ECON G011 (Public Microeconomics) 2011
Guidance Notes on Answers

1. Individuals within a population have wages w distributed according to the known distribution function $F(w)$. Preferences over consumption c and hours of work h are represented by utility function
$$u(c, h) \ = \ c - \frac{1}{2}h^2.$$
There is a linear tax on labour income at rate $0 \leq t \leq 1$ with revenue returned to individuals through a uniform lump sum grant G so that optimising individuals have utilities
$$v(w, t, G) \ = \ \max_h \left[w(1-t)h + G - \frac{1}{2}h^2 \right]$$
and $t \int wh \, \mathrm{d}F(w) \ = \ G$.

(a) Show that an individual with wage w has welfare given by

$$V(w,t) \;=\; \frac{1}{2}w^2 + \left(\mu_2 - w^2\right)t + \left(w^2 - 2\mu_2\right)\frac{1}{2}t^2$$

where $\mu_2 = \int w^2 \, \mathrm{d}F(w)$ and that their preferred tax rate is therefore

$$t^*(w) = \begin{cases} \left(\mu_2 - w^2\right) / \left(2\mu_2 - w^2\right) & \text{if } \; w \le \sqrt{\mu_2} \\ 0 & \text{if } \; w \ge \sqrt{\mu_2} \end{cases}$$

Ans: The first order condition for choice of hours gives $w(1-t) = h$. *From the revenue constraint therefore* $G = t(1-t)\mu_2$. *Substituting into* $v(w,t,G)$ *gives* $V(w,t) = \frac{1}{2}w^2(1-t)^2 + \mu_2 t(1-t)$. *Differentiating gives* $\partial V/\partial t = (\mu_2 - w^2) + (w^2 - 2\mu_2)t$. *If* $\mu_2 < w^2$ *then the optimum is at* $t = 0$, *otherwise it is as given.*

(b) Suppose that the government's social preferences over tax rates $0 \leq t \leq 1$ are decided by majority voting. That is to say, one tax rate is socially preferred to another if and only if it is preferred by more individuals. Explain why social preferences are complete and transitive.

Ans: Completeness is obvious. Transitivity holds because preferences are single peaked - either with a peak above 0 if $\mu_2 > w^2$ or everywhere decreasing if not.

(c) Explain what a Condorcet winner is. Does there exist a Condorcet winner among tax rates? Does it exceed zero?

Ans: The Condorcet winner exists and is the preferred tax rate of the person with median wage, w_m. It is positive only if $w_m < \sqrt{\mu_2}$.

(d) Comment on the relevance of this sort of analysis to setting of tax rates.

Ans: Actual tax setting is more complex than exhaustive majority voting over single tax rates. In particular there are many tax parameters, other issues and plurality voting.

2. A certain population consists of individuals with fixed pretax incomes y distributed according to the known distribution function $F(y)$. The government assesses pretax social welfare by the total income of those at or below mean income

$$W_0(F) = \int_0^{\mu} y \, \mathrm{d}F(y)$$

where $\mu = \int_0^{\infty} y \, \mathrm{d}F(y)$.

(a) Explain why social judgments made on this basis are

- weakly Paretian (ie weakly favour increasing incomes)
  Ans: Increases in any income raises μ and also raises the integrand if it happens at $y \leq \mu$
- weakly Schur-concave (ie weakly favour Pigou-Dalton transfers)
  Ans: Pigou Dalton transfers have no impact if happening on the same side of the mean and increase the integral if happening across μ.
- homothetic (ie preserve indifference across uniform scaling of incomes)
  Ans: Scaling all incomes by λ scales μ by λ and all individual y so that W_0 is also scaled by λ

(b) Explain what is meant by the equally-distributed equivalent income, ξ, and derive an expression for it. Hence find an expression for the corresponding relative inequality index of the Atkinson-Kolm-Sen kind, $\mathcal{I} = 1 - \xi/\mu$.

Ans: W_0 already is equally distributed equivalent income: $\xi = \int_0^\mu y \, \mathrm{d}F(y)$ and $\mathcal{I} = 1 - \int_0^\mu y \, \mathrm{d}F(y)/\mu$

The government can redistribute income by imposing a proportional tax at rate t and paying out a uniform grant G so that after-tax income is $(1-t)y+G$. However administrative costs mean that resources are lost in the process so that the value of the grant affordable is only $G=\phi t\mu$ where $\phi<1$. The government assesses posttax social welfare by the total income of those at or below mean posttax income.

(c) Explain why posttax social welfare is

$$W(t) = \begin{cases} (1-t)\int_0^\mu y \, \mathrm{d}F(y) + \phi t \mu F(\mu) & \text{if } 0 \le t < 1 \\ \phi\mu & \text{if } t = 1 \end{cases}$$

and show that

- $W(1) > W(0)$ iff $\phi > 1 - \mathcal{I}$
- $W'(0) > 0$ iff $\phi > (1-\mathcal{I})/F(\mu)$.

Ans: Someone with mean pretax income also has mean posttax income, given linearity of taxes. If $t = 1$ everyone is at or below mean so there is a discontinuity.

$W(1) = \phi\mu$ *and* $W(0) = \int_0^\mu y \, \mathrm{d}F(y)$ *hence* $W(1) > W(0)$ *iff* $\phi\mu > \int_0^\mu y \, \mathrm{d}F(y)$. $W'(0) = -\int_0^\mu y \, \mathrm{d}F(w) + \phi\mu F(\mu)$.

(d) Interpret these conditions and discuss the suitability of such a social welfare function for capturing aversion to inequality.

Ans: The desirability of redistribution involves a comparison of the efficiency losses against the extent of inequality before taxes. The dependence of the social welfare function on the comparison of income to the threshold μ *creates the discontinuity which means you could have* W *decreasing in* t *for all* $t < 1$ *but* $W(1) > W(0)$.

3. Individuals in a certain country consume K goods $q = \{q_i\}_{i=1,\ldots,K}$ at prices $p = \{p_i\}_{i=1,\ldots,K}$ and supply labour h for wages w. Goods prices are the same for all individuals but wages vary according to distribution function $F(w)$. Goods are measured in units such that all pretax prices are 1 and the government raises funds by taxing goods at rates $t = \{t_i\}_{i=1,\ldots,K}$ so that $p_i = 1 + t_i$, $i = 1, \ldots, K$. A uniform lump sum grant of G is provided out of the commodity tax revenue. Individuals have no source of income other than G and labour income.

Suppose individual preferences are such that utilities are

$$v(w, p, G) \;=\; \max_h \left[\phi\left(\frac{wh + G - a(p)}{b(p)}\right) \;+\; \psi\,(1-h)\right],$$

where $a(p)$ and $b(p)$ are homogeneous of degree one. Let individual labour supplies be $\eta(w, p, G)$ and individual goods demands be $f(w, p, G)$.

(a) Show that individual commodity demands can be written linearly as

$$f_i(w,p,G) \;=\; \alpha_i(p)\left[w\eta(w,p,G)+G\right]+\beta_i(p), \qquad i=1,\ldots,K$$

for some functions $\alpha_i(p)$ and $\beta_i(p)$, $i=1,\ldots,K$.

Ans: Using the envelope theorem, $v_G = \phi'/b$ and

$$v_i = \left[-a_i/b - (w\eta+G-a)b_i/b^2\right]\phi'.$$

Thus by Roy's identity $f_i = a_i + (w\eta+G-a)b_i/b$.

A government choosing t and G to maximise social welfare

$$W(t, G) \;=\; \int v(w, p, G)\, \mathrm{d}F(w)$$

subject to revenue constraint

$$\sum_{i=1,\ldots,K} \left[t_i \int f_i(w, p, G)\, \mathrm{d}F(w) \right] = G$$

would set optimum commodity tax rates t^* which are uniform,

$$t_i^* = \tau \qquad i = 1, \ldots, K.$$

(b) To what extent would this conclusion be affected by relaxing assumptions of

- unavailability of a linear tax on labour income
- additive separability between goods and leisure
- linearity of Engel curves
- optimal setting of G?

Ans: The uniformity result is due to Deaton. The question does not ask for a proof but for understanding to be shown of the conditions under which the uniform optimum holds.

The unavailability of a linear tax on income is simply a normalisation and cannot be material to the result. Additive separability can be relaxed to weak separability without affecting the argument. The optimal setting of G and linearity of Engel curves are both, however, essential.

4. Suppose that individuals in a population consume goods q at prices p. One of these goods, books q_0, is currently exempt from taxation. The government is considering ending this exemption and wants to understand the distributional implications. You are approached to conduct a study. The minister concerned informs you that the government thinks that books possibly constitute a small fraction of the spending of poor households but one that rises rapidly with income at low incomes before flattening off at higher incomes.

(a) You consider modelling spending using a demand system corresponding to the following class of expenditure functions

$$\ln e(u,p) \;=\; \ln A(p) + \frac{uB(p)}{1+uC(p)}$$

where u is utility and $A(p)$, $B(p)$ and $C(p)$ are appropriate functions of prices. What properties must the functions $A(p)$, $B(p)$ and $C(p)$ have for this to constitute a well-specified expenditure function?

Ans: A must be homogeneous of degree one and both B and C must be homogeneous of degree zero.

Furthermore

$$
\begin{aligned}
\ln A(p) &= \alpha_0 + \sum_i \alpha_i \ln p_i + \sum_i \sum_j \beta_{ij} \ln p_i \ln p_j \\
\ln B(p) &= \sum_i \gamma_i \ln p_i \\
\ln C(p) &= \sum_i \delta_i \ln p_i
\end{aligned}
$$

(b) Derive Hicksian and Marshallian budget share functions corresponding to these preferences.

Ans: Hicksian budget shares are found from Shephards' lemma to be

$$
\begin{aligned}
w_i = \ & \frac{\partial \ln e}{\partial \ln p_i} = \ \frac{\partial \ln A}{\partial \ln p_i} + \ln\left(\frac{e}{A}\right)\frac{\partial \ln B}{\partial \ln p_i} + \ln\left(\frac{e}{A}\right)^2\left(\frac{C}{B}\right)\frac{\partial \ln C}{\partial \ln p_i} \\
= \ & \alpha_i + \sum_j \frac{1}{2}[\beta_{ij} + \beta_{ji}] \ln p_j + \gamma_i \ln\left(\frac{e}{A}\right) + \delta_i \ln\left(\frac{e}{A}\right)^2
\end{aligned}
$$

Marshallian budget shares follow by replacing e *with total budget* y.

(c) Comment upon the ability of this demand system to capture the pattern of spending postulated by the minister if

- $B(p)$ and $C(p)$ are both constant
- $C(p)$ alone is constant

Ans: If B and C constant then preferences are homothetic and budget shares constant. If C is constant then Engel curves have Working-Leser form but no curvature in the income term.

(d) The same minister tells you that the government believes that the fraction of income spent on books by the poor is likely to be fairly unresponsive to prices but that richer households may be much more sensitive. How well do you think the demand specification adopted would be able to capture this? Suggest ways of enhancing the flexibility with which it could capture the variation of price responsiveness across the income distribution.

Ans: The main price terms in the budget share equation do not interact with income at all. This could be introduced by making either B or C also quadratic in ln *prices.*

5. Suppose a population consists of two equally numerous types of individual, A and B. Utility for each depends on consumption of m goods q and supply of labour L according to utility function $u(q, L)$. Pretax goods prices are p. Wages are w^A and w^B where $w^B/w^A = \phi < 1$. The government, which needs to raise revenue R, can observe both consumption of goods and earnings, but not wages, at the individual level and can therefore set taxes based on individual goods choices and earnings. Although the government does not know which specific individuals are the more productive it does know the value of ϕ. By choice of tax functions on earnings and goods choices it can determine the values of q^A, L^A, q^B and L^B chosen by individuals subject to the constraint that individuals reveal their types. Assume its objective is the sum of utilities and that looking at first order conditions is good enough to characterise the optimum.

(a) The Lagrangean for its problem is therefore:

$$\begin{aligned}
u(q^A, L^A) \; &+ \; u(q^B, L^B) \\
&- \; \lambda \left[R + p' \left(q^A + q^B \right) - w^A L^A - w^B L^B \right] \\
&+ \; \mu \left[u(q^A, L^A) - u(q^B, \phi L^B) \right]
\end{aligned}$$

Explain the interpretation of the two constraints.

Ans: The first is a revenue constraint and the second an incentive compatibility constraint.

First order conditions are:

$$
\begin{aligned}
[1+\mu]\frac{\partial u^A}{\partial q_i} \; - \; \lambda p_i \; &= 0 \qquad i = 1, \ldots, m \\
\frac{\partial u^B}{\partial q_i} - \mu\frac{\partial \tilde{u}}{\partial q_i} \; - \; \lambda p_i \; &= 0 \qquad i = 1, \ldots, m \\
[1+\mu]\frac{\partial u^A}{\partial L} \; + \; \lambda w^A \; &= 0 \\
\frac{\partial u^B}{\partial L} - \mu\phi\frac{\partial \tilde{u}}{\partial L} \; + \; \lambda w^B \; &= 0
\end{aligned}
$$

where u^A denotes $u\left(q^A, L^A\right)$, u^B denotes $u\left(q^B, L^B\right)$ and $\tilde{u}$ denotes $u\left(q^B, \phi L^B\right)$.

(b) Explain the significance of each of the following facts regarding the optimum:

i. $\dfrac{\partial u^A}{\partial q_i} \Big/ \dfrac{\partial u^A}{\partial q_j} = p_i/p_j$

ii. $\dfrac{\partial u^A}{\partial L} \Big/ \dfrac{\partial u^A}{\partial q_j} = -w^A/p_j$

iii. $\dfrac{\partial u^B}{\partial L} \Big/ \dfrac{\partial u^B}{\partial q_j} \neq -w^B/p_j$ unless $\dfrac{\partial u^B}{\partial L} \Big/ \dfrac{\partial u^B}{\partial q_j} = \phi \, \dfrac{\partial \tilde{u}}{\partial L} \Big/ \dfrac{\partial \tilde{u}}{\partial q_j}$

iv. $\dfrac{\partial u^B}{\partial q_i} \Big/ \dfrac{\partial u^B}{\partial q_j} \neq p_i/p_j$ unless $\dfrac{\partial u^B}{\partial q_i} \Big/ \dfrac{\partial u^B}{\partial q_j} = \dfrac{\partial \tilde{u}}{\partial q_i} \Big/ \dfrac{\partial \tilde{u}}{\partial q_j}$

Ans: The first two state that marginal rates of substitution between goods and between goods and leisure are equal to pretax prices for the more able. Neither their goods choices nor their labour supply are therefore distorted. The third shows that the labour supply of the less able is generally distorted since the required equality will not generally hold, both because the two MRSs are evaluated at different levels of labour supply and because of the premultiplying factor ϕ*. The fourth shows that the goods choices of the less able may or may not be distorted: if goods and leisure are separable then the MRSs will be equal and there will be no distortion but if not then goods choice of the less able will also be distorted.*

(c) What do we learn from this about the usefulness of commodity taxation as a tool for redistribution?

Ans: If there is available an optimum nonlinear labour income tax then the case for nonuniform commodity taxation can only rest on nonseparability.

6. Suppose preferences are such that an individual working L hours and consuming c has utility $u(c, L)$. Individuals have pretax wages w distributed according to distribution function $F(w)$.

Suppose the government sets a tax function T on labour income $y = wL$ from the class $\mathfrak{T}$ of functions which are increasing and linear above a threshold Z

$$T(y) \begin{cases} = t_0(y) & \text{if } y \leq Z \\ = t_0(Z) + \tau\,[y - Z] & \text{if } y \geq Z \end{cases}$$

where t_0 is an increasing function and τ is a positive scalar.

Let

$$v(w; T) = \max_L u(wL - T(wL), L)$$
$$V(w, \tau, G) = \max_L u(w(1-\tau)L + G, L).$$

Suppose the government objective is to maximise social welfare

$$W(T) \;=\; \int_0^\infty v(w;T)\,\mathrm{d}F(w)$$

subject to a revenue requirement

$$\int_0^\infty T(wL)\,\mathrm{d}F(w) \geq 0.$$

Let $\Psi(R)$ denote the maximum contribution to social welfare of those with pretax incomes below Z if revenue of R is raised from those with incomes above Z

$$\begin{aligned}\Psi(R) \;=\; \max_{T\in\mathfrak{T}} \int_{wL\leq Z} v(w;T)\,\mathrm{d}F(w) \\ \text{subject to } \int_{wL\leq Z} T(wL)\,\mathrm{d}F(w) \geq -R.\end{aligned}$$

Optimum choice of τ therefore solves

$$\max_{\tau} \int_{wL \geq Z} V\left(w, \tau, G(\tau)\right) \, \mathrm{d}F(w) + \Psi\left(\tau \int_{wL \geq Z} (wL - Z) \, \mathrm{d}F(w)\right)$$

where $G(\tau) = \tau Z - t_0(Z)$.

(a) Show that

$$\frac{\tau}{1-\tau} = \frac{\int_z^\infty \left(wL/Z - 1\right)\left(1 - \theta/\lambda\right)\ \mathrm{d}F(w)}{\int_z^\infty \left(wL\eta/Z - \phi\right)\ \mathrm{d}F(w)}$$

where

$$\theta = \partial V\left(w, \tau, G(\tau)\right)/\partial G \qquad \lambda = \Psi'\left(\tau \int_{wL \geq Z} (wL - Z)\,\mathrm{d}F(w)\right)$$

$$\eta = \frac{\partial \ln L}{\partial w(1-\tau)} \qquad \phi = \frac{w(1-\tau)L}{G}\frac{\partial \ln L}{\partial \ln G}$$

Ans: First order condition wrt τ is

$$\begin{aligned}
0 = & \int \left(V_\tau + ZV_G\right)\, \mathrm{d}F + \Psi' \left[\int (wL - Z)\,\mathrm{d}F + \tau \int w\left(L_\tau + ZL_G\right)\, \mathrm{d}F\right] \\
= & -\int \theta\left(wL - Z\right)\, \mathrm{d}F + \Psi' \left[\int (wL - Z)\,\mathrm{d}F + \tau \int w\left(L_\tau + ZL_G\right)\, \mathrm{d}F\right] \\
= & -\int \theta\left(wL - Z\right)\, \mathrm{d}F + \\
& \qquad \Psi' \left[\int (wL - Z)\,\mathrm{d}F + \frac{\tau}{1-\tau} \int \left(wL\eta + Z\phi\right)\, \mathrm{d}F\right]
\end{aligned}$$

which can be rearranged to give the required expression.

(b) Suppose that θ, η and ϕ are roughly constant above high enough Z and that preferences and the income distribution are such that

$$\int_{wL \geq Z} wL \, \mathrm{d}F(w)/Z(1 - F(Z)) \to \gamma \qquad \text{as} \qquad Z \to \infty.$$

What happens to the optimum marginal tax rate on the highest incomes?

Ans: Marginal tax rates on the most able tend to

$$\frac{\tau}{1-\tau} = \frac{(\gamma - 1)(1 - \theta/\lambda)}{\gamma\eta - \phi}$$

(c) Discuss the usefulness of this sort of approach in thinking about setting of top marginal tax rates.

Ans: This is Saez' argument. The idea is that all terms except θ/λ can be empirically determined and θ/λ is a question of value judgment.

7. A population consists of N individuals with incomes y_i, $i = 1, \ldots, N$. Each has additive preferences over their consumption of a private good q_i and collective consumption of a public good Q

$$u(q_i, Q) = \phi(q_i) + \Phi(Q)$$

where ϕ and Φ are concave functions. Both public and private goods are measured in units such that their prices are 1.

Consider the Nash equilibrium in which each individual chooses to allocate their private budget between spending on their private good q_i and contribution towards spending on the public good $Q_i = y_i - q_i$ with $Q = \sum_i Q_i$ so as maximise their own utility subject to the spending decisions of the other individuals in the population

$$\max_{q_i} \left[\phi(q_i) + \Phi\left(y_i - q_i + \sum_{j \neq i} Q_j \right) \right]$$

(a) Let M be the number of individuals who choose to contribute to the public good and Y denote their total income. Show that the equilibrium public good provision solves

$$Y = Q + M\psi\left(\Phi'\left(Q\right)\right)$$

where ψ is the inverse of the derivative of ϕ (or in other words the function defined by $\phi'\left(\psi\left(U\right)\right) = U$).

Ans: First order condition is $\phi'(q_i) = \Phi'(Q)$ from which it follows that $q_i = \psi(\Phi'(Q))$. Adding across the contributors gives $Y - Q = M\psi(\Phi(Q))$.

(b) Hence explain why

- the quantity of the public good chosen is unaffected by changes in the distribution of income among these contributors, provided that all continue to contribute
- all contributors enjoy the same standard of living whatever the difference in their incomes y_i and this standard of living is higher than that of noncontributors
- an increase in the number of contributors with mean contributor income Y/M unchanged increases public good provision
- an increase in the number of contributors with total contributor income Y unchanged decreases public good provision.

Ans: The condition $Y - Q = M\psi(\Phi(Q))$ *implicitly defines* Q *as a function of* Y *alone so the internal distribution of income among contributors is irrelevant. The contributors are those with an interior solution which requires* $\psi(\Phi'(Q)) \leq y_i$*. Those for whom this holds all consume* $\psi(\Phi'(Q))$ *of the private good and those for whom it does not enjoy the same quantity of the public good but less of the private good. Differentiating, holding* Y/M *constant, gives* $dQ/dM = \left(Y/M - \psi\right) / \left(1 + M\psi'\Phi''\right) > 0$*. Differentiating, holding* Y *constant, gives* $dQ/dM = -\psi / \left(1 + M\psi'\Phi''\right) < 0$.

(c) Comment on the relevance of these facts to analysis of individual welfare in multi-member households.

Ans: If households behave noncooperatively and income is fairly equal within then the welfare of the household members is unaffected by the internal distribution, all are equally well off and an appropriate equivalence scale lies between 1 and M.

8. Individuals have preferences over a private good (bread) q, a public good (circuses) Q and leisure H given by utility function

$$u(q, Q, H) = 4q(1 - H) \; + \; \ln(2Q) \, .$$

Individual wages are w, distributed according to distribution function $F(w)$, and producer prices of both goods are 1. The government can raise revenue by taxing bread at rate t, so that its consumer price is $1 + t$, and uses the revenue raised to provide circuses.

(a) Show that individuals choosing q and H to maximise utility given t and Q choose

$$q = \frac{w}{2(1+t)} \qquad H = \frac{1}{2}$$

and that therefore

$$Q = t \int_0^\infty q \, \mathrm{d}F(w) = \frac{\bar{w}t}{2(1+t)}.$$

Ans: Preferences are Cobb-Douglas given Q. H is chosen to maximise $wH(1-H)/(1+t)$ which is solved at $H = 1/2$ and $q = w/2(1+t)$ follows immediately from the budget constraint. The expression for Q follows from substitution.

(b) Show that the preferred tax rate of an individual with wage w therefore satisfies $t/(1+t) = 1/w$.

Ans: Utility is $V(w,t) = w/(1+t) + \ln\left(\bar{w}t/(1+t)\right)$. *Differentiating gives* $\partial V/\partial t = -w/(1+t)^2 + 1/t(1+t)$.

Suppose the government is concerned about poverty. It judges anyone to be poor if their utility fails to reach a minimum acceptable standard captured in a poverty line $\ln z$,

$$V(w,t) = \max_{q,H} \{u(q,Q,H) \,|wH \geq (1+t)q\} < \ln z.$$

(c) Show that the fraction of the population who are poor is

$$\mathcal{H}(t) = F(w^*) \qquad \text{where} \qquad w^* = (1+t)\ln\left[\frac{z(1+t)}{\bar{w}t}\right]$$

and that this is minimised at tax rate t_H where $t_{\mathcal{H}}/(1+t_{\mathcal{H}}) = 1/w^*$.
Ans: An individual is poor if $V(w,t) = w/(1+t) + \ln(\bar{w}t/(1+t)) < \ln z$ *which is so iff* $w < w^*$. *The number of poor is therefore* $F(w^*)$. *This is minimised by minimising* w^* *and* $\partial w^*/\partial t = \ln\left[\frac{z(1+t)}{\bar{w}t}\right] + (1+t)\left[1/(1+t) - 1/t\right] = w^*/(1+t) - 1/t$. *Minimising the number of poor must involve maximising the utility of the person at the poverty line.*

(d) The aggregate shortfall is

$$\begin{aligned}\mathcal{G}(t) &= \int_0^{w^*} \left[\ln z - V(w,t)\right] \, \mathrm{d}F(w) \\ &= F(w^*) \ln \left[\frac{z(1+t)}{\bar{w}t}\right] - \frac{1}{1+t} \int_0^{w^*} w \, \mathrm{d}F(w).\end{aligned}$$

Noting that $\partial V(w, t_H)/\partial t > 0$ for $w < w^*$, explain why $\mathcal{G}(t)$ is minimised at a tax rate higher than $t_{\mathcal{H}}$.

Ans: $\partial \mathcal{G}/\partial t = [\ln z - V(w^*, t)]\, f(w^*) - \int_0^{w^*} \partial V(w, t_H)/\partial t \, \mathrm{d}F(w) < 0$ *since* $[\ln z - V(w^*, t)] = 0$.

(e) Discuss the merits of $\mathcal{H}(t)$ and $\mathcal{G}(t)$ as targets for policy.

Ans: $\mathcal{H}(t)$ is insensitive to depth of poverty and would approve redistributing from the very poor to those near the line if it could take some of the latter out of poverty. $\mathcal{G}(t)$ does not have this problem.

ECON G011 (Public Microeconomics) 2012 Answers

1. Individuals within a population have wages w distributed according to the known distribution function $F(w)$. Preferences over consumption c, hours of work h and spending on a public good G are represented by utility function

$$u(c,h) \;=\; \sqrt{c(1-h)} + \sqrt{G}$$

There is a linear tax on labour income at rate $0 \leq t \leq 1$ with all revenue spent on the public good so that optimising individuals have utilities

$$v(w,t,G) \;=\; \max_h \sqrt{w(1-t)h(1-h)} + \sqrt{G}$$

and $t \int wh \,\mathrm{d}F(w) \;=\; G$.

(a) Show that an individual with wage w has welfare given by

$$V(w,t) \;=\; \frac{\sqrt{w(1-t)}}{2} + \sqrt{\frac{\mu_1 t}{2}}$$

where $\mu_1 = \int w \,\mathrm{d}F(w)$.

Ans: Maximising $\sqrt{w(1-t)h(1-h)}$ leads to $h = \frac{1}{2}$ at all wages and therefore $G = \frac{1}{2}t\mu_1$. Substituting into $v(w,t,G)$ gives the required result.

(b) Explain what it means for preferences to be single peaked. Show that preferences over tax rate t are single peaked.

Ans: If preferences are single peaked then either they are monotonically increasing or declining in t or there exists some t^* such that on either side of t^* the individual prefers values of t closer to t^*. For these preferences

$$\frac{\partial^2 V}{\partial t^2} \;=\; -\frac{1}{4}\left[\frac{\sqrt{w}}{2}\frac{1}{(1-t)\sqrt{1-t}} + \sqrt{\frac{\mu_1}{2}}\frac{1}{t\sqrt{t}}\right] < 0$$

so the function is concave and therefore single peaked.

(c) Show that the preferred tax rate of an individual with wage w is $t^*(w) = 2\mu_1/\left(w + 2\mu_1\right)$.

Ans: First order condition is

$$-\frac{1}{2}\frac{\sqrt{w}}{2}\frac{1}{\sqrt{1-t}} + \frac{1}{2}\sqrt{\frac{\mu_1}{2\sqrt{t}}} \;=\; 0$$

which implies $t/(1-t) \;=\; 2\mu_1/w$.

(d) Explain what a Condorcet winner is. Explain why a Condorcet winning tax rate t^C exists for these preferences. What is it?

Ans: A Condorcet winner beats any other tax rate in pairwise voting. For single peaked preferences the median preferred tax rate is a Condorcet winner. Hence the Condorcet winner is $t^C = 2\mu_1/(2\mu_1 + m)$ *where* m *is median wage.*

(e) What is the tax rate t^U that maximises utilitarian social welfare $\int V(w,t)\,\mathrm{d}F(w)$?

Ans: Social welfare is $\frac{\mu_{1/2}\sqrt{(1-t)}}{2} + \sqrt{\frac{\mu_1 t}{2}}$ *which is maximised at* $t^U = 2\mu_1/(2\mu_1 + \mu_{1/2}^2)$.

(f) What determines which is higher, t^C or t^U?

Ans: Which is higher, m *or* $\mu_{1/2}^2$.

2. A finite population consists of n individuals. Individuals have incomes $y_i,\ i = 1, \ldots, n$ and are ordered so that

$$(y_i - y_j)\,(i - j) \geq 0.$$

The population income vector is therefore

$$\mathbf{y} \;=\; (y_1,\ y_2,\ \ldots,\ y_n)\,.$$

(a) It is proposed to measure inequality by the Gini coefficient

$$G(\mathbf{y}) \;=\; \frac{2}{n^2\bar{y}} \sum_{i=1}^{n} i\,(y_i - \bar{y})$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ denotes mean income. Demonstrate and interpret each of the following:

i. $G(\bar{y},\ \bar{y}, \ldots,\ \bar{y},\ \bar{y}) = 0$
Ans: All terms in the sum $\sum_i i\,(y_i - \bar{y})$ are zero. The Gini coefficient is zero if there is perfect equality.

ii. $G(0,\ 0, \ldots, 0,\ n\bar{y}) = 1 - 1/n$
Ans:

$$\begin{aligned} G(0,\ 0, \ldots, 0,\ n\bar{y}) &= \frac{2}{n^2\bar{y}} \left[-\sum_i i\bar{y} + n(n\bar{y}) \right] \\ &= \frac{2}{n^2\bar{y}} \left[-\frac{1}{2}n^2 - \frac{1}{2}n + n^2 \right] \bar{y} \\ &= \frac{n-1}{n} \end{aligned}$$

The Gini coefficient takes a value which approaches 1 for large n in the case of maximum inequality.

iii. $G(y_1, y_2, \ldots,\ y_i + \delta, \ldots,\ y_j - \delta, \ldots,\ y_{n-1},\ y_n)$
$< G(y_1, y_2, \ldots,\ y_i, \ldots,\ y_j, \ldots,\ y_{n-1},\ y_n)$ for small enough δ
Ans:

$$\left.\frac{\partial G(y_1, y_2, \ldots,\ y_i + \delta, \ldots,\ y_j - \delta, \ldots,\ y_{n-1},\ y_n)}{\partial \delta}\right|_{\delta = 0} \;=\; \frac{2}{n^2\bar{y}} \sum_i (i - j) < 0$$

Small enough transfers from rich to poor reduce the Gini in accordance with the Pigou-Dalton principle.

iv. $G(\lambda \mathbf{y}) = G(\mathbf{y})$ for any $\lambda > 0$.
Ans: $G(\lambda \mathbf{y}) = \frac{2}{n^2 \lambda \bar{y}} \sum_i i\lambda (y_i - \bar{y}) = G(\mathbf{y})$. *The Gini index is a relative inequality index in the sense that it is invariant to scaling up or down the income vector.*

You may want to use the fact that $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$.

(b) Suppose that individuals require a minimal subsistence income z to survive which is non-negligible relative to $\bar{y}$.

i. Show that $G(\mathbf{y})$ cannot exceed $1 - z/\bar{y}$.
Ans: Individual incomes cannot fall short of z *so maximum inequality is*

$$\begin{aligned} G(z,\ z, \ldots, z,\ n(\bar{y} - z)) &= \frac{2}{n^2 \bar{y}} \left[-\sum_i i\,(z - \bar{y}) + n(n\bar{y} - (n+1)z) \right] \\ &= (1 - 1/n) - (1 + 1/n) z/\bar{y} < 1 - z/\bar{y} \end{aligned}$$

ii. It is suggested instead to use $\Gamma(\mathbf{y}) = \bar{y} G(\mathbf{y})/(\bar{y} - z)$ to measure inequality. How would you interpret this index?
Ans: This is a rescaling relative to the maximum possible value. Γ *is* G *as a proportion of its maximum possible value. Alternatively note that* $\Gamma(\mathbf{y}) = G(\mathbf{y} - z\mathbf{1})$ *so that it can be interpreted as the inequality in the excess of* y *over subsistence income* z.

iii. Explain why $\Gamma(\mathbf{y})$ is a Schur-convex measure bounded between 0 and 1 and show that

$$\Gamma(\lambda \mathbf{y} + (1 - \lambda) z \mathbf{1}) = \Gamma(\mathbf{y})$$

where $\mathbf{1} = (1, 1, \ldots, 1, 1)$.
Ans: Given the earlier results Γ *is zero when there is perfect equality and obeys the Pigou-Dalton criterion (and is therefore Schur-convex). It was shown previously that* G *cannot exceed* $1 - z/\bar{y}$ *so* Γ *cannot exceed 1. The final results follows from* $\Gamma(\lambda \mathbf{y} + (1-\lambda) z\mathbf{1}) = G(\lambda [\mathbf{y} - z\mathbf{1}]) = G(\mathbf{y} - z\mathbf{1}) = \Gamma(\mathbf{y})$.

iv. Discuss the suitability of $\Gamma(\mathbf{y})$ as a measure of inequality.
Ans: Γ *has all the regular properties of an inequality index except invariance to scaling of incomes. It is however invariant to scaling of* $\mathbf{y} - z\mathbf{1}$ *which is a natural and defensible property in context.*

3. Individuals have preferences over consumption c and hours of work h given by utility functions $u(c,h) = \ln c - h$. Suppose government is constrained to consider linear tax schedules so that $c = w(1-t)h + G$ where w is individual wage rate and t and G are tax parameters. Wages are distributed in the population according to distribution function $F(w)$ where $F(w_{min}) = 0$ for some $w_{min} > 0$. Assume throughout the question that w_{min} is high enough that all individuals choose to work at tax rates under consideration. The government revenue constraint requires that $\int wth \, \mathrm{d}F(w) = G$.

   (a) Show that if individuals choose optimally then $G = \mu_1 t(1-t)$ where $\mu_1 = \int w \, \mathrm{d}F(w)$.
   Ans: First order condition for choice of hours h is $w(1-t)/(w(1-t)h + G) = 1$ so $h = 1 - G/w(1-t)$. By the government revenue constraint $G = t\mu_1 - Gt/(1-t)$ from which $G = \mu_1 t(1-t)$.

   (b) Show that individual utilities are therefore

$$V(w,t) = \ln w + \ln(1-t) + t\mu_1/w - 1.$$

   Ans: $w(1-t)h + G = w(1-t)$ so $V(w,t) = \ln[w(1-t)] - 1 + \mu_1 t/w$.

(c) Suppose the government objective is to maximise social welfare given by $\Omega(t) = \int V(w,t)\, \mathrm{d}F(w)$. Show that the optimum tax rate is

$$t^* = \frac{\mu_1\mu_{-1} - 1}{\mu_1\mu_{-1}}$$

where $\mu_{-1} = \int (1/w)\, \mathrm{d}F(w)$
Ans: $\Omega(t) = \int \ln w\, \mathrm{d}F(w) + \ln(1-t) - 1 + t\mu_1\mu_{-1}$. *First order condition for maximising is* $1/(1-t) = \mu_1\mu_{-1}$ *so* $t^* = \frac{\mu_1\mu_{-1}-1}{\mu_1\mu_{-1}}$.

(d) Let $\mu_\alpha = \int w^\alpha\, \mathrm{d}F(w)$ for any α. Use the inequality

$$\mu_\beta\mu_\gamma \geq \mu^2_{(\beta+\gamma)/2} \qquad \text{for any } \beta,\ \gamma$$

to show that $0 \leq t^* \leq 1$.
Ans: By the inequality $\mu_1\mu_{-1} \geq 1$ *since* $\mu_0 = 1$.

(e) Suppose wages follow a Pareto distribution with parameters $a > 2$ and $b > 0$ so that

$$\mu_\alpha = \frac{ab^\alpha}{a - \alpha}$$

for $\alpha < a$. Derive an expression for t^* as a function of a and b.
Ans: $\mu_1 = ab/(a-1)$ *and* $\mu_{-1} = a/b(a+1)$ *so* $\mu_1\mu_{-1} = a^2/(a^2-1)$ *and* $t^* = a^{-2}$.

4. (a) Individuals consume two goods, food, q_1, and clothing, q_2, and supply labour, L. Utilities are given by $U = u(q_1, q_2, L)$. Individuals face the same pretax prices for goods, p_1 and p_2, but differ in unobserved productivity w, which is known to be distributed according to distribution function $F(w)$ (with associated density function $f(w)$). Government can raise revenue by nonlinear taxation of earned income $Y = wL$ and spending on clothing $X_2 = p_2 q_2$. Let the tax function be $T(Y, X_2)$.

Show that the rate of change of utility with w is

$$\frac{\mathrm{d}U}{\mathrm{d}w} = \frac{\partial u}{\partial L}\frac{L}{w}.$$

Ans:

$$\frac{\mathrm{d}U}{\mathrm{d}w} = \frac{\partial u}{\partial q_1}\frac{\partial q_1}{\partial w} + \frac{\partial u}{\partial q_2}\frac{\partial q_2}{\partial w} + \frac{\partial u}{\partial L}\frac{\partial L}{\partial w}$$

But from individual optimisation

$$p_1 = -w\frac{\partial u/\partial q_1}{\partial u/\partial L} \qquad p_2 = -w\frac{\partial u/\partial q_2}{\partial u/\partial L}$$

and from the individual budget constraint

$$0 = p_1\frac{\partial q_1}{\partial w} + p_2\frac{\partial q_2}{\partial w} - w\frac{\partial L}{\partial w} - L$$

from which

$$\frac{\mathrm{d}U}{\mathrm{d}w} = \frac{\partial u}{\partial L}\frac{L}{w}.$$

(b) Suppose the government wishes to choose the tax function to maximise average utility $\int U \,\mathrm{d}F(w)$ subject to the constraint of raising a revenue requirement per head of R, $\int [wL - p_1 q_1 - p_2 q_2]\ \mathrm{d}F(w) > R$.

Setting this up as an optimal control problem with Hamiltonian

$$\mathcal{H} = \{U + \lambda [wL - p_1 q_1 - p_2 q_2]\} f(w) + \mu \frac{\partial u}{\partial L}\frac{L}{w},$$

state variable U and controls L and q_2, show that first order conditions require

$$\lambda\left[w - p_1 \left.\frac{\partial q_1}{\partial L}\right|_U\right] f(w) + \mu\left[\left[\frac{\partial^2 u}{\partial L^2} + \frac{\partial^2 u}{\partial L \partial q_1}\left.\frac{\partial q_1}{\partial L}\right|_U\right]\frac{L}{w} + \frac{\partial u}{\partial L}\frac{1}{w}\right] = 0$$

$$\lambda\left[-p_2 - p_1 \left.\frac{\partial q_1}{\partial q_2}\right|_U\right] f(w) + \mu\left[\frac{\partial^2 u}{\partial L \partial q_2} + \frac{\partial^2 u}{\partial L \partial q_1}\left.\frac{\partial q_1}{\partial q_2}\right|_U\right]\frac{L}{w} = 0$$

Ans: These are $\frac{\partial\mathcal{H}}{\partial L}\Big|_U = 0$ *and* $\frac{\partial\mathcal{H}}{\partial q_2}\Big|_U = 0$.

(c) Under which of the following preferences will the optimum involve a zero marginal tax rate on clothing, $\partial T/\partial X_2 = 0$?

i. $u(q_1, q_2, L) = q_1^\alpha q_2^\beta (1 - L)$

ii. $u(q_1, q_2, L) = q_1^\alpha + q_2^\beta (1 - L)$

iii. $u(q_1, q_2, L) = q_1^\alpha + q_2^\beta + (1 - L)$

Explain.

Ans: If q_1 *and* q_2 *are separable from leisure,* $u(q_1, q_2, L) = \upsilon(\phi(q_1, q_2), L)$, *then*

$$\left[\frac{\partial^2 u}{\partial L \partial q_2} + \frac{\partial^2 u}{\partial L \partial q_1} \frac{\partial q_1}{\partial q_2}\bigg|_U\right] = \frac{\partial^2 \upsilon}{\partial L \partial \phi}\left[\frac{\partial \phi}{\partial q_2} + \frac{\partial \phi}{\partial q_1} \frac{\partial q_1}{\partial q_2}\bigg|_U\right] = 0$$

in which case the first order condition requires

$$-\frac{p_2}{p_1} = \frac{\partial q_1}{\partial q_2}\bigg|_U = -\frac{p_2(1 + \partial T/\partial X_2)}{p_1}$$

and therefore $\partial T/\partial X_2 = 0$. *Without separability this will not generally be true. Separability is therefore the critical condition and is satisfied by the first and third preferences.*

5. (a) Consider a population of individuals with incomes x distributed according to $F_x(x)$. An individual is considered poor if $z \geq x$ where z is a poverty line. Let $\tilde{x} = \min[x, z]$ denote income capped at z.

The following are all possible measures of poverty.

- The headcount index is $\mathcal{H}(F_x) = F_x(z)$.
- The shortfall index is $\mathcal{S}(F_x) = 1 - \int \tilde{x}\ \mathrm{d}F_x(x)/z$.
- Let $\psi(x)$ be a concave function and define $\xi(F_x)$ by $\psi(\xi(F_x)) = \int \psi(\tilde{x})\ \mathrm{d}F_x(x)$. The Clark-Hemming-Ulph index is $\mathcal{P}(F_x) = 1 - \xi(F_x)/z$.

i. Discuss the advantages of these three indices.

Ans: $\mathcal{H}$ increases if there is any fall in income taking someone across the poverty line. It is insensitive however to falls in the incomes of those already poor. $\mathcal{S}$ is sensitive to falls in incomes of the poor but not to transfers of income from the poorer poor to the less poor poor. $\mathcal{P}$ has none of these disadvantages.

ii. Let $\mathcal{I}(F_x)$ be the Atkinson-Kolm-Sen inequality index calculated on $\tilde{x}$. Show that

$$\mathcal{P}(F_x) = 1 - (1 - \mathcal{I}(F_x))\,(1 - \mathcal{S}(F_x))$$

Ans: $\mathcal{I} = 1 - \xi/\int \tilde{y}\ \mathrm{d}Fy = 1 - (1 - \mathcal{P})z/(1 - \mathcal{S})z.$

(b) A population has pretax incomes y distributed according to $F_y(y)$ on closed support $[y_{min}, y_{max}]$ where $y_{min} < z < y_{max}$. Suppose there is a proportional tax at rate t, with a proportion $1 - \phi$ of revenue being lost to administrative costs and the remainder returned as a lump sum grant $\phi t\bar{y}$ so that posttax incomes are $Y = (1-t)y + \phi t\bar{y}$ where $\bar{y}$ is mean pretax income $\int y\ \mathrm{d}F_y(y)$. You may assume there is no behavioural response to taxation.

Let $t_{\mathcal{H}}$ be the lowest tax rate in the interval [0,1) that minimises $\mathcal{H}(F_Y)$, $t_{\mathcal{S}}$ be the lowest tax rate in the interval [0,1) that minimises $\mathcal{S}(F_Y)$ and $t_{\mathcal{P}}$ be the lowest tax rate in the interval [0,1) that minimises $\mathcal{P}(F_Y)$.

Show that

i. if $z < \phi\bar{y}$ then

$$t_{\mathcal{H}} = t_{\mathcal{S}} = t_{\mathcal{P}} = \frac{z - y_{min}}{\phi\bar{y} - y_{min}}$$

Ans: Poverty can be eliminated. The given tax rate is the smallest to raise someone on y_{min} to z

ii. if $z > \phi\bar{y}$ then

- $t_{\mathcal{H}} = 0$
  Ans: The breakeven point $y = \phi\bar{y}$ is below the poverty line z. Raising t therefore reduces the income of someone at the poverty line and increases the number of poor. The number of poor is therefore minimised at $t = 0$.
- $t_{\mathcal{S}} > 0$ only if $\mathcal{S}(F_y) \; > \; \mathcal{H}(F_y)\,(1 - \phi\bar{y}/z)$
  Ans: Although the number of poor is increased by $t > 0$ the incomes of individuals below $\phi\bar{y}$ are increased and the shortfall may therefore be reduced.

$$\mathcal{S} = \mathcal{H} - \frac{1}{z}\int_{y_{min}}^{(z-\phi t\bar{y})/(1-t)} (y(1-t) + \phi t\bar{y}) \; \mathrm{d}F$$

so

$$\begin{aligned} \left.\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t}\right|_{t=0} &= \frac{1}{z}\int_{y_{min}}^{z} (y - \phi\bar{y}) \; \mathrm{d}F \\ &= [\mathcal{H}(F_y) - \mathcal{S}(F_y)] - \frac{\phi\bar{y}}{z}\mathcal{H}(F_y) \end{aligned}$$

and $\mathcal{S}$ is decreasing in t at $t = 0$ iff $\mathcal{S}(F_y) \; > \; \mathcal{H}(F_y)\,(1 - \phi\bar{y}/z)$

Does this illustrate anything regarding the suitability of these indices as guides to policy?

Ans: The headcount puts too much emphasis on what is happening to individuals close to the poverty line at the cost of insensitivity to the effects of policy on those deeper in poverty.

6. (a) A population consumes a public and a private good. The ith individual consumes q_i of the private good and the population collectively consumes Q of the public good. There are two types of individuals, one type with preferences

$$u_i = q_i + \ln Q$$

and the other with preferences

$$u_i = q_i + Q.$$

Suppose there are N_1 individuals of the first type and N_2 individuals of the second type. The marginal rate of transformation between the private and public good is constant at P.

Show that optimum production of the public good, assuming $P > N_2$, is

$$Q \;=\; N_1 / \left(P - N_2\right).$$

Ans: From the Samuelson condition $P = N_1/Q + N_2$

(b) Explain what Lindahl prices are and how they could be used to decentralise public good provision if known. What are the Lindahl prices in this case and what incentives do individuals have to reveal them?

Ans: Lindahl prices are equal to individual marginal rates of substitution between the public and private good. For individuals in the first group Lindahl prices are $1/Q$ *and in the second* 1. *Individuals have no incentive to reveal these accurately if they expect to be charged according to them - they will claim to be of whichever type has the lower price which depend upon whether* Q *is greater than or less than one.*

(c) Consider a newcomer. Suppose the individual is asked to reveal their type by declaring a value δ, $0 \le \delta \le 1$, and told that they will be charged

$$T \;=\; P\hat{Q}(\delta) - N_1 \ln \hat{Q}(\delta) - N_2 \hat{Q}(\delta)$$

where $\hat{Q}(\delta) = (N_1 + \delta)/(P - N_2 - 1 + \delta)$.

Show that the individual will reveal their type truly and discuss the general principles which this exemplifies.

Ans: Let income be y. *Utility is* $y - [P - N_2]\hat{Q} + (N_1 + 1)\ln \hat{Q}$ *if of type 1 which is maximised by choosing* δ *so that* $\left[(N_1 + 1)/\hat{Q} + N_2 - P\right] \mathrm{d}\hat{Q}/\,\mathrm{d}\delta = 0$; *utility is* $y - [P - N_2 - 1]\hat{Q}(\delta) + N_1 \ln \hat{Q}$ *if of type 1 which is maximised* *by choosing* δ *so that* $\left[N_1/\hat{Q}(\delta) + N_2 + 1 - P\right]\ \mathrm{d}\hat{Q}/\ \mathrm{d}\delta = 0$. *In each case the condition is satisfied by choosing* δ *to reflect the individual's type accurately.*

This is a Clarke-Groves-Ledyard type scheme based on the mechanism design literature.

7. (a) Consider an individual with hourly wage w and unearned income m who works hours L. This individual pays tax $T(wL)$ which is zero if pretax income $wL+m$ is below a threshold E but equal to $t(wL+m-E)$ if it is above.

Suppose the individual has convex preferences captured in utility function $u(c,L,\epsilon)$ where $c = wL - T(wL)$ is consumption and ϵ is a taste parameter. Let $h(w,m,\epsilon)$ denote utility-maximising choice of hours for someone facing no taxation. Explain why utility-maximising choice of hours for someone facing the tax schedule described is

$$H(w,m,t,E,\epsilon) \;=\; \begin{cases} h(w,m,\epsilon) & \text{if} \quad h(w,m,\epsilon) \leq ((E-m)/w \\[2ex] h(w(1-t),m+tE,\epsilon) & \text{if} \quad \begin{array}{c} h(w(1-t),m+tE,\epsilon) \\ \geq (E-m)/w \end{array} \\[2ex] (E-m)/w & \text{if} \quad \begin{array}{c} h(w,m,\epsilon) > (E-m)/w \\ > h(w(1-t),m+tE,\epsilon) \end{array} \end{cases}$$

Ans: The budget set is strictly convex so any optimum is unique and occurs in one or other tax bracket or at the kink.

(b) Suppose a cross-sectional sample of n individuals facing such a tax function with wages w_i and unearned incomes m_i, $i = 1,\ldots,n$, is to be used to estimate labour supply elasticities for assessment of the deadweight loss of taxation. It is decided to fit a model of the form

$$L_i \;=\; H(w_i,m_i,t,E,\epsilon_i) + \eta_i$$

by maximum likelihood where ϵ_i and η_i are to be treated as independent stochastic error terms with distributions of known form.

Explain why it is empirically useful to allow for two error terms and how it is possible for the distributions of the two error terms ϵ_i and η_i to be identified. Discuss the merits of this idea.

Ans: With the preference error only there is too much clustering predicted at the kink; with the optimisation error only there is too little. The two distributions can be separately identified because the preference error unlike the optimisation error does not locally affect choice if the optimum is at the kink. The merits and disadvantages are discussed at length in the lecture notes.

(c) Suppose the tax structure were such that no tax were paid below the threshold E but that individuals above the threshold paid twL rather than $t(wL-E)$. How would that change the analysis and the applicability of the suggested procedure?

Ans: The budget set would no longer be convex. Uniqueness of local optima is no longer guaranteed and solving the optimisation requires searching for all possible optima and comparing utility. The approach with two types of error remains applicable.

8. (a) Suppose the government raises revenue through a tax on income X which has a constant marginal rate t in a top bracket starting at earnings of ξ. Let R denote the revenue from tax in the top bracket.

   i. Show that

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left[E(X|X \geq \xi) - \frac{t}{1-t}E(X\eta|X \geq \xi) - \xi\right]$$

   where $\eta = \partial \ln X/\partial \ln(1-t)$ is the elasticity of taxable income X to the net-of-tax rate $1-t$ and it is assumed that the threshold ξ is held constant.

   Ans: Revenue is $R = t\,(E(X|X \geq \xi) - \xi)$ so by straightforward differentiation $\mathrm{d}R/\,\mathrm{d}t = (E(X|X \geq \xi) - \xi) - tE(\,\mathrm{d}X/\,\mathrm{d}t|X \geq \xi)$.

   ii. Suppose γ is the social marginal value of consumption by individuals in the top tax bracket measured relative to government revenue. Explain why the optimum value for t is given by

$$\frac{t}{1-t} = \frac{(1/\xi)E(X|X \geq \xi) - 1}{(1/\xi)E(X\eta|X \geq \xi)}(1-\gamma).$$

   Ans: By the envelope theorem, we can ignore behavioural responses in evaluating marginal welfare effects on the rich. In revenue terms, the marginal welfare loss is therefore $\gamma\,(E(X|X \geq \xi) - \xi)$. Equating this to the revenue gain gives the formula.

(b) The following quotation is from Emmanuel Saez ("Using elasticities to derive optimal income tax rates", *Review of Economic Studies* 2001):

> "[O]ptimal income taxation has interested mostly theorists and has not changed the way applied public finance economists think about the equity-efficiency tradeoff. Though behavioural elasticities are the key concept in applied studies, there has been no

> systematic attempt to derive results in optimal taxation which could be easily used in applied studies. As a result, optimal income tax theory is often ignored and tax reform discussions are centred on the concept of deadweight burden."

Assess this evaluation of the optimum income tax literature and the value of formulae such as that derived above as a practical guide to tax setting, discussing how the method might be extended to the determination of rates in lower tax brackets. Hence evaluate the contribution of this sort of analysis to changing the practical relevance of optimum tax theory to applied analysis.

Ans: Everything except γ *in the above formula is in principle estimable. In particular it may be illuminating to consider the formula as* $\xi \to \infty$*. Extending to lower tax brackets is less easy because of the need to consider welfare and revenue effects on those in higher brackets but is still possible. This sort of method has had several recent applications.*

ECONG011 Public Microeconomics

Revision Session

1. You are undertaking a study of inequality in a population of n individuals where the vector of incomes is $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

(a) Outline properties suitable for a measure of inequality.

Ans: Homogeneity, Schur convexity

(b) It is suggested that inequality be measured by

$$\mathfrak{I}(\mathbf{y}) = \frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{\bar{y}}{y_i}\right)$$

Discuss the properties of the index as they are relevant to its suitability as a measure of inequality.

Ans: It is clearly homogeneous and also Schur convex since

$$\left[\frac{d\mathfrak{I}}{dy_i} - \frac{d\mathfrak{I}}{dy_j}\right][y_i - y_j] \;=\; \left[\frac{1}{y_j} - \frac{1}{y_i}\right][y_i - y_j] > 0$$

(c) It is also proposed that social welfare be measured by $\mathfrak{W}(\mathbf{y}) = \bar{y} \exp(-\mathfrak{I}(\mathbf{y}))$. Justify this as a social welfare measure.

Ans: $\mathfrak{I} \in [0, \infty)$ *so* $1 - e^{-\mathfrak{I}} \in [0, 1]$ *is an inequality measure in the unit interval and* $\bar{y}e^{-\mathfrak{I}}$ *can be interpreted as the corresponding equally distributed equivalent income. In fact this is simply geometric mean income.*

(d) Government considers introducing a proportional income tax at rate t. A fraction γ of the proceeds would be absorbed by administrative costs and the remainder would be returned as a uniform lump sum payment $(1-\gamma)t\bar{y}$. What is the largest value for γ compatible with such a scheme for redistribution raising social welfare?

Ans: Noting that social welfare is just geometric mean income, social welfare varies with $\frac{1}{n}\sum \ln\left[(1-t)y_i + (1-\gamma)t\bar{y}\right]$. *This is increasing in* t *at* $t=0$ *only if* $(1-\gamma)\frac{1}{n}\sum \bar{y}/y_i > 1$ *hence the largest* γ *is* $1 - 1/\frac{1}{n}\sum \bar{y}/y_i$.

2. Individuals in a certain population of N individuals consume a private and a public good. An individual i consuming q_i of the private good and Q of the public good enjoys utility

$$U_i = q_i - \max\{\delta_i - Q, 0\}$$

where δ_i is an individual preference threshold. There is a constant marginal rate of transformation between the two goods so that producing a quantity Q of the public good requires forgoing PQ units of the private good where $N > P > 1$.

(a) Explain the Samuelson condition for efficient supply of the public good Q^* and apply it to this case.

Ans: The Samuelson condition says that the sum of MRSs equals P. In this case, $\#_i\{\hat{\delta}_i > Q\} = P$.

(b) Suppose supply of the public good is left to voluntary provision. What condition would you expect to be satisfied by the chosen supply Q°. How much will be supplied in equilibrium given these preferences?

Ans: Only those with MRS of at least P contribute. In this case, that is no-one.

(c) Suppose supply of the public good is decided instead by majority voting with provision funded by equal per-head tax payments PQ/N. Explain what a Condorcet winner is, show that a Condorcet winner $Q^\dagger$ exists in this case and identify it. Is $Q^\dagger$ greater than or less than Q^*?

Ans: A Condorcet winner beats any other option in pairwise majority voting. In this case one exists since preferences over Q are single peaked and the winner is the median δ_i. In other words $\#_i\{\delta_i > Q^\dagger\} = \frac{1}{2}N$. Whether $Q^\dagger$ exceeds Q^* depends upon whether $P > \frac{1}{2}$ or not.

(d) Suppose government asks individuals to declare a value of the preference parameter $\hat{\delta}_i$ and announces that it will provide according to the Samuelson condition assuming reported preferences to be accurate while charging each individual a tax

$$T_i = [P - \#\{j \mid j \neq i,\ \hat{\delta}_j > Q\}]Q$$

where $\#\{X\}$ denotes the number of individuals in the set X. Discuss the advantages of such a system of provision.

Ans: This is a Clark-Groves scheme. It incentivises truthful preference revelation but may not satisfy budget balance.

3. The residents of a certain city, N, need to choose between three possible public projects, x, y and z. Individual preferences among those residents fall into four classes

$$\begin{array}{ll} x \succ_i y \succ_i z & i \in A \\ z \succ_i y \succ_i x & i \in B \\ y \succ_i x \succ_i z & i \in C \\ y \succ_i z \succ_i x & i \in D \end{array}$$

where $A \cup B \cup C \cup D = N$. The number of residents is at least three.

Suppose social preference, $\succsim^*$, is decided by majority voting

$$u \succsim^* v \qquad \Leftrightarrow \qquad \#\{i \mid u \succ_i v\} \geq \#\{i \mid u \prec_i v\}$$
$$u,\ v \in \{x, y, z\}$$

where $\#\{X\}$ denotes the number of individuals in the set X.

(a) Show that social preference satisfies

(i) completeness

Ans: Individual preferences are complete so numbers can always be counted

(ii) transitivity

Ans: If either A or B has a majority then their preferences dictate majority preference which is therefore transitive. If neither has a majority then y is majority-preferred to both x and z so there cannot be a social preference cycle.

(iii) independence of irrelevant alternatives

Ans: Social preference is only ever reliant on individual preference between the pair under comparison.

(iv) the Pareto principle

Ans: If everyone has a preference then this must be the majority preference.

(v) nondictatorship

Ans: No-one's preference determines social preference without support from others.

explaining in each case what that means.

(b) Suppose instead that, in addition to the four groups described above, a fifth preference group exists

$$x \succ_i z \succ_i y \qquad i \in E$$

Show that social preference need no longer be transitive.

Ans: Suppose E, B and C each constitute a third of the population. Then there is a cycle in majority preference $z \succ^* y \succ^* x \succ^* z$

(c) Explain the general principles exemplified by this example and discuss how these principles can be extended to cases of choice over a continuum of alternatives.

Ans: The important thing is that y *is no-one's least preferred option in* A*,* B*,* C *or* D*. This is how cycles are avoided with single-peaked preferences which is the natural extension to a continuous choice space.*

4. Let y denote individual income, distributed according to cumulative distribution function $F(y)$, and z be the poverty line. Let $\zeta(y) = \min(y, z)$, $u(y)$ be a strictly increasing, strictly concave function and $\psi(y) = u(\zeta(y))$.

(a) Define $m(F)$ and $\chi(F)$ by:

$$m(F) = \int_0^\infty \zeta(y)\, \mathrm{d}F(y)$$
$$\psi(\chi(F)) = \int_0^\infty \psi(y)\, \mathrm{d}F(y)$$

Interpret $m(F)$ and $\chi(F)$ and show that $z \geq m(F) \geq \chi(F)$.

Ans: m is the truncated mean. χ is the equally distributed equivalent of the truncated distribution. It is obvious that $z \geq m$. By Jensen's inequality, $\psi(m) \geq \int \psi dF = \psi(\chi)$.

(b) Define:

$$
\begin{aligned}
\mathcal{H}(F) &= F(z) \\
\mathcal{S}(F) &= 1 - m(F)/z \\
\mathcal{I}(F) &= 1 - \chi(F)/m(F) \\
\mathcal{P}(F) &= 1 - \chi(F)/z
\end{aligned}
$$

Interpret each of these and discuss their suitability as measures of poverty.

Ans: $\mathcal{H}$ *is the headcount which increases as more people become poor but is insensitive to decreases in the incomes of the poor.* $\mathcal{S}$ *is the shortfall which is sensitive to decreases in the incomes of the poor but is insensitive to transfer of income between poor people.* $\mathcal{I}$ *is an Atkinson-Sen index of inequality in the truncated distribution and is not in itself a poverty measure.* $\mathcal{P}$ *is the Clark-Hemming-Ulph index which is sensitive to depth of poverty and inequality among the poor.*

(c) Suppose government has a windfall gain equal per person to $\mathcal{W}$ where $0 < \mathcal{W} < \mathcal{S}(F)z$. It wishes to redistribute the resources as income-contingent grants to alleviate poverty. How should it do so to minimise

(i) $\mathcal{H}(F)$ (ii) $\mathcal{S}(F)$ (iii) $\mathcal{P}(F)$?

Ans: To minimise $\mathcal{H}$ give to the least poor of the poor because they are cheaper to raise to the line. To minimise $\mathcal{S}$ give to any of the poor. To minimise $\mathcal{P}$ give to the poorest of the poor.

5. A population consists of two types, those with high productivity w_1 and those with low productivity $w_2 < w_1$. Individuals know their own type but government does not. There are known to be equal numbers of the two types.

Utility is $U(c, L) = u(c) - v(L)$ where u is a concave function of consumption c and v is a convex function of hours worked L.

(a) The government's objective is to maximise $\Psi(c_1, L_1) + \Psi(c_2, L_2)$ where $\Psi(c, L) = \min\left(U(c, L), \bar{U}\right)$ subject to a government budget constraint $c_1 + c_2 \leq w_1 L_1 + w_2 L_2$ and to an incentive compatibility constraint $U(c_1, L_1) \geq U(c_2, w_2 L_2 / w_1)$. Explain why the latter constraint has this form.

Ans: The constraint says that the more able have no higher utility if revealing their type than if taking the (c, wL) *combination offered to the less able and working the fewer hours necessary to earn the appropriate pretax income.*

(b) Suppose that at the optimum $U(c_1, L_1) < \bar{U}$. Explain why it is the case that at the optimum:

(i)

$$U(c_1, L_1) > U(c_2, L_2)$$

Ans: This follows easily from the incentive constraint. The more able cannot be persuaded to reveal their type without being better off than the less able.

(ii)

$$-\frac{\partial U(c_1, L_1)/\partial L_1}{\partial U(c_1, L_1)/\partial c_1} = w_1$$

Ans: Write a Lagrangean

$$U(c_1, L_1)+U(c_2, L_2)-\lambda\left[c_1 + c_2 - w_1 L_1 - w_2 L_2\right]+\mu\left[U(c_1, L_1) - U(c_2, w_2 L_2/w_1)\right]$$

First order conditions include $(1+\mu)U_{c_1} = \lambda$ *and* $(1+\mu)U_{L_1} = -\lambda w_1$ *from which the results follows. Labour supply of the more able is undistorted at the margin.*

(iii)

$$-\frac{\partial U(c_2, L_2)/\partial L_2}{\partial U(c_2, L_2)/\partial c_2} < w_2$$

Ans: First order conditions also include $(1-\mu)U_{c_2} = \lambda$ *and* $U_{L_1} - \mu(w_2/w_1)U_{w_2L_2/w_1} = -\lambda w_2$ *from which the result follows. Labour supply of the less able is distorted as a consequence of compliance with incentive compatibility.* and interpret each condition.

(c) Suppose that at the optimum $U(c_2, L_2) < \bar{U} < U(c_1, L_1)$. Is it still true that: (i)

$$\frac{\partial U(c_1, L_1)/\partial L_1}{\partial U(c_1, L_1)/\partial c_1} = w_1$$

(ii)

$$\frac{\partial U(c_2, L_2)/\partial L_2}{\partial U(c_2, L_2)/\partial c_2} < w_2?$$

Discuss.

Ans: Set up another Lagrangean

$$\bar{U} + U(c_2, L_2) - \lambda\left[c_1 + c_2 - w_1 L_1 - w_2 L_2\right] + \mu\left[U(c_1, L_1) - U(c_2, w_2 L_2 / w_1)\right]$$

First order conditions include $\mu U_{c_1} = \lambda$ *and* $\mu U_{L_1} = \lambda w_1$ *so that the nondistortion result is still true for the rich. The first order conditions and results relating to the less able are unchanged. Even this quite radically egalitarian change to the objective does not change these results.*

6. In a certain population, individuals consume a bundle of n goods q, where $n > 2$, and supply labour L, the income from which is untaxed. Wages w are distributed according to known cumulative distribution function F. Goods are measured in units such that all pretax prices are unity. Revenue can be raised only through taxes on goods t so that post-tax prices are $p = e + t$ where e is the unit vector. Individual utilities are determined by w, p and by total budgets y according to indirect utility functions denoted $V(w, p, y)$ and individual Marshallian demand functions for goods are denoted $f(w, p, y)$. Individuals have no source of incomes other than labour income or a uniform government grant G funded by the receipts of the taxes on commodities.

The government chooses commodity taxes with the objective of maximising social welfare $\int V(w,p,G)dF(w)$ subject to its revenue constraint

$$\int \left[\sum_{i=1}^{n} t_i f_i(w,p,G) - G \right] dF(w) = 0$$

(a) Explain the general principles of optimum commodity taxation in this setting.

Ans: Explain the modified Ramsey rules

(b) Two alternative models are proposed for individual preferences:

$$\text{(Model A)} \qquad V(w,p,y) = \max_L \phi\left(L, A(p)\left[wL + y\right] + B(p)\right)$$

$$\text{(Model B)} \qquad V(w,p,y) = \max_L \phi\left(L, A(p)\ln\left[wL + y\right] + B(p)\right)$$

where ϕ is in each case some function decreasing in its first argument and increasing in its second.

In which, if either, of the two cases would optimum taxes be uniform? Explain.

Ans: Model A has linear Engel curves and separability of leisure. By Deaton's results, this is sufficient for linearity to be optimal. Model B has Working Leser Engel curves and optimum taxes will not be uniform.

(c) Are the two cases empirically distinguishable? If so, how?

Ans: The two differ in the shape of Engel curves. Under Model A but not Model B, $\partial q/\partial wL$ *is constant. This is testable in principle.*

(d) Is it restrictive to assume unavailability of: (i) a linear tax (ii) a nonlinear tax on labour income?

Ans: Unavailability of a linear tax is simply a normalisation and does not restrict the solution at all. If a nonlinear tax is possible ,however, then optimum taxes will generally be uniform given separability of leisure, even under Model B.

7. The government is considering introducing a tax on sugary drinks. The minister in charge of the policy is concerned to have a good understanding of the distributional impacts of such a policy. There is data available from a repeated budget survey which is sufficiently disaggregated to contain information regarding spending on sugary drinks and other products. The minister engages you to conduct an empirical study using this dataset.

Let p denote the vector of after-tax prices and y denote total budget. You consider basing your investigation on the following form for indirect utility functions

$$V(p,y) = \frac{\ln Y}{B(p) + C(p)\ln Y} \qquad \text{where } Y = \frac{y}{A(p)}$$

(a) Establish the general form of the Marshallian budget share equations for such a specification.

Ans: Using Roy's identity

$$V_y = \frac{1}{y}\frac{B}{(B + C\ln y)^2}$$

$$V_p = -\frac{1}{A}\frac{A_p}{(B + C\ln y)} - \frac{\ln Y}{(B + C\ln y)^2}\left[B_p + C_p \ln Y - CA_p/A\right]$$

from which budget shares are

$$w = -\frac{pV_p}{yV_y} = \frac{pA_p}{A} + \frac{pB_p}{B}\ln Y + \frac{pC_p}{B}(\ln Y)^2$$

so budget shares are quadratic in $\ln Y$

(b) What would be suitable degrees of homogeneity to impose on the functions A, B and C? Discuss suitable forms which have been adopted for these functions in empirical practice.

Ans: A should be linearly homogeneous while B and C should be homogeneous of degree zero. Typically $\ln A$ *is made quadratic in* $\ln$ *prices while* $\ln B$ *and* $\ln C$ *are linear. The latter two can be made quadratic if one wants to introduce interactions of the* $\ln Y$ *terms with prices.*

(c) How would you go about estimating the parameters of such a specification?

Ans: Impose the required restrictions to ensure homogeneity and symmetry. Estimate budget shares by linear regression techniques, instrumenting the total budget terms. Update the price indexes iteratively until convergence.

(d) Under what restrictions would (i) preferences be homothetic? (ii) budget shares be linear in $\ln y$?

How restrictive do you think either of these would be given the purpose for which the minister has asked you to conduct the investigation?

Ans: Homotheticity requires B *and* C *independent of* p *whereas linear budget shares requires* C *independent of* p*. These are undesirably strong given the aim of investigating distributional impact.*

8. Suppose a government sets a tax schedule which is linear in each of several tax brackets defined by ranges of pretax income. Consider the specific problem of choosing the optimal tax rate for the topmost income bracket. Explain the considerations which ought to influence the choice of rate, paying particular attention to

(a) the labour supply elasticity of high earners

(b) the relative social value of income to top earners and to the rest of the population

(c) the distribution of income within the top bracket.

For each of these, explain where you would find information relevant to the calculations you would wish to make and explain briefly but carefully the options open to you for estimating necessary parameters and the main difficulties needing to be overcome in doing so.

Ans: Saez-type optimum income tax analysis suggest a formula for top tax rate τ

$$\frac{\tau}{1-\tau} = (1-\gamma)\frac{E(X-\zeta|X \geq \zeta)}{E(X|X \geq \zeta)}\frac{1}{\eta}$$

where γ *is the welfare weight on top bracket incomes,* X *is income,* ζ *is the beginning of the top bracket and* η *is labour supply elasticity. Here* η *can be estimated from labour supply behaviour, taking particular account of nonlinearities in budget constraints either by relatively structural or non-structural methods;* γ *is a value judgement;* $\frac{E(X-\zeta|X\geq\zeta)}{E(X|X\geq\zeta)}$ *is the so-called Pareto parameter and can be estimated from data on income distribution.*

ECON G011 (Public Microeconomics)

Examination time allowed: TWO hours

Answer THREE questions.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

1. A population of N individuals supply labour L_i for wages w_i taxed at rate τ, receive a lump sum grant G and consume $c_i = w_i(1-\tau)L_i + G$, $i = 1, \ldots, N$. Utilities are given by $U_i = \ln c_i \; - \; L_i/\alpha$. for some preference parameter α.

   (a) Show that individuals choose $L_i = \alpha - G/w_i(1-\tau)$, assuming $w_i \geq G/\alpha(1-\tau)$.

   Ans: Substitute from budget constraint to get $U_i = \ln(w_i(1-\tau)L_i + G) - L_i/\alpha_i$. *Maximise wrt* L_i *to get* $w_i(1-\tau)/(w_i(1-\tau)L_i + G) = 1/\alpha_i$ *from which the expression follows.*

   (b) If τ and G are set so that all individuals choose to supply positive hours of work then show that budget balance, assuming no other source of revenue or expenditure, implies $G = \alpha\bar{w}\tau(1-\tau)$ where $\bar{w}$ is mean wage and hence show that utilities are

   $$U_i \; = \; \ln(\alpha w_i(1-\tau)) \; + \; \bar{w}\tau/w_i \; - \; 1$$

   Ans: Equate mean tax receipt $\alpha\bar{w}\tau - \tau G/(1-\tau)$ *to* G *and solve for* G*. Substitute into expression for* U_i*.*

   (c) Derive an expression for the optimum utilitarian tax rate.

   Ans: Maximising $\bar{U}$ *is equivalent to maximising* $\ln(1-\tau) + \bar{w}\tau/\hat{w}$ *where* $\hat{w}$ *is harmonic mean wage (ie* $\int dF/w = 1/\hat{w}$*). Hence the optimal rate is* $\tau = 1 - \hat{w}/\bar{w}$*, assuming* $\hat{w} < \bar{w}$*.*

(d) Discuss how you think the optimum tax rate might compare with the tax rate determined by voting.

Ans: Majority voting would give $\tau = 1 - w_m/\bar{w}$ *where* w_m *is median wage so it depends which of* $\hat{w}$ *and* w_m *is greater. (Plurality voting would give* $\tau = 1 - w_0/\bar{w}$ *where* w_0 *is modal wage.)*

2. Individuals in a population consume two goods, q_1 and q_2, at after-tax prices, p_1 and p_2. They supply hours of work h at wage w and also receive an unconditional government grant G. The cumulative distribution function for wages is F. Suppose their utilities U are given by

$$U \;=\; V(G, w, p_1, p_2) \;=\; \frac{1}{2}\frac{w^2}{p_1 p_2} + \frac{G}{\sqrt{p_1 p_2}}$$

Suppose before-tax prices of both goods are 1 and the government raises revenue by a per-unit tax of t on the first good.

(a) Show that revenue per person from the tax is

$$R \;=\; \frac{t}{2(1+t)}\left[\frac{\mu_2}{\sqrt{1+t}} + G\right]$$

where μ_2 is the population mean of w^2, $\mu_2 = \int w^2 \,\mathrm{d}F$, and that therefore

$$G \;=\; \frac{2t\mu_2}{(2+t)\sqrt{1+t}}$$

if the government budget balances.

Ans: By Roy's identity, $q_1 = \frac{1}{2p_1}[w^2/\sqrt{p_1 p_2} + G]$. *If* $p_1 = 1+t$, $p_2 = 1$ *then* $tp_1q_1 = \frac{t}{2(1+t)}\left[\frac{w^2}{\sqrt{1+t}} + G\right]$ *and mean revenue is as given. The expression for G comes from now setting* $R = G$ *and solving.*

(b) Individual utility is therefore given by

$$\frac{U}{\mu_2} \;=\; \left[\frac{(2+t)w^2/\mu_2 + 2t}{(1+t)(2+t)}\right]$$

The right hand side of this equation as a function of t over the range $t \geq 0$ has a maximum at $t = 0$ for $w^2 \geq \mu_2$ and a maximum at $t > 0$ if $w^2 < \mu_2$.

Suppose the government has social objective to maximise $\int \frac{1}{\epsilon}U^{\epsilon}\,\mathrm{d}F$ where $\epsilon \leq 1$ is a parameter capturing social aversion to inequality.

(i) Suppose all individuals in the population have the same wage. What is the optimum tax rate? Explain.

Ans: The optimum is clearly zero. There is no redistributive motive for the government and taxes are set to minimise deadweight loss by raising all revenue through G.

(ii) Suppose that wages differ but $\epsilon = 1$ What is the optimum tax rate? Explain.

Ans: Again the optimum is zero. Although wages differ there is still no redistributive incentive since $\epsilon = 1$.

(iii) Suppose that wages differ and $\epsilon < 1$. Would you expect the optimum tax rate to be greater than 0? Explain.

Ans: Now the optimum tax will be positive. The government has a reason to want to redistribute and $t > 0$ is the only way to raise higher taxes from the rich.

(iv) Suppose the government could raise revenue through a tax on the other good too. With these preferences, goods and labour are weakly separable and Engel curves are linear. Would the government tax the goods at different rates in any of these scenarios?

Ans: No, by a result of Deaton.

3. (a) Suppose that individuals in a population of size N have preference orderings $\succsim_i$, $i = 1, \ldots, N$, over a social choice set $\mathcal{X}$. A social preference relation $\succsim^*$ needs to be derived from the individual preference orderings. Explain what each of the following criteria require:

- ($\mathcal{U}$) Universal domain
- ($\mathcal{P}$) Pareto principle
- ($\mathcal{I}$) Independence of irrelevant alternatives

Ans: ($\mathcal{U}$) says that social preference should be complet and transitive for all possible individual preferences. ($\mathcal{P}$) says that social preference should respect unanimous individual preference. ($\mathcal{I}$) says that social preference should depend only on individual; preference between the goods in question.

(b) Consider the following two social preference relations:

- Majority voting rule: $x \succsim^* y$ if and only if $\# \{i : x \succsim_i y\} \geq \# \{i : y \succsim_i x\}$
- Plurality voting rule: $x \succsim^* y$ if and only if $\# \{i : x \succsim_i z \text{ for all } z \in \mathcal{X}\} \geq \# \{i : y \succsim_i z \text{ for all } z \in \mathcal{X}\}$

where $\# \{i : \mathcal{A}_i\}$ is the number of individuals for whom $\mathcal{A}_i$ is true, $i = 1, \ldots, N$. Which of the above criteria do these social preference relations satisfy and which do they fail? Illustrate failures with examples.

Ans: Majority voting fails $\mathcal{U}$ because of the existence of majority voting cycles. Plurality voting fails $\mathcal{I}$ because of sensitivity to the choice set.

(c) Suppose that ($\mathcal{U}$) is relaxed as follows. For any three options $\{x, y, z\}$ in $\mathcal{X}$ it is known that it will be the case that there will be one option, say y, which will be no individual's least preferred option among the three. Discuss how this modifies, if at all, the acceptability of the majority voting and plurality voting rules. Sketch a proof of any claims you make.

Ans: If this is true it is no longer possible to construct a majority voting cycle so majority voting is no longer problematic. Plurality voting continues to have the same problem.

(d) Suppose that social choice is with regard to the level of spending on a public good Q and that individual preferences, all things considered, are captured by utility functions $u_i(Q) = -(Q - \xi_i)^2$ where ξ_i is an individual preference parameter, $i = 1, \ldots, N$. Discuss the application of the above arguments to this situation.

Ans: These preferences are single peaked so the middle option of any triple will never be least preferred meaning that the situation in (c) applies.

4. Individuals in a population fall into M different productivity types, w_i, $i = 1, \ldots, M$, unobservable by the government except by inference from individual earnings. Individuals have utilities

$$U_i \;=\; \ln c_i + \ln(1 - L_i), \qquad i = 1, \ldots, M$$

where c_i is consumption, and L_i is hours of work. The government wants to choose a tax system for earnings such as to maximise aggregate utility $\sum_i U_i$ subject to raising a given revenue R per person, $\sum_i \pi_i \left[w_i L_i - c_i\right] \geq R$ where π_i is the fraction of the population of the ith type.

(a) Explain why policy optimisation needs to respect the constraints

$$U_i \geq U_j - \ln(1 - L_j) + \ln(1 - w_j L_j / w_i) \qquad i, j = 1, \ldots, M$$

and why, as the distribution of productivity types tends to a continuum with $w_i - w_j \to 0$ for close productivity types, this can be replaced with a constraint

$$\mathrm{d}U/\mathrm{d}w \;=\; L/w(1 - L).$$

Offer an interpretation of this constraint.

Ans: These are incentive compatibility constraints recognising that the more able need to be persuaded to reveal their type. The derivative form can be seen as the same as requiring individuals to choose optimally on their budget constraints

(b) Explain how the Hamiltonian

$$\mathcal{H} = \left[U + \lambda\left[wL - \frac{e^U}{1-L} - R\right]\right]\pi + \mu\frac{L}{w(1-L)}$$

would be used in representation of the optimum tax problem as an optimum control problem, discussing the terms in the expression and characterisation of the optimum. (You do not need to solve the problem, just discuss method and interpretation).

Ans: $\int U\pi dw$ *is the objective,* λ *is Lagrange multiplier on the revenue constraint* $\int [wL - c - R]dw$ *and* μ *is costate variable attached to the derivative constraint*

$dU/dw = L/w(1 - L)$. The optimum requires $d\mathcal{H}/dL = 0$, $d\mathcal{H}/dU = d\mu/dw$ and endpoint constraints.

(c) Discuss insights about the form of the optimum tax function that can be derived in general by such an approach.

Ans: Marginal tax rates are higher where labour supply is less sensitive to after tax wages and where such distortions are less costly in terms of lost revenue. Marginal tax rates are zero at the top of the distrinbution.

5. (a) A population has incomes y distributed across individuals according to cumulative distribution function F. An individual is poor if their income falls below a poverty line z. It is proposed to measure poverty by an index $\mathcal{P}(F, z)$. Discuss the properties which you would expect such a measure to satisfy.

Ans: It should increase as incomes of the poor fall and, arguably, as income is transferred from more to less poor.

(b) The following three measures are proposed:

$$\begin{aligned}\mathcal{H}(F, z) &= F(z) \\ \mathcal{S}(F, z) &= \int_0^z (1 - y/z)\,\mathrm{d}F \\ \mathcal{R}(F, z) &= \int_0^z (1 - y/z)^2\,\mathrm{d}F\end{aligned}$$

Discuss the advantages and disadvantages of these three measures.

Ans: The headcount $\mathcal{H}$ fails both criteria. The shortfall $\mathcal{S}$ satisfies the first but not the second. The Foster, Greer, Thorbecke index $\mathcal{R}$ satisfies both.

(c) Government is considering introducing a tax system under which it takes a proportion τ of pretax income and returns the proceeds, net of a fraction lost in administration $1 - \phi$, as a uniform grant so that posttax income is

$$Y = (1 - \tau)y + \phi\tau \int_0^\infty y\mathrm{d}F.$$

Discuss the optimum choice of τ for the purpose of poverty reduction under each of the three measures.

Ans: Suppose y_{min} is the lowest pretax income and $\bar{y}$ is mean income. The pretax income at which an individual is poor after tax is $Z = (z - \phi\tau\bar{y})/(1 - \tau)$. If $\phi\bar{y} > z$ then $dZ/d\tau < 0$ and poverty can be eliminated by setting $\tau = (z - y_{min})/(\phi\bar{y} - y_{min})$. If this is not true then $dZ/d\tau > 0$ and any $\tau > 0$ raises the headcount $\mathcal{H}$ so the optimum τ is zero.

The shortfall and FGT indices are

$$\mathcal{S} = \frac{1}{z}\int_0^Z (z - (1 - \tau)y - \phi\tau\bar{y})\,dF$$

$$\mathcal{R} = \frac{1}{z^2}\int_0^Z (z-(1-\tau)y-\phi\tau\bar{y})^2\,dF$$

Thus

$$\frac{d\mathcal{S}}{d\tau} \;=\; \int_0^Z (y-\phi\bar{y})dF + \frac{dZ}{d\tau}(z-(1-\tau)Z-\phi\tau\bar{y}) \;=\; \int_0^Z (y-\phi\bar{y})dF$$

so that $d\mathcal{S}/d\tau < 0$ *at* $\tau = 0$ *under the weaker condition that* $\phi\bar{y} > \int_0^z ydF/F(z)$. *Because* $\mathcal{S}$ *is sensitive to depth of poverty it can be desirable to have a positive* τ *despite it increasing* $\mathcal{H}$ *if it raises the incomes of the poorest enough. Similarly* $d\mathcal{R}/d\tau$ *can be negative at* $\tau = 0$ *under even weaker conditions.*

6. (a) A population consists of N individuals whose economic positions are summarised in values of a variable x, x_i, $i = 1, \ldots, N$, with mean value $\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i$ and mean squared value $\sigma_x = \frac{1}{N}\sum_{i=1}^{N} x_i^2$.

The Gini coefficient for x

$$\mathcal{G}_x = \frac{2}{n(n+1)\mu_x}\sum_{i=1}^{N} i\,(x_i - \mu_x)$$

is a relative measure of inequality in the distribution of x. Explain and demonstrate the properties which make it appropriate for this purpose.

Ans: It is Schur-convex and homogeneous of degree zero.

(b) These individuals have pretax wages w_i and work hours h_i with no unearned private income so that pretax incomes are $y_i = w_i h_i$. Government taxes earnings at rate τ and returns the revenues through a uniform lump sum grant B so that posttax incomes are $Y_i = (1-\tau)y_i + B$. Individuals choose hours to maximise utility $U_i = Y_i - \frac{1}{2}h_i^2$.

Show that individuals choose $h_i = w_i(1-\tau)$ and that pretax and posttax incomes are therefore

$$y_i = w_i^2(1-\tau) \qquad Y_i = w_i^2(1-\tau)^2 + B$$

Ans: Solving $\max_{h_i} w_i(1-\tau)h_i + B - \frac{1}{2}h_i^2$ *gives first order condition* $w_i(1-\tau) = h_i$*. Pretax income is* $w_i h_i = w_i^2(1-\tau)$ *and posttax income* $Y_i = w_i(1-\tau)h_i + B = w_i^2((1-\tau)^2 + B$*.*

(c) Assuming that government budget balances explain why $B = \sigma_w \tau(1-\tau)$ and therefore .

$$Y_i = w_i^2(1-\tau)^2 + \sigma_w \tau(1-\tau)$$

Ans: To balance the budget requires $B = \tau\frac{1}{N}\sum_{i=1}^{N} y_i = \tau(1-\tau)\sigma_w$*.* $Y_i = w_i^2(1-\tau)^2 + \tau(1-\tau)\sigma_w$ *follows by substitution.*

(d) Show therefore that, if $\mathcal{G}_0$ is the inequality of income when $\tau = 0$,

$$\mathcal{G}_y = \mathcal{G}_0 \qquad \mathcal{G}_Y = (1-\tau)\mathcal{G}_0 \qquad \mathcal{G}_y - \mathcal{G}_Y = \tau\mathcal{G}_0$$

and use your results to discuss the redistributive effect of taxation. How far do you think any observations you have would generalise to other preferences?

Ans: Inequality of income when $\tau = 0$ is inequality of w_i^2

$$\mathcal{G}_0 = \frac{2}{n(n+1)\sigma_x} \sum_{i=1}^{N} i \left(w_i^2 - \sigma_x\right)$$

Taxation just causes all individuals to scale down hours and therefore earnings equiproportionally so the Gini for pretax income, being homogeneous, is unchanged. This conclusion is very specific to these preferences; generally taxation will change pretax inequality. However $\mu_Y = \sigma_w(1-\tau)$ while income gaps are reduced by a factor $(1-\tau)^2$ so

$$\mathcal{G}_Y = \frac{2}{n(n+1)\sigma_x(1-\tau)} \sum_{i=1}^{N} i(1-\tau)^2 \left(w_i^2 - \sigma_x\right) = (1-\tau)\mathcal{G}_0$$

(Actually it is obvious that since taxes are linear and $\bar{y} = \bar{Y}$, $\mathcal{G}_Y = (1-\tau)\mathcal{G}_y$). Hence posttax inequality is reduced. The extent of redistribution is greater the greater is τ and the greater is pretax inequality $\mathcal{G}_0$.

7. A population consists of N people who consume a private good q and a public good Q. They have endowments of the private good y^i, $i = 1, \ldots, N$, from which they make contributions to a collective fund which can be converted into quantities of the collectively consumed public good at a constant marginal rate of transformation $1/P$. Let Y denote aggregate endowment $\sum_i y^i$ so that $Y = \sum_i q^i + PQ$. Individuals have preferences captured in utility functions $u^i = \ln q^i + \alpha_i \ln Q$ where α_i is a preference parameter.

   (a) What condition describes Pareto efficient allocations within this economy?

   Ans: Individual MRS between the two goods is $\alpha_i q_i/Q$. Efficiency requires that these should add up to the MRT, $\sum_i \alpha_i q_i = PQ$.

   (b) Suppose individuals contribute independently to the public good according to their own preferences.

   (i) Find the Nash equilibrium in voluntary contributions, assuming private endowments are such that all choose to contribute positively.

   Ans: Each individual contributes until their own MRS equals P. Hence $\alpha_i q_i = PQ$ for each i. But then $\sum_i = q_i = Y - PQ = PQ \sum_i 1/\alpha_i$ and therefore $PQ = Y/[1 + \sum_i 1/\alpha_i]$

   (ii) How does the Nash equilibrium allocation depend on the distribution of income across individuals?

   Ans: It is invariant, provided all continue to contribute.

   (iii) Is the Nash equilibrium efficient?

   Ans: No, the sum of MRSs is NP, not P.

   (c) Suppose that provision is financed by a uniform lump sum tax of PQ/N and the level of provision is decided by plurality voting. That is to say, suppose the chosen provision is the most preferred outcome of most people.

   (i) Describe the resulting allocation.

   Ans: Individuals choose such that $\alpha_i(y_i - PQ/N) = PQ/N$ so individuals prefer $PQ = y_i N/[1 + 1/\alpha_i]$. Plurality voting will select the modal value.

(ii) Is the outcome efficient?

Ans: This will not typically be efficient

8. Suppose the government is intending to introduce a tax on sugar content of foodstuffs and wants to know the distributional impact of the implied price changes.

   (a) The following indirect utility function is suggested as the basis for modelling consumer demand behaviour where y is total budget, $p = (p_1,\ p_2,\ \ldots)$ is the posttax price vector and $a(p)$ and $b(p)$ are appropriate price vectors

   $$v(y, p) \ = \ \frac{\ln\left(y/a(p)\right)}{b(p)}$$

   Derive the general form of Marshallian demands for goods, corresponding to these preferences. Why is this a popular specification in applied analysis?

   Ans: Use Roy's identity to show

   $$w_i(y, p) = \frac{\partial \ln a}{\partial \ln p_i} + \frac{\partial \ln b}{\partial \ln p_i} \ln(y/a)$$

   Linear budget shares in $\ln y$ *fit data reasonably well.*

   (b) The following specifications are suggested for the two price indices

   $$\begin{aligned} \ln a(p) &= \alpha_0 + \sum_i \alpha_i \ln p_i + \sum_i \sum_j \gamma_{ij} \ln p_i \ln p_j \\ \ln b(p) &= \beta_0 + \sum_i \beta_i \ln p_i + \sum_i \sum_j \delta_{ij} \ln p_i \ln p_j \end{aligned}$$

   (i) What restrictions on parameters are required to satisfy integrability restrictions?

   Ans: Adding up: $\sum_i \alpha_i = 1$, $\sum_i \beta_1 = 0$, $\sum_i \gamma_{ij} = \sum_i \delta_{ij} = 0$; *Homogeneity:* $\sum_i \alpha_1 = 1$, $\sum_j \gamma_{ij} = 0$; *Symmetry:* $\gamma_{ij} = \gamma_{ji}$, $\delta_{ij} = \delta_{ji}$; *Negativity less straightforward, has to be checked*

   (ii) How would you impose these restrictions in practice?

   Ans: Adding up is automatic. Express prices relative to one good to impose homogeneity. Impose symmetru by cross-equation methods in estimation or after estimation by minimum distance.

(iii) For each of the four types of parameters – α_i, β_i, γ_{ij}, δ_{ij}, $i, j = 1, 2, \ldots$ – explain what role it plays in enhancing flexibility of the demand specification.

Ans: α_i sets base budget shares, γ_{ij} sets price responses, β_i sets income responses (departure from homogeneity, δ_{ij} allows interaction between price responses and income.

(c) Are there any feasible ways you would wish to modify this specification to push its flexibility further?

Ans: Can extend to allow greater flexibility in income response eg QUAIDS

ECON G011 (Public Microeconomics)

Summer 2015

Examination time allowed: TWO hours

Answer THREE questions.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

1. Individuals in a population have pretax wages w distributed according to cumulative distribution function F. All have preferences over consumption c and hours of work h given by utility function

$$u = c - \frac{1}{2}h^2.$$

The government imposes a linear labour income tax with rate t and lump sum grant G and individuals have no other source of income so that $c = w(1-t)h + G$.

(a) Show that individuals choose to supply hours of work $h = w(1-t)$.

Ans: Individual solve $\max_h w(1-t)h + G - \frac{1}{2}h^2$. *First order condition is* $w(1-t) - h = 0$.

(b) If the government is required to balance its budget then show that $G = \mu_2 t(1-t)$ where $\mu_2 = \int w^2 \, \mathrm{d}F(w)$.

Ans: Tax raised through proportional labour tax is $\int wth \, \mathrm{d}F = t(1-t) \int w^2 \, \mathrm{d}F = \mu_2 t(1-t)$ *which finances* G *per head.*

(c) Hence show that individual utility is given by $V(w,t) = \frac{1}{2}w^2(1-t)^2 + \mu_2 t(1-t)$.

Ans: Substitute into $w(1-t)h + G - \frac{1}{2}h^2$

(d) Suppose that the government's objective is to maximise social welfare measured as

$$\Omega(t) = \int (1-F(w))V(w,t) \, \mathrm{d}F(w).$$

Find the optimum tax rate t^* as a function of μ_2 and $\phi_2 = \int(1-F(w))w^2 \, \mathrm{d}F(w)$. Interpret the relation between wage inequality and t^*, given that $G_2 = 1 - 2\phi_2/\mu_2$ is the Gini coefficient of squared wages.

Ans: Social welfare is

$$\frac{1}{2}(1-t)^2 \int (1-F)w^2 \, \mathrm{d}F + \mu_2 t(1-t) \int (1-F) \, \mathrm{d}F \;=\; \frac{1}{2}\phi_2(1-t)^2 + \frac{1}{2}\mu_2 t(1-t)$$

First order condition for maximising wrt t is $-\phi_2(1-t) + \frac{1}{2}\mu_2(1-2t) + 0$ *which implies*

$$t \;=\; \frac{\frac{1}{2}\mu_2 - \phi_2}{\mu_2 - \phi_2} \;=\; \frac{G_2}{1+G_2}$$

Greater pretax inequality in wages leads to a higher optimum tax rate.

2. (a) Explain what is meant by the three following criteria for a social choice rule as proposed by Arrow: *Universal Domain* ($\mathcal{U}$), *the Pareto Principle* ($\mathcal{P}$), *Independence of Irrelevant Alternatives* ($\mathcal{I}$)

   Ans: $\mathcal{U}$*: Must cover all possible profiles of individual preferences;* $\mathcal{P}$*: Must agree with unanimous individual preference;* $\mathcal{I}$*: Social preference between two outcomes must only depend on individual preferences between those two outcomes.*

   (b) Suppose a population consisting of two groups A and B has to decide between two social alternatives x and y. Suppose that everyone in A strictly prefers x to y whereas everyone in B strictly prefers y to x. If x is socially preferred to y then show that, if social choice satisfies $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{I}$, then social preferences must follow the preferences of A over *any* pair of social outcomes over which the members of A are unanimous.

   Ans: By $\mathcal{U}$*, suppose social preferences are that all in* A *prefer* a *to* x *to* y *to* b *and everyone in* B *prefers* y *to* b *to* a *to* x*. By* $\mathcal{P}$*,* a *is socially preferred to* x *and* y *to* b*. But then* a *is socially preferred to* b *by transitivity because* x *is social preferred to* y*. But, by* $\mathcal{I}$*, preferences over* x *and* y *cannot have mattered to this. So* A *is decisive over any pair of outcomes whenever opposed by everyone in* B*.*

   But now, by $\mathcal{U}$*, suppose everyone in* A *prefers* x *to* y *to* a *and everyone in* B *prefers* y *to* x *and* a*. By* $\mathcal{P}$ y *is socially preferred to* a *and by transitivity* x *is therefore socially preferred to* a*. But, by* $\mathcal{I}$*, preferences between* y *and* a *cannot have mattered to this and preferences in* B *between* x *and* a *have not been specified so it is irrelevant whether preferences of* A *are opposed.*

   (c) Show, again using $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{I}$ as and if required, that under these criteria there must be a smaller group within A whose unanimous preferences are socially decisive.

   Ans: By $\mathcal{U}$*, suppose* A *is split into two groups such that everyone in* A_1 *prefers* x *to* y *to* z *and everyone in* A_2 *prefers* z *to* x *to* y *whereas everyone in* B *prefers* y *to* z *to* x*. If* x *is socially preferred to* z *then* A_1 *is a decisive subgroup. If* z *is socially preferred to* x *then by transitivity* z *is preferred to* y *and* A_2 *is decisive.*

   (d) Discuss the significance of these results for the possibility of a social choice rule.

Ans: By repeated application of the argument there must be a dictator. This is Arrow's Theorem.

3. Individuals in a population have additive utility over consumption c and hours worked h, $U = u(c) + v(1-h)$ for some increasing, concave functions u and v. Pretax wages w are distributed according to cumulative distribution function F with associated density f. The government wants to set a tax function T on earnings wh (so that $c = wh = T(wh)$) with the intention of maximising social welfare $\int \phi(U)\, \mathrm{d}F$ subject to a revenue requirement $\int T\, \mathrm{d}F = 0$ where ϕ is some increasing concave transformation.

(a) Explain why the Hamiltonian for the problem $\mathcal{H}$ takes the form

$$\mathcal{H} \;=\; \phi(U)f \;+\; \lambda\,[wh - c]\,f \;+\; \mu v'(1-h)\frac{h}{w}$$

where λ is a Lagrange multiplier and μ is a costate variable. Explain the significance and form of each of the three terms:

(i) $\phi(U)f$

Ans: This comes simply from the objective.

(ii) $\lambda\,[wh - c]\,f$

Ans: This is from the revenue constraint

(iii) $\mu v'(1-h)h/w$

Ans: This covers the equation of motion for U and can be seen as capturing incentive compatibility. Differentiating, $\mathrm{d}U/\,\mathrm{d}w = u'\,\mathrm{d}c/\,\mathrm{d}w - v'\,\mathrm{d}h/\,\mathrm{d}w$. *But from the household budget constraint,* $\mathrm{d}c/\,\mathrm{d}w = (1-T')[h + w\,\mathrm{d}h/\,\mathrm{d}w]$. *Hence, by substitution,* $\mathrm{d}U/\,\mathrm{d}w = [u'w(1-T') - v']\,\mathrm{d}h/\,\mathrm{d}w + hu'(1-T') = hv'/w$ *since* $u'w(1-T') = v'$.

(b) Explain why the solution to the problem requires

$$\begin{aligned} -\,\mathrm{d}\mu/\,\mathrm{d}w \;&=\; \phi'(U)f - \lambda f/u'(c) \\ 0 \;&=\; \lambda w T'(wh) f \;-\; \mu u'(c)\left[1 - T'(wh)\right]\left[\frac{v''(1-h)}{v'(1-h)}h - 1\right] \end{aligned}$$

and interpret these conditions.

Ans: Optimisation requires $\mathrm{d}\mathcal{H}/\,\mathrm{d}U = -\,\mathrm{d}\mu/\,\mathrm{d}w$ *which gives the first line and* $\mathrm{d}\mathcal{H}/\,\mathrm{d}h = 0$ *which gives the second. The first determines the evolution of* μ *across the wage distribution.*

The second can be rearranged to give

$$\frac{T'(wh)}{1-T'(wh)} = \frac{\mu u'(c)}{\lambda w f}\left[\frac{v''(1-h)}{v'(1-h)}h - 1\right]$$

which relates, albeit only implicitly, optimal marginal tax rate to distributional concerns (through $\mu u'$) and efficiency concerns (through $1 + v''/v'h$).

(c) Discuss how changes in the concavity of the social welfare transformation ϕ would feed through into changes in the optimum tax function T.

Ans: Greater concavity (ie greater aversion to inequality) would mean more rapidly diminishing μ and since μ always ends at zero that means higher μ and therefore higher tax rates on middle incomes.

4. Suppose that individuals in a population, $i = 1, \ldots, N$, have incomes y_i and that individuals are ordered such that $y_i \geq y_j$ if $i > j$. Let social welfare be measured by $\sum_{i=1}^{N} \phi(i) y_i$ where ϕ is a decreasing function. Define $\Phi_N = \sum_{i=1}^{N} \phi(i)$ and let $\mu = \frac{1}{N} \sum_{i=1}^{N} y_i$ be mean income.

(a) What is meant by the *equally distributed equivalent income*? Find an expression for the equally distributed equivalent income ξ for this population and given this social welfare function.

Ans: ξ is that income which is such that if every individual had an income of ξ then social welfare would be the same as it is with the actual distribution. Here $\xi = \sum_{i=1}^{N} \frac{\phi(i)}{\Phi(N)} y_i$

(b) What properties should a *relative inequality index* have? Consider the measure $\mathfrak{I} = 1 - \xi/\mu$. Discuss its suitability as a relative inequality index given the requirements you have listed.

Ans: It ought to be homogeneous of degree zero and Schur-convex. $\mathfrak{I}$ is clearly homogeneous of degree one and also Schur convex since $[\partial\mathfrak{I}/\partial y_i - \partial\mathfrak{I}/\partial y_j] [i - j] = [\phi'(i) - \phi'(j)] [i - j] < 0$.

(c) Suppose that income is the sum of K components, $y_i = \sum_{k=1}^{K} y_i^k$, and that the components are similarly ordered across individuals: $y_i^k \geq y_j^k$ if $i > j$ for $k = 1, \ldots, K$.

Show that

$$\mathfrak{I} = \sum_{k=1}^{K} \pi_k \mathfrak{I}_k$$

where $\pi_k = \mu_k/\mu$ and $\mathfrak{I}_k$, μ_k and so on have the obvious definitions: $\mathfrak{I}_k = 1 - \xi_k/\mu_k$, $\mu_k = \frac{1}{N} \sum_{i=1}^{N} y_i^k$ and ξ_k is the equally distributed equivalent value of the kth component.

Ans: Clearly $\mu = \sum_{k=1}^{K} \mu_k$ *and* $\xi = \sum_{k=1}^{K} \xi_k$ *so*

$$\mathfrak{I} = 1 - \xi/\mu = 1 - \sum_{k=1}^{K} \mu_k (1 - \mathfrak{I}_k)/\mu = \sum_{k=1}^{K} \pi_k \mathfrak{I}_k$$

(d) How would this generalise to the case where some components of income are not similarly ordered in the population to total income?

Ans: Formula still true but $\mathfrak{I}_k$ need to be interpreted as indices of concentration rather than inequality.

(e) Suppose disposable income Y_i is original income X_i less tax liability T_i plus benefit receipt B_i. Use the above results to relate the inequality of final income to the inequality of original incomes and the distribution of tax liabilities and benefit receipts.

Ans: $\mu_Y \Im_Y = \mu_X \Im_X - \mu_T \Im_T + \mu_B \Im_B$

5. A household of n members with total budget y and facing prices p is taken to make goods purchases in accordance with indirect utility function

$$v(y, p, n) = \frac{\ln(y/a(p, n))}{b(p) + \phi(p)\ln(y/a(p, n))}$$

where $a(p, n)$, $b(p)$ and $\phi(p)$ are price indices.

(a) What would be appropriate homogeneity properties for the three price indices?

Ans: $a(p, n)$ should be linearly homogeneous in p; $b(p)$ and $\phi(p)$ should be homogeneous of degree zero. Otherwise $v(y, p, n)$ cannot be homogeneous of degree zero in y and p.

(b) Establish the implied form of Marshallian budget share equations.

Ans: Use Roy's identity

$$w_i(y, p, n) = -\frac{\partial v/\partial \ln p_i}{\partial v/\partial \ln y}$$

and

$$\frac{\partial v}{\partial \ln p_i} = -\frac{b\,\partial \ln a/\partial \ln p_i}{(b + \phi\ln(y/a))^2} - \frac{\ln(y/a)}{(b + \phi\ln(y/a))^2}\left(\partial b/\partial \ln p_i + \ln(y/a)\partial\phi/\partial \ln p_i\right)$$

$$\frac{\partial v}{\partial \ln y} = \frac{b}{(b + \phi\ln(y/a))^2}$$

to show

$$w_i(y, p, n) = \frac{\partial \ln a}{\partial \ln p_i} + \frac{\partial \ln b}{\partial \ln p_i}\ln(y/a) + \frac{1}{b}\frac{\partial \phi}{\partial \ln p_i}\ln(y/a)^2$$

(c) Suppose

$$\begin{aligned}
\ln a(p, n) &= \sum_i \alpha_i \ln p_i + \sum_i\sum_j \beta_{ij}\ln p_i \ln p_j + \sum_i \gamma_i \ln p_i \ln n \\
\ln b(p) &= \sum_i \delta_i \ln p_i \\
\ln \phi(p) &= \sum_i \eta_i \ln p_i
\end{aligned}$$

Write down budget share equations appropriate for estimation and describe how you would go about estimating these in practice.

Ans: Applying the earlier budget share expression

$$w_i \;=\; \alpha_i + \gamma_i \ln n + \sum_j \left[\beta_{ij} + \beta_{ji}\right] \ln p_j + \delta_i \ln(y/a) + \eta_i(\phi/b) \ln(y/a)^2$$

Start with approximations for indices a, b and ϕ. Estimate parameters by linear methods, update indices and iterate until convergence. Impose homogeneity by taking one good as numéraire and symmetry either by 3SLS or by MDE.

6. Individuals in a population, $i = 1, \ldots, N$, consume a private good q^i while government provides a public good Q. Individual preferences are captured in utility function

$$u^i(q^i, Q) \;=\; q^i - \pi^i/Q, \qquad i = 1, \ldots, N$$

where π_i is an individual-specific probability of using the publicly provided service. There is a constant marginal rate of transformation between the two goods such that each unit of the public good costs P units of the private good.

(a) Explain Samuelson's condition for optimum supply of a public good and apply it to this case to determine the optimum provision of Q.

Ans: Sum of marginal rates of substitution should equal marginal rate of transformation. So $\sum_i \pi_i/Q^2 = P$ and $Q = \sqrt{\sum_i \pi_i/P}$.

(b) Explain the concept of Lindahl equilibrium and calculate individual Lindahl prices for the social optimum.

Ans: Individual prices equal to individual willingness to pay and adding up to P. Lindahl prices here are π_i/Q^2.

(c) Suppose the individual probabilities π_i are known to individuals but not to the government and the government decides to ask individuals to declare their values of π_i as the basis for calculating optimum provision and charging Lindahl prices. Explain why individuals will have incentive to underdeclare.

Ans: Individuals expecting to be charged according to their own declared willingness to pay will have obvious incentive to underdeclare. If declared π_i is $\hat{\pi}_i$ (and income is y_i) then utility would be $y_i - \hat{\pi}_i/\hat{Q} - \pi_i/\hat{Q}$ where $\hat{Q} = \sqrt{\sum_i \hat{\pi}_i/P}$. This is not maximised at $\hat{\pi}_i = \pi_i$.

(d) Explain the concept of a Clarke-Groves-Ledyard scheme for incentivising accurate revelation. What prices would this involve charging in this particular case?

Ans: Individuals are charged the cost of provision net of the utility gain to others. In this case individuals would pay $P\hat{Q} + \sum_{j \neq i} \hat{\pi}_j/\hat{Q} + K$ for some K set to ensure costs are covered.

7. Suppose that individuals in a population with after-tax wages W and unearned income M have indirect utility functions $V(W, M) = W^2 + aM$.

   (a) Establish the form of Marshallian labour supply functions.

   Ans: By Roy's identity

$$h = V_W/V_M = 2W/a$$

   (b) Suppose pretax wages are w, there is a linear tax at rate t and a uniform government grant G and no other source of income so that $W = w(1-t)$ and $M = G$. Pretax wages are distributed according to cumulative distribution function F. Suppose also that the government is required to balance its budget. Show that $G = 2\mu_2 t(1-t)/a$ where $\mu_2 = \int w^2 \, \mathrm{d}F(w)$ and that individual utilities are therefore $U = w^2(1-t)^2 + 2\mu_2 t(1-t)$.

   Ans: Tax raised is $\int wth \, \mathrm{d}F - G = 2t(1-t)\int w^2 \, \mathrm{d}F/a - G = 0$. *Then substitute into* $V(W, M)$.

   (c) The government's objective is to minimise poverty. It regards an individual as poor if $U < z^2$ where $z = \zeta\mu_1$ is the pretax wage at which an individual would be poor in the absence of redistribution, set as a fraction ζ of the mean pretax wage $\mu_1 = \int w \, \mathrm{d}F(w)$. Show that the proportion of the population who are poor after taxation is

$$H(t) = F\left(\sqrt{\left(\frac{\zeta\mu_1}{1-t}\right)^2 - \frac{2\mu_2 t}{1-t}}\right)$$

   .

   Ans: Individuals are poor if $w^2(1-t)^2 + 2\mu_2 t(1-t) < z^2$ *or, equivalently,* $w < w^* = \sqrt{(z/(1-t))^2 - 2\mu_2 t/(1-t)}$. *Hence the proportion poor is* $F(w^*)$.

   (d) What is the tax rate which minimises $H(t)$? (Assume that $H(t)$ is minimised at some value between 0 and 1.) Comment on how it varies with μ_2/μ_1^2.

   Ans: The tax rate which minimises $H(t)$ *is the tax rate which minimises* w^*. *First order condition is* $2z^2/(1-t)^3 - 2\mu_2/(1-t)^2 = 0$ *so optimal* t *is* $t = 1 - z^2/\mu_2 = 1 - \zeta^2\mu_1^2/\mu_2$. *Greater* μ_2/μ_1^2, *in other words greater wage inequality, means a higher poverty minimising tax rate.*

(e) How good a measure of poverty is the headcount $H(t)$ as a guide to policy? Suggest other measures.

Ans: It takes no account of depth of poverty and this can create perverse policy incentives. A better measure might be based around mean shortfall.

8. Suppose that a population consists of identical individuals consuming three goods – bread q_1, cheese q_2 and pickle q_3 – at prices $p = (p_1, p_2, p_3)$. Each has endowed income of y and preferences follow the indirect utility function

$$V(y, p) \;=\; \frac{y}{p_1} - \frac{p_3}{p_1} + 2\ln p_1 - \ln p_2 - \ln p_3$$

Pretax prices of all goods are equal to 1 and the only revenue-raising possibilities for the government are specific taxes t_2 and t_3 on cheese and pickle.

(a) Find demands for cheese and pickle and therefore revenue $R(t_2, t_3)$ as a function of t_2 and t_3.

Ans: Use Roy's identity. $V_2 = -1/p_2$, $V_3 = -1/p_1 - 1/p_3$ *and* $V_y = 1/p_1$ *so* $q_2 = p_1/p_2$ *and* $q_3 = 1 + p_1/p_3$. *Thus revenue is* $R = t_2/(1+t_2) + t_3 + t_3/(1+t_3)$.

(b) Suppose the government aims to maximise utility of a representative individual subject to raising required revenue per head $\bar{R}$.

$$\max_{t_1, t_3} V(y, 1, 1+t_2, 1+t_3) \qquad \text{s.t.} \qquad R(t_2, t_3) \geq \bar{R}$$

Show that at the optimum

$$\frac{1}{1+t_3} + 1 \;=\; (1+t_2)\left[\left(\frac{1}{1+t_3}\right)^2 + 1\right]$$

and therefore $t_2 \neq t_3$ unless $\bar{R} = 0$.

Ans: Utility is $y - (1+t_3) + 2 - \ln(1+t_2) - \ln(1+t_3)$. *So first order conditions are* $-1/(1+t_2) + \lambda/(1+t_2)^2 = 0$ *and* $-[1 + 1/(1 = t_3)] + \lambda[1 + 1/(1+t_3)^2] = 0$. *From the first,* $\lambda = 1 + t_2$, *which can be substituted into the second to give the expression given. If* $t_2 = t_3 = t$ *then this requires* $1 + 1/(1+t) = 1 + t + 1/(1+t)$ *which implies* $t = 0$ *and* $R = 0$ *so uniformity is optimal only if no revenue is required.*

(c) Suppose $\bar{R} = 5/3$. Show that $t_2 = 1/5$ and $t_3 = 1$ are the optimum tax rates. Why is it optimal to tax pickle more heavily than cheese?

Ans: These values raise the right revenue since $R = 1/6 + 1 + 1/2 = 5/3$ *and satisfy the optimality condition since* $1 + 1/(1+t_3) = 1 + 1/2 = 3/2$ *and* $(1+t_2)\left[\left(\frac{1}{1+t_3}\right)^2 + 1\right] =$

$(1 + 1/5)[1/4 + 1] = 3/2$. *Pickle is more heavily taxed because it is more inelastically demanded.*

ECON G011 (Public Microeconomics)

Summer 2017

Examination time allowed: TWO hours

Answer THREE questions.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

1. A population of N individuals consumes a private good q_i, $i = 1, \ldots, N$ and a public good Q. The population consists of $N_1 = \pi N$ individuals with preferences described by utility function

$$u_i = q_i + \ln Q$$

and $N_2 = (1 - \pi)N$ individuals with preferences described by utility function

$$u_i = q_i - 1/Q.$$

Individuals have fixed endowments of the private good, ω_i, which can be transformed into the public good according to a technology with fixed marginal rate of transformation P so that $PQ = \sum_i^N (\omega_i - q_i)$. Assume $P < N$ and assume that endowments are large enough that individuals always want positive quantities provided of both goods.

(a) Suppose the public good is provided by voluntary private contribution and that each individual decides how much to contribute taking the contributions of others as given. Find the Nash equilibrium provision, $\breve{Q}$.

Ans: Marginal willingness to pay is $1/Q$ for the first type and $1/Q^2$ for the second. Only one type will contribute. If $P > 1$ then only the second type will do so until $\breve{Q} = 1/\sqrt{P}$ and if $P < 1$ then only the first type will do so until $\breve{Q} = 1/P$.

(b) Explain why this level of provision is Pareto inefficient and show that the unique Pareto efficient level of provision, $\hat{Q}$, solves

$$\frac{\pi}{\hat{Q}} + \frac{1-\pi}{\hat{Q}^2} - \frac{P}{N} = 0$$

Why is $\hat{Q}$ greater the greater is π?

Ans: Samuelson condition requires sum of marginal willingness to pay $N_1/Q + N_2/Q^2$ to equal P. The more of type 1 there are the higher is the sum since they have the higher willingness to pay.

(c) Suppose the government knows individual preferences. Show that if it were to propose to charge individuals of the first type a levy $Q/\hat{Q}$ and to charge individuals of the second type a levy $Q/\hat{Q}^2$ then

(i) individuals would vote unanimously for provision of $Q = \hat{Q}$

(ii) the scheme would exactly finance provision of the public good.

Ans: Individuals face prices $1/\hat{Q}$ and $1/\hat{Q}^2$. Equating these to marginal willingness to pay $1/Q$ and $1/Q^2$ obviously leads both to choose $\hat{Q}$. Sum of contributions would be $\hat{Q}\left(N_1/\hat{Q} + N_2/\hat{Q}^2\right)$ which equals $P\hat{Q}$ by the definition of $\hat{Q}$. (These are standard properties of Lindahl equilibrium.)

(d) Suppose instead that the government knows that individual preferences are of these two types but does not know which individuals are of which type and does not know π. Explain why asking individuals to declare their type and implementing the scheme just described would not be feasible.

Ans: Not incentive compatible. Type 1 will not volunteer to pay more.

(e) Describe a Clarke-Groves-Ledyard scheme for this case and explain how it overcomes the preference revelation problem.

Ans: Ask individuals to pay the cost of provision less the benefits to others (plus some constant) $PQ - n_1 \ln Q + n_2/Q + K$ where n_1 and n_2 are the number of others declaring themselves to be of the two types and K is a constant. If others declare accurately then each individual chooses the social optimum so true declaration is a Nash equilibrium.

2. Individuals in a population have fixed total budgets y and have preferences over two goods, food q_1 and fuel q_2, as captured in indirect utility functions

$$v(y, p_1, p_2) \;=\; (y/p_1) \;+\; \ln(y/p_2)$$

where p_1 and p_2 are the prices of the two goods.

(a) Explain why demands are

$$q_1 \;=\; \frac{y^2}{p_1\,(y+p_1)} \qquad q_1 \;=\; \frac{yp_1}{p_2\,(y+p_1)}$$

Ans: Use Roy's identity and $v_1 = -y/p_1^2$, $v_2 = -1/p_2$, $v_y = 1/p_1 + 1/y$.

Suppose that individual total budgets y are made up of pretax incomes Y plus a uniform lump sum grant G, $y = Y + G$, and that pretax incomes are distributed over the interval $[A, B]$ according to cumulative distribution function F. Suppose that the grant is funded by a proportional tax at rate t on fuel spending and that, for administrative reasons, no other taxes are possible. Pretax prices of both goods are equal to one so $p_1 = 1$, $p_2 = 1 + t$.

(b) Show that government budget balance requires

$$\frac{t}{1+t}\int_A^B \frac{Y+G}{Y+G+1}dF \;=\; G$$

and that therefore dG/dt satisfies

$$\left(\frac{1}{1+t}\right)^2 \int_A^B \frac{Y+G}{Y+G+1}dF \;=\; \left[1 - \frac{t}{1+t}\int_A^B \frac{dF}{(Y+G+1)^2}\right]\frac{dG}{dt}$$

Ans: Demand for q_2 *is* $(Y+G)/(1+t)(Y+G+1)$ *so mean cost is* $\frac{t}{1+t}\int_A^B \frac{Y+G}{Y+G+1}dF$. *The next expression follows by differentiating.*

(c) Suppose that the government wants to maximise welfare of the poorest individual. Explain why the optimum subsidy t^* requires

$$\frac{1}{1+t^*} \;=\; \left[\frac{A+G+1}{A+G}\right]\frac{dG}{dt}$$

Ans: Welfare of the worst off person is $v(A+G, 1, 1+t) \;=\; A + G + \ln(A+G) - \ln(1+t)$. *Differentiating and setting* $dv/dt \;=\; 0$ *gives the expression.*

(d) By substitution

$$t^* = \left[\frac{A-G+1}{A-G}\int_A^B \frac{Y+G}{Y+G+1}dF - 1\right] \Big/ \left[1 - \int_A^B \frac{dF}{(Y+G+1)^2}\right]$$

so that $t^* > 0$ unless $B = A$. Discuss the considerations underlying choice of t^*.

Ans: Numerator and denominator are both positive since $(Y+G)/(Y+G+1) > (A+G)(A+G+1)$ *for* $Y > A$ *and* $1/(Y+G+1)^2 < 1$ *for all* $Y > 0$. *Fuel is still a normal good even if it is a necessity so taxing it and paying out a uniform grant redistributes to the least well off. There is a deadweight loss since only one good is being taxed but the redistributive gain outweighs this for small enough* t.

3. Suppose a population of N individuals has incomes $\mathbf{y} = (y_1, y_2, \ldots, y_N)$. Assume that individuals are numbered in order of income so that $y_i \geq y_j$ if $i > j$. Let $\mu(\mathbf{y})$ denote mean income. Consider a Schur-concave social welfare function $W(\mathbf{y})$ which is increasing in each income and homothetic so that if

$$W\left(\mathbf{y}^A\right) = W\left(\mathbf{y}^B\right)$$

then

$$W\left(\lambda \mathbf{y}^A\right) = W\left(\lambda \mathbf{y}^B\right)$$

for any positive λ.

(a) (i) Explain the concept of the equally distributed equivalent income, $\xi(\mathbf{y})$, associated with $W(\mathbf{y})$.

Ans: It is that income such that social welfare would be the same if everyone had it as it is under the existing distribution: $W(\xi, \xi, \ldots) = W(\mathbf{y})$.

(ii) Outline appropriate properties for a measure of relative inequality and explain why the index $I(\mathbf{y}) = 1 - \xi(\mathbf{y}) / \mu(\mathbf{y})$ has these properties. *Ans: Since social welfare is Schur-concave, Pigou-Dalton transfers raise ξ without affecting μ so I is Schur-convex. Also since W is homothetic, ξ is linearly homogeneous and I is homogeneous of degree zero.*

(iii) Suppose social welfare takes the form $W(\mathbf{y}) = \sum_i^N \ln y_i$. Justify

$$1 - \frac{\prod_i^N y_i^{1/N}}{(1/N) \sum_i^N y_i}$$

as a measure of inequality.

Ans: This is just I for this W, one minus the ratio of the geometric to the arithmetic mean.

(b) Suppose z denotes the minimum income needed to satisfy basic needs so that an individual is poor if $y_i < z$. Suppose that H individuals are poor. Let $\tilde{\mathbf{y}} = (y_1, y_2, \ldots, y_H, z, \ldots, z)$ be the income distribution truncated at the poverty line.

(i) Outline appropriate properties for a measure of poverty and explain why the index $P(\mathbf{y}) = 1 - \xi(\tilde{\mathbf{y}})/z$ has these properties.

Ans: The index increases if the income of any poor person decreases while being invariant to the incomes of the non-poor. It also decreases if there is a Pigou-Dalton transfer between poor people.

(ii) Justify

$$1 - \frac{\prod_i^H y_i^{1/N}}{z^{H/N}}$$

as a measure of poverty.

Ans: This is P for the W introduced earlier.

4. (a) (i) A population needs to decide a public policy question involving choice between multiple options from a choice set X. Suppose there is an odd number of individuals and no-one is indifferent between any two options. Suppose preferences are such that the following is true: if you take any three options from X then there is always one of those options which no-one thinks is the worst of the three. Explain why pairwise majority voting defines a complete and transitive social ordering.

Ans: Preferences can be of four types: A $x \succ y \succ z$, B $z \succ y \succ x$, C $y \succ x \succ z$, D $y \succ z \succ x$. If A or B in majority then majority voting returns their preference ordering. If neither is in a majority then y is preferred by majority voting to both x and z so there can be no cycle.

(ii) Suppose the options lie along a single dimension. Explain what it means for preferences to be single peaked and explain why such preferences satisfy the property described above.

Ans: Each voter has a most preferred option, say ξ. For any two options on the same side of ξ that one which is nearest to ξ is most preferred. Any three options can be placed on the single dimension and the middle one will be no-one's least preferred.

(iii) What is meant by a Condorcet winner? Which option will be the Condorcet winner if preferences are single peaked?

Ans: Condorcet winner beats all others in pairwise voting. If preferences are single peaked then the median most preferred option is a Condorcet winner.

(b) (i) Suppose that voting is over the tax rate on earned income τ. Preferences over consumption c and hours worked H follow utility function $u = c - \frac{1}{2}H^2$. Individual budget constraints imply $c = w(1-\tau)H + G$ where w is individual wage and τ and G are tax parameters linked through a government budget constraint. It can be shown that individual utilities can therefore be written as

$$u = \frac{1}{2}w^2(1-\tau)^2 + \mu_2\tau(1-\tau)$$

where μ_2 is the mean squared wage. Assume that most of the population have a wage below $\sqrt{\mu_2}$. What is the Condorcet winning tax rate?

Ans: Preferences are quadratic so single peaked. Preferred points solve first order conditions $\left(w^2 - 2\mu_2\right) t - \left(w^2 - \mu_2\right) = 0$. *Hence Condorcet winner is* $\left(m^2 - \mu_2\right) / \left(m^2 - 2\mu_2\right)$ *where* m *is median wage.*

(ii) Discuss the weaknesses of this as a model of the political economy of actual democratic tax setting. *Ans: Taxes do not need to be linear so tax setting is actually many-dimensional. Decided jointly with other issues. Voting is not by exhaustive pairwise majority votes so may not return Condorcet winner.*

5. Suppose that individuals face a tax system with a constant marginal tax rate on earned income so that their marginal posttax wages w are independent of hours worked H and posttax incomes are $y = wH + m$ where m denotes unearned income (inclusive of any fixed taxes). They consume goods q, purchased at posttax prices p. Indirect utility takes the form

$$\begin{aligned} V(w, p, m) &= \frac{1}{B(p)}\left[\ln\left(\frac{w}{A(p)B(p)C}\right) - 1\right] + \frac{Cm}{w} && \text{if } w \geq mB(p)C \\ &= \frac{1}{B(p)}\ln\left(\frac{m}{A(p)}\right) && \text{if } w < mB(p)C \end{aligned}$$

where $A(p)$ is a linearly homogeneous price index, $B(p)$ is a homogeneous-of-degree-zero price index and C is a preference parameter.

(a) Show that individuals choose incomes $y = w/B(p)C$ if $w \geq mB(p)C$ and $y = m$ if not.

Ans: Use Roy's identity. $V_w = (1/wB) - (Cm/w^2)$ *and* $V_m = C/w$ *if* $w \geq mBC$ *whereas* $V_w = 0$ *otherwise. Thus* $H = 1/BC - m/w$ *if* $w \geq mBC$ *and* $H = 0$ *otherwise.*

(b) Show that individuals choose budget shares for consumer goods

$$\frac{p_i q_i}{y} = \frac{\partial \ln A}{\partial \ln p_i} + \frac{\partial \ln B}{\partial \ln p_i}\ln\left(\frac{y}{A(p)}\right) \qquad i = 1, \ldots, M$$

where M is the number of goods.

Ans: Again, use Roy's identity. $V_i = -(1/B)A_i/A - (B_i/B^2)\ln(w/ABC)$ *if* $w \geq mBC$ *and* $V_i = -(1/B)A_i/A - (B_i/B^2)\ln(m/A)$ *otherwise. In each case,* $p_i q_i = -p_i V_i/v_m = (p_i A_i/A)y + (p_i B_i/B)y\ln[y/A]$

(c) Suppose

$$\begin{aligned} \ln A(p) &= \alpha_0 + \sum_i \alpha_i \ln p_i + \frac{1}{2}\sum_i \gamma_{ij} \ln p_i \ln p_j \\ \ln B(p) &= \sum_i \beta_i \ln p_i \end{aligned}$$

for suitable parameters α_0, α_i, β_i, γ_{ij}, $i, j = 1, \ldots, M$.

Discuss practicalities of estimating such a system of demands.

Ans: Budget shares would take the form $p_i q_i/y = \alpha_i + \sum_j \gamma^*_{ij}\ln p_j + \beta_i \ln\left(\frac{y}{A(p)}\right)$ *where* $\gamma^*_{ij} = \frac{1}{2}(\gamma_{ij} + \gamma_{ji})$. *Adding up would require* $\sum_i \alpha_i = 1$, $\sum_i \beta_i = 0$, $\sum_i \gamma^*_{ij} = 0$. *Homogeneity* *would require also $\sum_i \gamma^*_{ij} = 0$ and symmetry would require $\gamma^*_{ij} = \gamma^*_{ji}$. To estimate, impose homogeneity by expressing all nominal values relative to price of one of the goods and estimate iteratively by linear methods, updating $A(p)$ at each iteration. If $B(p)$ known then C estimable from hours regression.*

6. (a) Suppose a population consists of two types of individuals in equal numbers. Individuals work L hours and consume a vector of goods q. All individuals have the same preferences as captured in separable utility function $u(q) - v(L)$ where u and v have the appropriate properties. Individuals of type A have higher productivity than individuals of type B.

The government sets nonlinear taxes on earned income and consumption of individual goods so as to maximise aggregate utility

$$u\left(q^A\right) - v\left(L^A\right) + u\left(q^B\right) - v\left(L^B\right)$$

subject to an aggregate resource constraint

$$p'\left[q^A + q^B\right] \leq w^A L^A + w^B L^B$$

where p denotes a common pretax price vector and w^A and w^B are pretax wages of the two types. An incentive compatibility constraint requires

$$u\left(q^A\right) - v\left(L^A\right) \geq u\left(q^B\right) - v\left(w^B L^B / w^A\right)$$

Show that

$$\begin{aligned} u_i\left(q^A\right)/u_j\left(q^A\right) &= u_i\left(q^B\right)/u_j\left(q^B\right) = p_i/p_j \\ v_L\left(L^A\right)/u_i\left(q^A\right) &= w^A/p_i \\ v_L\left(L^B\right)/u_i\left(q^B\right) &> w^B/p_i \end{aligned}$$

for any i and j, where $u_i\left(q\right)$ denotes $\partial u\left(q\right)/\partial q_i$ and $v_L\left(L\right)$ denotes $\partial v\left(L\right)/\partial L$.

Interpret each of these.

Ans: All standard results based on manipulation of first order conditions. The first implies that MRS between any two goods is equated to the pretax price ratio so goods choice is undistorted. The second implies the MRS between leisure and consumption of any good equals the pretax product wage for the more able, so their labour supply is undistorted. The last shows that this is not true for the less able. Distortion to their labour supply is the cost of satisfying incentive compatibility. [NB: The inequality is accidentally the wrong way round in the question. One student spots this.]

(b) In the light of these results, comment on the following quotation and its relevance to practical tax reform:

> If indirect taxes are considered in isolation, then they appear to be a classic battleground between efficiency (tax more heavily goods that are inelastic in demand) and equity (exempt necessities) . . . But once we introduce the possibility of levying direct taxes, then the role of indirect taxes changes. Indeed, if the only source of income differences is earning capacity, and there are no restrictions on the government's ability to levy nonlinear direct taxes on earnings, then – under certain conditions – optimality can be achieved without differential rates of indirect tax.
> *[Source: Anthony B. Atkinson "The Mirrlees Review and the State of Public Economics" Journal of Economic Literature 2012, 50(3), p.773]*

Ans: The preferences of the earlier part are a case in which the 'certain conditions' apply. The separability between goods and leisure mean that the MRS between goods is independent of hours worked and distorting goods choice is of no help in satisfying the incentive compatibility constraint. This could be weakened from additive separability to weak separability and this would still be true. In practice this means that exemptions from VAT, for example, have a tenuous justification in terms of redistributive objectives.

ECON G011 (Public Microeconomics)

Summer 2018

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Answer THREE questions.

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1. Individuals in a population supply labour L and consume c. Labour is supplied at pretax wages w which is taxed at a proportional rate τ with the proceeds used to fund a uniform lump sum grant G so that $c = w(1-\tau)L + G$. Wages are distributed on the interval $[w_0, w_1]$ according to cumulative distribution function F. Individuals choose c and L so as to maximise utility function

$$u(c, L) \;=\; c - \frac{1}{2}L^2.$$

   (a) Find chosen labour supply and hence, using government budget balance, find an expression for the grant which can be afforded for each level of the tax rate τ and show that individual utilities are therefore

$$V \;=\; \frac{1}{2}w^2(1-\tau)^2 + \mu_2\tau(1-\tau)$$

   where $\mu_2 = \int w^2 \, \mathrm{d}F$ is the mean squared wage.

   Ans: Utility is $u \;=\; w(1-\tau)L + G - \frac{1}{2}L^2$. The first order condition for maximisation with respect to L is $w(1-\tau) - L = 0$. Hence tax revenue is $\mu_2\tau(1-\tau)$ which is the value of the grant G. Substitute the expression for G and L into utility. **15 marks**

   (b) Find the tax rate that maximises utility of the person with lowest wage w_0 and find their utility under this tax rate u_0^*.

Ans: First order condition is $-w_0^2(1-\tau)+\mu_2(1-2\tau)=0$ *which is solved by* $\tau=(\mu_2-w_0^2)/(2\mu_2-w_0^2)$. *Substitute into utility to get*

$$u_0^* = \frac{1}{2}\frac{\mu_2^2}{2\mu_2-w_0^2}.$$

30 marks

(c) Suppose an individual is regarded as poor if utility is below some threshold $\frac{1}{2}z^2$ and $u_0^* < \frac{1}{2}z^2$. Find the tax rate which minimises the proportion poor

$$\mathcal{H} = F\left(\sqrt{\left(\frac{z}{1-\tau}\right)^2 - \frac{2\mu_2\tau}{1-\tau}}\right).$$

Is this tax rate the same as that which maximises utility of the person with lowest wage?

Ans: Poverty minimisation requires minimisation of the wage at which an individual escapes poverty which is achieved by minimising $\left(\frac{z}{1-\tau}\right)^2-\frac{2\mu_2\tau}{1-\tau}$. *First order condition is* $2\frac{z^2}{(1-\tau)^3}-\frac{2\mu_2}{(1-\tau)^2}=0$ *which is solved by* $\tau=1-z^2/\mu_2$. *Since* $\frac{1}{2}z^2>\frac{1}{2}(\mu_2^2)/(2\mu_2-w_0^2)$ *this is less than the tax rate* $(\mu_2-w_0^2)/(2\mu_2-w_0^2)$ *minimising the utility of the lowest waged.* **30 marks**

(d) Discuss weaknesses in $\mathcal{H}$ as a measure of poverty.

Ans: The headcount takes no account of depth of poverty. It arguably puts too much weight on increasing incomes of those close to the poverty line (who are easy to take out of poverty) relative to those far from the poverty line (who are less easy). The previous part showed the conflict with minimising the depth of poverty of the poorest. **25 marks**

2. Individuals supply labour L at wages w and consume goods q_1 and q_2 at prices p_1 and p_2 respectively. Their preferences are captured in indirect utility function

$$V(y, p_1, p_2, w) \;=\; \frac{y+w}{p_1} + 2\ln p_1 - \ln p_2 - \ln w$$

where y denotes unearned income. Chosen labour supply and demand for goods are therefore given by

$$L \;=\; 1 - p_1/w \qquad q_1 \;=\; (y+w)/p_1 - 2 \qquad q_2 \;=\; p_1/p_2.$$

Suppose pretax prices for both goods equal 1 and the government sets tax rates on goods at t_1 and t_2 so that $p_1 = 1 + t_1$ and $p_2 = 1 + t_2$. Labour income is untaxed and revenue from taxing the two goods is returned to individuals in the form of a lump sum grant G. Individuals have no other income than earnings and the grant so $y = G$. Wages are distributed on the interval $[w_0, w_1]$ and mean wage is $\bar{w}$. Assume throughout that wages and prices are such that all individuals supply positive labour and demand positive quantities of both goods.

(a) Show that

$$G = \bar{w}t_1 + (1+t_1)^2\frac{t_2}{1+t_2} - 2t_1(1+t_1).$$

Hence show that individual utility U can be written

$$U \;=\; \frac{t_2}{1+t_2}(1+t_1) - 2t_1 + \frac{\bar{w}t_1 + w}{1+t_1} + 2\ln(1+t_1) - \ln(1+t_2) - \ln w.$$

Ans: Mean revenue raised given unit pretax prices is mean $t_1q_1 + t_2q_2$ *so*

$$G \;=\; t_1\frac{G+\bar{w}}{1+t_1} - 2t_1 + t_2\frac{1+t_1}{1+t_2}$$

which implies

$$\frac{G}{1+t_1} \;=\; t_1\frac{\bar{w}}{1+t_1} - 2t_1 + t_2\frac{1+t_1}{1+t_2}$$

from which the expression follows. Now substitute for G *in indirect utility.* **20 marks**

(b) Suppose that the government wants to set taxes such as to maximise mean utility $\bar{U}$. Show that $t_1 = t_2 = 0$. Comment.

Ans: Mean utility is $\frac{t_2}{1+t_2}(1+t_1) - 2t_1 + 2\ln(1+t_1) - \ln(1+t_2) + \bar{w} - \overline{\ln w}$. *First order conditions are*

$$\begin{aligned}\left(\frac{t_2}{1+t_2} - 2\right) + \frac{2}{1+t_1} &= 0 \\ \frac{1+t_1}{(1+t_2)^2} - \frac{1}{1+t_2} &= 0\end{aligned}$$

From the second of these $t_1 = t_2 = t$, *say. Then from the first* $t = 0$. *Since utility is linear in* y *and the objective is mean utility there is no redistributive motive and there is no revenue requirement. Taxation of goods merely induces deadweight loss so the optimum is no taxation.* **35 marks**

(c) Suppose that the government wants to set taxes such as to maximise utility of the worst off person U_0. Show that $t_1 = t_2 > 0$. Comment.

Ans: The objective is now $\frac{t_2}{1+t_2}(1+t_1) - 2t_1 + 2\ln(1+t_1) - \ln(1+t_2) + \frac{\bar{w}t_1 + w_0}{1+t_1} - \ln w_0$. *First order conditions are*

$$\begin{aligned}\left(\frac{t_2}{1+t_2} - 2\right) + \frac{2}{1+t_1} + \frac{\bar{w} - w_0}{(1+t_1)^2} &= 0 \\ \frac{1+t_1}{(1+t_2)^2} - \frac{1}{1+t_2} &= 0\end{aligned}$$

From the second of these it is still true that $t_1 = t_2 = t$, *say. But* $t = 0$ *no longer solves the first. Rearranging gives* $t(1+t) = \bar{w} - w_0$ *and since* $\bar{w} > w_0$, *assuming there is not perfect equality,* $t > 0$. *The objective now incorporates a redistributive incentive. Positive taxes raise revenue which can be redistributed to the poorest worker. However uniformity is still optimal in this case. (Labour is separable from goods and Engel curves are linear so this follows from Deaton but answers need not go this far).* **45 marks**

3. A population $\mathcal{N}$ needs to choose a policy x from some choice set $\mathcal{X}$. Let $\#$ denote the function returning the number of elements in a set and assume $\#(\mathcal{N})$ and $\#(\mathcal{X})$ are both finite.

Individual preferences over those policies are given by individual-specific utility functions $U^i(x)$, $i \in \mathcal{N}$. The government knows the individual preference orderings and seeks to establish a social preference relation $\succsim^*$ based on the individual utilities.

The following social preference relations are suggested:

- *Majority voting:* $\succsim^* = \succsim^A$ where $x \succsim^A y$ iff $\#\{i|U^i(x) \geq U^i(y),\ i \in \mathcal{N}\} \geq \#\{i|U^i(y) \geq U^i(x),\ i \in \mathcal{N}\}$
- *Borda voting:* $\succsim^* = \succsim^B$ where $x \succsim^B y$ iff $\sum_{i\in\mathcal{N}} \#\{z \in \mathcal{X}|U^i(x) \geq U^i(z)\} \geq \sum_{i\in\mathcal{N}} \#\{z \in \mathcal{X}|U^i(y) \geq U^i(z)\}$
- *Dictatorship:* $\succsim^* = \succsim^C$ where $x \succsim^C y$ iff $U^I(x) \geq U^I(y)$ for some named individual $I \in \mathcal{N}$
- *Maximin welfarism:* $\succsim^* = \succsim^D$ where $x \succsim^D y$ iff $\min_{i\in\mathcal{N}} U^i(x) \geq \min_{i\in\mathcal{N}} U^i(y)$
- *Benthamite utilitarianism:* $\succsim^* = \succsim^E$ where $x \succsim^E y$ iff $\sum_{i\in\mathcal{N}} U^i(x) \geq \sum_{i\in\mathcal{N}} U^i(y)$
- *Logarithmic utilitarianism:* $\succsim^* = \succsim^F$ where $x \succsim^F y$ iff $\sum_{i\in\mathcal{N}} \ln U^i(x) \geq \sum_{i\in\mathcal{N}} \ln U^i(y)$

(a) Social judgments are required to be invariant to transformations within a class Φ

$$\succsim^* \left(U^1, U^2, \ldots; \mathcal{X}\right) = \succsim^* \left(\phi^1(U^1), \phi^2(U^2), \ldots; \mathcal{X}\right)$$

for all $\phi^1, \phi^2, \dots \in \Phi$.

Consider the following specifications for Φ:

- Φ_0 contains all individual-specific increasing functions ϕ^i, $i \in \mathcal{N}$
- Φ_1 contains all common increasing functions $\phi^i(U) = \phi(U)$
- Φ_2 contains all increasing affine functions with individual-specific intercept $\phi^i(U) = a^i + bU$, $i \in \mathcal{N}$, $b > 0$
- Φ_3 contains all increasing affine functions with common parameters $\phi^i(U) = a + bU$, $b > 0$

- Φ_4 contains all increasing power functions with individual-specific scale $\phi^i(U) = a_i U^b$, $i \in \mathcal{N}$, $a_i, b > 0$
- Φ_5 contains all increasing power functions with common parameters $\phi^i(U) = aU^b$, $a, b > 0$
- Φ_6 contains all increasing common scaling $\phi^i(U) = aU$, $a > 0$

For each of the six suggested social preference relations explain which of these classes of transformations the ordering is invariant within.

Ans: $\succsim^A$, $\succsim^B$ and $\succsim^C$ all use the information in the individual orderings only and need no assumptions on cardinality and comparability so are invariant within all classes. Identification of the minimum utility is vulnerable to any sorts of individual specific transformations so $\succsim^D$ is invariant only within Φ_1, Φ_3, Φ_5 and Φ_6. Utilitarian social welfare comparison are unaffected by additive components to individidual utility that do not depend on policy so $\succsim^E$ is invariant within Φ_2, Φ_3 and Φ_6. The final social preference relation works similarly with log utility so $\succsim^F$ is invariant within Φ_4, Φ_5 and Φ_6. **50 marks**

(b) The social preference relation is also required to be complete and transitive for any individual preferences, to satisfy the Pareto principle and irrelevance of independent alternatives. Explain your understanding of what these requirements are and say which of these six suggested social preference relations satisfy these requirements?

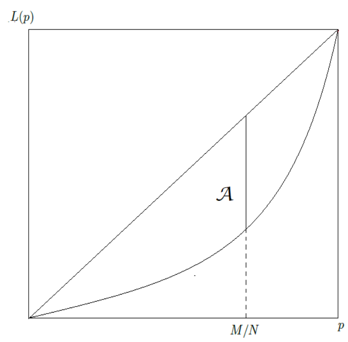
Ans: All satisfy the Pareto principle. $\succsim^A$ is majority voting which is not transitive for certain individual preferences because of the possibility of Condorcet cycles. $\succsim^B$ is the Borda count which violates independence because the preference ordering is sensitive to what is in $\mathcal{X}$. **50 marks**

4. (a) A A population of N individuals have incomes y_i, $i = 1, \ldots, N$. Individuals are ranked by income so that $y_i \geq y_j$ if $i > j$. The Lorenz curve, $L(p)$, defined as the piecewise linear curve joining the points

$$L(i/N) \;=\; \sum_{j=1}^{i} \frac{y_j}{N\bar{y}}, \qquad i = 1, \ldots, N$$

where $\bar{y} = \frac{1}{N}\sum_{j=1}^{N} y_j$ is mean income, is illustrated in Figure 1.

Figure 1: Lorenz curve



Describe and justify the way in which comparisons of Lorenz curves are used in assessing inequality in income distributions.

Ans: The more bowed-out it is the greater is inequality. This means that wherever you cut the distributions the poorer part has a greater share of income in the one judged more equal. If one distribution has a higher Lorenz curve than another with the same mean then it is possible to get between them by a series of Pigou-Dalton transfers and social welfare is higher for any concave symmetric utilitarian social welfare function. **20 marks**

(b) Noting that the slope at any point p is $y_I/\bar{y}$ where $I - 1 < pN < I$, explain why the maximum vertical distance between the Lorenz curve and the diagonal ($\mathcal{A}$ in Figure 1) lies at $p = M/N$ where the Mth individual is the individual with the highest income that is no greater than $\bar{y}$ and that

$$\mathcal{A} \;=\; M/N - \sum_{i=1}^{M} y_i/N\bar{y}.$$

Ans: The distance between the Lorenz curve and the diagonal is increasing as long as the slope of the former is less than one which is true as long as income is below the mean. The maximum is therefore at the point stated. The height of the diagonal is M/N and the Lorenz curve is $\sum_{i=1}^{M} y_i/N\bar{y}$. **25 marks**

(c) It is suggested that $\mathcal{A}$ be used as a relative inequality index. Discuss whether it has the appropriate properties.

Ans: It is clearly homogeneous of degree zero. It is weakly Schur convex but not strictly Schur convex. A Pigou Dalton transfer reduces the index only if it goes from someone above the mean to someone below. **25 marks**

(d) The so-called Hoover index is the proportion of total income which would need to be redistributed in order to achieve perfect equality. How is $\mathcal{A}$ related to the Hoover index?

Ans: To eliminate inequality the excess of incomes above the mean needs to be transferred to those below the mean. The amount needing to be redistributed is therefore the difference between the share of population of those below the mean M/N and their share of income $\sum_{i=1}^{M} y_i/N\bar{y}$. The Hoover index is therefore equal to $\mathcal{A}$. **10 marks**

(e) Explain the concept of the equally distributed equivalent income corresponding to an inequality measure and discuss how you might interpret the corresponding equally distributed equivalent income in this case

$$\xi_{\mathcal{A}} \;=\; \bar{y}(1-\mathcal{A}) \;=\; \frac{1}{N}\sum_{i=1}^{M} y_i + (1-\frac{M}{N})\bar{y}$$

Ans: Social welfare is the mean of incomes if those above the mean are capped at $\bar{y}$. **20 marks**

5. Wages w in a population are distributed according to cumulative distribution function F with corresponding density function f. Individuals consume c, supply labour L and utilities are given by $U(c, L) = u(c) - v(L)$ where u is increasing and concave and v is increasing and convex. Government wants to design a tax function on earned income $T(wL) = wL - c$ so as to maximise utilitarian social welfare $\int_0^\infty U(c, L)\ \mathrm{d}F$ subject to balancing its budget $\int_0^\infty T\ \mathrm{d}F = 0$.

(a) Explain why incentive compatibility requires a dynamic constraint $\mathrm{d}U/\ \mathrm{d}w = Lv'(L)/w$.

Ans: The incentive compatibility constraint can be derived either from the condition that no worker should have any incentive to pose as someone of lower ability or from the condition for optimum labour supply choice $v'(L)/u'(c) = w(1 - T'(wL))$. **20 marks**

(b) The optimally designed policy requires

$$\frac{T'(wL)}{1 - T'(wL)} = P \cdot Q \cdot R$$

where

$$P = 1 + \frac{Lv''(L)}{v'(L)} \qquad Q = \frac{1 - F(w)}{wf(w)} \qquad R = \frac{\int_w^\infty [\lambda - u'(c)]\ \mathrm{d}F}{[1 - F(w)]\lambda}$$

and λ is the Lagrange multiplier on the government budget constraint. Interpret the role of each of the terms P, Q and R.

Ans: P captures the efficiency costs of distorting labour supply. Q reflects the shape of the income distribution. R captures the evolution of redistributive incentives according to the shape of social welfare. **30 marks**

(c) Suppose that as $w \to \infty$

$$\frac{Lv''(L)}{v'(L)} \to \frac{1 - e}{e} \qquad 1 - F(w) \to \left(\frac{w_0}{w}\right)^a \qquad u'(c) \to 0$$

for certain constants e, a and w_0. Show that $T'(wL) \to 1/(1 + ae)$.

Ans: $P \to 1/e$, $Q \to 1/a$, $R \to 1$. Hence $T'(wL) \to 1/(1 + ae)$. **25 marks**

(d) "Economists tend to assume that it is e (the elasticity) that is the core of their subject, but equally central should be a (the distribution)."

A. B. Atkinson, *Journal of Economic Literature*, 2012

Discuss how economists might find information on e and a.

Ans: Labour supply elasticities usually estimated from labour force surveys fitting labour supply models to data on hours and wages, taking account of complications raised by taxation and so on. However e here is related to labour supply elasticity at top of the wage distribution where data may be poor and nature of hours contracts may differ. Could look at tax data at top end. Information on a comes from income surveys but again this applies specifically to top end. **25 marks**

6. Individual preferences over a public good Q and private good q are captured in utility function $u(q, Q, a) = q + 2a\sqrt{Q}$ where a is an individual-specific preference parameter distributed according to cumulative distribution function F. Individuals have endowments of the private good ω and the cost per person of producing the public good is PQ. Suppose $P = 1$ and a is distributed over support $[0, 2]$ according to $F(a) = \frac{1}{4}a^2$.

   (a) Explain the condition for optimal supply, show that it solves

   $$P = \int_0^2 a\sqrt{\frac{1}{Q}}dF(a)$$

   and hence find the unique Pareto optimal provision Q^*. *Ans: Optimality requires* $\int(u_Q/u_q)dF(a) = P$. *In this case that means* $1 = \int_0^2 a\sqrt{1/Q}dF(a) = (1/2\sqrt{Q})\int_0^2 a^2 da = (4/3\sqrt{Q})$ *and therefore* $Q^* = 16/9$. **30 marks**

Government cannot observe individual preferences a. It decides to cover the cost of supplying the public good by a uniform lump sum tax PQ so that individual utilities are $\omega - PQ + 2a\sqrt{Q}$.

   (b) What is meant by a Condorcet winner? Why does a Condorcet winning supply of Q exist for these preferences and what is it?

   Ans: Condorcet winner beats all others in majority voting. Single peaked preferences guarantee absence of Condorcet cycles so majority voting defines a social preference ordering and therefore a Condorcet winner which is equal to the median preference. Preferred Q *solves* $P = a/\sqrt{Q}$ *so* $Q = a^2$. *Median* a *is* $\sqrt{2}$ *and corresponding* Q *is* 2. **30 marks**

   (c) Suppose individuals vote on their most preferred outcome by a system of plurality voting (which is to say, they select the most preferred option of the highest number of voters). What supply of Q wins?

   Ans: Density $f(a)$ *reaches a maximum at* $a = 2$ *and corresponding* Q *is* 4. **15 marks**

   (d) The government wants to implement a scheme which will encourage individuals to accurately reveal their preferences so that it can supply Q^*. Discuss the design of such a scheme.

Ans: Clarke-Groves schemes provide correct incentives by charging according to excess of cost of provision over loss of willingness to pay by others. **25 marks**

ECON0061 (Public Microeconomics)

Summer 2019

Examination time allowed: TWO hours

Answer THREE questions.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

1. (a) Explain the concept of the Lorenz curve and how it is used to compare inequality in income distributions. Explain how the Lorenz dominance criterion is linked to criteria for the comparison of inequality framed in terms of income transfers and in terms of comparisons of aggregate utility.

   Ans: Lorenz curve plots cumulated shares of income against population shares. One distribution is regarded as more equal than another if it has a higher Lorenz curve. If mean incomes are the same, then one Lorenz curve is above another iff you can get from the second distribution to the first by a series of progressive transfers. If mean incomes are the same, then one Lorenz curve is above another iff it is preferred by all concave utilitarian social welfare functions.

   (b) A population consists of H individuals with incomes $\mathbf{y} = (y_1, y_2, \ldots, y_H)$ who have been ranked so that $y_i \geq y_j$ if $i > j$. Social welfare is assessed by an increasing, homothetic, Schur-concave function $W(\mathbf{y})$. Let $I(\mathbf{y})$ be defined by

$$W(\mathbf{y}) = W\left(\bar{y}(1 - I(\mathbf{y}))\mathbf{1}_H\right)$$

   where $\bar{y}$ is mean income $\frac{1}{H}\sum_{h=1}^{H} y_h$ and $\mathbf{1}_H = (1, 1, \ldots, 1)$ is a vector of H ones. Justify $I(\mathbf{y})$ as a measure of relative inequality.

Ans: A measure of relative inequality should be homogeneous of degree zero and Schur convex. Since social welfare is Schur concave $I(\mathbf{y})$ must be Schur convex. Since social welfare is homothetic the measure is homogeneous of degree zero. It can be interpreted as the fraction by which income could be scaled down without loss of social welfare if inequality were eliminated.

(c) Suppose that $W(\mathbf{y})$ takes the form

$$W(\mathbf{y}) \;=\; \frac{1}{H}\sum_{h=1}^{H}(H-h)y_h.$$

Show that this is indeed increasing, homothetic and Schur-concave.

Ans: $(\partial W(\mathbf{y})/\partial y_i - \partial W(\mathbf{y})/\partial y_j)\,(i-j) = -(i-j)^2/H < 0$ *so it is Schur concave.* $W(\lambda\mathbf{y}) = \lambda W(\mathbf{y})$ *so it is also homothetic. It is also obviously increasing.*

(d) Suppose $W(\mathbf{y})$ takes this form and that H is large so that $\sum_{h=1}^{H} h \simeq \frac{1}{2}H^2$. Show that

$$I(\mathbf{y}) \simeq \frac{2}{H^2\bar{y}}\sum_{h=1}^{H} h\,(y_h - \bar{y})\,.$$

Ans: $W(\bar{y}(1-I(\mathbf{y}))\mathbf{1}_H) \;=\; \bar{y}(1-I(\mathbf{y}))\frac{1}{H}\sum_{h=1}^{H}(H-h) \simeq \frac{1}{2}H\bar{y}(1-I(\mathbf{y})) \simeq \frac{1}{H}\sum_{h=1}^{H}(H-h)y_h = H\bar{y} - \frac{1}{H}\sum_{h=1}^{H} hy_h$ *and therefore* $1-I(\mathbf{y}) \simeq \frac{2}{H^2\bar{y}}\sum_{h=1}^{H} hy_h$.

(e) Given this form for $W(\mathbf{y})$, how does $I(\mathbf{y})$ relate to the Lorenz curve for the income distribution?

Ans: $W(\mathbf{y})$ *is the area under the generalised Lorenz curve and* $I(\mathbf{y})$ *is the Gini coefficient, which is twice the area between the Lorenz curve and the diagonal.*

2. Individuals in an economy have preferences between consumption c and hours of work L described by utility function

$$u(c, L) \;=\; c + \ln(1 - L).$$

Pretax wages w are distributed according to cumulative distribution function F. Government imposes a linear income tax with parameters t and G such that $c = w(1-t)L + G$ and where $1 \geq t \geq 0$. Suppose that the government needs to raise no revenue so that $t \int wL \, \mathrm{d}F = G$.

(a) Derive individual labour supply, show that $G = \bar{w}t - \frac{t}{1-t}$ where $\bar{w} = \int w \, \mathrm{d}F$ is mean wage and hence show that utility of an individual with wage w is

$$\bar{w}t + w(1-t) - \ln w - \frac{1}{1-t} - \ln(1-t).$$

Ans: Utility is $G + w(1-t)L + \ln(1-L)$. This is maximised at $L = 1 - 1/w(1-t)$. Thus $G = t \int wL \, \mathrm{d}F = \bar{w}t - t/(1-t)$. Utility is $G + w(1-t) - 1 - \ln(w(1-t))$. Substitute the expression for G.

(b) Show that preferences over taxes are single peaked and show that the most preferred tax rate of an individual with wage w solves

$$\bar{w} - w \;=\; \frac{t}{(1-t)^2}$$

if $w < \bar{w}$ and $t = 0$ if $w \geq \bar{w}$.

Ans: Derivative of utility with respect to t is $\bar{w} - w - t/(1-t)^2$. Since this is decreasing in t utility is single peaked. If $w \geq \bar{w}$ then utility is decreasing in t for all $t \geq 0$. If $w < \bar{w}$ then the first order condition requires $\bar{w} - w = t/(1-t)^2$.

Suppose that wages follow a Pareto distribution with parameters w_0 and α such that $F(w) = 1 - (w_0/w)^\alpha$. For such a distribution, mean wage is $\alpha w_0/(\alpha - 1)$, median wage is $2^{1/\alpha} w_0$ and modal wage is w_0 where $\alpha/(\alpha - 1) > 2^{1/\alpha} > 1$.

(c) Compare the tax rates set by the government if it is set

(i) to maximise mean utility

(ii) to maximise the utility of the least well off individual

(iii) as the Condorcet winning tax rate

(iv) by plurality vote

Ans: The tax rate maximising mean utility is $t = 0$. *Since the lowest wage is also the modal wage for this distribution, maximising the utility of the least well off individual leads to the same tax rate as plurality voting. The Condorcet winning tax rate is the preference of the median voter which lies inbetween since the median wage is below the mean wage for this distribution.*

3. Individuals in a population of size H consume a private good q^h, $h = 1, \ldots, H$, and a public good Q. Preferences, captured in identical utility functions, $u(q^h, Q)$, are convex and homothetic so that

$$\frac{\partial u/\partial Q}{\partial u/\partial q^h} = F\left(\frac{q^h}{Q}\right)$$

for some increasing function F.

Individuals have endowments of the private good ω^h, $h = 1, \ldots, H$, which can be converted into the public good according to a technology with constant marginal rate of transformation $1/P$ so that $PQ = Y - \sum_h q^h$ where Y is aggregate endowment of the private good, $Y = \sum_h \omega^h$.

(a) Suppose that individuals contribute voluntarily to production of the public good and suppose there exists a Nash equilibrium in which all individuals choose to contribute.

(i) Show that the equilibrium is unique and

$$Q = \frac{Y}{HF^{-1}(P) + P}$$

Ans: If all individuals voluntarily contribute then $F(q^h/Q) = P$ and therefore $q^h = F^{-1}(P)Q$, for all h. Substituting into the production constraint implies $HF^{-1}(P)Q + PQ = Y$.

(ii) Why does the level of public provision not depend on the distribution of endowments among contributors?

Ans: Income pooling is a standard result for Nash equilibrium in voluntary contributions.

(b) (i) Show that the unique Pareto efficient allocation in which all individuals consume the same quantity of the private good has

$$Q = Q = \frac{Y}{HF^{-1}(P/H) + P}$$

Ans: The condition for optimum supply is that $\sum_h F(q^h/Q) = P$. If all indi- viduals have the same private consumption then $q^h = F^{-1}(P/H)Q$ for all h and $HF^{-1}(P/H)Q + PQ = Y$.

(ii) Do there exist Pareto efficient allocations in which individuals consume different quantities of the private good?

Ans: Yes, provided $\sum_h F(q^h/Q) = P$ and $\sum_h q^h + PQ = Y$.

(c) Suppose that the government chooses to fund public good provision by a proportional tax on endowments at rate t so that $q^h = (1-t)\omega^h$ and $PQ = tY$ and the rate t is to be decided by majority vote.

(i) Find each individuals preferred level of public provision and hence establish the Condorcet winning tax rate and level of public provision.

Ans: Utility is $u((1-t)\omega^h, tY/P)$ and this is maximised where $-u_q^h\omega^h + u_Q^h Y/P = 0$. So $F((1-t)P\omega^h/tY) = P\omega^h/Y$ and most preferred tax rate satisfies $(1-t)/t = YF^{-1}(P\omega^h/Y)/P\omega^h$. Hence preferred level of public spending is

$$Q = \frac{Y}{\frac{Y}{\omega^h}F^{-1}(\frac{\omega^h}{Y}P) + P}$$

The Condorcet winning provision is the preferred level of the median endowed individual.

(ii) Is the resulting allocation likely to be Pareto efficient?

Ans: No, not in general. The unique efficient level of provision subject to this tax scheme is that corresponding to the tax rate solving

$$\sum_h F\left(\left[\frac{1-t}{t}\right]\frac{\omega^h}{Y}P\right) \;=\; P$$

and there is no obvious reason why this should be the preference of the median voter.

4. A population consists of H individuals each of whom supplies the same fixed quantity of labour and who earn incomes $y^h,\ h = 1, \ldots, H$ (where all $y^h > 1$). Individuals consume three goods, cheese, fruit, and bread, at prices, p_1, p_2 and p_3. Preferences are described by identical indirect utility functions

$$V(y^h, p_1, p_2, p_3) \;=\; \frac{y^h}{p_3} - \frac{1}{2}\ln p_1 - \frac{1}{2}\ln p_2 + \ln p_3.$$

Pretax prices for all goods are unity and posttax prices are $p_i = 1 + t_i,\ i = 1, 2, 3,$ where t_1, t_2 and t_3 are tax rates. Define $\tau_i = t_i/(1 + t_i)$ as the tax rate expressed as a fraction of the posttax price. Taxes on goods are the government's only source of revenue.

(a) Show that revenue raised per person is

$$R \;=\; \frac{\tau_1 + \tau_2 - 2\tau_3}{2(1 - \tau_3)} + \tau_3\bar{y}$$

where $\bar{y}$ is mean income.

Ans: Use Roy's identity to find demands. $V_y = 1/p_3$, $V_1 = -1/2p_1$, $V_2 = -1/2p_2$, $V_3 = -y^h/p_3^2 + 1/p3$. *So* $q_1 = (1+t_3)/2(1+t_1)$, $q_2 = (1+t_3)/2(1+t_2)$ *and* $q_3 = y^h/(1+t_3) - 1$. *Revenue is then*

$$R = t_1q_1 + t_2q_2 + t_3q_3 = \frac{\frac{1}{2}t_1/(1+t_1) + \frac{1}{2}t_2/(1+t_2)}{1/(1+t_3)} - t_3 + \frac{t_3\bar{y}}{1+t_3}$$

(b) Suppose the government wishes to set τ_1, τ_2 and τ_3 to maximise mean utility

$$\frac{1}{H}\sum V^h \;=\; (1-\tau_3)\bar{y} + \frac{1}{2}\ln(1-\tau_1) + \frac{1}{2}\ln(1-\tau_2) - \ln(1-\tau_3)$$

subject to raising a specified revenue $\bar{R}$ per person. Show that the solution is to set $\tau_1 = \tau_2 = \tau_3 = \bar{R}/\bar{y}$. Discuss.

Ans: Write down a Lagrangean $\frac{1}{H}\sum_h V_h - \lambda[R - \bar{R}]$. *First order conditions are*

$$-\frac{1}{2}\frac{1}{1-\tau_1} + \frac{1}{2}\frac{\lambda}{1-\tau_3} = 0$$
$$-\frac{1}{2}\frac{1}{1-\tau_2} + \frac{1}{2}\frac{\lambda}{1-\tau_3} = 0$$

$$-\bar{y}+\frac{1}{1-\tau_3}+\lambda\left[\bar{y}-\frac{1}{1-\tau_3}+\frac{\tau_1+\tau_2-2\tau_3}{2(1-\tau_3)^2}\right]=0$$

If $\tau_1 = \tau_2 = \tau_3$ then $\lambda = 1$ and all three conditions are satisfied. The common value is equal to $\bar{R}/\bar{y}$ by substitution into the revenue constraint. There is no redistributive motive because utility is linear in income and the objective is utilitarian. So taxes are set to minimise deadweight loss.

(c) Suppose there is a powerful bread industry lobby which persuades the government to set $\tau_3 = 0$. The government now wishes to set τ_1 and τ_2 to maximise mean utility subject to raising the same specified revenue $\bar{R}$ per person. Show that the solution is to set $\tau_1 = \tau_2 = \bar{R}$. Discuss.

Ans: From the first two first order conditions $\tau_1 = \tau_2$ and the common value equals $\bar{R}$ from the revenue constraint given $\tau_3 = 0$. Preferences are symmetric in the first two goods so the tax rate on them is the same.

(d) Suppose instead that there is a powerful cheese industry lobby which persuades the government to set $\tau_1 = 0$. The government now wishes to set τ_2 and τ_3 to maximise mean utility subject to raising the same specified revenue $\bar{R}$ per person. Show that the solution is not to set $\tau_2 = \tau_3$. Discuss.

Ans: If $\tau_1 = \tau_3 = \tau$, say, and $\tau_2 = 0$ then $\lambda = 1$ from the first condition and the left hand side of the final condition becomes $-\tau/2(1-\tau)^2$ which is not zero. Without the option for uniform taxation we want to tax the high elasticity good at a lower rate.

5. There are two individuals in a population $\mathcal{P}$, a religious devotee R and a militant atheist A, and a single copy of a religious tract between them. A social preference relation $\succsim^*$ needs to be defined over sets of possible social states, $\mathcal{X}$, where the elements of $\mathcal{X}$ are drawn from the three possible social states, r (the devotee reads the tract), a (the atheist reads the tract) and 0 (the tract is unread). The atheist disapproves so strongly of what he sees as the devotee's impressionability that his preference ordering is $0 \succ_A a \succ_A r$. The devotee is so convinced that the atheist can be redeemed by exposure to the tract that his preference ordering is $a \succ_R r \succ_R 0$.

For any set $\mathcal{A}$, let $\#\{\mathcal{A}\}$ denote the number of elements in $\mathcal{A}$. Consider the following possible bases for the social preference relation:

- *Majority voting*: For any two social states $x, y \in \mathcal{X}$, $x \succsim^* y$ iff

$$\#\{i \in \mathcal{P} | x \succsim_i y\} \geq \#\{i \in \mathcal{P} | y \succsim_i x\}$$

- *Plurality voting*: For any two social states $x, y \in \mathcal{X}$, $x \succsim^* y$ iff

$$\#\{i \in \mathcal{P} | x \succsim_i z \text{ for all } z \in \mathcal{X}\} \geq \#\{i \in \mathcal{P} | y \succsim_i z \text{ for all } z \in \mathcal{X}\}$$

- *Borda count*: For any two social states $x, y \in \mathcal{X}$, $x \succsim^* y$ iff

$$\sum_{i \in \mathcal{P}} \#\{z \in \mathcal{X} | x \succsim_i z\} \geq \sum_{i \in \mathcal{P}} \#\{z \in \mathcal{X} | y \succsim_i z\}$$

- *Libertarianism*: For any two social states $x, y \in \mathcal{X}$, $x \succsim^* y$ iff either (i) there is some $i \in \mathcal{P}$ such that x and y differ only in what is read by i and $x \succsim_i y$ or (ii) there is some social state z such that by repeated application of (i) or (ii) $x \succsim^* z$ and $z \succsim^* y$

If $\mathcal{X} = \{r, a, 0\}$, then these four rules give the following social preference relations:

- Majority voting: $a \succ^* r$, $a \sim^* 0$, $r \sim^* 0$
- Plurality voting: $a \succ^* r$, $a \sim^* 0$, $0 \succ^* r$
- Borda count: $a \succ^* r$, $a \succ^* 0$, $0 \succ^* r$

- Libertarianism: $r \succ^* a$, $0 \succ^* a$, $r \succ^* 0$

For the smaller subsets of possible social states, the social preference relations are as follows.

If $\mathcal{X} = \{r, 0\}$, then

- Majority voting, Plurality voting, Borda count: $r \sim^* 0$
- Libertarianism: $r \succ^* 0$

If $\mathcal{X} = \{a, 0\}$,

- Majority voting, Plurality voting, Borda count: $a \sim^* 0$
- Libertarianism: $0 \succ^* a$

If $\mathcal{X} = \{r, a\}$,

- Majority voting, Plurality voting, Borda count: $a \succ^* r$
- Libertarianism: No basis for social preference.

(a) A social preference relation $\succsim^*$ is complete iff, for any $\mathcal{X}$ and any $x, y \in \mathcal{X}$, either $x \succsim^* y$ or $y \succsim^* x$. Which of these four bases leads to a social preference relation which is complete?

Ans: All are complete except Libertarianism which fails if $\mathcal{X} = \{r, a\}$

(b) A social preference relation $\succsim^*$ is transitive iff, for any $\mathcal{X}$ and any $x, y \in \mathcal{X}$, $x \succsim^* y$ and $y \succsim^* z$ implies $x \succsim^* z$. Which of these four bases leads to a social preference relation which is transitive?

Ans: All except Majority voting. If $\mathcal{X} = \{r, a, 0\}$ *then* $r \succsim^* 0$ *and* $0 \succsim^* a$ *but* $a \succ^* r$.

(c) A social preference relation $\succsim^*$ satisfies the Pareto principle iff, for any $\mathcal{X}$ and any $x, y \in \mathcal{X}$, $x \succsim^* y$ whenever $x \succsim_i y$ for all $i \in \mathcal{P}$. Which of these four bases leads to a social preference relation which satisfies the Pareto principle?

Ans: All except Libertarianism. If $\mathcal{X} \supseteq \{r, a\}$ *then both prefer* a *to* r *but Libertarianism either judges* $r \succ^* a$ *or makes no judgement.*

(d) A social preference relation $\succsim^*$ satisfies independence of irrelevant alternatives iff,for any $\mathcal{X}$ and any $x, y \in \mathcal{X}$, $x \succsim^* y$ only if it is also true that $x \succsim^* y$ for any $\mathcal{X} \supseteq \{x, y\}$. Which of these four bases leads to a social preference relation which satisfies independence of irrelevant alternatives?

Ans: Only Majority voting. For Plurality voting and Borda count, preference between r *and* 0 *can be altered by adding or removing* a *from the choice set. For Libertarianism preference between* a *and* r *can be made indeterminate by removing* 0*.*

6. A population consists of two types, those able to command high wages w_A for their hours of work L_A and those able to command low wages w_B for their hours of work L_B. A high wage type can pose as a low wage type but a low wage type cannot pose as a high wage type.

   The government knows the proportions of each type π_A and $\pi_B = 1 - \pi_A$ but cannot identify who is of which type. It sets a tax schedule determining after-tax consumption c as a function of before-tax earnings wL so that high wage types choose consumption c_A and earnings $w_A L_A$ and low wage types choose consumption c_B and earnings $w_B L_B$.

   Individuals of all types have preferences over consumption and hours of work described by utility function $u(c) - v(L)$ where u is concave and v is convex.

   The government aims to maximise aggregate utility

$$\pi_A[u(c_A) - v(L_A)] + \pi_B[u(c_B) - v(L_B)]$$

   subject to meeting a revenue requirement

$$R + \pi_A[c_A - w_A L_A] + \pi_B[c_B - w_B L_B] = 0$$

   and to an incentive compatibility constraint

$$[u(c_A) - v(L_A)] \geq [u(c_B) - v(w_B L_B / w_A)]$$

   (a) Explain the nature of the incentive compatibility constraint.

   Ans: High ability types must be offered at least the utility they could get by consuming the consumption-earnings combination offered to low ability types.

   (b) Show that at the social optimum

$$v'(L_A) = w_A u'(c_A) \qquad \text{and} \qquad v'(L_B) < w_B u'(c_B)$$

   and interpret these conditions.

Ans: Set up Lagrangean

$$\begin{aligned}&\pi_A[u(c_A) - v(L_A)] + (1 - \pi_A)[u(c_B) - v(L_B)]\\&\quad - \lambda[R + \pi_A[c_A - w_A L_A] + (1 - \pi_A)[c_B - w_B L_B]]\\&\quad + \mu[u(c_A) - v(L_A) - u(c_B) + v(w_B L_B/w_A)].\end{aligned}$$

First order conditions for c_A and L_A require $u'(c_A)(\pi_A+\mu) = \lambda$ and $v'(L_A)(\pi_A+\mu) = w_A\lambda$. From this follows the first equality, $v'(L_A) = w_A u'(c_A)$, which says that the MRS between consumption and labour should be undistorted for the more able. Because there is noone more able who might want to pretend to be them the incentive compatibility constraint does not require a positive marginal tax rate.

First order conditions for c_B and L_B require $u'(c_B)(1 - \pi_A - \mu) = \lambda$ and $(1 - \pi_A)v'(L_B) + \mu w_B/w_A v'(w_B L_B/w_A) = w_B\lambda$. From this, noting $w_A > w_B$, follows the second inequality, $v'(L_B) < w_B u'(c_B)$, which says that the MRS between consumption and labour should be distorted for the less able.

(c) Discuss the extension of such a model to cover a greater finite number of types and a continuum of types.

Ans: For each new type added you increase the number of incentive compatibility constraints by one as you require each type to have no incentive to pose as the next least able. (Other incentive compatibility constraints are redundant.) In the limit, as differences between types go to zero, the incentive compatibility constraints become a differential equation describing how utility has to increase with type.

(d) The following comment, taken from J. A. Mirrlees, *Welfare, Incentives and Taxation*, 2006, relates to the results of his 1971 paper on optimum income taxation:

> "Since that paper was published, a number of new results have been found ... The first, and most notorious, is the proposition that the

> marginal tax rate on the highest income (if there is one) should be zero. My paper only considered models with an unbounded distribution of wage rates, and therefore did not have such a result. I believe that was the right strategy. ... The zero rate is ... practically irrelevant. Nor is it a good approximation to tax rates within, say, ten per cent of the highest possible."

Discuss the opinions expressed in these comments, in particular as they relate to the results of the earlier part of this question.

Ans: The result which says that the most able have undistorted labour supply is true only at the very top since there are incentive compatibility constraints which bite everywhere else. If there is no known highest possible ability then the result holds nowhere. As the quote says, simulations suggest the relevance of the result to practical tax setting is low.

SUMMER TERM 2020

24-HOUR ONLINE EXAMINATION

ECON0061 (Public Microeconomics)

All work must be submitted anonymously. Please ensure that you add your **candidate number** *and the module code to the template answer sheet provided. Note that the candidate number is a combination of four letters plus a number, e.g. ABCD9. You can find your candidate number in your PORTICO account, under "My Studies" then the "Examinations" container. Please, note that the candidate number is NOT the same as your student number (8 digits), which is printed on your UCL ID card. Submitting with your student number will delay marking and when your results might be available.*

Answer THREE questions.

Questions carry one third of the total mark each.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

Allow enough time to submit your work. Waiting until the deadline for submission risks facing technical problems when submitting your work, due to limited network or systems capacity.

By submitting this assessment, **I pledge my honour** *that I have not violated UCL's Assessment Regulations which are detailed in*

`https://www.ucl.ac.uk/academic-manual/chapters/chapter-6-student-casework-framework/section-9-student-academic-misconduct-procedure`,

which include (but are not limited to) plagiarism, self-plagiarism, unauthorised collaboration between students, sharing my assessment with another student or third party, accessing another student's assessment, falsification, contract cheating, and falsification of extenuating circumstances.

1. (a) State and explain the Pigou-Dalton criterion for comparison of inequality between two income distributions.

   Ans: Transfers from richer to poorer reduce inequality.

   (b) State and explain the properties of a relative inequality index.

   Ans: Homogeneous of degree zero and Schur-convex.

   (c) A population of H individuals have incomes $\mathbf{y} = (y_1, y_2, \ldots, y_H)$ where individuals are ordered so that $y_i > y_j$ if $i > j$.

   (i) Consider the function $\mathcal{V}_1(\mathbf{y})$ defined by

$$\mathcal{V}_1(\mathbf{y}) \;=\; \ln\left[\frac{1}{H}\sum_h y_h^2\right] - 2\ln\left[\frac{1}{H}\sum_h y_h\right].$$

   Find an expression for $\partial\mathcal{V}_1(\mathbf{y})/\partial y_i$ and thus show that a small transfer of income from individual i to individual j reduces $\mathcal{V}_1(\mathbf{y})$ whenever $i > j$.
   Ans: $\partial\mathcal{V}_1(\mathbf{y})/\partial y_i = \frac{2}{H}y_i/\frac{1}{H}\sum_h y_h^2 - \frac{2}{H}/\frac{1}{H}\sum_h y_h$. *This is plainly increasing in* i.

   (ii) Consider the function $\mathcal{V}_2(\mathbf{y})$ defined by

$$\mathcal{V}_2(\mathbf{y}) \;=\; \frac{1}{H}\sum_h(\ln y_h)^2 - \left[\frac{1}{H}\sum_h \ln y_h\right]^2.$$

   Find an expression for $\partial\mathcal{V}_2(y)/\partial y_i$ and thus show that a small transfer of income from individual i to individual j reduces $\mathcal{V}_2(\mathbf{y})$ only if

$$\frac{\ln(y_i/\tilde{y})}{y_i/\tilde{y}} > \frac{\ln(y_j/\tilde{y})}{y_j/\tilde{y}}$$

   where $\ln\tilde{y} = \frac{1}{H}\sum_h \ln y_h$. Hence show that a small transfer from a richer to a poorer person increases $\mathcal{V}_2(\mathbf{y})$ if both giver and receiver have incomes greater than $\ln\tilde{y} + 1$. Comment.
   Ans: $\partial\mathcal{V}_2(\mathbf{y})/\partial y_i = \frac{\frac{2}{H}\ln(y_i/\tilde{y})}{y_i}$. *So* $\partial\mathcal{V}_2(\mathbf{y})/\partial y_i > \partial\mathcal{V}_2(\mathbf{y})/\partial y_j$ *if and only if*

$$\frac{\ln(y_i/\tilde{y})}{y_i/\tilde{y}} > \frac{\ln(y_j/\tilde{y})}{y_j/\tilde{y}}.$$

Since $\ln x/x$ is decreasing for $1 < \ln x$, a progressive transfer increases $\mathcal{V}_2(\mathbf{y})$ if both $ln(y_i/\tilde{y})$ and $ln(y_j/\tilde{y})$ are above 1. The variance of logs, which is what $\mathcal{V}_2$ is, is not Schur-convex.

(iii) Suppose incomes are drawn from a lognormal distribution, $\ln y \sim \mathcal{N}(\mu, \sigma^2)$. For a lognormal distribution, $\ln \mathcal{E}(y) = \mu + \frac{1}{2}\sigma^2$, $\ln \mathcal{E}(y^2) = 2\mu + 2\sigma^2$, $\mathcal{E}(\ln y) = \mu$, $\mathcal{E}((\ln y)^2) = \mu^2 + \sigma^2$. Suppose that H is large enough that we can assume moments in the population are equal to these values. Show that $\mathcal{V}_1(\mathbf{y}) = \mathcal{V}_2(\mathbf{y})$ and that comparisons of inequality using the two measures should coincide. Comment.

Ans: Both equal σ^2. Two lognormal distributions never differ in a way that involves the sort of inequality-reducing change that has a perverse effect on $\mathcal{V}_2(\mathbf{y})$. The variance of logs can be a suitable index of inequality if comparing within a restricted class of income distributions.

2. A population of H individuals have wages w_h, supply hours of work L_h and consume a private good q_h and a public good Q, $h = 1, 2, \ldots H$. Preferences are described by utility function

$$u_h = q_h + \alpha \ln Q - \frac{1}{2} L_h^2$$

The marginal cost of producing the public good is constant at P units of the private good so that

$$\sum_{h=1}^{H} [w_h L_h - q_h] = PQ$$

(a) Suppose that resources for spending on the public good are raised through voluntary private contributions and a Nash equilibrium exists in which some or all individuals contribute. Show that the level of provision is $Q^N = \alpha/P$.

Ans: For anyone contributing $MRS = \alpha/Q = P$ so $Q = \alpha/P$.

(b) Suppose that resources for spending on the public good are raised through a uniform lump sum tax T so $Q = HT/P$. Mean utility is therefore

$$W = \frac{1}{H} \sum_h \max_{L_h} \left[w_h L_h - \frac{1}{2} L_h^2 \right] - T + \alpha \ln(TH/P)$$

Show that the optimum level of provision is $Q^U = H\alpha/P$. Comment.

Ans: Optimum provision requires

$$\sum_h MRS = H\alpha/Q = P$$

so $Q = H\alpha/P$. The efficient provision is larger than the equilibrium voluntary provision.

(c) Suppose that resources for spending on the public good are raised through a proportional tax at rate t on labour income so $Q = t \sum_h w_h L_h / P$. Each individual regards themself as small relative to the aggregate and therefore decides labour supply to maximise $w_h(1-t)L_h - \frac{1}{2}L_h^2$ without regard to the implications for supply of the public good. Find optimum labour supply and hence show that mean utility is

$$W = \frac{1}{2}\mu_2(1-t)^2 + \alpha \ln(\mu_2 t(1-t)H/P)$$

where $\mu_2 = \frac{1}{H}\sum_h w_h^2$.

Ans: Utility is maximised at $L_h = w_h(1-t)$. This raises revenue $\sum_h t w_h L_h = H\mu_2 t(1-t)$ which funds $Q = H\mu_2 t(1-t)/P$ Hence mean utility is

$$\frac{1}{H}\sum_h \left[w(1-t)L_h - \frac{1}{2}L_h^2\right] + \alpha \ln(\mu_2 t(1-t)H/P) = \frac{1}{2}\mu_2(1-t)^2 + \alpha \ln(\mu_2 t(1-t)H/P).$$

(d) Show that the optimum level of t, say t^*, satisfies $\mu_2 t^*(1-t^*) = \alpha\left(1 - \frac{t^*}{1-t^*}\right)$ and therefore the optimum level of provision is

$$Q^D = \frac{H\alpha}{P}\left(1 - \frac{t^*}{1-t^*}\right).$$

Comment.

Ans: Differentiate utility with respect to t and set to zero to get

$$\mu_2(1-t^*) = \alpha\frac{1-2t^*}{t^*(1-t^*)}$$

Therefore

$$\mu_2 t^*(1-t^*) = \alpha\left(1 - \frac{t^*}{1-t^*}\right)$$

and, since $Q = H\mu_2 t(1-t)/P$, $Q^D = H\alpha\left(1 - \frac{t^*}{1-t^*}\right)/P$. Thus $Q^D < Q^U$ since $t^* > 0$. The distortionary cost of raising funds for public spending reduces the optimum supply.

3. A population of individuals varies in their pretax wage w, distributed according to distribution function F and corresponding density function f. Individuals supply labour L and consume a vector of goods $q = (q_0, q_1, q_2, \dots)$ with common pretax prices $p = (p_0, p_1, p_2, \dots)$. Utility is additively separable, $U = u(q) - v(L)$. The government wants to design nonlinear taxes so as to maximise aggregate utility $\int Uf \, \mathrm{d}w$ subject to a revenue constraint

$$\int [wL - p'q]f \, \mathrm{d}w \ = \ \bar{R}$$

where $\bar{R}$ is required revenue per person. The government knows the distribution F but cannot observe individual wages and can therefore base taxation only on observed earnings and goods purchases.

(a) Discuss the nature of the incentive compatibility constraint restricting the government's tax choice and explain why it is captured by the requirement

$$\frac{\mathrm{d}U}{\mathrm{d}w} \ = \ \frac{Lv'(L)}{w}$$

Ans: Individual's cannot have incentive to choose an earnings-consumption combination intended for those with lower wage. So $u(q^A) - v(L^A) \geq u(q^B) - v(w^B L^B / w^A)$ *if* $w^A > w^B$*, or equivalently* $U(w^A) - U(w^B) \geq v(w^B L^B / w^A) - v(w^B)$*. If we let* $w^A - w^B \to 0$ *then we get* $\mathrm{d}U/\, \mathrm{d}w \ = \ Lv'(L)/w$*. This is the same as saying individuals will choose labour supply to maximise their own utility subject to the budget constraint offered. Differentiating the expression for utility*

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}w} &= \sum_i u'(q_i) \frac{\mathrm{d}q_i}{\mathrm{d}w} - v'(L) \frac{\mathrm{d}L}{\mathrm{d}w} \\
&= \frac{v'(L)}{w} \left[\sum_i \frac{p_i(1+t_i)}{(1-T')} \frac{\mathrm{d}q_i}{\mathrm{d}w} - w \frac{\mathrm{d}L}{\mathrm{d}w} \right] \\
&= Lv'(L)/w
\end{aligned}$$

using the individual optimising condition $w(1-T')/p_i(1+t_i) \ = \ v'(L)/u'(q_i)$ *and the budget constraint.*

Represent the government's problem as an optimum control problem with Hamiltonian

$$\mathcal{H} \;=\; Uf \;+\; \lambda[wL - p'q - \bar{R}]f \;+\; \mu L v'(L)/w$$

where λ is a Lagrange variable and μ is a costate variable.

(b) Explain why the solution involves the conditions

$$\lambda\left[w - p_0 \frac{\partial q_0}{\partial L}\bigg|_U\right] f + \frac{\mu}{w}\left[v'(L) + Lv''(L)\right] \;=\; 0$$

$$\lambda\left[-p_0 \frac{\partial q_0}{\partial q_i}\bigg|_U \;-\; p_i\right] f \;=\; 0$$

and discuss the implications for

(i) distortion of labour supply decisions in optimum policy

(ii) the role of differentiated commodity taxation in optimum policy.

Ans: Treat $q_1, q_2, \ldots$ and L as control variables and U as state variable. First order conditions require $\partial\mathcal{H}/\partial q_i = 0$ and $\partial\mathcal{H}/\partial L = 0$ which are the two conditions here. If the consumer is optimising then $\partial q_0/\partial L|_U = \; w(1-T')/p_0(1+t_0)$ and $\partial q_0/\partial q_i|_U = \; -p_i(1+t_i)/p_0(1+t_0)$. Substituting the first into the first expression gives

$$\lambda w\left[\frac{t_0 + T'}{1+t_0}\right] f = \frac{\mu}{w}\left[v'(L) + Lv''(L)\right]$$

from which $t_0 + T' \neq 0$ and labour supply is distorted. Substituting the second into the second expression gives

$$\lambda p_i \frac{t_i - t_0}{1+t_0} = 0$$

from which $t_i \;=\; t_0$ and commodity choice is undistorted.

(c) What determines how μ varies across the wage distribution?

Ans: At the optimum

$$\partial\mu/\partial w \;=\; -\,\partial\mathcal{H}/\partial U \;=\; \left[1 - \lambda\,\partial q_0/\partial U|_{L,q_1,q_2,\ldots}\right] f$$

4. Consider a population of individuals, $h = 1, \dots, H$ consuming m commodities q^h at prices p. Each works the same fixed hours for an earned income y^h. Preferences over goods are identical and homothetic and described by indirect utility functions

$$V\left(y^h, p\right) \; = \; \ln y^h - \ln A(p)$$

where $A(p)$ is a linearly homogeneous function of prices.

Suppose pretax prices are all equal to 1 and posttax prices are $p = 1 + t$ where t is a vector of commodity tax rates.

(a) Show that goods demands are

$$q_i^h \; = \; \frac{A_i}{A} \, y^h \qquad i = 1, \dots, m$$

where $A_i \; = \; \partial A(p)/\partial p_i$ and that revenue is therefore

$$R \; = \; \sum_i \frac{t_i A_i}{A} \sum_h y^h.$$

Ans: Use Roy's identity. $f_i(y^h, 1+t) = \; -V_i/V_y$, $V_i = -A_i/A$ *and* $V_y = 1/y^h$. *Revenue is* $R = \sum_h \sum_i t_i \cdot 1 \cdot q_i^h$

(b) Consider the problem of maximising the sum of utilities $\sum_h V(y^h, 1+t)$ subject to raising a required share of total income as revenue $R/\sum_h y^h \geq \bar{R}$. Set up a Lagrangean

$$\frac{1}{H} \sum_h V(y^h, 1+t) \; + \; \lambda \left[R/\sum y^h - \bar{R} \right]$$

and show that

$$1 \; = \; \lambda \left[1 + \sum_j t_j \left(\frac{A_{ji}}{A_i} - \frac{A_j}{A} \right) \right] \qquad i = 1, \dots, m$$

where $A_{ij} \; = \; \partial^2 A(p)/\partial p_i \partial p_j$.

Ans: First order conditions are

$$-\frac{A_i}{A} + \lambda \left[\frac{A_i}{A} + \sum_j t_j \left(\frac{A_{ji}}{A} - \frac{A_j A_i}{A^2} \right) \right] \; = \; 0$$

Dividing through by A_i/A *gives the given condition.*

(c) Use the homogeneity properties of A, $\sum_j (1+t_j)A_j = A$ and $\sum_j (1+t_j)A_{ij} = 0$, to show that these equations can be satisfied with uniform taxes, $t_i = \bar{R}$, $i = 1, \ldots, m$.

Ans: Suppose $t_j = \tau$ for all j. Then

$$
\begin{aligned}
\left[1 + \sum_j t_j \left(\frac{A_{ji}}{A_i} - \frac{A_j}{A}\right)\right] &= \left[1 + \sum_j \frac{t_j}{1+t_j}\left(\frac{(1+t_j)A_{ji}}{A_i} - \frac{(1+t_j)A_j}{A}\right)\right] \\
&= \left[1 + \frac{\tau}{1+\tau}\left(\frac{\sum_j (1+t_j)A_{ij}}{A_i} - \frac{\sum_j (1+t_j)A_j}{A}\right)\right] \\
&= \left[1 + \frac{\tau}{1+\tau}\left(\frac{0}{A_i} - \frac{A}{A}\right)\right] \\
&= \frac{1}{1+\tau}
\end{aligned}
$$

So all m conditions are satisfied if $\lambda = 1 + \tau$. Then $\tau = \bar{R}$ from the revenue constraint.

(d) Comment on the generality of this result.

This can be considerably generalised. Homotheticity can be relaxed to linearity of Engel curves. Fixed labour supply can be generalised to separability of goods and leisure in preferences.

5. A population consists of individuals with preferences between consumption c and hours of work L described by utility function

$$u(c, L) \;=\; \alpha \ln c + (1-\alpha)\ln(1-L)$$

where α is a parameter reflecting taste for work. Pretax wages w are distributed according to cumulative distribution function F. The government imposes a linear income tax with parameters t and G such that $c = w(1-t)L + G$ and where $1 \geq t \geq 0$. Suppose that the government's revenue constraint is $t \int wL \,\mathrm{d}F = G$.

(a) Derive individual labour supply, show that

$$G \;=\; \frac{\alpha \bar{w} t(1-t)}{1-\alpha t}$$

where $\bar{w} = \int w \,\mathrm{d}F$ is mean wage and hence show that utility of an individual with wage w is

$$U \;=\; \ln\left[w + \frac{\alpha \bar{w} t}{1-\alpha t}\right] + \alpha \ln(1-t) - (1-\alpha)\ln w + \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha).$$

Ans: Utility is $\alpha \ln(G + w(1-t)L) + (1-\alpha)\ln(1-L)$. In other words, preferences are Cobb-Douglas with full income $G + w(1-t)$. So budget shares are constant and $L = 1 - (1-\alpha)[G + w(1-t)]/w(1-t)$. Thus, from the revenue constraint, $G = t\int wL \,\mathrm{d}F = \alpha\bar{w}t - (1-\alpha)Gt/(1-t)$ and therefore $G = \alpha\bar{w}t(1-t)/(1-\alpha t)$. So full income is $G + w(1-t) = (1-t)[w + \alpha\bar{w}t/(1-\alpha t)]$ and

$$\begin{aligned} U &= \ln(G + w(1-t)) - (1-\alpha)\ln(w(1-t)) + \alpha\ln\alpha + (1-\alpha)\ln(1-\alpha) \\ &= \ln\left[w + \frac{\alpha\bar{w}t}{1-\alpha t}\right] + \alpha\ln(1-t) - (1-\alpha)\ln w + \alpha\ln\alpha + (1-\alpha)\ln(1-\alpha) \end{aligned}$$

(b) These preferences over taxes are single peaked. Show that

$$\frac{\mathrm{d}U}{\mathrm{d}t} \;=\; \alpha\left[\frac{\bar{w} - w}{w + \alpha t(\bar{w} - w)} - \frac{(1-\alpha)t}{(1-t)(1-\alpha t)}\right]$$

and thereby show that the most preferred tax rate of an individual with wage w is positive if and only if $w < \bar{w}$.

Ans: The derivative of utility with respect to t is

$$-\frac{\alpha}{1-t} + \frac{\alpha}{1-\alpha t} + \alpha\frac{\bar{w}-w}{w+\alpha t(\bar{w}-w)} = \alpha\left[\frac{\bar{w}-w}{w+\alpha t(\bar{w}-w)} - \frac{(1-\alpha)t}{(1-t)(1-\alpha t)}\right]$$

Since $(1-\alpha)t/(1-t)(1-\alpha t) > 0$ for $1 \geq t \geq 0$ there is no interior solution unless $w < \bar{w}$.

(c) It can be shown that $\bar{w}\int(1/w)\,\mathrm{d}F > 1$. Use this to show that the tax rate which maximises average utility is positive.

Ans: Derivative of mean utility with respect to t at $t = 0$ is
$\mathrm{d}\bar{U}/\,\mathrm{d}t = \alpha\left[\int \bar{w}/w\,\mathrm{d}F - 1\right] > 0.$

(d) Suppose that the tax rate is selected which beats all others in pairwise majority votes. What must be true for the selected tax rate to be positive?

Ans: The selected tax rate will be the preference of the individual with median wage. Hence it will be positive only if median wage is below $\bar{w}$.

6. (a) Consider a population with preferences over a set of policy alternatives.

   (i) Why does majority voting fail to satisfy the requirements for a social preference relation specified in Arrow's General Possibility Theorem?

   Ans: It violates U (universal domain) because of the possibility of Condorcet cycles and violation of transitivity.

   (ii) What does it mean for preferences to be single-peaked?

   Ans: There exists an ordering of the alternatives such that for every voter if we take any two options on the same side of their most preferred alternative then the one nearest to the most preferred is the more preferred of the two.

   (iii) What is meant by a Condorcet cycle? What does single-peakedness of preferences imply for the possibility of a Condorcet cycle?

   Ans: Violation of transitivity in majority voting. This is impossible if preferences are single-peaked.

(b) In March 2019, 5,000 UK voters were polled over their preferences regarding four policy alternatives: R (remaining in the EU), S (soft Brexit), D (Theresa May's Brexit deal), N (leaving the EU with no deal). The percentages of those polled agreeing with each of the 24 possible preference orderings are given in the table below. (Source: Christina Pagel)

$D \succ N \succ R \succ S$	1	$N \succ D \succ R \succ S$	1	$R \succ D \succ N \succ S$	1	$S \succ D \succ N \succ R$	4
$D \succ N \succ S \succ R$	6	$N \succ D \succ S \succ R$	15	$R \succ D \succ S \succ N$	5	$S \succ D \succ R \succ N$	2
$D \succ R \succ N \succ S$	0	$N \succ R \succ D \succ S$	1	$R \succ N \succ D \succ S$	1	$S \succ N \succ D \succ R$	4
$D \succ R \succ S \succ N$	1	$N \succ R \succ S \succ D$	1	$R \succ N \succ S \succ D$	2	$S \succ N \succ R \succ D$	1
$D \succ S \succ N \succ R$	4	$N \succ S \succ D \succ R$	9	$R \succ S \succ D \succ N$	29	$S \succ R \succ D \succ N$	2
$D \succ S \succ R \succ N$	1	$N \succ S \succ R \succ D$	2	$R \succ S \succ N \succ D$	6	$S \succ R \succ N \succ D$	1

   (i) Consider the conventional ordering of these four alternatives according to strength of pro-EU sentiment: R then S then D then N. Explain why only eight of the possible preference orderings are single-peaked along that dimension. Identify those eight preference orderings and calculate the percentage of the polled population who have single-peaked preferences along that dimension.

Ans: If R or N are most preferred then preferences need to decline continually as we move away from them. So only $R \succ S \succ D \succ N$ and $N \succ D \succ S \succ R$ are allowed. If S is most preferred then D must be preferred to N and if D is most preferred then S must be preferred to R. So only $S \succ R \succ D \succ N$, $S \succ D \succ R \succ N$, $S \succ D \succ N \succ R$ and $D \succ N \succ S \succ R$, $D \succ S \succ N \succ R$, $D \succ S \succ R \succ N$ re allowed. These eight are 29+15+2+2+4+6+4+1=63 per cent of voters.

(ii) If we consider majority votes over the six possible binary pairings based on the information in the table above then we find the results in the following table

$D \succ N$	62	$N \succ D$	38	$N \succ R$	49	$R \succ N$	51
$D \succ R$	48	$R \succ D$	52	$N \succ S$	40	$S \succ N$	60
$D \succ S$	43	$S \succ D$	57	$S \succ R$	51	$R \succ S$	49

Does there exist any Condorcet cycle? If not, which option is Condorcet winner? Comment.

Ans: There is no Condorcet cycle. S beats all three others, R beats both D and N and D beats N. So S is Condorcet winner (despite having least first preferences of any option).