SUMMER TERM 2021

24-HOUR ONLINE EXAMINATION

ECON0061 (Public Microeconomics)

All work must be submitted anonymously. Please ensure that you add your **candidate number** and the module code to the template answer sheet provided. Note that the candidate number is a combination of four letters plus a number, e.g. ABCD9. You can find your candidate number in your PORTICO account, under "My Studies" then the "Examinations" container. Please, note that the candidate number is NOT the same as your student number (8 digits), which is printed on your UCL ID card. Submitting with your student number will delay marking and when your results might be available.

Answer THREE questions.

Questions carry one third of the total mark each.

In cases where a student answers more questions than requested by the examination rubric, the policy of the Economics Department is that the student's first set of answers up to the required number will be the ones that count (not the best answers). All remaining answers will be ignored.

Allow enough time to submit your work. Waiting until the deadline for submission risks facing technical problems when submitting your work, due to limited network or systems capacity.

As a guide, this online examination is intended to be equivalent to what would be a two hour invigilated examination and it would be reasonable for the length of your answers to reflect that.

By submitting this assessment, **you pledge your honour** *that you have not violated UCL's Assessment Regulations which are detailed in*

https://www.ucl.ac.uk/academic- manual/chapters/chapter-6-student-casework-framework/section-9-student-academic-misconduct-procedure *which include (but are not limited to) plagiarism, self-plagiarism, unauthorised collaboration between students, sharing your assessment with another student or third party, accessing another student's assessment, falsification, contract cheating, and falsification of extenuating circumstances.*

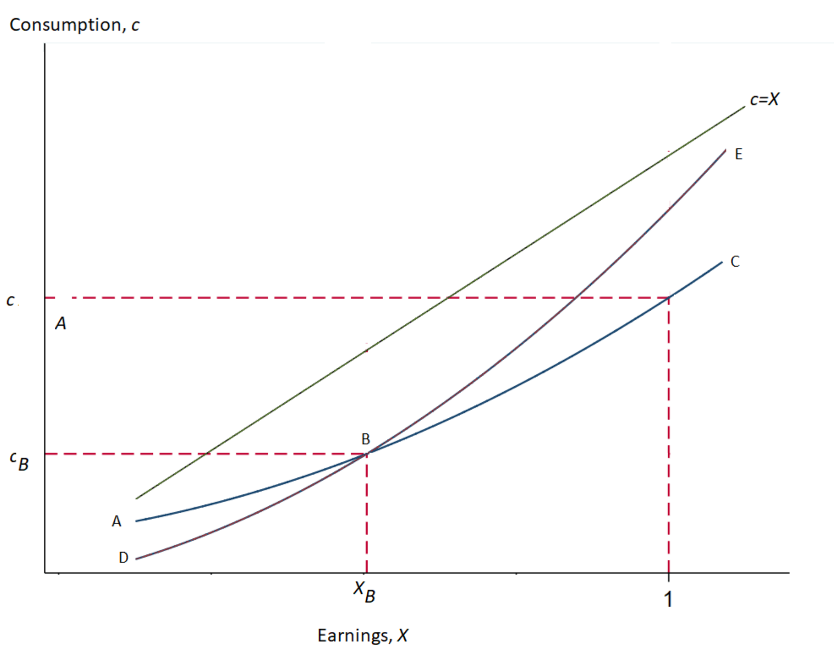
1. A population consists of individuals who supply labour L and consume c. Preferences are identical and described by utility functions

$$v(c, L) \; = \; c - \frac{1}{2}L^2$$

Individuals consist of two types, differing according to their productivity. More able individuals (type A) produce 1 unit of the consumption good for each unit of labour whereas less able types (type B) produce 0.8 units. Earnings before tax of the two types X are therefore $X_A = L_A$ and $X_B = 0.8L_B$ respectively, where L_A and L_B are their labour supplies. There are equal numbers of individuals of each type.

Figure 1 shows an indifference curve for each of the two types in the space of consumption c and earnings X. The indifference curve ABC is for the more able type and the indifference curve DBE is for the less able type.

Figure 1: Indifference curves



(a) Explain why indifference curves of the less able type always cross those of the more able type from below, as in the diagram.

Ans: The slope for the more able type is $-X$ *and for the less able type is* $-X/0.64$

Government knows the number of each type in the economy but cannot identify which individuals are of which type. It can observe earnings and dictate what consumption is given to individuals at each earnings level by imposing tax payments $c - X$.

(b) Explain why tax revenue from each type is maximised given their utility if the more able supply $L_A = 1$ and the less able supply $L_B = 0.8$.

Ans: For the more able, $\upsilon_A + \frac{1}{2}L_A^2 - L_A$ *is maximised at* $L_A = 1$*. For the more able,* $\upsilon_B + \frac{1}{2}L_B^2 - 0.8L_B$ *is maximised at* $L_B = 0.8$*.*

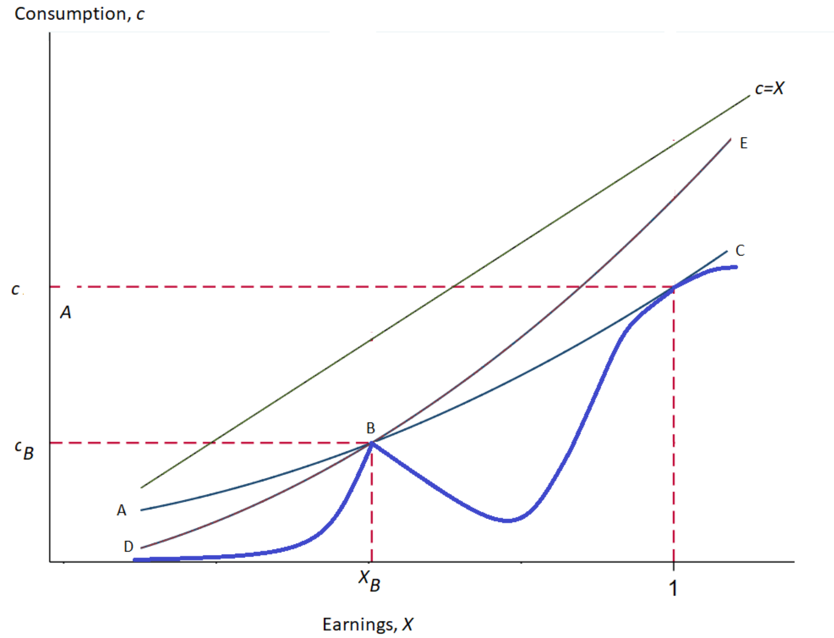
The government wants to design a tax scheme to maximise social welfare

$$W \;=\; \frac{1}{\alpha}\upsilon_A^\alpha \;+\; \frac{1}{\alpha}\upsilon_B^\alpha$$

where υ_A and υ_B are utilities of the two types and $\alpha \leq 1$ is a social welfare parameter, subject to raising a certain required revenue per person.

(c) Explain the nature of the incentive compatibility constraint which must link the consumption-earnings choices of the two types given the information available to the government. Explain why this constraint holds if the more able individuals choose $X_A = 1$ and the less able choose X_B on Figure 1. Draw a figure to represent a tax scheme implementing this outcome.

Ans: The more able must have no incentive to pose as the less able so $x_B - \frac{1}{2}(L_B/0.8)^2 \leq x_A - \frac{1}{2}L_A^2$*. So the choice of the less able in* c, X*-space must be on the indifference curve of the more able. A tax scheme implementing this is shown below.*



(d) Explain why the social optimum under these information constraints involves $L_A = 1$ but $L_B < 0.8$.

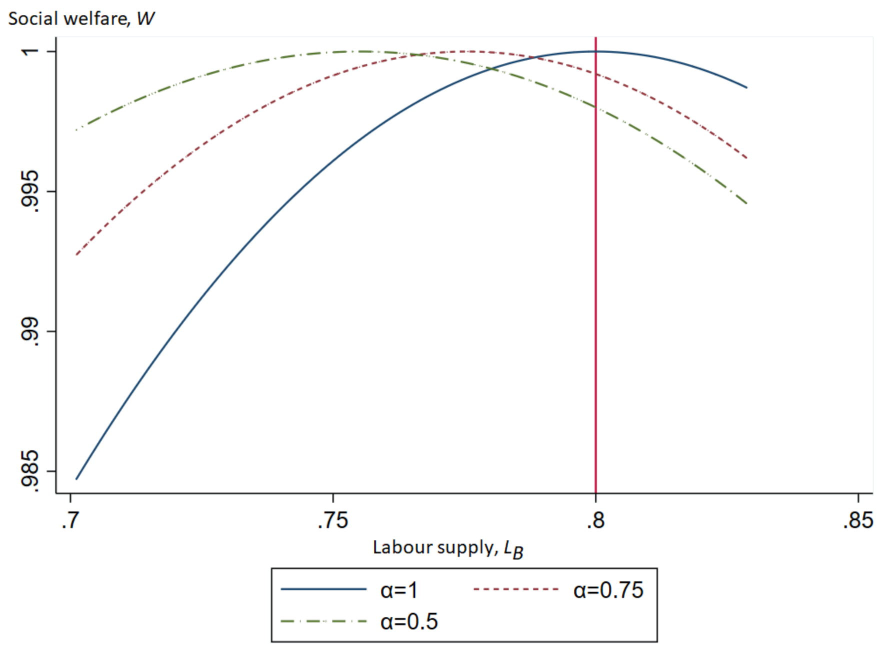
Ans: The labour supply decision of the more able is undistorted at the constrained optimum but that of the less able is distorted because of the incentive compatibility constraint.

Figure 2 shows social welfare plotted against labour supply of the less able given that $L_A = 1$, incentive compatibility is satisfied and revenue per person is 0.25. Social welfare is shown for three values of α and in each case social welfare has been scaled so that the maximum value is 1.

(e) Comment.

Ans: The more averse to inequality is social welfare the more the distortion to labour supply of the less able at the optimum.

Figure 2: Social welfare



2. A population consists of H individuals with incomes $\mathbf{y} = (y_1, y_2, \ldots, y_H)$. Assume that the population is ordered by incomes so that $(y_i - y_j)(i - j) > 0$ for all $i, j = 1, \ldots, H$. Let $\mu(\mathbf{y}) = \frac{1}{H}\sum_i y_i$ denote mean income.

Suppose social welfare is measured by an increasing, homothetic, Schur-concave social welfare function $W(\mathbf{y})$.

(a) Explain what the equally distributed equivalent income function $\xi(\mathbf{y})$ is and what its properties are. Explain why

$$I(\mathbf{y}) \;=\; 1 - \xi(\mathbf{y})/\mu(\mathbf{y})$$

has the right properties to be regarded as a relative inequality index.

Ans: Equally distributed equivalent income is that income which, if equally distributed, would give the same level of actual social welfare. It is increasing, linearly homogeneous and Schur-concave. A relative inequality index needs to be homogeneous of degree zero and Schur-convex. Both ξ and μ are homogeneous of degree one given homotheticity so I is homogeneous of degree zero. Since ξ is itself a measure of social welfare it is Schur-concave and I is therefore Schur-convex.

Suppose that social welfare has the form

$$W(\mathbf{y}) \;=\; \sum_i (H - i + \frac{1}{2}) y_i$$

(b) Show that this has the properties for a social welfare function described above (which is to say it is increasing, homothetic, and Schur-concave).

Ans: W is increasing since $i \leq H$. It is clearly homogeneous of degree one so social welfare is homothetic. It is Schur-concave since $\partial W/\partial y_i = H - i + \frac{1}{2}$ is decreasing in i.

(c) Find the form of the corresponding equally distributed equivalent income function and associated relative inequality index as described above. You may find it helpful to use the rule $\sum_i i = \frac{1}{2}H(H+1)$. What common inequality index is this?

Ans: $\frac{1}{2}H^2\xi = H(H+\frac{1}{2})\mu - \sum_i iy_i$ *so*
$I = 1 - \xi/\mu = -(1+\frac{1}{H}) - (2/H^2)\sum_i iy_i/\mu = (2/H^2\mu)\sum_i i(y_i - \mu)$. *This is the Gini coefficient.*

Suppose individuals are to be regarded as poor if income falls below a poverty line z and let $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_H)$ denote the vector of incomes truncated from above at z,

$$\tilde{y}_i \;=\; \max(y_i, z)\,.$$

Consider the following three proposals for measuring poverty

$$P_0(\mathbf{y}) \;=\; \frac{1}{H}\#(y_i < z)$$

$$P_1(\mathbf{y}) \;=\; 1 - \mu(\tilde{\mathbf{y}})/z$$

$$P_2(\mathbf{y}) \;=\; 1 - \xi(\tilde{\mathbf{y}})/z$$

(d) Discuss the drawbacks and advantages of each of these three measures.

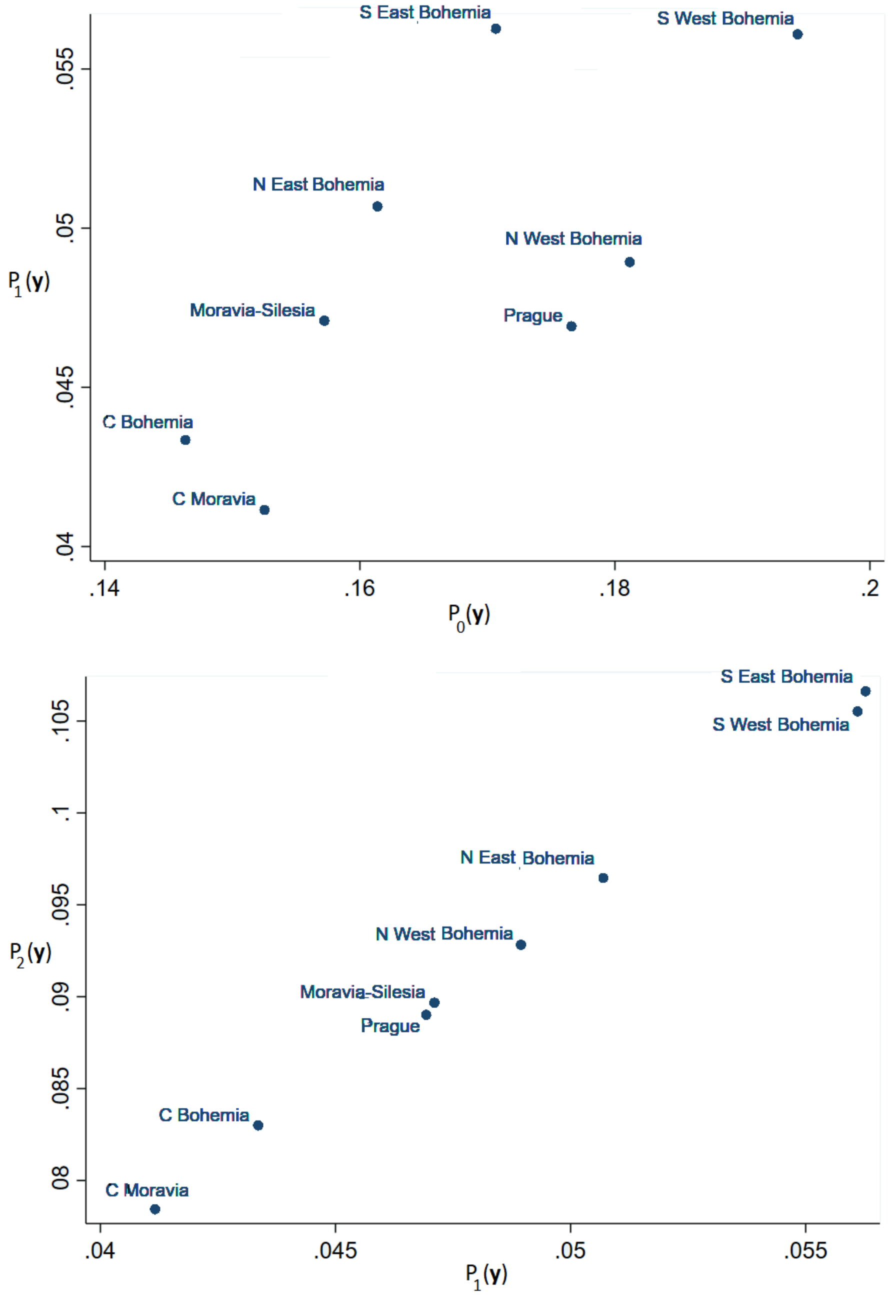
Ans: P_0 *is insensitive to depth of poverty of the poor.* P_1 *is sensitive to depth of poverty but not to inequality among the poor.* P_2 *is sensitive to both.*

Each of these three indices is calculated using data on incomes for eight regions in the Czech republic in the 1990s, using a poverty line set at half national median income and the social welfare function described above. Figure 3 plots $P_1(\mathbf{y})$ against $P_0(\mathbf{y})$ and $P_2(\mathbf{y})$ against $P_1(\mathbf{y})$.

(e) Comment on the differences between the indices in what they suggest about levels of poverty in the different regions.

Ans: P_1 *and* P_2 *order regions similarly. Taking account of inequality among the poor doesn't seem to add much useful information. However* P_1 *and* P_0 *do seem to order differently. Prague and N West Bohemia seem to have comparatively many poor but they are not so deeply so whereas S East Bohemia has fewer poor but they seem to be deeply so.*

Figure 3: Poverty indices for the Czech Republic



3. A choice needs to be made over three possible policies, x, y and z, for a particular population. The population is divided in its opinion as follows. A proportion p are conservatives and the remaining proportion $1-p$ are radicals. Among conservatives, a proportion q are reactionaries and the remaining $1-q$ are moderates. There are therefore three groups in the population whose proportions and preference orderings are:

- Reactionary conservatives pq: $x \succ y \succ z$
- Moderate conservatives $p(1-q)$: $y \succ z \succ x$
- Radicals $1-p$: $z \succ x \succ y$

(a) Explain what single-peaked preferences are.

Ans: Outcomes can be ordered such that preferences decline monotonically away on either side from the most preferred option for all voters

(b) In which, if any, of the following cases can preferences be regarded as single-peaked?

(i) $p = 0$, $0 < q < 1$

(ii) $p = 1$, $0 < q < 1$

(iii) $0 < p < 1$, $q = 0$

(iv) $0 < p < 1$, $q = 1$

In each case, say whether or not there exists a Condorcet winner among the policies and, if there is one, identify it.

Ans: All of them.

If $p = 0$ then everyone has the same preference ordering and z is obviously Condorcet winner.

If $p = 1$ then y is noone's least favourite so ordering with that option in the middle gives single peaked preferences. The Condorcet winner is x if $q > 1/2$ and y if $q < 1/2$.

If $q = 0$ then z is noone's least favourite so ordering with that option in the middle gives single peaked preferences. The Condorcet winner is y if $p > 1/2$ and z if $p < 1/2$.

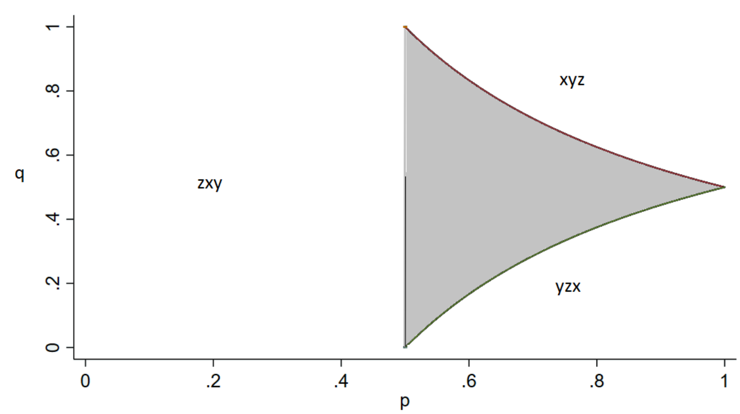
If $p = 1$ then x is noone's least favourite so ordering with that option in the middle gives single peaked preferences. The Condorcet winner is x if $p > 1/2$ and z if $p < 1/2$.

(c) Suppose that $p = 3/4$, $q = 1/2$. Are preferences single-peaked? Does there exist a Condorcet winner? Explain.

Ans: Now there is a majority voting cycle and no Condorcet winner.

(d) Identify the set of values for p and q which are such that no Condorcet winner exists.

There is no Condorcet winner if there is a majority voting cycle which happens if none of the three groups has a majority. This is true if $p > 1/2$, $pq < 1/2$ and $p(1 - q) < 1/2$. So the set of values is $p > 1/2$ and $1 - 1/2p < q < 1/2p$ (as illustrated below).



4. (a) Individuals supply labour L for after-tax wages W and consume $C = WL + m$ where m is unearned income. Preferences are described by indirect utility function

$$\begin{aligned} V(m, W) &= \frac{m}{W} + \frac{1}{\beta}W^{\beta} && \text{if } W^{1+\beta} \geq m \\ &= \frac{1+\beta}{\beta}m^{1+1/\beta} && \text{otherwise} \end{aligned}$$

where $\beta > 0$ is a preference parameter. Show that individuals will choose labour supply

$$L(m, W) = W^{\beta} - \frac{m}{W}$$

if $W^{1+\beta} \geq m$ and zero otherwise.

Ans: Use Roy's identity.

$L(m, w) = V_W(m, W)/V_m(m, W) = (-m/W^2 + W^{\beta-1})/(1/W)$ *if* $W^{1+\beta} \geq m$ *and zero otherwise.*

(b) Pretax wages w are distributed in the population according to distribution function F. Taxation is linear at rate τ and there is no source of income other than uniform government grant G so $c = w(1-\tau)L + G$ where w is pretax wage. Suppose the government has a revenue requirement of R per person (in addition to the cost of funding the uniform grant) and that R is large enough that even the lowest waged individual chooses to work at the value of G satisfying the government's revenue constraint. Show that

$$G = \mu_{\beta+1}(1-\tau)^{\beta+1}\tau - R(1-\tau)$$

where $\mu_a = \int w^a \, \mathrm{d}F$ denotes the ath moment of the distribution of wages.

Ans: Pretax earnings are $wL = (1-\tau)^{\beta}w^{1+\beta} - G/(1-\tau)$. *So the revenue constraint requires* $\tau(1-\tau)^{\beta}\mu_{\beta+1} - \tau G/(1-\tau) = G + R$.

(c) Suppose the government wishes to set taxes to maximise social welfare $\int V(G, w(1-\tau)) \, \mathrm{d}F$ subject to its revenue constraint. Assume that R is such that all individuals work at the social optimum. Show that the optimal tax rate is given by

$$\tau^* = \frac{1}{1+\beta}\left[1 - \frac{\mu_\beta}{\mu_{\beta+1}\mu_{-1}}\right].$$

Ans: Mean utility is $\mu_{\beta+1}\mu_{-1}(1-\tau)^\beta\tau - R\mu_{-1} + \frac{1}{\beta}\mu_\beta(1-\tau)^\beta$. *Maximising with respect to* τ *gives*

$$\mu_{\beta+1}\mu_{-1}(1-\tau)^\beta - \left(\beta\mu_{\beta+1}\mu_{-1}\tau + \mu_\beta\right)(1-\tau)^{\beta-1} = 0$$

and therefore $\tau = \frac{1}{1+\beta}\left[1 - \frac{\mu_\beta}{\mu_{\beta+1}\mu_{-1}}\right]$.

(d) Suppose the distribution of pretax wages is lognormal, $\ln w \sim \mathcal{N}\left(\mu, \sigma^2\right)$. For a lognormal distribution, μ_a is given by $\mu_a = \mathrm{e}^{a\mu + \frac{1}{2}a^2\sigma^2}$. Using this formula, discuss the way in which the optimal tax rate varies with

- the parameters of the wage distribution
- labour supply preferences

  Ans: $\frac{\mu_\beta}{\mu_{\beta+1}\mu_{-1}} = \mathrm{e}^{-2\beta\sigma^2}$ *so*

$$\tau^* = \frac{1 - \mathrm{e}^{-2\beta\sigma^2}}{1+\beta}.$$

  The optimal tax rate is increasing in wage inequality σ^2. *Labour supply preferences enter through* β *which determines the distortionary cost.*

(e) In what way would this analysis need to be adapted if some workers might choose not to work at the optimum?

Ans: Both formulae for G *and utility need adapting.*

5. Suppose that individuals, who all supply one unit of labour, are paid wages w distributed on the interval $[w_0, w_1]$ according to distribution function F. They consume two goods, food q_1 and fuel q_2. Both goods have pretax prices equal to 1, taxes are t_1 and t_2 and posttax prices are therefore $p_1 = 1 + t_1$ and $p_2 = 1 + t_2$. Preferences are captured in indirect utility function

$$V(w, p_1, p_2) \;=\; \frac{w}{p_1} + \beta \ln\left(\frac{p_1}{p_2}\right)$$

where $\beta > 0$ is a preference parameter. Assume $w_0 > p_1$.

(a) Show that revenue raised per person is

$$R \;=\; \frac{\bar{w}t_1}{1+t_1} \;+\; \beta\frac{t_2\,(1+t_1)}{1+t_2} - \beta t_1$$

where $\bar{w} \;=\; \int_{w_0}^{w_1} w \,\mathrm{d}F$ is mean wage.

Ans: Demands are $q_1 = w/p_1 - \beta$ *and* $q_2 = \beta p_1/p_2$*. So* $t_1q_1 + t_2q_2$ *is as given.*

Suppose that the government's objective is to choose tax rates to maximise mean utility $\bar{V} \;=\; \int_{w_0}^{w_1} V\left(w, 1+t_1, 1+t_2\right)\,\mathrm{d}F$ subject to a revenue requirement $R = \bar{R}$.

(b) Show that the optimum taxes are uniform, $t_1 = t_2$. Comment on the reasons for this.

Ans: The government problem is

$$\max_{t_1,\,t_2} \frac{\bar{w}}{1+t_1} + \beta \ln\left(\frac{1+t_1}{1+t_2}\right) \qquad \text{s.t.} \qquad \frac{\bar{w}t_1}{1+t_1} \;+\; \beta\frac{t_2\,(1+t_1)}{1+t_2} - \beta t_1 = \bar{R}$$

First order conditions are

$$\begin{aligned} 0 &= -\frac{\bar{w}}{(1+t_1)^2} + \beta\frac{1}{1+t_1} + \lambda\left[\frac{\bar{w}}{(1+t_1)^2} \;-\; \beta\frac{1}{1+t_2}\right] \\ 0 &= -\beta\frac{1}{1+t_2} + \lambda\left[\frac{\beta\,(1+t_1)}{(1+t_2)^2}\right] \end{aligned}$$

From the second of these $\lambda = 1$ *if* $t_1 = t_2$*. Substituting this into the first, shows that it is also satisfied if* $t_1 = t_2$*. Optimum tax setting is purely about reducing* *distortion since there is no redistributive social objective and uniform taxation is equivalent to a lump sum tax since labour is fixed.*

Suppose instead that the government's objective is to choose tax rates to maximise utility of the least well off $V_0 = V(w_0, 1+t_1, 1+t_2)$ subject to the same revenue requirement $R = \bar{R}$.

(c) Is it still true that optimal taxes are uniform? Comment on the reasons for this.

Ans: First order conditions are now

$$\begin{aligned} 0 &= -\frac{\bar{w_0}}{(1+t_1)^2} + \beta\frac{1}{1+t_1} + \lambda\left[\frac{\bar{w}}{(1+t_1)^2} - \beta\frac{1}{1+t_2}\right] \\ 0 &= -\beta\frac{1}{1+t_2} + \lambda\left[\frac{\beta(1+t_1)}{(1+t_2)^2}\right] \end{aligned}$$

From the second of these it is still true that $\lambda = 1$ if $t_1 = t_2$. But substituting this into the first and letting $t_1 = t_2 = t$ gives $(\bar{w} - w_0)/(1+t)^2 = 0$ which is untrue (unless there is no inequality, $w_0 = w_1$). There is now a redistributive incentive for government and the optimum will involve a higher tax on the luxury q_1.

6. A population of H individuals consume a private good in quantities q^h, $h = 1, \ldots, H$, and a public good Q. Individuals have endowments ω^h, $h = 1, \ldots, H$, of the private good and there is a constant marginal rate of transformation P such that production possibilities in the economy are described by

$$\sum_{h=1}^{H} q^h + PQ \ = \ \Omega$$

where $\Omega = \sum_{h=1}^{H} \omega^h$ denotes economy-wide resources. Individual preferences between the public and private good are expressed by utility functions

$$u^h(q^h, Q) \ = \ - \, \mathrm{e}^{-q^h} \ - \ \beta^h \mathrm{e}^{-Q}$$

where β^h is an individual specific preference parameter.

(a) Explain why Pareto efficiency is satisfied at any allocation satisfying

$$\sum_{h=1}^{H} q^h + P \ln \left(\sum_{h=1}^{H} \beta^h \mathrm{e}^{q^h} \right) \ = \ \Omega + P \ln P.$$

Does this condition tie down a unique level of public provision Q? Why is this?

Ans: MRS between the public and private good is $\beta^h \mathrm{e}^{q^h - Q}$. Pareto efficiency requires that these add up to P so $\sum_h \beta^h \mathrm{e}^{q^h} = P\mathrm{e}^Q = P\mathrm{e}^{\left(\Omega - \sum_h q^h\right)/P}$. Now rearrange. This does not tie down a unique Q because the distribution of private goods across individuals affects the sum of the MRSs.

(b) Show that the allocation maximising utilitarian social welfare $\sum_{h=1}^{H} u^h$ has public provision equal to

$$Q^* \ = \ \frac{\Omega + H \ln \left(H\bar{\beta}/P \right)}{H + P}$$

where $\bar{\beta} = \frac{1}{H} \sum_{h=1}^{H} \beta^h$.

Ans: Social welfare is Schur-concave in private consumption so the optimum involves equal q^h, say $q^h = q$. Substituting into the optimality condition, $(H + P)q + P \ln(H\bar{\beta}) = \Omega + P \ln P$ so $q = \left(\Omega - P \ln(H\bar{\beta}/P)\right)/(H + P)$. Substitute into $Q = (\Omega - Hq)/P$.

(c) Suppose that private endowments are such that there is a Nash equilibrium in voluntary private contributions at which all individuals choose to contribute a positive amount to provision of the public good. Show that provision of the public good will be lower than that at the social optimum. (You might want to use the general inequality: $\ln\left(\frac{1}{H}\sum_{h=1}^{H} X^h\right) > \frac{1}{H}\sum_{h=1}^{H}\ln X^h$).

Ans: If each individual contributes positively then each individual equates their own MRS to P. So $\beta^h \mathrm{e}^{q^h - Q} = P$ and $q^h = Q - \ln\left(\beta^h/P\right)$. Substituting into the national resource constraint, $(H+P)Q - \sum_h \ln\left(\beta^h/P\right) = \Omega$ and therefore $Q = \left(\Omega + \sum_h \ln\left(\beta^h/P\right)\right)/(H+P)$. This is smaller than Q^* if $\sum_h \ln\left(\beta^h/P\right) < H \ln \sum_h \left(\beta^h/P\right) = H\ln H + H \ln \frac{1}{H}\sum_h \left(\beta^h/P\right)$ which is certainly true if $H \geq 1$.

(d) What is meant by Lindahl equilibrium and what are Lindahl prices? What are Lindahl prices at the social optimum in this example? Are Lindahl prices equal across individuals? Comment on the difficulties of inducing preferences to be revealed so that Lindahl prices can be calculated.

Ans: Individuals face individual specific prices for the public good equal to their MRS $\beta^h \mathrm{e}^{q^h - Q}$. These differ since the q^h are equal and the β^h differ. Individuals will not want to reveal preferences which lead to them being charged higher.