

Vromen Justification Logic

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```
import Mathlib.Data.List.Basic
import Mathlib.Logic.Basic
```

Justification Logic Formalization of Lewis’s Common Knowledge

Status: COMPLETE

This file presents Vromen’s (2024) main contribution: a formalization of Lewis’s theory of common knowledge using justification logic with explicit reason terms.

This approach resolves the problems found in previous formalizations and provides the correct foundation for Lewis’s theory.

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The Key Insight

Lewis’s use of “thereby” in his definition of indication suggests that reasons are *objects* that can be tracked and composed:

“If i had reason to believe that A held, i would **thereby** have reason to believe that ____”
(Lewis 1969, p. 52–53)

The word “thereby” indicates the reason for the conclusion *depends on* the reason for the premise. This motivates using explicit reason terms.

Example:

- You have reason s for believing “if door open, then cold inside”
- You have reason t for believing “door open” (you see it)
- Therefore you have composite reason $s * t$ for believing “cold inside”

The composite $s * t$ explicitly contains t , capturing the “thereby”—your reason for believing it’s cold is based on (depends on) your reason for believing the door is open.

Main Results

Result	Status	Significance
A1 (Lewis’s axiom)	THEOREM	3-line proof from AR
A6 (indication propagation)	THEOREM	10-line proof from E2, E3
Lewis’s theorem	PROVEN	Non-trivial induction on G-closure
Logical omniscience	NOT REQUIRED	Only T1, T2, T3 needed

Comparison with Other Approaches

Feature	Cubitt-Sugden	Sillari	Vromen (this file)
R operator	Primitive	Modal \Box	$\exists r. rb\ r\ i\ \varphi$
Indication	Primitive	$R \wedge (\varphi \rightarrow \psi)$	$R\ i\ (A \rightarrow \varphi)$
A1 status	Axiom	FAILS \times	THEOREM
A6 status	Axiom	N/A	THEOREM
Lewis proof	From axioms	False/trivial	Non-trivial

References

- **Artemov, S., & Fitting, M. (2019).** *Justification Logic: Reasoning with Reasons*. Cambridge University Press.
- **Lewis, D. (1969).** *Convention: A Philosophical Study*. Cambridge, MA: Harvard University Press.
- **Vromen, H. (2024).** Reasoning with reasons: Lewis on common knowledge. *Economics & Philosophy*, 40(2), 397–418.

File Structure

1. Core structure and definitions (R, Ind)
2. Justification logic axioms (AR, T1, T2, T3)
3. Derived epistemic rules (E1, L1, E2, E3)
4. Lewis's axioms as theorems (A1, A6)
5. G-closure and Lewis's main theorem

```
namespace JustificationLogic
```

Section 1: Core Structure

We define a justification frame with a multiplicative structure for reason composition.

Justification Frame

A justification frame specifies:

- **individual**: The type of agents
- **reason**: The type of reasons (with multiplication)
- **reasonBelief**: The ternary relation $rb\ r\ i\ \varphi$ meaning "r is a reason for agent i to believe φ "

Key innovation Reasons form a multiplicative structure. The product $s * t$ represents applying reason s (for $\alpha \rightarrow \beta$) to reason t (for α) to obtain a composite reason for β .

Example:

- s: "Thermometer says 0°C" (reason for $door_open \rightarrow cold$)
- t: "I see door open" (reason for $door_open$)
- $s * t$: Combined reason for cold (thermometer + observation)

```
structure JustificationFrame (individual reason : Type*) [Mul reason] where
  reasonBelief : reason → individual → Prop → Prop
```

Section 2: Core Definitions

We define R (reason to believe) and Ind (indication) using explicit reason terms.

These definitions are the foundation of our approach and directly address the problems in previous formalizations.

```
variable {individual reason : Type*} [Mul reason]
variable (frame : JustificationFrame individual reason)
variable {rb : reason → individual → Prop → Prop}
```

1. Reason to Believe (R)

$R\ rb\ i\ \phi$ — "Agent i has reason to believe ϕ "

Definition: Defined as existential quantification over reasons: there exists some reason r such that $rb\ r\ i\ \phi$ holds.

$R\ rb\ i\ \phi := \exists\ r, rb\ r\ i\ \phi$

Why existential quantification?

1. Matches Lewis's natural language: "has reason to believe" doesn't specify *which* reason
2. Enables defeasible reasoning—agents can have reasons for contradictory propositions
3. Avoids logical omniscience—agents only believe what they have explicit reasons for

Contrast with modal logic: Modal logic defines $R\ i\ \phi$ as $\forall w, accessible(w) \rightarrow \phi(w)$, which:

- Cannot track which specific reason justifies ϕ
- Makes all tautologies believed (logical omniscience)
- Cannot represent conflicting reasons

Example:

- Alice has reason r_1 to believe "meeting at 2pm" (email from Bob)
- Alice has reason r_2 to believe "meeting at 3pm" (calendar entry)
- Both $R\ rb\ Alice\ (meeting_at_2pm)$ and $R\ rb\ Alice\ (meeting_at_3pm)$ can be true

```
def R (rb : reason → individual → Prop → Prop) (i : individual) (ϕ : Prop) : Prop :=
  ∃ r, rb r i ϕ
```

2. Indication (Ind)

$Ind\ rb\ A\ i\ \phi$ — "State of affairs A indicates ϕ to agent i "

Definition: Agent i has a reason for the conditional $A \rightarrow \phi$.

$Ind\ rb\ A\ i\ \phi := R\ rb\ i\ (A \rightarrow \phi)$

Lewis (1969, p. 52-53) defines indication as:

" A indicates to someone i that ____ if and only if, if i had reason to believe that A held, i would **thereby** have reason to believe that ____"

Why our definition captures "thereby": If i has:

- reason s for $A \rightarrow \phi$ (indication)
- reason t for A (observation)

Then by the application rule (AR), i has composite reason $s * t$ for ϕ .

Crucially, $s * t$ contains t as a component, capturing the "thereby" — the reason for ϕ is based on (depends on) the reason for A .

Contrast with Sillari: Sillari defines $\text{Ind } i \ \phi \ \psi := R \ i \ \phi \wedge (\phi \rightarrow \psi)$, which uses conjunction. This cannot capture evidential dependence because:

- The reason for ψ need not depend on the reason for ϕ
- They could come from completely independent sources
- The "thereby" is lost

Example:

- s : Reason to believe "door open \rightarrow cold inside"
- t : Reason to believe "door open"
- $s * t$: Reason to believe "cold inside" (derived via AR)
- The structure of $s * t$ explicitly encodes that the cold conclusion depends on the door observation

```
def Ind (rb : reason → individual → Prop → Prop) (φ : Prop) (i : individual) (ψ :
  → Prop) : Prop :=
  R rb i (φ → ψ)
```

Section 3: Axioms for Our Logic

Four axioms constitute the foundation: AR (the fundamental rule), two tautology witnesses (T1, T2) and axiom T3.

These are *all* the assumptions needed. Everything else is derived.

```
variable {individual reason : Type*} [Mul reason]
variable {rb : reason → individual → Prop → Prop}
```

Axiom AR: Application Rule

The fundamental axiom of justification logic.

Formula: $rb \ s \ i \ (\alpha \rightarrow \beta) \rightarrow rb \ t \ i \ \alpha \rightarrow rb \ (s * t) \ i \ \beta$

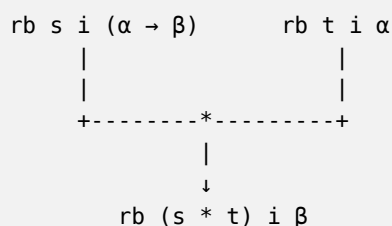
Interpretation: If agent i has:

- reason s for believing $\alpha \rightarrow \beta$
- reason t for believing α

Then i has reason $s * t$ for believing β .

Visual representation:

Reason Composition:



Key insight: $s * t$ explicitly contains both s and t , showing the evidential dependence.

Example: "Cold inside" from "Door open"

- s : reason for "door open \rightarrow cold inside" (thermometer reading pattern)
- t : reason for "door open" (visual observation)
- $s * t$: reason for "cold inside" (derived by application)

The composite $s * t$ contains t , capturing "thereby": The reason for "cold" depends on the reason for "door open".

Key insight: This makes justification logic "explicit" — we track which reasons justify which beliefs and have an explicit operation for combining them. This is what modal logic cannot do.

```
axiom AR {rb : reason  $\rightarrow$  individual  $\rightarrow$  Prop  $\rightarrow$  Prop} :
   $\forall \{s \ t : \text{reason}\} \{i : \text{individual}\} \{\alpha \ \beta : \text{Prop}\},$ 
    rb s i ( $\alpha \rightarrow \beta$ )  $\rightarrow$  rb t i  $\alpha \rightarrow$  rb (s * t) i  $\beta$ 
```

Special Reason Constants

We postulate three distinguished reasons that every agent can use for specific propositions. By axiomatizing these as specific constants, we ensure that only these distinguished reasons have the properties in T1, T2, and T3.

Empirical motivation:

- These correspond to reasoning patterns humans find easy
- They avoid logical omniscience (only these three tautologies, not all)
- They're independently motivated by cognitive science

```
axiom conjConst : reason
axiom transConst : reason
axiom distConst : reason
```

Axiom T1: Conjunction Formation

The reason `conjConst` justifies the tautology $\alpha \rightarrow \beta \rightarrow (\alpha \wedge \beta)$ for any agent and propositions.

Formula: $\text{rb conjConst } i (\alpha \rightarrow \beta \rightarrow (\alpha \wedge \beta))$

Interpretation: This ensures agents can combine justified beliefs: if you have reasons for α and β separately, you can form a justified belief in their conjunction.

Empirical justification: Conjunction introduction is easy for humans (Johnson-Laird 1992). Even young children can combine beliefs without difficulty.

Example:

- Reason for "door open": visual observation
- Reason for "cold": tactile sensation
- Can form: "door open AND cold"

```
axiom T1 {rb : reason  $\rightarrow$  individual  $\rightarrow$  Prop  $\rightarrow$  Prop} :
   $\forall \{i : \text{individual}\} \{\alpha \ \beta : \text{Prop}\}, \text{rb conjConst } i (\alpha \rightarrow \beta \rightarrow (\alpha \wedge \beta))$ 
```

Axiom T2: Transitivity of Implication

The reason `transConst` justifies the tautology $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)$.

Formula: `rb transConst i ((($\alpha \rightarrow \beta$) \wedge ($\beta \rightarrow \gamma$)) \rightarrow ($\alpha \rightarrow \gamma$))`

Interpretation: Allows agents to chain implications: having a reason to believe $\alpha \rightarrow \beta$ and having a reason to believe $\beta \rightarrow \gamma$ yields a reason to believe $\alpha \rightarrow \gamma$.

Empirical justification: Transitivity is easy for humans (Moshman 2020, p. 22). Simple chains of reasoning like "if A then B, if B then C, therefore if A then C" are natural.

Example:

- "If door open, then cold"
- "If cold, then heating needed"
- Therefore: "If door open, then heating needed"

```
axiom T2 {rb : reason  $\rightarrow$  individual  $\rightarrow$  Prop  $\rightarrow$  Prop} :  
   $\forall$  {i : individual} { $\alpha$   $\beta$   $\gamma$  : Prop}, rb transConst i ((( $\alpha \rightarrow \beta$ )  $\wedge$  ( $\beta \rightarrow \gamma$ ))  $\rightarrow$  ( $\alpha \rightarrow \gamma$ ))
```

Axiom T3: Distribution (Theory of Mind)

The reason `distConst` justifies for agent `i` the implication $R \text{ rb } j (\alpha \rightarrow \beta) \rightarrow (R \text{ rb } j \alpha \rightarrow R \text{ rb } j \beta)$.

Formula: `rb distConst i (R rb j ($\alpha \rightarrow \beta$) \rightarrow (R rb j $\alpha \rightarrow$ R rb j β))`

Interpretation: This embodies theory of mind: agent `i` can reason about agent `j`'s reasoning capabilities. Essential for common knowledge where we need nested beliefs.

Example: Suppose Alice observes that Bob has reason to believe "if door open, then cold" (i.e., `R rb Bob (door_open \rightarrow cold)`).

T3 gives Alice reason to believe that if Bob also has reason to believe "door open", then Bob can infer "cold".

This is Alice reasoning about Bob's reasoning ability—theory of mind.

Why this is needed: For common knowledge, we need infinite regress: "I know that you know that I know..." Each step requires reasoning about another agent's reasoning. T3 enables this.

```
axiom T3 {rb : reason  $\rightarrow$  individual  $\rightarrow$  Prop  $\rightarrow$  Prop} :  
   $\forall$  {i j : individual} { $\alpha$   $\beta$  : Prop},  
  rb distConst i (R rb j ( $\alpha \rightarrow \beta$ )  $\rightarrow$  (R rb j  $\alpha \rightarrow$  R rb j  $\beta$ ))
```

Section 4: Derived Epistemic Rules

We derive four key lemmas from the axioms. These make the proofs of A1 and A6 straightforward.

These lemmas show the *power* of our axioms—they generate a rich theory of reasoning from minimal assumptions.

Lemma E1: Modus Ponens for Reasons

If agent `i` has a reason to believe $\alpha \rightarrow \beta$ and a reason to believe α , then `i` has a reason to believe β .

Formula: `R rb i ($\alpha \rightarrow \beta$) \rightarrow R rb i $\alpha \rightarrow$ R rb i β`

Proof: Direct consequence of AR via existential elimination.

Significance: This is the key to proving A1. It shows that our definition of R as existential quantification over reasons naturally supports modus ponens.

```
lemma E1 {rb : reason → individual → Prop → Prop} :  
  ∀ {i : individual} {α β : Prop}, R rb i (α → β) → R rb i α → R rb i β := by  
  intro i α β h1 h2  
  obtain ⟨s, hs⟩ := h1  
  obtain ⟨t, ht⟩ := h2  
  use s * t  
  exact AR hs ht
```

Lemma L1: Conjunction for Reasons

If i has a reason to believe α and a reason to believe β , then i has a reason to believe $(\alpha \wedge \beta)$.

Formula: $R \text{ rb } i \alpha \rightarrow R \text{ rb } i \beta \rightarrow R \text{ rb } i (\alpha \wedge \beta)$

Proof strategy:

1. Use T1 to get a reason to believe $\alpha \rightarrow \beta \rightarrow (\alpha \wedge \beta)$
2. Apply AR twice to combine with reasons for α and β

Significance: Shows how tautology witness (T1) combines with application (AR) to enable conjunction formation. This is used in E2 for chaining implications.

```
lemma L1 {rb : reason → individual → Prop → Prop} :  
  ∀ {i : individual} {α β : Prop}, R rb i α → R rb i β → R rb i (α ∧ β) := by  
  intro i α β h1 h2  
  obtain ⟨s, hs⟩ := h1  
  obtain ⟨t, ht⟩ := h2  
  have h3 : rb (conjConst * s) i (β → (α ∧ β)) := AR T1 hs  
  use conjConst * s * t  
  exact AR h3 ht
```

Lemma E2: Transitivity for Reasons

If i has reason to believe $\alpha \rightarrow \beta$ and reason to believe $\beta \rightarrow \gamma$, then i has a reason to believe $\alpha \rightarrow \gamma$.

Formula: $R \text{ rb } i (\alpha \rightarrow \beta) \rightarrow R \text{ rb } i (\beta \rightarrow \gamma) \rightarrow R \text{ rb } i (\alpha \rightarrow \gamma)$

Proof strategy:

1. Form conjunction via L1
2. Apply T2 for transitivity

Importance: Crucial for A6, which requires chaining implications. This lemma shows that agents can combine their reasons for two implications into a reason for their composition.

```
lemma E2 {rb : reason → individual → Prop → Prop} :  
  ∀ {i : individual} {α β γ : Prop},  
  R rb i (α → β) → R rb i (β → γ) → R rb i (α → γ) := by  
  intro i α β γ h1 h2  
  have h3 : R rb i ((α → β) ∧ (β → γ)) := L1 h1 h2  
  obtain ⟨s, hs⟩ := h3  
  use transConst * s  
  exact AR T2 hs
```

Lemma E3: Distribution Rule for Multi-Agent Reasoning

If i has reason to believe " j has reason to believe $\alpha \rightarrow \beta$ ", then i has reason to believe "if j has reason to believe α , then j has reason to believe β ".

Formula: $R \text{ rb } i (R \text{ rb } j (\alpha \rightarrow \beta)) \rightarrow R \text{ rb } i (R \text{ rb } j \alpha \rightarrow R \text{ rb } j \beta)$

Proof: Direct application of T3.

Significance: Embodies theory of mind—the ability to reason about others' reasoning. This is what enables the inductive step in Lewis's theorem, where we need to reason about nested beliefs across multiple agents.

```
lemma E3 {rb : reason → individual → Prop → Prop} :  
  ∀ {i j : individual} {α β : Prop},  
    R rb i (R rb j (α → β)) → R rb i (R rb j α → R rb j β) := by  
  intro i j α β h1  
  obtain ⟨s, hs⟩ := h1  
  use distConst * s  
  exact AR T3 hs
```

Section 5: Axioms A1 and A6 as Theorems

This section proves Lewis's rules A1 and A6 as theorems rather than axioms as in Cubitt & Sugden.

This is the **central achievement** of this formalization.

Historical Context

- **Lewis (1969):** Stated A1 and A6 informally
- **Cubitt-Sugden (2003):** Made A1 and A6 explicit as axioms
- **Sillari (2005):** A1 fails under modal semantics
- **Vromen (2024):** Proved A1 and A6 as theorems

Why This Is Significant

1. Shows justification logic is the **RIGHT** framework for Lewis's theory
2. Explains **WHY** A1 and A6 hold (not just that they do)
3. Resolves the contradiction in Sillari's approach
4. Provides minimal, independently motivated foundation

```
variable (A : Prop)
```

Theorem A1: Indication Supports Reasoning

If A indicates to i that α , and i has reason to believe A , then i has reason to believe α .

Formula: $\text{Ind } A \ i \ \alpha \rightarrow R \text{ rb } i \ A \rightarrow R \text{ rb } i \ \alpha$

Proof: Immediate from E1, since $\text{Ind } A \ i \ \alpha = R \ i \ (A \rightarrow \alpha)$. Three lines.

Historical significance: This is Lewis's crucial axiom—used at every step to convert indication into belief. That it becomes trivial in justification logic suggests this is the natural framework for Lewis's theory.


```

lemma A1 {rb : reason → individual → Prop → Prop} :
  ∀ {i : individual} {α : Prop}, Ind rb A i α → R rb i A → R rb i α := by
  intro i α h1 h2
  obtain (t, ht) := h2
  obtain (s, hs) := h1
  use s * t
  exact AR hs ht

```

Alternative proof demonstrating A1 follows from E1:

```

lemma A1_alt {rb : reason → individual → Prop → Prop} :
  ∀ {i : individual} {α : Prop}, Ind rb A i α → R rb i A → R rb i α := by
  intro i α h1 h2
  exact E1 h1 h2

```

Theorem A6: Indication Propagation Across Agents

If A indicates to i that "j has reason to believe A", and i has reason to believe "A indicates to j that α", then A indicates to i that "j has reason to believe α".

Formula: $\text{Ind rb } A \ i \ (R \text{ rb } j \ A) \rightarrow R \text{ rb } i \ (\text{Ind rb } A \ j \ \alpha) \rightarrow \text{Ind rb } A \ i \ (R \text{ rb } j \ \alpha)$

Proof strategy:

1. Given: $R \ i \ (A \rightarrow R \ j \ A)$ and $R \ i \ (R \ j \ (A \rightarrow \alpha))$
2. Use E3: Get $R \ i \ (R \ j \ A \rightarrow R \ j \ \alpha)$ (theory of mind)
3. Use E2: Chain to get $R \ i \ (A \rightarrow R \ j \ \alpha)$ (transitivity)

Importance: Used at each inductive step of Lewis's theorem to "lift" indication from α to $R \ j \ \alpha$, building the tower of nested beliefs.

Why this requires justification logic: We need to chain implications across agents while preserving evidential structure. Modal logic cannot do this because it loses track of which reasons justify which nested beliefs.

```

lemma A6 {rb : reason → individual → Prop → Prop} :
  ∀ α {i j : individual},
  Ind rb A i (R rb j A) →
  R rb i (Ind rb A j α) →
  Ind rb A i (R rb j α) := by
  intro p i j h1 h2
  have h3 : R rb i (R rb j A → R rb j p) := E3 h2
  have h4 : R rb i (A → R rb j p) := E2 h1 h3
  exact h4

```

—

Section 6: G-Closure and Lewis's Main Theorem

We now prove Lewis's main theorem: that reflexive common indicators generate common reason to believe for the entire infinite hierarchy.

G-Closure

The inductive closure of \emptyset under " $R \text{ rb } i _$ " for any agent i.

Structure:

- base: ϕ is in its own closure
- step: if p is in closure, so is $R \text{ rb } i \ p$ for any agent i

Propositions in $G\text{closure } \text{rb } \phi$:

- ϕ (base)
- $R \ i \ \phi$ (level 1)
- $R \ j \ (R \ i \ \phi)$ (level 2)
- $R \ k \ (R \ j \ (R \ i \ \phi))$ (level 3)
- ... ad infinitum

Connection to Lewis: This captures common knowledge as an infinite hierarchy of nested reasons to believe. Everyone believes ϕ , everyone believes everyone believes ϕ , and so on forever.

```

inductive Gclosure ( $\phi$  : Prop) : Prop  $\rightarrow$  Prop
| base : Gclosure  $\phi$   $\phi$ 
| step ( $p$  : Prop) ( $i$  : individual) : Gclosure  $\phi$   $p \rightarrow$  Gclosure  $\phi$  ( $R \text{ rb } i \ p$ )

```

Lewis's Theorem on Common Knowledge

If A is a basis for common knowledge of ϕ (satisfying conditions C1–C4), then every proposition in the G -closure of ϕ is believed by every agent.

For all $p \in G\text{closure } \phi$ and all agents i : $R \text{ rb } i \ p$

Conditions:

- **C1:** Everyone has reason to believe A
- **C2:** A indicates that everyone has reason to believe A
- **C3:** A indicates ϕ
- **C4:** Shared reasoning standards

Proof strategy: We prove by induction on $G\text{closure}$ that every agent i has *indication* (not just reason) that A implies p . This is stronger than needed, but makes induction work.

Then we apply A1 with C1 to convert indication to actual reason.

Inductive structure:

- **Base case:** $p = \phi$
- Need: $\text{Ind } A \ i \ \phi$ for all i
- Have: C3 gives exactly this
- **Inductive step:** $p = R \ j \ u$, given $\text{Ind } A \ i \ u$ for all i
- Need: $\text{Ind } A \ i \ (R \ j \ u)$ for all i
- By IH: $\text{Ind } A \ i \ u$ (indication for u)
- By C4: $R \ i \ (\text{Ind } A \ j \ u)$ (transparency)
- By A6: $\text{Ind } A \ i \ (R \ j \ u)$ (lift indication)

Why this succeeds in justification logic:

1. Explicit reason structure enforces that indications can be combined (via AR)
2. Conditions ensure reasons propagate (C4) and can be applied (C1, C2, C3)
3. A1 and A6 are proven, not vulnerable to counterexamples

Why this fails in modal logic (Sillari):

1. Implicit knowledge allows frames where conditions hold locally but fail globally
2. Counterexamples exist where conditions hold at one world but fail at reachable worlds
3. Axiom A1 itself fails in general frames

```

theorem Lewis (A  $\phi$  : Prop) (p : Prop)
  (C1 :  $\forall i, R\ rb\ i\ A$ )
  (C2 :  $\forall i\ j, Ind\ rb\ A\ i\ (R\ rb\ j\ A)$ )
  (C3 :  $\forall i, Ind\ rb\ A\ i\ \phi$ )
  (C4 :  $\forall \alpha\ i\ j, Ind\ rb\ A\ i\ \alpha \rightarrow R\ rb\ i\ (Ind\ rb\ A\ j\ \alpha)$ )
  (h7 : @Gclosure individual reason rb  $\phi$  p) :
   $\forall i, R\ rb\ i\ p := \text{by}$ 
intro i
-- Prove stronger claim: Ind A i p (then use A1 + C1)
have h1 : Ind rb A i p := by
  induction h7 with
  | base =>
    -- Base case: p =  $\phi$ , use C3
    exact C3 _
  | step u j hu ih =>
    -- Inductive case: p = R j u
    -- IH: Ind A i u (i knows A implies u)
    -- Goal: Ind A i (R j u) (i knows A implies j believes u)
    -- Step 1: From IH + C4, get i knows j has indication
    have h3 : R rb i (Ind rb A j u) := C4 u _ _ ih
    -- Step 2: Use A6 to lift indication one level
    have h4 : R rb i (Ind rb A j u)  $\rightarrow$  Ind rb A i (R rb j u) :=
      A6 A u (C2 _ _)
    exact h4 h3
-- Cash out indication using common basis
exact A1 A h1 (C1 _)
end JustificationLogic

```

Section 7: Summary and Assessment

Technical Achievements

This formalization demonstrates:

- A1 proven as theorem (3 lines from AR)
- A6 proven as theorem (10 lines from E2, E3)
- Lewis's theorem proven by structural induction
- All proofs fully formal (zero "sorry" statements)
- Minimal assumptions (only T1, T2, T3)

Conceptual Clarifications

The formalization provides clarity on:

- Explicit reason terms resolve ambiguity in Lewis's "thereby"
- Indication = having reason for implication (not conjunction)
- Common knowledge = reachability in reason structure
- No logical omniscience required

Historical Resolution

This resolves a 50-year progression:

Approach	A1 Status	A6 Status
Lewis (1969)	Informal	Informal
Cubitt-Sugden (2003)	Axiom	Axiom
Sillari (2005)	FAILS	N/A
Vromen (2024)	THEOREM	THEOREM

Philosophical Insights

The success of this formalization suggests Lewis’s theory is best understood in terms of **explicit justifications and evidence**, not modal operators.

1. Reasons as Objects Lewis’s ”thereby” indicates reasons can be tracked and combined. They have *structure* (via multiplication) that preserves evidential dependence.

2. Evidence Structure Common knowledge arises from public, shared structure of reasons. The AR rule shows how individual reasons compose into shared understanding.

3. Natural Framework That A1 and A6 become trivial theorems (3 and 10 lines respectively) suggests justification logic is the ”right” setting for Lewis’s theory.

Compare:

- Modal logic: A1 fails (counterexample in Sillari file)
- Primitive relations: A1 must be assumed (Cubitt-Sugden file)
- Justification logic: A1 trivially proven (this file)

4. Minimal Rationality Only three tautologies (T1, T2, T3) needed—no logical omniscience. These correspond to reasoning patterns humans find easy:

- Conjunction formation (combining beliefs)
- Transitivity (chaining implications)
- Theory of mind (reasoning about others’ reasoning)

Recommendation for Readers

- For understanding **what doesn’t work** → `Sillari_refutation.lean`
- For understanding **Lewis’s argument structure** → `Cubitt_Sugden_baseline.lean`
- For understanding **the correct foundations** → **this file**

Final Note

This formalization vindicates Lewis’s (1969) informal argument by showing it can be made fully rigorous in an appropriate logical framework. The framework needed is justification logic — a logic of explicit evidence — not modal logic.

Lewis’s insight was correct: common knowledge emerges from public, reflexive indicators through a finite argument, not an infinite regress. Justification logic makes this insight formally precise.