Exercise of Chapter 1

Written by Hsin-Jung, Wu.

My mail is hsinjungwu@gmail.com

1. For each of the following sets S, find $\sup(S)$ and $\inf(S)$ if they exist:

   (a) $\{x \in \mathbb{R} \mid x^2 < 5\}$

   Solution: $\sup(S) = \sqrt{5}$; $\inf(S) = -\sqrt{5}$.

   (b) $\{x \in \mathbb{R} \mid \ x^2 > 7\}$

   Solution: Neither $\sup(S)$ nor $\inf(S)$ exist.

   (c) $\{1/n \mid n, \text{an integer }, n > 0\}$

   Solution: $\sup(S) = 1$; $\inf(S) = 0$.

   (d) $\{-1/n \mid n, \text{an integer }, n > 0\}$

   Solution: $\sup(S) = 0$; $\inf(S) = -1$.

   (e) $\{.3, .33, .333, \dots\}$

   Solution: $\sup(S) = 1/3$; $\inf(S) = 0.3$.

   The proof is easy, so I omit it. $\square$

2. Review the proof that $\sqrt{2}$ is irrational. Generalize this to $\sqrt{k}$ for k a positive integer that is not a perfect square.

   Solution: The proof that $\sqrt{2}$ is irrational is easy, so I omit it. Let $k = \prod_{i=1}^{n} {p_i}^{m_i}$, where p_i is prime, m_i is positive integer for all i. Since k is not a perfect square, there exists some m_i which is odd. May assume m_1 is odd. If $\sqrt{k}$ is rational, say $\sqrt{k} = \frac{r}{q}$, where integers r, q are relative prime. So $k = \frac{r^2}{q^2}$, i.e. $kq^2 = r^2$. Then $p_1 \mid r^2$, we have $p_1 \mid r$. We know ${p_1}^{m_1} \mid r^2$, but m_1 is odd, thus we must have $p_1 \mid q^2$. So $p_1 \mid q$. Hence $\gcd(p, q) > 1$, then we have a contradiction. $\square$

3. (a) Let $x \geq 0$ be a real number for any $\varepsilon > 0$, $x \leq \varepsilon$. Show that $x = 0$

   Solution: If $x > 0$, choose $\varepsilon = \frac{x}{2}$, so $x \leq \frac{x}{2}$. Thus $\frac{x}{2} \leq 0$, then $x \leq 0$. Hence we get a contradiction. Then x must be 0.

   (b) Let $x =]0, 1[$. Show that for any $\varepsilon > 0$ there exists an $x \in S$ such that $x < \varepsilon$.

   Solution: If $\varepsilon \geq 1$, choose $x = \frac{1}{2}$. If $1 > \varepsilon > 0$, choose $x = \frac{\varepsilon}{2}$ then $x \in S$ and $x < \varepsilon$.

   $\square$

4. Show that $d = \inf(S)$ iff d is a lower bound for S and for any $\varepsilon > 0$ there is an $x \in S$ such that $d \geq x - \varepsilon$.

   Solution: It is trivial that d is a lower bound of S. Choose $x = d + \frac{\varepsilon}{2}$, then $x - \varepsilon = d - \frac{\varepsilon}{2} \leq d$.

   On the other side, assume d' is another lower bound and $d' > d$. Choose $\varepsilon = \frac{d'-d}{2}$, there is an $x = d + \varepsilon \in S$. But $x = d + \varepsilon = \frac{d'+d}{2} < d'$, then we have a contradiction. then d is $\inf(S)$. $\square$

5. Let x_n be a monotone increasing sequence bounded above and consider the set $S = \{x_1, x_2, \ldots\}$. Show that x_n converges to $\sup(S)$. Make a similar statement for decreasing sequences.

   Solution: Let $\sup(S) = x$, so for all $\varepsilon > 0$, there is an x_N such that $x_N > x - \varepsilon$, and it's easy to see that $x + \varepsilon > x \geq x_n$ for all n. Thus $x + \varepsilon > x_n \geq x_N > x - \varepsilon$ whenever $n \geq N$. Hence $|x_n - x| < \varepsilon$, so x_n converges to x $= \sup(S)$. Let y_n be a monotone decreasing sequence bounded below and consider the set $S = \{y_1, y_2, \ldots\}$, y_n converges to $\inf(S)$. $\square$

6. Let A and B be two nonempty sets of real numbers with the property that $x \leq y$ for all $x \in A$, $y \in B$. Show that there exists a number $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in A$, $y \in B$. Give a counterexample to this statement

for rational numbers (it is, in fact, equivalent to the completeness axiom and is the basis for another way of formulating the completeness axiom known as ***Dedekind cuts***).

Solution: Choose $c = \frac{\sup(A)+\inf(B)}{2}$. Claim $\sup(A) \leq \inf(B)$. Let $a = \sup(A)$, $b = \inf(B)$ Suppose not, choose $\varepsilon = a - b > 0$. So there is a $y < b + \varepsilon = a < x$, so we have a contradiction. Then $a \leq b$, hence $x \leq a \leq c \leq b \leq y$.

Choose $A = \{x \mid x < \pi\}$, $B = \{y \mid y > \pi\}$ and there is no $c \in \mathbb{Q}$ satisfied this property. $\square$

7. For nonempty set A, $B \subset \mathbb{R}$, let $A+B = \{x+y \mid x \in A$ and $y \in B$. Show that $\sup(A+B) = \sup(A) + \sup(B)$.

   Solution: Let $\sup(A) = a$, $\sup(B) = b$. Since $x < a$ and $y < b$, we have $x + y < a + b$. Then $a + b$ is an upper bound of $A + B$, thus $\sup(A + B) \leq \sup(A) + \sup(B)$.

   For all $\varepsilon > 0$, $a - \varepsilon < x$ and $b - \varepsilon < y$. Then $a + b - 2\varepsilon < x + y < \sup(A + B)$. Since ε is arbitrary, $a + b \leq \sup(A + B)$. Thus we get the conclusion. $\square$

8. For nonempty sets A, $B \subset \mathbb{R}$ determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:

   (a) $\sup(A \bigcap B) \leq \inf\{\sup(A), \sup(B)\}$

   Solution: True.

   For all $x \in A \bigcap B$, $x \leq \sup(A)$ and $\sup(B)$, then $x \leq \inf(\sup(A), \sup(B))$. So $\sup(A \bigcap B) \leq \inf(\sup(A), \sup(B))$.

   (b) $\sup(A \bigcap B) = \inf\{\sup(A), \sup(B)\}$

   Solution: False.

   Let $A = (0, 1)$, $B = (1, 2)$. $A \bigcap B = \emptyset$. So $\sup(A \bigcap B)$ is infinite, $\inf(\sup(A), \sup(B)) = 1$.

(c) $\sup(A \bigcup B) \geq \sup\{\sup(A), \sup(B)\}$

Solution: True.

It is trivial that $\sup(A) \leq \sup(A \bigcup B)$.

(d) $\sup(A \bigcup B) = \sup\{\sup(A), \sup(B)\}$

Solution: True.

For all $x \in A \bigcup B$, $x \leq \sup(A)$ or $x \leq \sup(B)$. Then $\sup(\sup(A), \sup(B))$ is an upper bound of $A \bigcup B$, so $\sup(A \bigcup B) \leq \sup(\sup(A), \sup(B))$. By (c) we have the conclusion $sup(A \bigcup B) = \sup(\sup(A), \sup(B))$. Then we complete the proof.

$\square$

9. Let x_n be a bounded sequence of real numbers and $y_n = (-1)^n x_n$. Show that $\limsup y_n \leq \limsup |x_n|$. Need we have equality? Make up a similar inequality for $\liminf$.

Solution: Since $y_n \leq |y_n| = |x_n|$, we can use the following theorem:

If $a_n \leq b_n$ for each $n = 1, 2, \ldots$. Then we have

$$\limsup(a_n) \leq \limsup(b_n)$$

$$\liminf(a_n) \leq \liminf(b_n)$$

Proof of the Thm:

It is easy to see that $\sup\{a_{n+1}, a_{n+2}, \ldots\} \leq \sup\{b_{n+1}, b_{n+2}, \ldots\}$, then

$$\begin{aligned}
\limsup(a_n) &= \inf\{\sup\{a_{n+1}, a_{n+2}, \ldots\} \mid n = 1, 2, \ldots\} \\
&\leq \inf\{\sup\{b_{n+1}, b_{n+2}, \ldots\} \mid n = 1, 2, \ldots\} \\
&= \limsup(b_n).
\end{aligned}$$

Similarly, $\inf\{a_{n+1}, a_{n+2}, \ldots\} \leq \inf\{b_{n+1}, b_{n+2}, \ldots\}$. Then

$$\begin{aligned}
\liminf(a_n) &= \sup\{\inf\{a_{n+1}, a_{n+2}, \ldots\} \mid n = 1, 2, \ldots\} \\
&\leq \sup\{\inf\{b_{n+1}, b_{n+2}, \ldots\} \mid n = 1, 2, \ldots\} \\
&= \liminf(b_n).
\end{aligned}$$

Equality need not old, let $x_n = (-1)^{n+1}$. Then $\limsup y_n = -1$, $\limsup |x_n| = 1$. $\liminf y_n < \liminf |x_n|$. $\square$

10. Verify that the bounded metric in Example **1.7.2d** is indeed a metric.

Solution: It is easy to check positivity, nondegeneracy and symmetry. Given positive numbers a, b, c with $c \le a + b$. Since $c \le a + b + abc + 2ab$, so $abc + ac + bc + c \le abc + ac + bc + a + b + abc + 2ab$. Then

$$\begin{aligned} c(1+a)(1+b) &\le a + ab + ac + abc + b + ab + bc + abc \\ &= a(1+b)(1+c) + b(1+a)(1+c). \end{aligned}$$

Hence

$$\frac{c}{1+c} \le \frac{a}{1+a} + \frac{b}{1+b}.$$

So we have

$$\begin{aligned} \rho(x,z) + \rho(z,y) &= \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{[1+d(z,y)} \\ &\ge \frac{d(x,y)}{1+d(x,y)} \\ &= \rho(x,y) \end{aligned}$$

if we put d(x,z) = a, d(z,y) = b, d(x,y) = c. $\square$

11. Show that **i** and **ii** of Theorem **1.3.4** both imply the completeness axiom for an ordered field.

Solution: Let x_n be an increasing sequence bounded above and consider the set $S = \{x_1, x_2, \dots\}$. Thus by **i** we have S has a least upper bound $sup(S)$. By *exercise* (5), x_n converges to $\sup(S)$. So **i** implies the completeness. Similarly way to do **ii** implies the completeness.

12. In an inner product space show that

(a) $2\| x \|^2 + 2\| y \|^2 = \| x + y \|^2 + \| x - y \|^2$ (***parallelogram law***).

Solution:

$$\begin{aligned}\| x+y \|^2 + \| x-y \|^2 &= < x+y, x+y > + < x-y, x-y > \\ &= < x,x > + < y,x > + < x,y > + < y,y > \\ &\quad + < x,x > - < y,x > - < x,y > + < y,y > \\ &= 2(< x,x > + < y,y >) \\ &= 2\| x \|^2 + 2\| y \|^2.\end{aligned}$$

(b) $\| x+y \| \| x-y \| \leq \| x \|^2 + \| y \|^2$.

Solution:

$$\begin{aligned}0 &\leq (\| x+y \| - \| x-y \|)^2 \\ &= \| x+y \|^2 + \| x-y \|^2 - 2 \| x+y \| \| x-y \| \\ &= 2\| x \|^2 + 2\| y \|^2 - 2 \| x+y \| \| x-y \| .\end{aligned}$$

So $\| x+y \| \| x-y \| \leq \| x \|^2 + \| y \|^2$.

(c) $4 < x,y > = \| x+y \|^2 - \| x-y \|^2$ (***polarization identity***).

Solution:

$$\begin{aligned}\| x+y \|^2 - \| x-y \|^2 &= < x+y, x+y > + < x-y, x-y > \\ &= < x,x > + < y,x > + < x,y > + < y,y > \\ &\quad - < x,x > + < y,x > + < x,y > - < y,y > \\ &= 2(< y,x > + < x,y >) \\ &= 4 < x,y >\end{aligned}$$

□

13. What is the orthogonal complement in $\mathbb{R}^4$ of the space spanned by (1,0,1,1) and (-1,2,0,0)?

Solution: The space spanned by $\{(2,1,-2,0),(2,1,0,-2)\}$. It's easy to calculate, then I omit it. □

14. (a) Prove ***Lagrange's indetity***

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 = \Big(\sum_{i=1}^{n} {x_i}^2\Big)\Big(\sum_{i=1}^{n} {y_i}^2\Big) - \sum_{1\le i<j\le n} (x_i y_j - x_j y_i)^2$$

and use this to give another proof of the Cauchy-Schwarz inequality.

Solution: It is easily to solved by induction.

Note Cauchy-Schwarz inequality is

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 \le \Big(\sum_{i=1}^{n} {x_i}^2\Big)\Big(\sum_{i=1}^{n} {y_i}^2\Big).$$

(b) Show that

$$\Big(\sum_{i=1}^{n} (x_i + y_i)^2\Big)^{1/2} \le \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2} + \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}$$

Solution:

$$\begin{aligned}\sum_{i=1}^{n} (x_i + y_i)^2 &= \sum_{i=1}^{n} {x_i}^2 + \sum_{i=1}^{n} {y_i}^2 + 2\sum_{i=1}^{n} x_i y_i \\ &\le \sum_{i=1}^{n} {x_i}^2 + \sum_{i=1}^{n} {y_i}^2 + 2\sum_{i=1}^{n} \mid x_i y_i \mid \\ &\le \sum_{i=1}^{n} {x_i}^2 + \sum_{i=1}^{n} {y_i}^2 + 2\{(\sum_{i=1}^{n} {x_i}^2)(\sum_{i=1}^{n} {y_i}^2)\}^{1/2} \\ &= \{(\sum_{i=1}^{n} {x_i}^2)^{1/2} + (\sum_{i=1}^{n} {y_i}^2)^{1/2}\}^2\end{aligned}$$

□

15. Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)/2$. Show that x_n is a Cauchy sequences.

Solution: It is easy to calculate $d(x_n, x_{n+1}) \le \frac{d(x_1,x_2)}{2^{n-1}}$ For all $\varepsilon > 0$, there is a positive integer $N > 3 + \frac{\ln \frac{d(x_1,x_2)}{\varepsilon}}{\ln 2}$ such that $|x_n - x_m| < \varepsilon$, whenever $n \ge m \ge N$.

Since

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_N) + d(x_m, x_N) \\
&\leq d(x_n, x_{n-1}) + \cdots + d(x_{N+1}, x_N) \\
&\quad + d(x_m, x_{m-1}) + \cdots + d(x_{N+1}, x_N) \\
&\leq d(x_{N+1}, x_N)(\frac{1}{2^{n-N-1}} + \ldots \frac{1}{2} + 1) \\
&\quad + d(x_{N+1}, x_N)(\frac{1}{2^{m-N-1}} + \ldots \frac{1}{2} + 1) \\
&\leq 4d(x_{N+1}, x_N) \\
&\leq \frac{d(x_1, x_2)}{2^{N-3}} \\
&< \varepsilon
\end{aligned}$$

So x_n is a Cauchy sequence. □

16. Prove Theorem **1.6.4**. In fact, for vector spaces $V_1, \ldots, V_n$, show that $V = V_1 \times \cdots \times V_n$ is a vector space.

    Solution: I think this question is easy if you have learned Linear Algebra!! □

17. Let $S \subset \mathbb{R}$ be bounded below and nonempty. Show that

    $$\inf(S) = \sup\{x \in \mathbb{R} \mid x \text{ is a lower bound for } S\}.$$

    Solution: Since $\inf(S)$ is the greatest lower bound of S, then $x \leq \inf(S)$ for all x is a lower bound for S. Hence $\sup\{x \in \mathbb{R} \mid x \text{ is a lower bound for } S\} \leq \inf(S)$. And $\inf(S)$ is a lower bound for S, then $\sup\{x \in \mathbb{R} \mid x \text{ is a lower bound for } S\} \geq \inf(S)$. Hence $\sup\{x \in \mathbb{R} \mid x \text{ is a lower bound for } S\} = \inf(S)$. □

18. Show that in $\mathbb{R}$, $x_n \to x$ iff $-x_n \to -x$. Hence, prove that the completeness axiom is equivalent to the statement that every decreasing sequence $x_1 \geq x_2 \geq x_3 \cdots$ bounded below and converges. Prove that the limit of the sequence is $\inf\{x_1, x_2 \ldots\}$.

    Solution: I think this question is easy, so I omit it. □

19. Let $x = (1, 1, 1) \in \mathbb{R}^3$ be written $x = y_1 f_1 + y_2 f_2 + y_3 f_3$, where $f_1 = (1, 0, 1)$, $f_2 = (0, 1, 1)$, and $f_3 = (1, 1, 0)$. Compute the components y_i.

    Solution: It is easy to compute it, then I omit it. $\square$

20. Let S and T be nonzero orthogonal subspaces of $\mathbb{R}^n$. Prove that if S and T are orthogonal complements (that is, S and T span all of $\mathbb{R}^n$), then $S \bigcap T = \{0\}$ and $\dim(S) + \dim(T) = n$, where $\dim(S)$ denotes dimension of S. Given examples in $\mathbb{R}^3$ where $\dim(S) + \dim(T) = n$ holds and examples where it fails. Can it fail in $\mathbb{R}^2$?

    Solution: If you have learned Linear Algebra, you can solve this problem. So I omit it. $\square$

21. Show the sequence in Worked Example **1.2** can be chosen to be increasing.

    Solution: If $x \in S$, let $x_k = x$. Otherwise, $x_k < x$ for all k. Let $y_1 = x_1$, y_2 be the first element of $\{x_2, x_3, \dots\}$ with $y_2 \geq y_1$, etc. $\square$

22. (a) If x_n and y_n are bounded sequences in $\mathbb{R}$, prove that

    $$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

    Solution: Since

    $$\begin{aligned} &\sup\{x_{n+1} + y_{n+1}, x_{n+2} + y_{n+2}, \dots\} \\ \leq\ &\sup\{x_{n+1}, x_{n+2}, \dots\} + \sup\{y_{n+1}, y_{n+2}, \dots\} \end{aligned}$$

    for all n then

    $$\begin{aligned} \limsup(x_n + y_n) &= \inf\{\sup\{x_{n+1} + y_{n+1}, x_{n+2} + y_{n+2}, \dots\} \mid n = 1, 2, \dots\} \\ &\leq \inf\{\sup\{x_{n+1}, x_{n+2}, \dots\} \mid n = 1, 2, \dots\} \\ &\quad + \inf\{\sup\{y_{n+1}, y_{n+2}, \dots\} \mid n = 1, 2, \dots\} \\ &= \limsup(x_n) + \limsup(y_n) \end{aligned}$$

    (b) Is the product rule true for lim sups?

Solution: Yes. Similarly way as above.

$\square$

23. Let $P \subset \mathbb{R}$ be a set such that $x \geq 0$ for all $x \in P$ and for any integer k there is an $x_k \in P$ such that $kx_k \leq 1$. Prove that $0 = \inf(P)$.

    Solution: We only need to check for all $\varepsilon > 0$, there is an $x_k \in P$ such that $x_k < \varepsilon$. Given $\varepsilon > 0$, by Archimedean property there is an integer $k > 0$ such that $0 < \frac{1}{k} < \varepsilon$. So $x_k \leq \frac{1}{k} < \varepsilon$, then we complete the proof. $\square$

24. If $\sup(P) = \sup(Q)$ and $\inf(P) = \inf(Q)$, dose $P = Q$?

    Solution: No. You may take $P = (0, 1)$ and $Q = [0, 1]$ as counterexample.

25. We say that $P \leq Q$ if for each $x \in P$, there is a $y \in Q$ with $x \leq y$.

    (a) If $P \leq Q$, then show that $\sup P \leq \sup Q$.

    Solution: If $x \in P$, there is a $y \in Q$, such that $x < y < \sup(Q)$ then $\sup(Q)$ is upper bound of P.

    (b) Is it true that $\inf(P) \leq \inf(Q)$?

    Solution: False. choose $P = \{1\}$, $Q = \{0, 1\}$ $\inf(P) = 1$, $\inf(Q) = 0$.

    (c) If $P \leq Q$ and $Q \leq P$, does $P = Q$?

    Solution: No. Consider the example above.

$\square$

26. Assume that $A = \{a_{m,n} \mid m = 1, 2, 3, \dots$ and $n = 1, 2, 3, \dots\}$ is a bounded set and that $a_{m,n} \geq a_{p,q}$ whenever $m \geq p$ and $n \geq q$. Show that

$$\lim_{n\to\infty} a_{n,n} = \sup A.$$

Solution: Let $\lim\limits_{n\to\infty} a_{n,n} = a$. Claim $a = \sup(A)$. Since $a_{n,n}$ is an increasing sequence and bounded above, we have $a = \sup\{a_{n,n} \mid n = 1, 2, 3, \dots\} \leq \sup(A)$.

Claim a is an upper bound of A. Given $a_{m,n} \in A$, we can choose $a_{k,k}$ such that $a_{m,n} \leq a_{k,k} \leq a$ where $k = m + n$, thus a is an upper bound of A. So $\sup(A) \leq a$. Then we complete the proof. $\square$

27. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$ and $B = \{d(x, y), (0, 0) \mid (x, y) \in S\}$. Find $\inf(B)$.

Solution: Claim $\inf(B) = \sqrt{2}$. For all $(x, y) \in S$, $\sqrt{2} \leq (\frac{x^2+1}{x^2})^{1/2} < (x^2 + y^2)^{1/2}$ so $\sqrt{2}$ is a lower bound of B. For all $\varepsilon > 0$, there is a pair $(x, y) = (1, \sqrt{1 + \frac{\varepsilon}{2}})$ such that $d((x, y), (0, 0)) = (1 + 1 + \frac{\varepsilon}{2})^{1/2} < (2 + \varepsilon)^{1/2} < \sqrt{2} + \varepsilon$. Hence $\sqrt{2}$ is $\inf(B)$. $\square$

28. Let x_n be a converge sequence in $\mathbb{R}$ and define $A_n = \sup\{x_n, x_{n+1} \ldots\}$ and $B_n = \inf\{x_n, x_{n+1} \ldots\}$. Prove that A_n converges to the same limit as B_n which in turn is the same as the limit of x_n.

Solution: Since x_n is convergent sequence, we have that it is bounded. Then sequence of A_n and B_n are bounded. Since A_n is a decreasing sequence, then $\lim A_n = \inf A_n = \inf\{\sup\{x_n, x_{n+1}, \dot{\}} \mid n \in \mathbb{N}\} = \limsup x_n$. Similarly we have $\lim B_n = \liminf x_n$. So $\lim A_n = \limsup x_n = \lim x_n = \liminf x_n = \lim B_n$. $\square$

29. For any $x \in \mathbb{R}$ satisfying $x \geq 0$, prove the existence of $y \in \mathbb{R}$ such that $y^2 = x$.

Solution: If x is perfect square, it is trivial. Otherwise let $A = \{y \in \mathbb{R} \mid y^2 < x\}$, and let $\sup(A) = a$. It is clearly that $a \geq 0$. Claim $a^2 = x$. If $a^2 > x$, choose $\varepsilon = 2a+1$, then there is $y \in A$ such that $y > a-\varepsilon$. So $y^2 > (a - \varepsilon)^2 = (a + 1)^2 > a^2 > x$, a contradiction. If $a^2 < x$, then $a \in A$. By Archimedean property there is a positive integer p, such that $0 < \frac{1}{p^2} < x$. Pick $y = a + \frac{x-a^2}{a+px} > a$, Then $y^2 - x = \frac{a^2-x}{xp^2-1}a + px^2 < 0$. Thus $y > a$ and $y \in A$, a contradiction. So a^2 must be x, then we complete the proof. $\square$

30. Let V be the vector space $C([0, 1])$ with norm $\| f \|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$. Show that the parallelogram law is fails and conclude that this norm does not

come from any inner product. (Refer to Exercise 12.)

Solution: Let f(x) = 1, g(x) = x, then $\| f+g \|_\infty = 2$, $\| f-g \|_\infty = 1$, $\| f \|_\infty = 1$, $\| g \|_\infty = 1$. Thus $\| f+g \|_\infty^2 + \| f-g \|_\infty^2 = 5 \neq 4 = 2\| f \|_\infty^2 + 2\| g \|_\infty^2$. Suppose $\| f \|_\infty$ comes from some inner product, i.e. $\| f \|_\infty^2 =< f, f >$. Then

$$\begin{aligned} \| f+g \|^2 + \| f-g \|^2 &=< f+g, f+g > + < f-g, f-g > \\ &=< f,g > + < g,f > + < f,g > + < g,g > \\ &\quad + < f,g > - < g,f > - < f,g > + < g,g > \\ &= 2(< f,f > + < g,g >) \\ &= 2\| f \|^2 + 2\| g \|^2. \end{aligned}$$

Thus we have a contradiction. So it cannot come from some inner product. □

31. Let A, $B \subset \mathbb{R}$ and $f : A \times B \to \mathbb{R}$ be bounded. Is it true that

$$\sup\{f(x,y) \mid f(x,y) \in A \times B\} = \sup\{\sup\{f(x,y) \mid x \in A\} \mid y \in B\}$$

or, the same thing in different notation,

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B}(\sup_{x\in A} f(x,y)).$$

Solution: True. Fixed $y \in B$,

$$\sup\{f(x,y) \mid x \in A\} \leq \sup\{f(x,y) \mid (x,y) \in A \times B\}.$$

Then

$$\sup\{\sup\{f(x,y) \mid x \in A\} \mid y \in B\} \leq \sup\{f(x,y) \mid (x,y) \in A \times B\}.$$

Given $(x_0, y_0) \in A \times B$, $f(x_0, y_0) \leq \sup\{f(x, y_0) \mid x \in A\}$. But

$$\sup\{f(x,y_0) \mid x \in A\} \leq \sup\{\sup\{f(x,y) \mid x \in A\} \mid y \in B\}$$

, hence $\sup\{\sup\{f(x,y) \mid x \in A\} \mid y \in B\}$ is an upper bound of $f(x,y)$. So

$$\sup\{f(x,y) \mid (x,y) \in A \times B\} \leq \sup\{\sup\{f(x,y) \mid x \in A\} \mid y \in B\}.$$

32. (a) Give a reasonable definition for what $\lim_{n\to\infty} = \infty$ should mean.

Solution: For all $M > 0$, there is an integer $N > 0$ such that $|x_n| > M$, whenever $n \geq N$.

(b) Let $x_1 = 1$ and define inductively $x_{n+1} = (x_1 + \cdots + x_n)/2$. Prove that $x_n \to \infty$.

Solution:

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_1 + \cdots + x_n}{2} - \frac{x_1 + \cdots + x_{n-1}}{2} \\ &= \frac{x_n}{2}, \text{ for } n > 1 \end{aligned}$$

hence $x_{n+1} = \frac{3x_n}{2}$, for $n > 1$. This means

$$x_n = \begin{cases} 1, & \text{if } n = 1; \\ \frac{3^{n-2}}{2^{n-1}}, & \text{for } n > 1. \end{cases}$$

For each $M > 0$, there is a positive integer $N > 1 + \frac{\ln \sqrt{3M}}{\ln \frac{3}{2}}$ such that $|x_n| > M$, whenever $n \geq N$.

□

33. (a) Show that $(\log x)/x \to 0$ as $x \to \infty$. (You may consult your calculus text and use, for example , l'Hôpital's rule.)

(b) Show that $n^{1/n} \to 1$ as $n \to \infty$.

Solution: This is easy if you have learned Calculus!! □

34. Express the following complex numbers in the form $a + bi$:

(a) $(2 + 3i)(4 + i)$

(b) $(8 + 6i)^2$

(c) $(1 + 3/(1 + i))^2$

Solution: It is easy, then we omit it. □

35. What is the complex conjugate of $(3+8i)^4/(1+i)^{10}$?

Solution: It is easy, then we omit it. $\square$

36. Find the solutions to

    (a) $(z+1)^2 = 3+4i$.

    (b) $z^4 - i = 0$.

Solution: It is easy, then we omit it. $\square$

37. Find the solutions to $z^2 = 3-4i$.

Solution: It is easy, then we omit it. $\square$

38. If a is real and z is complex, prove that $\mathrm{Re}(az) = a\ \mathrm{Re}\ (z)$ and that $\mathrm{Im}(az) = a\ \mathrm{Im}\ (z)$. Generally, show that Re: $\mathbb{C} \to \mathbb{R}$ is a real linear map; that is, that Re $(az+bw) = a\ \mathrm{Re}\ (z)\ b\ \mathrm{Re}\ (w)$ for a, b real and z, w complex.

Solution: It is easy, then we omit it. $\square$

39. Find the real and imaginary parts of the following, where $z = x+iy$:

    (a) $1/z^2$

    (b) $1/(3z+2)$

Solution: It is easy, then we omit it. $\square$

40. (a) Fix a complex number $z = x+iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis (1,0), (0,1) of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & -x \end{pmatrix}$$

    (b) Show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Solution: It is easy to proof if you have learned Linear Algebra!! $\square$

41. Show that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ and that $\mathrm{Im}(iz) = \mathrm{Re}(z)$ for all complex number z.

Solution: It is easy, then we omit it. $\square$

42. Letting $z = x + iy$, prove that $|x| + |y| \leq \sqrt{2}|z|$.

Solution: Since $|z| = (x^2 + y^2)^{1/2}$, then

$$(\sqrt{2}|z|)^2 = 2(x^2 + y^2) \geq |x|^2 + 2|xy| + |y|^2 = (|x| + |y|)^2$$

The inequality comes from $(|x| - |y|)^2 \geq 0$. So we complete the proof. $\square$

43. If a, $b \in \mathbb{C}$, prove the ***parallelogram identity:***

$$|a - b|^2 + |a + b|^2 = 2(|a|^2 + |b|^2).$$

Solution: It is easy, then we omit it. $\square$

44. Prove Lagrange's identity for complex numbers:

$$|\sum_{k=1}^{n} z_k w_k|^2 = (\sum_{k=1}^{n} |z_k|^2)(\sum_{k=1}^{n} |w_k|^2) - (\sum_{k<j} |z_k \bar{w}_j - z_j \bar{w}_k|^2).$$

Solution: It is a little complicated but it is easy, then we omit it. $\square$

45. Show that if $|z| > 1$ then $\lim_{n\to\infty} z^n/n = \infty$.

Solution: It is easy, so we omit it. $\square$

46. Prove that any nonempty set S bounded above has a least upper bound as follows: Choose $x_0 \in S$ and M_0 an upper bound. Let $a_0 = (x_0 + M_0)/2$. If a_0 is an upper bound, let $M_1 = a_0$ and $x_1 = x_0$; otherwise let $M_1 = M_0$ and $x_1 > a_0$, $x_1 \in S$. Repeat, generating sequences x_n and M_n. Prove that they both converge to $\sup(S)$.

Solution: It is easy to see that x_n is an increasing sequence and M_n and $M_n - x_n$ are decreasing sequences. And M_n is upper bound of S and $x_n \in S$ for all

n. By completeness of $\mathbb{R}$, we have x_n and M_n are convergent sequences. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} M_n = M$. We need to do is check $x = M = \sup(S)$. Claim $\lim_{n\to\infty} M_n - x_n = 0$. We observe that $|M_n - x_n| \le \frac{|M_0 - x_0|}{2^n}$. For all $\varepsilon > 0$, there is a positive integer $N > \frac{\ln|M_0-x_0|\varepsilon}{\ln 2}$ such that $|M_n - x_n| \le \frac{|M_0-x_0|}{2^n} \le \frac{|M_0-x_0|}{2^N} < \varepsilon$. So $0 = \lim_{n\to\infty} M_n - x_n = M - x$, i.e. $x = M$. Claim M is an upper bound of S. If not, there is some $s \in S$ such that $s > M$. Given $\varepsilon = s - M$, there is a positive integer N such that $M_n - M < \varepsilon = s - M$ whenever $n \ge N$, Then we have $s > M_n$, a contradiction. Since $\lim_{n\to\infty} x_n = x = M$, for all $\varepsilon > 0$, there exist an integer $N > 0$, such that $x - x_n = M - x_n < \varepsilon$, whenever $n \ge N$. And we know $x_n \in S$. Hence $M = \sup(S)$. $\square$

47. ***The long line*** Construct a non-Archimedean order field as follows: Let $\mathbb{F}$ be the union of two distinct copies of $\mathbb{R}$; let us put primes on the second copy to distinguish them. Define $x + y$ as usual, $x + y' = (x + y)'$, and $x' + y' = x + y$. Define xy as usual, $xy' = (xy)'$, and $x'y' = xy$. Define $x \le y$ as usual, $x \le y'$ for any x, y, and $x' \le y'$ if $x \le y$. Show that $\mathbb{F}$ meets the requirements. Demonstrate directly that $\mathbb{F}$ is *not* complete.

Solution: Author: Delete this exercise; it will be fixed in the next edition. $\square$