The Vibrating Guitar String

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Introduction

Figure 1 shows a standard acoustic guitar with strings E B G D A E, and diameters 0.012 0.016 0.020 0.032 0.042 0.054 [in]. The "stretched length" of each string is $60cm$, and they are made of steel (elastic modulus of $200GPa$ and density $8000kg/m^3$). Each string is under $100N$ tension.

Knowing the frequency spectrum of each note on a guitar is important to keep the instrument in tune. We need a way to estimate these frequencies.

(i)

Modelling a guitar string as a single stretched spring, write the Lagrangian and find the turning points for a guitar string plucked half way along its length, assuming the displacement y_0 is small compared to the string length $2L$. The natural length of the string is $2L_0$.

(ii)

Substitute the relevant values and calculate the frequency f_1 of each string's vibrations, assuming the string is plucked 1cm upwards $t = 0$. The following formula may be useful in modelling the string as a spring:

$$F = \frac{EA}{L_0}\Delta L \tag{1}$$

Where E, A, L_0 are the elastic modulus, cross sectional area and initial length of the given object, and ΔL is the longitudinal change in material length.

You just calculated the "fundamental frequency" f_1 of each string, which is the most noticeable tone when the string is plucked. With this frequency and the initial conditions, we can make an approximation to the strings equation of motion:

$$y_1(t) = y_0 cos(2\pi f_1 t) \tag{2}$$

However, there are (infinitely) many other frequencies present that are integer multiples of the fundamental, and a better model of the vibrating string would identify more than just the first.

So we look for solutions of the form:

$$y(t) = \sum_{n=1}^{\infty} a_n cos(2\pi n f_1 t) = a_1 cos(2\pi f_1 t) + a_2 cos(4\pi f_1 t) + a_3 cos(6\pi f_1 t) + ... \quad (3)$$

Note how the approximation in (1) is simply the first term in this cosine expansion, where the coefficient a_1 comes from inspection of the initial conditions.

(iii)

What should the next highest frequency be? In other words, what is the lowest possible $n > 1$ such that $a_n \neq 0$ in (3). Plotting a few cosine curves should help. Can you guess the pattern for all higher frequencies?