# Assignment 1

## 1. The Solow Model

**a.** The marginal product of capital is

$$f'(k) = \alpha \left( \frac{1}{k^{1-\alpha}} \right)$$

Given that $\alpha \in (0, 1)$ we have $\lim_{k \to 0} f'(k) = +\infty$ and $\lim_{k \to \infty} f'(k) = 0$.

**b.** In steady state we have

$$sk^{\alpha} = \delta k \quad \Rightarrow \quad k = \left( \frac{s}{\delta} \right)^{1/(1-\alpha)}$$

The steady state level of per-capita capital is decreasing in the depreciation rate $\delta$.

**c.** *see the Excel file for problem set 1.*

**d.** The Golden Rule savings rate $s^{GR}$ is the savings rate that maximises steady state per-capita consumption

$$s^{GR} = \arg\max_{s} \{(1 - s) f(k(s))\} = \arg\max_{s} \{f(k(s)) - \delta k(s)\}$$

where the last equality makes use of the definition of the steady state $sf(k) = \delta k$. Given the concavity of the problem, $s^{GR}$ is implicitly characterised by the first order condition

$$f'(k(s^{GR}))k'(s^{GR}) - \delta k'(s^{GR}) = 0 \quad \Rightarrow \quad f'(k(s^{GR})) = \delta$$

With $f(k) = k^\alpha$ we have

$$k(s) = \left(\frac{s}{\delta}\right)^{1/(1-\alpha)} \quad \text{and} \quad f'(k(s)) = \frac{\alpha\delta}{s}$$

It follows that $s^{GR} = \alpha$. The intuition for the Golden Rule comes from $f'(k^{GR}) = \delta$. If the steady state capital stock is increased marginally (by raising $s$ marginally) total available output for consumption or investment increases by $f'(k)dk$. In steady state investment has to increase by $\delta dk$ in order to keep the capital stock constant. So as long as $f'(k)dk > \delta dk$ expanding the steady state capital stock by increasing $s$ increases steady state consumption. Likewise steady state consumption is decreased if $f'(k)dk < \delta dk$.

**e. and f.** *see the Excel file for problem set 1.*

## 2. A three-period model of intertemporal optimisation

**a.** Given that there is no fourth period and the marginal utility of consumption is always positive (for finite values of c) then $b_2 > 0$ can never be optimal. We necessarily have $b_2 = 0$. The Lagrangian for the problem is then

$$\mathcal{L} = U(c_0, c_1, c_2) + \lambda_0(e_0 - c_0 - b_0) + \lambda_1(e_1 + (1+r_0)b_0 - c_1 - b_1) + \lambda_2(e_2 + (1+r_1)b_1 - c_2)$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + (1+r_t)\lambda_{t+1} = 0 \;\; , \;\; t = 0, 1$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^{t-1}\, c_t^{-\theta} - \lambda_t = 0 \;\; , \;\; t = 0, 1, 2$$

as well as $\partial\mathcal{L}/\partial\lambda_t = 0$ for $t = 0, 1, 2$ which yields the three period budget constraints.

The Lagrange multiplier $\lambda_t$ represents the shadow value of wealth. To see this note that $\partial\mathcal{L}/\partial e_t = \lambda_t$. Thus $\lambda_t$ tells you by how much the maximised value of the objective function increases if the endowment is increased marginally in period $t$.

**b.** We get the Euler Equations from combining the first order conditions for $c_t$ and $b_t$

$$c_t^{-\theta} = \beta(1 + r_t)c_{t+1}^{-\theta} \ , \quad t = 0, 1$$

The Euler Equation tells us that at the optimal solution, the utility loss from consuming one less unit today must be equal to the utility gain from saving that unit and consuming it next period.
Note that as long as there is a bond that allows us to move consumption between any two adjacent periods (and hence defines an EE between period t and t+1) we can recursive substitute out to solve for the EE between any $t$ and $t + j$ period.

**c.** We need to adjust the first and third period budget constraint. Call $l_0$ the units of long-term bonds purchased in period 0. Then

$$c_0 + b_0 + l_0 = e_0 \ \text{ and } \ c_2 = e_2 + (1 + r_1)b_1 + (1 + r_{0,2})l_0$$

Re-writing the Lagrangian again we get

$$\begin{aligned}\mathcal{L} = U(c_0, c_1, c_2) + \lambda_0(e_0 - c_0 - b_0 - l_0) + \lambda_1(e_1 + (1 + r_0)b_0 - c_1 - b_1) + \\ \lambda_2(e_2 + (1 + r_1)b_1 + (1 + r_{0,2})l_0 - c_2)\end{aligned}$$

Comparing it to the previous Lagrangian we can see that the FOC for $\{c_t, b_t\}$ do not change. But we now have an additional first order condition with respect to $l_0$:

$$\frac{\partial \mathcal{L}}{\partial l_0} = -\lambda_0 + (1 + r_{0,2})\lambda_2 = 0$$

and the corresponding Euler Equation is

$$c_0^{-\theta} = \beta^2(1 + r_{0,2})c_2^{-\theta}$$

**d.** To rule out arbitrage we need

$$1 + r_{0,2} = (1 + r_0)(1 + r_1)$$

To see this suppose you borrow a unit of period 0 consumption financed by issuing a long-term bond and then lend on this unit in exchange for a short term bond. The short term bonds repays $1 + r_0$ in period 1 which you then reinvest into a short term bond yielding a total of $(1+r_0)(1+r_1)$ units of consumption goods in period 2. In period 2 you have to repay $1+r_{0,2}$ units for your long-term bond. If $1+r_{0,2} < (1+r_0)(1+r_1)$ you can finance infinite amounts of consumption by going short in the long-term bond and long in short-term bonds. Similarly if $1+r_{0,2} > (1+r_0)(1+r_1)$ you can finance infinite levels of consumption by going short in short-term bonds and long in long-term bonds.

# Assignment 2

## 1 . An economy with two agents and two periods

**a.** To solve for the optimum substitute the resource constraints of the economy, $c_1 + d_1 = 1$ and $c_2 + d_2 = g$, into the objective function

$$\lambda[\log c_1 + \beta \log c_2] + (1-\lambda)[\log(1-c_1) + \beta \log(g-c_2)]$$

Now maximise this with respect to $c_1$ and $c_2$. You get the following two first order conditions

$$\frac{\lambda}{c_1} = \frac{1-\lambda}{1-c_1}$$

$$\beta\frac{\lambda}{c_2} = \beta\frac{1-\lambda}{g-c_2}$$

Solving for $c_1$ and $c_2$ yields

$$c_1 = \lambda\ ,\ c_2 = \lambda g$$

We can now get $d_1$ and $d_2$ from the resource constraints

$$d_1 = 1 - c_1 = 1 - \lambda\ ,\ d_2 = g - c_2 = (1-\lambda)g$$

We can easily see that the growth rates of consumption are the same for both consumers

$$\frac{c_2}{c_1} - 1 = \frac{d_2}{d_1} - 1 = g - 1$$

**b.** The marginal rate of substitution for consumer 1 is

$$\frac{\partial u(c_1,c_2)/\partial c_1}{\partial u(c_1,c_2)/\partial c_2} = \frac{1/c_1}{\beta/c_2} = \frac{c_2}{\beta c_1}$$

For the optimal allocation this yields the relative shadow price

$$\frac{p_1}{p_2} = \frac{c_2}{\beta c_1} = \frac{g}{\beta}$$

The same is true for consumer 2

$$\frac{\partial u(d_1, d_2)/\partial d_1}{\partial u(d_1, d_2)/\partial d_2} = \frac{1/d_1}{\beta/d_2} = \frac{d_2}{\beta d_1} = \frac{g}{\beta}$$

The implied real interest rate is the opportunity cost of consuming one unit of the consumption good today rather than tomorrow measured in units of consumption goods. If I do not consume the unit today I save $p_1$. I will be able to buy $p_1/p_2$ units of tomorrow's consumption goods instead. Thus the opportunity cost of consuming today measured in consumption goods is

$$r = \frac{p_1}{p_2} - 1$$

Here the interest rate is

$$r = \frac{g}{\beta} - 1$$

This expression makes intuitive sense. The higher the growth rate $g$, the more abundant is the consumption good in period 2. Consumption in period 1 becomes more valuable as compared to consumption in period 2 ( the marginal utility of consumption is higher ). The interest rate has to increase in order to prevent consumers from consuming too much in period 1 violating the economy's resource constraint. We can also reshuffle the equation such that:

$$\beta(1 + r) = g$$

**c.** A competitive equilibrium is a price vector $(p_1, p_2)$, and an allocation $\{(c_1, c_2), (d_1, d_2)\}$, such that
1) the allocation $\{(c_1, c_2), (d_1, d_2)\}$ maximises consumers' utility subject to their budget constraint given prices $(p_1, p_2)$.
2) all market clear, that is $c_1 + d_1 = 1$, and $d_2 + c_2 = g$.

Now let's check these requirements in turn. First, consumer maximisation: consumer 1 will maximise her utility over the choice of $c_1$ and $c_2$ subject to her budget constraint $p_1c_1 + p_2c_2 = p_1\lambda + p_2\lambda g$. The Lagrangian for this problem is

$$\mathcal{L} = \log c_1 + \beta \log c_2 - \mu(p_1c_1 + p_2c_2 - p_1\lambda - p_2\lambda g)$$

The first order conditions for this problem are

$$\frac{1}{c_1} = \mu p_1 \text{ and } \frac{\beta}{c_2} = \mu p_2$$

Combining these two we get

$$\frac{p_1}{p_2} = \frac{c_2}{\beta c_1}$$

Likewise from consumer 2's maximisation problem we get

$$\frac{p_1}{p_2} = \frac{d_2}{\beta d_1}$$

We can now see that at the intertemporal price $p_1/p_2 = \frac{g}{\beta}$, the allocation derived in (a) does indeed maximise consumers' utility. Check this for consumer 1:

$$\frac{p_1}{p_2} = \frac{g}{\beta} = \frac{c_2}{\beta c_1} \Rightarrow c_2 = g c_1$$

Plug this into the consumer's budget constraint

$$c_1 + \frac{p_2}{p_1} c_2 = \lambda + \frac{p_2}{p_1} \lambda g \Rightarrow c_1 + \frac{\beta}{g} g c_1 = \lambda + \frac{\beta}{g} \lambda g \Rightarrow c_1 = \lambda$$

and then as $c_2 = g c_1$ it follows that $c_2 = \lambda g$. This is the optimal allocation for consumer 1 found in (a). You can check yourself that the same is true for consumer 2.
This verifies the first element of the definition of a competitive equilibrium, consumer maximisation. Now we would have to check that at the intertemporal price $p_1/p_2 = \frac{g}{\beta}$, the allocation also does not violate the economy's resource constraint. But by the way we have solved the problem in (a) we already now that for the allocation $\{(\lambda, \lambda g), (1 - \lambda, (1 - \lambda) g)\}$ this will be the case.

The correspondence between the social planner solution and the competitive market outcome should not come as a surprise to those of you who are trained in economics. The allocation achieved in a competitive market is a Pareto Optimum ( remember the Edgeworth Box ? ). What the social planner solution does is to establish the Pareto Frontier of the economy. If we vary the weights $\lambda$ from zero to one we can trace out the whole Pareto Frontier of the economy, i.e. all allocations at which no individual can be made better off without making the other individual worse off ( if not, the allocation would not maximise the welfare function ). Each choice of $\lambda$ in the social welfare function corresponds to a competitive equilibrium with a specific distribution of total wealth between the two consumers. So the right choice of $\lambda$ is closely linked to the individual endowments of the consumers.

## 2. An infinite-horizon model with durables

**a.** The household's lifetime utility is given by

$$V(\{c_t, d_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t \left(\log c_t + \gamma \log d_t\right)$$

The marginal rate of substitution between $d_t$ and $c_t$ is given by

$$\text{MRS}_t^{c,d} = -\frac{\partial V/\partial d_t}{\partial V/\partial c_t}$$

We have

$$\frac{\partial V}{\partial d_t} = \sum_{j=0}^{\infty} \beta^{t+j} \frac{\partial u(c_{t+j}, d_{t+j})}{\partial d_{t+j}} \frac{\partial d_{t+j}}{\partial d_t} = \sum_{j=0}^{\infty} \beta^{t+j} \frac{\gamma}{d_{t+j}} (1-\delta)^j$$

and

$$\frac{\partial V}{\partial c_t} = \beta^t \frac{1}{c_t}$$

It follows that

$$\text{MRS}_t^{c,d} = -\sum_{j=0}^{\infty} \beta^j (1-\delta)^j \gamma \left(\frac{c_t}{d_{t+j}}\right)$$

Given a consumption stream $\{c_t, d_t\}_{t=0}^{\infty}$ a household is willing to give up at most $\mid \text{MRS}_t^{c,d} \mid$ units of consumption goods in exchange for receiving one more unit of the durable good.

**b.** The constrained optimisation problem for the household is given by the following Lagrangian

$$\mathcal{L}_0 = \sum_{t=0}^{\infty} \{\beta^t (\ln c_t + \gamma \ln d_t) + \lambda_t [y_t + (1+r_{t-1})b_{t-1} - c_t - q_t x_t - b_t]$$
$$+\mu_t [(1-\delta)d_{t-1} + x_t - d_t]\}$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial d_t} = \beta^t \frac{\gamma}{d_t} - \mu_t + (1-\delta)\mu_{t+1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_t} = -\lambda_t q_t + \mu_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + (1+r_t)\lambda_{t+1} = 0$$

We can combine these to get two Euler Equations. For consumption

$$\frac{1}{c_t} = \beta(1+r_t)\frac{1}{c_{t+1}}$$

and for durables

$$q_t \left(\frac{1}{c_t}\right) = \frac{\gamma}{d_t} + \beta(1-\delta) q_{t+1} \left(\frac{1}{c_{t+1}}\right).$$

**c.** Consider the Euler Equation for durables

$$q_t \left(\frac{1}{c_t}\right) = \frac{\gamma}{d_t} + \beta(1-\delta)q_{t+1}\left(\frac{1}{c_{t+1}}\right).$$

It has an interpretation in terms of marginal cost and marginal benefits of buying an additional unit of durables instead of using the resources for consumption. The LHS gives the cost of this readjustment : to buy one unit of the durable we have to give up $q_t$ units of the consumption good. The marginal loss of consumption utility is evaluated at the marginal utility of consumption. The marginal benefit appears on the RHS : one additional unit of durables yields an instantaneous benefit equal to the marginal utility of durable consumption. This is the first term. As the good is durable, there will still be $1-\delta$ units left in the next period. These can be sold off in exchange for $q_{t+1}(1-\delta)$ consumption goods. The additional utility from consuming these units is given by the second term.

Alternatively we can directly consider the first order condition for $d_t$

$$\mu_t = \beta^t \frac{\gamma}{d_t} + (1-\delta)\mu_{t+1} = 0$$

and treat it as a first order difference equation in $\mu_t$. This can be solved forward to yield

$$\mu_t = \beta^t \sum_{j=0}^{\infty}(1-\delta)^j \beta^j \frac{\gamma}{d_{t+j}}$$

which together with the first order condition for $c_t$ and $x_t$ yields

$$q_t\left(\frac{1}{c_t}\right) = \sum_{j=0}^{\infty}(1-\delta)^j\beta^j\left(\frac{\gamma}{d_{t+j}}\right)$$

The marginal cost of buying one more unit of durables has to equal the discounted stream of utility derived from this additional unit if it is never resold.

**d.** A competitive equilibrium is a sequence of prices $\{q_t, r_t\}_{t=1}^{\infty}$ and an allocation $\{\{(c_t(i), d_t(i), b_t(i))\}_{t=1}^{\infty}\}_{i\in[0,1]}$ such that
(i) $\{(c_t(i), d_t(i), b_t(i))\}_{t=1}^{\infty}$ is optimal for household $i$ given the price sequence $\{q_t, r_t\}_{t=1}^{\infty}$ and its budget constraint.
(ii) The allocation clears all markets, meaning :

$$\int_0^1 c_t(i)\,di = \int_0^1 y_t(i)\,di \;\text{ for all}\, t$$

$$\int_0^1 d_t(i)\,di = 1 \;\text{ and }\; \int_0^1 b_t(i)\,di = 0 \;\text{ for all}\, t$$

**e.** Given that all households are identical market clearing immediately gives us $c_t(i) = c_t = y_t$, $d_t(i) = d_t = 1$ and $b_t(i) = b_t = 0$. The consumption Euler Equation then pins down the equilibrium interest rate in terms of the growth rate of consumption endowments

$$1 + r_t = \frac{1}{\beta}\left(\frac{y_{t+1}}{y_t}\right)$$

Now take the Euler Equation for durables which, as has been shown in (c), can be expressed as

$$q_t\left(\frac{1}{c_t}\right) = \sum_{j=0}^{\infty}(1-\delta)^j\beta^j\left(\frac{\gamma}{d_{t+j}}\right)$$

With $d_t = 1$ this yields the equilibrium price of durables

$$q_t = \frac{\gamma\, y_t}{1-\beta(1-\delta)}$$

It follows that the growth rate of durables prices can be expressed as

$$\frac{q_{t+1}}{q_t} = \frac{y_{t+1}}{y_t} = \beta(1+r_t)$$

where the last line follows from the consumption Euler Equation.

# Solutions Assignment 3, G022

November 5, 2011

## 1 . Solving a Neoclassical Growth Model

**a.** The value function for the problem is given by

$$V(k_t) = \max_{c_t, k_{t+1}} \left\{ \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta\, V(k_{t+1}) + \lambda_t (k_t^\alpha + (1-\delta)k_t - c_t - k_{t+1}) \right\}$$

We have two first order conditions. One for $c_t$ :

$$c_t^{-\sigma} - \lambda_t = 0$$

and one for $k_{t+1}$ :

$$\beta V'(k_{t+1}) - \lambda_t = 0$$

which combined yield

$$c_t^{-\sigma} = \beta V'(k_{t+1})$$

The Envelope Condition for this problem is

$$V'(k_t) = \lambda_t \left(\alpha k_t^{\alpha-1} + 1 - \delta\right) = \left(\alpha k_t^{\alpha-1} + 1 - \delta\right) c_t^{-\sigma}$$

Shifting this one period ahead and substituting into our first order condition yields the Euler Equation :

$$c_t^{-\sigma} = \beta \left(\alpha k_t^{\alpha-1} + 1 - \delta\right) c_{t+1}^{-\sigma}$$

This Euler Equation together with the resource constraint

$$c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t$$

characterises the optimal consumption-investment path $\{c_t, k_{t+1}\}_{t=1}^{\infty}$ given $k_0$.

In steady state both $c_t$ and $k_{t+1}$ are constant for all $t$. The Euler Equation in steady state is then

$$1 = \beta \left(\alpha k^{\alpha-1} + 1 - \delta\right)$$

which directly pins down the steady state capital stock $k$ :

$$k = \left[\frac{1}{\alpha}\left(\frac{1-\beta}{\beta} + \delta\right)\right]^{1/(\alpha-1)}$$

and steady state consumption can then be solved for from the resource constraint

$$c = k^\alpha - \delta k$$

**b.** Now with full depreciation and logarithmic utility the Euler Equation is

$$\frac{1}{c_t} = \beta\alpha\, k_{t+1}^{\alpha-1}\left(\frac{1}{c_{t+1}}\right)$$

and the resource constraint is

$$c_t + k_{t+1} = k_t^\alpha$$

Now make the guess $c_t = \gamma k_t^\alpha$ which together with the resource constraint implies $k_{t+1} = (1-\gamma)k_t^\alpha$. Plug this into the Euler Equation to verify the guess

$$\frac{1}{\gamma k_t^\alpha} = \beta\alpha\, k_{t+1}^{\alpha-1}\left(\frac{1}{\gamma k_{t+1}^\alpha}\right) \quad\Rightarrow\quad k_{t+1} = \alpha\beta\, k_t^\alpha$$

Now use the substitute out $k_{t+1}$ with the result we got from the resource constraint. We see that the initial guess can only be true if $\gamma = 1 - \alpha\beta$.
The policy functions are then

$$c_t = (1-\alpha\beta)k_t \;\text{ and }\; k_{t+1} = \alpha\beta k_t$$

which together with $k_0$ yield the optimal path $\{c_t, k_{t+1}\}_{t=0}^\infty$.

**c.** The linearised resource constraint is

$$c + \hat{c}_t + k + \hat{k}_{t+1} = k^\alpha + \alpha k^{\alpha-1}\hat{k}_t \quad\Rightarrow\quad \hat{c}_t + \hat{k}_{t+1} = \alpha k^{\alpha-1}\hat{k}_t$$

where $\hat{x}_t = x_t - x$. Now from the Euler Equation in steady state we have $\alpha k^{\alpha-1} = 1/\beta$. The linearised resource constraint can then be written as

$$\hat{c}_t + \hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t$$

The linearised Euler Equation is

$$c + \hat{c}_{t+1} = \beta\alpha\, k^{\alpha-1}c + (\alpha-1)\beta\alpha\, k^{\alpha-2}c\,\hat{k}_{t+1} + \beta\alpha\, k^{\alpha-1}\,\hat{c}_t$$

Again using the Euler Equation in steady state to simplify this expression we get

$$\hat{c}_{t+1} - \hat{c}_t = -(1-\alpha)\frac{c}{k}\,\hat{k}_{t+1}$$

From the resource constraint in steady state we have

$$\frac{c}{k} + 1 = k^{\alpha-1} = \frac{1}{\alpha\beta}$$

where the last equality comes from the Euler Equation. It follows that $c/k = (1 - \alpha\beta)/\alpha\beta$. We can use this in the linearised Euler Equation :

$$\hat{c}_{t+1} - \hat{c}_t = -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\,\hat{k}_{t+1}$$

Now make the guess $\hat{k}_{t+1} = \gamma\hat{k}_t$. [1] The linearised resource constraint then implies

$$\hat{c}_t = \left(\frac{1}{\beta} - \gamma\right)\hat{k}_t$$

Plug this together with the guess for $\hat{k}_{t+1}$ into the Euler Equation to find

$$\hat{k}_t\left[\left(\frac{1}{\beta} - \gamma\right)\gamma - \left(\frac{1}{\beta} - \gamma\right) + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\gamma\right] = 0$$

[1] Alternatively you could make a guess for both policy functions : $\hat{k}_{t+1} = \eta_1\hat{k}_t$ and $\hat{c}_t = \eta_2\hat{k}_t$. If you plug these guesses into the Euler Equation and the resource constraint you get

$$\begin{aligned}(\eta_2\eta_1 - \eta_2)\hat{k}_t &= -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\eta_1\hat{k}_t \\ (\eta_2 + \eta_1)\hat{k}_t &= \frac{1}{\beta}\hat{k}_t\end{aligned}$$

Both equations can only hold for all $\hat{k}_t$ if

$$\begin{aligned}\eta_2\eta_1 - \eta_2 &= -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\eta_1 \\ \eta_2 + \eta_1 &= \frac{1}{\beta}\end{aligned}$$

From the last equation we have $\eta_2 = 1/\beta - \eta_1$. Plug this into the first equation to get

$$\left(\frac{1}{\beta} - \eta_1\right)\eta_1 - \left(\frac{1}{\beta} - \eta_1\right) = -\frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\eta_1$$

a quadratic equation in $\eta_1$, the same we will find below for $\gamma$.

This has to hold for all possible values of $\hat{k}_t$, but this is only possible if the term inside the brackets is zero. This condition yields a quadratic equation in $\gamma$ :

$$\gamma^2 - \left[\frac{1}{\beta} + 1 + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha\beta}\right]\gamma + \frac{1}{\beta} = 0 \quad \Rightarrow \quad \gamma^2 - \left(\frac{1}{\alpha\beta} + \alpha\right)\gamma + \frac{1}{\beta} = 0$$

The two roots of this equation are given by

$$\gamma_{1,2} = \frac{1}{2}\left(\frac{1}{\alpha\beta} + \alpha\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{\alpha\beta} + \alpha\right)^2 - \frac{1}{\beta}}$$

Now consider the discriminant

$$\frac{1}{4}\left(\frac{1}{\alpha\beta} + \alpha\right)^2 - \frac{1}{\beta} = \frac{1}{4}\left[\left(\frac{1}{\alpha\beta}\right)^2 - \frac{2}{\beta} + \alpha^2\right] = \frac{1}{4}\left(\frac{1}{\alpha\beta} - \alpha\right)^2$$

So the two roots are

$$\gamma_1 = \alpha \;\text{ and }\; \gamma_2 = \frac{1}{\alpha\beta}$$

As $0 < \alpha < 1$ and $0 < \beta < 1$ we have $|\gamma_1| < 1$ and $|\gamma_2| > 1$. To get the unique stable solution of the system we pick the first root $\gamma = \alpha$. The solution to our system is then given by

$$\hat{k}_{t+1} = \alpha\,\hat{k}_t \;\text{ and }\; \hat{c}_t = \left(\frac{1}{\beta} - \alpha\right)\hat{k}_t \;\;,\; \hat{k}_0 = k_0 - k \text{ given}$$

**d.** By definition we have $\hat{x}_t = x_t - x$. To get $k_{t+1}$ in levels as a function of $k_t$

$$k_{t+1} = k + \hat{k}_{t+1} = k + \gamma\hat{k}_t = k + \gamma(k_t - k) = \gamma k_t + (1-\gamma)k$$

and for $c_t$ we have

$$c_t = c + \hat{c}_t = c + \left(\frac{1}{\beta} - \gamma\right)\hat{k}_t = c + \left(\frac{1}{\beta} - \gamma\right)(k_t - k)$$

**e. & f.** *see Matlab code*

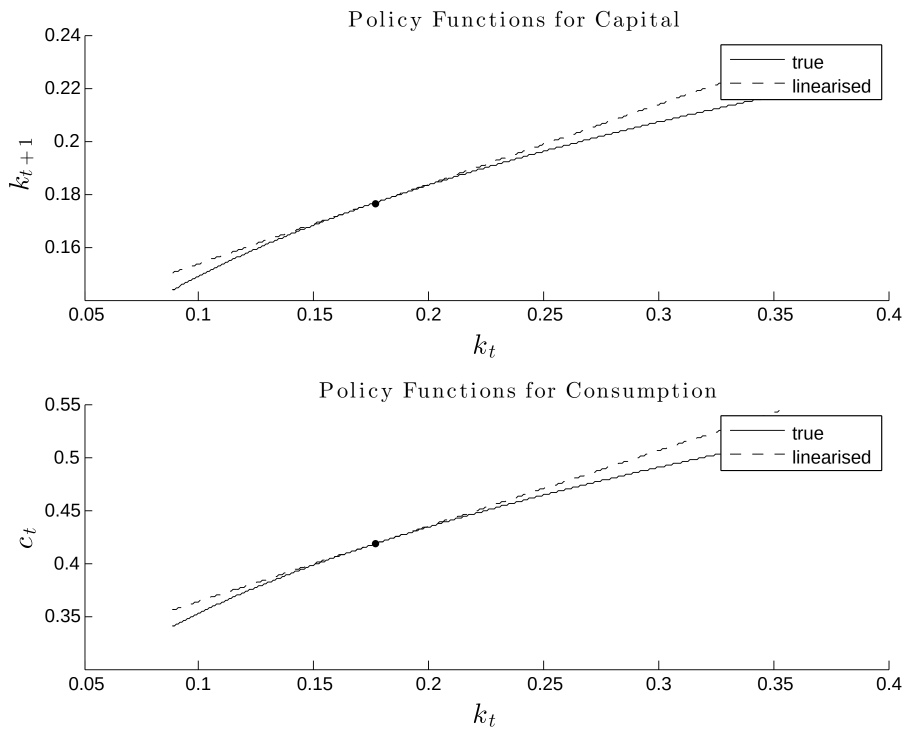



Figure 1: Graph for exercise 1(e)

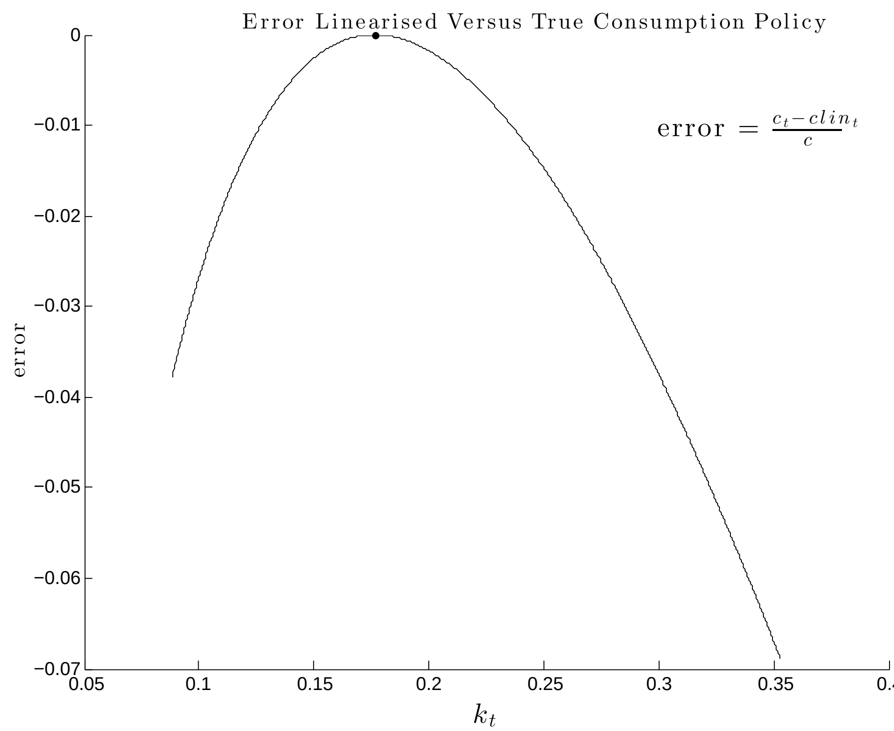



Figure 2: Graph for exercise 1(f)

# Assignment 4

## 1 Investment-Specific Technology Shocks

**a.** We normalize the price of the general output good to be one, $P_{Y,t} = 1$.

The profit maximization problem for the general output producer is then given by

$$\Pi_{i,t} = \max_{\{K_{i,t}, L_{i,t}\}} \{K_{i,t}^{\alpha} L_{i,t}^{1-\alpha} - r_t K_{i,t} - w_t L_{i,t}\}$$

We can see from these expressions that labour's share of total revenue is $1 - \alpha$ while capital's share is $\alpha$ and therefore the general output producer is making zero profits in equilibrium ($\Pi_{i,t} = 0$).

The FOC of this problem give use the factor demands as a function of the rental price and the wage.

$$\alpha \left(\frac{K_{i,t}}{L_{i,t}}\right)^{\alpha-1} = \alpha \frac{Y_{i,t}}{K_{i,t}} = r_t \quad \text{and} \quad (1-\alpha)\left(\frac{K_{i,t}}{L_{i,t}}\right)^{\alpha} = (1-\alpha)\frac{Y_{i,t}}{L_{i,t}} = w_t$$

The profit maximization problem for the consumption good firm is given by

$$\Pi_{j,t} = \max_{\{C_{j,t}\}} \{P_{C,t} C_{j,t} - C_{j,t}\} = \max_{\{C_{j,t}\}} \{(P_{C,t} - 1) C_{j,t}\}$$

The profit that the firm makes for each unit it sells is constant and equal to $P_{C,t} - 1$ and so the supply of the consumption good by firm $j$ will thus depend on the price of the good as following

$$C_{j,t} = \begin{cases} +\infty & if\ P_{C,t} > 1 \\ [0, +\infty) & if\ P_{C,t} = 1 \\ 0 & if\ P_{C,t} < 1 \end{cases}$$

The profit maximization for the investment goods producer $k$ is given by

$$\Pi_{k,t} = \max_{\{I_{k,t}\}} \left\{ P_{I,t} I_{k,t} - \frac{I_{k,t}}{\psi_t} \right\} = \max_{\{I_{k,t}\}} \left\{ \left(1 - \frac{1}{\psi_t}\right) I_{k,t} \right\}$$

As $\psi_t > 0$ the supply of investment goods of firm $k$ as a function of the price for investment goods and $\psi_t$ is then given by

$$I_{k,t} = \begin{cases} +\infty & if\ P_{I,t} > 1/\psi_t \\ [0, +\infty) & if\ P_{I,t} = 1/\psi_t \\ 0 & if\ P_{I,t} < 1/\psi_t \end{cases}$$

We can directly see from the profit maximisation problem of consumption and investment good firms, that in order to have a non-zero and finite supply of consumption and investment goods we need

$$P_{C,t} = 1 \quad and \quad P_{I,t} = \frac{1}{\psi_t}$$

The household problem in recursive form is given by:

$$\begin{aligned} V(K_{l,t}, \psi_t) &= \max_{K_{l,t+1}, C_{l,t}, L_{l,t}} \left\{ \frac{1}{1-\eta} C_{l,t}^{1-\eta} + \log(1 - L_{l,t}) + \beta E_t V(K_{l,t+1}, \psi_t) \right\} \\ s.t. \quad & r_t K_{l,t} + (1-\delta) P_{I,t} K_{l,t} + w_t L_{l,t} = P_{C,t} C_{l,t} + P_{I,t} K_{l,t+1} \end{aligned}$$

The first order conditions for $C_{l,t}, L_{l,t}, K_{l,t+1}$ are respectively

$$\begin{aligned} C_{l,t}^{-\eta} - \lambda_t P_{C,t} &= 0 \\ -\left(\frac{1}{1 - L_{l,t}}\right) + \lambda_t w_t &= 0 \\ -\lambda_t P_{I,t} + \beta E_t V'_{K_{l,t+1}}(K_{l,t+1}, \psi_{t+1}) &= 0 \end{aligned}$$

and the Envelope Condition is

$$V'_{K_{l,t}}(K_{l,t}, \psi_t) = \lambda_t[(1-\delta) P_{I,t} + r_t]$$

The first order condition for $C_{l,t}, L_{l,t}$ give us the labour supply equation

$$\frac{1}{1-L_{l,t}} = \frac{w_t}{P_{C,t}} C_{l,t}^{-\eta}$$

The first order conditions for $C_{l,t}, K_{l,t+1}$ and the Envelope Condition give us the Euler Equation

$$\frac{P_{I,t}}{P_{C,t}} C_{l,t}^{-\eta} = \beta E_t \left\{ \left( \frac{(1-\delta)P_{I,t+1} + r_{t+1}}{P_{C,t+1}} \right) C_{l,t+1}^{-\eta} \right\}$$

**b.** A Competitive Equilibrium is a collection of Policy Functions for each agent $\{C_l(K_l,\Psi), L_l(K_l,\Psi), K_l'(K_l,\Psi), I_l(K_l,\Psi)\}_{l\in(0,1)}$, $\{K_i(r,w), L_i(r,w)\}_{i\in(0,1)}$, $\{C_j(P_C)\}_{j\in(0,1)}$ , $\{I_k(P_I)\}_{k\in(0,1)}$ together with a set of price functions $\{r(K,\Psi), w(K,\Psi), P_C(K,\Psi), P_I(K,\Psi)\}$ such that

1. $\{C_l(K_l,\Psi), L_l(K_l,\Psi), K_l'(K_l,\Psi), I_l(K_l,\Psi)\}_{l\in(0,1)}$ maximizes each agents $l$ expected lifetime utility given their budget constraint and taking prices as given.

2. $\{\{\mathrm{K}_i(r,w), L_i(r,w)\}_{i\in(0,1)}, \{C_j(P_C)\}_{j\in(0,1)}$ , $\{I_k(P_I)\}_{k\in(0,1)}$ maximizes firm profits given prices.

3. All markets clear.

Writing down the market clearing conditions in full we have the following:

$$\int_0^1 C_{j,t}\, dj = \int_0^1 C_{l,t}\, dl$$
$$\int_0^1 I_{k,t}\, dk = \int_0^1 [K_{l,t+1} - (1-\delta)K_{l,t}]\, dl$$
$$\int_0^1 L_{i,t}\, di = \int_0^1 L_{l,t}\, dl$$
$$[\int_0^1 K_{i,t}\, di = \int_0^1 K_{l,t}\, dl$$
$$\int_0^1 K_{i,t}^{\alpha} L_{i,t}^{1-\alpha}\, di = \int_0^1 C_{j,t} dj + \frac{1}{\psi_t}\int_0^1 I_{k,t}\, dk$$

**c.** We have already solved for the price of the investment good as a function of the investment-specific productivity and so we know that following a positive shock in productivity the $P_{I,t}$ will go down. Output, on the other hand is an endogenous variable that we have not solved for yet. However, we know from the Euler equation and the fact that

the positive technological shock is expected to die out over time at rate $1-\rho$ that the agent would prefer to save more to take advantage of the improved investment returns.
This should lead to higher hours worked and more output. Therefore we predict that in light of this model $Corr(P_{I,t}, Y_t)$ is negative.

**d.** To solve for the Policy Function for $\hat{L}_t$ use the FOC for labour supply.

$$(1-L_t)^{-1} = w_t C_t^{-\eta}$$

where we have substituted out for the fact that in equilibrium $P_{C,t} = 1$ We can re-write this using the log of the variables as

$$\left(1-e^{lnL_t}\right)^{-1} = e^{lnw_t - \eta lnC_t}$$

Applying a Taylor approximation from the *deterministic steady state* we get

$$(1-L)^{-1} + (1-L)^{-2} L\hat{L}_t = wC^{-\eta} + wC^{-\eta}\left(\hat{w}_t - \eta\hat{C}_t\right)$$

The constant terms cancel each other since in steady state we have that $(1-L)^{-1} = wC^{-\eta}$ and after some re-arrangement we get that

$$\hat{L}_t = \frac{1-L}{L}\left(\hat{w}_t - \eta\hat{C}_t\right)$$

We now use the law of motion for $\hat{w}_t$ and $\hat{C}_t$ given in the question to solve for the law of motion of $\hat{L}_t$.

$$\hat{L}_t = \frac{1-L}{L}\left((\gamma_5 - \eta\gamma_1)\hat{K}_t + (\gamma_6 - \eta\gamma_2)\hat{\Psi}_t\right)$$

To solve for $\hat{Y}_t$ use the production function definition

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

Note that if we take logs on both sides we get

$$lnY_t = \alpha lnK_t + (1-\alpha) lnL_t$$

while in Steady State this simplifies to

$$lnY = \alpha lnK + (1-\alpha)lnL$$

Taking the difference between these two equation we arrive at the expression for the log deviations

$$\hat{Y}_t = \alpha\hat{K}_t + (1-\alpha)\hat{L}_t$$

It follows that

$$\hat{Y}_t = \left[\alpha + (1-\alpha)\left(\frac{1-L}{L}\right)(\gamma_5 - \eta\gamma_1)\right]\hat{K}_t + \left[(1-\alpha)\left(\frac{1-L}{L}\right)(\gamma_6 - \eta\gamma_2)\right]\hat{\Psi}_t$$

For $\hat{I}_t$ we simply use the definition of investment

$$I_t = K_{t+1} - (1-\delta)K_t$$

$$e^{lnI_t} = e^{lnK_{t+1}} - (1-\delta)e^{lnK_t}$$

$$I + I\hat{I}_t = K - (1-\delta)K + K\hat{K}_{t+1} - (1-\delta)K\hat{K}_t$$

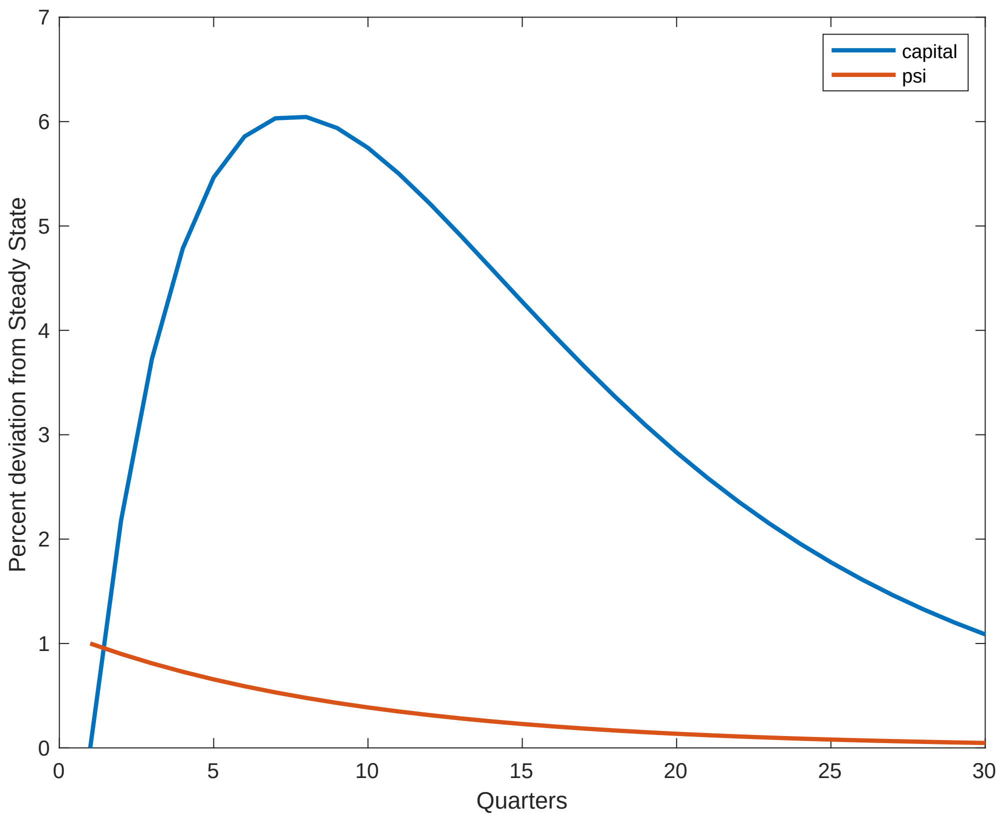
Using the steady state definition to cancel the constant and re-arrange we get

$$\hat{I}_t = \frac{1}{\delta}\hat{K}_{t+1} - \left(\frac{1-\delta}{\delta}\right)\hat{K}_t$$

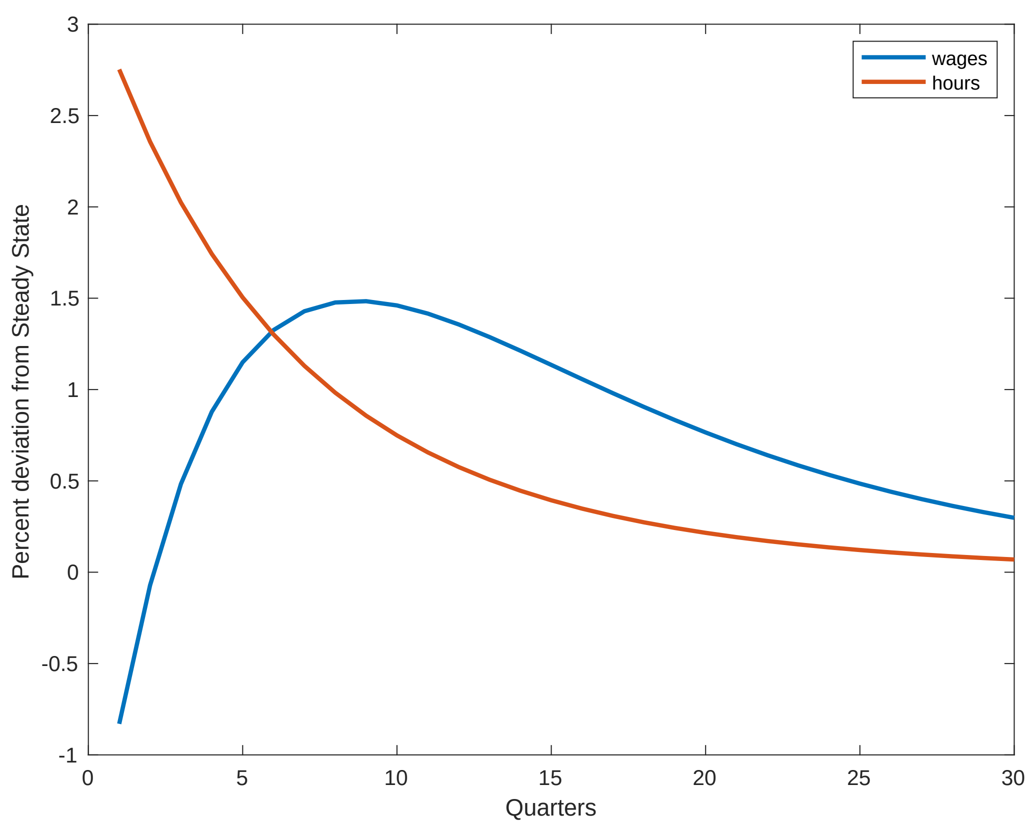
The law of motion for investment is then

$$\hat{I}_t = \left(\frac{\gamma_3}{\delta} - \frac{1-\delta}{\delta}\right)\hat{K}_t + \frac{\gamma_4}{\delta}\hat{\Psi}_t$$

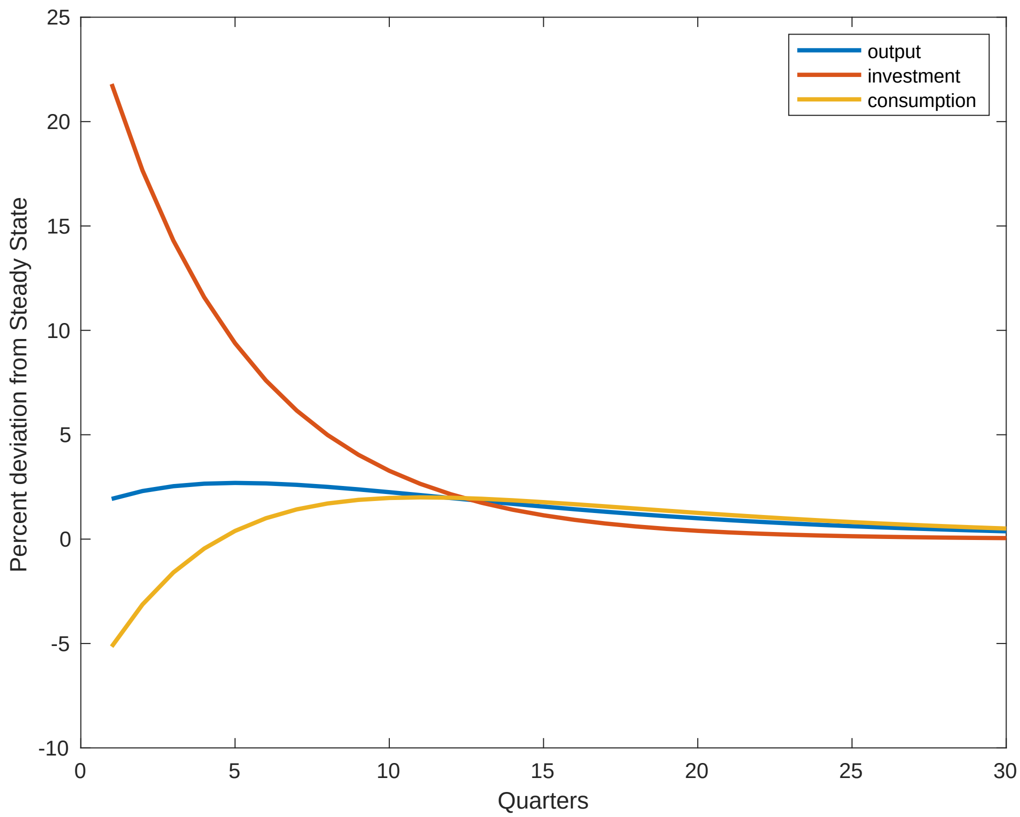
**e.** IRFs for a shock that increases the investment specific technology variable $\Psi_t$ by 1% on impact.



capital
psi
Percent deviation from Steady State
Quarters
0
1
2
3
4
5
6
7
0
5
10
15
20
25
30



wages
hours
Percent deviation from Steady State
Quarters
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30



output
investment
consumption
Percent deviation from Steady State
Quarters
25
20
15
10
5
0
-5
-10
0
5
10
15
20
25
30

**f.** The correlation matrix for the HP-filtered variables with a standard deviation of the shock given by $\sigma^2 = 0.01$ is

$$\begin{pmatrix} Y & C & I & L & w & K \\ 1.0 & -0.30302 & 0.72988 & 0.8067 & 0.33477 & 0.74391 \\ -0.30302 & 1.0 & -0.8726 & -0.80762 & 0.79655 & 0.41144 \\ 0.72988 & -0.8726 & 1.0 & 0.99276 & -0.39979 & 0.086144 \\ 0.8067 & -0.80762 & 0.99276 & 1.0 & -0.28681 & 0.20518 \\ 0.33477 & 0.79655 & -0.39979 & -0.28681 & 1.0 & 0.87876 \\ 0.74391 & 0.41144 & 0.086144 & 0.20518 & 0.87876 & 1.0 \end{pmatrix}$$

The standard deviation of the HP-filtered variables relative to the standard deviation of the HP-filtered output are

$$\begin{pmatrix} \sigma_c/\sigma_Y & \sigma_I/\sigma_Y & \sigma_L/\sigma_Y & \sigma_w/\sigma_Y \\ 1.9599 & 7.5896 & 0.97544 & 0.60321 \end{pmatrix}$$

Compare this with the empirical data moments given in the slides.

# Assignment 5

## 1 Job Security

**a.** Once unemployed the agent cannot get any more job offers but will receive a time-invariant unemployment benefit $u$ so that the value function in this case is simply

$$V^u = \sum_{t=0}^{\infty} \beta^t u(b) = b \sum_{t=0}^{\infty} \beta^t = \frac{b}{1-\beta}$$

**b.** Let's start first with the value function of the agent that is in his second job with some wage $w_2$. We are told that $V^{e2}(w_2)$ denotes the value function *at the beginning of the period*, i.e. before the agent has received this period's wage so therefore we have

$$V^{e2}(w_2) = w_2 + \beta \left((1-p_x)V^{e2}(w_2) + p_x V^u\right)$$

Note that the continuation value given by the second term has no *max operator* because the agent has no further decision to make once in the $e2$ state. He faces an exogenous probability $p_x$ of loosing his job and becoming unemployed while with the complementary probability $1 - p_x$ he gets to keep his job and enjoy $V^{e2}(w_2)$.

This allows us to directly solve for $V^{e2}(w_2)$:

$$[1-\beta(1-p_x)]V^{e2}(w_2) = w_2 + \beta p_x V^u$$

$$V^{e2}(w_2) = \frac{1}{1-\beta(1-p_x)} \left\{ w_2 + \frac{\beta}{1-\beta} p_x b \right\}$$

$$V^{e2}(w_2) = \frac{1}{1-\beta} \left\{ w_2 - \frac{\beta}{1-\beta(1-p_x)} p_x (w_2 - b) \right\}$$

The value function for the first employment state $e1$ will also be made up of the utility that the agent gets from getting his current wage $w_1$ and the continuation value at the end of the period. In this case, the agent draws a new job offer with some associated wage $w_2$ and so has to decide optimally whether to keep his first job and hence get $V^{e1}(w_1)$ or switch to a second job and get $V^{e2}(w_2)$.

$$V^{e1}(w_1) = w_1 + \beta \int_0^{w_2^{\max}} \max\left\{V^{e1}(w_1), V^{e2}(w_2')\right\} dF(w_2')$$

Under the assumption that there exists a threshold $\bar{w}_2$ such that the agent accept the new only if $w_2 \geq \bar{w}_2$ we can further simplify the expression above to

$$V^{e1}(w_1) = w_1 + \beta \left( \int_0^{\bar{w}_2} V^{e1}(w_1) dF(w_2') + \int_{\bar{w}_2}^{w_2^{\max}} V^{e2}(w_2') dF(w_2') \right)$$

**c.** If the agent never accepts a new job offer, then the value function will be simply $V^{e1}(w_1) = \frac{w_1}{1-\beta}$. For this to be optimal it must be that at his initial wage $w_1$

$$\max\left\{V^{e1}(w_1), V^{e2}(w_2)\right\} = V^{e1}(w_1) \quad \text{for all } w_2 \in [0, w_2^{\max}]$$

In particular, we know from the solution of $V^{e2}(w_2)$ that the value function increases with $w_2$ so that the condition will be the most binding at the highest possible wage offer on the support of the distribution $F(\cdot)$. Hence, we can re-write the above condition as

$$V^{e1}(w_1) \geq V^{e2}(w_2^{\max})$$

To solve for the threshold $\tilde{w}_1$ we must therefore solve for the wage $w_1$ at which the agent is exactly indifferent between always keeping the first job of accepting the highest possible offer.

$$V^{e1}(\tilde{w}_1) = V^{e2}(w_2^{\max})$$

$$\frac{\tilde{w}_1}{1-\beta} = \frac{1}{1-\beta}\left\{ w_2^{\max} - \frac{\beta}{1-\beta(1-p_x)} p_x (w_2^{\max} - b) \right\}$$

$$\tilde{w}_1 = w_2^{\max} - \frac{\beta}{1-\beta(1-p_x)} p_x (w_2^{\max} - b)$$

**d.** The agent switches from job one to job two only if he gets offered a strictly higher wage $w_2 > w_1$. That is because switching to the second job implies that the agent now faces a non-zero probability $p_x$ of loosing his job and getting an unemployment benefit $b < w_1$ from that point onward and no opportunity for another job later. Intuitively, the agent is trading a fixed stream of income $w_1$ with one where he will initially earn higher but then eventually fall to a strictly lower level b.
So whatever makes the present value of this second (stochastic) option lower will also lower the threshold $\tilde{w}_1$ above which the agent will always keep his fixed income $w_1$. Specifically, a lower unemployment benefit and a higher probability of getting fired are going to push down the value of switching to job 2 and thus will lower $\tilde{w}_1$.

# Assignment 6

## Long-run unemployment in the DMP Model

**a.** We first start by writing down the value functions for the employed and unemployed agent respectively.

$$V_t^E = w_t + \Pi_t - T_t + \beta(1-\rho)\,\mathbb{E}_t\,V_{t+1}^E + \beta\rho\,\mathbb{E}_t\,V_{t+1}^U \tag{1}$$

$$V_t^U = \bar{b} + \Pi_t - T_t + \beta(1-f_t)\,\mathbb{E}_t\,V_{t+1}^U + \beta f_t\,\mathbb{E}_t\,V_{t+1}^E \tag{2}$$

We can now take the difference of these two equations to get an expression for $V_t^E - V_t^U$ as a function of next period's match surplus.

$$\begin{aligned} V_t^E - V_t^U &= w_t - \bar{b} + \beta(1-\rho-f_t)\,\mathbb{E}_t\,V_{t+1}^E + \beta(\rho - 1 + f_t)\,\mathbb{E}_t\,V_{t+1}^U \\ &= w_t - \bar{b} + \beta(1-\rho-f_t)\left[\mathbb{E}_t\,V_{t+1}^E - \mathbb{E}_t\,V_{t+1}^U\right] \end{aligned}$$

We know from the solution of the Nash Bargaining game that the workers value surplus is just a fixed share of the total joint surplus $S_t$ created by the worker-firm match. This is true for every time t and so we can substitute both the LHS and RHS in the equation above to get:

$$\phi S_t = w_t - \bar{b} + \beta(1-\rho-f_t)\phi\,\mathbb{E}_t\,S_{t+1}$$

Re-shuffling the equation we get the required expression.

$$w_t = \bar{b} + \phi S_t - \beta(1-\rho-f_t)\phi\,\mathbb{E}_t\,S_{t+1} \tag{3}$$

**b.** We start by writing down the definition of the joint surplus.

$$S_t = \left(V_t^E - V_t^U\right) + (F_t - J_t)$$

Using the fact that there are no barries to entry and so the value of an unmatched firm is always exactly zero $J_t = 0$ we can derive $F_t$ as

$$
\begin{aligned}
F_t &= A_t - w_t + \beta(1-\rho)\,\mathbb{E}_t\, F_{t+1} + \beta\rho\,\mathbb{E}_t\, J_{t+1} \\
&= A_t - w_t + \beta(1-\rho)\,\mathbb{E}_t\, F_{t+1}
\end{aligned}
$$

Similarly, we have already derived the employed agent's surplus as:

$$V_t^E - V_t^U = w_t - \bar{b} + \beta(1-\rho-f_t)\,\mathbb{E}_t\, S_{t+1}$$

Summing up the two surpluses we get:

$$
\begin{aligned}
S_t &= A_t - \bar{b} + \beta(1-\rho-f_t)\left[\mathbb{E}_t\, V_{t+1}^E - \mathbb{E}_t\, V_{t+1}^U\right] + \beta(1-\rho)\,\mathbb{E}_t\, F_{t+1} \\
&= A_t - \bar{b} + \beta(1-\rho-f_t)\left[\mathbb{E}_t\, V_{t+1}^E - \mathbb{E}_t\, V_{t+1}^U + \mathbb{E}_t\, F_{t+1}\right] + \beta[1-\rho-(1-\rho-f_t)]\,\mathbb{E}_t\, F_{t+1} \\
&= A_t - \bar{b} + \beta(1-\rho-f_t)\,\mathbb{E}_t\, S_{t+1} + \beta f_t\,\mathbb{E}_t\, F_{t+1}
\end{aligned}
$$

**c.** We are going to use the expression derived in part (a) and part (b) to get the solution for $w_t$. In particular, we can re-write equation (3) as:

$$
\begin{aligned}
w_t &= \bar{b} + \phi\left(S_t - \beta(1-\rho-f_t)\,\mathbb{E}_t\, S_{t+1}\right) \\
&= \bar{b} + \phi\left(A_t - \bar{b} + \beta f_t\,\mathbb{E}_t\, F_{t+1}\right) \\
&= \bar{b} + \phi\left(A_t - \bar{b} + \beta f_t \frac{\vartheta}{\beta g_t}\right) \\
&= (1-\phi)\bar{b} + \phi\left(A_t + \frac{\vartheta f_t}{g_t}\right) \\
&= (1-\phi)\bar{b} + \phi\left(A_t + \vartheta\theta_t\right)
\end{aligned}
$$

Note that in the third line we made use of the zero-profit condition that states that the cost of posting a vacancy ($\vartheta$) must be equal to the expected benefit $\beta g_t\,\mathbb{E}_t\, F_{t+1}$. Re-writing this condition allows us to derive $\mathbb{E}_t\, F_{t+1} = \frac{\vartheta}{\beta g_t}$.
To the market tightness variable $\theta_t$ in the last line, simply use the definition for $\{f_t, g_t\}$.

$$f_t = \frac{m_t}{u_t}\ ,\ g_t = \frac{m_t}{v_t} \quad \longrightarrow \quad \frac{f_t}{g_t} = \frac{v_t}{u_t} \equiv \theta_t$$

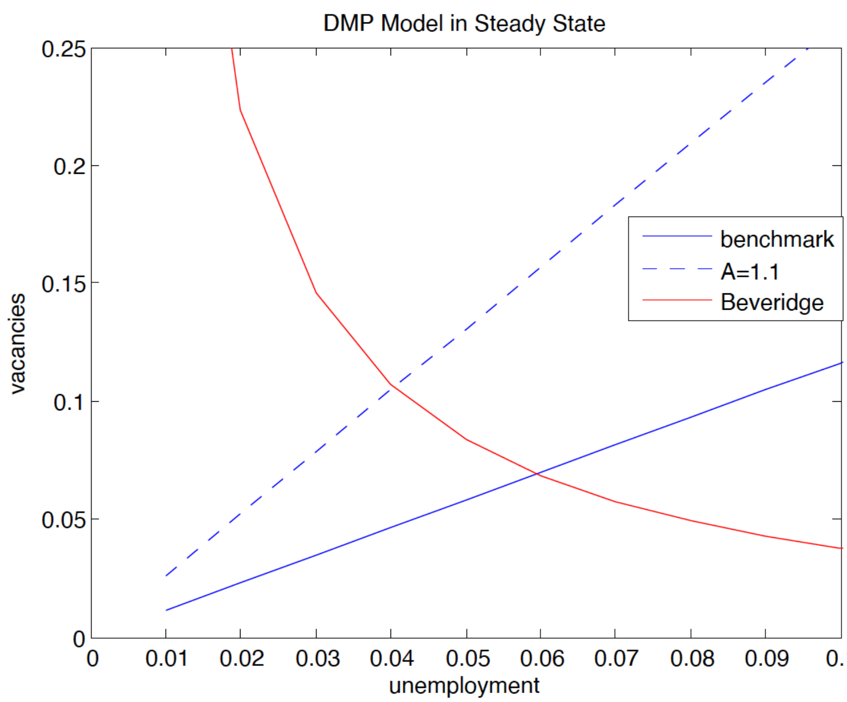
**d.** If the firm receives all the surplus of the match then the bargaining weight of the worker ($\phi$) is zero. This implies that the workers wage will always be equal to his outside option ($\bar{b}$) and hence it will not depend on other aggregate variables like productivity.

**e.** See Excel File.
In the special case where firms hold all bargaining power, the JC equation simplifies to:

$$\frac{\vartheta}{\mu}\theta^{\gamma} = \beta\frac{A - \bar{b}}{1 - \beta(1 - \rho)}$$

This is particularly convenient because we can now calculate the steady state value of market tightness ($\theta$) from this expression alone, simply by plugging in the values of the parameters.
Finally, we can use the Beveridge Curve to get the unemployment level in steady state.

**f.** A permanent increase in productivity makes the job creation curve steeper and it does not affect the BC. Note that this effect is bigger the smaller is $\phi$. Intuitively, the larger is the increase in profits due to the productivity increase, the more vacancies will need to be posted in equilibrium for the Zero-Profit condition to hold and hence the higher the increase in the steepness of the JC curve. In this example where workers have no bargaining power, the wage does not increase at all following the productivity rise. This implies that the new steady state equilibrium entails higher vacancies, less unemployment and hence higher labour market tightness. See Excel File for how to plot.


DMP Model in Steady State
0.25
0.2
0.15
0.1
0.05
0
vacancies
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
unemployment
benchmark
A=1.1
Beveridge

# Solutions Assignment 7

December 9, 2019

## 1. Random Walk Theory

**a.** The period $t$ budget constraint for the household is given by

$$c_t + b_{t+1} = y_t + (1+r)b_t$$

We can therefore write the household's maximisation problem in terms of the following Bellman Equation

$$V(b_t) = \max_{c_t} \left\{ E\frac{c_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t V(y_t + (1+r)b_t - c_t) \right\}$$

The first order condition for this problem with respect to $c_t$ is

$$c_t^{-\gamma} - \beta E_t V'(b_{t+1}) = 0$$

and the Envelope Condition is

$$V'(b_t) = (1+r)\beta E_t V'(b_{t+1}) = (1+r)c_t^{-\gamma}$$

where the last equality follows from the first order condition for $c_t$. Shifting the Envelope Condition ahead by one period and substituting into the first order condition for $c_t$ yields the familiar Euler Equation

$$c_t^{-\gamma} = \beta(1+r)E_t\, c_{t+1}^{-\gamma}$$

**b. & c.** First write the Euler Equation in terms of $\ln c_t$ :

$$e^{-\gamma \ln c_t} = \beta(1+r)E_t e^{-\gamma \ln c_{t+1}}$$

Now make use of the provided hint

$$e^{-\gamma \ln c_t} = \beta(1+r)\, e^{-\gamma E_t \ln c_{t+1}}\, e^{(1/2)\gamma^2\sigma^2}$$

Take logs to get

$$-\gamma \ln c_t = \ln \beta(1+r) - \gamma E_t \ln c_{t+1} + \frac{1}{2}\gamma^2\sigma^2$$

Rearranging and using the fact that for $r$ close to zero we have $\ln(1+r) \approx r$ we get

$$E_t \ln c_{t+1} = \frac{\ln \beta + r}{\gamma} + \frac{1}{2}\gamma\sigma^2 + \ln c_t$$

or

$$\ln c_{t+1} = \frac{\ln \beta + r}{\gamma} + \frac{1}{2}\gamma\sigma^2 + \ln c_t + u_{t+1}$$

where $u_{t+1}$ is a zero-mean forecast error that is uncorrelated with any information available up to period $t$. It follows that consumption is a random walk with drift.

**d.** An increase in the interest rate leads to an increase in expected consumption growth. The price of today's consumption has gone up compared to the price of consumption tomorrow. The household reacts by backloading consumption.
Likewise an increase in the variance of income leads to an increase in expected consumption growth. This is due to precautionary savings. Income risk has gone up, the household reacts by increasing its savings in order to self-insure against the higher volatility. The fact that we have precautionary savings is a result of the marginal utility of consumption being convex :

$$u'''(c) = (\gamma+1)\gamma c^{-\gamma-2} > 0$$

## 2. Habit Formation

**a.** The intertemporal utility function of the household is

$$U = \sum_{t=0}^{\infty} \beta^t \frac{(c_t - \alpha c_{t-1})^{1-\sigma} - 1}{1-\sigma}$$

The marginal rate of substitution between period $t$ and period $t+1$ consumption is

$$\text{MRS}_{t,t+1} = -\frac{\partial U/\partial c_{t+1}}{\partial U/\partial c_t} = \frac{\beta\left[(c_{t+1} - \alpha c_t)^{-\sigma} - \alpha\beta(c_{t+2} - \alpha c_{t+1})^{-\sigma}\right]}{(c_t - \alpha c_{t-1})^{-\sigma} - \alpha\beta(c_{t+1} - \alpha c_t)^{-\sigma}}$$

Define $\hat{c}_t = c_t - \alpha c_{t-1}$ as net-consumption ( the level of present consumption that actually yields positive consumption utility ). Then the MRS can be written as

$$\text{MRS}_{t,t+1} = -\frac{\beta\left[\hat{c}_{t+1}^{-\sigma} - \alpha\beta\,\hat{c}_{t+2}^{-\sigma}\right]}{\hat{c}_t^{-\sigma} - \alpha\beta\,\hat{c}_{t+1}^{-\sigma}}$$

To derive the Euler Equation note that now we have three state variables: our current bond holdings $b_t$, last period's consumption level $c_{t-1}$ and current income realization $y_t$[1]. The Bellman Equation is then

$$V(b_t, c_{t-1}; y_t) = \max_{c_t} \left\{ \frac{(c_t - \alpha c_{t-1})^{1-\sigma} - 1}{1-\sigma} + \beta E_t V(y_t + (1+r_{t-1})b_t - c_t, c_t; y_{t+1}) \right\}$$

having substituted out $b_{t+1}$ with the period $t$ budget constraint $y_t + (1 + r_{t-1})b_t = c_t + b_{t+1}$.

The first order condition with respect to $c_t$ is

$$(c_t - \alpha c_{t-1})^{-\sigma} - \beta E_t V_b(b_{t+1}, c_t) + \beta E_t V_c(b_{t+1}, c_t) = 0$$

The Envelope Condition for $b_t$ is

$$V_b(b_t, c_{t-1}) = (1 + r_{t-1})\beta E_t V_b(b_{t+1}, c_t)$$

and the Envelope Condition for $c_{t-1}$ is

$$V_c(b_t, c_{t-1}) = -\alpha(c_t - \alpha c_{t-1})^{-\sigma}$$

For both envelope conditions we used the chain rule. We can use the Envelope Condition for $c_t$ to substitute out $V_c$ in the first oder condition

$$V_b(b_{t+1}, c_t) = \frac{1}{\beta}\left[(c_t - \alpha c_{t-1})^{-\sigma} - \alpha\beta E_t(c_{t+1} - \alpha c_t)^{-\sigma}\right]$$

Now shift the Envelope Condition for $b_t$ ahead by one period and substitute in the above expression for $V_b$ to find the Euler Equation

$$(c_t - \alpha c_{t-1})^{-\sigma} - \alpha\beta E_t(c_{t+1} - \alpha c_t)^{-\sigma} = \beta E_t\left\{(1 + r_t)\left[(c_{t+1} - \alpha c_t)^{-\sigma} - \alpha\beta(c_{t+2} - \alpha c_{t+1})^{-\sigma}\right]\right\}$$

Or in terms of net-consumption

$$\hat{c}_t^{-\sigma} - \alpha\beta E_t \hat{c}_{t+1}^{-\sigma} = \beta E_t\left\{(1 + r_t)\left[\hat{c}_{t+1}^{-\sigma} - \alpha\beta\hat{c}_{t+2}^{-\sigma}\right]\right\}$$

**b.** A positive output shock makes the households in the model richer, hence they would like to consume more. However, with internal habits, more consumption today adversely affects utility tomorrow. This implies that consumption does not immediately jump up in the wake of a positive output shock. In contrast, consumption adjusts more slowly to a shock that increases output, since households are particularly concerned with consumption smoothing over time.

[1]Note that, differently from $b_t$ and $c_{t-1}$, income is an exogenous state variable. For this reason, we list it in the value function arguments after a semicolon. Just to simplify notation, we will omit it in the rest of the exercise.

**c.** If $0 < \alpha < 1$, utility today is negatively affected by last period's aggregate consumption. Households are concerned with how far away they are from the aggregate level of consumption. In contrast, when $\alpha < 0$, the household enjoys higher utility the higher the level of aggregate consumption.

**d.** Now the Bellman Equation for the household is given by

$$V(b_t; c_{t-1}^{agg}, y_t) = \max_{c_t} \left\{ \frac{(c_t - \alpha c_{t-1}^{agg})^{1-\sigma} - 1}{1-\sigma} + \beta E_t V(y_t + (1 + r_{t-1})b_t - c_t; c_t^{agg}, y_{t+1}) \right\}$$

The household takes aggregate consumption $c_t^{agg}$ as given[2] and thus does not take into account that its choice of $c_t$ will have an effect on $c_t^{agg}$. The first order condition with respect to $c_t$ is then

$$(c_t - \alpha c_{t-1}^{agg})^{-\sigma} - \beta E_t V_b(b_{t+1}) = 0$$

and the Envelope Conditions for bonds is

$$V_b(b_t) = (1 + r_{t-1})\beta E_t V_b(b_{t+1})$$

If we combine these two conditions we get the Euler Equation [3]

$$(c_t - \alpha c_{t-1}^{agg})^{-\sigma} = \beta E_t \left\{ (1 + r_t)(c_{t+1} - \alpha c_t^{agg})^{-\sigma} \right\}$$

Suppose there is a measure one of identical households in the economy. Then in equilibrium we have $c_t = c_t^{agg}$. The Euler Equation in equilibrium is then

$$(c_t - \alpha c_{t-1})^{-\sigma} = \beta E_t \left\{ (1 + r_t)(c_{t+1} - \alpha c_t)^{-\sigma} \right\}$$

or in terms of net-consumption

$$\hat{c}_t^{-\sigma} = \beta E_t \left\{ (1 + r_t)\hat{c}_{t+1}^{-\sigma} \right\}$$

**e.** They will not hold in the case of external habits as here we have externalities. The individual consumption choice of a household influences the choices of other households through the aggregate level of consumption. A social planner would internalise such spill-overs and implement an allocation that differs from the competitive equilibrium allocation.

[2] As explained in footnote 1, we separate endogenous and exogenous state variables with a semicolon. Again in the derivations we only keep endogenous state variables to keep notation simpler

[3] If we subsitute the FOC into the Envelope Condition we have

$$V_b(b_t) = (1 + r_{t-1})(c_t - \alpha c_{t-1}^{agg})^{-\sigma}$$

Now shift this ahead by one period and subsitute it back into the FOC.

**f.** The Euler Equation with external habits is given by

$$c_t^{-\sigma}\left(c_t^{agg}\right)^{\sigma\alpha} = \beta E_t\left\{(1+r_t)c_{t+1}^{-\sigma}\left(c_{t+1}^{agg}\right)^{\sigma\alpha}\right\}$$

Again with a measure 1 of identical households in equilibrium we have $c_t = c_t^{agg}$. Thus in equilibrium the Euler Equation is

$$c_t^{-\sigma(1-\alpha)} = \beta E_t\left\{(1+r_t)c_{t+1}^{-\sigma(1-\alpha)}\right\}$$

Th Euler Equation with standard CRRA preferences is

$$c_t^{-\rho} = \beta E_t\left\{(1+r_t)c_{t+1}^{-\rho}\right\}$$

In both case bond market clearing requires $b_t = 0$ and consumption goods market clearing requires $c_t = y_t = \int_0^1 y_t(i)di$. If we substitute this into the respective Euler Equations we have

$$y_t^{-\sigma(1-\alpha)} = \beta E_t\left\{(1+r_t)y_{t+1}^{-\sigma(1-\alpha)}\right\}$$

and

$$y_t^{-\rho} = \beta E_t\left\{(1+r_t)y_{t+1}^{-\rho}\right\}$$

We can immediately see that if $\rho = \sigma(1-\alpha)$ then the equilibrium sequence of interest rates has to be identical under the two preference specifications. And from bond and consumption goods market clearing we know that the equilibrium allocation has to be the same in the two cases.

## 1. Aggregate Stock of Money in Utility Function

**a.** The Bellman Equation for the household problem is given by

$$V(k_t, b_{t-1}, m_{t-1}) = \max_{c_t, k_{t+1}, b_t, m_t} \{\ln c_t + v(m_t) + w\left(m_t^{agg}\right) + \beta V(k_{t+1}, b_t, m_t)$$
$$+\lambda_t \left[ f(k_t) + (1-\delta)k_t + \frac{1+r_{t-1}}{1+\pi_t} b_{t-1} + \frac{1}{1+\pi_t} m_{t-1} + \tau_t - c_t - k_{t+1} - b_t - m_t \right] \Bigg\}$$

The first order conditions for $c_t, k_{t+1}, b_t$ and $m_t$ are respectively

$$\frac{1}{c_t} = \lambda_t$$
$$\beta V_k(k_{t+1}, b_t, m_t) = \lambda_t$$
$$\beta V_b(k_{t+1}, b_t, m_t) = \lambda_t$$
$$v'(m_t) + \beta V_m(k_{t+1}, b_t, m_t) = \lambda_t$$

The Envelope Conditions for $k_t, b_{t-1}$ and $m_{t-1}$ are

$$V_k(k_t, b_{t-1}, m_{t-1}) = \lambda_t [f'(k_t) + 1 - \delta]$$
$$V_b(k_t, b_{t-1}, m_{t-1}) = \lambda_t \left( \frac{1+r_{t-1}}{1+\pi_t} \right)$$
$$V_m(k_t, b_{t-1}, m_{t-1}) = \lambda_t \left( \frac{1}{1+\pi_t} \right)$$

If we combine these seven conditions we find the Euler Equation for capital

$$\frac{1}{c_t} = \beta [f'(k_{t+1}) + 1 - \delta] \frac{1}{c_{t+1}}$$

the Euler Equation for bonds

$$\frac{1}{c_t} = \beta \left( \frac{1+r_t}{1+\pi_{t+1}} \right) \frac{1}{c_{t+1}}$$

and the Euler Equation for real money

$$\frac{1}{c_t} = v'(m_t) + \beta \left( \frac{1}{1+\pi_{t+1}} \right) \frac{1}{c_{t+1}}$$

These three Euler Equations together with the households budget constraint characterise the household's optimal choice of $c_t, k_{t+1}, b_t$ and $m_t$ in every period $t$ given $k_0, b_1$ and $m_0$ and given the sequence of interest and inflation rates $\{r_{t-1}, \pi_t\}_{t=0}^{\infty}$ as well as the path of transfers $\tau_t$.

Here a competitive equilibrium is an allocation $\{c_t, k_{t+1}, b_t, m_t\}_{t=0}^{\infty}$, a sequence of interest and inflations rates $\{r_{t-1}, \pi_t\}_{t=0}^{\infty}$ such that
(i) $\{c_t, k_{t+1}, b_t, m_t\}_{t=0}^{\infty}$ is optimal for the households given the sequence of interest and inflation rates.
(ii) The allocation clears all markets.

We have four markets : a consumption goods market, a capital market, a bond market and a money market. In each of these markets aggregate demand has to equal aggregate supply, that is in the consumption goods market consumption plus investment has to be equal total output

$$c_t + k_{t+1} - (1-\delta)k_t = f(k_t)$$

Capital market clearing is trivially satisfied as households own the production technology and the capital stock. They both demand and supply capital. Bond market clearing requires that aggregate net demand for bonds equal zero in every period $t$. If we assume that all households are identical this implies $b_t = 0$. Lastly money market clearing requires that the supply of real money equal the demand for real money. The supply of real money is given by last period's nominal money stock divided by the current price level $M_{t-1}/P_t = m_{t-1}/(1+\pi_t)$ as well as the current government transfer $\tau_t$. The demand for real money $m_t$ comes from the household optimisation problem. We have

$$m_t = \frac{m_{t-1}}{1+\pi_t} + \tau_t$$

**c.** In steady state the Euler Equation for capital pins down the steady state capital stock $k$

$$1 = \beta[f'(k) + 1 - \delta]$$

Once we know $k$ we also have steady state consumption from the aggregate resource constraint in steady state

$$c = f(k) - \delta k$$

But this shows that the real side of the economy, capital and consumption, are not affected at all by the monetary variables in the steady state.

For the real stock of money to be constant the ratio of nominal money $M_t$ to price level $P_t$ has to be constant. Nominal money grows at rate $\theta$. It directly follows that the steady state rate of inflation $\pi$ has to be $\theta$ as well for otherwise the real money stock cannot be constant.

The steady state nominal interest rate is now determined by the Euler Equation for bonds in steady state

$$1 = \beta \left( \frac{1+r}{1+\pi} \right) \quad \Rightarrow \quad r = \frac{1+\theta-\beta}{\beta}$$

Lastly, the steady state real money stock is determined by the Euler Equation for real money

$$\frac{1}{c} = v'(m) + \left( \frac{\beta}{1+\theta} \right) \frac{1}{c}$$

If we combine this with the steady state interest rate we obtain

$$c\, v'(m) = \frac{r}{1+r}$$

As steady state consumption is not affected by monetary variables, in order to choose the socially optimal steady state money growth rate we should choose the growth rate that maximises $v(m) + w(m) = 2\, v(m)$. Given the specification of $v$, the real money stock that maximises this expression is given by $m = \hat{m}$. But at this level we have $v'(\hat{m}) = 0$. It follows that at the social optimum we need $r = 0$, the steady state nominal interest rate has to be zero. But this is only possible if the steady state growth rate of nominal money is $\theta = -(1-\beta) < 0$. This is equal to the steady state inflation rate, so it is socially optimal to have deflation in steady state.

**b.** The fundamental theorems of welfare do not apply to this model since the aggregate stock of money in the utility function constitutes an externality faced by the agent.

**d.** The conditions characterising the steady state of the economy are the same as in the previous setup. Thus we still have super-neutrality and the socially optimal level of steady state real money $m$ maximises $v(m) + w(m) = 2w(m) + \mu m$. Since $\mu < 0$ The socially optimal stock of steady state money will be smaller than before, that is $\grave{m} < \hat{m}$.

**e.** The condition for the socially optimal money stock is $w'(\grave{m}) = -\frac{\mu}{2}$. At this point $\grave{m}$, we have $v'(\grave{m}) = \frac{\mu}{2} < 0$. Recall the previously derived relationship between steady state real money and the steady state nominal interest rate

$$cv'(m) = \frac{r}{1+r} \tag{1}$$

It follows that the socially optimal nominal interest rate is negative, $r < 0$.This result is not likely to be observed in reality.

# Assignment 9

## 1. Money Growth in the OLG Model

**a.** $\tau_t$ is the real value ( in terms of units of period $t$ consumption goods ) of seignorage. The monetary authority prints $M_t - M_{t-1}$ units of (nominal) money between periods $t-1$ and $t$, the real value of which is

$$\frac{M_t - M_{t-1}}{p_t} = g\frac{M_{t-1}}{p_t} = \tau_t$$

**b.** (1) The utility maximisation problem of a household born in $t$ is given by [1]

$$\mathcal{L} = \ln c_t^y + \ln c_{t+1}^o + \lambda_t^y \left( e^y - c_t^y - \frac{m_t}{p_t} \right) + \lambda_{t+1}^o \left( e^o + \tau_{t+1} + \frac{m_t}{p_{t+1}} - c_{t+1}^o \right)$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t^y} = \frac{1}{c_t^y} - \lambda_t^y = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}^o} = \frac{1}{c_{t+1}^o} - \lambda_{t+1}^o = 0$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = -\frac{\lambda_t^y}{p_t} + \frac{\lambda_{t+1}^o}{p_{t+1}} = 0 \quad \Rightarrow \quad \frac{\lambda_t^y}{\lambda_{t+1}^o} = \frac{p_t}{p_{t+1}}$$

Combining these first order conditions we get the usual condition for optimal household behaviour, the marginal rate of substitution between period $t$ and $t+1$ consumption has to equal the price ratio

$$\frac{c_{t+1}^o}{c_t^y} = \frac{p_t}{p_{t+1}} = \frac{1}{1+\pi_{t+1}}$$

(2) A competitive equilibrium is an allocation $\{c_t^o, c_t^y, m_t\}_{t=0}^{\infty}$ and a price sequence $\{p_t\}_{t=0}^{\infty}$ such that given the initial stock of money $M_{-1}$ and the sequence of government transfers

[1] In principle we would should also consider that there is a non-negativity constraint on $m_t$. We will ignore this constraint here, implicitly allowing $m_t < 0$ in the solutions. In equilibrium the market clearing conditions for the money market will make sure that prices $p_t$ and $p_{t+1}$ are such that households' money demand is always strictly positive.

$\{\tau_t\}_{t=0}^{\infty}$

(i) The allocation is optimal for each generation given the equilibrium price sequence. This means for the initially old that $c_0^o = e^o + M_{-1}/p_0 + \tau_0$ and for all generations born in $t$, $(c_t^y, c_{t+1}^o, m_t)$ is a solution to the household problem we considered in (1) given $p_t$ and $p_{t+1}$.

(ii) The allocation clears all markets. Here we have two markets in every period $t$, a consumption goods market and a money market. In both markets aggregate demand has to equal aggregate supply. For the consumption goods market this means

$$c_t^o + c_t^y = e^o + e^y \text{ for all } t \geq 0$$

For the money market this means

$$m_t = \frac{M_t}{p_t} = (1+g)\, m_{t-1}$$

**c.** We know that $M_{t+1} = (1+g)M_t$, in real terms we have

$$\frac{M_{t+1}}{p_{t+1}} = (1+g)\frac{p_t}{p_{t+1}}\frac{M_t}{p_t} = \left(\frac{1+g}{1+\pi_{t+1}}\right)\frac{M_t}{p_t}$$

So for real money $M_t/p_t$ to be constant in steady state we need $\pi = g$.

From the household's Euler Equation in steady state we then get

$$\frac{c^o}{c^y} = \frac{1}{1+\pi} = \frac{1}{1+g}$$

The higher is the growth rate of nominal money the lower is the real return on holding money. Households will front-load consumption, the growth rate of consumption decreases.

**d.** As the planner gives each generation $t$ equal weight, he maximises the sum of total utilities. As this sum can be infinite, we will use equal welfare weights for each generation such that the total weighted sum of utilities is bounded. This can be done as follows. Let $W$ be the welfare function of the planner

$$W(\{c_t^o, c_t^y\}_{t=0}^{\infty}) = \lim_{T\to\infty}\frac{1}{T}\left\{\ln c_0^o + \sum_{t=0}^{T}(\ln c_t^y + \ln c_{t+1}^o)\right\}$$

So the planner maximises the average utility of generations over the consumption allocation $\{c_t^o, c_t^o\}_{t=0}^{\infty}$ such that the allocation is feasible, i.e. $c_t^o + c_t^y = e^o + e^y$ for every $t \geq 0$. Now notice that as the choice of $c_t^o$ and $c_t^y$ for a given period $t$ does not have an impact on past or future resource constraints, this maximisation is equivalent to solving the problem

$$\max_{\{c_t^o, c_t^y\}} \ln c_t^o + \ln c_t^y$$

subject to

$$c_t^o + c_t^y = e^o + e^y$$

for each period $t$ individually the first order condition of which is

$$\frac{1}{c_t^o} = \frac{1}{c_t^y}$$

This implies $c_t^y = c_t^o = (e^o + e^y)/2$. It follows that the social planner solution with equal welfare weights is equal consumption for young and old in each period.

**e.** We know that in the social planner solution $c_t^y = c_{t+1}^o$. In every competitive equilibrium the Euler Equation of the household has to hold

$$\frac{c_{t+1}^o}{c_t^y} = \frac{1}{1 + \pi_{t+1}}$$

It follows that the above derived social planner solutions can only be implemented as a c.e. if equilibrium prices are constant, i.e. $\pi_{t+1} = 0$ for all $t$.

*This was not asked, but obviously the above condition is only a necessary condition for implementation, not a sufficient one. We would still need to show that a competitive equilibrium with constant price exists. The following steps show under which conditions this is the case.*

Let $P$ be the ( constant ) equilibrium price. From the young household's budget constraint in $t$ we have

$$c_t^y + \frac{m_t}{P} = e^y$$

It follows directly that $m_t$ has to be constant in any c.e. with constant prices and constant consumption $c_t^y = (e^o + e^y)/2$. This is only possible if $g = 0$.
Next note that for the optimal consumption level we need

$$\frac{m}{P} = \frac{e^y - e^o}{2}$$

As both $m$ and $P$ have to be non-negative, we also need $e^y \geq e^o$. If both this condition and $g = 0$ hold, then there exists a c.e. with constant prices that implements the optimal allocation $c_t^y = c_t^o = (e^o + e^y)/2$. The equilibrium price is

$$P = \frac{2m}{e^y - e^o}$$

$c_t^o = c_t^y$ follows from the Euler Equation and the fact that prices are constant.

## 2. Population Growth in the OLG Model

**a.** The period $t$ resource constraint of the economy is

$$n_t^y c_t^y + n_{t-1}^y c_t^o = n_t^y e^y + n_{t-1}^y e^o \quad \Rightarrow \quad c_t^y + \frac{c_t^o}{1+g} = e^y + \frac{e^o}{1+g}$$

**b.** The household problem for generation $t$ is given by

$$\max_{c_t^y, c_{t+1}^o} \ln c_t^y + \ln c_{t+1}^o$$

subject to [2]

$$p_t c_t^y + p_{t+1} c_{t+1}^o = p_t e^y + p_{t+1} e^o$$

The first order condition for this problem is given by MRS equals price ratio, i.e.

$$\frac{c_{t+1}^o}{c_t^y} = \frac{p_t}{p_{t+1}}$$

which together with the intertemporal budget constraint characterises the optimal consumption bundle for the household as a function of the intertemporal price ratio $p_t/p_{t+1}$.

Here a competitive equilibrium is an allocation $\{c_t^y, c_t^o\}_{t=0}^{\infty}$ and a price sequence $\{p_t\}_{t=0}^{\infty}$ such that
(i) The allocation is optimal for all generation given prices, that is $c_0^o = e^o$ and $\{c_t^y, c_{t+1}^o\}$ solves the household problem for the generation born in $t$ given $p_t/p_{t+1}$.
(ii) The allocation clears all markets. Here we only have one market per period, the consumption goods market. Market clearing requires

$$c_t^y + \frac{c_t^o}{1+g} = e^y + \frac{e^o}{1+g} \quad \text{for all } t \geq 0$$

Now notice that as the old in period 0 consume $c_0^o = e^o$, consumption goods market

[2] This is assuming that there is a futures market for consumption goods in period $t+1$ that is open in period $t$. Otherwise, without any asset that can be used for savings, we would simply have two period budget constraints : $c_t^y = e^y$ and $c_{t+1}^o = e^o$.

clearing in period 0 requires $c_0^y = e^y$. But then from generation 0's budget constraint we have $c_1^o = e^o$ which implies $c_1^y = e^y$ for the period 1 market to clear ... It follows that the only possible equilibrium allocation is autarky, that is $\{c_t^o = e^o, c_t^y = e^y\}_{t=0}^{\infty}$.

**c.** *You were only supposed to show (i).*

(i) Let us start with the 'only if' part. What we have to show is that if $u_1'(e^y, e^o) < (1+g)u_2'(e^y, e^o)$, then the c.e. allocation $\{c_t^o = e^o, c_t^y = e^y\}_{t=0}^{\infty}$ can never be Pareto optimal. To prove this all we have to do it so find a feasible consumption allocation $\{c_t^o, c_t^y\}_{t=0}^{\infty}$ that Pareto dominates the autarkic allocation $\{c_t^o = e^o, c_t^y = e^y\}_{t=0}^{\infty}$. Feasible means that $c_t^o + c_t^y = e^o + e^y$ all $t \geq 0$. Now consider the following allocation : Every young generation consumes $e^y - \varepsilon$, every old generation consumes $e^o + (1+g)\varepsilon$. If you go back to the resource constraint in (a) you see that such a reallocation is certainly feasible. We only have to show that it makes every generation better off. The old in period 0 are certainly better off as they only get the transfer from the young but do not have to give up anything in exchange. Then consider the young generation in period t. For small $\varepsilon$ giving up $\varepsilon$ consumption units in period $t$ reduces their utility by $\varepsilon u_1'(e^y, e^y)$ given that they are currently consuming their endowments in each period. The additional transfer in period $t+1$ increases their utility by $(1+g)u_2'(e^y, e^o)$. Thus they are better off if the gain in consumption utility in period $t+1$ is higher than the loss in period $t$, i.e.

$$\varepsilon u_1'(e^y, e^o) < (1+g)\varepsilon u_2'(e^y, e^o) \ \ \Rightarrow \ \ u_1'(e^y, e^o) < (1+g)u_2'(e^y, e^o)$$

which is what we wanted to show.

(ii) Now let us do the 'if' part. We have to show that if $u_1'(e^y, e^o) \geq (1+g)u_2'(e^y, e^o)$ then there is no feasible consumption allocation $\{c_t^o, c_t^y\}_{t=0}^{\infty}$ that Pareto dominates the autarkic allocation $\{c_t^o = e^o, c_t^y = e^y\}_{t=0}^{\infty}$. We can express any feasible consumption allocation in reference to the autarkic allocation as follows : $\{c_t^o = e^o + \varepsilon_t, c_t^y = e^y - \varepsilon_t/(1+g)\}_{t=0}^{\infty}$ with $\varepsilon_t \leq (1+g)e^y$ for all $t$. First note that for the sequence of transfers $\{\varepsilon_t\}_{t=0}^{\infty}$ to Pareto dominate the autarkic allocation we need $\varepsilon_0 \geq 0$ otherwise we make the old generation in period 0 worse off. For the young generation born in $t$ transfers have to be such that

$$u(e^y - \varepsilon_t/(1+g), e^o + \varepsilon_{t+1}) \geq u(e^y, e^o)$$

Now note that because the utility function is strictly concave, if we take a first order Taylor approximation around $(e^y, e^o)$ we have

$$u(e^y, e^o) - u_1'(e^y, e^o)\varepsilon_t/(1+g) + u_2'(e^y, e^o)\varepsilon_{t+1} > u(e^y - \varepsilon_t/(1+g), e^o + \varepsilon_{t+1})$$

for $\varepsilon_t, \varepsilon_{t+1} > 0$. From the two above two inequalities it follows that the transfer $(-\varepsilon_t/(1+g), \varepsilon_{t+1})$ can only make generation $t$ better off if

$$u_1'(e^y, e^o) < (1+g)\frac{\varepsilon_{t+1}}{\varepsilon_t}u_2'(e^y, e^o)$$

But because $u_1'(e^y, e^o) \geq (1+g)u_2'(e^y, e^o)$ by assumption, this implies that $\varepsilon_{t+1}/\varepsilon_t > 1$ for all $t \geq j$ where $j$ is the first period such that the transfer $\varepsilon_j$ is strictly positive. But this means that there is an $\eta > 0$ such that

$$\frac{\varepsilon_{t+1}}{\varepsilon_t} \geq 1 + \eta \text{ with } \eta > 0.$$

Which in turn implies $\varepsilon_t \geq (1+\eta)^{t-j}\varepsilon_j$ for all $t \geq j$. It follows that if there is a $j$ such that $\varepsilon_j > 0$, then $lim_{t\to\infty}\varepsilon_t = +\infty$. But this is infeasible as by the resource constraint $\varepsilon_t \leq (1+g)e^y$. It follows that there cannot be a feasible consumption allocation that Pareto dominates the autarkic allocation.