Quantum Gauge Theory

Reading Notes of

M. Henneaux and C. Teitelboim's *Quantization of Gauge Systems*

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The first-time encounter: Gauge in classical electrodynamics

The first time I encountered the term "gauge", precisely the "gauge transformation" and the "gauge fixing," is when I studied the Maxwell equations. The Maxwell equations in vacuum without magnetic monopole can be written as

$$\begin{aligned} \operatorname{div} E &= \frac{1}{\epsilon_0}\rho & \operatorname{curl} E &= -\frac{\partial B}{\partial t} \\ \operatorname{div} B &= 0 & \operatorname{curl} B &= \mu_0 J + \mu_0\epsilon_0 \frac{\partial B}{\partial t}. \end{aligned}$$

The gauge transformation here can be summarized as the following statement.

Theorem 0 Let the gradient, divergence, and curl be taken with respect to $x \in \mathbb{R}^3$. Let $A : (x,t) \mapsto \mathbb{R}^3$, $V : (x,t) \mapsto \mathbb{R}$, and $\lambda : (x,t) \mapsto \mathbb{R}$ be C^∞ functions, then the map

$$\begin{cases} A(x,t) \mapsto A(x,t) + \operatorname{grad} \lambda(x,t) \\ V(x,t) \mapsto V(x,t) - \dfrac{\partial \lambda(x,t)}{\partial t} \end{cases}$$

preserves $B = \operatorname{curl} A$ and $E = -\operatorname{grad} V - \dfrac{\partial A}{\partial t}$. The map is called the **gauge transformation**.

Proof $\operatorname{curl}(A + \operatorname{grad}\lambda) = \operatorname{curl} A + \operatorname{curl}\operatorname{grad}\lambda = \operatorname{curl} A = B$;

$$-\operatorname{grad}\left(V - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial}{\partial t}(A + \operatorname{grad}\lambda) = -\operatorname{grad} V + \frac{\partial(\operatorname{grad}\lambda)}{\partial t} - \frac{\partial A}{\partial t} - \frac{\partial(\operatorname{grad}\lambda)}{\partial t} = -\operatorname{grad} V - \frac{\partial A}{\partial t} = E. \qquad \blacksquare$$

In this example,

(a) The gauge transformation transforms the gauge, so $A(x,t)$ and $V(x,t)$ are roughly called the "gauge";
(b) $A(x,t)$ and $V(x,t)$ are a kind of reference frame;
(c) the physically variables B and E are independent of the choice of the local reference frames; for example, you can set $V = 0$ at any surface but still get the same E;
(d) the physically variables B and E are invariant under the change of the local reference frames, as the preceding theorem proved.

Based on these statements, it is easy to understand the first two paragraphs in Section 1.1:

> A gauge theory may be thought of as one in which the dynamical variables are specified with respect to a "reference frame" whose choice is arbitrary at every instant of time. The physically important variables are those that are independent of the choice of the local reference frame. A transformation of the variables induced by a change in the arbitrary reference frame is called a gauge transformation. Physical variables ("observables") are then said to be gauge invariant.
>
> In a gauge theory, one cannot expect that the equations of motion will determine all the dynamical variables for all times if the initial conditions are given because one can always change the reference frame in the future, say, while keeping the initial conditions fixed. A different time evolution will then stem from the same initial conditions. Thus, it is a key property of a gauge theory that *the general solution of the equations of motion contains arbitrary functions of time*.

We will shown later that

> … the presence of arbitrary functions of time in the general solution of the equations of motion implies that the canonical variables are not all independent. Rather, there are relations among them called constraints. Thus, *a gauge system is always a constrained Hamiltonian system*.

In short,

Logic in general

"reference frame" is arbitrary at every t

$\Downarrow$

the solution contains arbitrary functions of t

$\Downarrow$

canonical variables are not all independent.

gauge system $\underset{\not\Leftarrow}{\Longrightarrow}$ constrained Hamiltonian system.

Lagrangian formalism: the primary constraints

Consider q^n, $n = 1, 2, \dots, N$. By the chain rule we have

$$\begin{aligned} \frac{\partial L}{\partial q^n} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^n}\right) = \frac{\mathrm{d}q^{n'}}{\mathrm{d}t}\left(\frac{\partial}{\partial q^{n'}}\frac{\partial L}{\partial \dot{q}^n}\right) + \frac{\mathrm{d}\dot{q}^{n'}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}^{n'}}\frac{\partial L}{\partial \dot{q}^n}\right) \\ \Rightarrow \quad \ddot{q}^{n'}\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n} &= \frac{\partial L}{\partial q^n} - \dot{q}^{n'}\frac{\partial^2 L}{\partial q^{n'}\partial \dot{q}^n} \qquad \text{for every } n, n' = 1, 2, \dots, N \end{aligned}$$

If $\det\left(\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n}\right) \neq 0$, invertible, then $\ddot{q}^{n'}$ can be uniquely determined and thus the equations of motion can be uniquely determined with the initial conditions. If $\det\left(\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n}\right) = 0$, non-invertible, then $\ddot{q}^{n'}$ can not be uniquely determined. This is what we are interested. Introducing $p_n = \frac{\partial L}{\partial \dot{q}^n}$, then we are interested in

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n}\right) = \det\left(\frac{\partial p_n}{\partial \dot{q}^{n'}}\right) = 0 \qquad n, n' = 1, 2, \dots, N \tag{1}$$

This equation actually means there are some relations, or say constraints, among different q_n, $n = 1, 2, \dots, N$. For example, if only one relation $p_2 = 2p_5$ exists, which is the constraints $\phi(p_1, \dots, p_N) = p_2 - 2p_5 = 0$, then the second and the fifth row of this matrix are *d*ependent and so the rank is $N - 1$; if two constraints exist, then the rank will be $N - 2$. In general, we have

Claim Let $q = (q^1, q^2, \dots, q^N)$ and $p = (p_1, p_2, \dots, p_N)$. If there are M constraints among p_n, namely the

$$\phi_m(q, p) = 0, \qquad m = 1, 2, \dots, M \tag{2}$$

then

$$\operatorname{rank}\left(\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n}\right) = N - M'$$

where M' is the number of independent constraints, $M' \leqslant M$.

Definition These $\phi_m(q, p) = 0$, $m = 1, 2, \dots, M$ caused the singularity of the matrix $\frac{\partial^2 L}{\partial \dot{q}^{n'}\partial \dot{q}^n}$ is called the **primary constraints**, and they form the primary constraints surface which is a $2N - M'$ submanifold in the phase space (q, p).

What the preceding arguments said is that the definition of canonical momentum $p_n = \frac{\partial L}{\partial \dot{q}^n}$, as a map from $\dot{q}$ to p, is not one-to-one; some p_n are actually related to each other.

$$\begin{aligned} & q^1, q^2, \dots, q^N, \quad \dot{q}^1, \dot{q}^2, \dots, \dot{q}^N && 2N \text{ submanifold} \\ \mapsto\ & q^1, q^2, \dots, q^N, \quad \underbrace{\dots, \dots, p_n, \dots}_{N - M' \text{ independent } p_n} && 2N - M' \text{ submanifold} \end{aligned} \tag{3}$$

From this, we have

Claim There exists some $n \in \{1, 2, ..., N\}$ such that the fiber over the point (q^n, p_n) on the primary constraints surface forms a submanifold of dimension M' in $(q, \dot{q})$ space. In other words, $(q, \dot{q})$ **space is a fibre bundle whose base space is the primary constraints surface in** (q, p) **space**.

Example Lagrangian $L = \dfrac{1}{2}(\dot{q}_1 - \dot{q}_2)^2$, then $p_1 = \dot{q}_1 - \dot{q}_2$ and $p_2 = \dot{q}_2 - \dot{q}_1 = -p_1$. So,

$$\text{map: } (q_1, q_2, \dot{q}_1, \dot{q}_2) \mapsto (q_1, q_2, p_1, -p_1),$$
$$\text{constraint: } \phi(q^1, q^2, p_1, p_2) = p_1 + p_2 = 0$$

In other words, for each $c \in \mathbb{R}$, the fiber $f_c = \{(q^1, q^2, \dot{q}^1, \dot{q}^2) : \dot{q}_1 - \dot{q}_2 = c\}$ in $(q, \dot{q})$ space is mapped to a point $b_c = \{(q^1, q^2, c, -c)\}$ in (q, p) space. Let $F = \{f_c : c \in \mathbb{R}\}$ and $B = \{b_c : c \in \mathbb{R}\}$, then

$$(q, \dot{q}) \text{ space } = F \cong \{(f_c, b_c) : c \in \mathbb{R}\}$$

From this example, we can see that the map $p_n = \dfrac{\partial L}{\partial \dot{q}^n}$ is not surjective since it maps the $(q, \dot{q})$ space only on the proper subspace $\{(q^1, q^2, p_1, p_2) : p_1 + p_2 = 0\}$ of the (q, p) space. Of course, it is not injective.

Claim Due to the existence of primary constraints, the map

$$(q, \dot{q}) \mapsto (q, p) \quad \text{by } p_n = \frac{\partial L}{\partial \dot{q}^n} \text{ for each } n \in \{1, 2, ..., N\}$$

is neither injective nor surjective in general.

One may write the constraints in many seemingly same and redundant ways. For example $p_1 = 0$ can be written as

(a) $p_1 = 0$;
(b) $p_1^2 = 0$;
(c) $p_1 = 0, p_1^2 = 0$.

However, the way of writing the constraints is actually important. This is why we set two different numbers M and M' to describe the number of constraints in preceding discussion. What we want to use are those are independent, namely M'. The **regularity condition** is what we used to determine the independence of constraints. Suppose we have

$$\text{independent constraints: } \phi_{m'} = 0, \; m' = 1, 2, ..., M'$$
$$\text{dependent constraints: } \phi_{\overline{m}'} = 0, \; \overline{m}' = M' + 1, ..., M'$$

Theorem 1.1 If a smooth function G defined on the phase space (q, p) vanished on the surface $\phi_m(q, p) = 0$, $m = 1, 2, ..., M$, then $G = g^m \phi_m$ for some functions $g^m(q, p)$.

Theorem 1.2 If $\lambda_n \delta q^n + \mu^n \delta p_n = 0$ for arbitrary variations δq^n, δp_n tangent to the constraint surface, then

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial q^n} \qquad\qquad \mu^n = u^m \frac{\partial \phi_m}{\partial p_n}$$

for some u^m. The equalities here are equalities on the surface $\phi_m(q, p) = 0$, $m = 1, 2, ..., M$.

Hamiltonian formalism with primary constraints

The canonical Hamiltonian H is obtained by the Legendre transformation of Lagrangian L by using $p_n = \dfrac{\partial L}{\partial \dot{q}^n}$ for each $n = 1, 2, ..., N$.

$$H(q,p) := \dot{q}^n p_n(q,\dot{q}) - L(q,\dot{q}) \tag{4}$$

where $q = (q^1, q^2, ..., q^N)$, $\dot{q} = (\dot{q}^1, \dot{q}^2, ..., \dot{q}^N)$, and $p = (p_1, p_2, ..., p_N)$. To convince yourself that H is a function only about q and p, preform

$$\mathrm{d}H = p_n \mathrm{d}\dot{q}^n + \dot{q}^n \mathrm{d}p_n - \frac{\partial L}{\partial q^n}\mathrm{d}q^n - \frac{\partial L}{\partial \dot{q}^n}\mathrm{d}\dot{q}^n = \dot{q}^n \mathrm{d}p_n - \dot{p}_n \mathrm{d}q^n \tag{5}$$

where in the last step we use the Lagrangian mechanics.

A very important question need to answer at this point is: can be claim that $\dfrac{\partial H}{\partial p_n} = \dot{q}^n$ and $\dfrac{\partial H}{\partial q^n} = -\dot{p}_n$ for each n? The answer is no, because those q^n and p_n are related by the primary constraints $\phi_m(q,p) = 0$, $m = 1, 2, ..., M$. **Due to the primary constraints, we have to use Theorem 1.2 to find the appropriate Hamilton's equations.** Rewrite the formula as

$$\begin{aligned}
\mathrm{d}H - \dot{q}^n \mathrm{d}p_n + \dot{p}_n \mathrm{d}q^n &= 0 \\
\frac{\partial H}{\partial q^n}\mathrm{d}q^n + \frac{\partial H}{\partial p^n}\mathrm{d}p^n - \dot{q}^n \mathrm{d}p_n + \dot{p}_n \mathrm{d}q^n &= 0 \\
\left(\frac{\partial H}{\partial q^n} + \dot{p}_n\right)\mathrm{d}q^n + \left(\frac{\partial H}{\partial p^n} - \dot{q}^n\right)\mathrm{d}p_n &= 0
\end{aligned} \tag{6}$$

By Theorem 1.2, we have, for each n,

$$\frac{\partial H}{\partial q^n} + \dot{p}_n = -u^m \frac{\partial \phi_m}{\partial q^n} \qquad \frac{\partial H}{\partial p_n} - \dot{q}^n = -u^m \frac{\partial \phi_m}{\partial p_n}$$

where m runs from 1 to M. So, our conclusion is

Hamilton's equation with primary constraints If the primary constraints $\phi_m(q,p) = 0$, $m = 1, 2, ..., M$ exits, then the Hamilton's equation is

$$\dot{p}_n = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n} \qquad \dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n} \tag{7}$$

If the primary constraints do not exist, then the above equations reduce to

$$\dot{p}_n = -\frac{\partial H}{\partial q^n} \qquad \dot{q}^n = \frac{\partial H}{\partial p_n} \tag{8}$$

The definition of H actually causes two more crucial issues.

Problem (a) Since $p_n = \dfrac{\partial L}{\partial \dot{q}^n}$ is not a one-to-one map, the Legendre transformation from $L(q,\dot{q})$ and $H(q,p)$ is not invertibe;

Problem (b) Since some p_n are restricted by the primary constraints, H is a well-defined function only on the primary constraints surface in (q,p) space yet is not a well-defined function on whole (q,p) space since H can be extended arbitrarily off that manifold. In other words, the expression of H is not uniquely specified on the complement of the primary constraints surface.

For problem (b), it actually tells that any extension will not change the formalism; why not choose the replacement

$$H(q,p) \mapsto H(q,p) + c^m(q,p)\phi_m(q,p)$$

where m runs from 1 to M. To solve problem (a),

Idea We need to introduce extra degrees of freedom in order to make the Legendre transformation between Lagrangian and Hamiltonian invertible.

Again, based on the Hamilton's equations we got, H should be in the space (q, p, u), where $u := (u^1, u^2, ..., u^M)$ in which M' are independent. Since the dimension of (q, p) is $2N - M'$, we can claim that the space (q, p, u) is of dimension $(2N - M') + M' = 2N$. Now, the dimensions of the space $(q, \dot{q})$, in which Lagrangian exists, and the space (q, p, u), in which Hamiltonian exists, are the same, we can then have a invertible Legendre transformation.

Legendre transformation with primary constraints If the primary constraints $\phi_m(q, p) = 0$, $m = 1, 2, ..., M$ exits, then the Legendre transformation from the whole $(q, \dot{q})$ space to the primary constraint surface $\{(q, p, u) : \phi_m(q, p) = 0, m = 1, 2, ..., M\} = \{(q, p, u) : \text{independent } \phi_{m'}(q, p) = 0, m' = 1, 2, ..., M'\}$ in (q, p, u) space is defined as

$$\begin{cases} q^n = q^n \\ p_n = \dfrac{\partial L(q, \dot{q})}{\partial \dot{q}^n} \\ u^m = u^m(q, \dot{q}) \end{cases}$$

This transformation between spaces of the same dimension $2N$ is invertible and its inverse from the whole (q, p, u) space (actually is restricted to the primary constraints surface) to the whole $(q, \dot{q})$ space is

$$\begin{cases} q^n = q^n \\ p_n = \dfrac{\partial H(q, p, u)}{\partial \dot{q}^n} + u^m \dfrac{\partial \phi_m(q, p)}{\partial p_n} \\ \phi_m(q, p) = 0 \end{cases}$$

Secondary constraints