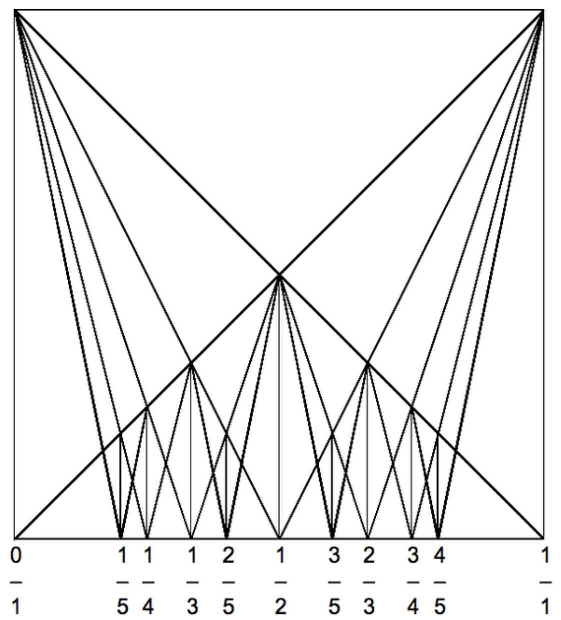
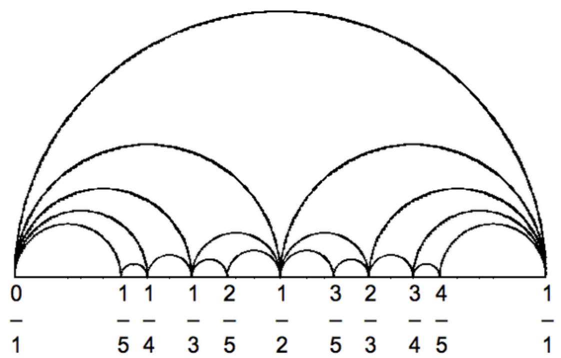
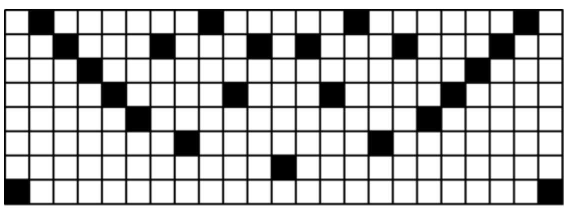
Farey sequence



Farey diagram to F_5.



Farey diagram to F_5.



Symmetrical pattern made by the denominators of the Farey sequence, F_8.

In mathematics, the **Farey sequence** of order n is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to n, arranged in order of increasing size.

Each Farey sequence starts with the value 0, denoted by the fraction ${}^0\!/_1$, and ends with the value 1, denoted by the fraction ${}^1\!/_1$ (although some authors omit these terms).

A Farey sequence is sometimes called a Farey series, which is not strictly correct, because the terms are not summed.

1 Examples

The Farey sequences of orders 1 to 8 are :

F_1 = { 0/1, 1/1 }

F_2 = { 0/1, 1/2, 1/1 }

F_3 = { 0/1, 1/3, 1/2, 2/3, 1/1 }

F_4 = { 0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1 }

F_5 = { 0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1 }

F_6 = { 0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1 }

F_7 = { 0/1, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1/1 }

F_8 = { 0/1, 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8, 1/1 }

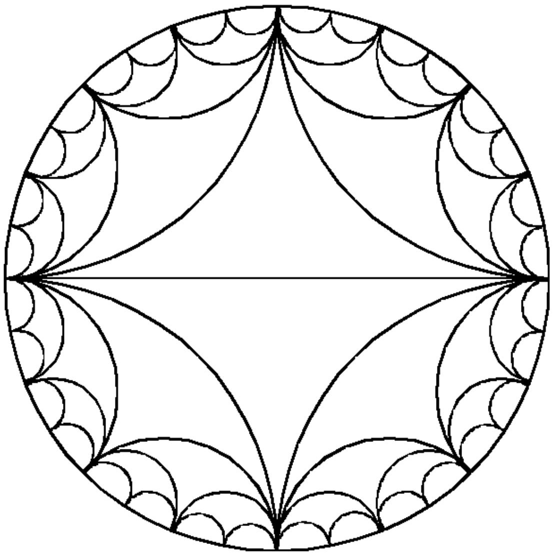
2 History

> *The history of 'Farey series' is very curious* — Hardy & Wright (1979) Chapter III[1]

> *... once again the man whose name was given to a mathematical relation was not the original discoverer so far as the records go.* — Beiler (1964) Chapter XVI[2]

Farey sequences are named after the British geologist John Farey, Sr., whose letter about these sequences was published in the *Philosophical Magazine* in 1816. Farey conjectured, without offering proof, that each new term in a Farey sequence expansion is the mediant of its neighbours. Farey's letter was read by Cauchy, who provided a proof in his *Exercices de mathématique*, and attributed this result to Farey. In fact, another mathematician, Charles Haros, had published similar results in 1802 which were not known either to Farey or to Cauchy.[2] Thus it was a historical accident that linked Farey's name with these sequences. This is an example of Stigler's law of eponymy.

3 Properties



3.1 Sequence length and index of a fraction

The Farey sequence of order *n* contains all of the members of the Farey sequences of lower orders. In particular *Fn* contains all of the members of Fn_{-1}, and also contains an additional fraction for each number that is less than *n* and coprime to *n*. Thus F_6 consists of F_5 together with the fractions 1/6 and 5/6.

The middle term of a Farey sequence *Fn* is always 1/2, for *n* > 1.

From this, we can relate the lengths of *Fn* and Fn_{-1} using Euler's totient function $\varphi(n)$:

$$|F_n| = |F_{n-1}| + \varphi(n).$$

Using the fact that $|F_1| = 2$, we can derive an expression for the length of *Fn* :

$$|F_n| = 1 + \sum_{m=1}^{n} \varphi(m).$$

We also have :

$$|F_n| = \frac{1}{2}\left(3 + \sum_{d=1}^{n} \mu(d) \left\lfloor \tfrac{n}{d} \right\rfloor^2\right),$$

and by a Möbius inversion formula :

$$|F_n| = \frac{1}{2}(n+3)n - \sum_{d=2}^{n} |F_{\lfloor n/d \rfloor}|,$$

where μ(*d*) is the number-theoretic Möbius function, and $\lfloor \frac{n}{d} \rfloor$ is the floor function.

The asymptotic behaviour of |*Fn*| is :

$$|F_n| \sim \frac{3n^2}{\pi^2}.$$

The index $I_n(a_{k,n}) = k$ of a fraction $a_{k,n}$ in the Farey sequence $F_n = \{a_{k,n} : k = 0, 1, \ldots, m_n\}$ is simply the position that $a_{k,n}$ occupies in the sequence. This is of special relevance as it is used in an alternative formulation of the Riemann hypothesis, see below. Various useful properties follow:

$$I_n(0/1) = 0,$$

$$I_n(1/n) = 1,$$
$$I_n(1/2) = (|F_n| - 1)/2,$$
$$I_n(1/1) = |F_n| - 1,$$
$$I_n(h/k) = |F_n| - 1 - I_n((k-h)/k).$$

3.2 Farey neighbours

Fractions which are neighbouring terms in any Farey sequence are known as a *Farey pair* and have the following properties.

If *a*/*b* and *c*/*d* are neighbours in a Farey sequence, with *a*/*b* < *c*/*d*, then their difference *c*/*d* − *a*/*b* is equal to 1/*bd*. Since

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd},$$

this is equivalent to saying that

bc − *ad* = 1.

Thus 1/3 and 2/5 are neighbours in F_5, and their difference is 1/15.

The converse is also true. If

bc − *ad* = 1

for positive integers *a*,*b*,*c* and *d* with *a* < *b* and *c* < *d* then *a*/*b* and *c*/*d* will be neighbours in the Farey sequence of order max(*b*,*d*).

If *p*/*q* has neighbours *a*/*b* and *c*/*d* in some Farey sequence, with

$a/b < p/q < c/d$

then p/q is the mediant of a/b and c/d — in other words,

$$\frac{p}{q} = \frac{a+c}{b+d}.$$

This follows easily from the previous property, since if bp-$aq = qc$-$pd = 1$, then $bp+pd = qc+aq$, $p(b+d)=q(a+c)$, $p/q = a+c/b+d$

It follows that if a/b and c/d are neighbours in a Farey sequence then the first term that appears between them as the order of the Farey sequence is increased is

$$\frac{a+c}{b+d},$$

which first appears in the Farey sequence of order $b + d$.

Thus the first term to appear between 1/3 and 2/5 is 3/8, which appears in F_8.

The *Stern-Brocot tree* is a data structure showing how the sequence is built up from 0 (= 0/1) and 1 (= 1/1), by taking successive mediants.

Fractions that appear as neighbours in a Farey sequence have closely related continued fraction expansions. Every fraction has two continued fraction expansions — in one the final term is 1; in the other the final term is greater than 1. If p/q, which first appears in Farey sequence Fq, has continued fraction expansions

$[0; a_1, a_2, \ldots, an - 1, an, 1]$
$[0; a_1, a_2, \ldots, an - 1, an + 1]$

then the nearest neighbour of p/q in Fq (which will be its neighbour with the larger denominator) has a continued fraction expansion

$[0; a_1, a_2, \ldots, an]$

and its other neighbour has a continued fraction expansion

$[0; a_1, a_2, \ldots, an - 1]$

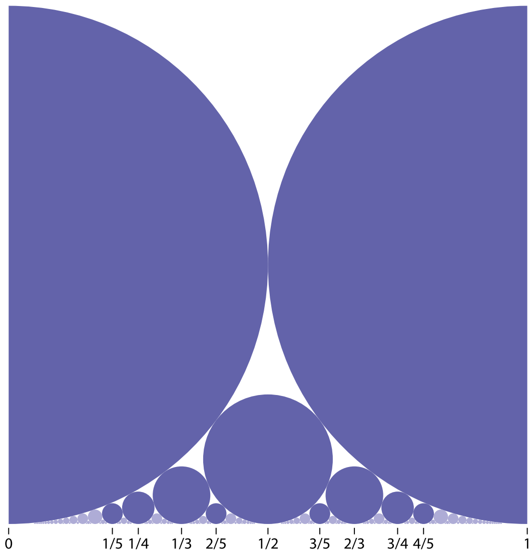
Thus 3/8 has the two continued fraction expansions [0; 2, 1, 1, 1] and [0; 2, 1, 2], and its neighbours in F_8 are 2/5, which can be expanded as [0; 2, 1, 1]; and 1/3, which can be expanded as [0; 2, 1].

3.3 Applications

Farey sequences are very useful to find rational approximations of irrational numbers .

In physics systems featuring resonance phenomena Farey sequences provide a very elegant and efficient method to compute resonance locations in 1D [3] and 2D [4]

3.4 Ford circles



Ford circles.

There is a connection between Farey sequence and Ford circles.

For every fraction p/q (in its lowest terms) there is a Ford circle C[p/q], which is the circle with radius $1/(2q^2)$ and centre at $(p/q,\ 1/(2q^2))$. Two Ford circles for different fractions are either disjoint or they are tangent to one another—two Ford circles never intersect. If $0 < p/q < 1$ then the Ford circles that are tangent to C[p/q] are precisely the Ford circles for fractions that are neighbours of p/q in some Farey sequence.

Thus $C[2/5]$ is tangent to $C[1/2]$, $C[1/3]$, $C[3/7]$, $C[3/8]$ etc.

3.5 Riemann hypothesis

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_n are $\{a_{k,n} : k = 0, 1, \ldots, m_n\}$. Define $d_{k,n} = a_{k,n} - k/m_n$, in other words $d_{k,n}$ is the difference between the kth term of the nth Farey sequence, and the kth member of a set of the same number of points, distributed evenly on the unit interval. In 1924 Jérôme Franel[5] proved that the statement

$$\sum_{k=1}^{m_n} d_{k,n}^2 = \mathcal{O}(n^r) \quad \forall r > -1$$

is equivalent to the Riemann hypothesis, and then Edmund Landau[6] remarked (just after Franel's paper) that the statement

$$\sum_{k=1}^{m_n} |d_{k,n}| = \mathcal{O}(n^r) \quad \forall r > 1/2$$

is also equivalent to the Riemann hypothesis.

4 Next term

A surprisingly simple algorithm exists to generate the terms of *Fn* in either traditional order (ascending) or non-traditional order (descending). The algorithm computes each successive entry in terms of the previous two entries using the mediant property given above. If a/b and c/d are the two given entries, and p/q is the unknown next entry, then $c/d = a + p/b + q$. Since c/d is in lowest terms, there must be an integer k such that $kc = a + p$ and $kd = b + q$, giving $p = kc - a$ and $q = kd - b$. If we consider p and q to be functions of k, then

$$\frac{p(k)}{q(k)} - \frac{c}{d} = \frac{cb - da}{d(kd - b)}$$

so the larger k gets, the closer p/q gets to c/d.

To give the next term in the sequence k must be as large as possible, subject to $kd - b \le n$ (as we are only considering numbers with denominators not greater than n), so k is the greatest integer $\le n + b/d$. Putting this value of k back into the equations for p and q gives

$$p = \left\lfloor \frac{n+b}{d} \right\rfloor c - a$$

$$q = \left\lfloor \frac{n+b}{d} \right\rfloor d - b$$

This is implemented in Python as:

def farey(n, asc=True): """Python function to print the nth Farey sequence, either ascending or descending.""" if asc: a, b, c, d = 0, 1, 1 , n # (*) else: a, b, c, d = 1, 1, n-1 , n # (*) print "%d/%d" % (a,b) while (asc and c <= n) or (not asc and a > 0): k = int((n + b)/d) a, b, c, d = c, d, k*c - a, k*d - b print "%d/%d" % (a,b)

Brute-force searches for solutions to Diophantine equations in rationals can often take advantage of the Farey series (to search only reduced forms). The lines marked (*) can also be modified to include any two adjacent terms so as to generate terms only larger (or smaller) than a given term.[7]

5 See also

- Stern-Brocot tree

6 References

[1] Hardy, G.H. & Wright, E.M. (1979) *An Introduction to the Theory of Numbers* (Fifth Edition). Oxford University Press. ISBN 0-19-853171-0

[2] Beiler, Albert H. (1964) *Recreations in the Theory of Numbers* (Second Edition). Dover. ISBN 0-486-21096-0. Cited in Farey Series, A Story at Cut-the-Knot

[3] A. Zhenhua Li, W.G. Harter, "Quantum Revivals of Morse Oscillators and Farey-Ford Geometry", arXiv: 1308.4470v1

[4] http://prst-ab.aps.org/abstract/PRSTAB/v17/i1/e014001

[5] "Les suites de Farey et le théorème des nombres premiers", Gött. Nachr. 1924, 198-201

[6] "Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel", Gött. Nachr. 1924, 202-206

[7] Norman Routledge, "Computing Farey Series," *The Mathematical Gazette*, Vol. **92** (No. 523), 55–62 (March 2008).

7 Further reading

- Allen Hatcher, Topology of Numbers
- Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition (Addison-Wesley, Boston, 1989); in particular, Sec. 4.5 (pp. 115–123), Bonus Problem 4.61 (pp. 150, 523–524), Sec. 4.9 (pp. 133–139), Sec. 9.3, Problem 9.3.6 (pp. 462–463). ISBN 0-201-55802-5.
- Linas Vepstas. *The Minkowski Question Mark, GL(2,Z), and the Modular Group.* http://linas.org/math/chap-minkowski.pdf reviews the isomorphisms of the Stern-Brocot Tree.
- Linas Vepstas. *Symmetries of Period-Doubling Maps.* http://linas.org/math/chap-takagi.pdf reviews connections between Farey Fractions and Fractals.
- Scott B. Guthery, *A Motif of Mathematics: History and Application of the Mediant and the Farey Sequence*, (Docent Press, Boston, 2010). ISBN 1-4538-1057-9.
- Cristian Cobeli and Alexandru Zaharescu, *The Haros-Farey Sequence at Two Hundred Years. A Survey*, Acta Univ. Apulensis Math. Inform. no. 5 (2003) 1–38, pp. 1–20 pp. 21–38
- A.O. Matveev, *A Note on Boolean Lattices and Farey Sequences II*, Integers 8(1), 2008, #A24
- A.O. Matveev, *Neighboring Fractions in Farey Subsequences*, arXiv:0801.1981

8 External links

- Alexander Bogomolny. Farey series and Stern-Brocot Tree at Cut-the-Knot
- Hazewinkel, Michiel, ed. (2001), "Farey series", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Weisstein, Eric W., "Stern-Brocot Tree", *MathWorld*.
- Farey Sequence from The On-Line Encyclopedia of Integer Sequences.

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