Solving Ordinary Differential Equations Using An Integrating Factor

Note: *This work is intended for informative and educational purposes only.*

1. Introduction

Last month we looked at one approach for solving ODES. In this month's issue, we continue in the same vein and consider how to solve one-dimensional ODE's using the integrating factor technique. This method has far reaching applications in fields such as Physics, Chemistry, Biology, and Finance.

2. Setup

Suppose we seek the function $y(x) := y : \mathbb{R} \longrightarrow \mathbb{R}$, that satisfies the following ODE,

$$\frac{dy}{dx} + g(x)y = h(x). \tag{1}$$

Where $g(x)$ and $h(x)$ are arbitrary functions of x only. Then we can use the integrating factor technique to solve the ODE.

3. Method

To solve this equation we will make use of the product rule from ordinary calculus. Before doing so, however, let us introduce a new term,

$$e^{G(x)} := e^{\int g(x)dx}. \tag{2}$$

Clearly, $g(x) = G'(x)$. This term is called the **Integrating Factor**.

By introducing this term we are able to take advantage of the results from differentiating exponential functions. Now, let us multiply our ODE (1) by this new term.

$$e^{G(x)}\frac{dy}{dx} + g(x)e^{G(x)}y = h(x)e^{G(x)}. \tag{3}$$

Examining the first two terms of this equation, we can see that this is precisely the product rule applied to the function $ye^{G(x)}$. That was lucky! Let's rewrite this equation once more,

$$\frac{d}{dx}\left[ye^{G(x)}\right] = h(x)e^{G(x)}. \tag{4}$$

This looks much more familiar. Let's integrate both sides.

$$\int \frac{d}{dx}\left[ye^{G(x)}\right] dx = \int h(x)e^{G(x)}. \tag{5}$$

But we know that an integral and a derivative 'cancel out', so finally we have,

$$y(x) = e^{-G(x)} \int h(x)e^{G(x)}. \tag{6}$$

4. Example

Let's consider a specific example. Find the general solution of the following ODE,

$$\frac{dy}{dx} + 3x^2 y = e^{-x^3}. \tag{7}$$

Let's start by working out the integrating factor,

$$e^{\int 3x^2 dx} = e^{x^3}. \tag{8}$$

So, multiplying through our ODE becomes,

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 1. \tag{9}$$

Which we can manipulate using the product rule,

$$\int \frac{d}{dx}\left[e^{x^3} y\right] dx = \int 1 dx. \tag{10}$$

Which becomes,

$$e^{x^3} y + c_1 = x + c_2. \tag{11}$$

And finally, we have our solution,

$$y = \frac{x + C}{e^{x^3}}. \tag{12}$$