Math 751 HW 1

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Due: September 18

Chapter 0.1: Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Proof. We define $\mathbb{T}^2$ as $[-1,1]\times[-1,1]/(-1,y)=(1,y),(x,-1)=(x,1)$. And, we denote this quotient map as p. Here we may as well just assume the point which is removed from $\mathbb{T}^2$ is $p((0,0))$. We first define a deformation retract $F(x,t)$ on $([-1,1]\times[-1,1]\backslash(0,0))\times[0,1]$ to $Bd([-1,1]\times[-1,1])$. Let $s(x,y)=\max(|x|,|y|)$. Let $F\big((x,y),t\big)=\Bigg((1-t)x+t\frac{x}{s(x,y)},(1-t)y+t\frac{y}{s(x,y)}\Bigg)$. Since $(x,y)\neq 0$, this is a continuous map. Also, we have $F\big((x,y),0\big)=id, F\big((x,y),1\big)=Bd([-1,1]\times[-1,1]), F\big((x,y),t\big)|_{Bd([0,1]\times[0,1])}=id$. So, this is a deformation retract. Then for every element $e\in\mathbb{T}^2$, we let $\tilde{F}=p\circ F\Bigg(\big(p^{-1}(e),t\big)\Bigg):\mathbb{T}^2\times[0.1]\to\mathbb{T}^2$. This is a well-defined and continuous map, which satisfies all the requirements of being a deformation retract. So, $\tilde{F}$ is a deformation retract from $\mathbb{T}^2$ to its longitude and meridian circles. □

Chapter 0.2: Construct an explicit deformation retraction of $\mathbb{R}^n-\{0\}$ onto S^{n-1}.

Proof. Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n\backslash(0,\cdots,0)$. Let $F(x,t):\mathbb{R}^n\backslash(0,\cdots,0)\to\mathbb{S}^{n-1}$ be $F(x,t)=(1-t)x+t\frac{x}{\|x\|}$, $\|x\|$ is x's euclidean norm. Then F is continuous and $F(x,0)=id$. Also, $F(x,1)=\mathbb{S}^{n-1}, F(x,t)|_{\mathbb{S}^{n-1}}=id$. So, this is a deformation retract from $\mathbb{R}^n\backslash(0,\cdots,0)$ to $\mathbb{S}^{n-1}$. □

Chapter 0.3:

(a) Show that the composition of homotopy equivalences $X\to Y$ and $Y\to Z$ is a homotopy equivalence $X\to Z$. Deduce that homotopy equivalence is an equivalence relation.

Proof. Let $f:X\to Y$, $g:Y\to X$, $g\circ f\simeq id|_X$, and the according homotopy is $F_1:X\times I\to X$. Also, $f\circ g\simeq id|_Y$, and the according homotopy is $F_2:Y\times I\to Y$. Let $f^{'}:Y\to Z$, $g^{'}:Z\to Y$, $g^{'}\circ f^{'}\simeq id|_Y$, and the according homotopy is $F_1^{'}:Y\times I\to Y$. Also, $f^{'}\circ g^{'}\simeq id|_Z$, and the according homotopy is $F_2^{'}:Z\times I\to Z$. Now we consider $f^{'}\circ f:X\to Z$ and $g\circ g^{'}:Z\to X$. First we prove a lemma. If $g:X\to Y$, $g^{'}:X\to Y$, $f:Y\to Z$ are continuous map. And, $g\simeq g^{'}$. We claim $f\circ g\simeq f\circ g^{'}$. Let $F:X\times I\to Y$ be the homotopy between g and $g^{'}$. Then, we let $G:X\times I\to Z=f\circ F$. So, this is a continuous map. And, $G|_{X\times\{0\}}=f\circ F(X,0)=f\circ g$. Also, $G|_{X\times\{1\}}=f\circ F(X,1)=f\circ g^{'}$. So, G is a homotopy between $f\circ g$ and $f\circ g^{'}$. We prove a second lemma. Let $f,g,h:X\to Y$ are continuous map, and $f\simeq g, g\simeq h$. We claim $f\simeq h$. Let $F:X\times I\to Y$ be the homotopy between f and g, and $G:X\times I\to Y$ be the homotopy between g and h. Let $H(x,t)=\begin{cases}F(x,2t) & 0\leq t\leq\frac{1}{2}\\ G(x,2t-1) & \frac{1}{2}\leq t\leq 1\end{cases}$. This is a continuous map. It is because $X\times[0,\frac{1}{2}],X\times[\frac{1}{2},1]$ are two closed subset of X, whose union is exactly X. And, we have $H|_{X\times[0,\frac{1}{2}]},H|_{X\times[\frac{1}{2},1]}$ are continuous. Also we have $H(X,0)=f,H(X,1)=h$. so, this is a homotopy between f and h. Now we go back to the main proof. By our lemmas, $g\circ g^{'}\circ f^{'}\circ f=g\circ(g^{'}\circ f^{'})\circ f\simeq g\circ id_Y\circ f=g\circ f\simeq id_X$. On the other hand, $f^{'}\circ f\circ g\circ g^{'}=f^{'}\circ(f\circ g)\circ g^{'}\simeq f^{'}\circ id_Y\circ g^{'}=f^{'}\circ g^{'}\simeq id_Z$. So, homotopy equivalence is an equivalence relation. □

(b) Show that the relation of homotopy among maps $X\to Y$ is an equivalence relation.

Proof. (i) We prove reflexivity: Let $f:X\to Y$. We define $F:X\times I\to Y$ as $F(x,t)=f(x)$ which is continuous. Also, $F(X,0)=F(X,1)=f(x)$.

(ii) Symmetry: Let $f,g:X\to Y$ be continuous map, and $F:X\times I\to Y$ be the homotopy between f and g. We define $G(x,t):X\times I\to Y=F(x,1-t)$, which is a homotopy between g and f.

(iii) Transitivity: we have proved it as the second lemma in the last question. □

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. Let $f : X \to Y, g : Y \to X$ be two continuous map. And, $f \circ g \simeq id_Y, g \circ f \simeq id_X$. Let $f^{'} \simeq f$. Then by the lemmas proved in question (a), we get $f^{'} \circ g \simeq id_Y, g \circ f^{'} \simeq id_X$. So, $f^{'}$ is also a homotopy equivalence. ☐

Chapter 0.4: **A deformation retraction in the weak sense** of a space X to a subspace A is a homotopy

$$f_t : X \to X$$

such that $f_0 = 1$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof. Let $\iota : A \to X$ be the inclusion map. Let $f = f_1 : X \to A, F(x,t) = f_t(x) : X \times I \to X$. Then F is a homotopy between id_X and $\iota \circ f$. Let $G = F|_{A \times I}$. Then it is a homotopy between id_A and $f \circ \iota$. So, ι is a homotopy equivalence. ☐

Chapter 0.9: Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space. Let A be its retract and $r : A \to X$ be the retract map. Since X is contractible, there exists a map $H : X \times I \to x_0 \in X,\ H(X,0) = id_X$. Let $H^{'} = r\big(H|_{A \times I}\big) : A \times I \to A$. And, we have $H^{'}(A,0) = id_A, H^{'}(X,1) = r(x_0)$. This means A can be contracted to $r(x_0)$, i.e. A is contractible. ☐

Chapter 0.10: Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Proof. (i) '$\longrightarrow$': Since X is contractible, we have $H : X \times I \to X, H(X,0) = id|_X, H(X,1) = x_0 \in X$. Let $G = f \circ H : X \times I \to Y$, then G is continuous. $G(X,0) = f \circ H(X,0) = f \circ id_X = f, G(X,1) = f \circ H(X,1) = f(x_0)$. So, f is nullhomotopic.
'$\longleftarrow$': Let $Y = X, f = id|_X$. Then there is a continuous map $H : X \times I \to X, H(X,0) = id_X, H(X,1) = x_0$. So, X is contractible.
(ii) '$\longrightarrow$': Since X is contractible, we have $H : X \times I \to X, H(X,0) = id|_X, H(X,1) = x_0 \in X$. We then construct a continuous map $F(y,t) = H(f(y),t)\ : Y \times I \to X$. Then, $F(y,0) = f(y), F(y,1) = x_0$. So, f is nullhomotopic.
'$\longleftarrow$': Like what we have done in (i), we only need to let $Y = X, f = id_X$. ☐

Chapter 0.11: Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq id$ and $hf \simeq id$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Proof. (i) We prove the first half of the problem. $g = id_X \circ d \simeq (h \circ f) \circ g = h \circ (f \circ g) \simeq h \circ id_Y = h$. So, $f \circ g \simeq f \circ h \simeq id_X$. So, f is a homotopy equivalence.
(ii) We prove the second half of the problem. If $f \circ g$ and $h \circ f$ are homotopy equivalence, there exists $k_1 : Y \to Y, k_2 : X \to X$ such that $(f \circ g) \circ k_1 = f \circ (g \circ k_1) \simeq id_Y,\ k_2 \circ (h \circ f) = (k_2 \circ h) \circ f \simeq id_X$. So, now we can use our first part of conclusion to get f is a homotopy equivalence. ☐

Chapter 0.20: Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Proof. Let X be the figure, and it intersects with itself through a circle C. First, we can retract the tube conncected to C to a line. This operation creates two point in X. First one is C, and another one is on the bottom of the klein bottle. Then, we attach these two points together. Now, we can see what lies below the 'point' is a sphere, and there is a circle and a 'neck' connected to the point. In the end, we can retract the 'neck' into a circle once again. So, we get a sphere and two circles who are attached to one point. That is $S^2 \vee S^1 \vee S^1$.

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Chapter 0.23: Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Proof. Let this CW complex be X, and its two subcomplexes be A, B. Let $C = A \cap B$. Then C is also a subcomplex of X, A, B. It is because first C is closed, and X is the disjoint union of arbitary open cells $\bigcup\limits_{\alpha \in I} e_\alpha, \{e_\alpha\} = \Gamma$, A, B are also disjoint union of some open cells from Γ, which makes C must be a disjoint union of some open cells from Γ. Since C is contractible, $X/C \simeq X, A/C \simeq A, B/C \simeq B$. Since $A/C, B/C$ are still CW complex, they are subcomplexes of X/C. Also, they are contractible because A, B are contractible. Then We have $X \simeq X/C \simeq (X/C)/(A/C)$. Let $p : X/C \to (X/C)/(A/C)$ be the quotient map. We claim that $f = p|_{B/C}$ is actually a homeomorphism. First of all, f is continuous. Since p is a quotient map, p is continuous which indicates $p|_{B/C} = f$ is also continuous. And, p is injective. It is because p actually means collapse A/C to one point, which means $p|_{X/C \backslash A/C}$ is injective. Since $B/C \cap A/C$ is acutually an one point set $\{x_0\}$, which means $f = P|_{B/C}$ is injective. Also, $f = p|_{B/C}$ is surjective by the definition of quotient map. Now we prove f^{-1} is also continuous. It suffices to prove f is an open map. Let U be an open set of B/C, then there exists an open subset $U^{'} \subset X/C$ such that $U^{'} \cap B/C = U$. We can rewrite $U^{'}$ as $U^{'} = U^{'} \cap (A/C \backslash B/C) \cup (U^{'} \cap B/C) = U^{'} \cap (A/C \backslash B/C) \cup U$, because $X/C = A/C \cup B/C$. Then $p(U^{'}) = p(U) \cup \{p(x_0)\}$. This is a disjoint union iff $x_0 \notin U, U^{'} \cap (A/C \backslash B/C) \neq \emptyset$. If $x_0 \notin U$, then $U = X/C \backslash A/C \cap U^{'}$. So, U itself is an open subset of X/C. So, $f(U) = p|_{B/C}(U) = p(U)$ is an open subset of $(X/C)/A/C$, and since $p(U) \subset p(B/C)$ it is also an open subset of $p(B/C)$. Then we assume $x_0 \in U$. So, $f(U) = p(U) = p(U^{'})$ by the discussion before. So, $p(U)$ is an open subset of $X/C = p(B/C)$ as $p(U^{'})$ is an open subset. Now, we can conclude that $(X/C)/(A/C)$ is contractible, since it is homeomorphic to B/C which is contractible($B/C \simeq B$). So, $X \simeq X/C \simeq (X/C)/(A/C)$ is also contracible. Last, we only need to verify homeomorphism is indeed a homotopy equivalence. It is trivial since a homeomorphism f always have $f \circ f^{-1} = id, f^{-1} \circ f = id$. □

Chapter 0.28: Show that if (X_1, A) satisfies the homotopy extension property, then so does every pair $(X_0 \amalg_f X_1, X_0)$ obtained by attaching X_1 to a space X_0 via a map $f : A \to X_0$.

Proof. Suppose we have a continuous map $g : X_1 \sqcup_f X_0 \to Y$, and a homotopy $H : X_0 \times I \to Y$ which satisfies $H(x_0, 0) = g|_{X_0}$. Let $p : X_1 \sqcup X_0 \to X_1 \sqcup_f X_0$ be the quotient map. And let $g^{'} = g \circ p$, and a homotopy $H^{'} = H(f(a), t) \; : A \times I \to Y$. Then $H^{'}(a, 0) = H(f(a), 0) = g(f(a)) = g^{'}|_A, a \in A$. By the homotopy extension property, we can get another homotopy $H^{''} : X_1 \times I \to Y$ which satisfies $H^{''}(a, t) = H^{'}(a, t)$. Now we need to glue these two homotopies together. We define $F(p(x), t) = \begin{cases} H^{''}(x, t), x \in X_1 \\ H(x, t), \; x \in X_0 \end{cases}$. This is well defined, and also continuous. Because we can check its continuity in $p(X_1) \times I, p(X_0) \times I$, who both are closed in $(X_1 \sqcup_f X_0) \times I$. □