The Vibrating Guitar String: Solutions

Holden Jones

(i) - Find Lagrangian

$\Delta y = 2(\sqrt{L^2 + y^2} - L_0) = 2L(\sqrt{1 + (y/L)^2} - L_0/L)$
$\approx 2L(\frac{(y/L)^2}{2} + (1 - L_0/L))$

So we approximate the potential energy:

$$U \approx 2kL^2(\frac{(y/L)^2}{2} + (1 - L_0/L))^2 \tag{1}$$

We ignore the gravitational contribution here as the displacement and mass are both small.

To find kinetic energy, we need to know the speed of the string's center of mass. The COM is at $y_{cm} = y/2$, so:

$$T = 1/2m(\dot{y}/2)^2 = m\dot{y}^2/8 \tag{2}$$

(ii) - Fundamental Frequency

At $t = 0$, $y = y_0$, $\dot{y} = 0$ so:

$$T + U_{t=0} = 2kL^2(\frac{(y_0/L)^2}{2} + (1 - L_0/L))^2 = E \tag{3}$$

This relation holds for all t:

$$T + U = m\dot{y}^2/8 + 2kL^2(\frac{(y/L)^2}{2} + (1 - L_0/L))^2 = E \tag{4}$$

Rearranging:

$$\dot{y} = -\sqrt{\frac{8(E - 2kL^2(\frac{(y/L)^2}{2} + (1 - L_0/L))^2)}{m}} \tag{5}$$

Separating and integrating, we get the time for the string to move between 2 points.

$$t = -\int_{y_1}^{y_2} \sqrt{\frac{m}{8(E - 2kL^2(\frac{(y/L)^2}{2} + (1 - L_0/L))^2)}} dy \quad (6)$$

Because we chose the negative root, this expression is only valid for the first half-period. We can still use it to get the full period:

$$\tau = 2\int_{-y_0}^{y_0} \sqrt{\frac{m}{8(E - 2kL^2(\frac{(y/L)^2}{2} + (1 - L_0/L))^2)}} dy \quad (7)$$

Constants

We need some numerical values before we proceed.
Cross sectional area: $A = \pi D^2/4$
Mass: $m = \rho LA = 8000 \times 0.3A$
Spring constant: $k = EA/L = 200 \times 10^9 A/0.3$
Unstretched length: $L_0 = L - F/k = 0.3 - 100/k$

(The last line uses Hooke's law, and F is the tension in the string).

Note that these expressions only depend on diameter, which varies for each string.

Now the integral (7) is ready to be evaluated, which is a job for a computer. See the attached python file for computation.

We can get the frequency from the period easily: $f = 1/\tau$

Calculated Frequencies	
String	Frequency (Hz)
e	454
b	347
g	285
d	197
a	165
E	144

Compare these to values from Wikipedia:

Measured Frequencies	
String	Frequency (Hz)
e	330
b	247
g	196
d	145
a	110
E	82

Most of these values are about 30 percent lower than predicted.

(iii) - Higher Order Terms

When we evaluate (7), we assume that the upward and downward motions are identical. This eliminates all the frequencies that are an even multiple of the fundamental frequency, as the “dips” wont add in phase.

Figure 1: y=cos(x) + cos(2x). Note that the range is not symmetrical about y=0, so these frequencies can not be present together in our model, so $a_2 = 0$. Similarly, we can deduce that all even multiples of f_0 must not be present in our solution

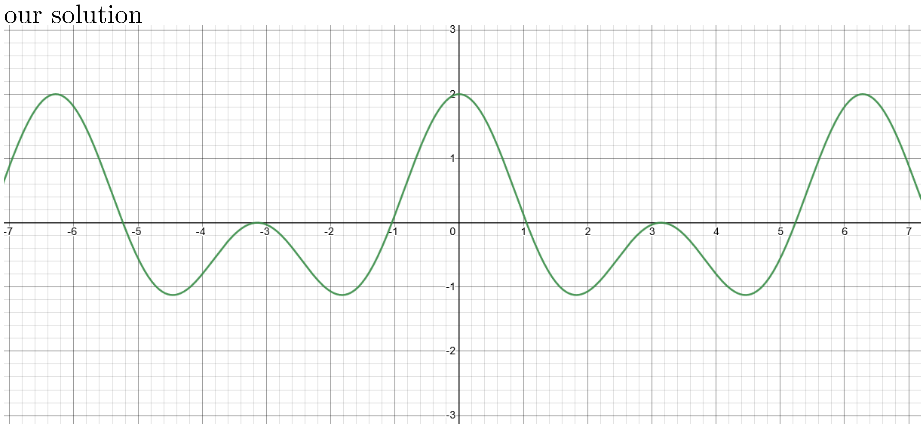
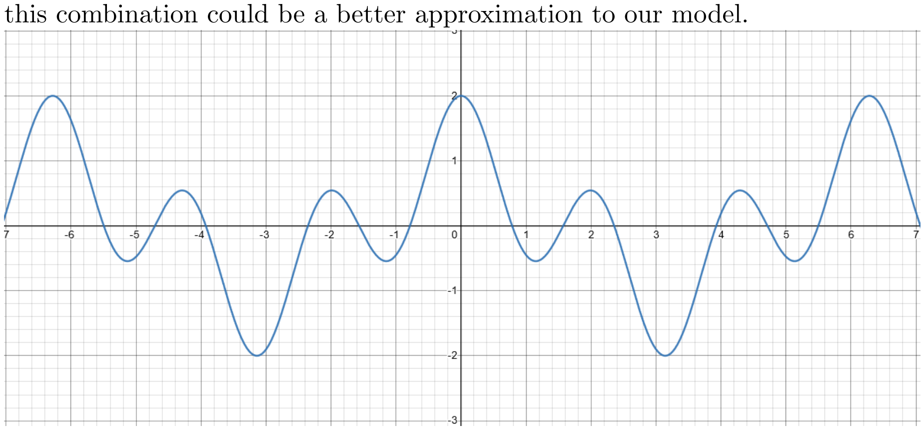


Figure 2: y=cos(x) + cos(3x). These y values are symmetrical about y=0, and this combination could be a better approximation to our model.



So the next lowest frequency is $f_3 = 3f_1$, and the rest of the higher frequencies will be odd multiples of the fundamental.