Exercise of Chapter 2

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1. Discuss whether the following sets are open or closed.

   (a) $]1,2[$ in $\mathbb{R}^1 = \mathbb{R}$

   (b) $[2,3]$ in $\mathbb{R}$

   (c) $\bigcap_{n=1}^{\infty} \in 1, 1/n[$ in $\mathbb{R}$

   (d) $\mathbb{R}^n$ in $\mathbb{R}^n$

   (e) A hyperplane in $\mathbb{R}^n$

   (f) $\{r \in]0,1[\mid r \text{ is rational}\}$ in $\mathbb{R}$

   (g) $\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$ in $\mathbb{R}^2$

   (h) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ in $\mathbb{R}^n$

   Solution: It is easy, then we omit it. □

2. Determine the interiors, closures, and boundaries of the sets in Exercise 1.

   Solution: It is easy, then we omit it. □

3. Let U be open in M and $U \subset A$. Show that $U \subset \operatorname{int}(A)$. What is the corresponding statement for closed sets?

   Solution: The proof you can use the definition to prove.

   Let V be closed in M and $A \subseteq V$, then $\operatorname{cl}(A) \subseteq V$. □

4. (a) Show that if $x_n \to x$ in a metric space M, then $x \in \operatorname{cl}\{x_1, x_2, \dots\}$. When is x an accumulation point?

Solution: It is easy to check x is an accumulation point of $\{x_1, x_2, \dots\}$, then we omit it. If $x \notin \{x_1, x_2, \dots\}$, and x is a cluster point of $\{x_1, x_2, \dots\}$, then x must be an accumulation point.

(b) Can a sequence have more than one accumulation point?

Solution: Consider the sequence

$$x_n = \begin{cases} \frac{1}{k} & \text{if } n = 2k-1 \\ \\ \frac{k+1}{k} & \text{if } n = 2k \end{cases}$$

then 1 and 0 are accumulation points of this sequence.

(c) If x is an accumulation point of a set A, prove that there is a sequence of *distinct* points of A converging to x.

Solution: Given $\varepsilon_1 = 1 > 0$, there is a point $x_1 \in A$ such that $0 < d(x, x_1) < 1$. Let $\varepsilon_2 = d(x, x_1)$, there is a point x_2 such that $0 < d(x, x_2) < d(x, x_1) = \varepsilon_2$. Let $\varepsilon_3 = d(x, x_2)$, repeat the progress, we will have a sequence x_n and $x_i \neq x_j$ if $i \neq j$, and x_n converges to x.

$\square$

5. Show that $x \in \operatorname{int}(A)$ iff there is an $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A$.

Solution: $x \in \operatorname{int}(A)$, there is an open set U contains x, such that $U \subseteq A$. Since U is open, there is an $\varepsilon > 0$, such that $D(x, \varepsilon) \subseteq U$, then $D(x, \varepsilon) \subseteq A$. On the other side, since $x \in D(x, \varepsilon) \subseteq A$, and $D(x, \varepsilon)$ is open, then $x \in \operatorname{int}(A)$. $\square$

6. Find the limits, if they exist, of these sequences in $\mathbb{R}^2$:

(a) $\left((-1)^n, \frac{1}{n}\right)$

(b) $\left(1, \frac{1}{n}\right)$

(c) $\left(\left(\frac{1}{n}\right)(\cos n\pi), \left(\frac{1}{n}\right)\left(\sin\left(n\pi + \frac{\pi}{2}\right)\right)\right)$

(d) $\left(\frac{1}{n}, n^{-n}\right)$

Solution: It is easy to calculate, then I omit it. $\square$

7. Let U be open in metric space M. Show that $U =$cl$(U)\backslash$bd(U). Is this true for every set in M.

   Solution: Suppose $x \in U$, then $x \in$ cl(U), since U is open, there exists an $\varepsilon > 0$, such that $D(x, \varepsilon) \subseteq U$, so $x \notin$ bd(U). Hence $U \subseteq$ cl$(U)\backslash$bd(U). Suppose $x \in$cl$(U)\backslash$bd(U), if $x \in U$, then done. Otherwise x is an accumulation point of U. Then for each $\varepsilon > 0$, there is a $y \in D(x, \varepsilon)$, where $y \in U$ and we know $x \in U^c$, then $x \in$ bd(A). Hence we get a contradiction, so cl$(U)\backslash$bd$(U) \subseteq U$. No, choose $U = \{0\}$, cl$(U) = U$, and bd$(U) = U$. But cl$(U)\backslash$bd$(U) = \emptyset$. □

8. Let $S \subset \mathbb{R}$ be nonempty, bounded below, and closed. Show that inf$(S) \in S$.

   Solution: Claim inf(S) is an accumulation point of S. Let inf$(S) = s$. For all $\varepsilon > 0$, there is $x \in S$ such that $s - \varepsilon < x < s + \varepsilon$. So we have s is an accumulation point of S, and S is closed, then $s \in S$. □

9. Show that

   (a) int$B = B\backslash$ bd$B$, and

   Solution: Suppose $x \in$ int(B), there exists an $\varepsilon > 0$, such that $D(x, \varepsilon) \subseteq B$, so $x \notin$bd(B). Then int$(B) \subseteq B\backslash$bd$(B)$.

   Suppose $x \in B\backslash$bd(B), there exists an $\varepsilon > 0$, such that $D(x, \varepsilon) \subseteq B$, so $x \in$int(B). Then $B\backslash$bd$(B) \subseteq$ int(B).

   (b) cl$(A) = M\backslash$int$(M\backslash A)$

   Solution: Suppose $x \in$ cl(A). If $x \in A$, then $x \in M\backslash(M\backslash A)$. So $x \in M\backslash$int$(M\backslash A)$. If x is an accumulation point of A, then for all $\varepsilon > 0$, there is a $y \in D(x, \varepsilon)$, where $y \in A$. So there is no $D(x, \varepsilon) \subseteq M\backslash A$, thus $x \notin$ int$(M\backslash A)$.

   Suppose $x \notin$ int$(M\backslash A)$, for all $\varepsilon > 0$, there is no $D(x, \varepsilon) \subseteq M\backslash A$. Then there is a $y \in D(x, \varepsilon)$, where $y \in A$. So x is an accumulation point of A if $x \neq y$, and $x \in A$ if $x = y$. Thus $x \in$ cl(A).

$\square$

10. Determine which of the following statements are true.

    (a) int(cl(A)) =int(A).

    Solution: False. Let $A = \mathbb{Q}$, $\text{int}(\text{cl}(\mathbb{Q})) = \text{int}(\mathbb{R}) = \mathbb{R}$. But $\text{int}(\mathbb{Q}) = \emptyset$.

    (b) $\text{cl}(A) \bigcap A = A$.

    Solution: True. It is trivial to see $\text{cl}(A) \bigcap A \subseteq A$. But $A \subseteq \text{cl}(A)$, then $A \subseteq \text{cl}(A) \bigcap A$.

    (c) $\text{cl}(\text{int}(A)) = A$.

    Solution: False. Let $A = (0, 1)$, $\text{cl}(\text{int}(A)) = [0, 1] \neq (0, 1) = A$.

    (d) $\text{bd}(\text{cl}(A)) = \text{bd}(A)$

    Solution: False. Let $A = \mathbb{Q}$, $\text{bd}(\text{cl}(A)) = \text{bd}(\mathbb{R}) = \emptyset$, but $\text{bd}(\mathbb{Q}) = \mathbb{R}$.

    (e) If A is open, then $\text{bd}(A) \in M \backslash A$.

    Solution: True. A is open implies $A = \text{int}(A)$. For all $x \in \text{bd}(A)$, for all $\varepsilon > 0$ there is a $y \notin A$, so $y \in D(x, \varepsilon)$ then x is not in $\text{int}(A) \backslash A$

$\square$

11. Show that in a metric space $x_m \to x$ iff for every $\varepsilon > 0$ there is an N such that $m \geq N$ implies $d(x_m, x) \leq \varepsilon$ (this differs from Position **2.7.2** in that here " $< \varepsilon$" replaced by " $\leq \varepsilon$").

    Solution: One side is trivial. The other side is that for every $\varepsilon > 0$, there is a positive integer N such that $m \geq N$ implies $d(x_m, x) \leq \frac{\varepsilon}{2} < \varepsilon$. So $x_m \to x$. $\square$

12. Prove the following properties for subsets A and B of a metric space:

    (a) $\text{int}(\text{int}(A)) = \text{int}(A)$.

    Solution: Since $\text{int}(A)$ is an open set, and we know if a set S is open, then $\text{int}(S) = S$. Hence $\text{int}(\text{int}(A)) = \text{int}(A)$.

(b) $\text{int}(A \bigcup B) \supset \text{int}(A) \bigcup \text{int}(B)$.

Solution: For all $x \in \text{int}(A) \bigcup \text{int}(B)$, may assume $x \in \text{int}(A)$. There exists an $\varepsilon > 0$, such that $D(x, \varepsilon) \in A \subseteq A \bigcup B$, hence $x \in \text{int}(A \bigcup B)$.

(c) $\text{int}(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B)$.

Solution: For all $x \in \text{int}(A \bigcap B)$

$\Longleftrightarrow$ there exist an $\varepsilon > 0$, $D(x, \varepsilon) \in A \bigcap B$

$\Longleftrightarrow$ there exist an $\varepsilon > 0$, $D(x, \varepsilon) \in A$ and $D(x, \varepsilon) \in B$

$\Longleftrightarrow x \in \text{int}(A)$ and $x \in \text{int}(B)$

$\Longleftrightarrow x \in \text{int}(A) \bigcap \text{int}(B)$

$\square$

13. Show that $\text{cl}(A) = A \bigcup \text{bd}(A)$.

Solution: Suppose x is an accumulation point of A and $x \in A^c$, then for each $\varepsilon > 0$, there exists $y \in A$ such that $y \in D(x, \varepsilon)$ and we know $x \in A^c$, then $x \in \text{bd}(A)$. So $\text{cl}(A) \subseteq A \bigcup \text{bd}(A)$.

Suppose $x \in \text{bd}(A) \backslash A$, for each $\varepsilon > 0$, there exists $y \in A$ such that $y \in D(x, \varepsilon)$, then x is an accumulation point of A. Hence $A \bigcup \text{bd}(A) \subseteq \text{cl}(A)$. $\square$

14. Prove the following for subsets of a metric space M:

(a) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

Solution: Since $\text{cl}(A)$ is an closed set, then $\text{cl}(\text{cl}(A) = \text{cl}(A)$.

(b) $\text{cl}(A \bigcup B) = \text{cl}(A) \bigcup \text{cl}(B)$.

Solution: Since $A \subseteq A \bigcup B$, then $\text{cl}(A) \subseteq \text{cl}(A \bigcup B)$. Similarly $\text{cl}(B) \subseteq \text{cl}(A \bigcup B)$. So $\text{cl}(A) \bigcup \text{cl}(B) \subseteq \text{cl}(A \bigcup B)$.

Since $A \bigcup B \subseteq \text{cl}(A) \bigcup \text{cl}(B)$ and $\text{cl}(A) \bigcup \text{cl}(B)$ is closed, then $\text{cl}(A \bigcup B) \subseteq \text{cl}(A) \bigcup \text{cl}(B)$.

(c) $\text{cl}(A \bigcap B) \subset \text{cl}(A) \bigcap \text{cl}(B)$.

Solution: Since $A\bigcap B \subseteq A$, $A\bigcap B \subseteq B$, then $\text{cl}(A\bigcap B) \subseteq \text{cl}(A)$ and $\text{cl}(A\bigcap B) \subseteq \text{cl}(B)$. So $\text{cl}(A\bigcap B) \subseteq \text{cl}(A)\bigcap \text{cl}(B)$.

$\square$

15. Prove the following for subsets of a metric space M:

    (a) $\text{bd}(A) = \text{bd}(M\backslash A)$.

    Solution: For all $x \in \text{bd}(A)$, choose $\varepsilon > 0$, there is a $y \in A$ and $z \in M\backslash A$ such that y and z both $\in D(x, \varepsilon)$. Then $x \in \text{bd}(M\backslash A)$.

    Conversely is the same.

    (b) $\text{bd}(\text{bd}(A)) \subset \text{bd}(A)$.

    Solution: For all $x \in \text{bd}(\text{bd}(A))$, choose $\frac{\varepsilon}{2} > 0$, there is a $y \in \text{bd}(A)$ such that $y \in D(x, \frac{\varepsilon}{2})$. Since $y \in \text{bd}(A)$, choose $\frac{\varepsilon}{2} > 0$, there is a $y' \in A$ and $z' \in M\backslash A$ such that y' and $z' \in D(y, \frac{\varepsilon}{2}) \subseteq D(x, \varepsilon)$, then $x \in \text{bd}(A)$.

    (c) $\text{bd}(A\bigcup B) \subset \text{bd}(A)\bigcup \text{bd}(B) \subset \text{bd}(A\bigcup B)\bigcup A\bigcup B$.

    Solution: For all $x \in \text{bd}(A\bigcup B)$, choose $\varepsilon > 0$, there is a $y \in A\bigcup B$ and $z \in M\backslash(A\bigcup B)$ such that y and $z \in D(x.\varepsilon)$. W.L.O.G. let $y \in A$, and $z \in M\backslash A$. Then $x \in \text{bd}(A) \subseteq \text{bd}(A)\bigcup \text{bd}(B)$.

    For all $x \in \text{bd}(A)\bigcup \text{bd}(B)$, W.L.O.G. let $x \in \text{bd}(A)$. If $x \in A\bigcup B$, then done. Otherwise $x \in M\backslash(A\bigcup B)$. Since $x \in \text{bd}(A)$, choose $\varepsilon > 0$, there is a $y \in A \subseteq A\bigcup B$, such that $y \in D(x.\varepsilon)$ and $x \in D(x.\varepsilon)$. Hence $x \in \text{bd}(A\bigcup B)$. So $x \in \text{bd}(A\bigcup B)\bigcup A\bigcup B$

    (d) $\text{bd}(\text{bd}(\text{bd}(A))) = \text{bd}(\text{bd}(A))$.

    Solution: $\text{bd}(\text{bd}(\text{bd}(A)))$ is a subset of $\text{bd}(\text{bd}(A))$ by (b).

    Conversely, for all $x \in \text{bd}(\text{bd}(A))$, choose $\varepsilon > 0$, there is a $y \in M\backslash \text{bd}(A) \subseteq M\backslash \text{bd}(\text{bd}(A))$. Then $x, y \in D(x, \varepsilon)$, so $x \in \text{bd}(\text{bd}(\text{bd}(A)))$.

16. Let $a_1 = \sqrt{2}$, $a_2 = (\sqrt{2})^{a_1}$, $\ldots$, $a_{n+1} = (\sqrt{2})^{a_n}$. Show that $a_n \to 2$ as $n \to \infty$.

(You may use any relevant facts from calculus.)

Solution: It is not difficulty, and I omit it. $\square$

17. If $\sum x_m$ converges absolutely in $\mathbb{R}^n$, show that $\sum x_m \sin m$ converges.

    Solution: Since $|x_m \sin m| \leq |x_m|$ for all i, and $\sum |x_m|$ converges. By Ratio comparison test, $\sum |x_m \sin m|$ converges. $\square$

18. If x, $y \in M$ and $x \neq y$, then prove that there exist open set U and V such that $x \in U$, $y \in V$, and $U \bigcap V = \emptyset$.

    Solution: Choose $U = D(x, \frac{d(x,y)}{3})$, $V = D(x, \frac{d(x,y)}{3})$. It is easy to see that U, V are both open and $x \in U$, $y \in V$. If $z \in U \bigcap V$, then $d(x,y) \leq d(x,z) + d(y,z) < \frac{2d(x,y)}{3}$, then we get a contradiction. So $U \bigcap V = \emptyset$. $\square$

19. Define a ***limit point*** of a set A in a metric space M to be a point $x \in M$ such that $U \bigcap A \neq \emptyset$ for every neighborhood U of x.

    (a) What is the difference between limits points and accumulation points? Give examples.

    Solution: x is a limit point then $(U \bigcap A) \backslash \{x\}$ may be empty set. x is an accumulation point then $(U \bigcap A) \backslash \{x\} \neq \emptyset$. Let $A = \{0\}$, then limit point of A is 0, but A has no accumulation points.

    (b) If x is a limit point of A, then show that there is a sequence $x_n \in A$ with $x_n \to x$.

    Solution: Given $\varepsilon_1 = 1 > 0$, there is a point $x_1 \in A$, such that $0 < d(x, x_1) < 1$. Let $\varepsilon_2 = d(x, x_1)$, there is a point x_2 such that $0 < d(x, x_2) < d(x, x_1) = \varepsilon_2$. Let $\varepsilon_3 = d(x, x_2)$, repeat the progress, we will have sequence x_n and $x_i \neq x_j if i \neq j$ and $x_n \to x$.

    (c) If x is an accumulation point of A, then show that x is a limit point of A. Is converse true?

Solution: By definition. No, consider $A = \{0\}$.

(d) If x is a limit point of A and $x \notin A$, then show that x is an accumulation point.

Solution: By definition.

(e) Prove: A set is closed iff it contains all of its limit points.

Solution: By (c) and (d), it is ok.

$\square$

20. For a set A in a metric space M and $x \in M$, let

$$d(x, A) = \inf\{d(x, y) \mid y \in A\},$$

and for $\varepsilon > 0$, let $D(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$.

(a) Show that $D(A, \varepsilon)$ is open.

Solution: If $x \in D(A, \varepsilon)\backslash A$, there must be an $y \neq x \in A$, such that $d(x, y) < \varepsilon$. Pick $\varepsilon - d(x, y) > 0$, for all $z \in D(x, \varepsilon - d(x, y))$, $d(z, A) \leq d(z, y) \leq d(x, z) + d(x, y) < \varepsilon - d(x, y) + d(x, y) = \varepsilon$.

If $x \in D(A, \varepsilon) \bigcap A$, for all $z \in D(x, \varepsilon)$, then $d(z, A) \leq d(z, x) < \varepsilon$.

(b) Let $A \subset M$ and $N_\varepsilon = \{x \in M \mid d(x, A) \leq \varepsilon\}$, where $\varepsilon > 0$. Show that N_ε is closed and that A is closed iff $A = \bigcap\{N_\varepsilon \mid \varepsilon > 0\}$.

Solution: For all accumulation point x of N_ε, claim $x \in N_\varepsilon$. Since x is an accumulation point of N_ε, for each $\delta > 0$, there is a $y \in N_\varepsilon$, such that $y \in D(x, \delta)$. Then $d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z) < \delta + \varepsilon$. Since δ is arbitrary, $d(x, A) \leq \varepsilon$. So we complete the first statement.

Since N_ε is closed for all ε, then $A = \bigcap\{N_\varepsilon \mid \varepsilon > 0\}$ is closed.

Conversely A is closed and $A \subseteq N_\varepsilon$ for all $\varepsilon > 0$, then we have $A \subseteq \bigcap\{N_\varepsilon \mid \varepsilon > 0\}$. For $x \in \bigcap\{N_\varepsilon \mid \varepsilon > 0\}$, $0 \leq d(x, A) \leq \varepsilon$. Since ε is arbitrary, then

$\inf d(x, y) = 0$ for all $y \in A$. Then x is an accumulation point of A. But A is closed, so $x \in A$.

$\square$

21. prove that a sequence x_k in a normed space is a Cauchy sequence iff for every neighborhood U of 0, there is an N such that $k, \ell \geq N$ implies $x_k = x_\ell \in U$.

    Solution: For every open set U contains 0, there exists $\varepsilon > 0$, such that $D(0, \varepsilon) \subseteq U$. Since x_k is a Cauchy sequence, there is a positive integer N, for $k, \ell \geq N$, such that $\| x_k - x_l \| < \varepsilon$. Then $x_k - x_l \in D(0, \varepsilon) \subseteq U$.

    Conversely given open set $D(0, \varepsilon)$, where $\varepsilon > 0$. There is a positive integer N, such that $k, \ell \geq N$ implies $x_k - x_l \in D(0, \varepsilon)$. So $\| x_k - x_l \| < \varepsilon$, then x_k is a Cauchy sequence. $\square$

22. Prove Proposition **2.3.2.** (Hint: Use Exercise 12 of the Introduction.)

    Solution: We can use Proposition 2.1.3 or induction. I think it is not difficulty to prove it, then I omit it. $\square$

23. Prove that the interior of a set $A \subset M$ is the union of all the subsets of A that are open. Deduce that A is open iff $A = \text{int}(A)$. Also, give a direct proof of the latter statement using the definitions.

    Solution: We need to proof is that $\text{int}(A) = \bigcup \{A_i \mid A_i \text{ is an open subset of } A\}$. Since $\text{int}(A)$ is an open subset of A, then LHS$\subseteq$ RHS. Given $x \in$ RHS, there exists $D(x, \varepsilon) \subseteq A_i \subseteq A$, hence $x \in \text{int}(A)$. A is open and $A \subseteq A$, so $A \subseteq \text{int}(A)$, but $\text{int}(A) \subseteq A$ is always true, then $A = \text{int}(A)$. If $A =$int(A) , and int(A) is open, so A is open. Suppose A is open, for $x \in A$, there exists an $\varepsilon > 0$ such that $D(x, \varepsilon) \subseteq A$, since $D(x, \varepsilon)$ is an open set, then $x \in \text{int}(A)$, and $\text{int}(A) \subseteq A$, so $\text{int}(A) = A$. Suppose $A =$int(A), and use *Exercise 5*, then we complete proof. $\square$

24. Identify $\mathbb{R}^{n+m}$ with $\mathbb{R}^n \times \mathbb{R}^m$. Show that $A \subset \mathbb{R}^{n+m}$ is open iff for each $(x, y) \in A$, with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, there exist open set $U \in \mathbb{R}^n$, $V \in \mathbb{R}^m$ with $x \in U$, $y \in V$ such that $U \times V \subset A$. Deduce that the product of open sets is open.

Solution: Suppose $A \subseteq \mathbb{R}^{n+m}$ is open, for $(x, y) \in A$, there exists an $\varepsilon > 0$, such that $D((x, y), \varepsilon) \subseteq A$. Let $D(x, \frac{\varepsilon}{2})$ be U and $D(y, \frac{\varepsilon}{2})$ be V. Claim $U \times V \subseteq D((x, y), \varepsilon)$. For $x' \in U$, $y' \in V$, $d((x, y), (x', y')) = (d(x, x')^2 + d(y, y')^2)^{1/2} < ((\frac{\varepsilon}{2})^2 + (\frac{\varepsilon}{2})^2)^{1/2} = \frac{\varepsilon}{\sqrt{2}} < \varepsilon$. So $U \times V \subseteq A$.

On the other hand, given $(x, y) \in A$, there exists $D(x, \varepsilon) \subseteq U$, and $D(y, \varepsilon) \subseteq V$, for some $\varepsilon > 0$. Claim $D((x, y), \varepsilon) \subseteq U \times V$. For $(x', y') \in D((x, y), \varepsilon)$, $d(x, x') \leq d((x, y), (x', y')) < \varepsilon$. So $x' \in D(x, \varepsilon) \subseteq U$, similarly $y' \in V$. Hence $(x', y') \in U \times V$, thus A is open.

Claim $U \times V$ is an open set if U and V are both open sets. Given $(x, y) \in U \times V$, there exists $D(x, \varepsilon) \subseteq U$ and $D(y, \varepsilon) \subseteq V$ for some $\varepsilon > 0$. And we know $D((x, y), \varepsilon) \subseteq U \times V$. So $U \times V$ is open. □

25. Prove that a set $A \subset M$ is open iff we can write A as the union of some family of ε-disks.

Solution: Suppose A is open, for each $x \in A$, there exists an $\varepsilon_x > 0$ such that $A = \bigcup_{x \in A} D(x, \varepsilon_x)$. Since $x \in A$, then $x \in D(x, \varepsilon_x) \subseteq \bigcup_{x \in A} D(x, \varepsilon_x)$. And $D(x, \varepsilon_x) \subseteq A$ for all x, then $\bigcup_{x \in A} D(x, \varepsilon_x) \subseteq A$. Suppose A is the union of some family of ε-disks. And we know ε-disk is open and union of open sets is still open, thus A is open. □

26. Define the sequence of numbers a_n by

$$a_0 = 1, a_1 = 1 + \frac{1}{1 + a_0}, \cdots, a_n = 1 + \frac{1}{1 + a_{n-1}}.$$

Show that a_n is a convergent sequence. Find the limit.

Solution: It is easy to check that $0 \leq a_n \leq 2$ for all n. Now we consider the subsequence a_{2n} and a_{2n+1}. We will show the former is strictly increasing and the latter is strictly decreasing. Since

$$a_{n+1} - a_{n-1} = \frac{-(a_n - a_{n-2})}{(1+a_n)(1+a_{n-2})} = \frac{a_{n-1} - a_{n-3}}{\text{something} > 1}$$

This shows $a_{n+1}-a_{n-1}$ has the same sign as $a_{n-1}-a_{n-3}$. Inductively, $a_{n+1}-a_{n-1}$ has the same sign as a_2-a_0 or a_3-a_1. In fact, the second equality above shows $a_{n+1}-a_{n-1}$ and a_n-a_{n-2} have opposite signs. Computing the first few terms, it follows that a_{2n} is strictly increasing while a_{2n+1} is strictly decreasing. Thus each of these subsequences converges, and they both obey the recurrence relation

$$a_{n+1} = 1 + \frac{1}{2 + \frac{1}{1+a_{n-1}}}$$

Now that we have established that the subsequences converge, it makes sense to take limits of the above equation. Since both subsequences satisfy the same recurrence relation,they converge to the same value(so a converges to this value).We easily compute that $\lim a_n = \sqrt{2}$. □

27. Suppose $a_n \geq 0$ and $a_n \to 0$ as $n \to \infty$. Given any $\varepsilon > 0$, show that there is a subsequence b_n of a_n such that $\sum\limits_{n=1}^{\infty} b_n < \varepsilon$.

Solution: Choose $a_{n_i} < \frac{\varepsilon}{2^i}$, it is possible since $a_n \to 0$. Let $a_{n_i} = b_i$, then we complete the proof. □

28. Given examples of:

(a) An infinite set in $\mathbb{R}$ with no accumulation points

Solution: $\mathbb{Z}$.

(b) A nonempty subset of $\mathbb{R}$ that is a contained in its set of accumulation points

Solution: $\mathbb{Q}$.

(c) A suset of $\mathbb{R}$ that has infinitely many accumulation points but contains none of them

Solution: $\{2n + \frac{1}{m} \mid n \in \mathbb{Z}, m = 2, 3, \ldots\}$

(d) A set A such that bd(A) =cl(A).

Solution: $\mathbb{Q}$.

You may check it by yourself. $\square$

29. Let A, $B \subset \mathbb{R}^n$ and x be an accumulation point of $A \bigcup B$. Must x be an accumulation point of either A or B?

Solution: Yes. If x is not an accumulation point of neither A nor B, then there is some $\varepsilon > 0$, such that $(D(x,\varepsilon)\backslash\{x\})\bigcap A = \emptyset$ and $\varepsilon' > 0$, such that $(D(x,\varepsilon')\backslash\{x\})\bigcap B = \emptyset$. Then choose $\delta = \min\{\varepsilon, \varepsilon'\}$ such that

$$(D(x,\delta)\backslash\{x\})\bigcap A \subset (D(x,\varepsilon)\backslash\{x\})\bigcap A = \emptyset$$

and

$$(D(x,\delta)\backslash\{x\})\bigcap B \subset (D(x,\varepsilon)\backslash\{x\})\bigcap B = \emptyset$$

Hence $(D(x,\delta)\backslash\{x\})\bigcap(A \bigcup B) = \emptyset$, so x is not an accumulation point of $A \bigcup B$, a contradiction. $\square$

30. Show that any open set in $\mathbb{R}$ is a union of disjoint open intervals. Is this sort of result true in $\mathbb{R}^n$ for $n > 1$, where we define an open interval as the Cartesian product of n open intervals, $]a_1, b_1[\times \cdot \times]a_n, b_n[$?

Solution: You may read ***Theorem 3.11 at page 51 of Mathematical Analysis, 2nd, Apostol.***

No. Consider the set $\Big((0,2) \times (0,2)\Big)\backslash\Big([1,2) \times [1,2)\Big)$. $\square$

31. Let A' denote the set of accumulation points of a set A. Prove that A' is closed. Is $(A')' = A'$ for all A?

Solution: Suppose x is an accumulation points of A', then given $\varepsilon > 0$, there is a $y \in A'$ such that $d(x,y) < \frac{\varepsilon}{2}$. Since y is an accumulation points of A, there is a $z \in A$, such that $d(z,y) < \frac{\varepsilon}{2}$. So there is a $z \in A$, such that

$$d(z,x) \le d(x,y) + d(z,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $x \in A'$, so A' is closed.

No. Take $A = \{1/n \mid n = 1, 2, \dots\}$, but $A' = \{0\}$, $(A')' = \emptyset$. $\square$

32. Let $A \subset \mathbb{R}^n$ be closed and $x_n \in A$ be a Cauchy sequence. Prove that x_n converges to a point in A

    Solution: Since A is closed, choose a subsequence of x_n converges to a point $x \in A$. Since a subsequence of a Cauchy sequence converges to a point, then the Cauchy sequence converges to the point. Hence x_n converges to the point $x \in A$. $\square$

33. Let s_n be a bounded sequence of real numbers. Assume that $2s_n \leq s_{n-1} + s_{n+1}$. Show that $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

    Solution: It is easy to see that $s_n - s_{n-1}$ is an increasing sequence. If there exists $s_{i+1} - s_i = a > 0$, then $s_{n+i} \geq s_i + na$. So $\lim_{n\to\infty} s_n = \infty$. So $s_{n+1} - s_n \leq 0$ for all n, then s_n is a decreasing sequence. Since $\mathbb{R}$ is a complete ordered field, then $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$. $\square$

34. Let $x_n \in \mathbb{R}^k$ and $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$, where $0 \leq r < 1$. Show that x_n converges.

    Solution: Tt is easy to check that x_n is a Cauchy sequence, thus it converges. $\square$

35. Show that any family of disjoint nonempty open sets of real numbers is countable.

    Solution: Since every open set A_i of real numbers contains a rational number, r_i. And $\mathbb{Q}$ is countable, then we have that $\bigcup r_i$ is countable, so we complete the proof. $\square$

36. Let A, $B \subset \mathbb{R}^n$ be closed sets. Does $A + B = \{x + y \mid x \in A, y \in B\}$ have to be closed?

Solution: No. Take $A = \{-1, -2, -3, \dots\}$ and $B = \{1 + \frac{1}{2}, 2 + \frac{1}{3}, 3 + \frac{1}{4}, \dots\}$ Thus 0 is accumulation point of $A + B$, but $0 \notin A + B$. $\square$

37. For $A \subset M$, a metric space, prove that

$$\mathrm{bd}(A) = [A \bigcap \mathrm{cl}(M \backslash A)] \bigcup [\mathrm{cl}(A) \backslash A]$$

Solution: Since $\mathrm{bd}(A) = \mathrm{cl}(A) \bigcap \mathrm{cl}(M \backslash A)$, then if $x \in \mathrm{bd}(A)$, $x \in \mathrm{cl}(A)$ and $x \in \mathrm{cl}(M \backslash A)$. If $x \in A$, then $x \in A \bigcap \mathrm{cl}(M \backslash A)$. If $x \notin A$, then $x \in \mathrm{cl}(A) \backslash A$. Hence LHS $\subseteq$ RHS.

If $x \in A \bigcap \mathrm{cl}(M \backslash A)$, then $x \in \mathrm{cl}(A) \bigcap \mathrm{cl}(M \backslash A)$. If $x \in \mathrm{cl}(A) \backslash A$, then $x \in M \backslash A$. So $x \in \mathrm{cl}(M \backslash A)$, hence $x \in \mathrm{cl}(A) \bigcap \mathrm{cl}(M \backslash A)$. Thus RHS $\subseteq$ LHS. $\square$

38. Let $x_k \in \mathbb{R}^n$ satisfy $\|x_k - x_\ell\| \leq 1/k + 1/\ell$. Prove that x_k converges.

Solution: For all $\varepsilon > 0$, there is an integer $N > \frac{2}{\varepsilon}$, such that for $k, \ell > N$, $\|x_k - x_\ell\| \leq 1/k + 1/\ell < 2/N < \varepsilon$. Then x_k converges. $\square$

39. Let $S \subset \mathbb{R}$ be bounded above and below. Prove that $\sup(S) - \inf(S) = \sup\{x - y \mid x \in S \text{ and } y \in S\}$.

Solution: For all x, $y \in S$, $x \leq \sup(S)$, $-y \leq -\inf(S)$, then $x - y \leq \sup(S) - \inf(S)$. So

$$\sup\{x - y \mid x \in S \text{ and } y \in S\} \leq \sup(S) - \inf(S).$$

For all $\varepsilon > 0$, there exists x and y such that $\sup(S) - \varepsilon/2 < x$, $-\inf(S) - \varepsilon/2 < -y$. So

$$\sup(S) - \inf(S) - \varepsilon < x - y \leq \sup\{x - y \mid x \in S \text{ and } y \in S\}.$$

Since ε is arbitrary, then

$$\sup(S) - \inf(S) \leq \sup\{x - y \mid x \in S \text{ and } y \in S\}$$

. $\square$

40. Suppose in $\mathbb{R}$ that for all n, $a_n \leq b_n$, $a_n \leq a_{n+1}$, and $b_{n+1} \leq b_n$. Prove that a_n converges.

Solution: Claim a_n is bounded. Since $a_0 \leq a_n \leq b_n \leq b_0$ for all n, then a_n is bounded and $\mathbb{R}$ is a complete ordered field. So a_n converges. $\square$

41. Let A_n be subsets of a metric space M, $A_{n+1} \subset A_n$, and $A_n \neq \emptyset$, but assume that $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$. Suppose $x \in \bigcap\limits_{n=1}^{\infty} \mathrm{cl}(A_n)$. Show that x is an accumulation point of A_1.

Solution: Suppose $x \in A_1$ but not accumulation point of A_1. Then x must be an accumulation point of A_k and $x \notin A_k$, for some $k > 1$. However it is impossible because $A_k \subset A_1$. Then x must be an accumulation point of A, since $x \in \mathrm{cl}(A)$.

$\square$

42. Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. Must there be a $z \in A$ such that $d(x, A) = d(x, z)$?

Solution: No. Let $A = \{1/n \mid \text{ for } n \in \mathbb{N}\}$. Choose $x = 0$, $d(x, A) = 0$, but there is no $z \in A$ such that $d(x, z) = 0$. $\square$

43. Let $x_1 = \sqrt{3}, \ldots, x_n = \sqrt{3 + x_{n-1}}$. Compute $\lim\limits_{n\to\infty} x_n$.

Solution: It is easy then I omit it. $\square$

44. A set $A \subset \mathbb{R}^n$ is said to be dense in $B \subset \mathbb{R}^n$ if $B \subset \mathrm{cl}(A)$. If A is dense in $\mathbb{R}^n$ and U is open, prove that $A \bigcap U$ is dense in U. Is this true if U is not open?

Solution: Since A is dense in $\mathbb{R}^n$, $\mathbb{R}^n \subset \mathrm{cl}(A) = A \bigcup \mathrm{acc}(A)$, for all $x \in U \subset \mathbb{R}$, if $x \in A$, then $x \in A \bigcap U \subset \mathrm{cl}(A \bigcap U)$, then done.

Otherwise, $x \in \mathrm{acc}(A)$. We know for all $\varepsilon > 0$, there is an $y \neq x$, $y \in A$ such that $y \in D(x, \varepsilon)$. Since A is open there is an $\varepsilon' > 0$, $D(x, \varepsilon') \subset U$ then for all $\varepsilon_i \leq \varepsilon'$, there is a point $y_i \in A \bigcap U$ and let $\varepsilon_0 = \varepsilon$. Otherwise, choose $y_0 \in D(x, \varepsilon) \subset D(x, \varepsilon_i)$. So $x \in \mathrm{acc}(A \bigcap U)$

No, $\mathbb{Q}$ is dense in $\mathbb{R}^n$, choose $U = \{\pi\}$. Then $U = \{\pi\}$, and $A \bigcap U = \emptyset =$ $\mathrm{cl}(A \bigcap U)$. $\square$

45. Show that $x^(\log x) = o(e^x)$ as $x \to \infty$ (see Worked Example **2.5**).

Solution: It's easy, then I omit it. $\square$

46. (a) If $f = o(g)$ and if $g(x) \to \infty$ as $x \to \infty$, then show that $e^{f(x)} = o(e^{g(x)})$.

    Solution: Since $(g - f)/g \to 1$ as $x \to \infty$, then $e^{(g-f)(x)} \sim e^{g(x)} = \infty$ as $x \to \infty$, so $e^{(f-g)(x)} = 0$ as $x \to \infty$. Hence $e^{f(x)} = o(e^{g(x)})$.

    (b) Show that $\lim\limits_{x\to\infty} (x \log x)/e^x = 0$ by showing that $x = o(e^{x/2})$ and that $\log x = o(e^{x/2})$.

    Solution: It is easy to proof that $x = o(e^{x/2})$ and $\log x = o(e^{x/2})$. Using $\lim\limits_{x\to\infty} f(x)g(x) = \Big(\lim\limits_{x\to\infty} f(x)\Big)\Big(\lim\limits_{x\to\infty} g(x)\Big)$, whenever $\lim f(x)$ and $\lim g(x)$ are both finite as $x \to \infty$.

    $\square$

47. Show that $\gamma = \lim\limits_{n\to\infty} \Big(1 + 1/2 + \cdots + 1/n - \log n\Big)$ exists by using the proof of the integral test (γ is called ***Euler's constant***).

Solution: Let $a_n = \Big(1 + 1/2 + \cdots + 1/n - \log n\Big)$, then $a_{n+1} - a_n = 1/(1 + n) - \log(n + 1) + \log n < 0$ and it is clearly that $a_n > 0$ (by graph), so a_n is converge. Thus γ exists. $\square$

48. (a) If $a_n > 0$ and $\limsup\limits_{n\to\infty} a_{n+1}/a_n < 1$, then $\sum a_n$ converges, and if $\liminf\limits_{n\to\infty} a_{n+1}/a_n >$ 1, then $\sum a_n$ diverges.

    (b) If $a_n \geq 0$ and if $\limsup\limits_{n\to\infty} \sqrt[n]{a_n} < 1$ (respectively, ¿1), then $\sum a_n$ converges (respectively, diverges).

    (c) In the ratio comparison test, can the limits be replaced by lim sup's.

Solution: a. and b. You may read ***p.65 p.67, Principle of Mathmatical Analysis, 3nd, Rudin.*** However, I don't know how to solve c. $\square$

49. Prove ***Raabe's test:*** If $a_n > 0$ and if $a_{n+1}/a_n \leq 1 - A/n$ for some fixed constant $A > 1$ and n sufficiently large, then $\sum a_n$ converges. Similarly, show that if $a_{n+1}/a_n > 1 - (1/n)$, then $\sum a_n$ diverges.

Use Raabe's test to prove convergence of the *textbfhypergeometric series* whose general term is

$$a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdots n\cdot\gamma\cdot(\gamma+1)\cdots(\gamma+n-1)}$$

where α, β, and γ are nonnegative integers, $\gamma > \alpha+\beta$. Show that the series diverges if $\gamma < \alpha + \beta$.

Solution: I don't know how to prove Raabe's test.

Use Raabe's test. Note that $a_{n+1}/a_n = (\alpha+n)(\beta+n)/(n+1)(n+\gamma)$, then it is easy to check. $\square$

50. Show that for x sufficiently large, $f(x) = (x\cos^2 x + \sin^2 x)e^{x^2}$ is monotonic and tends to $+\infty$, but that neither the ratio $f(x)/(x^{1/2}e^{x^2})$ nor its reciprocal is bounded.

Solution: It is easy to check $f(x)$ monotone and tends to ∞. Since $1/\sqrt{x} \leq f(x)/x^{1/2}e^{x^2} \leq \sqrt{x}$ and $1/\sqrt{x}$ is not bounded. Then we complete the proof. $\square$

51. (a) If $u_n > 0$, $n = 1, 2, \ldots$ show that

$$\liminf \frac{u_{n+1}}{u_n} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup \frac{u_{n+1}}{u_n}.$$

Solution: You may read ***p.68 $\sim$ p.69, Principle of Mathmatical Analysis, 3nd, Rudin***.

(b) Deduce that if $\lim(u_{n+1}/u_n) = A$, then $\limsup \sqrt[n]{u_n} = A$.

Solution: It is trivial, since $\limsup = \text{limit} = \liminf$.

(c) Show that the converse of part **b** is false by use of the sequence $u_{2n} = u_{2n+1} = 2^{-n}$.

Solution: You can check it by yourself.

(d) Calculate $\limsup \sqrt[n]{n!}/n$.

Solution: Let $u_n = n!/n^n$, and easy calculate $\lim\limits_{n\to\infty} u_{n+1}/u_n$.

$\square$

52. Test the following series for convergence.

(a) $\sum\limits_{n=0}^{\infty} \frac{e^{-k}}{\sqrt{k+1}}$ *Solution:* converge

(b) $\sum\limits_{n=0}^{\infty} \frac{k}{k^2+1}$ *Solution:* diverge

(c) $\sum\limits_{n=0}^{\infty} \frac{\sqrt{n+1}}{n^2-3n+1}$ *Solution:* converge

(d) $\sum\limits_{n=0}^{\infty} \frac{\log(k+1)-\log k}{\tan^{-1}(2/k)}$ *Solution:* diverge

(e) $\sum\limits_{n=0}^{\infty} \sin(n^\alpha)$, α real, ¿ 0 *Solution:* converge for $\alpha > 1$; diverge for $0 < \alpha \leq 1$

(f) $\sum\limits_{n=0}^{\infty} \frac{n^3}{3^n}$ *Solution:* converge

It is easy to check, then I omit it. $\square$

53. Given a set A in a metric space, what is the maximum number of distinct subsets that can be produced by successively applying the operation closure, interior, and complement to A (in any order)? Give an example of a set achieving your maximum.

Solution: The answer is 14. A, $\mathrm{int}(A)$, $\mathrm{cl}(A)$, $\mathrm{cl}(\mathrm{int}(A))$, $\mathrm{int}(\mathrm{cl}(A))$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ A^c, $\mathrm{int}(A^c)$, $\mathrm{cl}(A^c)$, $\mathrm{cl}(\mathrm{int}(A^c))$, $\mathrm{int}(\mathrm{cl}(A^c))$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(A^c)))$, $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A^c)))$. Just only need to check the following two statements.

$$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) = \mathrm{cl}(\mathrm{int}(A)) \ldots\ldots (1)$$

proof of (1): It is easy to see LHS $\subset$ RHS. Conversely claim $\mathrm{int}(A) \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. Suppose $x \in \mathrm{int}(A)$, there is a $D(x, \varepsilon) \subset A$, for some $\varepsilon > 0$. Then $D(x, \varepsilon) \subset \mathrm{int}(A)$, since $D(x, \varepsilon)$ is an open subset of A. So $D(x, \varepsilon) \subset$

$\mathrm{cl}(\mathrm{int}(A))$, hence $x \in \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. Then we have $\mathrm{cl}(\mathrm{int}(A)) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))))$. Then we complete the proof.

$$\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(A)) \ldots\ldots (2)$$

proof of (2): It is easy to see that RHS $\subset$ LHS. Conversely claim $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset \mathrm{cl}(A)$. Suppose $x \in \mathrm{int}(\mathrm{cl}(A))$, then $x \in \mathrm{cl}(A)$. Otherwise x is an accumulation point of $\mathrm{int}(\mathrm{cl}(A))$. Thus for all $\varepsilon > 0$, there exist $y \in \mathrm{int}(\mathrm{cl}(A)) \subset \mathrm{cl}(A)$, such that $y \in D(x, \varepsilon)$, then x is an accumulation point of $\mathrm{cl}(A)$. But $\mathrm{cl}(A)$ is closed, then we have that $x \in \mathrm{cl}(A)$. So we have $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{int}(\mathrm{cl}(A))$.

Let the metric space $M = \mathbb{R}$. Then the following sets are desired.

$A = [0,1) \bigcup (1,2] \bigcup \{3\} \bigcup \{\mathbb{Q} \bigcap [4,5]\}$

$\mathrm{int}(A) = (0,1) \bigcup (1,2)$

$\mathrm{cl}(A) = [0,2] \bigcup \{3\} \bigcup [4,5]$

$\mathrm{cl}(\mathrm{int}(A)) = [0,2]$

$\mathrm{int}(\mathrm{cl}(A)) = (0,2) \bigcup (4,5)$

$\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) = (0,2)$

$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = [0,2] \bigcup [4,5]$

$A^c = (-\infty,0) \bigcup \{1\} \bigcup (2,3) \bigcup (3,4) \bigcup \{\mathbb{Q}' \bigcap [4,5]\} \bigcup (5,\infty)$

$\mathrm{int}(A^c) = (-\infty,0) \bigcup (2,3) \bigcup (3,4) \bigcup (5,\infty)$

$\mathrm{cl}(A^c) = (-\infty,0] \bigcup \{1\} \bigcup [2,\infty)$

$\mathrm{cl}(\mathrm{int}(A^c)) = (-\infty,0] \bigcup [2,4] \bigcup [5,\infty)$

$\mathrm{int}(\mathrm{cl}(A^c)) = (-\infty,0) \bigcup (2,\infty)$

$\mathrm{int}(\mathrm{cl}(\mathrm{int}(A^c))) = (-\infty,0) \bigcup (2,4) \bigcup (5,\infty)$

$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A^c))) = (-\infty,0] \bigcup [2,\infty)$ $\square$