Auctions applied to course assignment

ENM140 Group 9

Johannes Hermjohannknecht*
Cristian Emanuel Boghiu†
Rikard Hjort‡

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*johhermj@student.chalmers.se
†boghiu@student.chalmers.se
‡hjortr@student.chalmers.se

1 Introduction

In this project, we study simple auction mechanisms and the evolution of different bidding strategies. The problem we are dealing with is framed as follows:

> A university has a number of elective courses available, each with a limited number of seats. Every student can get at most one such course for the semester. The university wants to allocate courses so that total utility is maximized, and decides to use an auction. How good is that auction for their purpose?

1.1 Problem statement

Universities, like Chalmers, need some way to assign students to limited seats in elective courses. Frequently there are fewer seats in a course than there are students who have it as their first choice. This makes the seats in courses scarce resources.[1] The goal of the universities is to allocate these scarce resources as well as possible among the students. Of course, this raises the question of what makes one allocation better than another. It seems reasonable that the university wants its students to get as much value out of their courses as possible, given that students have different utilities for each course. If the university knew the true utilities, they could, of course, find the best such allocation. However, if the students were to be asked directly, they would have every incentive to lie about their true utilities, in order to get their favorite course.

Realistically, the university faces some constraints. It could, conceivably, allow students to use their own money for bidding in the auction, which would simplify allocation by connecting bids to real utility. This is obviously not desirable. The university wants to give every student a chance to participate on the same terms. To do so, instead of bidding money, students will be offered "tokens" which they can use to place bids. These will be connected to real utility in some form. It could be that they can be used to buy coffee, get book discounts, or be saved for future auctions. The university offers the students to put real utility on the line, but it is likely that the utility they stand to lose by bidding is far less than what they stand to gain from taking interesting courses.

The university wants to introduce an auction to tackle this challenge in a fair way. Our goal is to suggest and analyze an auction.

2 Model and Method

We have codified the described situation into an auction and designed a clearing function ("clearing" refers to the process of assigning items to participants at the end of an auction). The auction design is mostly heuristic, based on what we deem realistic.

[1] All the seats in a course are *fungible*, meaning that it does not matter if a student gets seat number 1, 2, or a 100 – all seats in a course are the same from a utility standpoint. However, seats in different courses are non-fungible.

2.1 The Auction

The game consists of N players (or "students"), which bid on M items (or "courses"). All students place their bids simultaneously. Each course has a specific number of seats (sometimes called "capacity"), and the total capacity of all courses is greater than or equal to the number of students. There will be a maximum bid allowed for each course with the same max bid value for all courses.

Each student has a positive utility associated with each course. They place their bids using pre-allocated tokens (everyone has the same amount), which have positive non-zero utility. We denominate utility in units of tokens, so 1 token has 1 utility. To be able to evaluate the auction mechanism, we make the strong assumption that students derive the same utility from these tokens.[2]

After the students have placed their bids, a clearing function takes the bids and assigns each student to exactly one course and one price that they must pay. The students pay with the tokens they were given.

The clearing function uses the following algorithm: it takes the highest bid on any course (if there are several such bids, one is chosen at random), and allocates the student who placed that bid to the course the bid was for. It then decrements the capacity of the course by 1, removes all bids placed by that student, and starts over. If the course capacity is zero, it cannot receive students.

The price the student has to pay is calculated in the following manner. Assume the student got course c. Run the clearing function again, but without that student. Did a new student (who earlier had the "first reserve seat") get into course c? If so, the price the student should pay is the bid of the new student that got in. If no new student got in the price is 0. Therefore the price you pay is the least price you would have had to pay to get the course. The motivation behind this cost scheme is that a student pays (at least) the utility that they deprived some other student of, by taking their seat. This is similar to a second-price auction[3] which is an example of a more general Vickery-Clarke-Groves mechanism, where players are incentivized to bid truthfully.

2.2 Definitions: honest reporting and allocative efficiency

We want to analyze two aspects of the auction, if it incentivizes *honest reporting* and how *allocatively efficient* it is. Both these terms have well-defined meanings for simple auctions and free markets, but as our auction is neither a simple one nor a fully free market, we need to establish some reasonable, adapted definitions of the terms which make them meaningful to our analysis.

In a second-price auction, honest reporting means that each player bids their true value (their utility). As our auction is capped at a max-bid, often far below

[2] We conjecture that this assumption is not strong, and that the natural variance in the utility for a token can be made up for with scaling in utilities for courses. We have, however, not proven this.

[3] A second-price auction is one where the winner, the highest bidder, pays the second-highest bid for the item. A first-price auction is one where the winner, the highest bidder, pays their bid. For an introduction see [1, p.254–257].

the utilities for courses, we will modify the concept and require that the players signal their preference by bidding the max-bid when they cannot bid their true value.

Definition. *Players are reporting honestly if, and only if, on the course for which they have the highest utility, they bid* $\min(\textit{utility for the course}, \textit{max bid})$. *Honest reporting is incentivized if it is a Nash equilibrium for all players to report honestly.*

In economic terms, allocative efficiency roughly means that there is an equilibrium between supply and demand. Again, this does not fully apply to our auction, since it is not a market – seats on courses cannot be produced, at least in our scenario. However, the university has a single goal in mind: to make sure that the utility the students get from the courses is as high as possible. Note that this does not necessarily mean that the one who wants a course most should get it, but rather that the sum of utilities gained by getting courses is maximized.

Definition. *The allocative efficiency of an auction is the sum of the utility all players receive from items they won in an auction, neglecting the price that was paid for these items.*

2.3 Method

We have two theoretical approaches for the analysis. The first one is an analysis of the properties of honest reporting and allocative efficiency. The second one attempts to apply calculus of variations to a probabilistic description of an even more simplified auction. When that fails, we resort to simulating the auction using a co-evolutionary algorithm.

3 Results

Here we summarise the results from applying the methods described in Sec. 2.3, first formally analyzing simplified auctions, then simulating our main auction.

3.1 Honest reporting analysis

In this subsection, we try to understand whether our auction incentivizes honest reporting. We start with a known auction mechanism which has honest reporting (second-price auction) as a weakly dominant strategy and modify it step by step towards the proposed auction.

3.1.1 Second-price auction with one winner

Within second-price single-item auctions, bidding your true value is well-known to be a weakly dominant strategy. See Appendix A for a proof.

3.1.2 Second-price auction with multiple winners

The university's auction does not auction of a single item, but several. Even when a single course is auctioned, it has multiple seats. So next, we investigate a single auction where there are several *identical* items auctioned off, and each player can get only a single item. This would be like the university offering a single course, with fewer seats than there were students. We use the "highest loser" pay scheme of our auction, where all bidders pay the highest bid which did not get an item. For simplicity, we assume no students have the exact same utility for the course.

As with the second-price auction with one winner, it can be shown that bidding the true value is a weakly dominant strategy. The proof is as follows:

Assume all players bid their utility, or "true value", ν. No player with utility ν who did not get an item could gain from raising their bid, b. If they did, they would either still lose, getting utility 0, or they would win but have to pay the bid of the previous "lowest winner" b_{lw}. But since $b < b_{lw}$ and $b = \nu$, $\nu - b_{lw} < 0$, and they get negative utility in total. The player could also not gain from lowering their bid, as the resulting utility would stay at 0.

Next, we consider the winners, who are getting utility $\nu - b_{hl}$ from the auction, ν being their utility and b_{hl} being the bid of the "highest loser". Because of honest reporting, we know that $\nu - b_{hl} > 0$. They cannot gain from increasing their bid, b, as their pay and utility from getting the course stays the same. Neither can they gain from lowering their bid, as long as the bid is higher than b_{hl}. However, if they lower their bid below b_{lw}, their resulting utility drops to 0, which is less than what they gained from winning.

We conclude that bidding your true value is a weakly dominant strategy in this auction, therefore the auction incentivizes honest reporting.

3.1.3 Second-price auction with multiple winners and limited bids

Next, we introduce to the previous auction a max-bid. Is honest reporting still incentivized? We find that it is, by the following proof.

Assume everyone bids min(true value, max bid). If the final price is less than the max-bid, the proof is analogous to the proof in the previous section. If the final price is the max bid, then those who bid below it can again not gain from raising their bids, and those that bid the max-bid must all have a utility higher than the max-bid, and thus gain from bidding the max-bid, as this gives them non-zero probability[4] of getting a positive resulting utility, whereas if they lowered their bid, they would surely get utility 0.

3.1.4 Dependent auctions: Each player can only win one auction

Compared to the auction in the previous section, the university's auction has two more feature: it is auctioning off *both* identical and non-identical items, and every player gets an item. However, the players can still win only one item.

[4] Recall that the clearing function picks randomly for identical bids.

As soon as a player wins an item, all their other bids are discarded. Furthermore, higher bids take precedence, so if a player has the highest bid for a course, they must win it.

Given this auction mechanism, it can be shown that honest reporting is no longer necessarily incentivized. One can consider it in this way: there is now an opportunity cost to winning a course, namely not being able to win any other.

A simple example shows why honest reporting is not necessarily a dominant strategy. Assume we have two players, α and β, and two courses with one seat each. The players have the following utilities:

	Course 1	Course 2
Player α	100	50
Player β	95	50

Assume the max-bid is 90, and both players bid min(true value, max-bid). First, player α is assigned course 1, and pays 90, the bid of the "highest loser", for a resulting utility of $100 - 90 = 10$. Then player β gets course 2, and no bids remain due to the removal of α's bids, so the resulting utility is $50 - 0 = 50$.

Now, player α would gain from dropping their bid to below that of player β.[5] Denote the bid of player i for course j as $b_{i,j}$. Then if player α drops his bid $b_{\alpha,1}$ so that $b_{\alpha,1} < b_{\beta,1} = 90$, what happens? Player β would now win course 1, as their bid of 90 is now the highest bid, and α would get course 2 and pay 0, for a resulting utility of $50 - 0 = 50$. Player α has therefore gained utility by not reporting honestly, so reporting honestly is not a Nash equilibrium.

3.2 Allocation efficiency analysis

In this section, we investigate how allocatively efficiency the university's auction is, i.e., what the sum of utilities for items awarded is, disregarding prices paid. As in the previous section, we analyze a simple auction first, then and modify it step by step towards to approach the university's auction.

3.2.1 Second-price auction with one winner

Recall that honest reporting is a dominant strategy for the second-price auction. Given that each player reports honestly, the player who has the highest true value for a course will win the auction. This clearly means that the allocation efficiency is maximized.

3.2.2 Second-price auction with multiple winners

Again, as per Sec. 3.1.3, each player is incentivized to report their true value. The n players with the highest utilities should therefore win the auction, which, again, clearly maximizes allocative efficiency.

[5] In case player β is winning the first auction at random, since both bid 90, Player β's resulting utility is $95 - 90 = 5$ while player α's resulting utility is $50 - 0 = 50$. In that case player β could gain from dropping the bid below that of player α.

3.2.3 Second-price auction with multiple winner and limited bids

If the amount of a bid is restricted by a max-bid, optimal allocative efficiency cannot be guaranteed. As seen, bidding min (true value, max-bid) is a weakly dominant strategy. However, if there are n items and m of the players ($n < m$) have a utility larger than the max-bid, and they all bid the max-bid, the clearing mechanism would have to assign a random selection of n out of these m players items. This still outperforms purely random assignment, as players whose true value are below the max-bid are excluded. A lower max-bid means that fewer players are excluded, and allocative efficiency drops. However, if the limit is higher than the highest utility, min (true value, max-bid) = true value in all cases, and the auction is maximally efficient.

3.2.4 Dependent auctions: Each player can only win one auction

As this type of auction generalizes the one from the previous section, there is obviously no guarantee that allocative efficiency is maximized. Furthermore, as we have seen, players are no longer necessarily incentivized to report honestly. However, we find that dishonest reporting can actually increase allocative efficiency.

Consider 2 players α and β and 2 courses with one seat each, and the following utilities:

	Course 1	Course 2
Player α	100	70
Player β	80	20

Honest reporting would mean that α bids of $b_{\alpha,1} = 100$ for course 1 whereas player β places bid of $b_{\beta,1} = 80$.The clearing function would assign α to course 1 (at a price of 80) and β would receive course 2 (at a price of 0). The allocative efficiency $= 100 + 20 = 120$. But the players would both be better off if α shades their bid. If they bid $b_{\alpha,1} = 50$, $b_{\alpha,2} = 70$, $b_{\beta,1} = 80$ and $b_{\beta,2} = 20$. Then α gets utility 70, β gets 30, and the allocative efficiency $= 70 + 80 = 150$. This is the most efficient allocation for this simple example. We see later, as we perform a simulation in Sec. 3.4, that this type of bid shading actually presents itself.

3.3 Probabilistic framework for finding optimal strategies using calculus of variations

So far, we have carried over strategies from other auctions and analyzed whether they maintain their properties under our auction rules, but we have no systematical way of finding optimal strategies. In this section we have the following goals:

- Give a general systematical approach to finding optimal strategies using calculus of variations
- Show that the problem is difficult even with a simple auction

As a result of our work, we will provide a framework in which to formulate the problem and give formal equations the optimal strategy must satisfy, but due to the mathematical complexity we will not be able to solve them. Nevertheless, this approach provides a path for future research and also showcases the difficulty found in purely analytical approaches, motivating the use of simulations.

We will not go into specifics, but try to give a general overview of our approach and establish the basic mathematical language. The curious reader can look at Appendix B to see the equations in more detail.

We will start with the simplest auction possible to formally analyze and go increasing the degree of complexity. Beginning with the 1-player 1-course case, it is clearly trivial as he/she automatically wins any auction. Therefore we will move to a 2-player 1-course auction. This is essentially just a regular auction, therefore, all the results from known literature [3] hold. Also, having 1 single course does not fit our problem context, where we want the capacity (total number of seats) to equal the number of students, therefore we will go to a 2-player 2-course auction. This is the simplest case which is similar to our problem. Now we have to decide on the auction properties. We will use different properties than those explained in Section 2 because we will prioritize simplicity of analysis over having a realistic auction.

Auction properties. The players, labelled α and β, bid for courses 1 and 2. Let $\nu_{i,j}$ and $b_{i,j}$ denote for course j player i's utility/"hidden value" and bid respectively, as in previous sections. We have a maximum bid, meaning $0 < b_{\alpha,1}, b_{\alpha,2}, b_{\beta,1}, b_{\beta,2} < 1$. For the clearing, we will start arbitrarily with course 1.[6] We will use a modified *second-price* rule for analytical simplicity. The winner of course 1 (say, player α) is the one with the highest bid and pays the second price for the course, with payoff $u_{\alpha,\text{winner}=\alpha} = \nu_{\alpha,1} - b_{\beta,1}$. If α loses he wins course 2 instead (no students without courses) while paying the price (which might not be the second-price for course 2) of the other student, $u_{\alpha,\text{winner}=\beta} = \nu_{\alpha,2} - b_{\beta,2}$.[7]

In our approach, we try to model all the previous variables as random variables with their own probability distributions. The players' bid distributions or strategies[8] will be $f_\alpha(x, y), f_\beta(x, y)$, which give the probabilities of a player bidding x and y for course 1 and 2 (the support[9] of $f_\alpha(x, y), f_\beta(x, y)$ will be denoted by Ω). The expected payoff $\langle u_\alpha \rangle$ for player α is given as a function of

[6] Ordering the bids from highest to lowest as in our main proposal is quite involved to do analytically, so we choose a simpler scheme.

[7] This is the modification to the second-price rule we mentioned, that for course 2 the payment is just the other player's and not second highest, and it is done for analytical simplicity. See the comments in Appendix B Table 3 for how the model becomes more complicated by having a true second-price.

[8] The pure strategies in this game are the players always making the same bids. They are a subset of the mixed strategies, which are given by distributions.

[9] The support of the distribution is the possible values x and y can take.

his strategy f_{b_α} and his opponent's f_{b_β} by

$$\langle u_\alpha\rangle([f_{b_\alpha}],[f_{b_\beta}]) = P(\text{winner}{=}\alpha)\langle u_{\alpha,\text{winner}=\alpha}\rangle + P(\text{winner}{=}\beta)\langle u_{\alpha,\text{winner}=\beta}\rangle$$

where $P(\text{winner}{=}\alpha)$ denotes the probability of α winning course 1 and we have $P(\text{winner}{=}\beta) = 1 - P(\text{winner}{=}\alpha)$. See Appendix B Table 3 for how the terms in the above equation depend on $f_{b_\alpha}, f_{b_\beta}$. Note that by having written the payoff as a function of strategies we have formulated the auction in game theoretic terms.

We can try to optimize the payoff by varying the functions f_{b_α}, f_{b_α} over the set Ω. The technique used is calculus of variations, with an introduction here: [4]. The distributions have some constraints, namely, that they have to be normalized. We will incorporate these conditions via Lagrange multipliers[10]:

$$\mathcal{L}_\alpha([f_{b_\alpha}],[f_{b_\beta}],\lambda_{\alpha,1},\lambda_{\alpha,2}) = \langle u_\alpha\rangle + \lambda_{\alpha,1}\left(\int_\Omega f_{b_\alpha} - 1\right) + \lambda_{\alpha,2}\left(\int_\Omega f_{b_\beta} - 1\right)$$

where $\lambda_{\alpha,1}$ is the Lagrange multiplier associated with the normalization constraint for α and so forth. To get the expressions for β, we can simply switch the indices $\alpha \leftrightarrow \beta$ as they are arbitrary labels. Now we take the functional gradients and make them equal zero:

$$\frac{\delta \mathcal{L}_\alpha}{\delta f_{b_\alpha}} = 0 \qquad\qquad \frac{\delta \mathcal{L}_\beta}{\delta f_{b_\beta}} = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}_\alpha}{\partial \lambda_{\alpha,1}} = 0 \quad \frac{\partial \mathcal{L}_\alpha}{\partial \lambda_{\alpha,2}} = 0 \qquad \frac{\partial \mathcal{L}_\beta}{\partial \lambda_{\beta,1}} = 0 \quad \frac{\partial \mathcal{L}_\beta}{\partial \lambda_{\beta,2}} = 0 \tag{2}$$

We also need an initial value (see [4, Sec.5]), which is the value of the distributions on the boundary of Ω, denoted $\partial\Omega$, which can be set to be any function we wish

$$f_{b_\alpha}(x,y)|_{(x,y)\in\partial\Omega} = g_\alpha(x,y) \qquad f_{b_\beta}(x,y)|_{(x,y)\in\partial\Omega} = g_\beta(x,y)$$

We can take $g_\alpha(x,y) = g_\beta(x,y) = 0$ for simplicity.[11]

In Eq. (1) the first terms corresponds to α choosing a strategy which maximizes his average utility, therefore giving his best response to β's strategy. The second identity gives β's best response given α's strategy. By solving both these equations we get a condition for mutual best responses, which are the Nash equilibria. Note that this is a necessary but insufficient condition, as critical points can also be minima. We should go to second order variations [4, Sec.4] to check whether it is a maximum or minimum. And even if they are maxima, we are not assured of them being global maxima, therefore a rigorous proof requires a very careful analysis.

One can substitute, in theory, all the expressions in Appendix B Table 3 in the Lagrangian, take the functional gradients with respect to the bid distributions and hope the problem is tractable. The recipe for finding the functional gradients of, say $\mathcal{L}[f]$ with respect to f, is as follows:

[10] Lagrange multipliers are a common technique for incorporating constraints into an optimization problem.

[11] Although this implies the probabilities of bids lying on the boundary are zero, we can still get bids arbitrarily close to the boundary, therefore this should not pose a significant problem.

1. Substitute the function $f(x, y)$ in $\mathcal{L}[f]$ with $f(x, y) + \epsilon\eta(x, y)$.
2. Differentiate with respect to ϵ and evaluate at $\epsilon = 0$: $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{L}_\alpha[f(x, y) + \epsilon\eta(x, y)]|_{\epsilon=0}$.
3. Write the last expression in the form $\iint_\Omega G(x, y)\eta(x, y)\,\mathrm{d}x\,\mathrm{d}y$, where $G(x, y)$ is what we call the functional gradient: $G(x, y) = \frac{\delta\mathcal{L}_\alpha}{\delta f}$.

We have tried using this approach but we were not able to solve for the functional gradients. Check Appendix B to see the furthest expression we could get to.

3.4 Simulation

At this point, the analysis has come to a halt because of the sheer mathematical complexity encountered. However, we may still be able to test our hypotheses from Sections 3.1 and 3.2, even if we may not be able to prove them. To do this, we developed simulations which, based on a fixed auction with fixed utilities for each player, co-evolves strategies for the players. Our idea is that if there is an optimal strategy for our auction, then given enough time, our simulation should find it.

To simulate a somewhat realistic scenario, we create an auction with 3 courses, with a different number of seats. We create fewer students than the total number of seats, and simulate the game, with players co-evolving strategies. We then can vary different parameters to see if the behavior of the players changes.

The full details of the game are as follows. There are 3 courses, their capacities are 3 (course 1), 10 (course 2) and 6 (course 3). The players' utilities used can be found in Table 1. The utilities were generated at random in the range $[0, 200)$.

3.4.1 Optimizing strategies

The simulation is based on genetic algorithms[12] in a changing environment. The optimization is built on each player running their own genetic algorithm, trying to optimize their results against all the other players. Each player, in turn, gets to evolve an optimal response to the other players. If the players run into a stable state, where no one changes their bids much, we assume we have found a true equilibrium. The optimization algorithm also gives each player more and more time to optimize their own strategy. After all players have gotten to do one generation of optimization against all others, the auction shuffles the order of the players and let them each do two generations of optimization, then three, and so on.

The genetic algorithm is somewhat involved. Each player evolves their strategy according to a standard genetic algorithm, the bids for each course being genes in each individual, and each player having their own population. The fitness of an individual is their average utility (after deducting their payment) over 1000 runs of the auction. We need to take an average due to the random allocation for tie bids. In each iteration i of the optimization, $i \in \{1, 2, ..., 20\}$,

[12] For a general introduction see [8].

	Course 1 (Capacity: 3)	Course 2 (Capacity: 10)	Course 3 (Capacity: 6)
Player 1	164.7	30.1	81.0
Player 2	117.6	32.3	143.2
Player 3	195.3	47.8	156.5
Player 4	158.9	176.9	109.5
Player 5	94.9	75.8	162.5
Player 6	104.7	52.4	52.8
Player 7	154.6	136.9	1.9
Player 8	89.8	44.5	68.5
Player 9	143.4	161.6	106.6
Player 10	60.1	65.6	140.9
Player 11	92.0	67.7	14.8
Player 12	199.6	68.5	93.8
Player 13	5.8	141.6	159.1
Player 14	199.1	0.3	87.1
Player 15	142.1	17.0	71.4
Player 16	141.3	95.4	199.4
Player 17	47.6	155.7	25.6
Player 18	180.3	42.3	145.8

Table 1: The utilities of each player for each course. Course 1 has capacity 3, course 2 capacity 10, and course 3 capacity 6. The utilities were generated uniformly at random in the range $[0, 200)$.

population size	100
win probability of the better individual in selection tournament	0.75
number of players in selection tournament	2
crossover probability	0.5
mutation probability	0.1
elitism copies	1
initial range for genes	[0, 1)
creep mutation range for gene g	$(-0.1g, 0.1g)$

Table 2: Values for parameters used in the co-evolutionary algorithm.

the order of the players is shuffled, and they then get to, in turn, optimize their own population against the current best action profiles of the other players for i generations. Thus, early on (when the environment is more dynamic), only a few generations of optimization are applied. As the players co-evolve, they get to spend more generations optimizing each turn, giving less random noise to their optimization (as they get to evolve more performing populations and best individuals). This is then an ad-hoc strategy to achieve simulated annealing. Some other important parameters are included in Table 2.

Note the design of creep mutation, to allow the players to find their own utilities from below. The creep range increases as the gene size does, so that individuals may in a sense accelerate upwards, either forever (most likely until they reach the corresponding utility, and are pruned from the population by selection) or until they hit the ceiling of the max-bid, which of course bounds the genes.

The code for the simulation is available on GitHub[13].

Validation of the simulation. We have run a single-winner first-price auction using the simulation. There are three players, with utilities 2, 4, and 1 for the item. In this situation, the simulation does not find the Bayesian Nash equlibrium, because it is not a Bayesian game; the utilities are fixed. Therefore, the winner is incentivized to drive down their own bid to the point where they are barely winning. The other players can then take advantage of this. There is therefore no equlibrium, but a constant flux of bids. However, we also ran the same auction with a second-price mechanism. In this case, the player with the highest bid finds that they should bid truthfully and the other players that they can bid anything below that, which is a weak Nash equilibrium. We use these findings to argue that our simulation should perform well on the auction we are simulating, but that we should expect some fluctuation, since honest reporting is not necessarily incentivized, as per Sec. 3.1.4. However, since the number of players is large, we expect the fluctuations to be small, and indeed, this is what we observed when

[13] https://github.com/hjorthjort/TheAuctionGame

monitoring the run of the optimization algorithm.

3.4.2 Experimental findings

By running the optimization on the full university auction with the fixed utilities, we see the following results emerge.

Players should not always bid honestly. As expected, we may see some players not placing their highest bid on their favorite course. This occurs mainly when there is much competition for a course, and a player may gain more utility from saving some of their tokens and getting a lower valued course than to get a higher valued course and pay more. For example, when a player may gain 40 utility from getting a very popular course, but they would pay 25 utility for it, for a total of 15, they prefer getting a course for which they gain utility 20 and pay nothing.

Having a max bid does not affect allocative efficiency much. Perhaps the most satisfying finding from running the evolutionary optimization on a somewhat realistic auction is that the players, after a few rounds of optimizing, tend to get a total utility that is higher than the one they would get if they all reported their true utilities. The higher the max bid, the more efficient the allocation, but even with a diminishing max bid, the efficiency stays high and close to the "honest-reporting" allocation, where students would be assigned courses based on their true utilities, according to our clearing function.

In Figure 1, the optimization algorithm has been run several times for the same auction, with only the max bid changing. The auction was run for 20 iterations for each max-bid variation. As should be clear from the trend, the max bid does indeed have an effect on the efficiency, but there is no need for it to be as high as the actual utilities to get decent efficiency.

From this, we conclude that the university should not be too concerned that the tokens awarded have a utility that is much lower than the utility of the courses. But if the university can ensure that the tokens possess some non-negligible utility to the students, the auction will still be fairly allocatively efficient.[14]

Players will sometimes drive up prices, at no gain to themselves. This is likely because all Nash equilibria are weak: in the end, only a single bid is going to affect your own utility, namely the one for the course you were assigned. The others can fluctuate randomly, which may drive the prices up for other players, and in the end, even cause a shift out of the equilibrium.

[14] A related point is that a very, very low token utility could be interpreted as an opportunity for the players to simply rank their preferences, which still yields decent allocative efficiency.

X	Y
∞	2231
250	2320
200	2466
100	2446
50	2344
25	2261
10	2168
1	2115
10^{-2}	2056
10^{-7}	2018

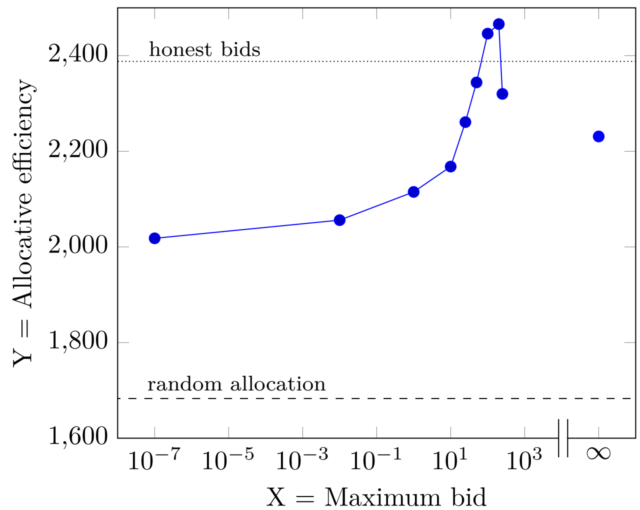

Figure 1: Effect of max bid on allocative efficiency. The allocative efficiency is based on players' strategies after 20 iterations of optimization. If all students were to report their true value as a bid (with infinite max bid), then the allocative efficiency would be 2388 (dashed line in the figure), and random assignment yields a utility of, on average, 1683 (dotted line). The efficiency is roughly peaked around 200, which is the maximum in the uniform random distribution on $[0, 200)$ which generated the players' utilities.

4 General discussion and outlook

Regarding the use of variational calculus, one may wonder though whether these techniques are valid. The answer is yes, as variational calculus lays the foundation to more general optimization principles, such as Pontryagin's maximum principle [6] which is heavily used in modern control theory.[15]

In the context of auctions, while a probabilistic approach is very common (see [3, Appendices] for examples), we have not found many instances of variational calculus, only one mention [5, Sec. 2.2]. After carrying out the analysis, we believe the reason might be the difficulty involved. Even if formally correct, we are not guaranteed to find solvable equations with this method. Even if we were, the example analyzed is a very simple auction with 2 players and 2 courses with very simplified properties. As a conclusion, for the analysis of more realistic auction we found it more feasible to use simulations. However, a possible future approach could be to use symbolic computation might make the calculations easier.

[15] This principle, a main result of optimal control theory, provides solutions to a wide course of optimization problems such as cubic splines, optimal surfaces, control of dynamical systems and more.

It is peculiar that max bids higher than the max utility in the simulation, such as infinity or 250, yield worse allocative efficiency than when the max-bid is 200 (Figure 1). Our hypothesis is that it is due to the larger search space that comes with a higher max-bid, and that it is easy for individuals to have their genes drift slightly higher than the max-bid, which is never a good strategy, but they may entertain for a while, leading to somewhat slower convergence.

An obvious way to get more reliable results would be to run the optimization for more iterations. However, since the run-time increases quadratically with the number of iterations, this is costly. Furthermore, individuals tend to converge on optimal strategies quickly.

Regarding the clearing function, a possible improvement to the mechanism would be to include some way to deter players from inflating prices, by having them pay something for the cost they incur to others. Encoding this well in the fitness function is, however, difficult, since we would aim to avoid "unnecessary damage", meaning damage a player incurs on another to no gain of their own, but to which the other player *could* retaliate, meaning that the first player is exposing themselves to unnecessary risk. Consider the following example of utilities:

	Course 1	Course 2
Player α	2	0
Player β	3	2

Player α bidding (2, 0) and player β bidding (0, 0) is a Nash equilibrium. However, it is a weak equilibrium, since player β could bid any values $(b_{\beta,1}, b_{\beta,2})$ and get the same utility, as long as $b_{\beta,1} < b_{\alpha,1} = 2$, and α would still not want to change his bid. However, having a high value of $b_{\beta,1}$, β is needlessly causing α a loss. Player α could retaliate (possibly at a loss to himself) by changing his bid $b_{\alpha,2}$ such that $b_{\beta,1} < b_{\alpha,2} < b_{\beta,2}$. It would be sensible by β to not expose himself to such a possible retaliation at all, and keep $b_{\beta,1}$ low. If such (and other) means of retaliation could be included in the fitness function, the bids may look somewhat less random. With the current fitness functions, players will happily bid in strange ways, as long as it does not change their outcome.

Another improvement to the clearing mechanism would be to take inspiration from solution approaches to the Stable Matching Problem. Our problem has some important differences, but also several similarities, and adapting the algorithms which solve them and similar problems to our situation is beyond our scope, but could be the basis for future work. We suggest going to [7] for inspiration in this direction.

5 Conclusion

In our project, we analyzed an auction for a university to assign courses. We analyzed it formally and using stochastic optimization algorithms. Our simulation revealed that indeed, this auction could perform well at allocating courses in an efficient way. This performance is, however, intimately connected to the allowed max-bid, and thus improving efficiency comes with a cost to the university, as well as to fairness – a higher monetary value of bids may benefit richer students, offsetting the benefit of using tokens. Still, even at low token utilities, the allocative efficiency is far above random allocation, and often close to a decent allocation based on honest reporting. Overall, we consider the results a moderate success and would recommend the university to adopt this auction, if possible.

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A Proof that second-price auctions have honest-reporting as a weakly dominant strategy

To prove that this assumption holds in second price auctions, we discuss two different cases:

First, the player would have won the auction given that he is bidding his true value and second the player would have lost the auction if he is bidding his true value.

In the case of winning the auction by bidding his true value, the player does not gain more utility by increasing his bid, neither by decreasing it slightly. The price is defined by the second highest bid and not by his own. In case the player reduces his bid to much, such that he is not the winner of the auction any longer he is giving up utility he would have gained as the winner of the auction.

Similar to that, we can investigate the second case in which the player would loose the auction by bidding his true value. If the player bid less than his true value, he will still not be the winner of the auction. If he bid slightly above his true value without becoming the highest bidder, the outcome for the player doe not change either. Only if the player increases his bid that much that he becomes the winner of the auction, the outcome changes. Since the price of the course is defined by the former winning bid (now second highest bid), which is higher than the players utility for the course, the player gains negative utility.

By this simple case analysis we can see that bidding their true value in a second price auction is a weakly dominant strategy.

B Mathematical details for the 2-player 2-course auction

This appendix contains all the mathematical details about the model from Sec. 3.3. We reintroduce the variables from the model again to make it more self-contained. See [2] for a source of all the probability concepts we will use.

Let us label the two players as α and β and the two courses with numbers 1 and 2. To establish notation, if ν is a random variable, let $f_\nu(x)$ be the probability distribution of this random variable naturally fulfilling $\int f_\nu(x)\,\mathrm{d}x = 1$, $f_\nu(x) \geq 0$. We define two distributions of "hidden values", $f_{\nu_1}(x)$ and $f_{\nu_2}(x)$, for course 1 and 2 respectively. These we will take to be bounded and take values only in the interval $[0, 1]$. Each player draws from these two probability distributions to get their value for the courses. Players now make bids but with a limited amount of "money" or tokens. If $b_{\alpha,1}$ represents the (non-negative) bid of player α on course 1 and so forth, then we simply have:

$$0 \leq b_{\alpha,1} + b_{\alpha,2} \leq 1 \qquad 0 \leq b_{\beta,1} + b_{\beta,2} \leq 1 \tag{3}$$

this being the split-bid model. See Figure 2 for a representation of the triangle Ω these inequalities define in the bid space. **See the "Important note"**

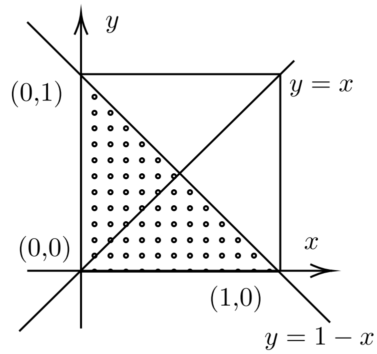

Figure 2: The dotted section is the set Ω which is the support of the bidding probability distribution in the case of **split-bids**. The set is defined by the inequalities $0 < x + y < 1$, $0 < x < 1$, $0 < y < 1$, representing bounded non-negative bids (x the bid for course 1 and y for course 2). Remove the condition $0 < x + y < 1$ for **max-bids** instead of split bids. In that case, Ω becomes a square of length 1. The line $y = x$ separates the points $x < y$ and $x > y$, which is useful in calculating the probability of winning a course.

text-box further on for comments on split-bid versus max-bid and our choice of analysing split-bids here.

Important note: As perhaps the reader has seen, this is different from the main report: we are solving the *split bid* where the constraints on the bids are $0 \leq b_{\alpha,1} + b_{\alpha,2} \leq 1, 0 \leq b_{\beta,1} + b_{\beta,2} \leq 1$ instead of the max bid where $0 < b_{\alpha,1}, b_{\alpha,2}, b_{\beta,1}, b_{\beta,2} < 1$, as explained in the main text. Why? Chronologically, we started investigating a split-bid model for our auction, but later on, we changed to a max-bid auction, and the mathematical work on the split-bid was done before the change. What changes between the two is only the support set Ω (becoming a square instead of a triangle as in Figure 2), meaning some integration bounds will change. As there are quite a few equations to change, the rest of the formalism remains the same and due to time constraints, we decided to leave the split-case.[16]

Note that the possible pure strategies the players have are to just pick two numbers satisfying the constraints and bid that always. We will analyse the

[16] Note that one might be tempted to say that in the case of a max-bid the strategy space is simpler by being able to consider two separate distributions for each course, instead of a joint-distribution for the bids. We argue that by doing so we remove all possible correlations between bidding for course 1 and course 2, and therefore, richness in strategy for player α and possible Nash equilibria. Since the case of separable distributions is a subset of the general distributions (it would be the case $f_{b_\alpha}(x, y) = f_{b_{\alpha,1}}(x) f_{b_{\alpha,2}}(y)$, i.e., we can separate the variables), we argue that it is best to use the more general joint distribution of bids $f_{b_\alpha}(x, y)$ rather than separate ones. With this we argue that the mathematical model is almost the same for both split-bids and max-bids.

mixed strategies instead (the pure are contained in the mixed) by choosing probability distributions over the bids which we will denote by $f_{b_\alpha}(x, y), f_{b_\beta}(x, y)$ (x and y represent bids for course 1 and 2 respectively). We can recover pure strategies by substituting the distributions with Dirac delta functions instead, defined by $\iint_{\mathbb{R}^2} \delta(x - x_0, y - y_0) f(x, y)\, \mathrm{d}x\, \mathrm{d}y = f(x_0, y_0)$.

Observe that we have defined a joint distribution for the two bids a single player makes. If we want to recover the distribution of bids for one course only $f_{b_{\alpha,1}}(x)$ from $f_{b_\alpha}(x, y)$, we need to take the marginal distribution, where we integrate over a section of Ω:

$$f_{b_{\alpha,1}}(x) = \int_0^{1-x} f_{b_\alpha}(x, y)\, \mathrm{d}y \qquad f_{b_{\alpha,2}}(y) = \int_0^{1-y} f_{b_\alpha}(x, y)\, \mathrm{d}x \tag{4}$$

and similarly for player β.

We will also use the following result: if X, Y denote continuous independent random variables with distributions f_X, f_Y, then the distribution of $Z = X - Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z + y) f_Y(y)\, \mathrm{d}y \coloneqq (f_X * f_Y)(z) \tag{5}$$

which we will define as *convolution*.[17] From here on we will restrict the support of the distributions f_X, f_Y to $[0, 1]$ as it is the case which is of relevance to us. Then we have $0 < z + y < 1$, $0 < y < 1$ and we can easily reason that $-1 < z < 1$. Depending on the sign of z, the value y lies in different intervals which will need to be taken into account when integrating. Now that we have defined all the mathematical baggage, we can proceed to analyse our auction.

We want to calculate the expected value of the payoffs for the players as a function of their bid distributions,

$$\langle u_\alpha \rangle([f_{b_\alpha}], [f_{b_\beta}]) = P(\text{winner=}\alpha)\langle u_{\alpha,\text{winner=}\alpha}\rangle + P(\text{winner=}\beta)\langle u_{\alpha,\text{winner=}\beta}\rangle \tag{6}$$

where $P(\text{winner=}\alpha)$ denotes the probability of α winning course 1, and the probability of losing is $P(\text{winner=}\beta) = 1 - P(\text{winner=}\alpha)$. If the winner is β, player α still gets some payoff $\langle u_{\alpha,\text{winner=}\beta}\rangle$ as he gets course 2. To get β's average payoff we just substitute the indices $\alpha \to \beta$, since they are arbitrary labels. All these terms, being sums and differences of random variables, will be convolutions. See Table 3 for details as how to calculate these terms. Once they are all calculated we have $\langle u_\alpha \rangle([f_{b_\alpha}], [f_{b_\alpha}])$ and we can perform the variational calculus mentioned in Sec. 3.3. Incorporating the normalisation of the distributions we get the Lagrange function

$$\mathcal{L}_\alpha([f_{b_\alpha}], [f_{b_\beta}]) = \langle u_\alpha \rangle + \lambda_{\alpha,1}\left(\int_\Omega f_{b_\alpha} - 1\right) + \lambda_{\alpha,2}\left(\int_\Omega f_{b_\beta} - 1\right)$$

[17] Technically speaking, a convolution in probability theory is the distribution of $Z = X + Y$ instead of $Z = X - Y$, but we will still use the term in order to avoid having to define extra terminology.

To get conditions for Nash equilibria as explained in the main text we need to take the functional gradient which is calculated using the following steps.

What are the functional gradients intuitively? In single variable calculus, it is a basic result that we can write $\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x_0 + \epsilon\eta)|_{\epsilon=0} = \sum_i \frac{\partial f(x_0)}{\partial x_i}\eta_i$ (see: Taylor series), giving us the rate of change of f as we go infinitesimally in direction η. If this change is zero for any η, then we are at a maximum, minimum or inflection point. Similarly, the "functional gradient" is found by substituting the sum for an integral and the point x by a function: $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{L}_\alpha[f(x,y) + \epsilon\eta(x,y)]|_{\epsilon=0} = \iint \frac{\delta\mathcal{L}_\alpha}{\delta f_{b_\alpha}}\eta(x,y)\,\mathrm{d}x\,\mathrm{d}y$, giving the rate of change of $\mathcal{L}_\alpha$ as we infinitesimally change the function $f(x,y)$ in "direction" $\eta(x,y)$. If this is zero for any $\eta(x,y)$, we are at an extremal point. The recipe for finding the functional gradients of, say $\mathcal{L}[f]$ with respect to f, then is as follows:

1. Substitute the function $f(x,y)$ in $\mathcal{L}[f]$ with $f(x,y) + \epsilon v(x,y)$.
2. Differentiate with respect to ϵ and evaluate at $\epsilon = 0$: $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{L}_\alpha[f(x,y) + \epsilon v(x,y)]|_{\epsilon=0}$.
3. Write the last expression in the form $\iint_\Omega G(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where $G(x,y)$ is what we call the functional gradient: $G(x,y) = \frac{\delta\mathcal{L}_\alpha}{\delta f}$.

When trying to apply this method to $\mathcal{L}_\alpha$ by varying f_{b_α} we arrive at the expression

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{L}_\alpha[f_{b_\alpha}(x,y) + \epsilon v(x,y)]|_{\epsilon=0} = \\
&\quad = \int_{z=0}^{1}\int_{x=0}^{1-z}\int\limits_{\substack{y_1\in[0,1-x-z]\\ y_2\in[0,1-x]}} C[f_{b_\beta}]\; v(z+x,y_1) f_{b_\beta}(x,y_2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2 + \\
&\qquad\qquad + \int_{x=0}^{1}\int_{y=0}^{1-x}(-\lambda_{1,\alpha})v(x,y)\,\mathrm{d}x\,\mathrm{d}y \stackrel{?}{=} \iint_\Omega G(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

where $C[f_{b_\beta}] = \langle v_{\alpha_1}\rangle - \langle v_{\alpha_2}\rangle + \langle b_{\beta_2}\rangle - \langle b_{\beta_1}\rangle$. The last equality has a "?" sign to denote the next step which we have not been able to do, which is writing the expression in that integral form to find $G(x,y) \equiv \frac{\delta\mathcal{L}_\alpha}{\delta f_{b_\alpha}}$. We guess that a change of variable might do the trick but we are not sure how to remove the dependence of $v(z+x,y_1)$ on y_1. If we do so, we will probably get a *family* of functions which satisfy the constraint. To only identify the maxima, we would need to go to second order variations and once there, look for the global maxima.

$P(\text{winner}{=}\alpha)$	The probability for α to win course 1 is the same as that of having his bid for course 1 be greater than β's: $$P(\text{winner}{=}\alpha) = P(b_{\alpha,1} > b_{\beta,1}) = P(b_{\alpha,1} - b_{\beta,1} > 0) = \\ = \int_0^1 (f_{b_{\alpha,1}} * f_{b_{\beta,1}})(z)\,\mathrm{d}z = \\ = \int\limits_{[0,1]} \mathrm{d}z \int\limits_{[0,1-z]} \mathrm{d}x \int\limits_{\substack{y_1\in[0,1-x-z]\\ y_2\in[0,1-x]}} \mathrm{d}y_1 \mathrm{d}y_2\ f_{b_\alpha}(x{+}z, y_1) f_{b_\beta}(x, y_2)$$ where $*$ denotes convolution, as in Eq. (5). Note that the bids of players α and β we assume to be independent as there is no collaboration between the two, thus we can use the convolution; else, we would need the joint probability distribution of the two.
$\langle u_{\alpha,\text{winner}=\alpha}\rangle$	The payoff is the player's hidden value minus the payment he makes if he wins: $u_{\alpha,\text{winner}=\alpha} = \nu_{1,\alpha} - b_{\beta,1}$. The average is given by $$\langle u_{\alpha,\text{winner}=\alpha}\rangle = \langle \nu_{1,\alpha}\rangle - \langle b_{\beta,1}\rangle$$ as the expected value is a linear function. We have $\langle \nu_{1,\alpha}\rangle = \int x f_{\nu,1}(x)\,\mathrm{d}x$ and $\langle b_{\beta,1}\rangle = \int_0^1 x f_{\beta,1}(x)\,\mathrm{d}x = \int_{x=0}^1 \int_{y=0}^{1-x} x f_\beta(x,y)\,\mathrm{d}x\,\mathrm{d}y$
$P(\text{winner}{=}\beta)$	They are complementary events therefore $P(\text{winner}{=}\beta) = 1 - P(\text{winner}{=}\alpha)$.
$\langle u_{\alpha,\text{winner}=\beta}\rangle$	If you lose, that means the other won and he drops out of the auction (part of the clearing rules). This means you win the second item: $u_{\alpha,\text{win}=\beta} = \nu_{2,\alpha} - b_{\beta,2}$ with average $$\langle u_{\alpha,\text{winner}=\beta}\rangle = \langle \nu_{2,\alpha}\rangle - \langle b_{\beta,2}\rangle$$ by the same reasoning as before. **Note:** Remember that this is not a *true* second-price auction as the price paid is not necessarily the second highest ($b_{\beta,2}$ can be greater than $b_{\alpha,2}$). To fix this we can substitute $\langle b_{\beta,2}\rangle \to P(b_{\alpha,2} > b_{\beta,2})\langle b_{\beta,2}\rangle + P(b_{\alpha,2} < b_{\beta,2})\langle b_{\alpha,2}\rangle$ where the probabilities are calculated as in the first row in the table. As this is more complicated, we use the modified second-price.

Table 3: Table with details as to how to calculate the different terms in Eq. (6).