Advanced Microeconomic Theory

Lecture 5: Applications of Bayesian Games II

Elaborate Information Structures and High-Order Beliefs

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February 2022

Plan of the Lecture

- The ex-ante formulation of Bayesian games
  - Application: Speculative trade
- The role of high-order beliefs: The investment game revisited with two information structures:
  - One humorous
  - The other (supposedly) serious

Ex-Ante Formulation of Bayesian Games

- Ex-ante perspective: Players commit ex-ante to a strategy
- At the ex-ante stage, there are no informational asymmetries.
- Reducing the model to a strategic game with complete information

Ex-Ante Formulation of Bayesian Games

- The set of player is $N = \{1, \ldots, n\}$
- For each player $i \in N$, the set of strategies is the set of all functions $s_i : T_i \to A_i$.
- Player i's utility from the strategy profile $(s_1, \ldots, s_n)$ is

$$U_i(s_1, \ldots, s_n) = \sum_{\omega \in \Omega} p(\omega) u_i\big(s_1(\tau_1(\omega)), \ldots, s_n(\tau_n(\omega)), \omega\big)$$

Ex-Ante Formulation of Bayesian Games

- A profile of strategies $(s_1, \dots, s_n)$ is a Nash equilibrium if for every player i, $U_i(s_i, s_{-i}) \geq U_i\left(s_i', s_{-i}\right)$ for every $s_i' \in S_i$.
  - Tests deviations at a hypothetical planning stage
  - Equivalent to the original, interim definition – thanks to the expected-utility assumption
  - Usually hard to work with because of large strategy space
  - Useful for general results about classes of games

Speculative Trade

- Trade motivated purely by differences in beliefs
- Arguably the predominant motive for trade in financial markets
- Can differences in beliefs that give rise to trade be entirely due to informational asymmetries?
- Traders’ strategic inferences from their counterparts’ willingness to trade is an impediment to trade

Speculative Trade: A Two-Player Example

- Consider a bet $f : \Omega \to \{-1,1\}$.
  - $f(\omega)$ is the amount that player 1 receives from player 2 in state ω.
- A trading game: Each player i chooses an action $a_i \in \{0,1\}$.

$$u_1(a_1, a_2, \omega) = a_1[a_2 f(\omega) - \varepsilon]$$

$$u_2(a_1, a_2, \omega) = a_2[-a_1 f(\omega) - \varepsilon]$$

  - $\varepsilon > 0$ is an arbitrarily small transaction cost.

Speculative Trade: A Two-Player Example

- $a_i = 1$ means agreeing to trade.
- The role of the transaction cost is to break ties.
- An arbitrary information structure $(\Omega, p, T_1, T_2, \tau_1, \tau_2)$
  - The prior p has full support.

Speculative Trade: A Two-Player Example

- A trivial example: $T_1 = T_2 = \{t\}$: Both traders are uninformed.
- Player i will play $a_i = 1$ only if $a_j = 1$, due to transaction cost.
- player 1 will play $a_1 = 1$ only if $\sum_\omega p(\omega) f(\omega) > 0$.
- But then player 2 doesn't want to trade!
- The only Nash equilibrium is $a_1 = a_2 = 0$ (no trade).

Speculative Trade: A Two-Player Example

- Another example: Player 1 knows ω , player 2 is uninformed.
- For trade to take place, we need $a_2 = 1$.
- player 1 will play $a_1 = 1$ if and only if $f(\omega) = 1$.
- But then player 2 earns a negative payoff. He can profitably deviate to $a_2 = 0$.
- The only Nash equilibrium involves no trade.

A “No-Trade Theorem”

Proposition: For any information structure, the unique Nash equilibrium in the induced Bayesian game is for each player i to play $s_i(t_i) = 0$ for every t_i.

- Speculative trade cannot be due to differential information, under the assumption that traders play Nash equilibrium.
- An example of a rich literature on “no-trade theorems”.

Proof (Using the Ex-ante Formulation)

- When $a_i = 0$ with certainty, $a_j = 0$ is a best-reply for player j regardless of his information, because trade doesn't occur anyway and playing 0 saves the transaction cost.
- Now consider a candidate Nash equilibrium in which each player sometimes plays 1.

Proof

- Each player can ensure an ex-ante payoff of 0 by always refusing to trade. This is a lower bound on his equilibrium payoff.
- By assumption, players incur the transaction cost with positive probability in the candidate equilibrium.
- Therefore, each player's ex-ante expected monetary transfer is strictly positive.

Proof

- Player 1's ex-ante monetary payoff:

$$\sum_{\omega} p(\omega) s_1\big(\tau_1(\omega)\big) s_2\big(\tau_2(\omega)\big) f(\omega) > 0$$

- Player 2's ex-ante monetary payoff:

$$-\sum_{\omega} p(\omega) s_1\big(\tau_1(\omega)\big) s_2\big(\tau_2(\omega)\big) f(\omega) > 0$$

- A contradiction!

Discussion

- Many culprits:
  - The common prior belief
  - Partitional information structures
  - Expected utility maximization
  - Rational expectations
- A common trick in the finance literature: “Noise traders”

High-Order Beliefs

- Information structures can express rich patterns of high-order beliefs ("my information about your information about my information…")
- The state space can have a dimensionality far beyond the payoff-relevant states.
- This richness can be strategically relevant.

The E-Mail Game

	I	NI
I	-1,-1	-1,0
NI	0,-1	0,0

Bad state

	I	NI
I	1,1	-1,0
NI	0,-1	0,0

Good state

- The investment game revisited
- A slight change in the payoff structure
- The probability of the bad state of Nature is $q > 0.5$.

The Information Structure

- Each player sits in front of a computer screen.
- When the state is good (and **only** then), player 1's computer sends an automatic message to player 2's computer.
- When player 2's computer receives the message, it sends a confirmation to player 1's computer, which sends a re-confirmation, and so forth…
- Each message gets lost with independent probability $\varepsilon > 0$.

The Information Structure

- The process terminates with probability one after finitely many rounds. Each player's computer screen displays the number of messages that the computer sent.
  - This number is the player's signal.
  - Players simultaneously take actions after receiving it.
- Ω is the set of all pairs of non-negative integers (t_1, t_2) for which $t_2 \in \{t_1 - 1, t_1\}$.

The Information Structure

$$p(0,0) = q$$

$$p(1,0) = (1 - q)\varepsilon$$

$$p(1,1) = (1 - q)(1 - \varepsilon)\varepsilon$$

$$p(2,1) = (1 - q)(1 - \varepsilon)^2\varepsilon$$

$$p(2,2) = (1 - q)(1 - \varepsilon)^3\varepsilon$$

$$p(3,2) = (1 - q)(1 - \varepsilon)^4\varepsilon$$

$$\vdots$$

The Information Structure

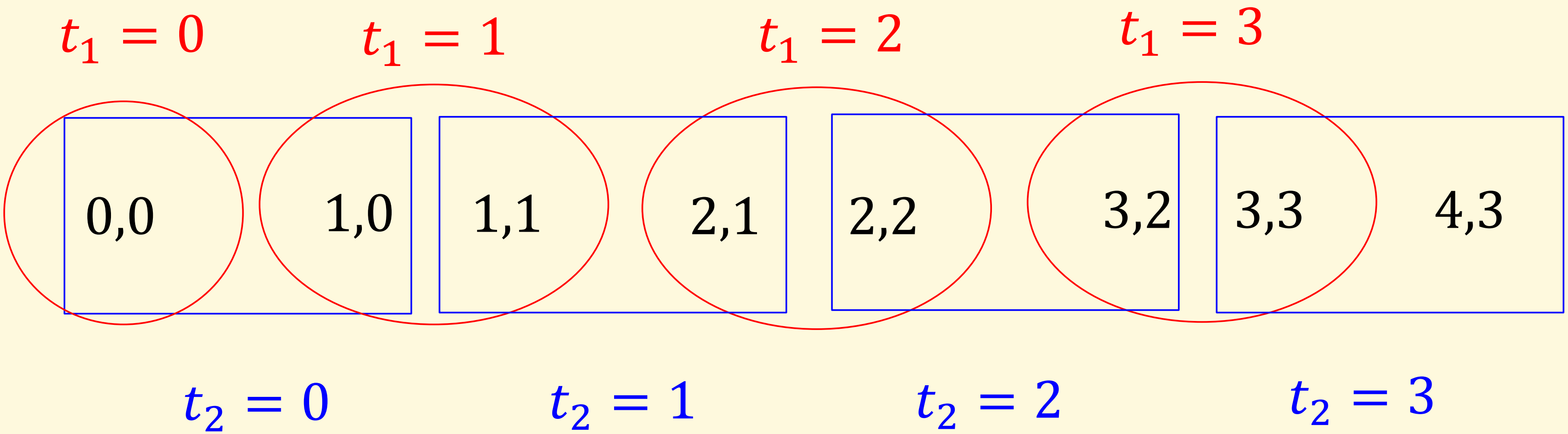
- $\tau_i(t_1, t_2) = t_i$ encodes player i's high-order knowledge regarding the state of Nature:
  - $t_i = 1$: The player knows it is good but doesn't know whether player j knows this.
  - $t_i = 2$: he knows the state is good and that player j knows it, but doesn't know whether j knows all this.

  $\vdots$

Almost Common Knowledge

- Small ε ensures that when the state of Nature is good, players are very likely to have a high degree of mutual knowledge of this event.
- However, common knowledge is never attained.
- What would **you** do if you saw a large number on your computer screen?

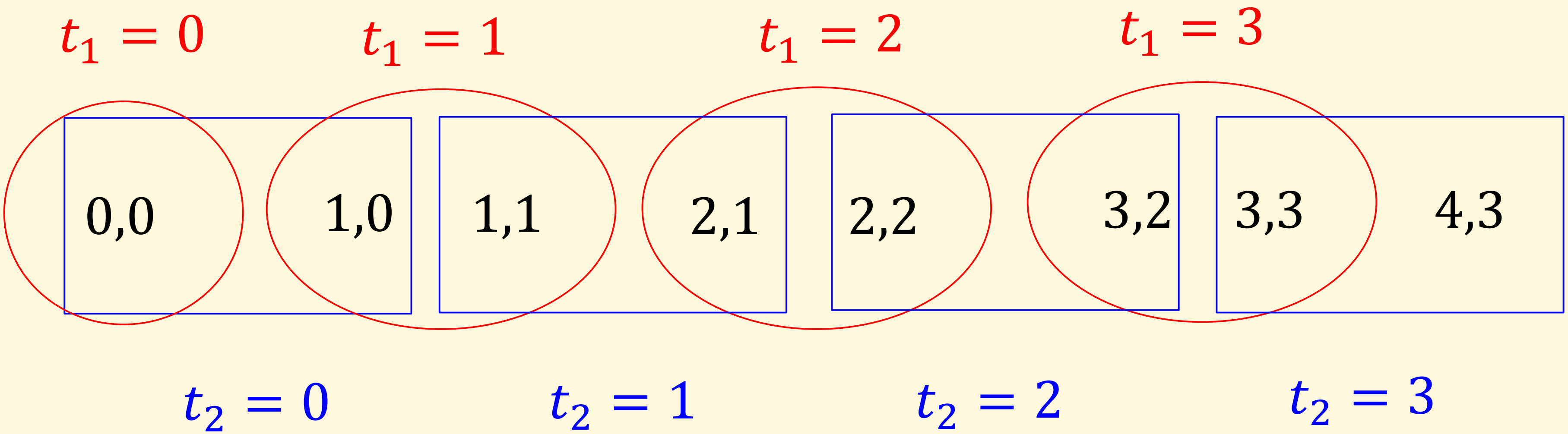
Diagrammatic Representation



- Interlocking information sets

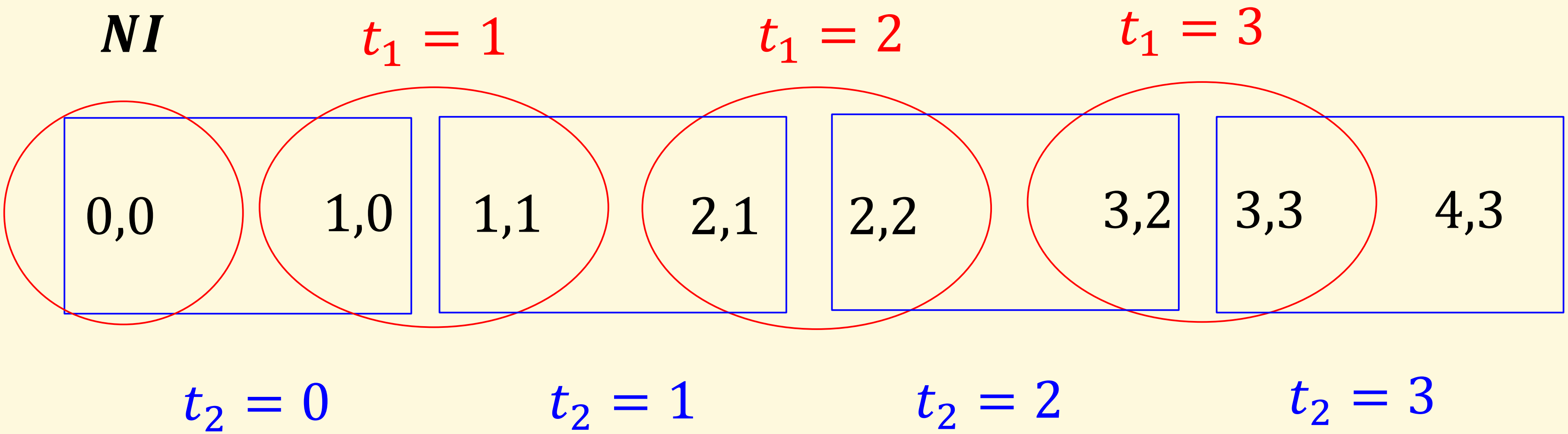
Proposition: The game has a unique Nash equilibrium. For every player $i = 1,2$, $s_i(t_i) = NI$ for every $t_i = 0,1,2,\ldots$

Proof



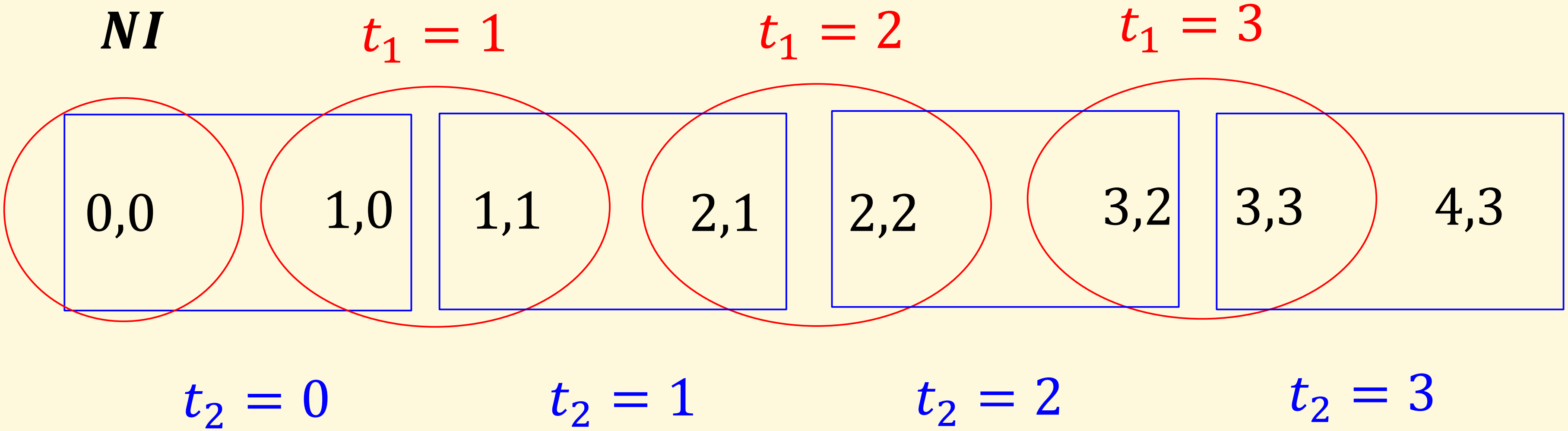
- The proof is by induction on the players' interlocking information sets: $t_1 = 0$, $t_2 = 0$, $t_1 = 1$, $t_2 = 1, \ldots$
- At $t_1 = 0$, NI is strictly dominant for player 1.

Proof



- At $t_2 = 0$, player 2 assigns probability $\frac{q}{q+(1-q)\varepsilon} > 0.5$ to the bad state of Nature.

Proof

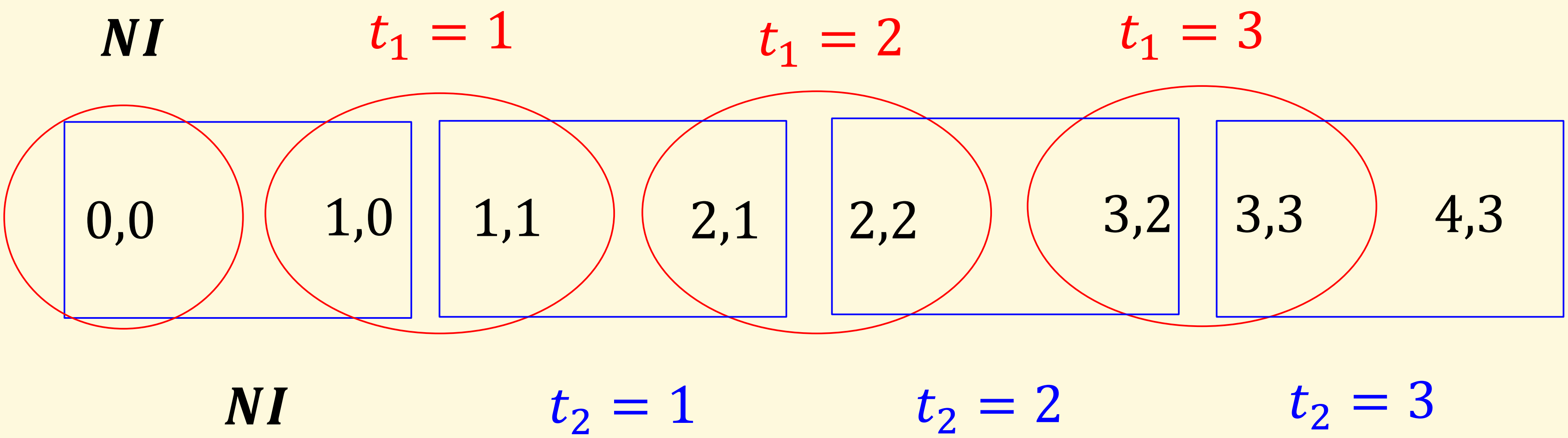


- The player's expected payoff from I is therefore at most

$$\frac{q}{q+(1-q)\varepsilon} \cdot (-1) + \frac{(1-q)\varepsilon}{q+(1-q)\varepsilon} \cdot 1 < 0.$$

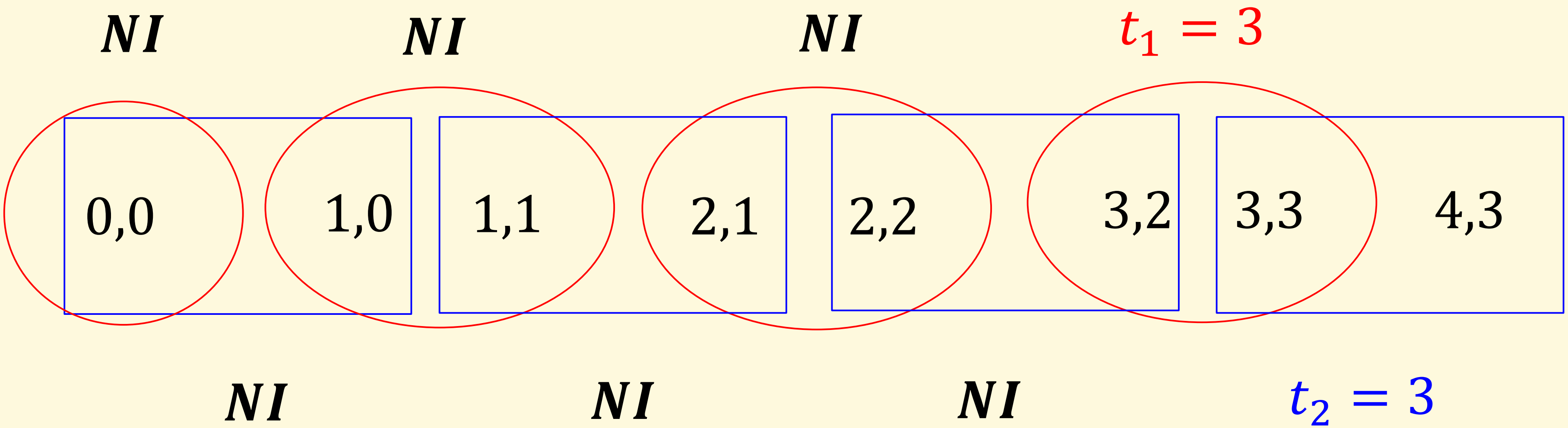
- Therefore, player 2 plays NI at $t_2 = 0$ in any Nash equilibrium.

Proof



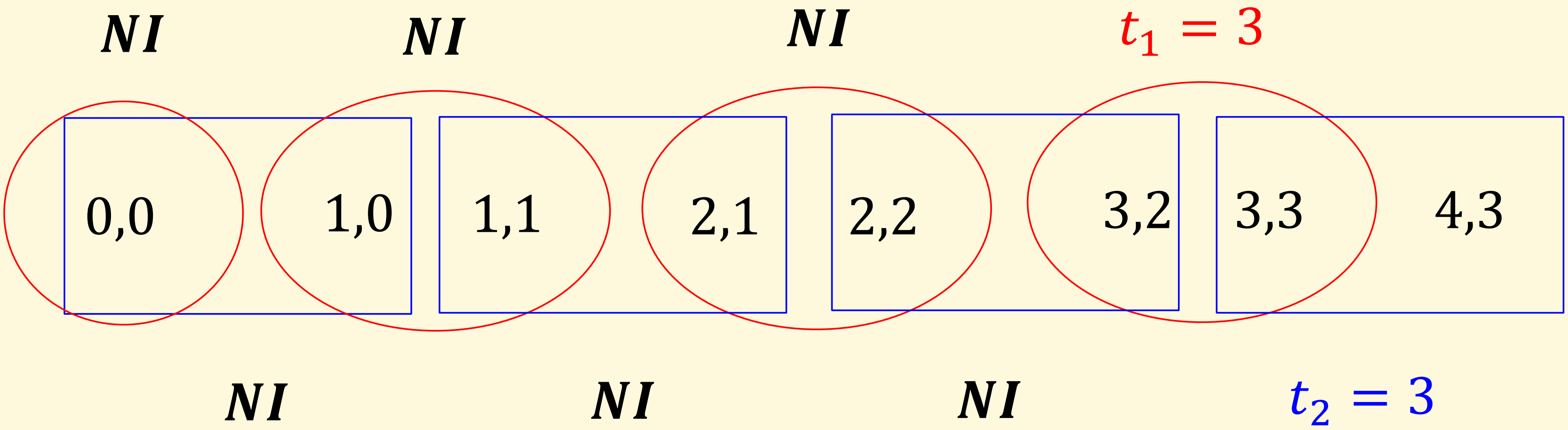
- We have established that $s_i(t_i = 0) = NI$ for both $i = 1{,}2$ in any Nash equilibrium.
- Now we'll put the inductive argument to work.

Proof



- Suppose we proved that players play NI in all information sets in the sequence up to some information set.
- In the diagram, that information set is $t_2 = 2$.

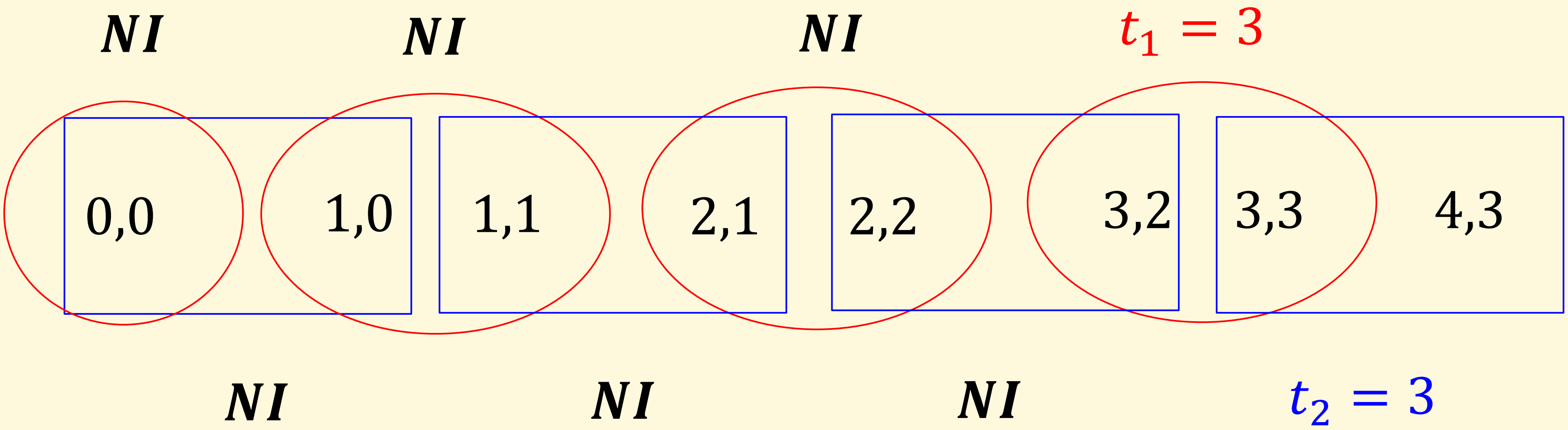
Proof



- At $t_1 = 3$, player 1's posterior probability the state $(3{,}2)$ is

$$\frac{(1-q)(1-\varepsilon)^4\varepsilon}{(1-q)(1-\varepsilon)^4\varepsilon + (1-q)(1-\varepsilon)^5\varepsilon} = \frac{1}{2-\varepsilon} > 0.5$$

Proof

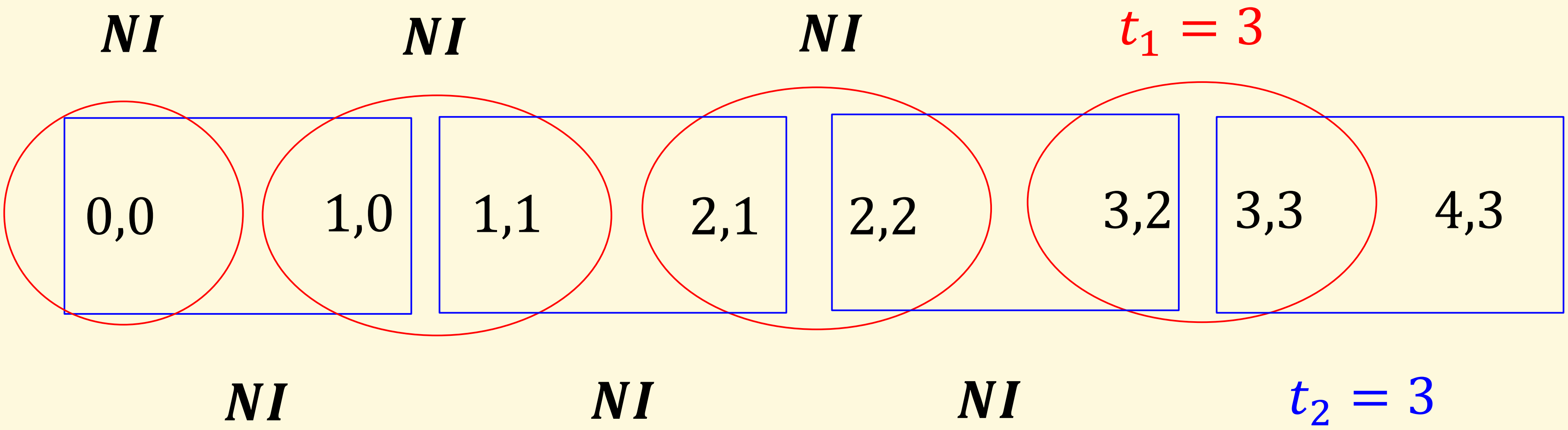


Intuition: Given that my last message hasn't been confirmed,

which scenario is more likely?

- My original message got lost (probability ε).
- The confirmation got lost (probability $\varepsilon(1-\varepsilon)$).

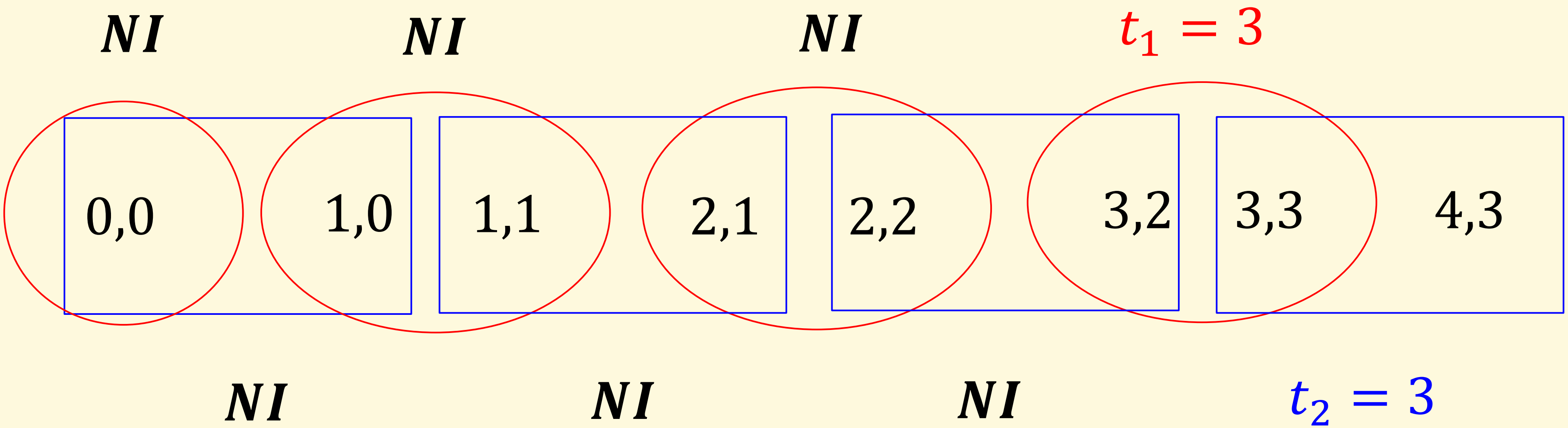
Proof



- Bayes' rule says: The first scenario is strictly more likely.
- Player 1's expected utility from I is therefore at most

$$\frac{1}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 2), good) + \frac{1-\varepsilon}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 3), good)$$

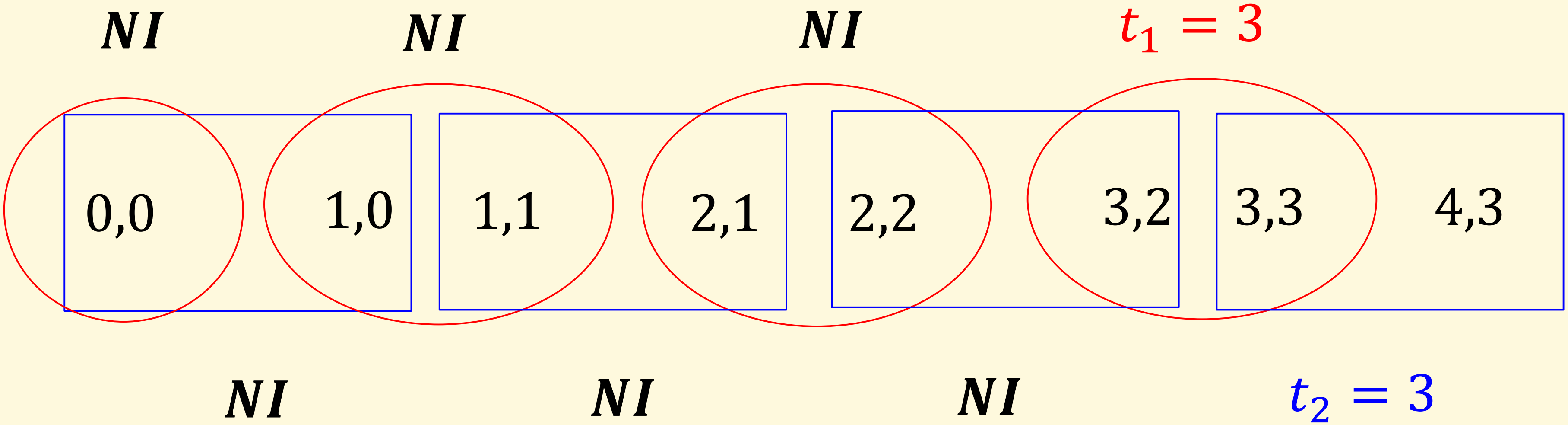
Proof



- Bayes' rule says: The first scenario is strictly more likely.
- Player 1's expected utility from I is therefore at most

$$\frac{1}{2-\varepsilon} \cdot u_1(I, NI, good) + \frac{1-\varepsilon}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 3), good)$$

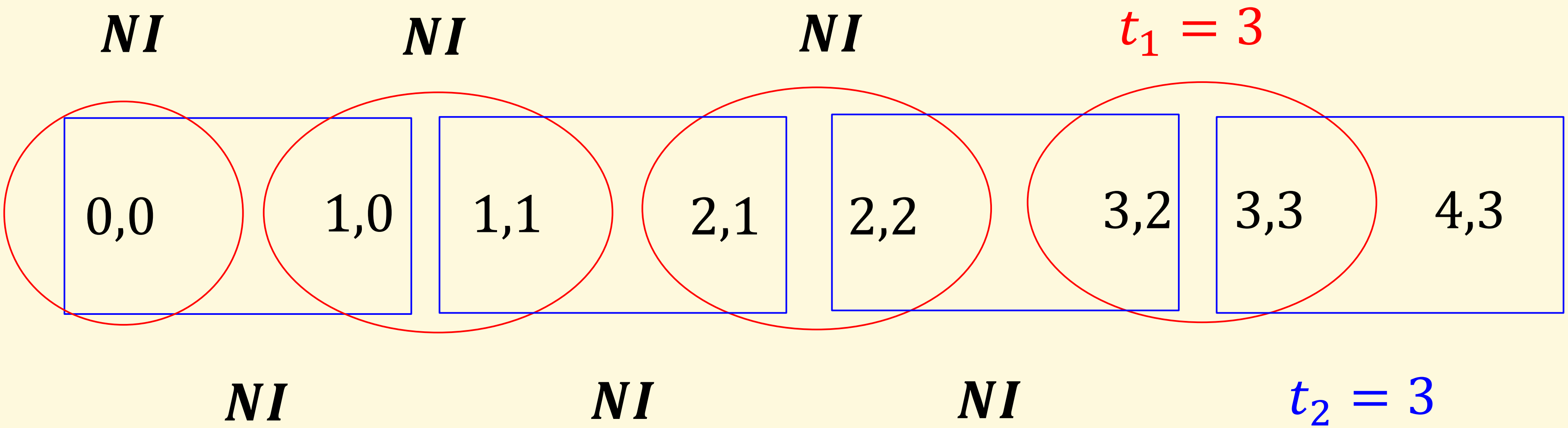
Proof



$$\frac{1}{2-\varepsilon} \cdot (-1) + \frac{1-\varepsilon}{2-\varepsilon} \cdot u_1(I, a_2(t_2 = 3), good)$$

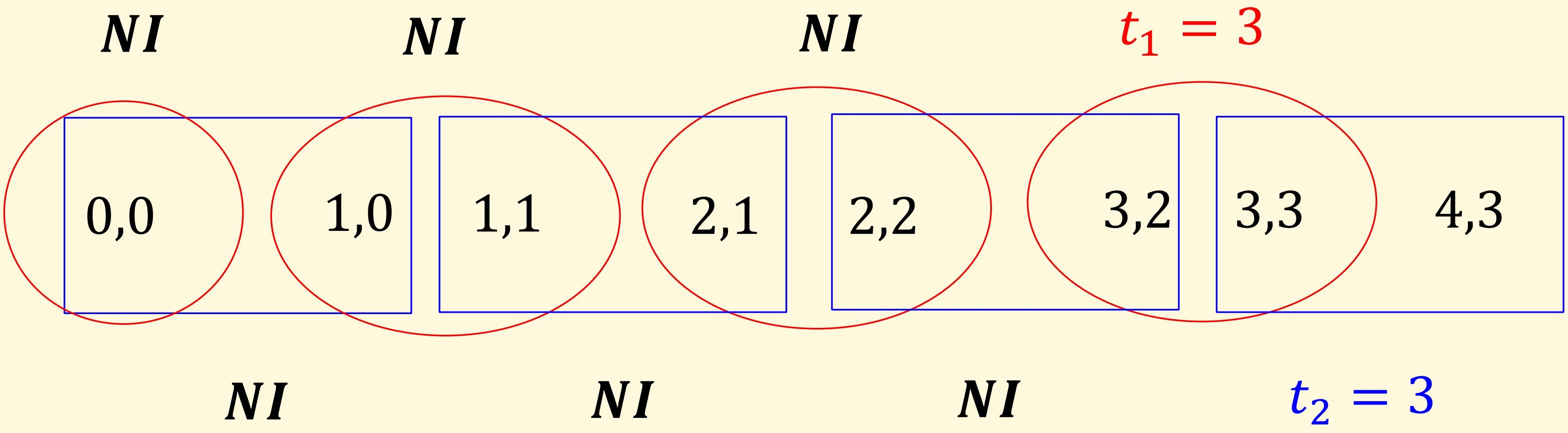
$$\leq \frac{1}{2-\varepsilon} \cdot (-1) + \frac{1-\varepsilon}{2-\varepsilon} \cdot 1 < 0$$

Proof



- Therefore, player 1's best-reply at $t_1 = 3$ in any Nash equilibrium is NI.
- Recall that we used $t_1 = 3$ for purely illustrative purposes.

Proof



- The same argument works for any information set $t_i > 0$.

- It follows that both players always play NI in Nash equilibrium.

  This completes the proof.

Discussion

- An example of “finite layers of strategic reasoning” paradoxes
- The difference between the states $(1,1), (2,1), (2,2), \ldots$ is in the players’ high-order beliefs.
- The inductive proof is fundamentally iterative elimination of strictly dominated strategies.
  - The Nash equilibrium is the unique **rationalizable** outcome.

Global Games

- The information structure of the E-mail game makes the players’ high-order beliefs explicit.
- However, it is “artificial” – not something that a self-respecting applied economist would assume…
- But very similar effects arise under more “respectable” information structures!

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- An example by Carlsson & can Demme (1993)
- θ is the state of Nature that indicates investment quality.
- Customary assumption: θ is distributed according to an "improper" uniform prior over $(-\infty, \infty)$.

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Player i's signal is $t_i = \theta + \varepsilon_i$, where ε_i is independently drawn according to $N(0, \sigma^2)$.
- $\omega = (\theta, \varepsilon_1, \varepsilon_2)$; $\tau_i(\theta, \varepsilon_1, \varepsilon_2) = \theta + \varepsilon_i$

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- When $\sigma^2 = 0$, θ is common knowledge.
  - $\theta > 1$ $\Longrightarrow$ I is a strictly dominant action.
  - $\theta < 0$ $\Longrightarrow$ NI is a strictly dominant action.
  - $\theta \in [0,1]$ $\Longrightarrow$ (I, I) and (NI, NI) are Nash equilibria.

Global Games

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

Proposition: When $\sigma^2 > 0$, there is an essentially unique Nash equilibrium. Each player i plays I whenever $t_i > 0.5$, and he plays NI whenever $t_i < 0.5$.

Discussion

- Slight incomplete-information perturbation of the complete information game leads to equilibrium selection.
  - Efficient coordination with near certainty when $\theta > 0.5$
  - Inefficient coordination when near certainty when $\theta < 0.5$
- Striking difference between the common knowledge and "almost common knowledge" environments

Why is it an Equilibrium?

- Suppose σ^2 is vanishing.
- At $t_i = 0.5$, player i believes $\theta \approx 0.5$.
- He also assigns probability 0.5 to $a_j = I$ because of his knowledge of player j's cut-off strategy.
- Therefore, he is indifferent between the two actions.
- When we raise (lower) t_i, the incentive to play I becomes stronger (weaker).

Why is it the Unique Equilibrium?

- Recall $\theta = t_i - \varepsilon_i$, $t_j = t_i - \varepsilon_i + \varepsilon_j$.
- Therefore, conditional on observing t_i, player i's posterior implies $\theta \sim N(t_i, \sigma^2)$ and $t_j \sim N(t_i, 2\sigma^2)$.
- When $t_i < 0$, $E(\theta | t_i) < 0$, and therefore NI is strictly dominant for player i.

$\Longrightarrow$ In any Nash equilibrium, $a_i = NI$ when $t_i < 0$.

Why is it the Unique Equilibrium?

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Now suppose that $t_i > 0$ but close to zero.
- By the previous argument, player i's posterior probability that $t_j < 0$ – and hence $a_j = NI$ – is close to 0.5.

Why is it the Unique Equilibrium?

	I	NI
I	θ,θ	θ-1,0
NI	0, θ-1	0,0

- Given that $E(\theta|t_i)$ is close to zero, NI is a best reply for i.
- And so in any Nash equilibrium, $a_i = NI$ also when t_i is positive but close to zero.

Why is it the Unique Equilibrium?

- The last argument was based on player i's second-order belief – i.e., his belief regarding player j's signal.
- We continue in this iterative manner, further expanding the range of signal realizations for which NI is a best reply for i.
- This iterative argument mirrors the inductive proof in the E-mail game.

Idea of the Proof

- The limit of this iterative argument is that in any Nash equilibrium, player i plays NI whenever $t_i < 0.5$.
- An analogous argument applies to the other side (starting with I being dominant when $t_i > 1$).
- When σ is small, players can be almost certain that investment is efficient and nevertheless they coordinate on the bad outcome because of lack of common knowledge.

Summary

- Despite the apparent gap between the whimsical E-mail game and the “applied look” of the last example, the game-theoretic analysis is very similar.
- But while the equilibrium in the E-mail game looks paradoxical, the cutoff strategies in the last example look natural.

Summary

- In Game Theory, there is a fine line between the applied and the paradoxical.
- An appropriate motto to conclude with…

THANK YOU!