Math 761 HW 1

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1. *Proof.* Let X be $[0,1] \times (-\frac{1}{2}, \frac{1}{2})$, and p be the quotient map from X to M First, we show M has a countable topological base. Let $\Gamma = \{U_i \quad i \in \mathbb{N}\}$ be X's countable topological base, and consider $\{\ p(U_i) \ \ i \in \mathbb{N}\}$. This is a family of open sets of M. For every open set $U^{'}$ of M, we have $p^{-1}(U^{'}) = \cup\, U_i$, for some $U_i \in \Gamma$. Then $U^{'} = p(\cup\, U_i) = \cup\ p(U_i)$, which means M has a countable topological base.
Let $\dot{X} = (0,1) \times (-\frac{1}{2}, \frac{1}{2})$, then p is a homeomorphism from $\dot{X}$ to $p(\dot{X})$. That is because this is obvious a bijection, and p, p^{-1} are continous according to the definition of a quotient map. So, for every element $x \in p(\dot{X}) \subset M$, we can make $(\ p(\dot{X}),\ p^{-1})$ be its chart. For the rest of M, we may as well choose $\{\ p(0,y), \ \ -\frac{1}{2} < y < \frac{1}{2}\}$ as their representative elements. For any $-\frac{1}{2} < y < \frac{1}{2}$, let δ be $\frac{min(\frac{1}{2}-y,\ y+\frac{1}{2})}{2}$, then $p(\{|x-(0,y)| < \delta,\ x \in X\} \cup \{|x-(1,-y)| < \delta,\ x \in X\})$ is an open neighborhood U_y of $p(0,y)$ in M. Also, we can observe that restricting this quotient map to $\{|x-(0,y)| < \delta,\ x \in X\} \cup \{|x-(1,-y)| < \delta,\ x \in X\}$ actually means we flip one semicircle and glue it to another semicircle along their diameters which is exactly an circle. So, $p(\{|x-(0,y)| < \delta,\ x \in X\} \cup \{|x-(1,-y)| < \delta,\ x \in X\})$ is homeomorphic to an open disk, and we may denote this homeomorphism by ϕ. So, we can make $(p(\{|x-(0,y)| < \delta,\ x \in X\} \cup \{|x-(1,-y)| < \delta,\ x \in X\}),\ \phi)$ is the chart of $p(0,y)$, and M is locally euclidean.

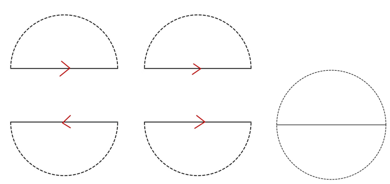

Figure 1: * (a) Figure 2: * (b) Figure 3: * (c)

Last, we need to prove M is Hausdorff, which is true, since Mobius Strip is a subspace of $\mathbb{R}^3$ (or $\mathbb{E}^3$). And $\mathbb{R}^3$ (or $\mathbb{E}^3$) is Hausdorff, which makes its subspace also Hausdorff. □

2. *Proof.* Here we use the definition of S^2 gluing its antipodal points, and let p be the quotient map. Also, we denote the antipodal mapping on S^2 by $x \to -x$. First, we show $\mathbb{RP}^2$ is second countable. S^2 is second countable since it is a subspace of $\mathbb{R}^3$. Let $\Gamma = \{U_i\ i \in \mathbb{N}\}$ be the topological base of S^2, then we claim that $\{p(U_i),\ i \in \mathbb{N}\}$ is a countable base of $\mathbb{RP}^2$. For every open set $U^{'} \subset \mathbb{RP}^2$, $p^{-1}(U^{'})$ is an open subset of S^2. So, $p^{-1}(U^{'}) = \cup U_i$, for some $U_i \in \Gamma$. So, $U^{'} = p(\cup U_i) = \cup p(U_i)$, which indicates $\{p(U_i), i \in \mathbb{N}\}$ is a topological base of $\mathbb{RP}^2$. So, $\mathbb{RP}^2$ is second countable.
Next, we prove $\mathbb{RP}^2$ is compact. Let $\{U_i\ ,\ \ i \in I\}$ is an open cover of $\mathbb{RP}^2$, then $\{p^{-1}(U_i)\ ,\ -p^{-1}(U_i)\ ,\ \ i \in I\}$ is an open cover of S^2. Since S^2 is compact, we have finite sub open cover $\{p^{-1}(U_i)\ ,\ -p^{-1}(U_i)\ \ 1 \leq i \leq n\}$ of S^2. Then $\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} p(p^{-1}(U_i) \cup -p^{-1}(U_i)) = p(\bigcup_{i=1}^{n} p^{-1}(U_i)) = p(S^2) = \mathbb{RP}^2$. So, $\mathbb{RP}^2$ is compact.
Last, we prove $\mathbb{RP}^2$ is Hausdorff. Let $p(x_1), p(x_2)\ , x_1\ x_2 \in S^2$ be two different element in $\mathbb{RP}^2$. Then x_1, x_2 must be two different elements in S^2. Since S^2 is Hausdorff, there exists two open neighborhhod $x_1 \in U_1\ , x_2 \in U_2\ , U_1 \cap U_2 = \emptyset\ , U_1 \subset S^2\ , U_2 \subset S^2$. So, $p(U_1)\ , p(U_2)$ are two open neighborhood of $p(x_1)\ , p(x_2)$. If $p(U_1) \cap p(U_2) \neq \emptyset$, then $-U_1 \cap U_2 \neq \emptyset$. Since $p(x_1) \neq p(x_2)$ in $\mathbb{RP}^3$, $-x_1 \neq x_2$. So, there exists another two open neighborhood of $-x_1\ , x_2$ in S^2, which are $U_1^{'}\ , U_2^{'}$, and the intersection of them

is $\emptyset$. Let $U_1^* = U_1 \cap -U_1^{'}$ and $U_2^* = U_2 \cap U_2^{'}$. They are all open sets of S^2, because antipodal mapping is actually a homeomorphism of S^2. So, $p(U_1^*)$ and $p(U_2^*)$ are open neighborhood of $p(x_1), p(x_2)$. If $p(U_1^*) \cap p(U_2^*) \neq \emptyset$, then $U_1^* \cap U_2^* \neq \emptyset$ or $-U_1^* \cap U_2^* \neq \emptyset$, which is impossible. So, $p(U_1^*) \cap p(U_2^*) = \emptyset$, which indicates $\mathbb{RP}^2$ is Hausdorff.

□

3. *Proof.* We continue to define $\mathbb{RP}^2$ as gluing the antipodal points of S^2, and we denote this quotient map by $\widetilde{p}$. We denote the upper half of S^2 by $\bar{S^2} = \{(x, y, z)|x^2 + y^2 + z^2 = 1, z \geq 0\}$. It is homeomorphic to $\bar{D^2}$ by sending $(x, y, z) \in \bar{S^2}$ to $(x, y) \in \bar{D^2}$, and we denote it by ϕ. Let p denote the quotient map from $\bar{D^2}$ to $X = \bar{D^2}/(\cos\theta, \sin\theta) \sim (\cos(\theta + \pi), \sin(\theta + \pi))$, namely gluing the opposite boundary points of $\bar{D^2}$. Then we have a quotient map $p \circ \phi : \bar{S^2} \to X$, so $\bar{S^2}/(\cos\theta, \sin\theta, 0) \sim (\cos(\theta + \pi), \sin(\theta + \pi), 0) \cong X$. Let $p^{'}$ denote gluing the antipodal points of the 'equator' of $\bar{S^2}$, and $p^{'}$ is also a quotient map from $\bar{S^2}$ to $\bar{S^2}/(\cos\theta, \sin\theta, 0) \sim (\cos(\theta + \pi), \sin(\theta + \pi), 0)$. Notice that $p^{'} = \widetilde{p}|_{\bar{S^2}}$, and $p^{'}(\bar{S^2}) = \widetilde{p}(S^2)$. We claim that $\bar{S^2}/p^{'} \cong S^2/\widetilde{p}$. First, the sets are the same, so we only need to verify if they have the same topology. Let U be an open set of $\bar{S^2}/p^{'}$, then $(p^{'})^{-1}(U)$ is an open set of $\bar{S^2}$. First We can assume that $(p^{'})^{-1}(U)$ doesn't contain any point of the equator, then $(p^{'})^{-1}(U)$ is also an open subset in S^2. So, U is also an open set in $\mathbb{RP}^2$. Then, we assume $(p^{'})^{-1}(U) \cap \{(x, y, 0)|x^2 + y^2 = 1\} \neq \emptyset$. We claim that $(p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)$ is an open set in S^2. For any $x \in ((p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)) \cap \{(x, y, 0)|x^2 + y^2 = 1\}^c$, we can find a open neighborhood of x in S^2, which is contained in $(p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)$. Let $x \in ((p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)) \cap \{(x, y, 0)|x^2 + y^2 = 1\}$, and we may as well let $x \in (p^{'})^{-1}(U)$. Then we can find a $\delta_1 > 0$ such that $\{y \in S^2, | \ \ |y - x| < \delta_1\} \cap (p^{'})^{-1}(U) \neq \emptyset$, since $((p^{'})^{-1}(U)$ is an open set of $\bar{S^2}$. (Here, $| \ \cdot \ |$ means the distance in S^2.) Also, $-x$ is in $(p^{'})^{-1}(U)$. So, we can find a $\delta_2 > 0$ such that $\{y \in S^2, | \ \ |y + x| < \delta_2\} \cap (p^{'})^{-1}(U) \neq \emptyset$. Let $\delta = min\{\delta_1 \ , \delta_2\}$, then $\{y \in S^2 \ , | \ \ |y - x| < \delta\} \subset (p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)$. So, $(p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U)$ is an open set in S^2. So, $U = p^{'}((p^{'})^{-1}(U)) = \widetilde{p}((p^{'})^{-1}(U) \cup -(p^{'})^{-1}(U))$ is an open set in $\mathbb{RP}^2$. On the other way, Let V be an open set of $\mathbb{RP}^2$. Then $\widetilde{p}^{-1}(V)$ is an open set in S^2. Then $\widetilde{p}^{-1}(V) \cap \bar{S^2}$ is an open set of $\bar{S^2}$. And it is easy to check that for any set $A \subset S^2$, we have $\widetilde{p}(A) = p^{'}(A \cap \bar{S^2})$, so we have $V = p^{'}(\widetilde{p}^{-1}(V) \cap \bar{S^2})$, which indicates V is an open set in $\bar{S^2}/p^{'}$. So, $\bar{S^2}/p^{'} \cong \mathbb{RP}^2$, which completes the proof.

□

4. *Proof.* First, we still need this 'M' to be second-countable and Hausdorff. Also, every point of M should have an open neighborhood homeomorphic to either an open set of $\mathbb{R}^n$ or $\mathbb{H}^n$. ($\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n)|x_n \geq 0\}$) We should define $x \in M$ is an interior interior point iff x has an open neighborhood in M which is homeomorphic to an open sets in $\mathbb{R}^n$. And, we should define $x \in M$ the boundary points of M iff x has an open neighborhood in M which is homeomorphic to an open sets in $\mathbb{H}^n$, and this homeomorphism sends x to the boundary of $\mathbb{H}^n$. And, the boundary of M should be a topological manifold of dimension $n - 1$. The easiest example we could come up is a simple compact curve in $\mathbb{R}^n$ with two ends. Then this curve should be a 1-dimensional topological manifold with boundary, and the two ends are its boundary. It is also easy to check they are 0-dimensional topological manifold. Another example is the upper half of a sphere S^2 ($\{(x, y, z)|x^2 + y^2 + z^2 = 1 \ , z \geq 0\}$) in $\mathbb{R}^3$. This should be a manifold with boundary points with S^1 as its boundary, which is a one dimensional manifold itself. □