Constant Acceleration Model in 3D space(9-dim)

$$x_{n+1} = Fx_n + u_n, \quad x_n = [x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}, z, \dot{z}, \ddot{z}]'$$

where the transistion matrix is

$$F = \begin{bmatrix} F_2 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

u_n is zero mean Gaussian with covariance

$$Q = \begin{bmatrix} u_2 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} T^4/4 & T^3/2 & T^2/2 \\ T^3/2 & T^2 & T \\ T^2 & T & 1 \end{bmatrix} q_2$$

Measurement Updata, we take the form from Li's book

$$z_n = h(x_n) + v_n$$

where

$$h = \begin{bmatrix} r \\ a1 \\ a2 \\ e \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ arctan(y/x) \\ arctan(y/x) \\ arctan(z/\sqrt{x^2 + y^2}) \end{bmatrix}$$

v_n is zero mean Gaussian with covariance

$$Q = diag(\sigma_r^2, \sigma_{a1}^2, \sigma_{a2}^2, \sigma_e^2)$$

Parameters Setting

$T = 1, q_2 = 1, (\sigma_r, \sigma_{a1}, \sigma_{a2}, \sigma_e) = (20, 7, 2, 2).$

In [1]:

```python
#---------------------------------------
# library loading
#---------------------------------------
import time
import numpy as np
import pandas as pd
import tensorflow as tf
from tensorflow import keras

#set suppress to not use scientific counting
np.set_printoptions(suppress=True)

#-----------------------------------------
# Initialization Math Model
#-----------------------------------------
cpu_start=time.perf_counter()

# q1: variance of the process noise modeling the acceleration
T=1; q2=1
# qa1, qa2: variance for azimuth from sensor 1,2 resp.
# qr, qe: variance for range and elevation resp.
qr=20; qa1=7; qa2=2;qe=2

dimX=9; dimY=4
N=1000; n0=50; N_sample= 1000

#Deterministic Matrix
F0=np.array([[1,T,T*T/2,0,0,0,0,0,0],
             [0,1,T,0,0,0,0,0,0],
             [0,0,1,0,0,0,0,0,0],
             [0,0,0,1,T,T*T/2,0,0,0],
             [0,0,0,0,1,T,0,0,0],
             [0,0,0,0,0,1,0,0,0],
             [0,0,0,0,0,0,1,T,T*T/2],
             [0,0,0,0,0,0,0,1,T],
             [0,0,0,0,0,0,0,0,1]])

x0=np.array([[21.68],
             [-0.3],
             [0],
             [10.8],
             [-3.9],
             [0],
             [0.4],
             [0],
             [0]])

#Covariance Matrix for random variable
#random variable u_n
Q0=np.array([[np.power(T,4)*q2/4,np.power(T,3)*q2/2,np.power(T,2)*q2/2,0,0,0,0,0,0],
             [np.power(T,3)*q2/2,np.power(T,2)*q2,T*q2,0,0,0,0,0,0],
             [np.power(T,2)*q2/2,T*q2,q2,0,0,0,0,0,0],
             [0,0,0,np.power(T,4)*q2/4,np.power(T,3)*q2/2,np.power(T,2)*q2/2,0,0,0],
             [0,0,0,np.power(T,3)*q2/2,np.power(T,2)*q2,T*q2,0,0,0],
             [0,0,0,np.power(T,2)*q2/2,T*q2,q2,0,0,0],
             [0,0,0,0,0,0,np.power(T,4)*q2/4,np.power(T,3)*q2/2,np.power(T,2)*q2/2],
```

```python
                 [0,0,0,0,0,0,np.power(T,3)*q2/2,np.power(T,2)*q2,T*q2],
                 [0,0,0,0,0,0,np.power(T,2)*q2/2,T*q2,q2]])

#random variable v_n/w_n
R0=np.diag((qr*qr,qa1*qa1,qa2*qa2,qe*qe))

# generate u_n, v_n
# 1-d Gaussian: np.random.default_rng().normal(mean, std, size)
# n-d Gaussian: np.random.default_rng().multivariate_normal(mean,cov,size)
# note to reshape multivariate normal random variable to column vector.
rng=np.random.default_rng()
u=[rng.multivariate_normal(np.zeros(dimX),Q0,1).reshape(dimX,1) for i in range(N)] #!!!
v=[rng.multivariate_normal(np.zeros(dimY),R0,1).reshape(dimY,1) for i in range(N+1)]
#!!!
u=np.array(u)
v=np.array(v)

#------------------------------
# Function Definition
#------------------------------
def f(x):
    return F0@x
def g(x):
    return np.eye(len(x))
def h(x):
    """
    input x is a 6-dim col vector,
    return a 4-dim col vector."""
    res=np.zeros((dimY,1))
    res[0]=np.sqrt(np.square(x[0])+np.square(x[3])+np.square(x[6]))
    res[1]=np.arctan(x[3]/x[0])
    res[2]=np.arctan(x[3]/x[0])
    res[3]=np.arctan(x[6]/np.sqrt(np.square(x[0])+np.square(x[3])))
    return res
def F(x):
    """Derivative of f """
    return F0
def G(x):
    """Derivative of g """
    return np.eye(dimX)
def H(x):
    """
    Derivative of h
    input a 6-dim col vector, return 4x6 matrix."""
    res=np.zeros((dimY,dimX))
    res[0][0]=x[0]/np.sqrt(x[0]*x[0]+x[3]*x[3]+x[6]*x[6])
    res[0][3]=x[3]/np.sqrt(x[0]*x[0]+x[3]*x[3]+x[6]*x[6])
    res[0][6]=x[6]/np.sqrt(x[0]*x[0]+x[3]*x[3]+x[6]*x[6])

    res[1][0]=-x[3]/(x[0]*x[0]+x[3]*x[3])
    res[1][3]=x[0]/(x[0]*x[0]+x[3]*x[3])

    res[2][0]=-x[3]/(x[0]*x[0]+x[3]*x[3])
    res[2][3]=x[0]/(x[0]*x[0]+x[3]*x[3])

    res[3][0]=-x[0]*x[6]/(np.sqrt(x[0]*x[0]+x[3]*x[3])*(x[0]*x[0]+x[3]*x[3]+x[6]*x[6]))
    res[3][3]=-x[3]*x[6]/(np.sqrt(x[0]*x[0]+x[3]*x[3])*(x[0]*x[0]+x[3]*x[3]+x[6]*x[6]))
```

```
    res[3][6]=np.sqrt(x[0]*x[0]+x[3]*x[3])/(x[0]*x[0]+x[3]*x[3]+x[6]*x[6])

    return res

#--------------------------------------
# Extended KF Monte Carlo
#--------------------------------------
def ekf_mc():
    x_raw=np.zeros((N+1,dimX,1)); x_raw[0]=x0
    y_raw=np.zeros((N+1,dimY,1))
    y_raw[0]=h(x_raw[0])+v[0]

    for k in range(N):
        x_raw[k+1]=F0@x_raw[k]+u[k]  #!!! here is u[k]
        y_raw[k+1]=h(x_raw[k+1])+v[k+1]

    return x_raw,y_raw

#--------------------------------------
# Extended Kalman Filter Algorithm
#--------------------------------------

def extended_kf(f,g,h,F,G,H,Q0,R0,x0,y_raw):
    """
    f,g,h,F,G,H are all functions.
    Q0: covariance matrix of u_n
    R0: covariance matrix of v_n """
    x_hat=np.zeros((N+1,dimX,1)); x_hat[0]=x0
    R=np.zeros((N+1,dimX,dimX)); R[0]=np.eye(dimX) #!!!!!!
    x_bar=np.zeros((N+1,dimX,1))
    x_bar[0]=x_hat[0]+R[0]@H(x_hat[0]).T@np.linalg.inv(H(x_hat[0])@R[0]@H(x_hat[0]).T+R
0)@(y_raw[0]-h(x_hat[0]))

    for k in range(N):
        x_hat[k+1]=f(x_bar[k])
        inv_pre=np.linalg.inv(H(x_hat[k])@R[k]@H(x_hat[k]).T+R0)
        R[k+1]=F(x_bar[k])@(R[k]-R[k]@H(x_hat[k]).T@inv_pre@H(x_hat[k])@R[k])@F(x_bar[k
]).T+G(x_bar[k])@Q0@G(x_bar[k]).T
        inv_pos=np.linalg.inv(H(x_hat[k+1])@R[k+1]@H(x_hat[k+1]).T+R0)
        x_bar[k+1]=x_hat[k+1]+R[k+1]@H(x_hat[k+1]).T@inv_pos@(y_raw[k+1]-h(x_hat[k+1]))

    return x_hat,x_bar


#-------------------------------------------------------------------------------------
---
# Generating tons of samples
#-------------------------------------------------------------------------------------
---

def sample_generator():

    datas=np.zeros(((N-n0+2)*N_sample,n0,dimY)) #for each sample path, we have N-n0+2 d
ata
    labels=np.zeros(((N-n0+2)*N_sample,dimX))

    x_bars=np.zeros(((N-n0+2)*N_sample,dimX)) #store Kalman filtering estimation value.
```

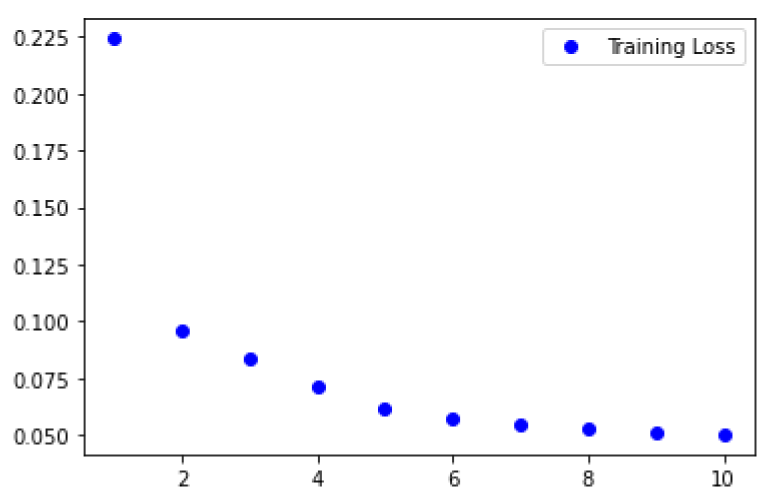
```
    x_hats=np.zeros(((N-n0+2)*N_sample,dimX))

    x_raws=np.zeros((N_sample, N+1, dimX, 1))
    y_raws=np.zeros((N_sample, N+1, dimY, 1))

    for i in range(N_sample):
        data=np.zeros((N-n0+2,n0,dimY)) #store data for each sample
        label=np.zeros((N-n0+2,dimX))
        # call ekf_mc function to generate sample
        x_raw,y_raw=ekf_mc()
        x_raws[i]=x_raw; y_raws[i]=y_raw

        # call extended_kf function to compute estimation
        # make sure here y_raw to be column vector
        x_hat, x_bar=extended_kf(f,g,h,F,G,H,Q0,R0,x0,y_raw)

        #---- convert x_raw, y_raw, x_hat, x_bar into row vector for each element------
---
        #----------------------------------------------------------------------------
---
        x_raw=x_raw.reshape(N+1,dimX)
        y_raw=y_raw.reshape(N+1,dimY)
        x_hat=x_hat.reshape(N+1,dimX)
        x_bar=x_bar.reshape(N+1,dimX)
        # make data and label for each sample
        for k in range(N-n0+2):
            data[k]=y_raw[k:k+n0]
            label[k]=x_raw[k+n0-1]

        # put data and label into datas and labels with i representing sample number
        datas[i*(N-n0+2):(i+1)*(N-n0+2)]=data
        labels[i*(N-n0+2):(i+1)*(N-n0+2)]=label
        x_hats[i*(N-n0+2):(i+1)*(N-n0+2)]=x_hat[n0-1:]
        x_bars[i*(N-n0+2):(i+1)*(N-n0+2)]=x_bar[n0-1:]

    return datas,labels,x_hats,x_bars,x_raws,y_raws

#------------------------------
# Data Preparation
#------------------------------
# call sample_generator function to generate sample
datas, labels, x_hats,x_bars,x_raws, y_raws=sample_generator()
datas=datas.reshape(((N-n0+2)*N_sample,dimY*n0))
# convert numpy array into pandas dataframe
datas=pd.DataFrame(datas)
labels=pd.DataFrame(labels)
x_hats=pd.DataFrame(x_hats)
x_bars=pd.DataFrame(x_bars)
```

In [2]:

```python
#### Data Normalization/Scaling

from sklearn.model_selection import train_test_split
seed=3
np.random.seed(seed)
training_data, test_data, training_label, test_label=train_test_split(datas,labels, tes
t_size=0.2, random_state=seed)

# Input normalization
data_mean=training_data.mean(axis=0)
data_std=training_data.std(axis=0)
training_data=(training_data-data_mean)/data_std
test_data=(test_data-data_mean)/data_std

# Output normalization
label_mean=training_label.mean(axis=0)
label_std=training_label.std(axis=0)
training_label=(training_label-label_mean)/label_std
test_label=(test_label-label_mean)/label_std

#-------------------------------
# Model building
#-------------------------------

from keras import models
from keras import layers
from keras import optimizers

def build_model():
    model=models.Sequential()
    model.add(layers.Dense(5,activation='relu',input_shape=(dimY*n0,)))
    model.add(layers.Dense(5,activation='relu'))
    model.add(layers.Dense(5,activation='relu'))
    model.add(layers.Dense(5,activation='relu'))
    model.add(layers.Dense(5,activation='relu'))
    model.add(layers.Dense(dimX))

    model.compile(optimizer=optimizers.SGD(lr=0.001),
                  loss='mean_squared_error',
                  metrics=[tf.keras.metrics.MeanSquaredError()])
    return model

model=build_model()
mymodel=model.fit(training_data,training_label, epochs=10, batch_size=16)

#-------------------------------
# Evaluation Performance
#-------------------------------

from sklearn.metrics import mean_squared_error

test_mse_score, test_mae_score=model.evaluate(test_data,test_label)

index=test_label.index.tolist()
kf_mse_err=mean_squared_error(x_bars.iloc[index],labels.iloc[index])
```

```python
cpu_end=time.perf_counter()

print("The mse of deep filtering is {:.3%}".format(test_mse_score))
print("The mse of Kalman Filtering is {:.3%}".format(kf_mse_err))
print("The CPU consuming time is {:.5}".format(cpu_end-cpu_start))


history_dict=mymodel.history
history_dict.keys()

loss_value=history_dict['loss']
#val_loss_value=history_dict['val_loss']
epochs=range(1,10+1)
import matplotlib.pyplot as plt
plt.plot(epochs, loss_value, 'bo',label='Training Loss')
#plt.plot(epochs, val_loss_value,'b',label='Validation Loss')
plt.legend()
plt.show()
```

```
Epoch 1/10
47600/47600 [==============================] - 19s 396us/step - loss: 0.2242 - mean_s
quared_error: 0.2242
Epoch 2/10
47600/47600 [==============================] - 19s 397us/step - loss: 0.0961 - mean_s
quared_error: 0.0961
Epoch 3/10
47600/47600 [==============================] - 19s 389us/step - loss: 0.0837 - mean_s
quared_error: 0.0837
Epoch 4/10
47600/47600 [==============================] - 19s 395us/step - loss: 0.0714 - mean_s
quared_error: 0.0714
Epoch 5/10
47600/47600 [==============================] - 19s 392us/step - loss: 0.0617 - mean_s
quared_error: 0.0617
Epoch 6/10
47600/47600 [==============================] - 19s 392us/step - loss: 0.0570 - mean_s
quared_error: 0.0570
Epoch 7/10
47600/47600 [==============================] - 19s 390us/step - loss: 0.0545 - mean_s
quared_error: 0.0545
Epoch 8/10
47600/47600 [==============================] - 19s 390us/step - loss: 0.0528 - mean_s
quared_error: 0.0528
Epoch 9/10
47600/47600 [==============================] - 19s 389us/step - loss: 0.0515 - mean_s
quared_error: 0.0515
Epoch 10/10
47600/47600 [==============================] - 19s 389us/step - loss: 0.0503 - mean_s
quared_error: 0.0503
5950/5950 [==============================] - 2s 344us/step - loss: 0.0497 - mean_squa
red_error: 0.0497
The mse of deep filtering is 4.968%
The mse of Kalman Filtering is 483896195876064.875%
The CPU consuming time is 619.08
```

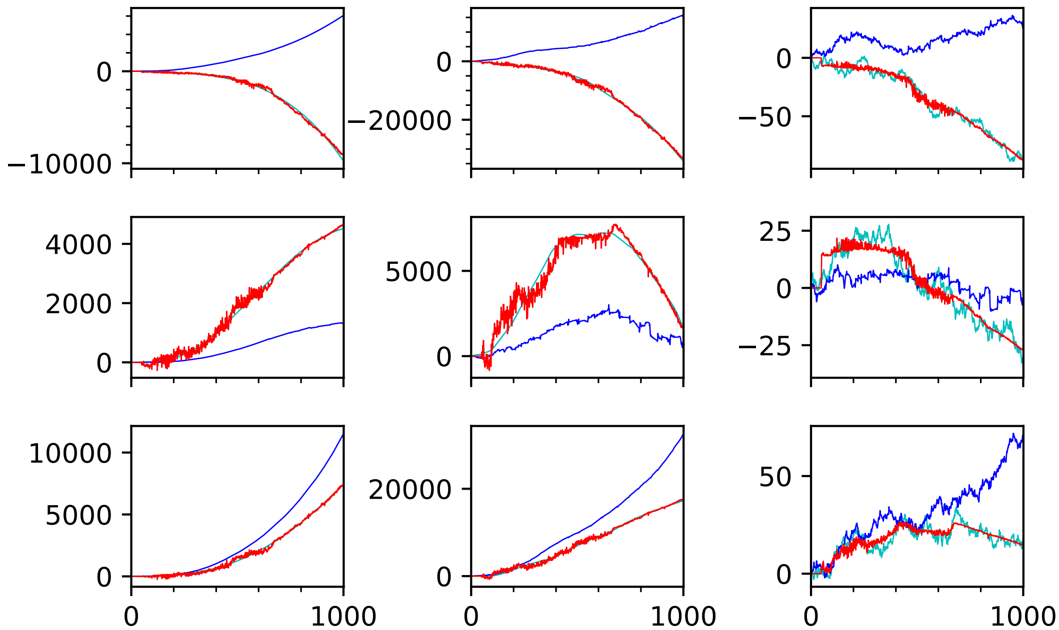


In [3]:

```
#--------------------------------------------------------------
# plot on new data
#--------------------------------------------------------------
x_new, y_new=ekf_mc()
x_hat_new, x_bar_new=extended_kf(f,g,h,F,G,H,Q0,R0,x0,y_new)
y_new=y_new.reshape(N+1,dimY)
data_new=np.zeros((N-n0+2,n0,dimY))
for k in range(N-n0+2):
    data_new[k]=y_new[k:k+n0]
# convert data to be consistent with deep learning.
data_new=data_new.reshape(N-n0+2,n0*dimY)
data_new=pd.DataFrame(data_new)

# Before predict, normalize data with training information.
data_new=(data_new-data_mean)/data_std
df_pred=model.predict(data_new)
for i in range(N-n0+2):
    # convect df results back to original scale.
    df_pred[i,:]=df_pred[i,:]*label_std+label_mean

df_new=[x0 for k in range(n0-1)]
df_new=np.array(df_new)
df_new=df_new.reshape(n0-1,dimX)
df_new=np.vstack((df_new,df_pred))

import matplotlib.pyplot as plt
%matplotlib inline
%config InlineBackend.figure_format = 'svg'

plt.figure(dpi = 600,figsize=[40,30])
axis=np.linspace(0,N+1,N+1)
fig,ax=plt.subplots(3,3,sharex=True)
plt.xlim((0,1))
ax[0][0].plot(axis, x_new[:,0]/1000,'c',axis,x_bar_new[:,0]/1000,'b',axis,df_new[:,0]/1
000,'r',linewidth=0.5)
ax[0][0].set_xlim((0,N+1))
ax[0][0].minorticks_on()
ax[0][1].plot(axis, x_new[:,1],'c',axis,x_bar_new[:,1],'b',axis,df_new[:,1],'r',linewid
th=0.5)
ax[0][1].minorticks_on()
ax[0][2].plot(axis, x_new[:,2],'c',axis,x_bar_new[:,2],'b',axis,df_new[:,2],'r',linewid
th=0.5)
#y
ax[1][0].plot(axis, x_new[:,3]/1000,'c',axis,x_bar_new[:,3]/1000,'b',axis,df_new[:,3]/1
000,'r',linewidth=0.5)
ax[1][1].plot(axis, x_new[:,4],'c',axis,x_bar_new[:,4],'b',axis,df_new[:,4],'r',linewid
th=0.5)
ax[1][2].plot(axis, x_new[:,5],'c',axis,x_bar_new[:,5],'b',axis,df_new[:,5],'r',linewid
th=0.5)
# z
ax[2][0].plot(axis, x_new[:,6]/1000,'c',axis,x_bar_new[:,6]/1000,'b',axis,df_new[:,6]/1
000,'r',linewidth=0.5)
ax[2][1].plot(axis, x_new[:,7],'c',axis,x_bar_new[:,7],'b',axis,df_new[:,7],'r',linewid
th=0.5)
ax[2][2].plot(axis, x_new[:,8],'c',axis,x_bar_new[:,8],'b',axis,df_new[:,8],'r',linewid
th=0.5)
```

```
fig.subplots_adjust(wspace=0.6, hspace=0.3)
plt.savefig('6dim-plot.pdf')
plt.show()
```

<Figure size 24000x18000 with 0 Axes>



In [ ]: