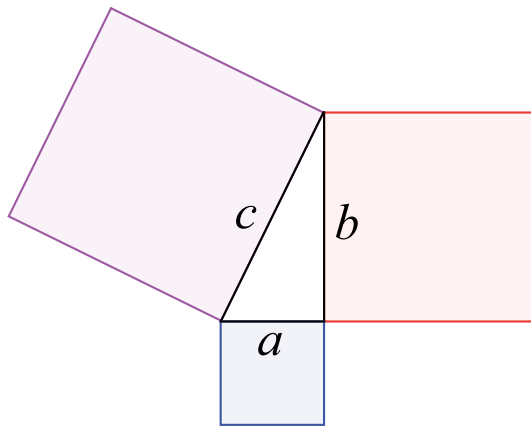# Pythagorean triple

*The Pythagorean theorem:* a² + b² = c²

Animation demonstrating the simplest case of the Pythagorean Triple: $3^2 + 4^2 = 5^2$.

A **Pythagorean triple** consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$. Such a triple is commonly written (a, b, c), and a well-known example is $(3, 4, 5)$. If (a, b, c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k. A **primitive Pythagorean triple** is one in which a, b and c are coprime. A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.

The name is derived from the Pythagorean theorem, stating that every right triangle has side lengths satisfying the formula $a^2 + b^2 = c^2$; thus, Pythagorean triples describe the three integer side lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. For instance, the triangle with sides $a = b = 1$ and $c = \sqrt{2}$ is right, but $(1, 1, \sqrt{2})$ is not a Pythagorean triple because $\sqrt{2}$ is not an integer. Moreover, 1 and $\sqrt{2}$ do not have an integer common multiple because $\sqrt{2}$ is irrational.

1 Examples
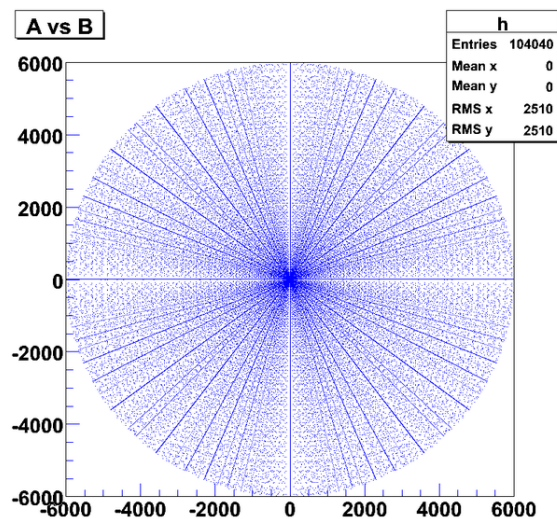
A scatter plot of the legs (a,b) *of the Pythagorean triples with* c *less than 6000. Negative values are included to illustrate the parabolic patterns in the plot more clearly.*
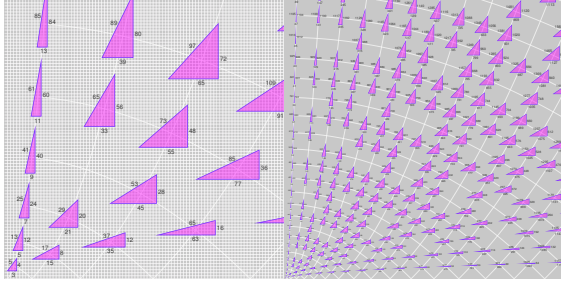
There are 16 primitive Pythagorean triples with $c \leq 100$:

Note, for example, that $(6, 8, 10)$ is *not* a primitive Pythagorean triple, as it is a multiple of $(3, 4, 5)$. Each one of these low-c points forms one of the more easily recognizable radiating lines in the scatter plot.

Additionally these are all the primitive Pythagorean triples with $100 < c \leq 300$:

2 Generating a triple

Main article: Formulas for generating Pythagorean triples
  Euclid's formula[1] is a fundamental formula for generating Pythagorean triples given an arbitrary pair of positive integers m and n with $m > n$. The formula states that

The primitive Pythagorean triples. The odd leg a *is plotted on the horizontal axis, the even leg* b *on the vertical. The curvilinear grid is composed of curves of constant* m - n *and of constant* m + n *in Euclid's formula.*
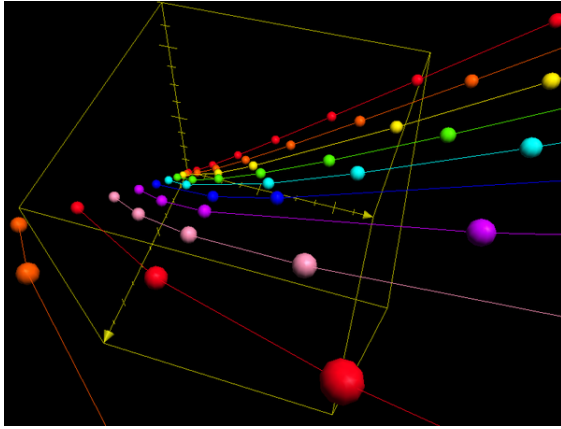
A plot of triples generated by Euclid's formula map out part of the $z^2 = x^2 + y^2$ *cone. A constant* m *or* n *traces out part of a parabola on the cone.*

the integers

$$a = m^2 - n^2, \ b = 2mn, \ c = m^2 + n^2$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are coprime and $m - n$ is odd. If both m and n are odd, then a, b, and c will be even, and so the triple will not be primitive; however, dividing a, b, and c by 2 will yield a primitive triple if m and n are coprime.[2]

Every primitive triple arises from a *unique pair* of coprime numbers m, n, one of which is even. It follows that there are infinitely many primitive Pythagorean triples. This relationship of a, b and c to m and n from Euclid's formula is referenced throughout the rest of this article.

Despite generating all primitive triples, Euclid's formula does not produce all triples. This can be remedied by inserting an additional parameter k to the formula. The following will generate all Pythagorean triples uniquely:

$$a = k \cdot (m^2 - n^2), \ b = k \cdot (2mn), \ c = k \cdot (m^2 + n^2)$$

where m, n, and k are positive integers with $m > n$, $m - n$ odd, and with m and n coprime.

That these formulas generate Pythagorean triples can be verified by expanding $a^2 + b^2$ using elementary algebra and verifying that the result coincides with c^2. Since every Pythagorean triple can be divided through by some integer k to obtain a primitive triple, every triple can be generated uniquely by using the formula with m and n to generate its primitive counterpart and then multiplying through by k as in the last equation.

Many formulas for generating triples have been developed since the time of Euclid.

2.1   Proof of Euclid's formula

That satisfaction of Euclid's formula by *a, b, c* is sufficient for the triangle to be Pythagorean is apparent from the fact that for positive integers m and n, $m > n$, the a, b, and c given by the formula are all positive integers, and from the fact that

$$a^2 + b^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 = c^2.$$

A simple proof of the *necessity* that *a, b, c* be expressed by Euclid's formula for any primitive Pythagorean triple is as follows.[3] All such triples can be written as (a, b, c) where $a^2 + b^2 = c^2$ and a, b, c are coprime, and where b and c have opposite parities (one is even and one is odd). (If c had the same parity as both legs, then if all were even the parameters would not be coprime, and if all were odd then $a^2 + b^2 = c^2$ would equate an even to an odd.) From $a^2 + b^2 = c^2$ we obtain $c^2 - a^2 = b^2$ and hence $(c - a)(c + a) = b^2$. Then $\frac{(c+a)}{b} = \frac{b}{(c-a)}$. Since $\frac{(c+a)}{b}$ is rational, we set it equal to $\frac{m}{n}$ in lowest terms. We also observe that $\frac{(c-a)}{b}$ equals the reciprocal of $\frac{b}{(c-a)}$ and hence equals the reciprocal of $\frac{(c+a)}{b}$, and thus equals $\frac{n}{m}$. Then solving

$$\frac{c}{b} + \frac{a}{b} = \frac{m}{n}, \ \frac{c}{b} - \frac{a}{b} = \frac{n}{m}$$

for $\frac{c}{b}$ and $\frac{a}{b}$ gives

$$\frac{c}{b} = \frac{m^2 + n^2}{2mn}, \ \frac{a}{b} = \frac{m^2 - n^2}{2mn}.$$

Since $\frac{c}{b}$ and $\frac{a}{b}$ are fully reduced by assumption, the numerators can be equated and the denominators can be equated if and only if the right side of each equation is fully reduced; given the previous specification that $\frac{m}{n}$ is fully reduced, implying that m and n are coprime, the right sides are fully reduced if and only if m and n have opposite parity (one is even and one is odd) so that the numerators are not divisible by 2. (And m and n *must* have opposite parity: if both were odd then dividing through $\frac{m^2+n^2}{2mn}$ by 2 would give the ratio of two odd numbers;

equating this ratio to $\frac{c}{b}$, which is a ratio of two numbers with opposite parities, would give different 2-adic orders for numbers supposedly equal.) So, equating numerators and equating denominators, we have Euclid's formula $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ with m and n coprime and of opposite parities.

A longer but more commonplace proof is given in Maor (2007)[4] and Sierpiński (2003).[5]:4-7

2.2 Interpretation of parameters in Euclid's formula

Suppose the sides of a Pythagorean triangle are $m^2 - n^2$, $2mn$, and $m^2 + n^2$, and suppose the angle between the leg $m^2 - n^2$ and the hypotenuse $m^2 + n^2$ is denoted as θ. Then a right triangle with legs m and n has the angle $\theta/2$ between the leg m and the (not necessarily rational) hypotenuse.[6]

3 Elementary properties of primitive Pythagorean triples

The properties of a primitive Pythagorean triple (a, b, c) with $a < b < c$ (without specifying which of a or b is even and which is odd) include:

- $(c - a)(c - b)/2$ is always a perfect square.[7] This is particularly useful in checking if a given triple of numbers is a Pythagorean triple, but it is only a necessary condition, not a sufficient one. The triple {6, 12, 18} passes the test that $(c - a)(c - b)/2$ is a perfect square, but it is not a Pythagorean triple. When a triple of numbers a, b and c forms a primitive Pythagorean triple, then (c minus the even leg) and one-half of (c minus the odd leg) are both perfect squares; however this is not a sufficient condition, as the triple {1, 8, 9} is a counterexample since $1^2 + 8^2 \neq 9^2$.

- At most one of a, b, c is a square.[8]

- The area of a Pythagorean triangle cannot be the square[9]:p. 17 or twice the square[9]:p. 21 of a natural number.

- Exactly one of a, b is odd; c is odd.[10]

- Exactly one of a, b is divisible by 3.[5]

- Exactly one of a, b is divisible by 4.[5]

- Exactly one of a, b, c is divisible by 5.[5]

- The largest number that always divides abc is 60.[11]

- All prime factors of c are primes of the form $4n + 1$.[12] Therefore c is of the form $4n + 1$.

- The area ($K = ab/2$) is an even congruent number.[13]

- In every Pythagorean triple, the radius of the incircle and the radii of the three excircles are natural numbers. Specifically, for a primitive triple the radius of the incircle is $r = n(m - n)$, and the radii of the excircles opposite the sides m^2–n^2, $2mn$, and the hypotenuse m^2+n^2 are respectively $m(m - n)$, $n(m + n)$, and $m(m + n)$.[14]

- As for any right triangle, the converse of Thales' theorem says that the diameter of the circumcircle equals the hypotenuse; hence for primitive triples the circumdiameter is $m^2 + n^2$, and the circumradius is half of this and thus is rational but non-integer (since m and n have opposite parity).

- When the area of a Pythagorean triangle is multiplied by the curvatures of its incircle and 3 excircles, the result is four positive integers $w > x > y > z$. Integers w, x, y, $-z$ satisfy Descartes's Circle Equation.[15]

- There are no Pythagorean triples in which the hypotenuse and one leg are the legs of another Pythagorean triple.[9]:p. 14

- Each primitive Pythagorean triangle has a ratio of area to squared semiperimeter that is unique to itself and is given by[16]

$$\frac{K}{s^2} = \frac{n(m - n)}{m(m + n)} = 1 - \frac{c}{s}.$$

- No primitive Pythagorean triangle has an integer altitude from the hypotenuse and is therefore indecomposable.[17]

In addition, special Pythagorean triples with certain additional properties can be guaranteed to exist:

- Every integer greater than 2 that is not congruent to 2 mod 4 (in other words, every integer greater than 2 which is *not* of the form $4n + 2$) is part of a primitive Pythagorean triple.

- Every integer greater than 2 is part of a primitive or non-primitive Pythagorean triple. For example, the integers 6, 10, 14, and 18 are not part of primitive triples, but are part of the non-primitive triples 6, 8, 10; 14, 48, 50 and 18, 80, 82.

- There exist infinitely many Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly one (such triples are necessarily primitive). One method to generate such triples is the relation $(2n+1)^2 + [2n(n+1)]^2 = [2n(n+1) + 1]^2$, leading to triples (3,4,5), (5,12,13), (7,24,25), etc. More generally, for every odd integer j, there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the even leg differ by j^2.

- There exist infinitely many primitive Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly two. Generalization: For every integer $k > 0$, there exist infinitely many primitive Pythagorean triples in which the hypotenuse and the odd leg differ by $2k^2$.

- There exist infinitely many Pythagorean triples in which the two legs differ by exactly one. For example, $20^2 + 21^2 = 29^2$.

- For each natural number n, there exist n Pythagorean triples with different hypotenuses and the same area.

- For each natural number n, there exist at least n different Pythagorean triples with the same leg a, where a is some natural number

- For each natural number n, there exist at least n different Pythagorean triples with the same hypotenuse.[5]:31

- There exist infinitely many Pythagorean triples with square numbers for both the hypotenuse c and the sum of the legs $a+b$. The **smallest** such triple[18] has $a = 4,565,486,027,761$; $b = 1,061,652,293,520$; and $c = 4,687,298,610,289$. Here $a+b = 2,372,159^2$ and $c = 2,165,017^2$. This is generated by Euclid's formula with parameter values $m = 2,150,905$ and $n = 246,792$.

- There exist Pythagorean triangles with integer altitude from the hypotenuse. Such Pythagorean triangles are known as decomposable since they can be split along this altitude into two separate and smaller Pythagorean triangles.[17]

The set of all primitive Pythagorean triples forms a rooted ternary tree in a natural way; see Tree of primitive Pythagorean triples.
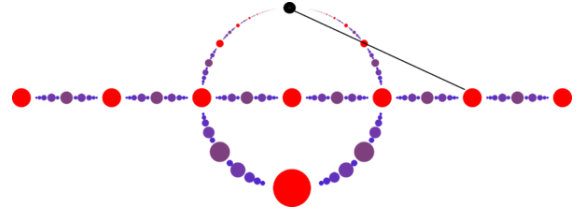
4   Geometry of Euclid's formula
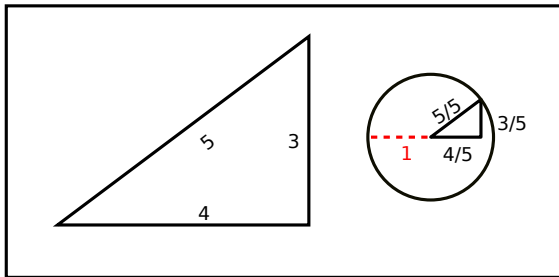
3,4,5 maps to x,y point (4/5,3/5) on the unit circle

Euclid's formulae for a Pythagorean triple

$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2$$

The rational points on a circle correspond, under stereographic projection, to the rational points of the line.

can be understood in terms of the geometry of rational number points on the unit circle (Trautman 1998). To motivate this, consider a right triangle with legs a and b, and hypotenuse c, where a, b, and c are positive integers. By the Pythagorean theorem, $a^2 + b^2 = c^2$ or, dividing both sides by c^2,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Geometrically, the point in the Cartesian plane with coordinates

$$x = \frac{a}{c}, \quad y = \frac{b}{c}$$

is on the unit circle $x^2 + y^2 = 1$. In this equation, the coordinates x and y are given by rational numbers. Conversely, any point on the unit circle whose coordinates x, y are rational numbers gives rise to a primitive Pythagorean triple. Indeed, write x and y as fractions in lowest terms:

$$x = \frac{a}{c}, \quad y = \frac{b}{c}$$

where the greatest common divisor of a, b, and c is 1. Then, since x and y are on the unit circle,

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \implies a^2 + b^2 = c^2,$$

as claimed.

There is therefore a correspondence between points on the unit circle with rational coordinates and primitive Pythagorean triples. At this point, Euclid's formulae can be derived either by methods of trigonometry or equivalently by using the stereographic projection.

For the stereographic approach, suppose that P' is a point on the x-axis with rational coordinates

$$P' = \left(\frac{m}{n}, 0\right).$$

Then, it can be shown by basic algebra that the point P has coordinates
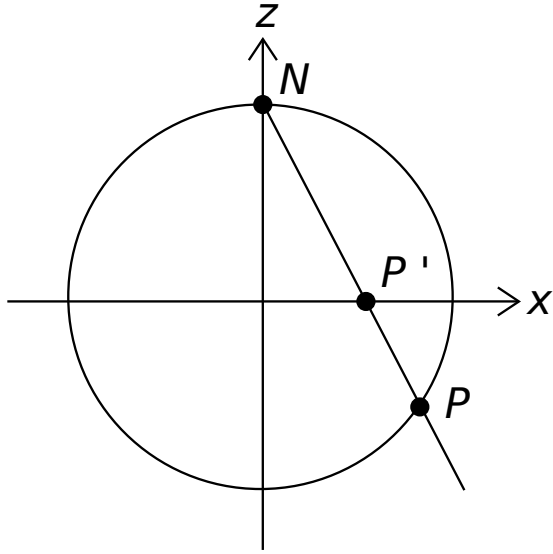
Stereographic projection of the unit circle onto the x-axis. Given a point P on the unit circle, draw a line from P to the point N = (0, 1) (the north pole). The point P′ where the line intersects the x-axis is the stereographic projection of P. Inversely, starting with a point P′ on the x-axis, and drawing a line from P′ to N, the inverse stereographic projection is the point P where the line intersects the unit circle.

$$P = \left(\frac{2\left(\frac{m}{n}\right)}{\left(\frac{m}{n}\right)^2 + 1}, \frac{\left(\frac{m}{n}\right)^2 - 1}{\left(\frac{m}{n}\right)^2 + 1} \right) = \left(\frac{2mn}{m^2 + n^2}, \frac{m^2 - n^2}{m^2 + n^2} \right).$$

This establishes that each rational point of the *x*-axis goes over to a rational point of the unit circle. The converse, that every rational point of the unit circle comes from such a point of the *x*-axis, follows by applying the inverse stereographic projection. Suppose that $P(x, y)$ is a point of the unit circle with *x* and *y* rational numbers. Then the point $P′$ obtained by stereographic projection onto the *x*-axis has coordinates

$$\left(\frac{x}{1 - y}, 0 \right)$$

which is rational.

In terms of algebraic geometry, the algebraic variety of rational points on the unit circle is birational to the affine line over the rational numbers. The unit circle is thus called a rational curve, and it is this fact which enables an explicit parameterization of the (rational number) points on it by means of rational functions.

5   Spinors and the modular group

Pythagorean triples can likewise be encoded into a matrix of the form

$$X = \begin{bmatrix} c + b & a \\ a & c - b \end{bmatrix}.$$

A matrix of this form is symmetric. Furthermore, the determinant of *X* is

$$\det X = c^2 - a^2 - b^2$$

which is zero precisely when (a,b,c) is a Pythagorean triple. If *X* corresponds to a Pythagorean triple, then as a matrix it must have rank 1.

Since *X* is symmetric, it follows from a result in linear algebra that there is a column vector $\xi = [m\ n]^\mathrm{T}$ such that the outer product

holds, where the *T* denotes the matrix transpose. The vector ξ is called a spinor (for the Lorentz group SO(1, 2)). In abstract terms, the Euclid formula means that each primitive Pythagorean triple can be written as the outer product with itself of a spinor with integer entries, as in (**1**).

The modular group Γ is the set of 2×2 matrices with integer entries

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with determinant equal to one: $\alpha\delta - \beta\gamma = 1$. This set forms a group, since the inverse of a matrix in Γ is again in Γ, as is the product of two matrices in Γ. The modular group acts on the collection of all integer spinors. Furthermore, the group is transitive on the collection of integer spinors with relatively prime entries. For if $[m\ n]^\mathrm{T}$ has relatively prime entries, then

$$\begin{bmatrix} m & -v \\ n & u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

where *u* and *v* are selected (by the Euclidean algorithm) so that $mu + nv = 1$.

By acting on the spinor ξ in (**1**), the action of Γ goes over to an action on Pythagorean triples, provided one allows for triples with possibly negative components. Thus if *A* is a matrix in Γ, then

gives rise to an action on the matrix *X* in (**1**). This does not give a well-defined action on primitive triples, since it may take a primitive triple to an imprimitive one. It is convenient at this point (per Trautman 1998) to call

a triple (*a*,*b*,*c*) **standard** if $c > 0$ and either (*a*,*b*,*c*) are relatively prime or (*a*/2,*b*/2,*c*/2) are relatively prime with *a*/2 odd. If the spinor [*m n*]^T has relatively prime entries, then the associated triple (*a*,*b*,*c*) determined by (**1**) is a standard triple. It follows that the action of the modular group is transitive on the set of standard triples.

Alternatively, restrict attention to those values of *m* and *n* for which *m* is odd and *n* is even. Let the subgroup Γ(2) of Γ be the kernel of the group homomorphism

$$\Gamma = \mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}_2)$$

where SL(2,$\mathbf{Z}_2$) is the special linear group over the finite field $\mathbf{Z}_2$ of integers modulo 2. Then Γ(2) is the group of unimodular transformations which preserve the parity of each entry. Thus if the first entry of ξ is odd and the second entry is even, then the same is true of *A*ξ for all $A \in \Gamma(2)$. In fact, under the action (**2**), the group Γ(2) acts transitively on the collection of primitive Pythagorean triples (Alperin 2005).

The group Γ(2) is the free group whose generators are the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Consequently, every primitive Pythagorean triple can be obtained in a unique way as a product of copies of the matrices *U* and *L*.

6 Parent/child relationships

Main article: Tree of Pythagorean triples

By a result of Berggren (1934), all primitive Pythagorean triples can be generated from the (3, 4, 5) triangle by using the three linear transformations T1, T2, T3 below, where *a*, *b*, *c* are sides of a triple:

If one begins with 3, 4, 5 then all other primitive triples will eventually be produced. In other words, every primitive triple will be a "parent" to 3 additional primitive triples. Starting from the initial node with $a = 3$, $b = 4$, and $c = 5$, the next generation of triples is

The linear transformations T1, T2, and T3 have a geometric interpretation in the language of quadratic forms. They are closely related to (but are not equal to) reflections generating the orthogonal group of $x^2 + y^2 - z^2$ over the integers. A different set of three linear transformations is discussed in Pythagorean triples by use of matrices and linear transformations. For further discussion of parent-child relationships in triples, see: Pythagorean triple (Wolfram) and (Alperin 2005).

7 Relation to Gaussian integers

Alternatively, Euclid's formulae can be analyzed and proven using the Gaussian integers.[19] Gaussian integers are complex numbers of the form $\alpha = u + vi$, where *u* and *v* are ordinary integers and *i* is the square root of negative one. The units of Gaussian integers are ±1 and ±i. The ordinary integers are called the rational integers and denoted as **Z**. The Gaussian integers are denoted as **Z**[*i*].. The right-hand side of the Pythagorean theorem may be factored in Gaussian integers:

$$c^2 = a^2 + b^2 = (a + bi)\overline{(a + bi)} = (a + bi)(a - bi).$$

A primitive Pythagorean triple is one in which *a* and *b* are coprime, i.e., they share no prime factors in the integers. For such a triple, either *a* or *b* is even, and the other is odd; from this, it follows that *c* is also odd.

The two factors $z := a + bi$ and $z^* := a - bi$ of a primitive Pythagorean triple each equal the square of a Gaussian integer. This can be proved using the property that every Gaussian integer can be factored uniquely into Gaussian primes up to units.[20] (This unique factorization follows from the fact that, roughly speaking, a version of the Euclidean algorithm can be defined on them.) The proof has three steps. First, if *a* and *b* share no prime factors in the integers, then they also share no prime factors in the Gaussian integers. (Assume $a = gu$ and $b = gv$ with Gaussian integers *g*, *u* and *v* and *g* not a unit. Then *u* and *v* lie on the same line through the origin. All Gaussian integers on such a line are integer multiples of some Gaussian integer *h*. But then the integer $gh \neq \pm 1$ divides both *a* and *b*.) Second, it follows that *z* and *z** likewise share no prime factors in the Gaussian integers. For if they did, then their common divisor δ would also divide $z + z^* = 2a$ and $z - z^* = 2ib$. Since *a* and *b* are coprime, that implies that δ divides $2 = (1 + i)(1 - i) = i(1 - i)^2$. From the formula $c^2 = zz^*$, that in turn would imply that *c* is even, contrary to the hypothesis of a primitive Pythagorean triple. Third, since c^2 is a square, every Gaussian prime in its factorization is doubled, i.e., appears an even number of times. Since *z* and *z** share no prime factors, this doubling is also true for them. Hence, *z* and *z** are squares.

Thus, the first factor can be written

$$a + bi = \varepsilon \left(m + ni\right)^2, \quad \varepsilon \in \{\pm 1, \pm i\}.$$

The real and imaginary parts of this equation give the two formulas

$$\varepsilon = +1, \quad a = m^2 - n^2, \quad b = 2mn;$$

$$\varepsilon = -1, \quad a = -\left(m^2 - n^2\right), \quad b = -2mn;$$

$$\varepsilon = +i, \quad a = -2mn, \quad b = m^2 - n^2;$$

$$\varepsilon = -i, \quad a = 2mn, \quad b = -\left(m^2 - n^2\right).$$

For any primitive Pythagorean triple, there must be integers m and n such that these two equations are satisfied. Hence, every Pythagorean triple can be generated from some choice of these integers.

## 7.1 As perfect square Gaussian integers

If we consider the square of a Gaussian integer we get the following direct interpretation of Euclid's formulae as representing a perfect square Gaussian integers.

$$(m + ni)^2 = (m^2 - n^2) + 2mni.$$

Using the facts that the Gaussian integers are a Euclidean domain and that for a Gaussian integer p $|p|^2$ is always a square it is possible to show that a Pythagorean triples correspond to the square of a prime Gaussian integer if the hypotenuse is prime.

If the Gaussian integer is not prime then it is the product of two Gaussian integers p and q with $|p|^2$ and $|q|^2$ integers. Since magnitudes multiply in the Gaussian integers, the product must be $|p||q|$, which when squared to find a Pythagorean triple must be composite. The contrapositive completes the proof.

8 Distribution of triples
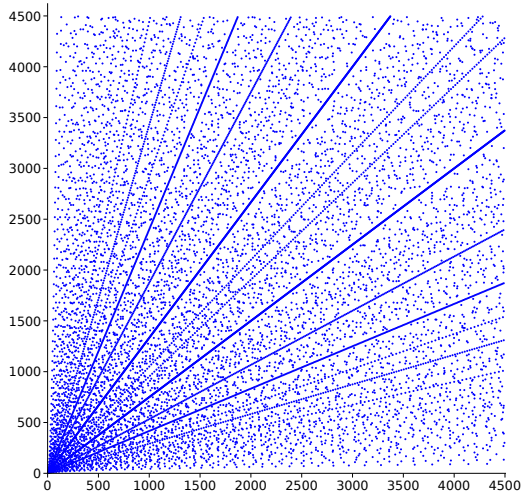
*A scatter plot of the legs (*a,b*) of the first Pythagorean triples with* a *and* b *less than 4500.*

There are a number of results on the distribution of Pythagorean triples. In the scatter plot, a number of obvious patterns are already apparent. Whenever the legs (*a,b*) of a primitive triple appear in the plot, all integer multiples of (*a,b*) must also appear in the plot, and this property produces the appearance of lines radiating from the origin in the diagram.

Within the scatter, there are sets of parabolic patterns with a high density of points and all their foci at the origin, opening up in all four directions. Different parabolas intersect at the axes and appear to reflect off the axis with an incidence angle of 45 degrees, with a third parabola entering in a perpendicular fashion. Within this quadrant, each arc centered around the origin shows that section of the parabola that lies between its tip and its intersection with its semi-latus rectum.

These patterns can be explained as follows. If $a^2/4n$ is an integer, then $(a, |n - a^2/4n| , n + a^2/4n)$ is a Pythagorean triple. (In fact every Pythagorean triple (*a*, *b*, *c*) can be written in this way with integer *n*, possibly after exchanging *a* and *b*, since $n = (b+c)/2$ and *a* and *b* cannot both be odd.) The Pythagorean triples thus lie on curves given by $b = |n - a^2/4n|$, that is, parabolas reflected at the *a*-axis, and the corresponding curves with *a* and *b* interchanged. If *a* is varied for a given *n* (i.e. on a given parabola), integer values of *b* occur relatively frequently if *n* is a square or a small multiple of a square. If several such values happen to lie close together, the corresponding parabolas approximately coincide, and the triples cluster in a narrow parabolic strip. For instance, $38^2 = 1444$, $2 \times 27^2 = 1458$, $3 \times 22^2 = 1452$, $5 \times 17^2 = 1445$ and $10 \times 12^2 = 1440$; the corresponding parabolic strip around $n \approx 1450$ is clearly visible in the scatter plot.

The angular properties described above follow immediately from the functional form of the parabolas. The parabolas are reflected at the *a*-axis at *a* = 2*n*, and the derivative of *b* with respect to *a* at this point is –1; hence the incidence angle is 45°. Since the clusters, like all triples, are repeated at integer multiples, the value 2*n* also corresponds to a cluster. The corresponding parabola intersects the *b*-axis at right angles at *b* = 2*n*, and hence its reflection upon interchange of *a* and *b* intersects the *a*-axis at right angles at *a* = 2*n*, precisely where the parabola for *n* is reflected at the *a*-axis. (The same is of course true for *a* and *b* interchanged.)

Albert Fässler and others provide insights into the significance of these parabolas in the context of conformal mappings.[21][22]

9 Special cases

9.1 The Platonic sequence

The case $n = 1$ of the more general construction of Pythagorean triples has been known for a long time. Proclus, in his commentary to the 47th Proposition of the first book of Euclid's Elements, describes it as follows:

> Certain methods for the discovery of triangles of this kind are handed down, one which they refer to Plato, and another to Pythagoras. (The latter) starts from odd numbers. For it

makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse.

...For the method of Plato argues from even numbers. It takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. ... Thus it has formed the same triangle that which was obtained by the other method.

In equation form, this becomes:

a is odd (Pythagoras, c. 540 BC):

$$\text{side}\,a : \ \text{side}\,b = \frac{a^2 - 1}{2} : \ \text{side}\,c = \frac{a^2 + 1}{2}.$$

a is even (Plato, c. 380 BC):

$$\text{side}\,a : \ \text{side}\,b = \left(\frac{a}{2}\right)^2 - 1 : \ \text{side}\,c = \left(\frac{a}{2}\right)^2 + 1$$

It can be shown that all Pythagorean triples are derivatives of the basic Platonic sequence $(x, y, z) = p$, $(p^2 - 1)/2$ and $(p^2 + 1)/2$ by allowing p to take non-integer rational values. If p is replaced with the rational fraction m/n in the sequence, the 'standard' triple generator $2mn$, $m^2 - n^2$ and $m^2 + n^2$ results. It follows that every triple has a corresponding rational p value which can be used to generate a similar (i.e. equiangular) triangle with rational sides in the same proportion as the original. For example, the Platonic equivalent of (6, 8, 10) is (3/2; 2, 5/2). The Platonic sequence itself can be derived by following the steps for 'splitting the square' described in Diophantus II.VIII.

9.2   The Jacobi-Madden equation

Main article: Jacobi-Madden equation

The equation,

$$a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$$

is equivalent to the special Pythagorean triple,

$$(a^2 + ab + b^2)^2 + (c^2 + cd + d^2)^2 = ((a+b)^2 + (a+b)(c+d) + (c+d)^2)^2$$

There is an infinite number of solutions to this equation as solving for the variables involves an elliptic curve. Small ones are,

$$a, b, c, d = -2634, 955, 1770, 5400$$

$$a, b, c, d = -31764, 7590, 27385, 48150$$

9.3   Equal sums of two fourth powers

Given two sets of Pythagorean triples,

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$$
$$(c^2 - d^2)^2 + (2cd)^2 = (c^2 + d^2)^2$$

the problem of finding equal products of a non-hypotenuse side and the hypotenuse,

$$(a^2 - b^2)(a^2 + b^2) = (c^2 - d^2)(c^2 + d^2)$$

is easily seen to be equivalent to the equation,

$$a^4 - b^4 = c^4 - d^4$$

and was first solved by Euler as $a, b, c, d = 133, 59, 158, 134$. Since he showed this is a rational point in an elliptic curve, then there is an infinite number of solutions. In fact, he also found a 7th degree polynomial parameterization.

9.4   Descartes' Circle Theorem

For the case of Descartes' circle theorem where all variables are squares,

$$2(a^4 + b^4 + c^4 + d^4) = (a^2 + b^2 + c^2 + d^2)^2$$

Euler showed this is equivalent to three simultaneous Pythagorean triples,

$$(2ab)^2 + (2cd)^2 = (a^2 + b^2 - c^2 - d^2)^2$$
$$(2ac)^2 + (2bd)^2 = (a^2 - b^2 + c^2 - d^2)^2$$
$$(2ad)^2 + (2bc)^2 = (a^2 - b^2 - c^2 + d^2)^2$$

There is also an infinite number of solutions, and for the special case when $a + b = c$, then the equation simplifies to,

$$4(a^2 + ab + b^2) = d^2$$

with small solutions as $a, b, c, d = 3, 5, 8, 14$ and can be solved as binary quadratic forms.

9.5   Almost-isosceles Pythagorean triples

These are right-angled triangles with integral sides for which the lengths of the non-hypotenuse edges differ by one, such as,

$$3^2 + 4^2 = 5^2$$

$$20^2 + 21^2 = 29^2$$

and an infinite number of others. They can be completely parameterized as,

$$\left(\tfrac{x-1}{2}\right)^2 + \left(\tfrac{x+1}{2}\right)^2 = y^2$$

where {*x*, *y*} are the solutions to the Pell equation $x^2 - 2y^2 = -1$.

When it is the longer non-hypotenuse leg and hypotenuse that differ by one, such as in

$$5^2 + 12^2 = 13^2$$

$$7^2 + 24^2 = 25^2$$

then the complete solution is

$$(2m+1)^2 + (2m^2+2m)^2 = (2m^2+2m+1)^2$$

which also shows that all odd numbers (greater than 1) appear in a primitive Pythagorean triple.

10   Generalizations

There are several ways to generalize the concept of Pythagorean triples.

10.1   Pythagorean quadruple

Main article: Pythagorean quadruple

A set of four positive integers *a*, *b*, *c* and *d* such that $a^2 + b^2 + c^2 = d^2$ is called a Pythagorean quadruple. The simplest example is (1, 2, 2, 3), since $1^2 + 2^2 + 2^2 = 3^2$. The next simplest (primitive) example is (2, 3, 6, 7), since $2^2 + 3^2 + 6^2 = 7^2$.

All quadruples are given by the formula

$$(m^2+n^2-p^2-q^2)^2+(2mq+2np)^2+(2nq-2mp)^2 = (m^2+n^2+p^2+q^2)^2.$$

10.2   Pythagorean *n*-tuple

Using the simple algebraic identity,

$$(x_1^2 - x_0)^2 + (2x_1)^2 x_0 = (x_1^2 + x_0)^2$$

for arbitrary x_0, x_1, it is easy to prove that the square of the sum of *n* squares is itself the sum of *n* squares by letting $x_0 = x_2{}^2 + x_3{}^2 + ... + xn^2$ and then distributing terms.[23] One can see how Pythagorean triples and quadruples are just the particular cases $x_0 = x_2{}^2$ and $x_0 = x_2{}^2 + x_3{}^2$, respectively, and so on for other *n*, with quintuples given by

$$(a^2-b^2-c^2-d^2)^2+(2ab)^2+(2ac)^2+(2ad)^2 = (a^2+b^2+c^2+d^2)^2.$$

Since the sum $F(k,m)$ of *k* consecutive squares beginning with m^2 is given by the formula,[24]

$$F(k,m) = km(k-1+m) + \frac{k(k-1)(2k-1)}{6}$$

one may find values (*k*, *m*) so that $F(k,m)$ is a square, such as one by Hirschhorn where the number of terms is itself a square,[25]

$$m = \tfrac{v^4-24v^2-25}{48}, \; k = v^2, \; F(m,k) = \tfrac{v^5+47v}{48}$$

and *v* ≥ 5 is any integer not divisible by 2 or 3. For the smallest case *v* = 5, hence *k* = 25, this yields the well-known cannonball-stacking problem of Lucas,

$$0^2 + 1^2 + 2^2 + \cdots + 24^2 = 70^2$$

a fact which is connected to the Leech lattice.

In addition, if in a Pythagorean *n*-tuple (*n* ≥ 4) all addends are consecutive except one, one can use the equation,[26]

$$F(k,m) + p^2 = (p+1)^2$$

Since the 2nd power of *p* cancels out, this is only linear and easily solved for as $p = \tfrac{F(k,m)-1}{2}$ though *k*, *m* should be chosen so that *p* is an integer, with a small example being *k* = 5, *m* = 1 yielding,

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 27^2 = 28^2$$

10.3   Fermat's Last Theorem

Main article: Fermat's Last Theorem

A generalization of the concept of Pythagorean triples is the search for triples of positive integers a, b, and c, such that $a^n + b^n = c^n$, for some n strictly greater than 2. Pierre de Fermat in 1637 claimed that no such triple exists, a claim that came to be known as Fermat's Last Theorem because it took longer than any other conjecture by Fermat to be proven or disproven. The first proof was given by Andrew Wiles in 1994.

10.4 $n - 1$ or n n^{th} powers summing to an nth power

Main article: Euler's sum of powers conjecture

Another generalization is searching for sequences of $n + 1$ positive integers for which the nth power of the last is the sum of the nth powers of the previous terms. The smallest sequences for known values of n are:

- $n = 3$: {3, 4, 5; 6}.

- $n = 4$: {30, 120, 272, 315; 353}

- $n = 5$: {19, 43, 46, 47, 67; 72}

- $n = 7$: {127, 258, 266, 413, 430, 439, 525; 568}

- $n = 8$: {90, 223, 478, 524, 748, 1088, 1190, 1324; 1409}

A slightly different generalization allows the sum of $(k + 1)$ nth powers to equal the sum of $(n - k)$ nth powers. For example:

- ($n = 3$): $1^3 + 12^3 = 9^3 + 10^3$, made famous by Hardy's recollection of a conversation with Ramanujan about the number 1729 being the smallest number that can be expressed as a sum of two cubes in two distinct ways.

There can also exist $n - 1$ positive integers whose n^{th} powers sum to an nth power (though, by Fermat's last theorem, not for $n = 3$); these are counterexamples to Euler's sum of powers conjecture. The smallest known counterexamples are[27][28][11]

- $n = 4$: (95800, 217519, 414560; 422481)

- $n = 5$: (27, 84, 110, 133; 144)

10.5   Heronian triangle triples

Main article: Heronian triangle

A **Heronian triangle** is commonly defined as one with integer sides whose area is also an integer, and we shall

consider Heronian triangles with *distinct* integer sides. The lengths of the sides of such a triangle form a **Heronian triple** (a, b, c) provided $a < b < c$. Clearly, any Pythagorean triple is a Heronian triple, since in a Pythagorean triple at least one of the legs a, b must be even, so that the area $ab/2$ is an integer. Not every Heronian triple is a Pythagorean triple, however, as the example (4, 13, 15) with area 24 shows.

If (a, b, c) is a Heronian triple, so is (ma, mb, mc) where m is any positive integer greater than one. The Heronian triple (a, b, c) is **primitive** provided a, b, c are pairwise relatively prime (as with a Pythagorean triple). Here are a few of the simplest primitive Heronian triples that are not Pythagorean triples:

(4, 13, 15) with area 24

(3, 25, 26) with area 36

(7, 15, 20) with area 42

(6, 25, 29) with area 60

(11, 13, 20) with area 66

(13, 14, 15) with area 84

(13, 20, 21) with area 126

By Heron's formula, the extra condition for a triple of positive integers (a, b, c) with $a < b < c$ to be Heronian is that

$$(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)$$

or equivalently

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

be a nonzero perfect square divisible by 16.

10.6   Application to cryptography

Primitive Pythagorean triples have been used in cryptography as random sequences and for the generation of keys.[29]

11   See also

- Diophantus II.VIII

- Eisenstein triple

- Euler brick

- Heronian triangle

- Hilbert's theorem 90

- Integer triangle

- Modular arithmetic

- Nonhypotenuse number

- Plimpton 322

- Pythagorean prime

- Pythagorean quadruple

- Tangent half-angle formula

- Trigonometric identity

12   Notes

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14   External links

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- Pythagorean Triples at cut-the-knot Interactive Applet showing unit circle relationships to Pythagorean Triples

- The Trinary Tree(s) underlying Primitive Pythagorean Triples at cut-the-knot

- Theoretical properties of the Pythagorean Triples and connections to geometry

- Clifford Algebras and Euclid's Parameterization of Pythagorean triples

- Pythagorean Triplets

- Discussion of Properties of Pythagorean triples, Interactive Calculators, Puzzles and Problems

- Generating Pythagorean Triples Using Arithmetic Progressions

- Parameterization of Pythagorean Triples by a single triple of polynomials

- Curious Consequences of a Miscopied Quadratic

- Solutions to Quadratic Compatible Pairs in relation to Pythagorean Triples

- The negative Pell equation and Pythagorean triples

- The Remarkable Incircle of a Triangle

- Interactive Calculator for Pythagorean Triples

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15.1 Text

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