G004: Time Series Econometrics

Exam for 2014-2015

Answer all questions. Each question carries equal weight (25 points).

Question 1

Consider the following model

$$\begin{aligned} Y_t &= a + bt + \phi Y_{t-1} + \varepsilon_t, \\ \varepsilon_t &\sim i.i.d.N(0,1). \end{aligned} \tag{1}$$

1. (5 points) Give conditions on the parameters that make the model covariance-stationary (briefly explain in words, no need to prove it formally).

   A: $b = 0$ so the model does not have a time trend, $|\phi| < 1$, so the root of the lag polynomial equation is outside the unit circle

2. (5 points) Write down the optimal two-steps-ahead forecast at time T, assuming a quadratic loss function.

   A: $a + b(T+2) + \phi(a + b(T+1)) + \phi^2 Y_T$

3. (5 points) Derive the autocorrelogram of ε_t.

   A: Since ε_t is white noise the autocorrelogram is $\rho_j = 0$ for all j

4. (5 points) Derive the autocorrelogram of Y_t when $b = 0$ and $\phi = .5$.

   A: $\rho_j = .5^j$

5. (5 points) Suppose you consider the residuals u_t from the regression $Y_t = a + bt + u_t$ and you run a regression of u_t on the variable $X_t = X_{t-1} + v_t$, with v_t independent of ε_t :

   $$u_t = \beta X_t + w_t.$$

   Discuss under which conditions is w_t is $I(0)$ or $I(1)$.

A: If $|\phi| < 1$ w_t is $I(0)$ because we are regressing an $I(0)$ variable over an $I(1)$ variable. If $\phi = 1$ we are regressing an $I(1)$ variable over an independent $I(1)$ variable so w_t is $I(1)$

CONTINUED

Question 2

Consider the model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-1} \cdot 1(W_{t-1} > 0) + \phi_3 Y_{t-2} + \varepsilon_t, \tag{2}$$

where $1(\cdot)$ is the indicator function taking value 1 when the argument is true, $\varepsilon_t \sim i.i.d.N(0,1)$ and W_t is an observable variable.

1. (6 points) Find the mean of Y_t, conditional on the information set at time $t-1$.

   A: $E[Y_t|\Omega_{t-1}] = \phi_1 Y_{t-1} + \phi_3 Y_{t-2}$ if $W_{t-1} \le 0$ and $E[Y_t|\Omega_{t-1}] = (\phi_1 + \phi_2) Y_{t-1} + \phi_3 Y_{t-2}$ otherwise

2. (6 points) Give a set of conditions on the parameters so that Y_t is covariance-stationary.

   $A:$ $\phi_2 = 0$, roots of $1 - \phi_1 x - \phi_3 x^2 = 0$ greater than one in absolute value (or $\phi_1 + \phi_3 < 1$)

3. (6 points) Explain how you would estimate the model.

   A: Regress Y_t on Y_{t-1}, $Y_{t-1} \cdot 1(W_{t-1} > 0)$ and Y_{t-2}

4. (7 points) Explain how you would go about forecasting Y_{T+1} and Y_{T+2} at time T.

   A: You can forecast Y_{T+1} one-step ahead as $E[Y_{T+1}|\Omega_T] = \phi_1 Y_T + \phi_2 Y_T \cdot 1(W_T > 0) + \phi_3 Y_{T-1}$ (3 points), but to forecast Y_{T+2} you need a forecast of $1(W_{T+1} > 0)$ which you are unable to forecast without more information about W_t. Since W_t is observable, you can try to specify a time series model for it and use it to forecast future values of the variable (4 points)

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Question 3

Let r_t^S be the short-term interest rate and r_t^L the long term interest rate, which are $I(1)$ processes. Suppose that they can be modelled as

$$\begin{aligned} \Delta r_t^S &= \alpha_1(r_{t-1}^L - \beta r_{t-1}^S) + \varepsilon_{1t} \\ \Delta r_t^L &= -\alpha_2(r_{t-1}^L - \beta r_{t-1}^S) + \varepsilon_{2t}, \end{aligned} \tag{3}$$

with ε_{1t} and ε_{2t} uncorrelated white noise processes.

1. (5 points) What is the long-run equilibrium relationship between r_t^S and r_t^L implied by the model?

   A: $r_t^L = \beta r_t^S$

2. (5 points) Consider a regression of r_t^L on r_t^S. What are the time series properties of the regression errors?

   A: They are $I(0)$ because the variables are cointegrated

3. (5 points) Consider a regression of r_t^L on r_t^S using a sample of size T. As $T \to \infty$, what does the OLS estimator converge to in probability and what is the rate of convergence?

   A: It converges to β (3 points) and at rate T (3 points) (superconsistent)

4. (5 points) Suppose $r_{t-1}^L - \beta r_{t-1}^S > 0$ and consider how the variables in the system will respond to this at time t. Under which conditions on the parameters would the short-term rate rise and the long-term rate fall?

   A: α_1 and $\alpha_2 > 0$

5. (5 points) What kind of restrictions does the model (3) impose on the coefficients of a VAR for the vector $\begin{pmatrix} r_t^S \\ r_t^L \end{pmatrix}$?

   A: It implies the model for $\begin{pmatrix} r_t^S \\ r_t^L \end{pmatrix}$ is a VAR(1) with coefficient matrix $\begin{pmatrix} 1-\alpha_1\beta & \alpha_1 \\ \alpha_2\beta & 1-\alpha_2 \end{pmatrix}$

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Question 4

Consider a SVAR(1) for $Y_t = [GDP_t, p_t, i_t, M_t]$ where the components represent, in order, output, prices, interest rates and money:

$$A_0 Y_t = A_1 Y_{t-1} + u_t,\ \ E u_t u_t' = \Omega_u \tag{4}$$

and consider the corresponding reduced-form VAR(1)

$$Y_t = \Phi Y_{t-1} + \varepsilon_t,\ \ E\varepsilon_t \varepsilon_t' = \Omega_\varepsilon. \tag{5}$$

1. (5 points) Suppose you have quarterly data and that you can plausibly assume that it takes longer than one quarter for output to react to shocks in the remaining variables. Write down what restrictions this implies for the structural coefficients.

   A: $A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & b & c \\ d & e & 1 & f \\ g & h & i & 1 \end{bmatrix}$

2. (5 points) How many more restrictions besides those considered in questions 4.1 do you need to identify the parameters of the SVAR?

   A: We need $n^2 - n = 12$ restrictions and we have 3, so we need 9 more (assuming Ω_u diagonal gives $n(n-1)/2 = 6$ restrictions so if that's already assumed we need 3 more).

3. (5 points) What is the one-step-ahead forecast of GDP_{T+1} at time T implied by the SVAR model?

   A: $E\left[GDP_{T+1}|\Omega_T\right] = A_{1,11}GDP_T + A_{1,12}p_T + A_{1,13}i_T + A_{1,14}M_T$

4. (5 points) In addition to the restrictions in question 4.1, suppose p_t, i_t and M_t do not Granger-cause GDP_t. Write down the model for GDP_t when imposing these additional restrictions.

   A: $GDP_t = A_{1,11}GDP_{t-1} + u_{1t}$ (or AR(1) model)

5. (5 points) Suppose you consider the squared residuals u_{1t}^2 from the model in question 4.4 and find that they follow an ARMA(2,1) model. Write down how you would modify the model for GDP in question 4.4 to account for this finding.

   A: An ARMA(2,1) for u_{1t}^2 implies a GARCH(2,1) model for GDP : $GDP_t = A_{1,11}GDP_{t-1} + u_{1t}$ with $\sigma_t^2 = E\left(u_{1t}^2|\Omega_{t-1}\right)$ and $\sigma_t^2 = w + \alpha u_{1t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2$

END OF PAPER

G004: Time Series Econometrics

Exam for 2015-2016

Answer all questions. Each question carries equal weight (25 points).

Question 1

Consider the regression

$$Y_t = \beta' X_t + u_t,\ t = 1, ..., T, \tag{1}$$

with u_t independent of X_t.

1. (5 points) Suppose u_t and X_t are i.i.d. Write down a consistent estimator of the asymptotic variance of the OLS estimator of β.

   A:Since $X_t u_t$ is iid, it's the usual estimator under homoskedasticity (because i.i.d. implies homoskedasticity): $\widehat{s}^2 \left(\sum X_t X_t'\right)^{-1}$ where $\widehat{s}$ is the sample variance of the OLS residuals

2. (6 points) Suppose $X_t u_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where ε_t is $i.i.d.(0, \sigma^2)$. Write down a consistent estimator of the asymptotic variance of the OLS estimator of β.

   A: HAC estimator with only one autocovariance term $\left(\frac{1}{T}\sum X_t X_t'\right)^{-1} \left(\frac{1}{T}\sum X_t X_t' \widehat{u}_t^2 + 2w_1\gamma_1\right) \left(\frac{1}{T}\sum\right.$ where $\gamma_1 = cov(X_t u_t, X_{t-1}u_{t-1}) = \theta\sigma^2$ and w_1 is a weight, such as the Newey-West weight (ok without weight)

3. (7 points) Suppose $correlation(X_t u_t, X_{t-j}u_{t-j}) = \phi^j$. Write down a time series model for $X_t u_t$ that is compatible with this assumption and discuss under which conditions (if they exist) the OLS estimator of β is consistent and asymptotically normal..

   A: $X_t u_t$ is an AR(1) with mean zero, $X_t u_t = \phi X_{t-1}u_{t-1} + \varepsilon_t$. If $|\phi| < 1$ there is a CLT and LLN that guarantee that the OLS estimator of β is consistent and asymptotically normal

4. (7 points) Suppose $covariance(X_t u_t, X_{t-j} u_{t-j}) = variance(X_{t-j} u_{t-j})$. Write down a time series model for $X_t u_t$ that is compatible with this assumption and discuss under which conditions (if they exist) the OLS estimator of β is consistent and asymptotically normal.

   A: This implies a random walk for $X_t u_t$ and thus the OLS estimator of β is super-consistent but not asymptotically normal

CONTINUED

Question 2

Consider the model

$$Y_t = c_1 + c_2 \cdot 1\left(W_{t-1} > 0\right) + \phi_1 Y_{t-1} + \phi_2 Y_{t-1} \cdot 1\left(W_{t-1} > 0\right) + \varepsilon_t, \tag{2}$$

where $1\left(\cdot\right)$ is the indicator function taking value 1 when the argument is true, $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$ and independent of Y_{t-1} and W_t is an observable variable.

1. (6 points) Find the mean of Y_t, conditional on the information set at time $t-1$.

   A: $E\left[Y_t | \Omega_{t-1}\right] = c_1 + \phi_1 Y_{t-1}$ if $W_{t-1} \leq 0$ and $E\left[Y_t | \Omega_{t-1}\right] = c_1 + c_2 + \left(\phi_1 + \phi_2\right) Y_{t-1}$ otherwise

2. (6 points) Find conditions on the parameters so that Y_t is covariance-stationary.

   $A: \;\; c_2 = \phi_2 = 0, \; |\phi_1| < 1$

3. (6 points) Explain how you would estimate the model.

   A: Regress Y_t on vector of 1, $1\left(W_{t-1} > 0\right), Y_{t-1}$ and $Y_{t-1} \cdot 1\left(W_{t-1} > 0\right)$

4. (7 points) Suppose $c_2 = \phi_2 = 0$. Explain how you would test for a break in the unconditional mean of Y_t at a known time T_0.

   A: $E\left[Y_t\right] = c_1/(1 - \phi_1)$ so must do a Chow's test of hypothesis that both coefficients changed at T_0, which can be implemented by creating a time dummy with 1 at T_0, substitute it for the dummy in (2) and then doing an F-test of $H_0 : c_2 = \phi_2 = 0$ (with homoskedastic standard errors because of assumptions) and compare it to χ^2_2

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Question 3

Consider the model

$$
\begin{aligned}
Y_t &= \varepsilon_t = \sigma_t z_t, &(3)\\
\sigma_t^2 &= w + \varepsilon_{t-1}^2 &(4)
\end{aligned}
$$

where $z_t \sim i.i.d.N(0,1)$.

1. (6 points) Show that the process is not stationary.

   A: This implies that the process for ε_t^2 has a unit root so it's not stationary

2. (6 points) Explain how you would test for non-stationarity.

   A: Dickey Fuller test of $\phi = 1$ in $Y_t^2 = w + \phi Y_{t-1}^2 + v_t$

3. (6 points) Derive the loglikelihood.

   A: $L(w) = -\frac{1}{2}\sum \frac{Y_t^2}{\sigma_t^2} - -\frac{1}{2}\sum \log \sigma_t^2$

4. (7 points) Explain how you would use model (3) to produce a one-step-ahead density forecast of Y_t (i.e., the density of Y_{t+1} conditional on the information set at time t).

   A: It is $N(0, \sigma_{t+1}^2)$, that is, $N(0, w + Y_t^2)$

CONTINUED

Question 4

Consider a SVAR(1) for $Y_t = [GDP_t, p_t, i_t]$ where the components represent, in order, output, prices and interest rates:

$$A_0 Y_t = A_1 Y_{t-1} + u_t, \ \ E u_t u_t' = \Omega_u \tag{5}$$

and consider the corresponding reduced-form VAR(1)

$$Y_t = \Phi Y_{t-1} + \varepsilon_t, \ \ E \varepsilon_t \varepsilon_t' = \Omega_\varepsilon. \tag{6}$$

1. (6 points) Find (sufficient) conditions on a matrix R such that u_t and its transformation $v_t = u_t R$ give rise to the same Ω_ε.

   A: Any orthonormal matrix $RR' = I$ will be such that $\Omega_\varepsilon = E\varepsilon_t\varepsilon_t' = A_0^{-1} E u_t u_t' A_0^{-1'} = A_0^{-1} E v_t v_t' A_0^{-1'}$

2. (6 points) How many restrictions do you need to identify the parameters of the SVAR?

   A: We need $n^2 - n = 6$

3. (6 points) Suppose you know that A_1 has all elements equal to 0. How many restrictions do you need to identify the parameters of the SVAR?

   A: We still need $n^2 - n = 6$ restrictions to identify elements of A_0 and Ω_u knowing only Ω_ε

4. (7 points) Write down an example of economically plausible set of restrictions that would give you identification of the SVAR parameters.

   A: E.g., causal ordering restrictions

END OF PAPER

G004: Time Series Econometrics

Exam for 2016-2017

Answer all questions. Each question carries equal weight (25 points).

Question 1

Consider the model

$$Y_t = \phi Y_{t-1} + X_t + \varepsilon_t, \ t = 1, ..., T, \tag{1}$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$.

1. (5 points) Find the conditional expectation of Y_{t+1} at time t, assuming you observe time series for both Y_t and X_t. Is the information provided enough to forecast Y_{t+1} at time t?

   A: $E\left[Y_{t+1}|Y_t, Y_{t-1}, ..., X_t, X_{t-1}, ...\right] = \phi Y_t + E\left[X_{t+1}|Y_t, Y_{t-1}, ..., X_t, X_{t-1}, ...\right]$ so it's not enough to observe past $X_t's$ as we would need to know the time series process for X_t to forecast.

2. (6 points) Suppose X_t is a white noise process, independent of ε_t. Find the resulting time series process for Y_t.

   A: $(1 - \phi L)Y_t = \varepsilon_t + X_t$ is an AR(1) because $\varepsilon_t + X_t$ is still white noise

3. (7 points) Suppose X_t is a random walk. Find the roots of the autoregressive lag polynomial equation for Y_t..

   A: $(1 - L) X_t = v_t$ so $(1 - \phi L)(1 - L) Y_t = v_t + (1 - L) \varepsilon_t$ so the roots are 1 and $1/\phi$.

4. (7 points) Suppose X_t is a random walk and that $\phi = 0$. Show that X_t and Y_t are cointegrated and write down the corresponding Error Correction Model.

   A: As we saw in point 3, Y_t has a unit root even when $\phi = 0$, so both X_t and Y_t have a unit root. Further we have $Y_t - X_t = \varepsilon_t$ which is white noise so (Y_t, X_t) are cointegrated with cointegrating vector $\beta = (1, -1)$ and the ECM is

$$\begin{bmatrix} \Delta Y_t \\ \Delta X_t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} [1 \quad -1] \begin{bmatrix} Y_{t-1} \\ X_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t + v_t \\ v_t \end{bmatrix}$$

CONTINUED

Question 2

Consider the model

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \underbrace{\sigma_t z_t}_{\varepsilon_t},\ t = 1, ..., T, \quad (2)$$

$$\sigma_t^2 = \begin{cases} w_1 + \phi_1 \varepsilon_{t-1}^2 & if\ W_{t-1} > 0 \\ w_2 + \phi_2 \varepsilon_{t-1}^2 & if\ W_{t-1} \leq 0 \end{cases}, \quad (3)$$

where $z_t \sim i.i.d.N(0,1)$ and W_t is an observable variable.

1. (6 points) Find the unconditional mean of Y_t

   A: $E[Y_t] = \frac{c}{1-\phi_1-\phi_2}$

2. (6 points) What is the model for ε_t^2?

   A: It is a threshold autoregression $\varepsilon_t^2 = \begin{cases} w_1 + \phi_1 \varepsilon_{t-1}^2 + v_t & if\ W_{t-1} > 0 \\ w_2 + \phi_2 \varepsilon_{t-1}^2 + v_t & if\ W_{t-1} \leq 0 \end{cases}$, where $v_t = \varepsilon_t^2 - \sigma_t^2$ is white noise since $E[v_t|\Omega_{t-1}] = 0$ (sufficient to show this condition)

3. (6 points) Derive the loglikelihood (it helps to use dummy variables).

   A: $L = -\frac{1}{2}\sum_{t=3}^{T} \frac{(Y_t - c - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2}{\sigma_t^2} - \frac{1}{2}\sum_{t=3}^{T} \log \sigma_t^2$ where

   $\sigma_t^2 = (w_1 + \phi_1 \varepsilon_{t-1}^2) 1 \left(W_{t-1} > 0\right) + (w_2 + \phi_2 \varepsilon_{t-1}^2) 1 \left(W_{t-1} \leq 0\right)$

4. (7 points) Explain how you would use model (2) to produce a one-step-ahead density forecast of Y_t (i.e., the density of Y_{t+1} conditional on the information set at time t).

   A: Since W_t is observable, the density forecast is $N(\phi_1 Y_t + \phi_2 Y_{t-1}, w_1 + \phi_1 (Y_t - c - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2)$ if $W_t > 0$ and $N(\phi_1 Y_t + \phi_2 Y_{t-1}, w_2 + \phi_2 (Y_t - c - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2)$ otherwise (fine to use ε_t^2 in the expression of the variance of the density forecast)

CONTINUED

Question 3

Consider a SVAR(0) for the 2×1 vector Y_t :

$$A_0 Y_t = \varepsilon_t, \ t = 1, ..., T, \tag{4}$$

where A_0 is invertible and $\varepsilon_t \sim i.i.d.N(0, I_2)$, with I_2 the identity matrix.

1. (6 points) Write down the impulse response function (the response of all variables to all shocks) for horizons $h = 0, 1, 2$, as a function of the model's parameters.

   A: The responses for $h = 0$ are the elements of A_0^{-1} and they equal zero for $h > 0$

2. (6 points) Show that Y_t is covariance stationary.

   A: Since ε_t is a vector white noise and $Y_t = A_0^{-1}\varepsilon_t$ we have $E[Y_t] = 0$, $Var[Y_t] = A_0^{-1}A_0^{-1\prime}$ and $cov(Y_t, Y_{t-j}) = 0$, which do not depend on time

3. (6 points) Briefly explain why A_0 is not identified. Then give an example of assumption(s) that would identify A_0 and explain in words its/their meaning.

   A: Because knowledge of $\Omega = var(Y_t)$ (3 elements) is not enough to pin down A_0 (4 elements). You thus need one identifying restriction, for example A_0 upper triangular, meaning that Y_1 does not react contemporaneously to ε_2

4. (7 points) Under the identifying restriction(s) you chose in question 3.3, explain how you would estimate the impulse response function for $Y_{2,t+h}$ with respect to a unit shock in ε_{1t}, for $h = 0, 1, 2$.

   A: Under the upper triangular identifying restrictions the impulse response for $h = 0$ is the (2,1) element of A_0^{-1}, where $A_0^{-1}$is estimated as the matrix B in the Cholesky decomposition $\Omega = var(Y_t) = BB'$ with B upper triangular. The impulse response is zero for $h > 0$

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Question 4

Consider the model

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t, \;\; t = 1, ..., T, \tag{5}$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$.

1. (6 points) Find the optimal 2-step ahead forecast for a quadratic loss.

   A: $E\left[Y_{t+2}|\Omega_t\right] = c + \phi E\left[Y_{t+1}|\Omega_t\right] = c + \phi(c + \phi Y_t) = (1+\phi)c + \phi^2 Y_t$

2. (6 points) What is the time-series process for the forecast error associated with the forecast you derived in question 4.1?

   A: Optimal $h - step$ ahead forecast errors are MA(h-1) so it's an MA(1) (don't need to show work)

3. (6 points) Suppose $c = 0$. What is the optimal $h-$step ahead forecast for $h \to \infty$?

   A: It equals $\lim_{h\to\infty} \phi^h Y_t = 0$ (the unconditional mean) when $|\phi| < 1$ and Y_t when $\phi = 1$

4. (7 points) Suppose you have a sample of size T and $c = k$ for $t \leq T/2$ and $c = -k$ for $t > T/2$, with $|\phi| < 1$. What do you think would happen if you tested the hypothesis $H_0 : \alpha = 0$ in the regression $Y_t = \alpha + error$ (it is ok to explain the reasoning in words)?

   A: α is estimated by the sample average of Y_t, and the true mean is $\frac{k}{1-\phi}$ for the first half of the sample and $\frac{-k}{1-\phi}$ for the second half, so the two cancel out and thus likely give estimates of α close to zero and failure to reject the null

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G004: Time Series Econometrics

Exam for 2017-2018

Answer all questions. Each question carries equal weight (25 points).

Question 1

Consider the model

$$y_t = \phi y_{t-1} + \varepsilon_t,\ t = 1, ..., T,$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$.

1. (5 points) Derive the autocorrelogram and the partial autocorrelogram of y_t as functions of the model's parameters.

   A: $\rho_j = \phi^j$; $\beta_1 = \phi$, $\beta_i = 0$ for $i > 1$

2. (6 points) Derive the mean $E(y_t)$, variance $Var(y_t)$ and first order autocovariance $cov(y_t, y_{t-1})$ when $\phi = 1$ and assuming $y_0 = 0$.

   A: $E(y_t) = E\left(\sum_{j=1}^{t} \varepsilon_j\right) = 0, Var\left(\sum_{j=1}^{t} \varepsilon_j\right) = t\sigma^2, cov\left(\sum_{j=1}^{t} \varepsilon_j, \sum_{j=1}^{t-1} \varepsilon_j\right) = (t-1)\sigma^2$

3. (7 points) Derive the conditions under which y_t can be written as an $MA(\infty)$ and derive the $MA(\infty)$ representation

   A: $|\phi| < 1$ $y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_j$

4. (7 points) Consider the regression

   $$y_t = \beta x_t + u_t$$

   where $x_t = x_{t-1} + v_t, v_t \sim i.i.d.N(0, \sigma_v^2)$ and v_t independent of ε_s for all t.and s. Derive the true value of β when $\phi = 1$ and assuming $y_0 = x_0 = 0$. (Hint: use the fact that $\beta = cov(y_t, x_t)/var(x_t)$)

   A: $y_t = \sum_{j=1}^{t} \varepsilon_j, x_t = \sum_{j=1}^{t} v_j$, so $\beta = cov(y_t, x_t)/var(x_t) = cov(\sum_{j=1}^{t} \varepsilon_j, \sum_{j=1}^{t} v_j)/var(\sum_{j=1}^{t} v_j$
   0

CONTINUED

Question 2

Consider the model

$$\begin{aligned} y_t &= \varepsilon_t = \sigma_t z_t, \\ \sigma_t^2 &= w + \alpha \varepsilon_{t-1}^2 \end{aligned}$$

with $z_t \sim i.i.d.N(0,1)$.

1. (6 points) Derive the conditional mean of y_t, $E[y_t|\Omega_{t-1}]$, and the unconditional mean $E[y_t]$ (show your work)

   A: $E[y_t|\Omega_{t-1}] = E[\sigma_t z_t|\Omega_{t-1}] = \sigma_t E[z_t|\Omega_{t-1}] = 0$ because of i.i.d. assumption, which implies $E[y_t] = 0$

2. (6 points) Explain how you can estimate the model by OLS

   A: Since the model implies that ε_t^2 is an AR(1) $\varepsilon_t^2 = w + \alpha\varepsilon_{t-1}^2 + v_t$ and in this case $\varepsilon_t = y_t$ is observable, can regress y_t^2 on an intercept and on y_{t-1}^2

3. (6 points) Find the autocorrelogram of y_t^2 when $|\alpha| < 1$

   A: Since the model implies a stationary AR(1) for y_t^2 the autocorrelogram is $\rho_j = \alpha^j$

4. (7 points) Find the first order autocorrelation of y_t, $corr(y_t, y_{t-1})$ (Hint: use the law of iterated expectations - i.e., for a variable x_t, $E[E[x_t|\Omega_{t-1}]] = E[x_t]$)

   A: $cov(y_t, y_{t-1}) = E[y_t y_{t-1}] = E[E[y_t y_{t-1}|\Omega_{t-1}]] = E[y_{t-1} E[y_t|\Omega_{t-1}]] = 0$ because $E[y_t|\Omega_{t-1}] = 0$ as shown above

CONTINUED

Question 3

Consider the SVAR model

$$A_0 y_t = A_1 y_{t-1} + u_t,$$

where y_t is a 2×1 vector with y_{1t} denoting GDP growth and y_{2t} denoting interest rate.

1. (6 points) How many restrictions do you need to identify the parameters?

   A: $n^2 - n = 2$

2. (6 points) Write down the system of equations that one can solve to find the structural parameters from the estimated reduced-form parameters (subject to identifying restrictions)

   A: $A_0^{-1}\Omega_u A_0^{-1\prime} = \Omega_\varepsilon$; $A_1 = A_0\Phi$

3. (6 points) Suppose $A_0 = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. Explain what the restriction imposed on A_0 implies regarding the contemporaneous relationship between the two variables

   A: GDP not affected by r_t but r_t affected by DGP

4. (7 points) Suppose $A_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} .8 & 0 \\ 0 & .8 \end{bmatrix}$. Derive the impulse response functions capturing the reaction of GDP growth to GDP shocks and to interest rate shocks

   A: $y_{1t} = .8y_{1t-1} + u_{1t}$ so it does not react to interest rate shocks. The reaction to GDP shocks at horizon h is$.8^h$ (from the MA(∞) representation)

CONTINUED

Question 4

Consider the model

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta\varepsilon_{t-1},$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$

1. (6 points) Find conditions under which the model is covariance stationary

   A: $|\phi_1 + \phi_2| < 1$ (ok to say roots of $1 - \phi_1 x - \phi_2 x^2 = 0$ outside unit circle)

2. (6 points) Find the two-step ahead forecast implied by the model (for a quadratic loss)

   A: $y_{t,2} = (\phi_1^2 + \phi_2)y_t + \phi_1\phi_2 y_{t-1} + \phi_1\theta\varepsilon_t$

3. (6 points) Find the variance of the two-step ahead forecast error

   A: $e_{t,2} = \varepsilon_{t+2} + (\theta + \phi_1)\varepsilon_{t+1}$ so variance is $\left(1 + (\theta + \phi_1)^2\right)\sigma^2$

4. (7 points) Suppose you have a sequence of out-of-sample two-step ahead forecast errors from this model, $\{e_{t,2}\}$.First, find the length of this sequence assuming that the total sample size is T and you used the first m observations for estimation. Second, explain how you would test forecast unbiasedness, stating in particular which standard errors you would use.

   A: It is of length $T-m-1$ (forecasts run from $m+2$ to T). To test unbiasedness regress $e_{t,2}$ on a constant and test that it is zero using a HAC estimator with 1 lag (because $e_{t,2}$ is MA(1) under the null)

END OF PAPER

ECON0058: Time Series Econometrics

Exam for 2018-2019

Answer all questions. Each question carries equal weight (25 points).

Question 1

Consider the model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t + \varepsilon_{t-1},\ t = 1, ..., T, \tag{1}$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$.

1. (5 points) Find $E[Y_t]$, $E[Y_t|\Omega_{t-1}]$ and $Var[Y_t|\Omega_{t-1}]$.

   A: $E[Y_t]$=0, $E[Y_t|\Omega_{t-1}] = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_{t-1}$ and $Var[Y_t|\Omega_{t-1}] = \sigma^2$.

2. (6 points) Discuss when the model is stationary and when it is invertible.

   A: It is stationary when roots of $1 - \phi_1 x - \phi_2 x^2$ are outside unit circle. It is not invertible because the MA polynomial $1 - x$ has a unit root.

3. (7 points) Suppose $X_t = -Y_t + v_t$, with $v_t \sim i.i.d.N(0, \sigma_v^2)$, and $Correlation(\varepsilon_t, v_t) = \rho$. First write down the conditions on the parameters so that Y_t and X_t are cointegrated and then write down the cointegrating vector.

   A: Both must have a unit root, so ϕ_1 and ϕ_2 must be such that the roots of $1 - \phi_1 x - \phi_2 x^2$ are outside the unit circle. Then X_t also has a unit root because it's the sum of a unit root and a stationary process so it has a unit root. The two variables are then cointegrated because $X_t + Y_t = v_t$ which is stationary. The value of ρ does not matter.

4. (7 points) Suppose $\phi_2 = 0$. Find the response of Y_{t+2} to a one-unit shock in ε_t.

   A: $Y_{t+2} = \phi_1 Y_{t+1} + \varepsilon_{t+2} + \varepsilon_{t+1} = \phi_1(\phi_1 Y_t + \varepsilon_{t+1} + \varepsilon_t) + \varepsilon_{t+2} + \varepsilon_{t+1}$
   $= \phi_1^2(\phi_1 Y_{t-1} + \varepsilon_t + \varepsilon_{t-1}) + \phi_1 \varepsilon_t + \varepsilon_{t+2} + (1 + \phi_1)\varepsilon_{t+1}$ so response is $\phi_1^2 + \phi_1$

CONTINUED

Question 2

Consider the model

$$Y_t = c + \rho Y_{t-1} + \underbrace{\sigma_t z_t}_{\varepsilon_t}, \; t = 1, ..., T, \tag{2}$$

$$\sigma_t^2 = \begin{cases} w_1 + \phi_1 \varepsilon_{t-1}^2 & if \; W_{t-1} > 0 \\ w_2 + \phi_2 \varepsilon_{t-1}^2 & if \; W_{t-1} \leq 0 \end{cases}, \tag{3}$$

where $z_t \sim i.i.d.N(0,1)$ and W_t is an observable variable.

1. (5 points) Derive the mean, variance and autocorrelation function of the standardized residuals, $\frac{Y_t - E[Y_t|\Omega_{t-1}]}{Var[Y_t|\Omega_{t-1}]}$.

   A: This questions contained a typo, as the square root is missing from the denominator of the standardized residual. Everyone was thus awarded 5 points for this question, whether they answered or not.

2. (6 points) Provide sufficient conditions for stationarity of Y_t.

   A: $|\rho| < 1$, $w_1 = w_2$, $\phi_1 = \phi_2 = \phi$ and $|\phi| < 1$

3. (7 points) Give a sufficient condition on the parameters c and ρ so that you can estimate the model by OLS, and explain how you would do so.

   A: if $c = \rho = 0$ you have $\epsilon_t = Y_t$ so you could estimate the model by OLS using dummy variables: $Y_t^2 = (w_1 + \phi_1 Y_{t-1}^2) 1 \left(W_{t-1} > 0\right) + (w_2 + \phi_2 Y_{t-1}^2) 1 \left(W_{t-1} \leq 0\right) + u_t$

4. (7 points) Derive the unconditional variance of Y_t under the assumptions: 1) stationarity; 2) W_t has a symmetric distribution (i.e., the probability that $W_t > 0$ is the same as the probability that $W_t < 0$); 3) W_{t-1} is uncorrelated with ϵ_{t-1}^2. (Hint: use the fact that, for a generic random variable X and a constant c, the probability that $X > c$ equals $E[1(X > c)]$, where 1 is the indicator function.)

   A: It is $\sigma^2/(1-\rho^2)$, where $\sigma^2 = E[\varepsilon_t^2] = E[(w_1 + \phi_1 \varepsilon_{t-1}^2) 1(W_{t-1} > 0) + (w_2 + \phi_2 \varepsilon_{t-1}^2) 1(W_{t-1} \leq 0)] = E[w_1 + \phi_1 \varepsilon_{t-1}^2] E[1(W_{t-1} > 0)] + E[w_2 + \phi_2 \varepsilon_{t-1}^2] E[1(W_{t-1} \leq 0)] = (w_1 + \phi_1 \sigma^2) 1/2 + (w_2 + \phi_2 \sigma^2)] 1/2$. Collecting terms, this gives

$$\sigma^2 = \frac{.5(w_1 + w_2)}{1 - .5\phi_1 - .5\phi_2} \tag{4}$$

CONTINUED

Question 3

Consider the model

$$Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \epsilon_{t-1},\ t = 1, ..., T, \tag{5}$$

with $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$.

1. (5 points) Find the optimal 2-step ahead forecast for a quadratic loss.

   A: $E[Y_{t+2}|\Omega_t] = \phi E[Y_{t+1}|\Omega_t] = \phi(\phi Y_t + \theta \epsilon_t) = \phi^2 Y_t + \phi\theta\epsilon_t$

2. (6 points) Find the variance of the optimal 2-step ahead forecast error.

   A: The error is $\epsilon_{t+2} + (\phi + \theta)\epsilon_{t+1}$ so its variance is $\sigma^2(1 + (\phi + \theta)^2)$

3. (5 points) Suppose $\theta = 0$. What is the optimal $h-$step ahead forecast for $h \to \infty$?

   A: It equals $\lim_{h\to\infty} \phi^h Y_t = 0$ (the unconditional mean) when $|\phi| < 1$ and Y_t when $\phi = 1$

4. (8 points) Briefly explain why it is not accurate to perform inference on ϕ by regressing Y_t on Y_{t-1} and using a HAC estimator of the asymptotic variance. Then briefly explain what is the proper way to conduct inference on ϕ.

   A: Because the residuals are correlated with the regressor (i.e., Y_{t-1} is correlated with ϵ_{t-1}) so the estimator of ϕ is inconsistent. The correct way is to estimate the model by ML and do standard inference in models estimated by ML.

CONTINUED

Question 4

Suppose $X_t = \Delta Z_t$, where $Z_t = .5Z_{t-1} + .5Z_{t-2} + \epsilon_t$ and $Y_t = \phi Y_{t-1} + v_t$, where ε_t and v_t are i.i.d.N(0,σ^2) and ϵ_t, v_t are uncorrelated with each other.

1. (6 points) Suppose you estimate the regression $Y_t = \beta X_t + u_t$. Explain how you would perform inference about β (depending on the value of ϕ).

   A: If $|\phi| < 1$ this is a standard regression involving two stationary variables, so perform standard inference using a HAC estimator. This should indicate that $\beta = 0$. If $\phi = 1$ the regression is unbalanced and the unit root in Y_t ends up in the residuals, so inference becomes non-standard

2. (6 points) Show that X_t is invertible.

   A: The model for X_t is $(1 + .5L)X_t = \epsilon_t$ and the root of $1 + .5L$ is greater than one in absolute value, so we have invertibility

3. (6 points) Write down the VAR model for (X_t,Y_t) and show under which conditions it is stationary.

   A: Since $X_t = -.5X_{t-1} + \epsilon_t$ it's a VAR(1) with diagonal coefficient matrix with elements -.5 and ϕ, so it is stationary if $|\phi| < 1$

4. (7 points) Use the VAR model your wrote down in question 4.3 to derive the impulse response functions of X_{t+h} with respect to shocks in Y_t and for Y_{t+h} with respect to shocks in X_t, for $h = 0, 1, 2, ...$

   A: This only makes sense if the VAR is stationary. The IRFs are all zero, because we have assumed that ϵ_t, v_t are uncorrelated with each other.

END OF PAPER

ECON0058

Exam for 2019-2020

Answer all questions. EXPLAIN YOUR DERIVATIONS AND REASONING. Each question carries equal weight (25 points).

Question 1

Supposed you have downloaded a time series for Morgan Stanley daily stock prices, MS_t, and a time series for the daily SP500 index, SP_t, over the same time period $t = 1, ..., T$. Consider the regression:

$$MS_t = \beta_1 + \beta_2 SP_t + u_t = \beta' x_t + u_t, \text{t=1,...,T}, \tag{1}$$

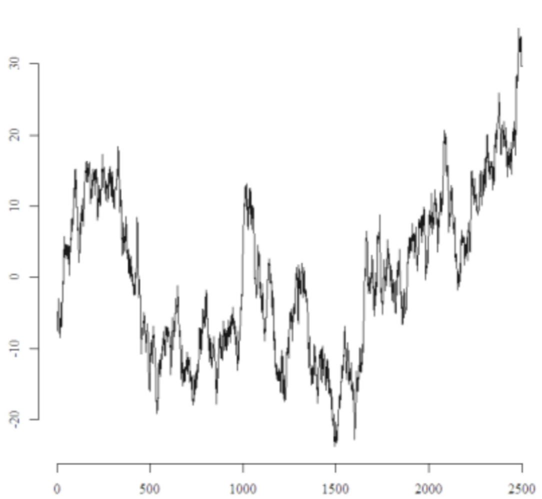
where x_t is a 2x1 vector with elements 1 and SP_t. Assume that $E[x_t u_t] = 0$.

1. (6 points) Suppose you obtain OLS estimates $\hat{\beta}_1 = -46$ and $\hat{\beta}_2 = .8$, with HAC standard errors respectively equal to 2.18 and 0.01. Discuss how/whether you would use this regression output to test the hypothesis $H_0 : \beta_2 = 1$, and describe any other analysis you would carry out to support your decision.

   A: Test for a unit root on each stock price. If reject both, this means that both dependent and independent variables are I(0). Together with the assumption $E[x_t u_t] = 0$ this implies that you can use the regression output to do a standard t-test using critical values from the standard normal. The test would reject because t-stat=0.2/0.01 >1.96. HAC standard errors are valid regardless of whether there is serial correlation in the regression residuals. If you find a unit root, inference is non-standard.

   3 points for one mistake (e.g., does not say to do unit root tests but otherwise everything else, including the test, is correct, or if only the test is incorrect). 0 points for more than 1 mistake (including saying something wrong even if the answer is correct)

2. (6 points) Answer question 1.1 again, in light of the fact that the time series of estimated residuals from regression (1), $\hat{u}_t$, looks as follows:



A: We suspect the presence of a unit root in the residuals (the time series has a visible trend, which is a violation of cov-stationarity), so we should perform a unit root test on this series to verify. If the test fails to reject, and the unit root tests you performed in the previous question indicate that the dependent and independent variables are I(1), this signals a spurious regression (and you should consider the regression in first differences to test hypotheses about the relationship between the two variables). If you reject, you are in the presence of cointegration, and an ECM would be the right model.

2 points if only says errors are I(1) without saying you must do a unit root test to confirm, but otherwise answer is right

0 points if it makes other mistakes

3. (6 points) Explain in detail how you would test whether returns on the SP500 index Granger-cause Morgan Stanley stock returns.

   A: Compute returns on both SP500 and MS by taking log price differences between day t and the previous day. Then regress MS returns on lags of MS returns and lags of the SP500 returns. Test if lags of SP500 are jointly significant by an F-test. If reject, there is no Granger causality.

   2 points if the test is correct but it uses prices instead of returns

4. (7 points) Discuss whether you think an Error Correction Model would be the right model for the vector $(MS_t, SP_t)'$, based on your answer to question 1.2.

   A: An ECM is the right model if MS_t and SP_t are both I(1) and are cointegrated. Since we have two variables, we can test for cointegration by testing if the residuals u_t have a unit root. If in question 1.2 we performed a unit root test on the residuals and the test did not reject, the prices are not cointegrated so the ECM is not the right model. If prices are I(1) and if a unit root test performed on the regression residuals in question 1.2 rejects the unit root, this is evidence of cointegration so an ECM is appropriate.

   0 points if answer is correct but not coherent with answer to question 1.2

CONTINUED

Question 2

Consider the model

$$y_t = -c_1 * 1(t \leq T/2) + c_2 * 1(t > T/2) + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}, \; t = 1, ..., T, \quad (2)$$

where $\varepsilon_t \sim i.i.d.N(0, 1)$ and $1(\cdot)$ equals 1 when the statement within parentheses is true and equals zero otherwise.

1. (6 points) Under which conditions on the parameters is the model stationary? Find the (unconditional) mean and variance of y_t under these conditions.

   A: When $-c_1 = c_2 = c$, in which case we have a MA(2) which is stationary with mean c and variance $1+\theta_1^2+\theta_2^2$

   3 points if mistake in any of the two

2. (6 points) Under which conditions is the model invertible?

   A: Roots of equation $1 + \theta_1 x + \theta_2 x^2 = 0$ greater than 1 in absolute value

3. (6 points) Given a quadratic loss function, what are the optimal 1-step ahead and 2-steps-ahead forecasts at time T based on model (2)?

   A: 1-step ahead is $c_2 + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1}$ and 2-steps ahead is $c_2 + \theta_2 \varepsilon_T$

   2 points if answers correct but uses c instead of c_2 (unless it explicitly says it is making the assumption) and uses T.

   2 points if answers correct but uses t instead of T (unless it explicitly says t=T)

4. (7 points) What is the impulse-response function, i.e., the response of y_{t+h} to a shock in ε_t, for $h \geq 0$?

   A: 1 for h=0, θ_1 for h=1, θ_2 for h=2 and 0 for $h > 2$

   2 points if only one is wrong

   0 points otherwise

CONTINUED

Question 3

Consider the model

$$y_t = \underbrace{\sigma_t z_t}_{\varepsilon_t}, \; t = 1, ..., T, \tag{3}$$

$$\sigma_t^2 = w + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 \tag{4}$$

where $z_t \sim i.i.d.N(0, 1)$.

1. (6 points) Derive $E[\varepsilon_t \varepsilon_{t-1}]$. (Hint: use the Law of Iterated Expectations, $E[x] = E[E[x|y]]$, for generic random variables x and y.)

   A: $E[\varepsilon_t \varepsilon_{t-1}] = E[\sigma_t z_t \sigma_{t-1} z_{t-1}] = E[E[\sigma_t z_t \sigma_{t-1} z_{t-1} | \Omega_{t-1}]] = E[\sigma_t \sigma_{t-1} z_{t-1} E[z_t | \Omega_{t-1}]] = 0$

   0 points if it shows that it doesn't understand LIE

2. (6 points) Explain how you would estimate the parameters by OLS (i.e., by running a regression).

   A: This is an ARCH(2), which is an AR(2) for ε_t^2. ε_t^2 can be approximated here (due to the lack of a conditional mean) with y_t^2, so estimate by regressing y_t^2 on y_{t-1}^2 and y_{t-2}^2

   A: This is an ARCH(2), which is an AR(2) for ε_t^2. ε_t^2 can be approximated here (since the conditional mean is zero) with y_t^2, so estimate the model by regressing y_t^2, t=3,...,T on a vector of 1's, y_{t-1}^2, t=2,...,T-1 and y_{t-2}^2 for t=1,...,T-2.

   2 points for partial credit (e.g., typo or messes up the t=3,...,T etc.)

   0 points if forgets the intercept or it is says to do regression for ε_t^2.

3. (6 points) Suppose that, after running the regression in question 3.2, you find that the residuals of the regression are autocorrelated. Explain how you would modify the model for y_t and how you would estimate the new model.

   A: You need to increase the order of the ARCH. E.g., start with an ARCH(3), which you can estimate by running the regression $y_t^2 = w + \alpha_1 y_{t-1}^2 + ... + \alpha_3 y_{t-3}^2 + v_t$, and then test if v_t is autocorrelated. If not, stop; if yes, increase the order of the ARCH until you no longer find residual autocorrelation. You can estimate all ARCH models by OLS. In alternative, you can do the same procedure considering GARCH models, which, being ARMA models for ε_t^2, could take care of the residual autocorrelation in a more parsimonious way (in this case you need to estimate them by MLE).

   Ok if says to estimate GARCH by MLE and still proposes doing the residual autocorrelation test

   2 points if all correct but does not say how to estimate the new models

0 points if not clear about which residuals it considers (it has to be v_t and not ε_t)

4. (7 points) Explain how you would construct an estimate of the standardized residual, $\hat{z}_t$, using results from the regression in question 3.2.

   A: $\hat{z}_t = \frac{y_t}{\hat{\sigma}_t}$ where $\hat{\sigma}_t = \sqrt{\hat{w} + \hat{\alpha}_1 y_{t-1}^2 + \hat{\alpha}_2 y_{t-2}^2}$

   0 points if it uses ε_t instead of y_t^2 or if it uses the unconditional variance

CONTINUED

Question 4

Consider the model

$$y_t = a + bt + y_{t-1} + \varepsilon_t \tag{5}$$

$\varepsilon_t \sim$ i.i.d.N(0,1); t = 1,...,T.

1. (5 points) Derive the autocorrelogram of ε_t^2.

   A: Since ε_t is i.i.d, so is ε_t^2 so the autocorrelogram equals zero at all $j > 0$

   2 points if it does not say that ε_t^2 is iid because ε_t is iid

2. (5 points) Derive the conditional density $f_{y_t|\Omega_{t-1}}$.

   A: $N(a + bt + y_{t-1}, 1)$

   2 points if variance is σ^2 instead of 1

3. (7 points) Suppose $b \neq 0$. Find the model for $v_t = \Delta z_t$, where $z_t = \Delta y_t$ (i.e., v_t is obtained by differencing y_t twice). Are v_t and/or z_t stationary?

   A: $z_t = a + bt + \varepsilon_t$ which is a white noise plus a time trend and is not stationary. Taking differences we have $v_t = b + \Delta\varepsilon_t$, which is an MA(1) thus stationary

   2 points if only first one is correct

4. (8 points) Derive the one-step-ahead forecast at time t implied by the model and compare it to the forecast made at the same time, but based on the model

   $$y_t = a + y_{t-1} + \varepsilon_t. \tag{6}$$

   Which model is more accurate (based on comparing the Mean Squared Forecast Errors for the two models?)

   A: The MSFE for the first model is $E[y_{t+1} - a - b(t+1) - y_t]^2 = E[\varepsilon_{t+1}^2] = 1$; the one for the second is $E[y_{t+1} - a - y_t]^2 = E[\varepsilon_{t+1}^2 + b^2(t+1)^2] = 1 + b^2(t+1)^2 > 1$ so the second model is always less accurate

   2 points if only one small mistake/typo. 0 for more than 1 mistake

END OF PAPER

UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**ECON0058**
ASSESSMENT PATTERN	:	**ECON0058**
MODULE NAME	:	**Time Series Econometrics**
LEVEL:	:	**Postgraduate**
DATE	:	**17 May 2021**
TIME	:	**Noon**

This paper is suitable for candidates who attended classes for this module in the following academic year(s):

Year
2020-21

Hall Instructions	.
Standard Calculators	Y
Non-Standard Calculators	

TURN OVER

SUMMER TERM 2021

24-HOUR ONLINE EXAMINATION

ECON0058: TIME SERIES ANALYSIS

All work must be submitted anonymously. Please ensure that you add your **candidate number** and the module code to the template answer sheet provided. Note that the candidate number is a combination of four letters plus a number, e.g. ABCD9. You can find your candidate number in your PORTICO account, under "My Studies" then the "Examinations" container. Please, note that the candidate number is NOT the same as your student number (8 digits), which is printed on your UCL ID card. Submitting with your student number will delay marking and when your results might be available.

Word count: 2000
The above word count is provided as guidance only on the expected total length of your submitted answer sheets. You will not be penalised if you exceed the word limit.

*Answer **ALL FOUR** questions*

Each of the FOUR questions carries the same weight (25 points)

Allow enough time to submit your work. Waiting until the deadline for submission risks facing technical problems when submitting your work, due to limited network or systems capacity.

*By submitting this assessment, **you pledge your honour** that you have not violated UCL's Assessment Regulations which are detailed in* https://www.ucl.ac.uk/academic-manual/chapters/chapter-6-student-casework-framework/section-9-student-academic-misconduct-procedure, *which include (but are not limited to) plagiarism, self-plagiarism, unauthorised collaboration between students, sharing my assessment with another student or third party, access another student's assessment, falsification, contract cheating, and falsification of extenuating circumstances.*

Question 1

Consider the following process:

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \tag{1}$$

where ε_t is a white noise process with variance σ^2.

(a) [5 points] Derive the conditions under which Y_t is stationary.

(b) [5 points] Derive the unconditional mean of this process. (Assume stationarity).

(c) [5 points] Derive the unconditional variance and the autocovariance function of this process. (Assume stationarity).

(d) [5 points] Assume that $\theta_1 = 0$. Derive the $\tau-$step-ahead forecast and show that as $\tau \to \infty$ it converges to the uncondional mean. (Assume stationarity).

(e) [5 points] Consider estimating the coefficient ϕ_1 using the linear regression model:

$$Y_t = \phi_1 Y_{t-1} + error_t. \tag{2}$$

Describe (with a brief explanation) the properties of the OLS estimator of the coefficient ϕ_1 in the regression model (2), under the following alternative parameterizations of the data generating process in (1):

(i) $\theta_1 = 0,\ \phi_1 = 1$

(ii) $\theta_1 = 0,\ \phi_1 = 1$ but adding Y_{t-2} as a regressor in (2).

(iii) $\theta_1 = 0,\ |\phi_1| < 1$

(iv) $\theta_1 \neq 0,\ |\phi_1| < 1$

Question 2

Suppose that two economists produced two alternative unbiased forecasts $f^A_{t+\tau|t}$ and $f^B_{t+\tau|t}$ for the variable $Y_{t+\tau}$. These forecasts lead to the forecast errors $e_{t+\tau|t,A} = Y_{t+\tau|t} - f^A_{t+\tau|t}$ and $e_{t+\tau|t,B} = Y_{t+\tau|t} - f^B_{t+\tau|t}$, respectively. Label the variances of these forecast errors as $\sigma^2_A = Var[e_{t+\tau|t,A}]$ and $\sigma^2_B = Var[e_{t+\tau|t,B}]$, and let $\rho = Cov(e_{t+\tau|t,A}, e_{t+\tau|t,B})/\sqrt{\sigma^2_A\sigma^2_B}$ be the correlation between the forecast errors.

(a) [5 points] How would you formally compare the two forecasts to establish which one performs better?

(b) Consider the combined forecasts:

$$f^*_t = \lambda f^A_{t+1|t} + (1-\lambda) f^B_{t+1|t} \tag{3}$$

where λ and $(1-\lambda)$ are the weights of the combination.

(i) [3 points] Derive the forecast error $e^*_{t+\tau|t}$ of the combined forecast.

(ii) [3 points] Derive the expected squared forecast error $E[e^{*2}_{t+\tau|t}]$ from the combined forecast.

(iii) [4 points] Show that the weights that minimize the expected squared forecast error $E[e^{*2}_{t+\tau|t}]$ are:

$$\lambda^* = \frac{\sigma^2_B - \rho\sigma_A\sigma_B}{\sigma^2_A + \sigma^2_B - 2\rho\sigma_A\sigma_B}.$$

That is, by setting $\lambda = \lambda^*$ in (3) the expected squared forecast error $E[e^{*2}_{t+\tau|t}]$ is minimized. HINT: study the first and second derivative of $E[e^{*2}_{t+\tau|t}]$ with respect to λ.

(c) [5 points] For simplicity, assume that the two forecasts $f^A_{t+1|t}$ and $f^B_{t+1|t}$ are mutually uncorrelated. Show that that if you combine the forecasts with the optimal weights λ^* the forecast error of the combined forecast is never larger than the forecast error of the individual forecasts.

(d) [5 points] Now assume that $\rho \neq 0$ but the two forecasts $f^A_{t+1|t}$ and $f^B_{t+1|t}$ have the same forecast error variance $\sigma^2 = \sigma^2_A = \sigma^2_B$. Combine the forecasts with the optimal weights λ^*. Show that a correlation of $\rho = 0.5$ reduces the forecast error variance by 25%. What happens in the extreme cases $\rho = 0$, $\rho = -1$ and $\rho = 1$?

Question 3

Consider the following ARCH model:

$$\begin{aligned} r_t &= \mu + \phi h_t + \eta_t; \\ \eta_t &= h_t \varepsilon_t; \quad \varepsilon_t \sim iidN(0,1) \\ h_t^2 &= \alpha_0 + \alpha_1 \eta_{t-1}^2 + \alpha_2 \eta_{t-2}^2 + \alpha_3 \eta_{t-3}^2 + \beta_1 h_{t-1}^2 \end{aligned}$$

where the coefficients are such that stationarity of η_t and non-negativity of h_t^2 are satisfied. Furthermore, assume that the fourth moment $E[\eta_t^4]$ exists and does not depend on time.

(a) [5 points] Compute the conditional mean and the conditional variance of r_t. (Conditioning on the information available in period $t-1$).

(b) [5 points] Show that η_t^2 follows an ARMA process.

(c) [5 points] Show that η_t features excess kurtosis, i.e.

$$\kappa = \frac{E[\eta_t^4]}{E[\eta_t^2]^2} > 3.$$

Which stylized fact in the data is consistent with this finding?

(d) [5 points] Set $\beta_1 = \alpha_2 = \alpha_3 = \phi = 0$.

- i Derive an expression for the fourth moment $E[\eta_t^4]$ as a function of α_1.
- ii Show that $\alpha_1^2 < 1/3$ is a necessary condition for the existence of the fourth moment.

(e) [5 points] Set $\beta_1 = \alpha_2 = \alpha_3 = \phi = 0$ and $\alpha_1^2 < 1/3$.

- i Derive the kurtosis of η_t as a function of α_1.
- ii Show that the kurtosis of η_t exceeds the kurtosis of ε_t by the quantity $6\alpha_1^2/\{1 - 3\alpha_1^2\}$.

Question 4

Consider the following data generating process:

$$x_t = u_{1t} \quad (4)$$

$$y_t = \gamma x_t + u_{2t} \quad (5)$$

Where $u_{1t} = u_{1t-1} + \epsilon_{1t}, u_{2t} = \rho u_{2t-1} + \epsilon_{2t}$, and the shocks $(\epsilon_{1t}, \epsilon_{2t})$ are serially uncorrelated, with $E[\epsilon_{1t}] = E[\epsilon_{2t}] = 0,\ Var[\epsilon_{1t}] = \sigma_1^2,\ Var[\epsilon_{2t}] = \sigma_2^2$,and $Cov[\epsilon_{1t}, \epsilon_{2t}] = \omega$.

(a) [5 points] What can we say about the degree of integratedness of the two series? Under what coefficient restrictions are x_t and y_t cointegrated?

(b) [5 points] Showing all the steps of the derivation, rewrite the model in the Vector Auto Regression (VAR) representation:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \gamma(1-\rho) & \rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \quad (6)$$

(c) [5 points] By making the appropriate assumptions on the coefficients γ and ρ, derive the Vector Error Correction (VEC) representation from (6).

(d) [5 points] Consider the parameterizations for which the series are both I(1), but not cointegrated. Derive the representation in first differences (DIFF).

(e) [5 points] Discuss the advantages/disadvantages of imposing the VEC or the DIFF representation on the data.