

Fourier Series

Henryk Nowakowski

July 12, 2017

1. The Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \quad n = 1, 2, 3, \dots$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used to transform the interval of integration from $[-\pi, \pi]$ to $[-L, L]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, \dots$$

2. Sine series:

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Letting the range go to L ,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

3. Cosine series:

$$f(x) \approx \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Letting the range go to L ,

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Useful formulas for solving integrals:

$$\cos \alpha \cos \beta = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$$

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

Euler formulas:

$$e^{ix} = \cos x + i \sin x$$

hence :

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{-i2}$$

4. Complex Fourier series:

$$f(x) \approx \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

For a function periodic in $[-\frac{L}{2}, \frac{L}{2}]$, these become

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{(2\pi nx)/L} A_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i(2\pi nx)/L}$$

Resources:

<http://mathworld.wolfram.com/FourierSeries.html>

<http://mathworld.wolfram.com/FourierSineSeries.html>

<http://mathworld.wolfram.com/FourierCosineSeries.html>