\varepsilon} > \frac{1}{2}$$

Therefore, even if player 2 believes in equilibrium that player 1 chooses $I$ at $K_1 = 1$, her expected payoff from taking the action $I$ is $p \cdot (-1) + (1-p) \cdot 1 < 0$, hence $I$ is not a best-reply. And obviously, if she expects player 1 to choose $NI$ at $K_1 = 1$, it is optimal for player 2 to play $NI$. Thus, player 2 chooses $NI$ at $K_2 = 0$.

Now assume that we have proved the claim for all states $(K_1, K_2)$ for which $K_1, K_2 \leq K$. Suppose that $K_1 = K+1$. She does not know whether $K_2 = K$ or $K_2 = K+1$. The posterior probability of the former possibility is

$$\frac{(1-p)(1-\varepsilon)^{2K}\varepsilon}{(1-p)(1-\varepsilon)^{2K}\varepsilon + (1-p)(1-\varepsilon)^{2K+1}\varepsilon} = \frac{1}{2-\varepsilon} > \frac{1}{2}$$

In other words, it is more likely that player 1 didn't get the $(K+1)^{th}$ confirmation because her $K^{th}$ message got lost. Player 1 knows - by the inductive step - that player 2's equilibrium action at $K_2 = K$ is $NI$. Thus, even if she expects player 2 to play $I$ at $K_2 = K+1$, she prefers to play $NI$ because it is more likely that $K_2 = K$. Thus, in equilibrium player 1 will play $NI$ at $K_1 = K+1$. Proving that player 2 plays $NI$ at $K_2 = K+1$ follows the same reasoning. This completes the proof.

Note that when $\varepsilon$ is very small, the players are very likely to have almost common knowledge that the project is profitable whenever it is. Nevertheless, the absence of common knowledge does not enable them to coordinate on the efficient action in equilibrium. This is paradoxical: intuitively, when the project's profitability is mutually known to a high order, it seems intuitive that the players should be able to coordinate on the efficient action.