Solving the One-Factor Interest Rate PDE: Extended Hull & White

Note: *This work is intended for informative and educational purposes only.*

1. Introduction

In this issue we continue solving the one-factor interest rate partial differential equation, this time considering one of the models put forth by Hull & White. The so called *Extended Hull & White Model*.

2. Setup

As always we have our underlying stochastic differential equation governing the interest rate,

$$dr = u(r,t)dt + w(r,t)dW_t. \tag{1}$$

Similarly, we have our one-factor **Bond Pricing Equation**,

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \tag{2}$$

In this model, Hull & White took the following functions,

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma r \tag{3}$$
$$w(r,t) = c. \tag{4}$$

Where $\gamma, c \in \mathbb{R}$. And the final condition is,

$$Z(r,T;T) = 1. \tag{5}$$

That is, at maturity the bond pays a par of 1.

3. Solution

As we have seen for the solution to the Ho & Lee and Vasicek solutions in the previous two issues, we can search for an *affine solution* of the form,

$$Z(r,t;T) = \exp[A(t;T) - rB(t;T)]. \tag{6}$$

Substituting the forms given by (3) and (4) into our BPE (2) we obtain,

$$\frac{\partial V}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 V}{\partial r^2} + [\eta(t) - \gamma r]\frac{\partial V}{\partial r} - rV = 0. \tag{7}$$

Let's compute the derivatives of our unknown solution (6) and substitute them into (7). We have,

$$\frac{\partial V}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\exp(A - rB) \tag{8}$$

$$\frac{\partial V}{\partial r} = -B\exp(A - rB) \tag{9}$$

$$\frac{\partial^2 V}{\partial r^2} = B^2\exp(A - rB). \tag{10}$$

Substituting these into our equation we get,

$$\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\exp(A - rB) + \frac{1}{2}c^2B^2\exp(A - rB) \\ - \eta(t)B\exp(A - rB) + \gamma rB\exp(A - rB) - r\exp(A - rB) = 0. \tag{11}$$

This is a rather long-winded expression. But notice that $\exp(A - rB)$ is multiplying every term in the equation, hence, we can divide through. In the interest of clarity, we can also rewrite the right hand side in a different way. Our equation simplifies to,

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}c^2B^2 - \eta(t)B + \gamma rB - r = 0 + 0r. \tag{12}$$

By writing $0 + 0r$, we can group the terms on the left hand side of (12) into those that are multiplied by r and those that aren't. This will then give us two *ordinary differential equations* to solve.

$$\left(\frac{\partial A}{\partial t} + \frac{1}{2}c^2B^2 - \eta(t)B\right) + r\left(-\frac{\partial B}{\partial t} + \gamma B - 1\right) = 0 + 0r. \tag{13}$$

Hence, we have

$$\frac{dB}{dt} = (\gamma B - 1) \tag{14}$$
$$\frac{dA}{dt} = \eta(t)B - \frac{1}{2}c^2B^2. \tag{15}$$

Finally, we need to convert our final condition from $Z(r,T;T)$ to $A(T)$ and $B(T)$. This is simple enough as for Z to equal 1, we must have that both $A(T) = B(T) = 0$.

Let's solve for $B(t)$ first.

$$\frac{dB}{dt} = (\gamma B - 1) \tag{16}$$

$$\int_t^T \frac{1}{\gamma B - 1} dB = \int_t^T 1 dt \tag{17}$$

$$\left[\frac{1}{\gamma} \ln|\gamma B - 1|\right]_t^T = (T - t) \tag{18}$$

$$\frac{1}{\gamma} \ln|\gamma B(T) - 1| - \frac{1}{\gamma} \ln|\gamma B(t) - 1| = (T - t) \tag{19}$$

$$\ln\left|\frac{\gamma B(T) - 1}{\gamma B(t) - 1}\right| = \gamma(T - t) \tag{20}$$

$$\ln\left|\frac{-1}{\gamma B(t) - 1}\right| = \gamma(T - t) \tag{21}$$

$$\ln|1 - \gamma B(t)|^{-1} = \gamma(T - t) \tag{22}$$

$$-\ln|1 - \gamma B(t)| = \gamma(T - t) \tag{23}$$

$$1 - \gamma B(t) = e^{-\gamma(T-t)} \tag{24}$$

$$B(t) = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}). \tag{25}$$

Now let's tackle $A(t)$.

$$\frac{dA}{dt} = \eta(t)B - \frac{1}{2}c^2B^2 \tag{26}$$

$$\frac{dA}{dt} = \eta(t)B(t) - \frac{1}{2}\frac{c^2}{\gamma^2}\left[1 - e^{-\gamma(T-t)}\right]^2 \tag{27}$$

$$\frac{dA}{dt} = \eta(t)B(t) - \frac{1}{2}\frac{c^2}{\gamma^2}\left[1 - 2e^{-\gamma(T-t)} + e^{-2\gamma(T-t)}\right] \tag{28}$$

$$\int_t^T 1dA = \int_t^T \eta(\tau)B(\tau)d\tau - \frac{1}{2}\frac{c^2}{\gamma^2}\int_t^T \left[1 - 2e^{-\gamma(T-\tau)} + e^{-2\gamma(T-\tau)}\right] d\tau \tag{29}$$

$$A(T) - A(t) = \int_t^T \eta(\tau)B(\tau)d\tau - \frac{1}{2}\frac{c^2}{\gamma^2}\left((T-t) - 2\left[\frac{1}{\gamma}e^{-\gamma(T-\tau)}\right]_t^T + \frac{1}{2\gamma}\left[e^{-2\gamma(T-\tau)}\right]_t^T\right) \tag{30}$$

$$-A(t) = \int_t^T \eta(\tau)B(\tau)d\tau - \frac{1}{2}\frac{c^2}{\gamma^2}\left((T-t) - \left[\frac{2}{\gamma} - \frac{2}{\gamma}e^{-\gamma(T-t)}\right] + \left[\frac{1}{2\gamma} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right]\right) \tag{31}$$

$$A(t) = -\int_t^T \eta(\tau)B(\tau)d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right) \tag{32}$$

Hence, we arrive out our full solution.

$$Z(r,t;T) = \exp\left[A(t;T) - rB(t;T)\right] \tag{33}$$

Where,

$$B(t;T) = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) \tag{34}$$

and

$$A(t;T) = -\int_t^T \eta(\tau)B(\tau)d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right). \tag{35}$$