Exercise of Chapter 3

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1. Which of the following sets are compact? Which are connected?

   (a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \le 1\}$

   Solution: Not compact but connected.

   (b) $\{x \in \mathbb{R}^n \mid \|x\| \le 10\}$

   Solution: Both compact and connected.

   (c) $\{x \in \mathbb{R}^n \mid 1 \le \|x\| \le 2\}$

   Solution: Both compact and connected.

   (d) $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$

   Solution: Neither compact nor connected.

   (e) A finite set in $\mathbb{R}$

   Solution: Compact but not connected.

   (f) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (distinguish between the cases $n = 1$ and $n \ge 2$)

   Solution: Compact. If $n = 1$ it is not connected, but if $n = 2$ it is connected.

   (g) Perimeter of the unit square in $\mathbb{R}^2$

   Solution: Both compact and connected.

   (h) The boundary of a bounded set in $\mathbb{R}$

   Solution: Compact but not connected.

   (i) The rationals in $[0, 1]$

   Solution: Neither compact nor connected.

(j) A closed set in $[0,1]$

Solution: Connected, but it cannot determined. Since $[0.2, 0.5]$ is connected, but $(0.2, 0.5) \bigcap [0,1]$ is not connected.

It is easy to check, then I omit it. $\square$

2. Prove that a set $A \subset \mathbb{R}^n$ is not connected iff we can write $A \subset F_1 \bigcup F_2$, where F_1, F_2 are closed, $A \bigcap F_1 \bigcap F_2 = \emptyset$, $F_1 \bigcap A \neq \emptyset$, $F_2 \bigcap A \neq \emptyset$.

   Solution: ($\Rightarrow$) Since A is not connect, there are two disjoint nonempty open sets U_1 and U_2 such that $A \subset U_1 \bigcup U_2$. Choose $F_1 = \text{cl}(U_1)$, $F_2 = \text{cl}(U_2)$, then $A \subset F_1 \bigcup F_2$. It is clearly to see that $A \bigcap F_1 \neq \emptyset$, $A \bigcap F_2 \neq \emptyset$. For $x \in A \bigcap F_1 \bigcap F_2$, since $x \in A \subset U_1 \bigcup U_2$, may assume $x \in U_1$. Thus there is an $\varepsilon > 0$ such that $D(x, \varepsilon) \subset U_1$, and we know $x \notin U_2$, so $x \in \text{acc}(U_2)$. Choose ε above, then $D(x, \varepsilon) \bigcap U_1 \bigcap U_2 \neq \emptyset$, a contradiction.

   ($\Leftarrow$) Choose an open set $O_i = \bigcup D(x, 1)$ for $x \in F_i$. Since $F_i \bigcap A \subset O_i \bigcap A$ then $O_i \bigcap A \neq \emptyset$. It is easy to see that $A \subset O_1 \bigcup O_2$. Claim $A \bigcap O_1 \bigcap O_2$ is empty. Given $y \in A \bigcap O_1$, since $A \subset F_1 \bigcup F_2$, then $y \in F_1$ or F_2. If $y \in F_1$, then $y \notin F_2$. Otherwise $y \in A \bigcap F_1 \bigcap F_2$. So $y \notin O_2$. Similarly if $y \in F_2$, so $A \bigcap O_1 \bigcap O_2 = \emptyset$. $\square$

3. Proof that in $\mathbb{R}^n$, a bounded infinite set A has an accumulation point.

   Solution: $\text{cl}(A)$ is bounded, since A is bounded. Since $\text{cl}(A)$ is closed, then it is compact. By *Section3.1Exercise1*, there must have an accumulation point of A. $\square$

4. Show that a set A is bounded iff there is a constant M such that $d(x, y) \leq M$ for all x, $y \in A$. Give a plausible definition of the diameter of a set and reformulate your result.

   Solution: Suppose A is bounded, pick $z \in A$, $d(z, y) < M$ for all $y \in A$. Then for all x, $y \in A$, choose $N = 2M$, $d(x, y) \leq d(x, z) + d(z, y) \leq 2M = N$. On the other hand it is trivial.

The diameter of the set is the $\sup\{d(x,y) \mid x, y \in A\}$. A set A is bounded iff the diameter of A is finite. $\square$

5. Show that the following set are not compact, by exhibiting an open cover with no finite subcover.

   (a) $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$

   Solution: Consider U_n be $1/(n+1)(n+2) < \|x\| < 1/n$.

   (b) $\mathbb{Z}$, the integers in $\mathbb{R}$

   Solution: Choose

$$U_n = D(k, 1/2) \begin{cases} & \text{if } n = 2k \text{ for } k \in \mathbb{N} \bigcup \{0\} \\ & \text{if } n = -2k - 1 \text{ for } k \in -\mathbb{N} \end{cases}$$

$\square$

6. Suppose that F_k is a sequence of compact nonempty sets satisfying the nested set property such that diameter$\{F_k\} \to 0$ as $k \to \infty$. Show that there is exactly one point in $\bigcap\{F_k\}$. (By definition, diameter$(F_k) = \sup\{d(x,y) \mid x, y \in F_k\}$).

   Solution: By nested property, there is at least one point in $\bigcap\{F_k\}$. If x_1 and x_2 in $\bigcap\{F_k\}$, then

$$0 \le d(x_1, x_2) \le \sup\{d(x,y) \mid x, y \in F_k\} = \text{diameter } (F_k)$$

   Let $k \to \infty$, then we have $0 \le d(x_1, x_2) \le 0$. So $x_1 = x_2$. $\square$

7. Let x_k be a sequence in $\mathbb{R}^n$ that converges to x and let $A_k = \{x_k, x_{k+1}, \dots\}$. Show that $\{x\} = \bigcap\limits_{k=1}^{\infty} \mathrm{cl}(A)$. Is this true in *any* metric space?

   Solution: It is easy to see $\mathrm{cl}(A_k) = \{x\} \bigcup A_k$. Hence $x \in \bigcap \mathrm{cl}(A_k)$. Suppose $y \neq x \in \bigcap \mathrm{cl}(A_k)$, then $y \in \mathrm{cl}(A_k)$ for all k. But $y \neq x$, then $y \in A_k$ for all k. So $y \in \bigcap A_k$, a contradiction. (Since $\bigcap A_k = \emptyset$.)

   Yes. It is true. $\square$

8. Let $A \subset \mathbb{R}^n$ be compact and let x_k be a Cauchy sequence in $\mathbb{R}^n$ with $x_k \in A$. Show that x_k converges to a point in A.

   Solution: Since x_k is a Cauchy sequence, then may assume $x_k \to x$, where $x \in \mathbb{R}^n$. Since A is compact, then A must be closed. By **proposition 2.7.6i**, we have $x \in A$. $\square$

9. Determine (by proof or counterexample) the truth or satisfy of the following statements:

   (a) (A is compact in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

   Solution: False; $A = \{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}$ is compact, but $\mathbb{R}^n \backslash A$ is not connected.

   (b) (A is connected in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

   Solution: False; same example as in **a.**

   (c) (A is connected in $\mathbb{R}^n$) $\Rightarrow$ (A is open or closed).

   Solution: False $]a, b]$ is connected but is neither open nor closed

   (d) ($A = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

   Solution: False for $n = 1$, true for $n \geq 2$.

   It is easy to check, then I omit it. $\square$

10. A metric space M is said to be ***locally path-connected*** if each point in M has a neighborhood U such that U is a path-connected. (This terminology differs somewhat from that of some topology books.) Show that (M is connected and locally path-connected) $\Longleftrightarrow$ (M is path-connected).

    Solution: ($\Leftarrow$) Trivial.

    ($\Rightarrow$) Given x_0, $y \in M$, if $y \in U_{x_0}$, then done. If $y \notin U_{x_0}$, choose $x_1 \in U_{x_0}$, check $y \in U_{x_1}$ or not. Repeat this progress, if $y \in U_{x_i}$ for some i, then done. Otherwise, $y \notin \bigcup U_{x_i}$. It is easy to see that $\bigcup U_y$ for $y \notin \bigcup U_{x_i}$. Then $M \subset$

$\left(\bigcup U_{x_i}\right) \bigcup \left(\bigcup U_y\right)$. Since $M \bigcap \left(\bigcup U_{x_i}\right)$ is not empty, and $M \bigcap \left(\bigcup U_y\right)$ is not empty. then $M \bigcap \left(\bigcup U_{x_i}\right) \bigcap \left(\bigcup U_y\right)$ is empty. Then M is not connected, a contradiction. $\square$

11. (a) Prove that if A is connected in a metric space M and $A \subset B \subset \mathrm{cl}(A)$.

    Solution: If B is disconnected, let open sets U and V separate B. Choose $x \in B \bigcap U$, claim $A \bigcap U$ is not empty. If $x \in A$, then $x \in A \bigcap U$. Otherwise x is an accumulation point of A. Since U is open, there exists $\varepsilon > 0$, such that $D(x, \varepsilon) \subset U$. But there must be some $x_n \in A$, such that $x_n \in D(x, \varepsilon)$. So $x_n \in A \bigcap U$. Hence $A \bigcap U$ is nonempty. Similarly $A \bigcap V$ is not empty. $A \bigcap U \bigcap V \subset B \bigcap U \bigcap V = \emptyset$, then $A \bigcap U \bigcap V = \emptyset$. So open sets U and V separate A, a contradiction.

    (b) Not solved.

    (c) Not solved.

    (d) Not solved.

    (e) Not solved.

$\square$

12. Let S be a set of real numbers that is nonempty and bounded above. Let $-S = \{x \in \mathbb{R} \mid -x \in S\}$. Prove that

    (a) $-S$ is bounded below.

    (b) $\sup S = -\inf(-S)$.

    Solution: It is easy, then I omit it. $\square$

13. Let M be a complete metric space and F_n be a collection of closed nonempty subsets (not necessarily compact) of M such that $F_{n+1} \subset F_n$ and diameter $(F_n) \to 0$. Prove that $\bigcap_{n=1}^{\infty} F_n$ consists of a single point; compare Exercise 6.

Solution: Pick $x_n \in F_n$, claim x_n is a Cauchy sequence. Given $\varepsilon > 0$, there is an integer $N > 0$ such that $\text{diam}(F_n) < \varepsilon$ whenever $n \geq N$, then $|x_k - x_m| \leq$ diam $(F_N) < \varepsilon$, for $k, m \geq N$. Since x_k and $x_m \in F_N$, and M is complete, may say $x_n \to x$. Let $S_n = \{x_i \mid i \geq n\} \bigcup \{x\}$, it is clearly that S_n is compact. Then $x \in \bigcap S_n \subset \bigcap F_n$. Let $y \in \bigcap F_n$, and $0 \leq d(x,y) \leq$ diam (F_n). Let $n \to \infty$, we have $d(x,y) = 0$. Hence $x = y$. $\square$

14. (a) A point $x \in A \subset M$ is said to be ***isolated*** in the set A if there is a neighborhood U of x such that $U \bigcap A = \{x\}$. Show that this is equivalent ti saying that there is an $\varepsilon > 0$ such that for all $y \in A$, $y \neq x$, we have *d(x,y)*$> \varepsilon$.

    Solution: ($\Leftarrow$) Since $U \bigcap A = \{x\}$, there there is an $\varepsilon > 0$ such that $D(x, 2\varepsilon) \subset U$, so $D(x, 2\varepsilon) \bigcap A = \{x\}$. Hence for all $y \neq x \in A$, $d(y,x) \geq 2\varepsilon > \varepsilon$.

    ($\Rightarrow$) Suppose there is an $\varepsilon > 0$ such that $d(x,y) > \varepsilon$ for $y \neq x$. Then choose $U = D(x, \varepsilon)$, so $U \bigcap A = \{x\}$.

    (b) A set is called ***discrete*** if all its points are isolated. Give some examples. Show that a discrete set is compact iff it is finite.

    Solution: Examples: $\mathbb{Z}$; $\mathbb{N}$.

    ($\Leftarrow$) Let $A = \{x_1, \ldots, x_n\}$ be a finite set, for all cover $\{U_i\}$ of A. we can choose subcover $\{U_{i_j} \mid 1 \leq j \leq n\}$ covers A, where U_{i_j} contains x_j. Hence A is compact.

    ($\Rightarrow$) Since A is compact, choose cover $\{U_x\}$ of A, where $U_x \bigcap A = \{x\}$ for all $x \in A$. Then we can find finite subcovers $\{U_{x_i}\}$ such that $A \subset \bigcup U_{x_i}$. Hence $\bigcup \{x_i\} \subset A \subset \left(\bigcup \{U_{x_i}\}\right) \bigcap A = \bigcup \{x_i\}$ then A is finite.

    $\square$

15. Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be path-connected (respectively, connected, com-

pact). Show that $K_{\times}K_2$ is path-connected (respectively, connected, compact) in $M_1 \times M_2$.

Solution: It is not difficulty, the I omit it. $\square$

16. If $x_k \to x$ in a normed space, prove that $\|x_k\| \to \|x\|$. Is the converse true? Use this to prove that $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$ is closed, using sequences.

    Solution: Since $x_k \to x$, then for each $\varepsilon > 0$, there is an integer $N > 0$ such that $\|x_k - x\| < \varepsilon$, for $k \geq N$.

$$\Big| \|x_k\| - \|x\| \Big| \leq \|x_k - x\| < \varepsilon$$

    then $\lim\limits_{k\to\infty} \|x_k\| = \|x\|$.

    Yes. Since $\|x_k\| \to \|x\|$, then

$$\lim_{k\to\infty} \|x_k\| \leq \lim_{k\to\infty} \|x_k - x\| + \lim_{k\to\infty} \|x\|.$$

    So $\|x\| \leq \lim\limits_{k\to\infty} \|x_k - x\| + \|x\|$, hence $\lim\limits_{k\to\infty} \|x_k - x\| = 0$

    The next question is not difficulty, then I omit it. $\square$

17. *Solution:* Not solved. $\square$

18. Let $F_n \subset \mathbb{R}^n$ be defined by $F_n = \{x \mid x \geq 0 \text{ and } 2 - 1/n \leq x^2 \leq 2 + 1/n\}$. Show that $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$. Use this to show the existence of $\sqrt{2}$.

    Solution: It is easy, then I omit it. $\square$

19. Let $V_n \subset M$ be open sets such that $\mathrm{cl}(V_n)$ is compact, $V_n \neq \emptyset$, and $\mathrm{cl}(V_n) \subset V_{n-1}$. Prove $\bigcap\limits_{n=1}^{\infty} V_n \neq \emptyset$.

    Solution: You may consider $\mathrm{cl}(V_{n-1}) \supset V_{n-1} \supset \mathrm{cl}(V_n)$. $\square$

20. Prove that a compact subset of a metric space must be closed as follows: Let x be in the complement of A. For each $y \in A$, choose disjoint neighborhoods U_y of y and V_y of x. Consider the open cover $\{U_y\}$ of A to show the complement of A is open.

Solution: Since A is compact, there exists finite subcover $\{U_{y_k}\}$ covers A. Since $\left(\bigcap\{V_{y_k}\}\right)$ is open and $\left(\bigcap\{V_{y_k}\}\right) \cap \left(\bigcup\{U_{y_k}\}\right)$ is empty, then $\left(\bigcap\{V_{y_k}\}\right) \subset A^c$. Hence A^c is open, then A is closed. $\square$

21. (a) Prove: a set $A \subset M$ is connected iff $\emptyset$ and A are the only subsets of A that are open and closed relative to A. (A set $U \subset A$ is called ***open relative to*** A if $U = V \bigcap A$ for some open set $V \subset M$; "closed relative to A" is defined similarly.)

    (b) Prove that $\emptyset$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.

    Solution: It is not hard to proof, then I omit it. $\square$

22. Find two subsets A, $B \subset \mathbb{R}^2$ and a point $x_0 \in \mathbb{R}^2$ such that $A \bigcup B$ is not connected but $A \bigcup B \bigcup \{x_0\}$ is connected.

    Solution: Choose $A = \{(x, 0)|x > 0\}$, $B = \{(x, 0)|x < 0\}$, $\{x\} = \{(0, 0)\}$ $\square$

23. Let $\mathbb{Q}$ denote the rationals in $\mathbb{R}$. Show that both $\mathbb{Q}$ and the irrationals $\mathbb{R}\backslash\mathbb{Q}$ are not connected.

    Solution: You may consider $\mathbb{Q}$ is dense in $\mathbb{R}$. $\square$

24. Prove that a set $A \subset M$ is not connected if we can write A as the disjoint union of two set B and C such that $B \bigcap A \neq \emptyset$, $C \bigcap A \neq \emptyset$, and neither of the set B or C has a point of accumulation belonging to the other set.

    Solution: ($\Rightarrow$) Since A is disconnected, choose B and C be disjoint open sets separate A. W.l.o.g. given $x \in B$, there is an $\varepsilon > 0$, such that $D(x, \varepsilon) \subset B$, but $D(x, \varepsilon) \bigcap C = \emptyset$. If $x \in C$. So x is not an accumulation point of C.

    ($\Leftarrow$) Choose $V = \bigcup D(x, d(x, C)/2)$ for $x \in B$, $U = \bigcup D(x, d(x, B)/2)$ for $x \in C$. It is clearly that $d(x, C) > 0$ for $x \in B$, since x is not an accumulation point of C. Then U, V both open, $U \bigcup V \supset B \bigcup C \supset A$, $U \bigcap A \supset B \bigcap A$ is nonempty, $V\bigcap A$ is also nonempty. And it is easy to see that $U\bigcap V$ is empty.

$\square$

25. Prove that there is a sequence of distinct integers $n_1, n_2, \cdots \to \infty$ such that $\lim\limits_{k\to\infty} \sin n_k$ exists.

    Solution: Look *Exercise (3).* $\square$

26. Show that the completeness property of $\mathbb{R}$ maybe replaced by the Nested Interval Property. If $\{F_n\}$ is a sequence of closed bounded intervals in $\mathbb{R}$ such that $F_{n+1}\subset F_n$ for all $n = 1, 2, 3, \ldots$, then there is at least one point in $\bigcap F_n$.

    Solution: Choose $x_i = \inf(F_i)$, since F_i is bounded and closed interval in $\mathbb{R}$, then $x_i \in F_i$. So $\{x_i\}$ is monotone increasing sequence. By completeness of $\mathbb{R}$, $x_i \to x$ for some $x \in \mathbb{R}$. And $x \in F_n$ for all n, since x is an accumulation point of $S_n = \{x_i | i \geq n\} \subset F_n$ and F_n is closed, then $x \in F_n$ for all n. Hence $\bigcap F_n$ is nonempty. $\square$

27. Let $A \subset \mathbb{R}$ be a bounded set. Show that A is closed iff for every sequence $x_n \in A$, $\limsup x_n \in A$ amd $\liminf x_n \in A$.

    Solution: It is not hard, then I omit it. $\square$

28. Let $A \subset M$ be connected and contain more than one point. Show that every point of A is an accumulation point of A.

    Solution: If x is not an accumulation point of A, choose $\{x\}$ and $A\backslash\{x\}$. Use *Exercise (26)*, where $\{x\} = B$, $A\backslash\{x\} = C$. $\square$

29. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1\}$. Show that A is compact. Is it connected?

    Solution: It is easy, then I omit it. $\square$

30. Let U_k be a sequence of open bounded sets in $\mathbb{R}^n$. Prove or disprove:

    (a) $\bigcup\limits_{k=1}^{\infty} U_k$ is open.

(b) $\bigcap\limits_{k=1}^{\infty} U_k$ is open.

(c) $\bigcap\limits_{k=1}^{\infty} \left(\mathbb{R}^n \backslash U_k\right)$ is closed.

(d) $\bigcap\limits_{k=1}^{\infty} \left(\mathbb{R}^n \backslash U_k\right)$ is compact.

Solution: It is easy, then I omit it. $\square$

31. Suppose $A \subset \mathbb{R}^n$ is not compact. Show that there exists a sequence $F_1 \supset F_2 \supset F_3 \cdots$ of closed sets such that $F_k \bigcap A \neq \emptyset$ for all k and

$$\left(\bigcap_{k=1}^{\infty} F_k\right) \bigcap A = \emptyset$$

Solution: If A is not closed, then choose $x \in \mathrm{cl}(A) \backslash A$, there exists $x_k \to x$ where $x_k \in A$ for all k. Choose $F_n = \{x_i | i \geq n\} \bigcup \{x\}$, then $F_n \supset F_{n+1}$ and $F_n \bigcap A$ is not empty for all n. But $\left(\bigcap\limits_{n=1}^{\infty} F_n\right) \bigcap A$ is empty.
If A is closed but not bounded, Choose $F_n = \{\|x\| \geq n\}$, then $F_n \supset F_{n+1}$ and $F_n \bigcap A$ is not empty for all n. Since if $F_n \bigcap A$ is empty for some n, then A is bounded. And it is clearly that $\left(\bigcap\limits_{n=1}^{\infty} F_n\right) \bigcap A$ is empty. $\square$

32. Let x_n be a sequence in $\mathbb{R}^3$ such that $\|x_{n+1} - x_n\| \leq 1/(n^2 + n)$, $n \geq 1$. Show that x_n converges.

Solution: It is easy, then I omit it. $\square$

33. *Solution:* Not Solved. $\square$

34. *Solution:* Not Solved. $\square$

35. Let $a \in \mathbb{R}$ and define the sequence $a_1, a_2, \ldots$ in $\mathbb{R}$ by $a_1 = a$, and $a_n = {a_{n-1}}^2 - a_{n-1} + 1$ if $n > 1$. For what $a \in \mathbb{R}$ is the sequence

(a) Monotone?

(b) Bounded?

(c) Convergent?

Compute the limits in the cases of convergence.

Solution: It is easy, then I omit it. $\square$

36. Let $A \subset \mathbb{R}^n$ be uncountable. Prove that A has an accumulation point.

    Solution: If A is bounded, then by *Exercise (3)*, we complete the proof. If A is unbounded, let $A_n = \{x \in A \mid \|x\| \leq n\}$. Since A is uncountable then there must be some A_k infinite but A_{k-1} is finite. By *Exercise (3)* again, A_k must has an accumulation point x. So x is the accumulation point of A. $\square$

37. Let A, $B \subset M$ with A compact, B closed, and $A \bigcap B = \emptyset$.

    (a) Show that there is an $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all $x \in A$ and $y \in B$,

    (b) Is **a** true if A, B are merely closed?

    Solution: You may look the *Appendix C*. $\square$

38. Show that $A \subset M$ is not connected iff there exist two *disjoint* open sets U, V such that $U \bigcap A \neq \emptyset$, $V \bigcap A \neq \emptyset$, and $A \subset U \bigcup V$.

    Solution: Look *Exercise (24)* $\square$

39. Let $F_1 = [0, 1/3] \bigcup [2/3, 1]$ be obtained from $[0, 1]$ by removing the middle third. Repeat, obtaining

$$F_2 = [0, 1/9] \bigcup [2/9, 1/3] \bigcup [2/3, 7/9] \bigcup [8/9, 1]$$

    In general, F_n is a union of intervals and F_{n+1} is obtained by removing the middle third of of these intervals. Let $C = \bigcap_{n=1}^{\infty} F_n$, the ***Cantor set***. Prove:

    (a) C is compact.

    (b) C has infinitely many points [Hint: look at the endpoints of F_n].

(c) $\text{int}(C) = \emptyset$.

(d) C is ***perfect***; that is , it is closed with no isolated.

(e) Show that C is ***totally disconnect*** ; that is, if x, $y \in C$ and $x \neq y$ then $x \in U$ and $y \in V$ where U and V are open sets that disconnected C.

Solution: You may look ***p.41*** $\sim$ ***p.42, Principle of Mathematical Analysis, 3nd, Rudin***. □

40. *Solution:* Not solved. □