Solving the One-Factor Interest Rate PDE: Vasicek

Note: *This work is intended for informative and educational purposes only.*

1. Introduction

In this month's issue we continue solving the one-factor interest rate partial differential equation, this time considering the model proposed by *Vasicek*.

2. Setup

As before we have our underlying stochastic differential equation governing the interest rate,

$$dr = u(r,t)dt + w(r,t)dW_t. \tag{1}$$

Likewise, we have our one-factor **Bond Pricing Equation**,

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0. \tag{2}$$

In this model, Vasicek took the following functions,

$$u(r,t) - \lambda(r,t)w(r,t) = a - br \tag{3}$$

$$w(r,t) = \sqrt{c}. \tag{4}$$

Where $a, b, c \in \mathbb{R}$. And the final condition is,

$$Z(r,T;T) = 1. \tag{5}$$

That is, at maturity the bond pays a par of 1.

3. Solution

As we saw in last month's issue for the solution of the Ho & Lee model, we can search for an *affine solution* of the form,

$$Z(r,t;T) = \exp\left[A(t;T) - rB(t;T)\right]. \tag{6}$$

Substituting the forms given by (3) and (4) into our BPE (2) we obtain,

$$\frac{\partial Z}{\partial t} + \frac{1}{2}c\frac{\partial^2 Z}{\partial r^2} + (a - br)\frac{\partial Z}{\partial r} - rZ = 0. \tag{7}$$

Let's compute the derivatives of our unknown solution (6) and substitute them into (7). We have,

$$\frac{\partial V}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\exp(A - rB) \tag{8}$$
$$\frac{\partial V}{\partial r} = -B\exp(A - rB) \tag{9}$$
$$\frac{\partial^2 V}{\partial r^2} = B^2\exp(A - rB). \tag{10}$$

Substituting these into our equation we get,

$$\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\exp(A - rB) + \frac{1}{2}cB^2\exp(A - rB) \\ - aB\exp(A - rB) + brB\exp(A - rB) - r\exp(A - rB) = 0. \tag{11}$$

Again, this is a long and unpleasant expression, but notice that $\exp(A - rB)$ is multiplying every term in the equation. We can cancel through once more and, in the interest of clarity, we can also rewrite the right hand side in a different way. Our equation simplifies to,

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}cB^2 - aB + brB - r = 0 + 0r. \tag{12}$$

By writing $0 + 0r$, we can group the terms on the left hand side of (12) into those that are multiplied by r and those that aren't. This will then give us two *ordinary differential equations* to solve.

$$\left(\frac{\partial A}{\partial t} + \frac{1}{2}cB^2 - aB\right) + r\left(-\frac{\partial B}{\partial t} + bB - 1\right) = 0 + 0r. \tag{13}$$

Hence, we have

$$\frac{dB}{dt} = (bB - 1) \tag{14}$$
$$\frac{dA}{dt} = aB - \frac{1}{2}cB^2. \tag{15}$$

Finally, we need to convert our final condition from $Z(r,T;T)$ to $A(T)$ and $B(T)$. This is simple enough as for Z to equal 1, we must have that both $A(T) = B(T) = 0$.

Let's solve for $B(t)$ first.

$$\frac{dB}{dt} = (bB - 1) \tag{16}$$

$$\int_t^T \frac{1}{bB-1} dB = \int_t^T 1 dt \tag{17}$$

$$\left[\frac{1}{b} \ln |bB(t) - 1|\right]_t^T = (T - t) \tag{18}$$

$$\ln |bB(T) - 1| - \ln |bB(t) - 1| = b(T - t) \tag{19}$$

$$\ln \left| \frac{bB(T) - 1}{bB(t) - 1} \right| = b(T - t) \tag{20}$$

$$\ln \left| \frac{-1}{bB(t) - 1} \right| = b(T - t) \tag{21}$$

$$\ln |1 - bB(t)|^{-1} = b(T - t) \tag{22}$$

$$-\ln |1 - bB(t)| = b(T - t) \tag{23}$$

$$1 - bB(t) = e^{-b(T-t)} \tag{24}$$

$$B(t) = \frac{1}{b}(1 - e^{-b(T-t)}). \tag{25}$$

Now let's tackle $A(t)$.

$$\frac{dA}{dt} = aB - \frac{1}{2}cB^2 \tag{26}$$

$$\frac{dA}{dt} = a\frac{1}{b}\left[1 - e^{-\gamma(T-t)}\right] - \frac{1}{2}\frac{c}{b^2}\left[1 - e^{-\gamma(T-t)}\right]^2 \tag{27}$$

$$\frac{dA}{dt} = \frac{a}{b}\left[1 - e^{-\gamma(T-t)}\right] - \frac{c}{2b^2}\left[1 - 2e^{-b(T-t)} + e^{-2b(T-t)}\right] \tag{28}$$

$$\int_t^T 1dA = \frac{a}{b}\int_t^T \left[1 - e^{-b(T-\tau)}\right] d\tau - \frac{c}{2b^2}\int_t^T \left[1 - 2e^{-b(T-\tau)} + e^{-2b(T-\tau)}\right] d\tau \tag{29}$$

$$A(T) - A(t) = \frac{a}{b}\left[(T-t) - \frac{1}{b}\left(1 - e^{-b(T-t)}\right)\right] \tag{30}$$

$$- \frac{c}{2b^2}\left((T-t) - 2\left[\frac{1}{b}e^{-b(T-\tau)}\right]_t^T + \frac{1}{2b}\left[e^{-2b(T-\tau)}\right]_t^T\right) \tag{31}$$

$$-A(t) = \frac{a}{b}\left[(T-t) - B(t)\right] - \frac{c}{2b^2}\left((T-t) + \left[\frac{2}{b} - \frac{2}{b}e^{-b(T-t)}\right] + \left[\frac{1}{2b} - \frac{1}{2b}e^{-2b(T-t)}\right]\right) \tag{32}$$

$$A(t) = \frac{a}{b}\left[B(t) - (T-t)\right] + \frac{c}{2b^2}\left((T-t) + \left[\frac{2}{b} - \frac{2}{b}e^{-b(T-t)}\right] + \left[\frac{1}{2b} - \frac{1}{2b}e^{-2b(T-t)}\right]\right). \tag{33}$$

As it turns out, we can show (after a lot of algebra...) that the expression $A(t)$ can be written in a neater form

$$A(t) = \frac{1}{b^2}\left[B(t;T) - (T-t)\right]\left[ab - \frac{1}{2}c\right] - \frac{cB(t;T)^2}{4b}. \tag{34}$$

Hence, we arrive out our full solution.

$$Z(r,t;T) = \exp\left[A(t;T) - rB(t;T)\right] \tag{35}$$

Where,

$$B(t;T) = \frac{1}{b}(1 - e^{-b(T-t)}). \tag{36}$$

and

$$A(t;T) = \frac{1}{b^2}\left[B(t;T) - (T-t)\right]\left[ab - \frac{1}{2}c\right] - \frac{cB(t;T)^2}{4b}. \tag{37}$$