Chapter 1 Local Theory of Curves

1. Definition of C^k function To say a function is C^k, $k = 0, 1, 2, ...$, means the function has continuous derivatives up to and including k-th derivative in its domain. For example, $f(x)$ is C^0 means f is continuous; $f(x)$ is C^1 means $f(x)$ and $f'(x)$ exists and are continuous; $f(x)$ is C^2 means $f(x)$, $f'(x)$, and $f''(x)$ exists and are continuous. To say a function is C^∞**, or smooth,** means the function has derivatives of any order in its domain.

2. Idea of parametric representation of curve The **classical differential geometry** only consider the curve in $\mathbb{R}^3$. The parametrized representation of a curve is raised from classical mechanics and takes the form $\boldsymbol{\beta}(t)$ where $\boldsymbol{\beta} : I \to \mathbb{R}^3$, I is an open interval in $\mathbb{R}$. Let the image of $\boldsymbol{\beta}$ in $\mathbb{R}^3$ be M, and the plot of M is just the shape of the curve $\boldsymbol{\beta}(t)$.

The parametric representation of a point set is not unique. For example, let the parameter runs the whole real line $-\infty < t < \infty$, then the plane curve $\{(x, y) \in \mathbb{R}^2 : x^2 - y = 0\}$ can be written as $\boldsymbol{\beta}(t) = (t, t^2)$ or $\boldsymbol{\beta}(t) = (2t, 4t^2)$, but it *cannot* be written as $\boldsymbol{\beta}(t) = (e^t, e^{2t})$ or $\boldsymbol{\beta}(t) = (\sin t, \sin^2 t)$ since each of them only gives a certain part of the curve, see figure below.

The parametric representation is not always "convenient" to describe a curve. For example, the algebraic plane curve $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, namely both x and y axis, does not have convenient differentiable parametric representation. So does an elliptic curve $\{(x, y) \in \mathbb{R}^2 : x^3 - 2x + 1 - y^2 = 0\}$, see figure below. This gives the opportunity for using **algebraic geometry** which has close relationships with number theory and arithmetic.

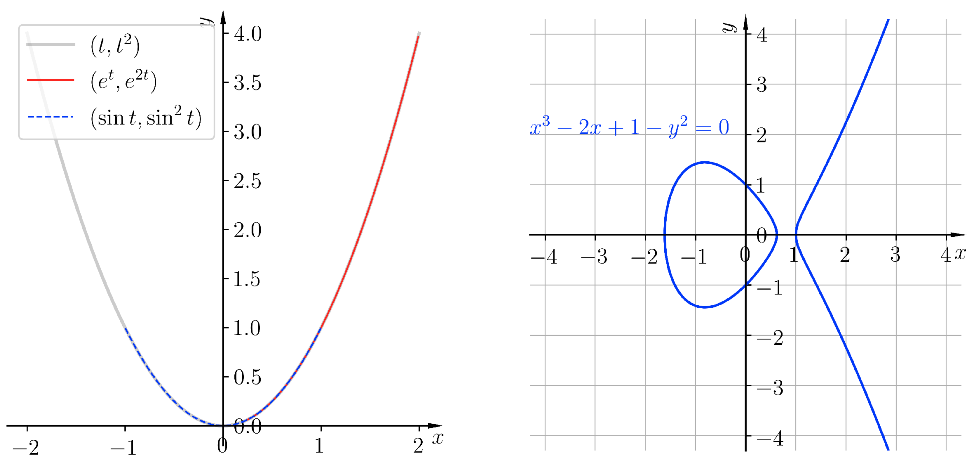

FIG. 1 (left) Parametric representation; (right) An elliptic curve

Algebraic geometry and differential geometry are two of many tools to study geometrical shapes such as curves. No one is more powerful than the other. For example, by utilizing **Fourier analysis**, the parametric representation can be used to fit the real graphs, and an example is the *curve of famous person*, see figure below. In addition, interesting curves also appears in the study of **dynamical systems**, such as Koch snowflake which is a fractal curve.

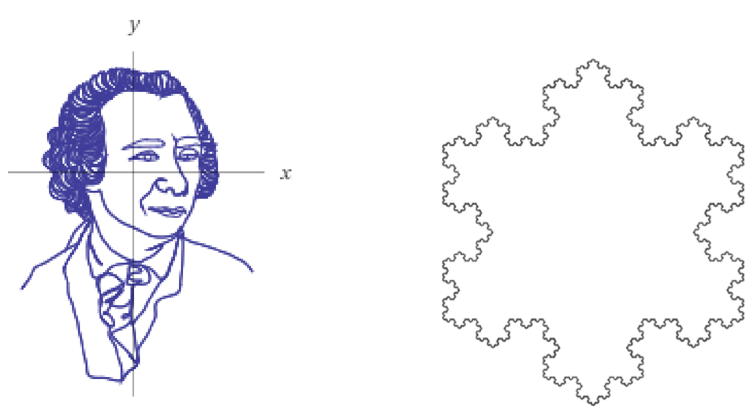

FIG. 2 (*left*) The curve of Leonhard Euler. Picture is retrieved from [1]. (*right*) Koch snowflake.

3. Definition of C^k parametric representation To say the mapping $\boldsymbol{\beta} : I \twoheadrightarrow M$ defined in Idea 2 is a C^k parametric representation means that it is a C^k function with $k = 1, 2, \ldots$ Note that $k \neq 0$, so it is at least C^1, and it is also called a differentiable parametrization. *We will focus on the study of differentiable parametrization, or the differentiable part of a parametrization.*

4. Definition of regular, simple C^k parametric representation In addition to Definition 3, to say the curve $\boldsymbol{\beta} : I \twoheadrightarrow M$ is simple means it is one-to-one and is regular means $\dot{\boldsymbol{\beta}}(t) \neq \mathbf{0}$ for every $t \in I$.

For example, consider the point set $\{(x, y) \in \mathbb{R}^2 : x^2 - y^3 = 0\}$. One parametric representation is $\boldsymbol{\beta}_1(t) = (t^3, t^2)$ and it is not regular since $\dot{\boldsymbol{\beta}}(0) = (0, 0)$. Another is $\boldsymbol{\beta}_2(t) = (t, |t|^{\frac{2}{3}})$ and it is not C^1 since $\dot{\boldsymbol{\beta}}(0)$ does not exist.

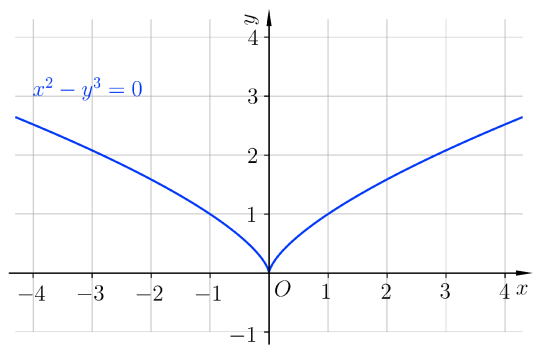

FIG. 3 The point set $\{(x, y) \in \mathbb{R}^2 : x^2 - y^3 = 0\}$.

5. Definition of regular C^k transformation To say the transformation $t : I^* \to \mathbb{R}$, $t = t(t^*)$ is a regular C^k transformation means it is a C^k function, $\dfrac{\mathrm{d}t}{\mathrm{d}t^*} \neq 0$ for all $t^* \in I^*$, and its range is I.

To say two regular C^k parametric representations are equivalent means there exists an regular C^k transformation of parameters which transforms the one into the other. For example, $\boldsymbol{\beta}_1(t) = (t, t^2)$ and $\boldsymbol{\beta}_2(t) = (2t, 4t^2)$ is equivalent due to $t \mapsto 2t$, or say $t \mapsto \dfrac{1}{2}t$, which is regular and smooth. So, $\boldsymbol{\beta}_1(t)$ and $\boldsymbol{\beta}_2(t)$ is equivalent and indeed they represent the same curve (Idea 2).

6. Definition of curve and Jordan arc An equivalence class of C^k parametric representation is called a C^k curve.
An equivalence class of regular C^k parametric representation is called a regular C^k curve.
An equivalence class of regular, simple C^k parametric representation is called a Jordan arc of a C^k curve.

7. Definition of a closed curve To say a curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ is closed means it is periodic: there exist a constant $T \in \mathbb{R}$ such that $\boldsymbol{\beta}(t + T) = \boldsymbol{\beta}(t)$ for every $t \in I$.

8. Definition of a plane curve To say a curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ is a plane curve means it is a subset of the plane $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x} - \boldsymbol{\beta}(t_0) \,|\, \mathbf{n} \rangle = 0\}$ where $t_0 \in I$ and $\mathbf{n}$ is a fixed unit vector. The image of $\boldsymbol{\beta}$ is in $\mathbb{R}^2$.

In terms of algebraic geometry, a real plane curve is defined as $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ for some polynomial $f \in \mathbb{R}[x, y]$. $\deg f = 1$ gives a straight line and $\deg f = 2$ gives a conic section: circle, ellipse, hyperbola, or parabola.

9. Definition of Jordan curve A simple, closed plane curve is called a Jordan curve. Every Jordan curve is homeomorphic to the unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

A brief review and classification of homeomorphism and homotopy. People should not be confused about them which are two completely different terms though both give a sense of continuous deformation.

Let X and Y be topological spaces. To say a function $\varphi : X \to Y$ is a homeomorphism means φ is one-to-one and continuous and φ^{-1} is continuous; in such case we say X is homeomorphic to Y. To say two continuous functions $f, g : X \to Y$ are homotopic means there exists a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and H is called a homotopy.

Here is an example. In $\mathbb{R}^2$, a straight line is homopotic to a point. By saying this, we treat the line and the point as two functions, more exactly the plots of two functions, and we mean that the two functions are homotopic (actually null-homotopic since the function whose plot is a point is a constant function). In contrast, a straight line is not homeomorphic to a point. By saying this, we treat the straight line and the point as two objects (in term of category theory), namely a subset of $\mathbb{R}$ and a real number. A function (morphism) between them cannot be one-to-one, so the function is not homeomorphism and the two objects are not homeomorphic.

Example: Newton's nodal cubic is the curve $\{(x,y)\in\mathbb{R}^2 : x^3+x^2-y^2=0\}$. See figure below.

From the point of view of algebraic geometry, set $y=0$, $x^3+x^2=0$ has two distinct real roots $x_1=x_2=0$ and $x_3=-1$; set $x=\lambda$, $y^2=\lambda^3+\lambda^2$ has no root when $\lambda<-1$, has one root $y_1=y_2=0$ when $\lambda=-1$ and 0, and has two distinct roots when $-1<\lambda<0$ and $\lambda>0$. Based on these information, we see that $(0,0)$ is a double point of the curve.

From the point of view of differential geometry, a smooth parametric representation $\boldsymbol{\beta}:\mathbb{R}\to\mathbb{R}^2$ is given by $\boldsymbol{\beta}(t)=(t^2-1,t-t^3)$, so $\dot{\boldsymbol{\beta}}(t)=(2t,1-3t^2)$. We see that $\boldsymbol{\beta}(1)=\boldsymbol{\beta}(-1)=(0,0)$ and $\dot{\boldsymbol{\beta}}(1)=(2,-2)$, $\dot{\boldsymbol{\beta}}(-1)=(-2,-2)$, and $\dot{\boldsymbol{\beta}(0)}\neq\mathbf{0}$ for every $t\in\mathbb{R}$. Based on these information, we see that $\boldsymbol{\beta}$ is a non-simple, regular curve; point (0,0) is a double point of the curve.

From either point of view, one can sketch the plot.

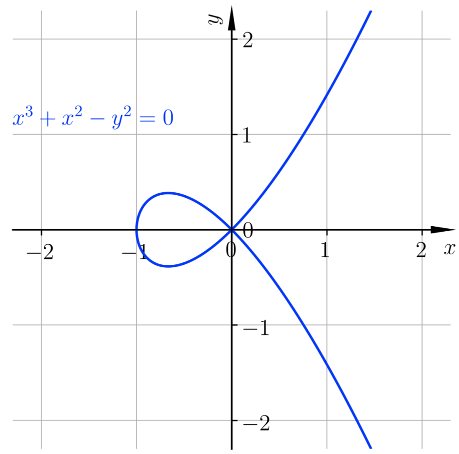

FIG. 4 Newton's nodal cubic

Example: Folium of Descartes (R. Descartes, 1638) $\boldsymbol{\beta}:\mathbb{R}\to\mathbb{R}^3$ is defined as $\boldsymbol{\beta}(t)=\left(\dfrac{3t}{1+t^3},\ \dfrac{3t^2}{1+t^3},\ 0\right)$.
By definition, it is a plane curve lying in $\mathbb{R}^2$; domain$\boldsymbol{\beta}=\mathbb{R}\setminus\{-1\}$.
$\lim\limits_{t\to-1^-}\boldsymbol{\beta}(t)=(+\infty,-\infty,0)$, $\lim\limits_{t\to-1^+}\boldsymbol{\beta}(t)=(-\infty,+\infty,0)$, and $\lim\limits_{t\to\pm\infty}\boldsymbol{\beta}(t)=(0,0,0)$. $\boldsymbol{\beta}(0)=(0,0,0)$.
$\dot{\boldsymbol{\beta}}(t)=\left(\dfrac{3(1-t^3)}{(1+t^3)^2},\ \dfrac{3t(2-t^3)}{(1+t^3)^2},\ 0\right)\neq\mathbf{0}$ for every $t\in$ domain$\boldsymbol{\beta}$, so it is a regular curve.
$\boldsymbol{\beta}$ is the algebraic plane curve $\{(x,y)\in\mathbb{R}^2 : x^3+y^3-3xy=0\}$.

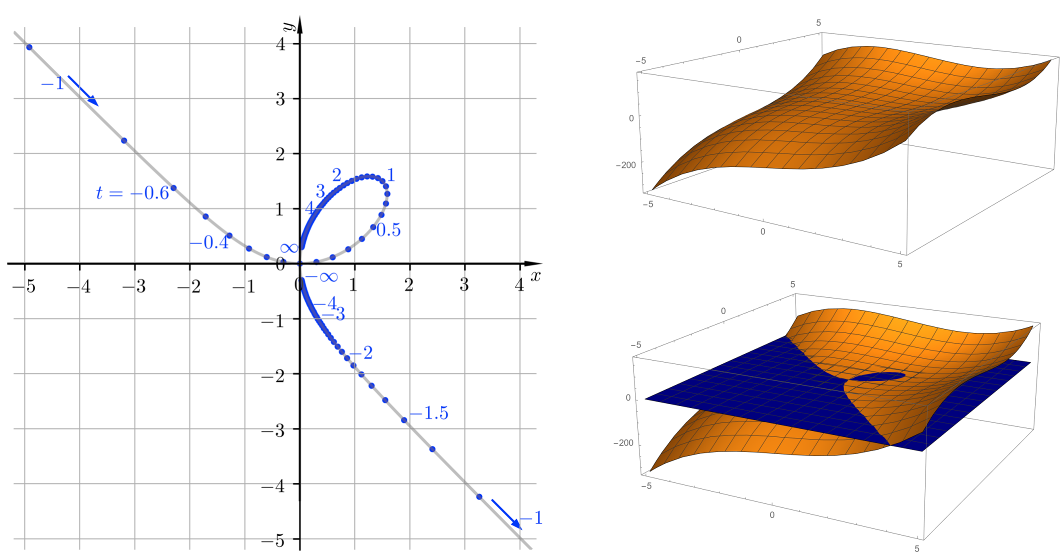

FIG. 5 Folium of Descartes. (left) The blue points are $\boldsymbol{\beta}$ at different values of parameters with step $\Delta t=0.1$; (right) the intersection of plot of $f(x,y)=x^3+y^3-3xy$ and the x-y plane is the folium of Descartes shown on the left.

Example: Circular helix $\boldsymbol{\beta}:\mathbb{R}\to\mathbb{R}^3$ is given by $\boldsymbol{\beta}(t)=(r\cos t, r\sin t, ct)$ where $r\neq 0$ and $c\neq 0$. It lies on the cylinder $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=r^2\}$ which is a two-dimensional surface (though it "exists" in three-dimensional space $\mathbb{R}^3$; we will prove this later). The curve is not a plane curve.

Claim The circular helix $\boldsymbol{\beta}$ is a regular simple smooth curve.
Proof Every component of $\boldsymbol{\beta}$ is smooth, so it is smooth. Since $\dot{\boldsymbol{\beta}}(t)=(-r\sin t, r\cos t, c)\neq\mathbf{0}$ for every $t\in\mathbb{R}$, it is a regular curve. To prove it is simple, we need to show $\boldsymbol{\beta}$ is injective (one-to-one on its image). If $\boldsymbol{\beta}(t_1)=\boldsymbol{\beta}(t_2)$, then $ct_1=ct_2$ and since $c\neq 0$ we have $t_1=t_2$; so it indeed is a one-to-one function. $\square$

If $c<0$, we call $\boldsymbol{\beta}$ a left circular helix; if $c>0$, we call $\boldsymbol{\beta}$ a right circular helix. See Figure 4.

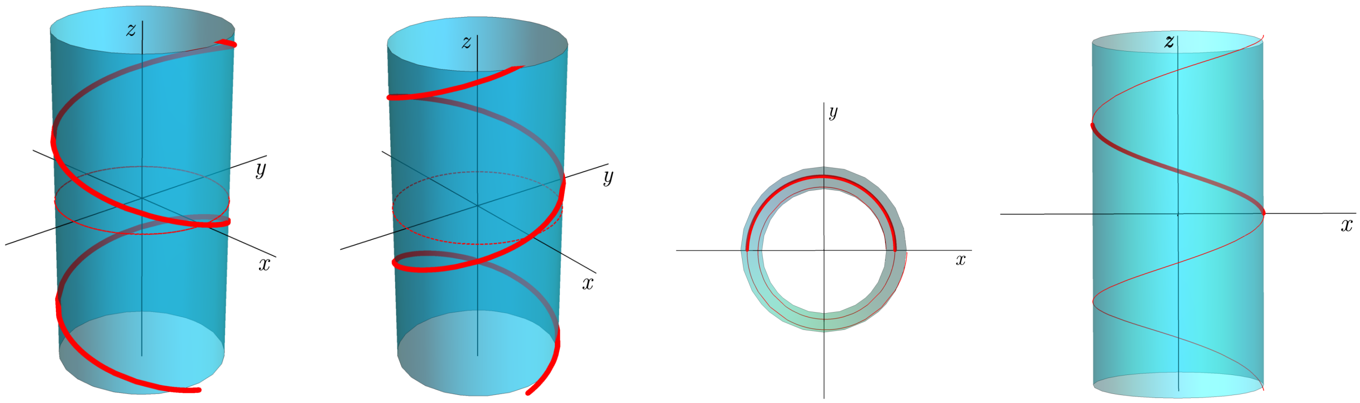

FIG. 6 (from left to right) Left circular helix, right circular helix, the x-y, x-z projections of right circular helix. The thick portion corresponds to $0<t<\pi$.

10. Projection Consider the curve $\boldsymbol{\beta}:I\to\mathbb{R}^3$, $\boldsymbol{\beta}(t)=(x(t),y(t),z(t))$. Let $t_0\in I$ and $\dot{\boldsymbol{\beta}}(t_0)\neq\mathbf{0}$, then $\dot{x}(t_0)\neq 0$. On a neighbourhood of t_0 the inverse function of $x(t)$ exists, that is $t(x)$. Then on the neighbourhood we have $y=y(x)$ and $z=z(x)$. They represent the projection of the portion of $\boldsymbol{\beta}$ on x-y plane and on x-z plane, respectively.

For example, consider the circular helix $\boldsymbol{\beta}(t)=(r\cos t, r\sin t, ct)$ where $r\neq 0$ and $c\neq 0$. On $0<t<\pi$, $\dot{x}(t)=-r\sin t\neq 0$ and $x(t)$ has inverse $t=\arccos\dfrac{x}{r}$. Then, $y=r\sin\left(\arccos\dfrac{x}{r}\right)=\sqrt{r^2-x^2}$ and $z=ct=c\arccos\dfrac{x}{r}$.

11. Orientation of a curve The sense corresponding to increase of parameter of a parametric representation is called the positive sense of the curve. Otherwise, it is called the negative sense of the curve. This actually defines the orientation of a curve.

12. Definition of arc of a curve Let J be a closed subinterval with non-zero length of I. The arc of the curve $\boldsymbol{\beta}:I\to\mathbb{R}^3$ is the point set $\{\boldsymbol{\beta}(t):t\in J\}$.

A brief review of interval in $\mathbb{R}$. Let $a,b\in\mathbb{R}$ and $a<b$, then an interval of $\mathbb{R}$ can only be one of (a,b), $[a,b)$, $(a,b]$, $[a,b]$, (a,∞), $[a,\infty)$, $(-\infty,a)$, $(-\infty,a]$, and $(-\infty,\infty)$. For example, $(1,2]$ is an interval of $\mathbb{R}$ but $(1,2]\cup(5,7)$ is not. In other words, one can prove that the subset $I\in\mathbb{R}$ is an interval if and only if $x<z<y$ and $x,y\in I$ then $z\in I$. Further, a connected subset of $\mathbb{R}$ can only be either an interval or a point.

13. Rectifiable arc and arc length Let $J=[c,d]$ be a closed subinterval with non-zero length of $I=[a,b]\subseteq\mathbb{R}$. Let the curve $\boldsymbol{\beta}:I\to\mathbb{R}^3$ be C^k, $k=1,2,\ldots$. To say the arc $\{\boldsymbol{\beta}(t):t\in J\}$ is rectifiable means for every equal partition $\mathcal{P}: c=t_0<t_1<\ldots<t_{i-1}<t_i<t_{i+1}<\ldots<t_N=d$ of J, $\displaystyle\sum_{i=0}^{N}\left|\boldsymbol{\beta}(t_i)-\boldsymbol{\beta}(t_{i-1})\right|<\infty$, or say converges. If converges, then the arc length s is defined as $\displaystyle\lim_{N\to\infty}\sum_{i=1}^{N}\left|\boldsymbol{\beta}(t_i)-\boldsymbol{\beta}(t_{i-1})\right|$. Further, let $\Delta t=t_i-t_{i-1}$, we have

$$s=\lim_{N\to\infty}\sum_{i=1}^{N}\left|\boldsymbol{\beta}(t_i)-\boldsymbol{\beta}(t_{i-1})\right|=\lim_{N\to\infty}\sum_{i=1}^{N}\left|\frac{\boldsymbol{\beta}(t_i)-\boldsymbol{\beta}(t_{i-1})}{\Delta t}\right|\Delta t=\lim_{N\to\infty}\sum_{i=1}^{N}|\dot{\boldsymbol{\beta}}(t_{i-1})|\Delta t=\int_c^d|\dot{\boldsymbol{\beta}}(t)|\mathrm{d}t$$

The length of the whole curve is $\displaystyle\int_a^b|\dot{\boldsymbol{\beta}}(t)|\mathrm{d}t$.

Claim The arch length is independent of the choice of regular C^k parametric representation.
Proof Let $t = t(t^*)$ be a C^k parametric representation such that $I^* : c^* \leqslant t^* \leqslant d^*$ corresponds to $I : c \leqslant t \leqslant d$ (which means $t(c^*) = c$, $t(d^*) = d$, and $\dfrac{\mathrm{d}t}{\mathrm{d}t^*} > 0$), then $\mathrm{d}t = \left|\dfrac{\mathrm{d}t}{\mathrm{d}t^*}\right|\mathrm{d}t^*$, and the arclength

$$\int_c^d |\dot{\boldsymbol{\beta}}(t)|\mathrm{d}t = \int_{c^*}^{d^*} |\dot{\boldsymbol{\beta}}(t)|\left|\frac{\mathrm{d}t}{\mathrm{d}t^*}\right|\mathrm{d}t^* = \int_{c^*}^{d^*} |\dot{\boldsymbol{\beta}}(t^*)|\mathrm{d}t^*$$ □

Notation Arc length as a function of t is $s(t) = \displaystyle\int_{t_0}^{t} |\dot{\boldsymbol{\beta}}(t)|\mathrm{d}t$, where $c < t_0 < d$.

14. Arc length as parameter If the arc length function $s(t)$ has inverse $t(s)$, then one can substitute t by s and make $\boldsymbol{\beta}$ as a function of s.

[1] https://www.wolframalpha.com/input/?i=person+curves