Pfaffian matrices

Rikard Hjort
hjortr@student.chalmers.se

today

1 Compute the determinant of a 6×6 skew symmetric matrix in which the elements above the diagonal are independent indeterminates over the ring of rational integers

A skew symmetric matrix is defined as a matrix M for which $M^T = -M$. If M is a 6×6 matrix, then

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix}$$

Each element above the diagonal is an integer, and for each element a_{ij} below the diagonal, $a_{ij} = -a_{ji}$. Further, all the elements on the diagonal are 0, since $a_{ii} = -a_{ii} \Leftarrow 2a_{ii} = 0 \Leftrightarrow a_{ii} = 0$ for the ring of integers.

To calculate the determinant, we use the fact that if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and D is singular, then

$$\det(M) = \det(D)\det(A - BD^{-1}C)$$

The following lemma will greatly simplify the calculations.

Lemma 1.1. *If M is a skew symmetric 4×4 matrix, then*

$$\det(M) = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2$$

Proof. This follows from developing the determinant in the usual fashion to find that

$$\begin{aligned}
\det(M) = & \\
& (a_{12}a_{34})^2 + (a_{13}a_{24})^2 + (a_{14}a_{23})^2 \\
& - 2a_{12}a_{13}a_{24}a_{34} + 2a_{12}a_{14}a_{23}a_{34} - 2a_{13}a_{14}a_{23}a_{24} \\
= & (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2
\end{aligned}$$

□

Now set

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} B = \begin{pmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{36} & a_{36} \\ a_{45} & a_{46} \end{pmatrix}$$

$$C = \begin{pmatrix} -a_{15} & a_{25} & -a_{35} & a_{45} \\ -a_{16} & a_{26} & -a_{36} & a_{46} \end{pmatrix} D = \begin{pmatrix} 0 & a_{56} \\ -a_{56} & 0 \end{pmatrix}$$

We now begin calculating $\det(D)\det(A - BD^{-1}C)$. First,

$$\det(D) = a_{56}^2 \tag{1}$$

We proceed with $A - BD^{-1}C$. However, we will first note that A is skew symmetric. It will so happen that $BD^{-1}C$ is also skew symmetric, so $A - BD^{-1}C$ is skew symmetric, too, and 4×4, so we can make use of lemma 1.1 to calculate $\det(A - BD^{-1}C)$.

$$D^{-1} = \begin{pmatrix} 0 & -\frac{1}{a_{56}} \\ \frac{1}{a_{56}} & 0 \end{pmatrix}$$

We can note immediately that this means the only valid value of a_{56} is 1 and -1. For any other value, a_{56} lacks a multiplicative inverse in the ring of integers. We will continue to use $\frac{1}{a_{56}}$ for generality, but one may note the calculations can be somewhat simplified by setting it to a_{56} directly, as 1 and -1 are their own inverses.[1]

$$\begin{aligned}
BD^{-1}C &= \begin{pmatrix} a_{15} & a_{16} \\ a_{25} & a_{26} \\ a_{36} & a_{36} \\ a_{45} & a_{46} \end{pmatrix} D^{-1} \begin{pmatrix} -a_{15} & a_{25} & -a_{35} & a_{45} \\ -a_{16} & a_{26} & -a_{36} & a_{46} \end{pmatrix} \\
&= \frac{1}{a_{56}}(a_{i5}a_{j6} - a_{i6}a_{j5}) \\
A - BD^{-1}C &= (a_{ij} - \frac{1}{a_{56}}(a_{i5}a_{j6} - a_{i6}a_{j5})) \\
&= \frac{1}{a_{56}}(a_{ij}a_{56} - a_{i5}a_{j6} + a_{i6}a_{j5}) \\
&\text{for every } 1 \leq i < j \leq 4
\end{aligned} \tag{2}$$

Note in equation (2) that for $i = j$ the entry is 0, and for $-i, -j$ the entry is the negative of that for i, j, so $BD^{-1}C$ is a skew symmetric matrix.

We are now ready to use lemma 1.1 to compute the determinant. We simply replace i and j in the result of equation (2) with the row and column numbers in the lemma.

$$\begin{aligned}
\det(M) =& \det(D)\det(A - BD^{-1}C) \\
=& a_{56}^2(\frac{1}{a_{56}}(a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25})\frac{1}{a_{56}}(a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}) \\
&- \frac{1}{a_{56}}(a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35})\frac{1}{a_{56}}(a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45}) \\
&+ \frac{1}{a_{56}}(a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45})\frac{1}{a_{56}}(a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35}))^2 \\
=& \frac{1}{a_{56}^2}((a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25})(a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}) \\
&- (a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35})(a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45}) \\
&+ (a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45})(a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35}))^2
\end{aligned} \tag{3}$$

By a final simplification, due to the note earlier about $\frac{1}{a_{56}} = a_{56} = \pm 1$, we get

$$\begin{aligned}
\det(M) &= \\
&= ((a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25})(a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}) \\
&\quad - (a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35})(a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45}) \\
&\quad + (a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45})(a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35}))^2
\end{aligned} \tag{4}$$

[1] How does this affect a general matrix over the ring of integers? Well, it hurts generalizability. But since we will be working with skew symmetric *adjacency* matrices, we can accept elements being restrained to -1 and 1. For a more general solution, we would use the definitition $\det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)})$, and solve from there.

2 Discuss its relation with all the perfect matchings of complete graph K_6

By looking at equation (2), we can see that every term of

$$A - BD^{-1}C = \pm(a_{ij}a_{56} - a_{i5}a_{j6} + a_{i6}a_{j5}) \quad \text{for all } 1 \leq i < j \leq 4$$

We may note that each of the three terms in in every element represents one way to combine i, j with $5, 6$. Thus, in terms of matchings, we may interpret each term as picking two edges in K_6. We may also note that for every term, it is not possible to pick i, j such that any of the edges share a vertix. This means that after picking these two edges, exactly 4 vertices have a picked edge into them, meaning two vertices remain, and by picking the dege between those two we obtain a perfect matching.

Thus, above 4×4 matrix enumerates all perfect matching in an interesting way. Each element not on the diagonal enumerates 3 matchings, with repetition under and over the diagonal. So there are 6 "unique" sets of matchings, giving a total of 18 matchings. However, among these, there will be 3 duplicates (when we pick the edges $\{5, 6\}$ and then $\{i, j\}$, this is identical to picking $\{5, 6\}$ and $\{1, 2, 3, 4\} \setminus \{5, 6, i, j\}$), and thus we have 15 perfect matchings. We can easily check that this is the correct number, by its correspondence with

$$\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$$

Closing notes I suspect that using the formula

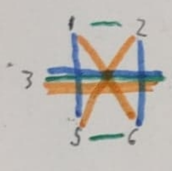
$$\det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)})$$

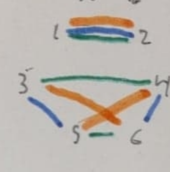
might have yielded more rewarding results, perhaps a polynomial directly containing the desired monomials representing the perfect matchings.

$$
\begin{aligned}
&(a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25})(a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}) \\
-&(a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35})(a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45}) \\
+&(a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45})(a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35})
\end{aligned}
$$

$a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25}$

$a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45}$





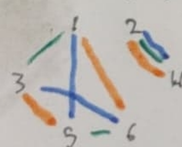
Missing: a_{34}

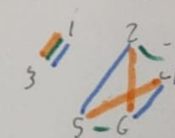
Green in common

$a_{12}(a_{56})^2 a_{34}$

$a_{13}a_{56} - a_{15}a_{36} + a_{16}a_{35}$

$a_{24}a_{56} - a_{25}a_{46} + a_{26}a_{45}$



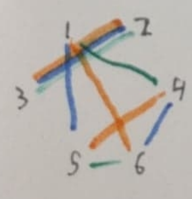


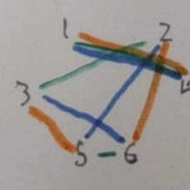
Green in common

$a_{13}(a_{56})^2 a_{24}$

$a_{14}a_{56} - a_{15}a_{46} + a_{16}a_{45}$

$a_{23}a_{56} - a_{25}a_{36} + a_{26}a_{35}$





Green in common

$a_{14}(a_{56})^2 a_{23}$