Math 760 HW 5

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1. Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F\colon M \to \mathbb{R}$.

   Proof. If $F : M \to \mathbb{R}$ is a smooth submersion, then F is an open map(Taught in class). So, $F(M)$ is an open set of $\mathbb{R}$. Since M is compact, and F is continuous, $F(M)$ is compact in $\mathbb{R}$. This indicating that $F(M)$ is at least a closed set, but $\mathbb{R}$ is connected. So, $F(M) = \mathbb{R}$, since it is not an empty set. This is a contradiction because $\mathbb{R}$ is not compact. $\square$

2. Let $M = \mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$. Show that M is a smooth manifold.

   Proof. Let $F = \det(\cdot) : \mathrm{M}_{n\times n}(\mathbb{R}) = \mathbb{R}^{n\times n} \to \mathbb{R}$, then F is a polynomial of the entries. So, F is smooth. Since $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $\mathrm{M}_{n\times n}(\mathbb{R})$, F is also smooth on $\mathrm{GL}(n, \mathbb{R})$. So, we only need to prove 1 is a regular value of F. Suppose $A \in M, A = (a_{ij})$. From the knowledge of linear algebra, we know $\frac{\partial F}{\partial a_{ij}} = C_{ij}$ which is the (i, j)-cofactor of A. Since $\det(A) \neq 0$, there must be a pair of (i, j) such that $\frac{\partial F}{\partial a_{ij}} = C_{ij} \neq 0$. This means dF_A has rank 1, indicating that 1 is a regular value of F, so $M = F^{-1}(1)$ is a submanifold. $\square$

3. Show that $F\colon \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}^3$ induces a smooth embedding of the Möbius strip in $\mathbb{R}^3$.

$$\begin{aligned} F(u, v) &= \Big(\left(1 + v\cos\tfrac{u}{2}\right)\cos u,\ \left(1 + v\cos\tfrac{u}{2}\right)\sin u,\ v\sin\tfrac{u}{2} \Big) \\ &= \left(1 + v\cos\tfrac{u}{2}\right)(\cos u,\ \sin u,\ 0)\ +\ (0,\ 0,\ v\sin\tfrac{u}{2}). \end{aligned}$$

   Note. You can think of the Möbius strip M as a quotient of $\mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$.

   Proof. Define a diffeomorphism $\sigma : X = \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}) \to X = \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$, $\quad \sigma(u, v) = (u + 2\pi, -v)$. Let $\Gamma = \langle\sigma\rangle \cong \mathbb{Z}$ act on X by $(u, v) \cdot n := \sigma^n(u, v)$. Let $\pi : X \longrightarrow X/\Gamma(X) = M$ be the quotient map. We shall build a 2-dimensional smooth structure on M with π being a *local diffeomorphism*. First, we have following facts: (i) If $\sigma^n(u, v) = (u, v)$, then $n = 0$: It is bacause from the first coordinate $u + 2\pi n = u$ we get $n = 0$. (ii) For every compact $K \subset X$, the set $\{\, n \in \mathbb{Z} \mid \sigma^n(K) \cap K \neq \varnothing \,\}$ is finite: If $(u', v') \in K$ and $\sigma^n(u', v') \in K$, then $u' + 2\pi n$ lies in the bounded set $\mathrm{proj}_u(K)$, so $|n|$ is uniformly bounded; hence only finitely many n occur. (iii) For each $x \in X$ there exists an open neighborhood $W_x \subset X$ such that $\{\sigma^n(W_x)\}_{n\in\mathbb{Z}}$ are pairwise disjoint: Choose a relatively compact open $K \ni x$. By (ii) only finitely many n satisfy $\sigma^n(K) \cap K \neq \varnothing$. Shrink K to W_x to separate all translates.
   From (iii) we have $\pi^{-1}\big(\pi(W_x)\big) = \bigsqcup_{n\in\mathbb{Z}} \sigma^n(W_x)$, so the restriction $\pi|_{W_x} : W_x \to \pi(W_x)$ is a homeomorphism (its inverse picks the representative in W_x). Consequently:

   - π is a *local homeomorphism*(π is a covering map), which makes it is also an open map.
   - M is Hausdorff and second countable (Since π is open and a covering map, and X is Hausdorff, second-countable) Hausdorff, second countable space and π is open).

   Now we need to prove the transition map is smooth to prove M is actually a smooth manifold.
   For each $x \in X$, set $U_x := \pi(W_x)$ and $\varphi_x := \big(\pi|_{W_x}\big)^{-1} :\ U_x \longrightarrow W_x \subset \mathbb{R}^2$. Let $\mathcal{A} = \{(U_x, \varphi_x) : x \in X\}$ $\mathcal{A}$ is a smooth 2-dimensional atlas on M: If (U_x, φ_x) and (U_y, φ_y) overlap, then for every $p \in U_x \cap U_y$ there exists a unique $n \in \mathbb{Z}$ with $\varphi_y \circ \varphi_x^{-1} = \sigma^n\big|_{\varphi_x(U_x\cap U_y)}$. Since $\sigma^n(u, v) = (u+2\pi n, (-1)^n v)$ is an affine diffeomorphism of $\mathbb{R}^2$, the transition maps are smooth. So far, we have proved this quotient map gives M a smooth structure. $F(u+2\pi, -v) = \big((1-v\cos(\frac{u}{2}+\pi))\cos(u),\ (1-v\cos(\frac{u}{2}+\pi)),\ -v\sin(\frac{u}{2}+\pi)\big) = \Big(\left(1 + v\cos\frac{u}{2}\right)\cos u,\ \left(1 + v\cos\frac{u}{2}\right)\sin u,\ v\sin\frac{u}{2} \Big) = F(u, v)$. So, F can induce a map $\tilde{F} : M \to \mathbb{R}^3$. Now, we prove $\tilde{F}$ is an smooth immersion. Since π is an open covering quotient map, which induce

the smooth structrue and $\tilde{F}$ is induced by F, we only need to check F on X. First, F is smooth. Let $e_1(u) = (\cos u, \sin u, 0), \quad e_2(u) = (-\sin u, \cos u, 0), \quad e_3 = (0, 0, 1)$. Let $E =$

$$\begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then e_1, e_2, e_3 forms a new basis of $\mathbb{R}^3$ because $\det(E) = -1 \neq 0$. Now we can calculate that $F_u = (1 + v\cos(\frac{u}{2}))e_2(u) - \frac{v}{2}\sin\frac{u}{2}e_1(u) + \frac{v}{2}\cos\frac{u}{2}e_3,\ F_v = \cos\frac{u}{2}e_1(u) + \sin\frac{u}{2}e_3$. If F_u, F_v is linear dependent, then there are $\alpha, \beta \in \mathbb{R}$ such that $\alpha F_u + \beta F_v = 0 \Rightarrow \alpha(1 + v\cos\frac{u}{2}) = 0, \beta(\cos\frac{u}{2}e_1 + \sin\frac{u}{2}e_3) = 0 \Rightarrow \alpha = \beta = 0$, because $-\frac{1}{2} < v < \frac{1}{2}; -1 \leq \cos\frac{u}{2} \leq 1$ and $\cos\frac{u}{2}; \sin\frac{u}{2}$ cannot be zero at the same time. So the jacobian matrix of F at (u, v) has contant rank 2, which indicates $\tilde{F}$ is an immersion. Now we prove $\tilde{F}$ is a topological embedding. Suppose we have $\tilde{F}(m) = \tilde{F}(m^{'});\ m, m^{'} \in M \Rightarrow F(u, v) = F(u^{'}, v^{'})$ $\Rightarrow (1 + v\cos\frac{u}{2})^2\cos^2 u + (1 + v\cos\frac{u}{2})^2\sin^2 u = (1 + v^{'}\cos\frac{u^{'}}{2})^2\cos^2 u^{'} + (1 + v^{'}\cos\frac{u^{'}}{2})^2\sin^2 u^{'} \Rightarrow$ $(1 + v\cos\frac{u}{2})^2 = (1 + v^{'}\cos\frac{u^{'}}{2})^2$. If $(1 + v\cos\frac{u}{2}) = (1 + v^{'}\cos\frac{u^{'}}{2}) \Rightarrow u = u^{'} + 2\pi \Rightarrow v = -v^{'} \Rightarrow m = m^{'}$. If $((1 + v\cos\frac{u}{2}) = -(1 + v^{'}\cos\frac{u^{'}}{2})) \Rightarrow -2 = v\cos\frac{u}{2} + v^{'}\cos\frac{u^{'}}{2}$ which is impossible. So, $\tilde{F}$ is injective and continuous, because we have shown $\tilde{F}$ is smooth. Now, we calculating the inverse of $\tilde{F}$. Define $G:\ \bar{F}(M) \longrightarrow M, \qquad G(x, y, z) := \pi\big(\ u = \theta,\ \ v = \cos\frac{\theta}{2}\,(r - 1) + \sin\frac{\theta}{2}\,z\ \big)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \operatorname{atan2}(y, x)$. First, we show it is well defined. Changing the branch $\theta \mapsto \theta + 2\pi$ gives $\cos\frac{\theta + 2\pi}{2} = -\cos\frac{\theta}{2}, \qquad \sin\frac{\theta + 2\pi}{2} = -\sin\frac{\theta}{2}$, hence $v \mapsto -v$ and $(\theta, v) \sim (\theta + 2\pi, -v)$ in M. Therefore G does not depend on the choice of the branch of θ. Let $\pi(u, v) \in M$ and set $(x, y, z) = \bar{F}(\pi(u, v)) = F(u, v)$. Then $r - 1 = v\cos\frac{u}{2}, \qquad z = v\sin\frac{u}{2}$, so $\cos\frac{u}{2}\,(r - 1) + \sin\frac{u}{2}\,z = v\big(\cos^2\frac{u}{2} + \sin^2\frac{u}{2}\big) = v$. Since $\theta \equiv u \pmod{2\pi}$, we get $G(x, y, z) = \pi(u, v)$ and thus $G \circ \bar{F} = \mathrm{id}_M$. Conversely, for $(x, y, z) \in \bar{F}(M)$ we have $\bar{F} \circ G(x, y, z) = (x, y, z)$ by construction, hence $\bar{F} \circ G = \mathrm{id}_{\bar{F}(M)}$.
Now we prove G is continuous. On the open set $\{(x, y) : r > 0\}$, $\operatorname{atan2}(y, x)$ is continuous. On overlaps of regions where different continuous branches of θ are chosen, the formulas differ by $(u, v) \sim (u + 2\pi, -v)$, which identifies the same point of M. Therefore the locally defined continuous expressions glue to a global continuous map G. Since $\bar{F}$ is continuous, bijective onto its image, and admits a continuous inverse G, it is a homeomorphism onto $\bar{F}(M)$. Hence $\bar{F}$ is a *topological embedding.* By now, we have shown that $\tilde{F}$ is an immersion and a topological embedding, so it is a smooth embedding.

$\square$

4. Note that $P(X, Y, Z, W) = X^2 + Z^2 - Y^2 - W^2$ is a homogeneous polynomial. Consider the hypersurface $S \subset \mathbb{RP}^3$ defined by $P(X, Y, Z, W) = 0$ (this makes sense since P is homogeneous). Prove that S is an embedded torus. *Hint.* Start with $\widetilde{S} \subset S^3 \subset \mathbb{R}^4$ defined by the same polynomial.

Proof. Here we may view $\mathbb{RP}^3$ as $S^3/v \sim -v,\ v \in S^3$, the image of an open covering quotient map, which induces the smooth structure of $\mathbb{RP}^3$. Also, we may view $\mathbb{T}^2$ as $(\mathbb{R}/\pi\mathbb{Z}) \times (\mathbb{R}/\pi\mathbb{Z})$, which is also an open covering quotient map inducing a smooth structure of $\mathbb{T}^2$. Now, we assume $\pi : S^3 \to \mathbb{RP}^3$ being the quotient map. First, we give a map f from $\mathbb{T}^2 \to S^3$ by sending $(\tilde{\theta}, \tilde{\phi}) \in \mathbb{T}^2$ to $\frac{1}{\sqrt{2}}(\cos\theta, \cos\phi, \sin\theta, \sin\phi) \in S^3$. This map is not well defined, but if we define an map g from $\mathbb{T}^2 \to \mathbb{RP}^3$ by $\pi \circ f$, we know g is well defined. First, we need to verify what is the image of g. We know $\frac{1}{2}(1 - 1) = 0 \Rightarrow f(\mathbb{T}^2) \subset P(X, Y, Z, W) = 0,\ (X, Y, Z, W) \in S^3$. On the other hand if $P(X, Y, Z, W) = 0, (X, Y, Z, W) \in S^3 \Rightarrow X^2 + Z^2 = Y^2 + W^2 = \frac{1}{2} \Rightarrow$ We can find some θ, ϕ such that $X = \cos\theta, Z = \sin\theta, Y = \cos\phi, W = \sin\phi \Rightarrow \{P(X, Y, Z, W) = 0,\ (X, Y, Z, W) \in S^3\} \subset f(\mathbb{T}^2)$ $\Rightarrow \frac{1}{2}(1 - 1) = 0 \Rightarrow f(\mathbb{T}^2) = \{P(X, Y, Z, W) = 0,\ (X, Y, Z, W) \in S^3\} \Rightarrow g(\mathbb{T}^2) = \{P(X, Y, Z, W) = 0,\ (X, Y, Z, W) \in \mathbb{RP}^3\}$. Now we may verify g is injective. Suppose $(\tilde{\theta_1}, \tilde{\phi_1}), (\tilde{\theta_2}, \tilde{\phi_2}) \in \mathbb{T}^2$ make $\pi\big(\frac{1}{\sqrt{2}}(\cos\theta_1, \cos\phi_1, \sin\theta_1, \sin\phi_1)\big) = \pi\big(\frac{1}{\sqrt{2}}(\cos\theta_2, \cos\phi_2, \sin\theta_2, \sin\phi_2)\big) \to \theta_1 = \theta_2 + k\pi, \phi_1 = \phi_2 + k\pi \Rightarrow$ $(\tilde{\theta_1}, \tilde{\phi_1}) = (\tilde{\theta_2}, \tilde{\phi_2}) \to g$ is injective. Now we prove g is immersion. Since π is an open quotient covering map which induces the smooth structure of $\mathbb{RP}^3$, by 5. π is a submersion. So, now we may calculate the rank of f. Also, we know $\mathbb{R}^2 \to \mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}/\pi\mathbb{Z} = \mathbb{T}^2$ is also an open quotient covering which induces the smooth structure of $\mathbb{T}^2$. So, we may caculate the rank of $F: \quad (\theta, \phi) \to \frac{1}{\sqrt{2}}(\cos\theta, \cos\phi, \sin\theta, \sin\phi)$. We can calculate that $F_\theta = \frac{1}{\sqrt{2}}(-\sin\theta, 0, \cos\theta, 0)$; $F_\phi = \frac{1}{\sqrt{2}}(0, \sin\phi, 0, \cos\phi)$. Since $\sin\theta, \cos\theta$ can not be zero at the same time; $\sin\phi, \cos\phi$ can not be zero at the same time, F_θ, F_ϕ are linear independent. $\Rightarrow$ f has rank 2. $\Rightarrow$ g is immersion. Also, We know $\mathbb{T}^2$ is compact, so g is acutually a smooth embedding. Since $g(\mathbb{T}^2) = \{P(X,Y,Z,W) = 0,\ (X,Y,Z,W) \in \mathbb{RP}^3\}$, $\{P(X,Y,Z,W) = 0,\ (X,Y,Z,W) \in \mathbb{RP}^3\}$ is an embedded torus in $\mathbb{RP}^3$. □

5. Let $\pi\colon \mathbb{K}^{n+1} \setminus \{0\} \to \begin{cases} \mathbb{RP}^n, & \mathbb{K} = \mathbb{R}, \\ \mathbb{CP}^n, & \mathbb{K} = \mathbb{C} \end{cases}$ be the canonical projection.

(a) Prove that π is a submersion.

Proof. (i) : Let $\mathbb{K} = \mathbb{R}$. Let $U_i = \{(x_1, \cdots, x_n, x_{n+1}) \in \mathbb{RP}^n | x_i \neq 0\}, \phi_i : (x_1, \cdots, x_i, \cdots, x_{n+1}) \to \frac{1}{x_i}(x_1, \cdots, \hat{x_i}, \cdots, x_{n+1}) \in \mathbb{R}^n$. Let $p = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} - 0$, we may assume $x_1 \neq 0$. Then we can find a neighborhood $p \in U \subset \mathbb{R}^{n+1} - 0$ such that $\forall p' = (x_1', \cdots, x_{n+1}') \in U, x_1' \neq 0$ since x_1 is a continuous map. Then we can find an open neighborhood $U_1 \subset \mathbb{RP}^n$ such that $\pi(U) \subset U_1$. Then we calculate the jacobian matrix of $\phi_1 \circ \pi$. It is $J =$

$$\begin{pmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} & 0 & \cdots & 0 \\ -\frac{x_3}{x_1^2} & 0 & \frac{1}{x_1} & 0\cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -\frac{x_{n+1}}{x_1^2} & 0 & \cdots & \cdots & \frac{1}{x_1} \end{pmatrix}$$

. So J has rank n. Since p is an arbitary point, we have showed π is a submersion.

(ii): $\mathbb{K} = \mathbb{C}$. Let f_i denote the natural homeomorphism from $\mathbb{C}^i \to \mathbb{R}^{2i}$. $U_i = \{(z_1, \cdots, z_n, z_{n+1}) \in \mathbb{CP}^n | Z_i \neq 0\}, \phi_i : (Z_1, \cdots, Z_i, \cdots, Z_{n+1}) \to \frac{1}{z_i}(z_1, \cdots, \hat{z_i}, \cdots, z_{n+1}) \in \mathbb{C}^n$. Then we know $(U_i, f_n \circ \phi_i)$ is a chart. Let $p = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} - 0$, we may assume $z_1 \neq 0$. Then we can find a neighborhood $p \in U \subset \mathbb{C}^{n+1} - 0$ such that $\forall p' = (z_1', \cdots, z_{n+1}') \in U, z_1' \neq 0$ since z_1 is a continuous map. Then we can find an open neighborhood $U_1 \subset \mathbb{CP}^n$ such that $\pi(U) \subset U_1$. Then we calculate the jacobian matrix of $\phi_1 \circ \pi$. It is $J =$

$$\begin{pmatrix} -\frac{z_2}{z_1^2} & \frac{1}{z_1} & 0 & \cdots & 0 \\ -\frac{z_3}{z_1^2} & 0 & \frac{1}{z_1} & 0\cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -\frac{z_{n+1}}{z_1^2} & 0 & \cdots & \cdots & \frac{1}{z_1} \end{pmatrix}$$

. So J has rank n. Since p is an arbitary point, we have showed $\phi_1 \circ \pi$ has constant rank n, which indicates $f_n \circ \phi_1 \circ \pi \circ f_{n+1}^{-1}$ has constant rank $2n$, which indicates π is a submersion. □

(b) Let π_0 be the restriction of π to the sphere S^n (for $\mathbb{K} = \mathbb{R}$) or S^{2n+1} (for $\mathbb{K} = \mathbb{C}$). Prove that π_0 is also a submersion.

Hint. To prove (b) using (a), it suffices to show that the kernel of $d\pi$ is not contained in the tangent space to the sphere.

Proof. Suppose $\mathbb{K} = \mathbb{R}$. Here, we consider the map $\pi_0 : S^n \to \mathbb{R}P^n$. Let $\iota : S^n \to \mathbb{R}^{n+1}$ be the natural smooth embedding. We know $\pi_0 = \pi \circ \iota \Rightarrow d\pi_0 = d\pi \circ d\iota$. We may view $T_p(S^n), p \in S^n \subset \mathbb{R}^{n+1}$ as a subspace of $T_p(\mathbb{R}^{n+1})$. Then from the knowledge of linear algebra, we only need to check $T_p(S^n) \cap \ker d\pi_p = 0$, $\forall p \in S^n$. We now need to calculate the kernel of $d\pi_p$. Let $p = (x_1 \cdots, x_i, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} - 0$ with $x_i \neq 0$. Then we can find $p \in U \subset \mathbb{R}^{n+1}$, $U_i \subset \mathbb{RP}^n$

such that $\pi(U) \subset U_1$ like what we did in (a). Our caculation shows the jacobian should be

$$\begin{pmatrix} \frac{1}{x_i} & 0 & \cdots & 0 & -\frac{x_1}{x_i^2} & 0\cdots & 0 & \\ 0 & \frac{1}{x_i} & 0 & \cdots & -\frac{x_2}{x_i^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{x_i} & -\frac{x_{i-1}}{x_i^2} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots & & \\ 0 & & & \cdots & -\frac{x_{n+1}}{x_i^2} & 0 & \cdots & \frac{1}{x_i} \end{pmatrix}$$

Noticing that $\mathbb{R}^{n+1}$ is a vector space, we have a canonical isomorphism between $T_p\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$. So, we can caculate that $\ker d\pi_p = \{\lambda(-\frac{x_1}{x_i}, \cdots, -\frac{x_{i-1}}{x_i}, 1, -\frac{x_{i+1}}{x_i}, \cdots, -\frac{x_{n+1}}{x_i}) | \lambda \in \mathbb{R}\} = \lambda p, \lambda \in \mathbb{R}$. If we also view $T_p(S^n)$ as a subspace of $\mathbb{R}^{n+1}$, we know $T_p(S^n) = \{v \cdot p = 0 | v \in \mathbb{R}^{n+1}\}$. So, we know $T_p(S^n) \perp \ker d\pi_p \Rightarrow T_p(S^n) \cap \ker d\pi_p = 0 \Rightarrow \pi_0$ is submersion, because p is an arbitary point.

(ii) : Suppose $\mathbb{K} = \mathbb{C}$. Let f_i denote the natural homeomorphism between $\mathbb{C}^i$ and $\mathbb{R}^i$. Let $U_i = \{(z_1, \cdots, z_n, z_{n+1}) \in \mathbb{CP}^n | Z_i \neq 0\}, \phi_i : (Z_1, \cdots, Z_i, \cdots, Z_{n+1}) \to \frac{1}{z_i}(z_1, \cdots, \hat{z_i}, \cdots, z_{n+1}) \in \mathbb{C}^n$. Then we know $(U_i, f_n \circ \phi_i)$ is a chart. Let $p = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} - 0$ with $z_i \neq 0$. Then we can find $p \in U \subset C^{n+1} - 0$ such that $\pi(U) \subset U_i$. Just like (i), we need to caculate the $\ker d\pi_p$ and show that its intersection with $T_p(S^{2n+1})$ is a one dimensional linear subspace of $\mathbb{R}^{2n+2}$. We have caculated that the jacobian is

$$\begin{pmatrix} \frac{1}{z_i} & 0 & \cdots & 0 & -\frac{z_1}{z_i^2} & 0\cdots & 0 & \\ 0 & \frac{1}{z_i} & 0 & \cdots & -\frac{z_2}{z_i^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{z_i} & -\frac{z_{i-1}}{z_i^2} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots & & \\ 0 & & & \cdots & -\frac{z_{n+1}}{z_i^2} & 0 & \cdots & \frac{1}{z_i} \end{pmatrix}$$

This is actually a matrix with entries belong to $\mathbb{C}$. We first find its kernel in $\mathbb{C}^{n+1}$, then we can transit it to $\mathbb{R}^{2n+2}$ in a natural way. Just like (i), its kernel in $\mathbb{C}^{n+1}$ is $\{\lambda(-\frac{z_1}{z_i}, \cdots, -\frac{z_{i-1}}{z_i}, 1, -\frac{z_{i+1}}{z_i}, \cdots, -\frac{z_{n+1}}{z_i}) | \lambda \in \mathbb{C}\} = \lambda p$. Then we know it is a 2 dimensional subspace of $\mathbb{R}^{2n+2}$ with basis p, ip. Just like (i), $T_p(S^{2n+1}) = \{v \cdot p = 0 | v \in \mathbb{R}^{2n+2}\}$. Suppose we have $ap + b(ip) \in T_p(S^{2n+1})$; $a, b \in \mathbb{R}$, then $ap + b(ip) \cdot_{\mathbb{R}} p = 0 \Rightarrow a + bRe\big(\sum_k p_k(-i\overline{p_k})\big) = a = 0$. This shows the intersection of $\ker d\pi_p$ and $\mathbb{T}_p(S^{2n+1})$ is one dimensional, which indicates that $\ker d\pi_p$ is not contained in $T_p(S^{2n+1})$. So, π_0 is a submersion. $\square$