# Homological Algebra

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[1]These notes are not endorsed by the lecturer, and I have modified them significantly after lectures. They are neither fully faithful nor essentially surjective to the accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

# Contents

# 1 Introduction

## 1.1 History of Homological Algebra

**1940s Leray spectral sequence.** The origins of homological methods trace back to Jean Leray's work during World War II, where he introduced spectral sequences in the study of sheaf cohomology. This provided a systematic way to compute cohomology groups through filtrations.

**1950s Cartan–Eilenberg** developed the formal foundations of homological algebra in their book *Homological Algebra* (1956), introducing derived functors such as Ext and Tor.

**Serre, *FAC*:** *Faisceaux Algébriques Cohérents* (1955) introduced coherent sheaves on complex manifolds and algebraic varieties, linking algebraic geometry and sheaf theory.

**Grothendieck's Tôhoku paper** (1957) unified homological algebra for abelian categories, generalizing the setting from modules to sheaves. This provided a categorical framework for derived functors and spectral sequences, combining Serre's geometric perspective with algebraic formalism.

**1960s Grothendieck and Verdier: derived categories.** Alexander Grothendieck's school, particularly Verdier's thesis (1963), formalized the notion of the *derived category* $D(\mathcal{A})$, allowing one to work with complexes up to quasi-isomorphism. This revolutionized the formulation of cohomological theories in algebraic geometry.

**1970s Triangulated categories.** Verdier's construction led to the concept of triangulated categories, providing the language for much of modern homological algebra, particularly in the study of sheaves and motives.

**1980s Quillen's model categories** (1967, influential in the 1980s) introduced a homotopical framework generalizing chain complexes, forming the foundation of modern homotopical algebra.

**Mukai's equivalences** (1981) established that for an abelian variety $A$, the derived categories of coherent sheaves satisfy

$$D^b(\mathrm{Coh}(A)) \simeq D^b(\mathrm{Coh}(\widehat{A})),$$

opening the study of derived equivalences in algebraic geometry.

**1990s Study of $D(X)$.** Derived categories of coherent sheaves on varieties became a central object of study (Bondal–Orlov). Ideas from mirror symmetry connected derived categories with symplectic geometry.

**Derived geometry.** Spaces themselves began to be "derived" — with work by Jacob Lurie and Bertrand Toën–Gabriele Vezzosi developing derived and higher algebraic geometry. These ideas eventually linked to the *geometric Langlands program* and higher categorical structures.

**2000s–Present Higher categories and derived geometry.** Lurie's *Higher Algebra* and *Spectral Algebraic Geometry* established $\infty$-categorical approaches to homological algebra. Derived categories evolved into *stable $\infty$-categories*, forming a modern unified language across topology, algebra, and geometry.

## 1.2 Abelian Categories

**Definition 1.2.1** (Additive Categories)**.** *A category $\mathcal{C}$ is said to be* ***additive*** *if it satisfies the following properties.*

1. *For each $A, B \in \mathcal{C}, \operatorname{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition (enriched over the category of abelian groups).*
2. *$\mathcal{C}$ has a zero object, denoted $0$ . (This is an object that is simultaneously an initial object and a final object.)*
3. *It has products of two objects.*

**Remark 1.2.2.** A functor between additive categories preserving the additive structure of Mor, is called an **additive functor**. For the following definitions, we assume the categories and functors are addictive.

**Definition 1.2.3** (Kernel)**.** *Let $f : B \to C$ be a morphism in an additive category. A* **kernel** *of $f$ is a morphism $i : \ker f \to B$ such that:*

$$f \circ i = 0,$$

*and for any morphism $a : A \to B$ with $f \circ a = 0$, there exists a unique morphism $a' : A \to \ker f$ making the following diagram commute:*

$$\begin{tikzcd}
\ker f \arrow[r, "i"] \arrow[rr, bend left, "0"] & B \arrow[r, "f"] & C \\
A \arrow[u, dashed, "a'"] \arrow[ur, "a"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

**Remark 1.2.4.** The $\ker f$ is the largest object mapping to $B$ that is sent to 0 under $f$. Equivalently, the kernel can be characterized *as a limit* of the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
B \arrow[r, "f"] & C
\end{tikzcd}$$

That is, $\ker f$ together with $i : \ker f \to B$ satisfies the universal property of the limit:

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \ker f \arrow[r] \arrow[d, "i"'] & 0 \arrow[d] \\
 & B \arrow[r, "f"'] & C
\end{tikzcd}$$

**Definition 1.2.5** (Cokernel)**.** *Let $f : C \to B$ be a morphism in an additive category. A* **cokernel** *of $f$ is a morphism $p : B \to \operatorname{coker} f$ such that:*

$$p \circ f = 0,$$

*and for any morphism $a : B \to A$ with $a \circ f = 0$, there exists a unique morphism $a' : \operatorname{coker} f \to A$ making the following diagram commute:*

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & B \arrow[r, "p"] \arrow[dr, "a"] & \operatorname{coker} f \arrow[d, dashed, "a'"] \\
 & & A
\end{tikzcd}$$

**Remark 1.2.6.** The coker $f$ is the smallest object mapping $B$ to 0 under $f$. Equivalently, the cokernel can be characterized *as a colimit* of the following diagram:

$$\begin{tikzcd} & 0 \\ B & C \arrow[u] \arrow[l, "f"'] \end{tikzcd}$$

$$\begin{tikzcd} A & & \\ & \operatorname{coker} f \arrow[ul, dashed, "\exists!"'] \arrow[r] & 0 \arrow[ull, bend right] \\ & B \arrow[u, "p"] \arrow[r, "f"'] \arrow[uul, bend left] & C \arrow[u] \end{tikzcd}$$

**Remark 1.2.7.**
- $\operatorname{im}(f) = \ker(\operatorname{coker} f)$
- If $i : A \to B$ is a monomorphism, then we say that $A$ is a subobject of $B$, where the map $i$ is implicit. A quotient object is defined dually to subobject.
- The cokernel of a monomorphism is called the quotient.

**Definition 1.2.8** (Abelian Categories)**.** *An* ***abelian category*** *is an additive category satisfying three additional properties.*

1. *Every map has a kernel and cokernel.*
2. *Every monomorphism is the kernel of its cokernel.*
3. *Every epimorphism is the cokernel of its kernel.*

**Theorem 1.2.9** (Mitchell Embedding Theorem)**.** *Every small abelian category admits a full, faithful and exact functor to the category* RMod *for some ring $R$.*

## 1.3 Homology

**Definition 1.3.1** (Exactness)**.** *In an abelian category, we say a sequence*

$$\ldots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \ldots$$

*is* ***exact*** *at $B$ if $\ker g = \operatorname{im} f$. A sequence is an* ***exact sequence*** *if it is exact at each term. A* ***short exact sequence*** *is an exact sequence with five terms, the first and last of which are zeros.*

**Definition 1.3.2** (Chain Complexes)**.** *A* ***chain complex*** *is a sequence of objects and homomorphisms in an abelian category*

$$\cdots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_0} \cdots$$

*such that*

$$d_i \circ d_{i+1} = 0$$

*for all $i$.*

**Definition 1.3.3** (Chain Maps)**.** *If $\left(C.,d^C_\cdot\right)$ and $\left(D.,d^D_\cdot\right)$ are chain complexes, then a* ***chain map*** *$C. \to D.$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that $d^D_n \circ f_n = f_{n-1} \circ d^C_n$. In other words, the following diagram has to commute for all $n$ :*

$$\begin{tikzcd}
C_n \arrow[r, "f_n"] \arrow[d, "d^C_n"'] & D_n \arrow[d, "d^D_n"] \\
C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1}
\end{tikzcd}$$

**Theorem 1.3.4.** *Let $\mathcal{A}$ be an Abelian category. The category of chain complexes in $\mathcal{A}$ is an Abelian category.*

**Definition 1.3.5** (Homology)**.** *The* ***homology*** *of a chain complex $C.$ is*

$$H_i(C.) = \frac{\ker\left(d_i : C_i \to C_{i-1}\right)}{\operatorname{im}\left(d_{i+1} : C_{i+1} \to C_i\right)}.$$

**Remark 1.3.6.** By using increasing superscript indices, and set $C^i = C_{-i}$, we get the notion of **cochain complexes** and **cohomologies**. Notice the cohomological notation is dual to homological notation and we prefer to using homological one in this course.

**Definition 1.3.7** (Quasi-isomorphism)**.** *A chain map*

$$f_\bullet : C_\bullet \to D_\bullet$$

*is called a* quasi-isomorphism *if*

$$H_i(f_\bullet) : H_i(C_\bullet) \xrightarrow{\sim} H_i(D_\bullet)$$

*is an isomorphism for all $i$.*

**Lemma 1.3.8.** *$H_i : Ch(\mathcal{A}) \to \mathcal{A}$ is a functor.*

**Theorem 1.3.9** (Snake Lemma)**.**

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

**Theorem 1.3.10** (Zig-Zag Lemma)**.** *Suppose we have a short exact sequence of complexes*

$$0 \longrightarrow A. \xrightarrow{i.} B. \xrightarrow{q.} C. \longrightarrow 0.$$

*Then there are maps*

$$\partial : H_n(C.) \to H_{n-1}(A.)$$

*such that there is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "q_*"] & H_n(C) \arrow[dll, "\partial_*"'] & \\ & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "q_*"] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

## 1.4 Differential Graded Categories

[1] A **differential graded (dg) object** $C$ is a $\mathbb{Z}$-graded object

$$C = \bigoplus_{i \in \mathbb{Z}} C_i,$$

together with a differential operator

$$d : C \longrightarrow C[1],$$

such that $d^2 = 0$.

The notation $C[1]$ denotes the *shifted* graded object, whose underlying object is still $C$, but with components shifted by one degree:

$$(C[1])_n = C_{n-1}, \qquad d_{C[1]} = (-1)^i \, d_C \text{ on } C^{i-1}.$$

Intuitively, "shifting by 1" means *moving the whole complex to the left by one.*

For cohomological notation:

$$C^i := C_{-i}, \qquad C[1]^i = C^{i+1}.$$

**Remark 1.4.1.** A chain complex of $R$-module is nothing but a dg $R$-module. Philosophy: consider everything as a dg thing.

**Definition 1.4.2** (Chain Homotopy)**.** *Let $\mathcal{A}$ be an additive category and let $(C_*, \partial_C)$ and $(D_*, \partial_D)$ be chain complexes with values in $\mathcal{A}$. Let $f = \{f_n\}_{n \in \mathbf{Z}}$ and $f' = \{f'_n\}_{n \in \mathbf{Z}}$ be chain maps from $C_*$ to $D_*$. A **chain homotopy** from $f$ to $f'$ is a collection of maps $h = \{h_n : C_n \to D_{n+1}\}$ which satisfy the identity*

$$f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C$$

*for every integer $n$.*

We say that $f$ and $f'$ are **chain homotopic** if there exists a chain homotopy from $f$ to $f'$. We will say that $f$ is a **chain homotopy equivalence** if there exists a chain map $g : D_* \to C_*$ such that $g \circ f$ and $f \circ g$ are chain homotopic to the identity morphisms $\mathrm{id}_{C_*}$ and $\mathrm{id}_{D_*}$, respectively. Clearly, a chain homotopy equivalence induces an isomorphism on homology.

[1] I have modified this section significantly, referring to Lurie's Kerodon.

**Definition 1.4.3** (Mapping Complexes)**.** *Let $(C_*, \partial_C)$ and $(D_*, \partial_D)$ be chain complexes with values in an Abelian category $\mathcal{A}$. For each integer d, we let $[C, D]_d$ denote the abelian group $\prod_{n\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(C_n, D_{n+d})$ consisting of maps from $C_*$ to $D_*$ which are homogeneous of degree d. These abelian groups can be organized into a chain complex*

$$\cdots \xrightarrow{\partial} [C, D]_2 \xrightarrow{\partial} [C, D]_1 \xrightarrow{\partial} [C, D]_0 \xrightarrow{\partial} [C, D]_{-1} \xrightarrow{\partial} [C, D]_{-2} \xrightarrow{\partial} \cdots,$$

*whose boundary operator $\partial : [C, D]_d \to [C, D]_{d-1}$ is given by the formula $\partial\{f_n : C_n \to D_{n+d}\}_{n\in\mathbf{Z}} = \{\partial_D \circ f_n - (-1)^d f_{n-1} \circ \partial_C\}_{n\in\mathbf{Z}}$. We will refer to $[C, D]_*$ as the* ***mapping complex*** *associated to the chain complexes $C_*$ and $D_*$.*

**Remark 1.4.4.**

- Chain maps from $C_*$ to $D_*$ can be identified with **0 -cycles** of the chain complex $[C, D]_*$ : that is, with elements $f = \{f_n\}_{n\in\mathbf{Z}} \in [C, D]_0$ satisfying $\partial(f) = 0$.
- Given a pair of chain maps $f, f' : C_* \to D_*$, a chain homotopy from $f$ to $f'$ is an element $h = \{h_n\}_{n\in\mathbf{Z}} \in [C, D]_1$ satisfying $\partial(h) = f' - f$. In particular, $f$ and $f'$ are **chain homotopic** if and only if they are **homologous** when viewed as 0 -cycles of the complex $[C, D]_*$, so $\mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*)$ can be identified with the 0th homology group of $[C, D]_*$.

**Definition 1.4.5** (Differential Graded Categories)**.** *A* ***differential graded category*** *$\mathcal{C}$ consists of the following data:*

1. *A collection $\mathrm{Ob}(\mathcal{C})$, whose elements we refer to as objects of $\mathcal{C}$.*
2. *For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a chain complex $(\mathrm{Hom}_{\mathcal{C}}(X, Y)_*, \partial)$. For each integer $n$, we refer to the elements of $\mathrm{Hom}_{\mathcal{C}}(X, Y)_n$ as morphisms of degree $n$ from $X$ to $Y$.*
3. *For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and every pair of integers $m, n \in \mathbf{Z}$, a function*

$$c_{Z,Y,X} : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_n \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_m \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_{m+n}$$

*which we will refer to as the composition law. Given a pair of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)_m$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)_n$, we will often denote the image $c_{Z,Y,X}(g, f) \in \mathrm{Hom}_{\mathcal{C}}(X, Z)_{m+n}$ by $g \circ f$ or $gf$.*

4. *For every object $X \in \mathrm{Ob}(\mathcal{C})$, a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)_0$, which we will refer to as the identity morphism.*

**Remark 1.4.6.** A dg category is a "category" enriched over $Ch(\mathcal{A})$ for some abelian category $\mathcal{A}$.

**Example 1.4.7.** Let $\mathcal{A}$ be an abelian category. $Ch(\mathcal{A})$ is a dg category enriched over $Ch(\mathbb{Z})$.

Let $\mathcal{A}$ be an abelian category.

**Definition 1.4.8** (Homotopy Categories)**.** *The* ***homotopy category*** *of chain complexes in $\mathcal{A}$, denoted*

$$\mathrm{HoCh}(\mathcal{A}) = H^0(\mathrm{Ch}(\mathcal{A})) = K(\mathcal{A}),$$

*is the category defined as follows:*

- *Objects are chain complexes in $\mathcal{A}$;*
- *Morphisms are chain maps, taken* ***modulo chain homotopy****.*

*That is,*

$$\mathrm{Hom}_{K(\mathcal{A})}(A., B.) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A., B.)/\textit{(chain homotopy)}.$$

**Remark 1.4.9.**

$$K(\mathcal{A}) \text{ is } \underline{\text{not}} \text{ an abelian category,}$$

but it carries a natural structure of a **triangulated category**.

# 2 Derived Functors

## 2.1 Exact Functors

**Definition 2.1.1** (Exact Functors)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a (covariant) functor between abelian categories.*

1. *$F$ is called **right exact** if for every short exact sequence*

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

   *in $\mathcal{C}$, the induced sequence*

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

   *is exact in $\mathcal{D}$.*

2. *$F$ is called **left exact** if for every short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

   *in $\mathcal{C}$, the induced sequence*

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC$$

   *is exact in $\mathcal{D}$.*

3. *$F$ is called **exact** if it is both left and right exact.*

**Remark 2.1.2.** If $F$ is *contravariant*, then $F$ is left exact if and only if for every exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC$$

is exact. i.e. we get a left exact sequence after applying $F$.

**Example 2.1.3** (Tensor is Left-exact)**.** Let $M \in R$-Mod. Then the functor

$$- \otimes_R M : \text{Mod-}R \longrightarrow \text{Ab}$$

is **right exact**, where ${}_{\mathbb{Z}}N_R \in \text{Mod-}R$ and ${}_RM_{\mathbb{Z}} \in R\text{-Mod}$, then

$${}_{\mathbb{Z}}N_R \otimes_R {}_RM_{\mathbb{Z}} \in \text{Ab}.$$

Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$, we get

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\simeq} \mathbb{Z}/2 \longrightarrow 0.$$

Hence the resulting sequence is *exact on the right* but not on the left — confirming that

$$- \otimes_R M \quad \text{is right exact (but not left exact).}$$

**Example 2.1.4** (Covariant Hom is Left-exact)**.** Let $M \in R$-Mod. Then the functor

$$\mathrm{Hom}(M, -) : R\text{-Mod} \longrightarrow \mathrm{Ab}$$

is **left exact**.

That is, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $R$-Mod, the induced sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$$

is exact in Ab.

Consider again the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}(\mathbb{Z}/2, -)$, we get:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow 0.$$

Since any map $\mathbb{Z}/2 \to \mathbb{Z}$ is zero, we obtain:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2.$$

Thus the sequence is exact on the left but not on the right, confirming that

$$\mathrm{Hom}(M, -) \text{ is left exact (but not right exact).}$$

**Example 2.1.5** (Contravariant Hom is Left-exact)**.** Let $M \in R$-Mod, the functor

$$\mathrm{Hom}(-, M) : R\text{-Mod} \longrightarrow \mathrm{Ab}$$

is **left exact**.

That is, for every short exact sequence in $R$-Mod,

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{v^*} \mathrm{Hom}(B, M) \xrightarrow{u^*} \mathrm{Hom}(A, M)$$

is exact in Ab.

(Note: this is the contravariant version of the Hom functor, hence the direction of arrows reverses.)

**Example 2.1.6** (Quotient is right exact)**.** Let $M \in R$-Mod and let $I \lhd R$ be an ideal.

Consider the functor

$$M \longmapsto M/IM.$$

We have the natural isomorphism

$$M/IM \;\cong\; M \otimes_R R/I.$$

Hence this functor can be viewed as the tensor functor

$$- \otimes_R R/I : R\text{-Mod} \longrightarrow \mathrm{Ab}.$$

Since the tensor functor $- \otimes_R N$ is **right exact** for any $N$, it follows that

$$M \longmapsto M/IM \quad \text{is right exact.}$$

**Example 2.1.7.** Consider the functor

$$(-)^G : \mathrm{Rep}_G \longrightarrow \mathrm{Vect},$$

which sends a representation $V$ of $G$ to its subspace of invariants

$$V^G = \{\, v \in V \mid g \cdot v = v \text{ for all } g \in G \,\}.$$

**Note.** This functor can be expressed as a Hom functor:

$$(-)^G \;\cong\; \mathrm{Hom}_{G\text{-rep}}(\mathbf{1}_G, -),$$

where $\mathbf{1}_G$ denotes the trivial representation of $G$.

Since any Hom functor of the form $\mathrm{Hom}(A, -)$ is **left exact**, it follows that

$$(-)^G : \mathrm{Rep}_G \longrightarrow \mathrm{Vect} \quad \text{is left exact.}$$

**Example 2.1.8** (Pushfoward is Left-exact)**.** Let $f : X \to Y$ be a continuous map of topological spaces (or morphism of schemes). Given a sheaf $\mathcal{F}$ of abelian groups on $X$, the **pushforward sheaf** (or **direct image sheaf**) of $\mathcal{F}$ along $f$ is the sheaf $f_*\mathcal{F}$on $Y$ defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)) \quad \text{for every open set } V \subseteq Y.$$

Pushforward is left exact. This gives a functor between categories of sheaves of abelian groups:

$$f_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y),$$

For a short exact sequence of sheaves on $X$:

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

applying $f_*$ gives

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

This sequence is exact on the left but not necessarily on the right. Thus $f_*$ is a **left exact functor.**

**Remark 2.1.9.** An addictive functor between abelian categories $F : \mathcal{C} \to D$ can always be lifted to $F : Ch(\mathcal{C}) \to Ch(\mathcal{D})$. However, many functors we are interested in are not exact, but only left- or right-exact. Therefore, they cannot be lifted to $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D})$ since $F(0) \ncong 0$. The derived versions of these functors we will talk about later will fix this problem.

## 2.2 Projective Objects

Let $\mathcal{C}$ be an abelian category.

**Definition 2.2.1** (Projective Objects)**.** *An object $P \in \mathcal{C}$ is called* ***projective*** *if for every epimorphism*

$$f : M' \twoheadrightarrow M$$

*and every morphism*

$$g : P \longrightarrow M,$$

*there exists a morphism $h : P \to M'$ such that*

$$f \circ h = g.$$

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
M' \arrow[r, two heads, "f"'] & M \arrow[r] & 0
\end{tikzcd}$$

*i.e. morphisms from $P$ lift along surjections.*

**Example 2.2.2** (Free Modules are Projective)**.** $R$ is a projective $R$-module. More generally, every **free** $R$-module is projective.

Indeed, given a surjection of $R$-modules

$$M' \twoheadrightarrow M \longrightarrow 0,$$

and any map from a free module $R^{(I)} \to M$, each **basis element** of $R^{(I)}$ can be lifted to $M'$ independently, so a lifting map $R^{(I)} \to M'$ exists.

$$\begin{tikzcd}
 & R^{(I)} \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
M' \arrow[r, two heads, "f"'] & M \arrow[r] & 0
\end{tikzcd}$$

**Example 2.2.3.** $\mathbb{Z}/2\mathbb{Z}$ is *not* a projective $\mathbb{Z}$-module.

Consider the surjection

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Suppose there were a lift of the identity map

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, dashed] \arrow[dr, "\mathrm{id}"'] & \mathbb{Z} \arrow[d, two heads, "\times 2"] \\
 & \mathbb{Z}/2
\end{tikzcd}$$

This would require a homomorphism $\mathbb{Z}/2 \to \mathbb{Z}$ such that composing with $\times 2$ gives the identity on $\mathbb{Z}/2$, which is impossible because $\mathbb{Z}$ has no element of order 2.

**Proposition 2.2.4** (Characterization of Projective Modules)**.** *Let $P$ be an $R$-module. The following statements are equivalent:*

*(1) $P$ is* ***projective****,*

*(2) The functor*

$$\mathrm{Hom}_R(P, -) : R\text{-Mod} \longrightarrow \mathrm{Ab}$$

*is* ***right-exact****, hence exact.*

*(3) $P$ is a* ***direct summand of a free module****, i.e. there exists a free $R$-module $F$ and a submodule $Q \subseteq F$ such that*

$$F = P \oplus Q.$$

*(4) Every short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

***splits****.*

**Theorem 2.2.5.** *Projective modules over local rings or PIDs are free.*

**Remark 2.2.6** (Projective Modules = Vector Bundles)**.** Geometrically, a vector bundle is locally free; algebraically, a projective module is locally a free module. These notions coincide on affine schemes.

Let $X = \mathrm{Spec}(R)$ be an affine scheme. There is an equivalence of categories

$$\mathrm{ProjMod}_R^{\mathrm{fg}} \;\simeq\; \mathrm{Vect}(X),$$

between finitely generated projective $R$-modules and vector bundles (locally free sheaves of finite rank or coherent sheaves with free stalks) on $X$.

## 2.3 Projective Resolution

**Definition 2.3.1.** *Let $M$ be an $R$-module (or an object in an abelian category). A* projective resolution *of $M$ is a chain complex*

$$P_\bullet = (\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0)$$

*together with a chain map $q : P_\bullet \to M[0]$ such that:*

1. *$P_i = 0$ for all $i < 0$;*
2. *$H_i(P_\bullet) = 0$ for all $i \neq 0$;*
3. *$H_0(P_\bullet) \cong M$;*
4. *each $P_i$ is projective.*

Equivalently, this means that we have an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{q} M \longrightarrow 0$$

with each $P_i$ projective.

Equivalently, the chain map between the projective chain complex $P_\bullet$ to $M[0]$

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "0"'] & P_0 \arrow[r] \arrow[d, "q"] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

If the abelian category $\mathcal{C}$ **has enough projectives** (every object is mapped surjectively by some projective object), then every object $M \in \mathcal{C}$ admits a (non-unique) projective resolution.

This can be done inductively as follows:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r, two heads] \arrow[d, no head] & K_0 \arrow[r] \arrow[d] & 0 \\
 & P_2 \arrow[u, two heads] \arrow[r] & P_1 \arrow[r] & P_0 \arrow[d, two heads] & \\
 & & & M \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

That is inductively finding a projective $P_{i+1}$ mapping onto the kernel $K_i$ of the last projection $P_i \to K_{i-1}$.

**Remark 2.3.2.** The category of $R$-modules $R$-**Mod** has enough free objects, i.e. $R^{(M)} \twoheadrightarrow M$. Therefore, any $R$-module admits a free projective resolution.

**Example 2.3.3** (Vector Spaces)**.** Over a field $R = k$, every module is free. Given $M$,

$$\cdots \to 0 \to 0 \to M \to 0 \to \cdots$$

is already a projective resolution.

**Example 2.3.4.** Over $\mathbb{Z}$, take $M = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] \arrow[d, two heads] & 0 \arrow[r] & 0 \\
 & & & \mathbb{Z}/2\mathbb{Z} & &
\end{tikzcd}$$

Notice the kernel $\mathbb{Z}$ is projective, so we are done.

**Example 2.3.5.** Over $k[x]$, let $M = k[x]/(x)$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & k[x] \arrow[r, "x"] & k[x] \arrow[r] \arrow[d, two heads] & 0 \\
 & & & k[x]/(x) &
\end{tikzcd}$$

**Example 2.3.6.** Over $k[x, y]$, let $M = k[x, y]/(x, y)$.

$$\begin{tikzcd}
0 \arrow[r] & k[x,y] \arrow[r, "\begin{pmatrix} y \\ -x \end{pmatrix}"] & k[x,y]^{\oplus 2} \arrow[r, "(x\ y)"] & k[x,y] \arrow[d] \\
 & & & k[x,y]/(x,y)
\end{tikzcd}$$

**Example 2.3.7.** Over $R = k[x,y]/(xy)$, let $M = R/(x,y)$.

$$\begin{tikzcd}
\cdots \arrow[r] & R\oplus R \arrow[r, "\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}"] & R\oplus R \arrow[r, "\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}"] & R\oplus R \arrow[r, "\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}"] & R\oplus R \arrow[r, "(x{,}y)"] & R \arrow[d] \\
 & & & & & \mathcal{M}
\end{tikzcd}$$

**Theorem 2.3.8** (Hilbert)**.** *If $R$ is a regular commutative local ring, then every minimal resolution of a module ends after* $\dim R$ *steps.*

**Theorem 2.3.9** (Serre)**.** *If $R$ is not regular, there exists an $R$-module with an infinite minimal resolution.*

**Theorem 2.3.10** (Eisenbud)**.** *If $R$ is a hypersurface ring, the infinite resolution is eventually* 2*-periodic.*

Question: How far from unique are projective resolution?

**Proposition 2.3.11.** *Let $R$ be a ring (or work in an abelian category). Suppose*

$$f : M \longrightarrow N$$

*is a morphism of $R$-modules, and let*

$$P_\bullet \to M, \qquad Q_\bullet \to N$$

*be projective resolutions of $M$ and $N$ respectively.*

*Then there exists a chain map*

$$\tilde{f} : P_\bullet \longrightarrow Q_\bullet$$

*lifting $f$, i.e.*

$$H_0(\tilde{f}) = f : M = H_0(P_\bullet) \longrightarrow H_0(Q_\bullet) = N.$$

*Furthermore, $\tilde{f}$ is not unique, but unique up to homotopy.*

That is, $\tilde{f}$ makes the following diagram commute up to homology:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\tilde{f}_1"'] & P_0 \arrow[r] \arrow[d, "\tilde{f}_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

and if $\bar{f}, \bar{f}' : P_\bullet \to Q_\bullet$ are two such liftings, then they are homotopic:

$$\tilde{f} \simeq \tilde{f}'.$$

Equivalently, there is a natural isomorphism

$$\mathrm{Hom}_R(M,N) \;\cong\; \mathrm{Hom}_{\mathcal{K}(R\text{-mod})}(P_\bullet, Q_\bullet).$$

**Remark 2.3.12.** 1. Any functor $F$ takes isomorphisms to isomorphisms.

2. Functors between abelian categories need *not* map quasi-isomorphisms to quasi-isomorphisms.

   For example:
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow 0$$
   is a complex, and the natural map
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$
   is a quasi-isomorphism, but there is no quasi-isomorphism back.

3. However, homotopic maps are always mapped to homotopic maps.

**Theorem 2.3.13** (Horseshoe Lemma)**.** *Let*
$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$
*be a short exact sequence of $R$-modules (or objects in an abelian category). Suppose we have projective resolutions*
$$\cdots \to P_1' \to P_0' \to A' \to 0, \qquad \cdots \to P_1'' \to P_0'' \to A'' \to 0.$$
*Then there exists a projective resolution*
$$\cdots \to P_1 \to P_0 \to A \to 0$$
*such that we have a short exact sequence of chain complexes*
$$0 \longrightarrow P_\bullet' \longrightarrow P_\bullet \longrightarrow P_\bullet'' \longrightarrow 0,$$
*and each $P_i$ fits into a short exact sequence*
$$0 \longrightarrow P_i' \longrightarrow P_i \longrightarrow P_i'' \longrightarrow 0.$$

We have the following diagram
$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & P_1' \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

**Remark 2.3.14.** This lemma shows that projective resolutions behave well with respect to short exact sequences.

## 2.4 Motivations of Derived Functors

Moral: If $F$ behaves badly on $X$, replace $X$ by something else on which $F$ behaves well.

We will discuss two examples from homotopy theory.

**Example 2.4.1** (Homotopy Fibre)**.** Let

$$f^{-1}(y) = X_y \longrightarrow X$$

be the fiber of a map

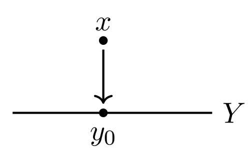
$$f : X \longrightarrow Y$$

over a point $y \in Y$. That is, the pullback

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{y\} \arrow[r, hook, "j"] & Y \end{tikzcd}$$

However, the process of "taking fibers" is *not homotopy invariant* — this is bad. For instance, consider a map $\{x\} \to Y$, $x \mapsto y_0$, then

$$X_y = \begin{cases} \varnothing, & y \neq y_0, \\ x, & y = y_0. \end{cases}$$




But "taking fiber" is well-defined up to homotopy for certain maps — namely, for fibrations. To obtain a homotopy-invariant notion, *replace $f$ by a homotopy equivalent fibration.*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

Here $X_f = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x)\}$ is called the mapping path space, $i$ is the inclusion and $p(x, \gamma) = \gamma(1)$. Then $i$ is a homotopy equivalence and $p$ is a fibration.

The *homotopy fiber $F_f$* of a map

$$f : X \to Y$$

is the ordinary fiber of the fibration $p$ replacing $f$. By the way, the homotopy fiber is the homotopy pullback (limit) of the following diagram

$$\underset{\longleftarrow}{\operatorname{holim}} \left( \begin{tikzcd} & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd} \right) \simeq F_f$$

Back to the example $x \mapsto y_0$. The mapping path space $X_f$ is the path space

$$P(Y, y_0) = \{\gamma \in Y^I \mid \gamma(0) = y_0\}$$

and the homotopy fibre is the loop space

$$\Omega(Y, y_0) = \{\gamma \in Y^I \mid \gamma(0) = \gamma(1) = y_0\}.$$

This is called a path fibration.

**Example 2.4.2** (Homotopy Quotient)**.** Let $X$ be a topological space with a (left) action of a topological group $G$. Consider the functor that assigns to $X$ the orbit space $X/G$. This functor is *not homotopy invariant.*

For example, let $X = \mathbb{R} \simeq *$, and let $\mathbb{Z}$ act on $\mathbb{R}$ by translation. Then

$$\mathbb{R}/\mathbb{Z} \cong S^1, \qquad */\mathbb{Z} \cong *.$$

The quotient $X/G$ is well-defined up to homotopy if the $G$-action on $X$ is *free*. Moreover, any continuous $G$-action on a space can be replaced by a homotopy equivalent free $G$-action.

To make the quotient construction homotopy invariant, replace $X$ by a free $G$-space $EG \times X$ where $EG$ is a contractible free $G$-space (the universal $G$-bundle). Define the *homotopy quotient*as

$$X_{hG} := EG \times_G X.$$

When the action of $G$ on $X$ is free, there is a homotopy equivalence $X_{hG} \simeq X/G$.

If $G$ acts on a single point $*$, we replace $*$ by $EG$ (which has a free $G$-action). Then

$$*_{hG} = EG/G =: BG,$$

where $BG$ is called the *classifying space* of $G$.

- To find $B\mathbb{Z}$, notice $E\mathbb{Z} = \mathbb{R}$ and we have seen $\mathbb{R}/\mathbb{Z} = S^1$. Therefore, $B\mathbb{Z} \simeq S^1$. On the other hand, $B\mathbb{Z} = K(\mathbb{Z}, 1)$ (true for all groups $G$) the Eilenberg–MacLane space, which is $S^1$ as well.
- To find $BS^1$, first notice $S^1 \simeq \mathbb{C}^* \cong GL(1, \mathbb{C})$. Therefore, $BS^1 \simeq BGL(1, \mathbb{C})$, which is the classifying space for complex line bundles $\mathbb{CP}^\infty$.

There is a natural fibration

$$\begin{tikzcd}
EG \times_G X \arrow[r] \arrow[d] & EG \arrow[d] \\
X/G \arrow[r] & BG
\end{tikzcd}$$

where the fiber of $EG \times_G X \to BG$ is homotopy equivalent to $X$.

For any coefficient ring $R$,

$$H^*(G; R) \cong H^*(BG; R).$$

More generally, if $G$ acts on a space $X$, the *$G$-equivariant cohomology* of $X$ is defined as

$$H^*_G(X; R) := H^*(EG \times_G X; R),$$

which is invariant under $G$-equivariant homotopy equivalence.

## 2.5 Left Derived Functors

**Definition 2.5.1** (Left Derived Functors). *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, and suppose $\mathcal{C}$ has enough projectives. Let*

$$F : \mathcal{C} \to \mathcal{D}$$

*be a right-exact functor. For an object $M \in \mathrm{Ob}(\mathcal{C})$, choose a projective resolution*

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

*Then the* left derived functors *of $F$ are defined by*

$$(L_i F)(M) := H_i(F P_\bullet),$$

*that is, we apply $F$ to the projective resolution and take homology.*

**Theorem 2.5.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a right-exact functor between abelian categories, and assume $\mathcal{C}$ has enough projectives. Then:*

*(a) $L_i F$ is a well-defined functor, i.e. it does not depend on the chosen projective resolution.*

*(b) $L_0 F \cong F$.*

*(c) For any short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*in $\mathcal{C}$, there is a natural long exact sequence in $\mathcal{D}$:*

$$\cdots \to (L_i F)(M') \to (L_i F)(M) \to (L_i F)(M'') \to (L_{i-1} F)(M') \to \cdots.$$

*(d) This long exact sequence is* functorial *with respect to short exact sequences in $\mathcal{C}$.*

**Remark 2.5.3.** To define the left derived functors, we must first assign a projective resolution to every object in this abelian category a priori, using the axiom of choice. Later we will see that the derived functor does not depend on the choice of resolution.

**Proposition 2.5.4.** *Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of the same module (or object) $M$. Then there exists a canonical isomorphism*

$$(L_i^P F)(M) \;\cong\; (L_i^Q F)(M)$$

*for all $i \geq 0$.*

Let $\mathcal{C} = R$-**Mod**, the category of left $R$-modules. Consider the functor

$$F = M \otimes_R - \;:\; R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab},$$

which is *right exact*.

**Definition 2.5.5** (Tor). *For $M \in R$-**Mod**, we define the* left derived functors *of $F$ as*

$$\mathrm{Tor}_i^R(M, -) \;:=\; L_i F(-).$$

**Remark 2.5.6.** We could have derived in the first variable instead of the second:

$$\mathrm{Tor}_i^R(-, N) \;=\; L_i(- \otimes_R N),$$

and the result would be the same up to canonical isomorphism.

**Example 2.5.7.** Let us compute $\mathrm{Tor}_i^R(M, R)$.

**(1) Find a projective resolution of $R$:** Since $R$ is a free (hence projective) module over itself,

$$0 \longrightarrow R \longrightarrow 0$$

is a projective resolution of $R$.

**(2) Apply the functor $M \otimes_R -$:**

$$0 \longrightarrow M \otimes_R R \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow M \longrightarrow 0.$$

**(3) Take homology:**

$$H_0 = M, \qquad H_1 = 0, \qquad H_2 = 0, \dots$$

Hence,

$$\mathrm{Tor}_i^R(M, R) = \begin{cases} M & i = 0, \\ 0 & i > 0. \end{cases}$$

**Example 2.5.8.** Assume $R$ is a commutative ring and $f \in R$ is *not* a zero-divisor. We want to compute

$$\mathrm{Tor}_i^R(M, R/(f)).$$

1. Find a projective resolution of $R/(f)$. Since $f$ is not a zero-divisor, we have a short exact sequence

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0,$$

which is a projective (in fact free) resolution of $R/(f)$.

2. Apply the functor $M \otimes_R -$ Tensoring with $M$ gives an exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow 0.$$

We denote this as a chain complex concentrated in degrees 1 and 0.

3. Compute homology

$$\begin{aligned} H_0 &= M/fM \;=\; M \otimes_R R/(f), \\ H_1 &= \ker(f : M \to M) = \{\, m \in M \mid fm = 0 \,\}, \text{ the } f\text{-torsion submodule.} \end{aligned}$$

Thus,

$$\mathrm{Tor}_i^R(M, R/(f)) = \begin{cases} M/fM & \text{if } i = 0, \\ \{\, m \in M \mid fm = 0 \,\} & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

i.e. $\operatorname{Tor}_1^R(M, R/(f))$ detects the $f$-torsion in $M$.

**Example 2.5.9.** Let $R = \mathbb{Z}$ and $M$ be a $\mathbb{Z}$-module. Take $N = \mathbb{Z}/k\mathbb{Z}$.

$$\operatorname{Tor}_0^{\mathbb{Z}}(M, N) = M/kM,$$
$$\operatorname{Tor}_1^{\mathbb{Z}}(M, N) = \{\, m \in M \mid km = 0 \,\},$$

the $k$-torsion submodule of $M$.

**Example 2.5.10.** Let $R$ be a principal ideal domain (PID). Then for any $R$-modules $M, N$,

$$\operatorname{Tor}_i^R(M, N) = 0 \quad \text{for all } i \geq 2.$$

The reason is every finitely generated $R$-module $N$ has a projective (even free) resolution of length at most 1:

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where $F$ is free. Fact: Over a PID, every submodule of a free module is free. Hence $K$ is free, and thus the resolution stops after one step. Therefore all higher Tor-groups ($i \geq 2$) vanish.

**Example 2.5.11.** Consider the short exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2, -)$ yields

$$\cdots \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow 0.$$

Since $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = 0$ for $i > 0$, this simplifies to

$$0 \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow 0,$$

so we again obtain

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2, \quad \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2.$$

**Example 2.5.12.** Let

$$R = k[x, y], \quad M = R/(x, y), \quad N = R/(x).$$

1. Projective Resolution of $N = R/(x)$

   A projective resolution of $N$ is:

   $$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow N \longrightarrow 0.$$

2. Tensor with $M$ Now apply $M \otimes_R -$:

   $$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow 0.$$

   But since $x$ acts by 0 on $M = R/(x, y)$, the map $\cdot x$ is the zero map.

3. Compute Homology

   We obtain

$$H_0 = M, \quad H_1 = M, \quad H_i = 0 \text{ for } i \geq 2.$$

   Therefore,

$$\operatorname{Tor}_i^R(M,N) = \begin{cases} M & i = 0, 1, \\ 0 & i \geq 2. \end{cases}$$

**Example 2.5.13.** Let

$$R = k[x,y], \quad M = R/(x,y), \quad N = R/(x,y).$$

1. Koszul Resolution:

$$0 \longrightarrow R \xrightarrow{(-y,x)^T} R^{\oplus 2} \xrightarrow{(x,y)} R \longrightarrow R/(x,y) \longrightarrow 0.$$

2. Tensoring with $M = R/(x,y)$:

$$0 \longrightarrow M \xrightarrow{(-y,x)^T} M^{\oplus 2} \xrightarrow{(0,y)} M \longrightarrow 0.$$

   Notice the maps are all zero.

3. Compute homology:

$$H_0 = H_1 = M, \quad H_2 = M^{\oplus 2}, \quad H_i = 0 \text{ for } i \geq 3.$$

**Example 2.5.14.** Let

$$R = \frac{k[x,y]}{(xy)}, \qquad M = N = \frac{R}{(x,y)}.$$

1. Projective Resolution of $N$.

   We use the periodic resolution

$$\cdots \longrightarrow R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R \longrightarrow N \longrightarrow 0,$$

2. Tensor with $M$.

$$M \otimes_R - : \quad \cdots \to M^{\oplus 2} \xrightarrow{0} M^{\oplus 2} \xrightarrow{0} M^{\oplus 2} \xrightarrow{0} M \to 0,$$

   since $x$ and $y$ both act as 0 on $M = R/(x,y)$.

3. Compute Homology. We then obtain

$$H_0 = M, \qquad H_i = M^{\oplus 2} \text{ for all } i > 0.$$

   Hence,

$$\operatorname{Tor}_i^R(M,N) = \begin{cases} M, & i = 0, \\ M^{\oplus 2}, & i > 0. \end{cases}$$

The module $N = R/(x,y)$ does not admit a finite projective resolution. This shows that $R = k[x,y]/(xy)$ is *not regular* — its residue field has infinite projective dimension.

Let $R$ be a ring, and let $M$ be a left $R$-module. Recall the functor

$$\mathrm{Hom}_R(-,M) : R\text{-Mod} \longrightarrow \mathbf{Ab}.$$

is **contravariant** and **left exact**. We can also use projective resolution to derive it.

**Definition 2.5.15** (Ext)**.** *For $N \in R$-Mod, the **Ext functors** are defined as*

$$\mathrm{Ext}^i_R(N,M) := (L_{-i}\,\mathrm{Hom}_R(-,M))(N) = H^i(\mathrm{Hom}_R(P_\bullet, M)),$$

*where $P_\bullet \to N \to 0$ is a projective resolution of $N$.*

**Example 2.5.16.**

$$\mathrm{Ext}_i^R(N,M) = \begin{cases} M, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

**Example 2.5.17.** Assume $R$ is a commutative ring, $M$ a $R$-module and $f \in R$ is not a zero divisor. We wish to compute

$$\mathrm{Ext}^i_R(R/(f), M).$$

1. Take a projective resolution of $R/(f)$:

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0.$$

2. Apply the functor $\mathrm{Hom}_R(-,M)$ to obtain

$$0 \longrightarrow \mathrm{Hom}_R(R,M) \xrightarrow{\cdot f} \mathrm{Hom}_R(R,M) \longrightarrow 0.$$

   Since $\mathrm{Hom}_R(R,M) \cong M$, this becomes

$$0 \longrightarrow M \xrightarrow{\cdot f} M \longrightarrow 0.$$

3. Take cohomology:

$$H^0 = \ker(f : M \to M) = \{\, m \in M \mid fm = 0 \,\},$$
$$H^1 = \mathrm{coker}(f : M \to M) = M/fM.$$

   Hence

$$\mathrm{Ext}^i_R(R/(f), M) = \begin{cases} \{m \in M \mid fm = 0\}, & i = 0, \\ M/fM, & i = 1, \\ 0, & i > 1. \end{cases}$$

   In particular:

$$\mathrm{Ext}^0_R(R/(f), M) = f\text{-torsion of } M, \quad \mathrm{Ext}^1_R(R/(f), M) = M/fM.$$

**Example 2.5.18.** Let $R = \mathbb{Z}$.

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/2, & i = 1. \end{cases}$$

## 2.6 Opposite Categories

Question: What to do with a left exact covariant functor?

Suppose we have a (covariant) functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}.$$

Then we can form the opposite functor

$$F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}},$$

which is still *covariant*. Recall that for morphisms in the opposite category,

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) \;=\; \mathrm{Hom}_{\mathcal{C}}(B, A).$$

Composition in $\mathcal{C}^{\mathrm{op}}$ is reversed:

$$f \circ_{\mathcal{C}^{\mathrm{op}}} g \;=\; g \circ_{\mathcal{C}} f.$$

That is, in $\mathcal{C}$ we have $f : A \to B$, while in $\mathcal{C}^{\mathrm{op}}$ the direction is reversed, but $F^{\mathrm{op}}$ still acts covariantly:

$$F^{\mathrm{op}}(f^{\mathrm{op}}) = (Ff)^{\mathrm{op}}.$$

Hence the operation $(-)^{\mathrm{op}}$ defines a functor

$$(-)^{\mathrm{op}} : \mathbf{AbCat} \longrightarrow \mathbf{AbCat},$$

sending an abelian category to its opposite abelian category.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a *left exact* functor. We claim that $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is *right exact.*

Given a short exact sequence in $\mathcal{C}^{\mathrm{op}}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

this corresponds to a short exact sequence in $\mathcal{C}$:

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Applying the left exact functor $F$ gives:

$$0 \longrightarrow FC \longrightarrow FB \longrightarrow FA$$

exact in $\mathcal{D}$.

Reversing arrows back to the opposite category, we get:

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

exact in $\mathcal{D}^{\mathrm{op}}$. Hence, $F^{\mathrm{op}}$ is right exact.

**Definition 2.6.1** (Right Derived Functors)**.** *Let $\mathcal{C}, \mathcal{D}$ be abelian categories. Let*

$$F : \mathcal{C} \to \mathcal{D}$$

*be a left-exact functor. We define the* right derived functors *of $F$ by*

$$R^i F := L_i(F^{\mathrm{op}}),$$

*provided that $\mathcal{C}^{\mathrm{op}}$ has enough projectives, (which is equivalent to saying that $\mathcal{C}$ has enough injectives).*

## 2.7 Adjunction

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let

$$F : \mathcal{C} \longrightarrow \mathcal{D}, \qquad G : \mathcal{D} \longrightarrow \mathcal{C}$$

be functors.

**Definition 2.7.1** (Adjunction)**.** *We say that $F$ is* ***left adjoint*** *to $G$ (and that $G$ is* ***right adjoint*** *to $F$), written*

$$F \dashv G,$$

*if there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(FX,\, Y) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(X,\, GY)$$

*for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.*

Naturality means the following diagrams commute.

Natural in $Y$: For any morphism $f : Y_1 \to Y_2$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FX, Y_1) & \xrightarrow{\Phi_{X,Y_1}} & \mathrm{Hom}_{\mathcal{C}}(X, GY_1) \\
{\scriptstyle f\circ-}\big\downarrow & & \big\downarrow{\scriptstyle Gf\circ-} \\
\mathrm{Hom}_{\mathcal{D}}(FX, Y_2) & \xrightarrow{\Phi_{X,Y_2}} & \mathrm{Hom}_{\mathcal{C}}(X, GY_2)
\end{array}$$

commutes for all $X \in \mathcal{C}$.

Natural in $X$: For any morphism $g : X_1 \to X_2$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FX_2, Y) & \xrightarrow{\Phi_{X_2,Y}} & \mathrm{Hom}_{\mathcal{C}}(X_2, GY) \\
{\scriptstyle -\circ Fg}\big\downarrow & & \big\downarrow{\scriptstyle -\circ g} \\
\mathrm{Hom}_{\mathcal{D}}(FX_1, Y) & \xrightarrow{\Phi_{X_1,Y}} & \mathrm{Hom}_{\mathcal{C}}(X_1, GY)
\end{array}$$

commutes for all $Y \in \mathcal{D}$.

**Example 2.7.2** (Free-Forget Adjunction)**.** Consider the categories

$$\mathcal{C} = \mathbb{Z}\text{-}\mathbf{Mod} \qquad \text{and} \qquad \mathcal{D} = \mathbf{Sets}.$$

There is a **forgetful functor**

$$\mathrm{Forget} : \mathbb{Z}\text{-}\mathbf{Mod} \longrightarrow \mathbf{Sets},$$

which sends an abelian group (i.e. a $\mathbb{Z}$-module) to its underlying set.

There is also a **free functor**

$$\mathrm{Free} : \mathbf{Sets} \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod},$$

which sends a set $S$ to the free abelian group

$$\mathrm{Free}(S) = \bigoplus_{s \in S} \mathbb{Z}\, e_s.$$

These two functors form an **adjoint pair**:

$$\text{Free} \dashv \text{Forget}.$$

That is, for all $S \in \mathbf{Sets}$ and $G \in \mathbb{Z}\text{-}\mathbf{Mod}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\text{-}\mathbf{Mod}}\big(\text{Free}(S), G\big) \;\cong\; \mathrm{Hom}_{\mathbf{Sets}}\big(S, \text{Forget}(G)\big),$$

natural in both $S$ and $G$.

Concretely, a group homomorphism $\text{Free}(S) \to G$ is completely determined by its values on the basis elements $e_s$, which correspond exactly to a function $S \to \text{Forget}(G)$.

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, bend left, "\text{Free}"] & \mathbb{Z}\text{-}\mathbf{Mod} \arrow[l, bend left, "\text{Forget}"] \end{tikzcd}$$

**Example 2.7.3** (Tensor Algebra)**.** Let

$$\mathcal{C} = k\text{-Alg}, \qquad \mathcal{D} = k\text{-Vect}.$$

We have the **forgetful functor**

$$\text{Forget} : k\text{-Alg} \longrightarrow k\text{-Vect},$$

which sends a (not necessarily commutative) $k$-algebra $A$ to its underlying vector space. This functor has a **left adjoint**, the *free algebra functor*

$$T : k\text{-Vect} \longrightarrow k\text{-Alg}.$$

For any $k$-vector space $V$, the free $k$-algebra on $V$ is given by the **tensor algebra**

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

with multiplication given by concatenation of tensors:

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

If $V = \langle x_1, \ldots, x_n \rangle$, then

$$T(V) \cong k\langle x_1, \ldots, x_n \rangle,$$

the noncommutative polynomial algebra in $n$ variables.

**Example 2.7.4** (Symmetric Algebra)**.** Let

$$\mathcal{C} = k\text{-ComAlg}, \qquad \mathcal{D} = k\text{-Vect}.$$

To make this Free-Forget adjunction work, the free object $\text{Free}(V)$ must be a commutative $k$-algebra. We take

$$\text{Free}(V) = S(V),$$

the **symmetric algebra** on $V$, defined as

$$S(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V),$$

which is the quotient of the tensor algebra $T(V)$ by the ideal generated by$v \otimes w - w \otimes v$.

If $V = \langle x_1, \ldots, x_n \rangle$, then
$$S(V) = k[x_1, \ldots, x_n].$$
Thus the polynomial algebra in $n$ variables is the free commutative algebra on an $n$-dimensional vector space.

**Example 2.7.5** (Extension-Restriction of scalars Adjunction)**.** Let $R \to S$ be a ring homomorphism. On the module level, we have two categories:

$$S\text{-Mod} \quad \text{and} \quad R\text{-Mod}.$$

- The **restriction of scalars** (or *forgetful functor*)

$$\text{Forget} : S\text{-Mod} \longrightarrow R\text{-Mod}$$

  regards an $S$-module as an $R$-module via the map $R \to S$.

- The **extension of scalars** functor

$$S \otimes_R - : R\text{-Mod} \longrightarrow S\text{-Mod}$$

  sends $M \mapsto S \otimes_R M$.

These form an *adjoint pair*:
$$S \otimes_R - \ \dashv \text{ Forget}.$$
That is, there is a natural isomorphism

$$\text{Hom}_{S\text{-Mod}}(S \otimes_R M, N) \ \cong \ \text{Hom}_{R\text{-Mod}}(M, \text{Forget}(N)).$$

On the scheme level, let $f : X = \text{Spec}(S) \to Y = \text{Spec}(R)$ be the induced morphism of affine schemes. Then on quasi-coherent sheaves, we have:

$$f^* : \text{QCoh}(Y) \longrightarrow \text{QCoh}(X), \qquad f_* : \text{QCoh}(X) \longrightarrow \text{QCoh}(Y),$$

and they satisfy
$$f^* \dashv f_*.$$

**Example 2.7.6.** Let $R, S$ be rings, $M$ an $(R, S)$-bimodule, and $N$ a right $S$-module. Then $\text{Hom}_S(M, N)$ is a right $R$-module, and the functor

$$\text{Hom}_S(M, -) : \text{mod-}S \longrightarrow \text{mod-}R$$

is **right adjoint** to
$$- \otimes_R M : \text{mod-}R \longrightarrow \text{mod-}S.$$
That is,
$$\text{Hom}_S(P, \text{Hom}_S(M, N)) \cong \text{Hom}_R(P \otimes_R M, N),$$
naturally in $P$ and $N$.

The universal property of tensor product is as follow. Given right $R$-module $P$ and left $R$-module $M$, we have:
$$\text{Maps}_{\mathbf{Ab}}(P \otimes_R M, N) = \{R\text{-balanced maps } f : P \times M \to N\},$$
that is, maps $f$ satisfying:

1. $f$ is a group homomorphism in each variable,
2. $f(pr, m) = f(p, rm)$ for all $p \in P, m \in M, r \in R$.

Each $R$-balanced map $f$ gives rise to an element of

$$\mathrm{Hom}_R(P, \mathrm{Hom}_S(M, N)).$$

i.e.

$$f \mapsto (p \mapsto (m \mapsto f(p, m))).$$

**Lemma 2.7.7.** *Let $\mathcal{C}, \mathcal{D}$ be abelian categories,*

$$F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$$

*such that $G \dashv F$. Then $G$ is **right exact** and $F$ is **left exact**.*

## 2.8 Injective Resolution

**Definition 2.8.1.** *Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is called* injective *if for every diagram with exact row*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

*there exists a morphism $B \to I$ making the diagram commute.*

Equivalently, $I$ is injective if and only if the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I)$$

is *exact*.

**Example 2.8.2.** $\mathbb{Z}$ is *not* an injective $\mathbb{Z}$-module.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}.$$

The following diagram cannot commute.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "1"'] & \mathbb{Z} \arrow[dl, dashed] \\ & \mathbb{Z} & \end{tikzcd}$$

Hence $\mathbb{Z}$ is not injective.

However, if we replace $\mathbb{Z}$ by $\mathbb{Q}$, we have:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "1"'] & \mathbb{Z} \arrow[dl, dashed] \\ & \mathbb{Q} & \end{tikzcd}$$

where the extension exists, given by $1 \mapsto \frac{1}{2}$. Thus $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

**Theorem 2.8.3** (Baer's Criterion)**.** *Let $R$ be a ring and $I$ an $R$-module. Then $I$ is injective if and only if for every (left) ideal $J \subseteq R$, any map $J \to I$ extends to a map $R \to I$.*

*That is, it suffices to check injectivity on diagrams of the form:*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

**Corollary 2.8.4.** *An abelian group $G$ (i.e. a $\mathbb{Z}$-module) is injective if and only if it is* divisible*, that is,*

$$\forall x \in G,\ \forall n \in \mathbb{Z}_{>0},\ \exists y \in G \text{ such that } ny = x.$$

*i.e. check injectivity on diagrams of the form:*

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[dl, dashed] \\ & G & \end{tikzcd}$$

**Example 2.8.5.** Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[d, "1"'] & \mathbb{Z} \arrow[dl, dashed, "1/n"] \\ & \mathbb{Q} & \end{tikzcd}$$

which shows that the map extends in $\mathbb{Q}$. Hence $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Similarly, $\mathbb{Q}/\mathbb{Z}$ is injective.

**Corollary 2.8.6.** *The category $\mathbb{Z}$-**Mod** has enough injectives.*

**Lemma 2.8.7.** *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, and let*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

*be an exact functor.*

*Assume that $F \dashv G$ for some functor*

$$G : \mathcal{D} \longrightarrow \mathcal{C}.$$

*Then for every injective object $I$ in $\mathcal{D}$, the object $G(I)$ is injective in $\mathcal{C}$.*

**Theorem 2.8.8. Mod**-*$R$ has enough injectives.*

**Corollary 2.8.9.** *For any left exact functor*

$$F : \text{Mod-}R \to \mathcal{C},$$

*we can define the right derived functors*

$$R^i F : \text{Mod-}R \to \mathcal{C}$$

*by*

$$(R^i F)(M) = H^i(F I^\bullet),$$

*where $I^\bullet$ is an injective resolution of $M$.*

**Example 2.8.10.** We compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ from the definition, as a functor in the second variable.

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i = 1. \end{cases}$$

1. Find an injective resolution of $\mathbb{Z}$.
   Since divisible abelian groups are injective, we use the short exact sequence
   $$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$
   where $\mathbb{Q}/\mathbb{Z}$ is injective. Hence this is an injective resolution of $\mathbb{Z}$:
   $$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

2. Apply the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$
   Applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to the above injective resolution, we obtain
   $$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$
   Since $\mathbb{Q}$ is torsion-free,
   $$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0.$$
   On the other hand,
   $$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \left\{ \tfrac{k}{n} \in \mathbb{Q}/\mathbb{Z} \,\middle|\, k \in \mathbb{Z} \right\} \simeq \mathbb{Z}/n\mathbb{Z}.$$

3. Compute cohomology
   The complex we obtained is
   $$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$
   so we get
   $$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0, \qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i = 1, \\ 0, & i > 1. \end{cases}$$

**Example 2.8.11.** Here is another way to compute

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}).$$

Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ yields a long exact sequence:

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to 0.$$

Since $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, we again obtain:

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}.$$

## 2.9 Distinguished Triangles

**Definition 2.9.1** (Mapping cone)**.** *Let $A^\bullet$ and $B^\bullet$ be chain complexes, and let*

$$f : A^\bullet \longrightarrow B^\bullet$$

*be a chain map. The* mapping cone *of $f$, denoted by* $\mathrm{Cone}(f)$*, is the chain complex defined by*

$$\mathrm{Cone}(f) := A[1] \oplus B,$$

*i.e.*

$$\mathrm{Cone}(f)^n = A^{n+1} \oplus B^n,$$

*with differential*

$$d_{\mathrm{Cone}(f)}(a, b) = (-d_A(a),\ f(a) + d_B(b)).$$

*In matrix form,*

$$d_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

Explicitly, the complex is:

$$\cdots \longrightarrow A^i \oplus B^{i-1} \xrightarrow{d_{\mathrm{Cone}(f)}} A^{i+1} \oplus B^i \xrightarrow{d_{\mathrm{Cone}(f)}} A^{i+2} \oplus B^{i+1} \longrightarrow \cdots$$

Recall that for the mapping cone

$$d_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

The composition

$$d^2_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} = \begin{pmatrix} d_A^2 & 0 \\ -f d_A + d_B f & d_B^2 \end{pmatrix},$$

is zero since $f$ is a chain map.

**Example 2.9.2** (Motivation from Topology)**.** Let $X, Y$ be topological spaces, and let

$$f : X \to Y$$

be a continuous map. The **mapping cone** of $f$ is the quotient space

$$\mathrm{Cone}(f) := (X \times [0,1] \sqcup Y)/\sim$$

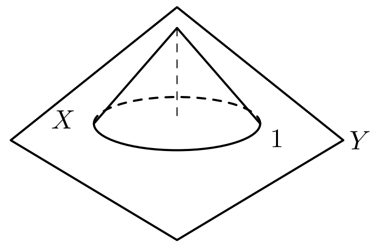
where the equivalence relation $\sim$ is given by

$$(x, 0) \sim (x', 0) \quad \text{for all } x, x' \in X, \qquad (x, 1) \sim f(x).$$

In words:

- all points $(x, 0)$ are identified together (forming the cone vertex),
- and each point $(x, 1)$ is glued to $f(x) \in Y$.

If $f$ is an inclusion $f : X \hookrightarrow Y$, then geometrically: $\mathrm{Cone}(f)$ is obtained by attaching a cone over $X$ to $Y$ along $X$.




There are canonical maps

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow \Sigma X,$$

where $\Sigma X = X \times [0,1]/(x,0) \sim (x',0), (x,1) \sim (x',1)$ is the suspension of $X$. The map $Y \to \mathrm{Cone}(f)$ is the inclusion and the map $\mathrm{Cone}(f) \to \Sigma X$ collapses $Y$ to a point.

**Definition 2.9.3.** *There are canonical chain maps*

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow A^\bullet[1].$$

*We call this a distinguished triangle, drawn as*

$$\begin{tikzcd} A^\bullet \arrow[rr, "f"] & & B^\bullet \arrow[dl] \\ & \mathrm{Cone}(f) \arrow[ul] & \end{tikzcd}$$

Notice the map $\mathrm{Cone}(f) \to A^\bullet$ is shfting degrees by 1, which is not an honest chain map.

| **Algebraic Topology** | **Homological Algebra** | **Homotopical Algebra** |
|---|---|---|
| Topological space | Chain complex | Object (0-morphism) |
| Continuous map | Chain map | Morphism (1-morphism) |
| Homotopy | Chain homotopy | Homotopy (2-morphism) |
| Homotopy equivalence | Chain homotopy equivalence | Homotopy equivalence |
| Homotopy group $\pi_n(X)$ | Homology group $H_n(X)$ | Homotopy group |
| Weak homotopy equivalence | Quasi-isomorphism | Weak equivalence |
| CW approximation | Projective/Injective resolution | Fibrant/Cofibrant replacement |
| Homotopy category hCW | Derived category $\mathcal{D}(R)$ | Homotopy category |
| Suspension / Loop $\Sigma \dashv \Omega$ | Shift $[1] \dashv [-1]$ | Suspension object / Loop object |
| Mapping cone, mapping cocone | Chain mapping cone, cocone | Homotopy fiber, homotopy cofiber |

Table 1: Analogy between Algebraic Topology, Homological Algebra, and Homotopical Algebra

**Proposition 2.9.4.** *For any chain map $f : A^\bullet \to B^\bullet$, we obtain a long exact sequence in cohomology:*

$$\cdots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(\mathrm{Cone}(f)) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow H^{i+1}(B^\bullet) \longrightarrow \cdots$$

*Proof.*
$$0 \to B^\bullet \to \mathrm{Cone}(f) \to A^\bullet[1] \to 0$$
is a short exact sequence. □

**Corollary 2.9.5.** *A chain map $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if*
$$\mathrm{Cone}(f) \simeq 0.$$

Fact: The homotopy category $\mathcal{K}(\mathcal{A})$ is *not* an abelian category.

However, *kernels* and *cokernels* are replaced by *cones*, which make sense in $K(\mathcal{A})$. Intuitively, we roughly have
$$\mathrm{Cone}(f) \sim (\ker f)[1] \oplus \mathrm{coker}(f).$$

Let $f : A \to B$ with $A, B \in \mathcal{A}$.

We may view $f$ as a morphism of complexes:
$$f : A[0] \longrightarrow B[0].$$

Then the cone is:
$$\mathrm{Cone}(f) : 0 \to A \xrightarrow{f} B \to 0 \to \cdots$$

Cohomology of the cone is
$$H^{-1}(\mathrm{Cone}(f)) = \ker f, \qquad H^0(\mathrm{Cone}(f)) = \mathrm{coker} f.$$

Hence we get an exact sequence:
$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \mathrm{coker} f \longrightarrow 0.$$

Actually the cone $\mathrm{Cone}(f)$ also encodes the $\mathrm{Ext}^2(\mathrm{coker} f, \ker f)$.

**Example 2.9.6.** If
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
is short exact, then
$$\mathrm{Cone}(f) \simeq C[0].$$

Thus, we obtain a distinguished triangle in $\mathcal{K}(\mathcal{A})$:

$$\begin{tikzcd} A^\bullet \arrow[rr, "f"] & & B^\bullet \arrow[dl] \\ & \mathrm{Cone}(f) \arrow[ul] \arrow[r, "\simeq"] & C^\bullet \end{tikzcd}$$

What we want is a distinguished triangle

$$\begin{tikzcd} A^\bullet \arrow[rr, "f"] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul] & \end{tikzcd}$$

which can be found in derived category $\mathcal{D}(\mathcal{A})$.

## 2.10 Balanced Bifunctors

In this section we want to show the Tor and Ext functors are balanced.

**Theorem 2.10.1** ($\mathrm{Tor}^{\mathrm{I}} \cong \mathrm{Tor}^{\mathrm{II}}$)**.** *Let $M, N$ be $R$-modules. Take projective resolutions*

$$P_\bullet \longrightarrow M, \qquad Q_\bullet \longrightarrow N.$$

*Then for every $i$ there is a natural isomorphism*

$$H_i(P_\bullet \otimes_R N) \;\cong\; H_i(M \otimes_R Q_\bullet)\,.$$

*Hence the two constructions of* Tor *agree:*

$$\mathrm{Tor}_i^R(M,N) := H_i(P_\bullet \otimes_R N) \;\cong\; H_i(M \otimes_R Q_\bullet) =: \mathrm{Tor}_i^R(M,N).$$

**Definition 2.10.2** (Bicomplex)**.** *A* bicomplex $(C_{\bullet,\bullet}, d^h, d^v)$ *consists of modules $C_{i,j}$ with horizontal and vertical differentials*

$$d^h : C_{i,j} \to C_{i-1,j}, \qquad d^v : C_{i,j} \to C_{i,j-1}$$

*such that*

$$(d^h)^2 = 0, \qquad (d^v)^2 = 0, \qquad d^h d^v = d^v d^h.$$

There are two standard ways to turn a bicomplex into a single complex.

**Definition 2.10.3** (Direct–sum and product totalizations)**.** *Given a bicomplex $C_{\bullet,\bullet}$, define complexes* $\mathrm{Tot}^{\oplus}(C)$ *and* $\mathrm{Tot}^{\Pi}(C)$ *by degree $n$:*

$$\mathrm{Tot}^{\oplus}(C)_n \;=\; \bigoplus_{i+j=n} C_{i,j}, \qquad \mathrm{Tot}^{\Pi}(C)_n \;=\; \prod_{i+j=n} C_{i,j},$$

*with differential $d = d^h + d^v$ (using the chosen sign convention). The direct–sum totalization takes* finite *sums along each diagonal; the product totalization allows* infinite *products.*

*Sketch proof of the theorem.* Consider the bicomplex $C_{\bullet,\bullet} = P_\bullet \otimes_R Q_\bullet$ with

$$C_{i,j} = P_i \otimes_R Q_j, \qquad d^h = d^P \otimes \mathrm{id}, \qquad d^v = (-1)^i \,\mathrm{id} \otimes d^Q \text{ on } C_{i,j}.$$

(Any consistent sign convention is fine; we use the commuting convention $d^h d^v = d^v d^h$. Another possible sign convention is to require squares to be anti-commutative.)

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i+1}\otimes Q_j \arrow[r, "d^h"] \arrow[d, "d^v"'] & P_i\otimes Q_j \arrow[r, "d^h"] \arrow[d, "d^v"'] & P_{i-1}\otimes Q_j \arrow[r] \arrow[d, "d^v"'] & \cdots \\
\cdots \arrow[r] & P_{i+1}\otimes Q_{j-1} \arrow[r, "d^h"] & P_i\otimes Q_{j-1} \arrow[r, "d^h"] & P_{i-1}\otimes Q_{j-1} \arrow[r] & \cdots
\end{tikzcd}$$

Notice the bicomplex lives in the second quadrant. Consider the augmented bicomplex on the right:

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & P_2\otimes Q_1 \arrow[r, "d^h"] \arrow[d, "d^v"'] & P_1\otimes Q_1 \arrow[r, "d^h"] \arrow[d, "d^v"'] & P_0\otimes Q_1 \arrow[r] \arrow[d, "d^v"'] & M \otimes Q_1 \arrow[r] \arrow[d, "d^v"'] & 0 \\
\cdots \arrow[r] & P_2\otimes Q_0 \arrow[r, "d^h"] & P_1\otimes Q_0 \arrow[r, "d^h"] & P_0\otimes Q_0 \arrow[r] & M \otimes Q_0 \arrow[r] & 0
\end{tikzcd}$$

denote the new bicomplex by $R_\bullet \otimes_R Q_\bullet$.

Check the following statements:

1. There is a natural map
$$\epsilon_n : \mathrm{Tot}^{\oplus}(P_\bullet \otimes_R Q_\bullet) \to M \otimes Q_\bullet.$$

2. $$\mathrm{Tot}^{\oplus}(R_\bullet \otimes_R Q_\bullet) = \mathrm{Cone}(\epsilon_M)[-1]$$

3. The rows of $R_\bullet \otimes_R Q_\bullet$ are acyclic because $R_\bullet$ is acyclic and $Q_i$ are projective (hence flat).

4. $\mathrm{Tot}^{\oplus}(R_\bullet \otimes_R Q_\bullet)$ is acyclic, by the acyclic assembly lemma or convergence theorem for spectral sequences.

Then $\epsilon_M$ is a quasi-isomorphism, i.e.
$$(P_\bullet \otimes_R Q_\bullet) \simeq M \otimes Q_\bullet.$$

Analogously,
$$(P_\bullet \otimes_R Q_\bullet) \simeq P_\bullet \otimes N.$$

□

**Remark 2.10.4.** One-sentence proof: The spectral sequences of the bicomplex $P_\bullet \otimes_R Q_\bullet$ under vertical and horizontal orientations converge to $H_\bullet(M \otimes_R Q_\bullet)$ and $H_\bullet(P_\bullet \otimes_R N)$ respectively.

**Theorem 2.10.5** ($\mathrm{Ext}^{\mathrm{I}} \cong \mathrm{Ext}^{\mathrm{II}}$)**.** Ext *is balanced. i.e.* $M, N \in R\text{-Mod}$. *Let* $P_\bullet \twoheadrightarrow M$ *be a projective resolution and* $N \hookrightarrow I^\bullet$ *an injective resolution.*
$$\mathrm{Ext}^i_R(M,N) \;\cong\; H^i\big(\mathrm{Hom}_R(P_\bullet, N)\big) \;\cong\; H^i\big(\mathrm{Hom}_R(M, I^\bullet)\big).$$

## 2.11 Acyclic Resolutions

[2]

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ a functor. Typically:

- If $F$ is left exact, one defines its right derived functors $R^iF$ using injective resolutions.
- If $F$ is right exact, one defines its left derived functors $L_iF$ using projective resolutions.

In many settings (e.g. sheaves), $\mathcal{A}$ does *not* have enough projectives, so $L_*F$ cannot be defined by projective resolutions. The remedy is to resolve by objects that are *acyclic for* $F$.

First we need a new definition independent of resolution.

**Definition 2.11.1** (Homological $\delta$-functor)**.** *A* homological $\delta$-functor *from* $\mathcal{A}$ *to* $\mathcal{B}$ *is a family* $(T_i)_{i\geq 0}$ *of functors* $T_i : \mathcal{A} \to \mathcal{B}$ *together with, for every short exact sequence* $0 \to X' \to X \to X'' \to 0$ *in* $\mathcal{A}$*,* connecting morphisms $\partial_i : T_i(X'') \to T_{i-1}(X')$ *producing a natural long exact sequence*
$$\cdots \to T_i(X') \to T_i(X) \to T_i(X'') \xrightarrow{\partial_i} T_{i-1}(X') \to \cdots \to T_0(X'') \to 0,$$
*natural in the short exact sequence and compatible with morphisms of short exact sequences.*

**Definition 2.11.2** (Left derived functors via universality)**.** *A homological $\delta$-functor* $(L_iF)_{i\geq 0}$ *is called the* left derived functor *of a right exact* $F$ *if:*

*(i)* $L_0F \simeq F$ *(naturally in* $X$*);*

[2] I have modified this section significantly, referring to Stack project.

(ii) *(Universality) For any homological $\delta$-functor $(T_i)$ and any natural transformation $\eta_0 : T_0 \Rightarrow F$, there exists a unique family $\eta_i : T_i \Rightarrow L_iF$ of natural transformations extending $\eta_0$ and compatible with all connecting morphisms. Equivalently, $(L_iF)$ is* initial *in the category of homological $\delta$-functors under $F$.*

Now assume $F : \mathcal{A} \to \mathcal{B}$ is right exact and we want $L_*F$, but $\mathcal{A}$ lacks enough projectives. We replace projectives by objects that are *left-acyclic for $F$*.

**Definition 2.11.3** ($F$-adapted / $F$-left-acyclic objects). *An object $A \in \mathcal{A}$ is* $F$-left-acyclic *if*

$$L_iF(A) = 0 \qquad \text{for all } i > 0.$$

*Equivalently, for any (bounded above) resolution $\cdots \to A_1 \to A_0 \twoheadrightarrow A$ by such objects, the homology of $F$ applied to the resolution vanishes in positive degrees.*

**Definition 2.11.4** (Left-acyclic resolution). *A (bounded above) morphism $A_\bullet \to X$ is an* $F$-left-acyclic resolution *if it is a quasi-isomorphism and each $A_n$ is $F$-acyclic.*

**Theorem 2.11.5** (Defining $L_*F$ via left-acyclic resolutions). *If every $X \in \mathcal{A}$ admits a bounded-above $F$-left-acyclic resolution $A_\bullet \twoheadrightarrow X$, then*

$$L_iF(X) \;:=\; H_i\big(F(A_\bullet)\big)$$

*is well defined (independent of choices), gives a universal $\delta$-functor, and agrees with the projective-resolution definition when $\mathcal{A}$ has enough projectives.*

**Example 2.11.6** (Tensor and flat resolutions). Let $F = - \otimes_R N$ (right exact). Even if $R$-Mod has few projectives, flat modules are $\otimes$-left-acyclic, and every module $M$ admits a flat resolution $F_\bullet \twoheadrightarrow M$. Therefore

$$L_i(- \otimes_R N)(M) \;\cong\; H_i(F_\bullet \otimes_R N) \;=\; \mathrm{Tor}_i^R(M, N).$$

Those constructions above work analogously for left exact functors.

**Definition 2.11.7** (Right derived functors via universality). *A homological $\delta$-functor $(R^iF)_{i\geq 0}$ is called the* right derived functor *of a left exact $F$ if:*

(i) *$R^0F \cong F$ (naturally in $X$);*

(ii) *(Universality) For any cohomological $\delta$-functor $(T^i)$ and any natural transformation $\eta^0 : F \Rightarrow T^0$, there exists a unique family $\eta^i : R^iF \Rightarrow T^i$ compatible with the long exact sequences and extending $\eta^0$.*

**Definition 2.11.8** ($F$-acyclic objects for left exact $F$). *Assume $F$ is left exact. An object $A \in \mathcal{A}$ is called* $F$-acyclic *if*

$$R^iF(A) = 0 \qquad \text{for all } i > 0.$$

*Equivalently, for any injective resolution $A \to I^\bullet$, the cochain complex $F(I^\bullet)$ has vanishing cohomology in positive degrees.*

**Definition 2.11.9** (Acyclic resolution). *A (bounded below) morphism $X \to A^\bullet$ is a* $F$-(right) acyclic resolution *of $X$ if it is a quasi-isomorphism and each $A^n$ is $F$-acyclic.*

**Theorem 2.11.10.** *Let $F : \mathcal{A} \to \mathcal{B}$ be left exact. If every $X \in \mathcal{A}$ admits a bounded-below $F$-acyclic resolution $X \to A^\bullet$, then*

$$R^i F(X) \ := \ H^i\big(F(A^\bullet)\big)$$

*is well defined (independent of the chosen resolution), yields a universal $\delta$-functor, and agrees with the usual definition via injective resolutions whenever $\mathcal{A}$ has enough injectives.*

**Example 2.11.11** (Sheaf cohomology)**.** For the global sections functor $\Gamma(X, -)$ (left exact), soft/flabby/fine sheaves are$\Gamma$-acyclic. Hence $R^i\Gamma$ can be computed from soft (or flabby/fine) resolutions, without using injective sheaves.

# 3 Derived Categories

## 3.1 Triangulated Categories

Start with an abelian category $\mathcal{A}$. We want a category whose objects are chain complexes in $\mathcal{A}$:

$$\mathrm{Ch}(\mathcal{A}).$$

Many constructions on complexes are not strictly unique, but are unique up to homotopy. This leads to the *homotopy category*

$$\mathcal{K}(\mathcal{A}) \ = \ \mathrm{Ch}(\mathcal{A})/(\text{chain homotopy}).$$

**Problem:** While $\mathcal{A}$ and $\mathrm{Ch}(\mathcal{A})$ are abelian (so we can speak of short exact sequences, kernels, cokernels), the homotopy category $\mathcal{K}(\mathcal{A})$ is *not* abelian.

**Theorem 3.1.1.** *The homotopy category $\mathcal{K}(\mathcal{A})$ is triangulated.*

**Definition 3.1.2** (Triangulated category)**.** *A triangulated category consists of:*

- *an additive category $\mathcal{C}$;*
- *an auto-equivalence (the* shift/suspension*)*

$$T : \mathcal{C} \xrightarrow{\cong} \mathcal{C}, \qquad X \mapsto TX;$$

- *a specified class of* distinguished triangles

$$(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA),$$

*subject to the Verdier axioms* TR1–TR4*:*

**TR1**

*(a) For every object $A$, the triangle $A \xrightarrow{\mathrm{id}} A \to 0 \to TA$ is distinguished.*

*(b) Every morphism $f : A \to B$ sits in some distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$, where $C$ is called a* ***cone*** *or* ***cofiber*** *of the morphism $f$.*

*(c) Any triangle isomorphic to a distinguished triangle is distinguished.*

**TR2** *(Rotation) If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ *is distinguished, then so are*

$$B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB \quad \text{and} \quad C \xrightarrow{h} TA \xrightarrow{-Tf} TB \xrightarrow{-Tg} TC.$$

**TR3** *(Functoriality up to homotopy) Given a morphism of the first two legs of distinguished triangles*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & TA \arrow[d, "Ta"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & TA' \end{tikzcd}$$

*there exists* $c : C \to C'$ *making the diagram a morphism of triangles.*

**TR4** *(Octahedral axiom) For composable maps* $A \xrightarrow{f} B \xrightarrow{g} C$*, the cones of* $f$*,* $g$*, and* $gf$ *fit into an "octahedron" relating the associated distinguished triangles.*

**Lemma 3.1.3.** *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ *is a distinguished triangle then* $g \circ f = 0$*.*

**Lemma 3.1.4** (Two-out-of-three for morphisms of triangles)**.** *Suppose we have a morphism of distinguished triangles*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, "c"'] & TA \arrow[d, "Ta"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & TA' \end{tikzcd}$$

*Then if any two of* $a, b, c$ *are isomorphisms, so is the third.*

**Lemma 3.1.5** (Exactness of $\mathrm{Hom}(D, -)$ on a distinguished triangle)**.** *Suppose*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

*a distinguished triangle. For any object* $D \in \mathrm{Ob}(\mathcal{C})$*, the sequence*

$$\mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, B) \xrightarrow{g_*} \mathrm{Hom}(D, C)$$

*is exact at* $\mathrm{Hom}(D, B)$*; i.e.* $\ker(g_*) = \mathrm{im}(f_*)$*.*

**Corollary 3.1.6** (Long exact sequence from a triangle)**.** *Let* $\mathcal{C}$ *be a triangulated category and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

*a distinguished triangle. For every object* $D \in \mathcal{C}$ *there is a natural long exact sequence*

$$\cdots \longrightarrow \mathrm{Hom}(D, A) \xrightarrow{f_*} \mathrm{Hom}(D, B) \xrightarrow{g_*} \mathrm{Hom}(D, C) \xrightarrow{h_*} \mathrm{Hom}(D, TA) \xrightarrow{T(f)_*} \mathrm{Hom}(D, TB) \xrightarrow{T(g)_*} \mathrm{Hom}(D, TC) \xrightarrow{T(h}$$

**Definition 3.1.7** (Homological and cohomological functors)**.** *Let* $\mathcal{C}$ *be a triangulated category and* $\mathcal{A}$ *an abelian category.*

- *A functor* $H : \mathcal{C} \to \mathcal{A}$ *is called* homological *if for every distinguished triangle*

  $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA \quad \text{in } \mathcal{C},$$

  *there is a natural long exact sequence in* $\mathcal{A}$

  $$\cdots \longrightarrow H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C) \xrightarrow{H(h)} H(TA) \xrightarrow{H(Tf)} H(TB) \xrightarrow{H(Tg)} H(TC) \xrightarrow{H(Th)} \cdots.$$

  *Equivalently,* $H$ *sends distinguished triangles to exact triangles of long exact sequences.*

- *A functor $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ is called* cohomological *if for every distinguished triangle as above there is a natural long exact sequence*

$$\cdots \longrightarrow H(T^{-1}C) \xrightarrow{H(T^{-1}g)} H(T^{-1}B) \xrightarrow{H(T^{-1}f)} H(T^{-1}A) \longrightarrow H(C) \xrightarrow{H(g)} H(B) \xrightarrow{H(f)} H(A) \longrightarrow \cdots .$$

**Example 3.1.8.** For any $D \in \mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(D, -) : \mathcal{C} \to \mathbf{Ab}$ and $\mathrm{Hom}_{\mathcal{C}}(-, D) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ are homological and cohomological, respectively.

**Theorem 3.1.9.** *Let $\mathcal{A}$ be an abelian category. The homotopy category $K(\mathcal{A})$ of chain complexes in $\mathcal{A}$ carries a natural triangulated structure.*

**Definition 3.1.10** (Shift and distinguished triangles on $K(\mathcal{A})$)**.** *On $K(\mathcal{A})$ define the* shift *(suspension) by*

$$X[1]^n := X^{n+1}, \qquad d_{X[1]} := -\, d_X.$$

*For a chain map $f : X \to Y$, let* $\mathrm{Cone}(f)$ *be the mapping cone*

$$\mathrm{Cone}(f)^n := Y^n \oplus X^{n+1}, \qquad d_{\mathrm{Cone}(f)} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}.$$

*Declare any triangle isomorphic in $K(\mathcal{A})$ to*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow X[1]$$

*to be a* distinguished triangle*. Equivalently, the class of distinguished triangles is the closure under isomorphism of these mapping–cone triangles.*

**Problem**: Given a short exact sequence in an abelian category $\mathcal{A}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{p} C \longrightarrow 0,$$

we would like the degree-0 complexes $A[0] \to B[0] \to C[0] \to A[1]$ to be a distinguished triangle in the homotopy category $K(\mathcal{A})$. This is *false in general.* What we do have is a canonical map $g : \mathrm{Cone}(f) \to C[0]$ that is a *quasi-isomorphism* (induces isomorphism on homology), but $g$ need not be an isomorphism in $K(\mathcal{A})$.

$$\begin{tikzcd}
A[0] \arrow[rr, "f"] & & B[0] \arrow[dl] \\
& \mathrm{Cone}(f) \arrow[ul] \arrow[r, "g\ \simeq"] & C[0]
\end{tikzcd}$$

**Example 3.1.11.** Take $\mathcal{A} = \mathbf{Ab}$ and the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Viewing $f : \mathbb{Z}[0] \to \mathbb{Z}[0]$ as a chain map, there is a quasi-isomorphism

$$g : \ \mathrm{Cone}(f) \ \xrightarrow{\simeq} \ \mathbb{Z}/2[0].$$

But $g$ is not necessarily an isomorphism in $K(\mathbf{Ab})$ (the cone and $\mathbb{Z}/2[0]$ need not be chain-homotopy equivalent). If we apply a degreewise functor such as

$$F = (\mathbb{Z}/2) \otimes_{\mathbb{Z}} - \ : \ K(\mathbf{Ab}) \to K(\mathbf{Ab}),$$

then typically $F(\mathrm{Cone}(f)) \not\simeq F(\mathbb{Z}/2[0])$ in $K(\mathbf{Ab})$ because $F$ does not preserve all quasi-isomorphisms ($\mathbb{Z}/2$ is not flat). This shows why we must formally force quasi-isomorphisms to be invertible.

### 3.2 Localization of a category

Given an abelian category $\mathcal{A}$, we want a triangulated category $D(\mathcal{A})$ (the *derived category of* $\mathcal{A}$) and, for any (left/right) exact functor $F : \mathcal{A} \to \mathcal{B}$, a *derived functor*

$$RF \text{ or } LF : \ D(\mathcal{A}) \longrightarrow D(\mathcal{B}).$$

The key construction: start with the homotopy category $K(\mathcal{A})$ and *invert all quasi-isomorphisms* to obtain $D(\mathcal{A})$.

Localizing a category works like localizing a ring $R$ at a multiplicative set $S$: we formally add inverses of the chosen elements. For categories we replace elements by morphisms and use *roofs* to record “fractions.”

**Definition 3.2.1** (Multiplicative system)**.** *Let $\mathcal{C}$ be a category and let $S$ be a class of arrows of $\mathcal{C}$. We say $S$ is a* multiplicative system *if:*

1. *For every object $X$ of $\mathcal{C}$, the identity $\mathrm{id}_X$ lies in $S$, and the composite of two composable arrows of $S$ lies in $S$.*
2. *Every solid diagram*

$$\begin{tikzcd} X \arrow[r, dotted, "g"] \arrow[d, dotted, "t"'] & Y \arrow[d, "s"] \\ Z \arrow[r, "f"'] & W \end{tikzcd}$$

*with $s \in S$ can be completed to a commutative square with $t \in S$.*

*Every solid diagram*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t"'] & Y \arrow[d, dotted, "s"] \\ Z \arrow[r, dotted, "f"'] & W \end{tikzcd}$$

*with $t \in S$ can be completed to a commutative dotted square with $s \in S$.*

3. *For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source $Y$ such that $s \circ f = s \circ g$ there exists a $t \in S$ with target $X$ such that $f \circ t = g \circ t$.*

*For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target $X$ such that $f \circ t = g \circ t$ there exists an $s \in S$ with source $Y$ such that $s \circ f = s \circ g$.*

**Definition 3.2.2** (Localization of a category)**.** *Let $\mathcal{C}$ be a category and let $S$ be a multiplicative system in $\mathcal{C}$. The* localization *$S^{-1}\mathcal{C}$ is defined as follows.*

*(1)* ***Objects.*** $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.

*(2)* ***Morphisms.*** *A morphism $X \to Y$ in $S^{-1}\mathcal{C}$ is an* equivalence class *of pairs (a “roof”)*

$$(r, s) : \qquad X \xleftarrow{s} X' \xrightarrow{r} Y \qquad (s \in S),$$

*which we also denote by $s^{-1}r : X \to Y$.*

*(3)* ***Equivalence of roofs.*** *Two roofs $(r_1, s_1) : X \xleftarrow{s_1} X_1 \xrightarrow{r_1} Y$ and $(r_2, s_2) : X \xleftarrow{s_2} X_2 \xrightarrow{r_2} Y$ are declared equivalent if there exist a third roof $(r_3, s_3) : X \xleftarrow{s_3} X_3 \xrightarrow{r_3} Y$ and morphisms $u : X_3 \to X_1$,*

*$v : X_3 \to X_2$ in $\mathcal{C}$ such that the following diagram commutes*

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "r_1"] & \\
X & X_3 \arrow[l, "s_3"'] \arrow[u, "u"] \arrow[d, "v"] \arrow[r, "r_3"'] & Y \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "r_2"'] &
\end{tikzcd}$$

*(4)* ***Composition.*** *Given classes of roofs $(r, s) : X \xleftarrow{s} X' \xrightarrow{r} Y$ and $(r', s') : Y \xleftarrow{s'} Y' \xrightarrow{r'} Z$, The composition is defined as the big roof below*

$$\begin{tikzcd}
 & & Z \arrow[dl, "s''"] \arrow[dr, "r''"] & & \\
 & X' \arrow[dl, "s"] \arrow[dr, "r"] & & Y' \arrow[dl, "s'"] \arrow[dr, "r'"] & \\
X & & Y & & Z
\end{tikzcd}$$

*(5)* ***Identities.*** *The identity $\mathrm{id}_X$ in $S^{-1}\mathcal{C}$ is the class of the roof $(\mathrm{id}_X, \mathrm{id}_X) : X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.*

There is a canonical functor

$$Q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}, \qquad Q(X) = X, \quad Q(f : X \to Y) = (f, \mathrm{id}_X),$$

which sends every $s \in S$ to an isomorphism.

**Theorem 3.2.3** (Universal property)**.** *$Q$ inverts every $s \in S$ and is universal with this property: for any category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$ sending all $S$-arrows to isomorphisms, there exists a unique $\overline{F} : S^{-1}\mathcal{C} \to \mathcal{D}$ with $F = \overline{F} \circ Q$.*

$$\begin{tikzcd}
 & S^{-1}\mathcal{C} \arrow[d, dashed, "\bar{F}"] \\
\mathcal{C} \arrow[ur, "Q"] \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

**Remark 3.2.4** (Ring localization as a special case)**.** If $\mathcal{C}$ is the one-object category of a ring $R$ and $S \subset R$ multiplicative, then $S^{-1}\mathcal{C}$ is the one-object category of the localized ring $S^{-1}R$. Roofs $(r, s)$ encode fractions $r/s$.

### 3.3 Derived Categories

**Definition 3.3.1** (Derived categories)**.** *Let $K(\mathcal{A})$ be the homotopy category of complexes in an abelian category $\mathcal{A}$, $W$ the class of* quasi-isomorphisms*. Then $W$ is a multiplicative system (satisfies a calculus of fractions), and we define*

$$D(\mathcal{A}) \;:=\; W^{-1}K(\mathcal{A}).$$

Morphisms in $D(\mathcal{A})$ are roofs $X \xleftarrow{\simeq} X' \to Y$ with the back arrow a quasi-isomorphism.

**Theorem 3.3.2** ($D(\mathcal{A})$ is triangulated)**.** *Let $\mathcal{A}$ be an abelian category. The derived category $D(\mathcal{A})$ inherits a natural triangulated structure from $K(\mathcal{A})$ as follows:*

- *The shift functor is the degree shift* $[1]$.
- *A triangle in $D(\mathcal{A})$ is* distinguished *iff it is isomorphic in $D(\mathcal{A})$ to the image under the localization $Q : K(\mathcal{A}) \to D(\mathcal{A})$ of a mapping-cone triangle in $K(\mathcal{A})$.*

**Lemma 3.3.3.** *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence in $\mathcal{A}$, then in $D(\mathcal{A})$ there is a distinguished triangle*

$$A[0] \longrightarrow B[0] \longrightarrow C[0] \longrightarrow A[1].$$

*Proof.* In $K(\mathcal{A})$ there is the cone triangle $A[0] \to B[0] \to \mathrm{Cone}(f) \to A[1]$. The canonical map $g : \mathrm{Cone}(f) \to C[0]$ is a quasi-isomorphism, so it becomes an isomorphism after localization to $D(\mathcal{A})$. □

**Question:** What is the map $C[0] \to A[1]$?

In $K(\mathcal{A})$ the composite $C[0] \to A[1]$ coming from the short exact sequence would be zero. In $D(\mathcal{A})$ it is the composition

$$C[0] \xleftarrow[\sim]{g} \mathrm{Cone}(f) \longrightarrow A[1],$$

where $g$ is the quasi-isomorphism above and the second arrow is the cone boundary. Thus the connecting morphism is defined via the cone.

## 3.4 Extension

**Fact:** For $X, Y \in \mathcal{A}$ and $i \geq 0$,

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) \;\cong\; \mathrm{Hom}_{D(\mathcal{A})}\big(X[0], Y[i]\big).$$

**Definition 3.4.1.** *Two extensions of $C$ by $A$ (i.e. short exact sequences $0 \to A \to B \to C \to 0$) are* equivalent *if there exists a map between them (i.e. a commutative diagram)*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

An extension of $C$: $0 \to A \to B \to C \to 0$ gives a long exact sequence by applying $\mathrm{Hom}(C, -)$:

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \xrightarrow{\delta} \mathrm{Ext}^1(C, A) \to \cdots$$

Notice there is a canonical element $\mathrm{id}_C \in \mathrm{Hom}(C, C)$, giving rise to $\xi := \delta(\mathrm{id}_C) \in \mathrm{Ext}^1(C, A)$.

**Theorem 3.4.2.** *The map $\Theta : (extension) \mapsto \xi(extension)$ is an isomorphism*

$$\{\textit{extensions}\}/\sim\, \cong\, \mathrm{Ext}^1(C, A).$$

**Example 3.4.3.** Work over a field $k$ (let $\mathcal{A} = \text{Mod-}A$, where $A$ is a $k$-algebra).

Assume $\mathrm{Ext}^1(C, A) = k$.

For each $\alpha \in k$ we get an extension

$$0 \longrightarrow A \longrightarrow B_\alpha \longrightarrow C \longrightarrow 0.$$

For $\alpha_1 = \lambda\, \alpha_2$ with $\lambda \neq 0$, we have $B_{\alpha_1} \cong B_{\alpha_2}$.

Hence all $B_\alpha$ are isomorphic for $\alpha \neq 0$, but $B_0 \cong A \oplus C$.

**Definition 3.4.4** (k-step extension)**.** *A $k$-step extension $(k \geq 1)$ of $C$ by $A$ is an exact sequence*

$$0 \longrightarrow A \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_k \longrightarrow C \longrightarrow 0.$$

*Two such extensions are* equivalent *if there is a morphism between them making the obvious diagram commute and which is the identity on the end terms $A$ and $C$:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_k \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_k \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

**Theorem 3.4.5.** *There is a bijection*

$$\big\{k\text{-step extensions of } C \text{ by } A\big\}/\sim \xrightarrow{\simeq} \mathrm{Ext}^k(C, A).$$

**Example 3.4.6** (Construction of 2-step extension)**.** Start with an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & X_1 \arrow[rr, "g"] \arrow[dr] & & X_2 \arrow[r, "h"] & C \arrow[r] & 0 \\
& & & \mathrm{coker}(f) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

and notice it splits into two short exact sequences. By applying $\mathrm{Hom}(C, -)$ to the first short exact sequence, we obtain a class

$$\mathrm{Ext}^1\big(C, \mathrm{coker}(f)\big) \longrightarrow \mathrm{Ext}^2(C, A)$$

sending the 1-step extension class to the 2-step extension class in $\mathrm{Ext}^2(C, A)$.

**Remark 3.4.7.** Extensions can be composed. Concatenating a $k$-step extension of $B$ by $A$ and a $l$-step extension of $C$ by $B$ gives a $k+l$-step extension of $C$ by $A$. This is called the Yoneda product:

$$\mathrm{Ext}^k(A, B) \times \mathrm{Ext}^l(B, C) \to \mathrm{Ext}^{k+d}(A, C)$$

Recall in $\mathcal{D}(\mathcal{A})$,

$$\mathrm{Ext}^l_{\mathcal{A}}(A, B) \;\cong\; \mathrm{Hom}_{D(\mathcal{A})}\big(A[0],\, B[l]\big).$$

Then $A \to B[k]$ and $B \to C[l]$ gives $A \to B[k] \to C[k+l]$.

## 3.5 Derived Functors in Derived Categories I

Start with an abelian category $\mathcal{A}$. Recall

$$K(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})/(\text{homotopies}).$$

There are full subcategories

$$K^-(\mathcal{A}), \qquad K^+(\mathcal{A}), \qquad K^b(\mathcal{A}),$$

which contain bounded-above, bounded-below and bounded complexes.

**Problem:** In $\mathcal{K}(\mathcal{A})$, there are complexes quasi-isomorphic to 0 but not isomorphic to 0.

**Example 3.5.1.** The complex

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

of free $\mathbb{Z}/4\mathbb{Z}$–modules is qiso to 0 quasi-isomorphic to 0 but not isomorphic to 0 in $\mathcal{K}(\mathbb{Z}/4\mathbb{Z}\text{-Mod})$.

**Theorem 3.5.2.** *Let $P^\bullet$ be a bounded-above complex of projectives in $\mathcal{A}$. If $P^\bullet$ is quasi-isomorphic to 0, then $P^\bullet$ is* null-homotopic*; in particular $P^\bullet \simeq 0$ in $K^-(\mathcal{A})$.*

**Corollary 3.5.3.** *If $P^\bullet, Q^\bullet$ are bounded-above complexes of projectives and $f : P^\bullet \to Q^\bullet$ is a quasi-isomorphism, then $f$ is an isomorphism in $K^-(\mathcal{A})$.*

*Proof.* $\mathrm{Cone}(f)$ is a bounded-above complex of projectives and is acyclic. By the theorem $\mathrm{Cone}(f) \simeq 0$ in $K^-(\mathcal{A})$, which implies $f$ is an isomorphism in $K^-(\mathcal{A})$ by 5-lemma. □

Write $\mathcal{P} \subset \mathcal{A}$ for projective objects and set

$$K^-(\mathcal{P}) \;:=\; \{\text{bounded-above complexes with all terms projective}\} \;\subset\; K^-(\mathcal{A}).$$

**Theorem 3.5.4.** *The natural map*

$$\big(W \cap K^-(\mathcal{P})\big)^{-1} K^-(\mathcal{P}) \cong K^-(\mathcal{P})$$

*is an equivalence.*

**Theorem 3.5.5** (Fully faithfulness)**.** *The localization functor restricts to a fully faithful functor*

$$Q :\; K^-(\mathcal{P}) \;\longrightarrow\; D^-(\mathcal{A}).$$

**Theorem 3.5.6** (Equivalence under enough projectives)**.** *If $\mathcal{A}$ has enough projectives, then*

$$Q :\; K^-(\mathcal{P}) \;\xrightarrow{\sim}\; D^-(\mathcal{A})$$

*is an equivalence of categories.*

*Proof.* Every bounded-above complex $A^\bullet$ admits a quasi-isomorphism $P^\bullet \to A^\bullet$ with $P^\bullet \in K^-(\mathcal{P})$ (e.g. via Cartan–Eilenberg resolutions). □

Let $F : \mathcal{A} \to \mathcal{B}$ be right exact between abelian categories.

**Definition 3.5.7** (left derived functor via $K^-(\mathcal{P})$)**.** *Assume $\mathcal{A}$ has enough projectives. The derived functor $LF$ is defined as*

$$\begin{tikzcd}
\mathcal{K}^-(\mathcal{P}_\mathcal{A}) \arrow[r, "F"] \arrow[d, hook, "\cong"] & \mathcal{K}^-(\mathcal{B}) \arrow[d, hook] \\
\mathcal{D}^-(\mathcal{A}) \arrow[r, dashed, "LF"] \arrow[u, dashed, bend left, "\exists"] & \mathcal{D}^-(\mathcal{B})
\end{tikzcd}$$

Dually, if $\mathcal{A}$ has enough injectives, we have an equivalence

$$Q : K^+(I) \xrightarrow{\sim} \mathcal{D}^+(\mathcal{A}),$$

and for a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ we obtain its *right derived functor*

$$RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}).$$

i.e.

$$\begin{tikzcd} \mathcal{K}^+(\mathcal{I}_\mathcal{A}) \arrow[r, "F"] \arrow[d, hook, "\cong"] & \mathcal{K}^+(\mathcal{B}) \arrow[d, hook] \\ \mathcal{D}^+(\mathcal{A}) \arrow[u, dashed, bend left, "\exists"] \arrow[r, dashed, "RF"] & \mathcal{D}^+(\mathcal{B}) \end{tikzcd}$$

For $A \in \mathcal{A}$,

$$(R^iF)(A) := H^i\big(RF(A[0])\big).$$

### 3.6 Derived Functors in Derived Categories II

Good definitions of derived functors should be given by a universal property.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let

$$F\colon \mathcal{K}^*(\mathcal{A}) \longrightarrow \mathcal{K}^*(\mathcal{B}),$$

where $* \in \{-, +, b, \emptyset\}$ indicates boundedness conditions (or none).

We have the localization functors

$$Q_\mathcal{A}\colon \mathcal{K}^*(\mathcal{A}) \to \mathcal{D}^*(\mathcal{A}), \qquad Q_\mathcal{B}\colon \mathcal{K}^*(\mathcal{B}) \to \mathcal{D}^*(\mathcal{B}).$$

**Definition 3.6.1** (Left derived functor via universal property)**.** *A* left derived functor *of $F$ is a functor*

$$LF\colon \mathcal{D}^*(\mathcal{A}) \longrightarrow \mathcal{D}^*(\mathcal{B})$$

*together with a natural transformation*

$$\varepsilon\colon LF \circ Q_\mathcal{A} \Longrightarrow Q_\mathcal{B} \circ F$$

*i.e.*

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_A"'] & K^*(\mathcal{B}) \arrow[d, "Q_B"] \\ D^*(\mathcal{A}) \arrow[r, "LF"'] \arrow[ur, Rightarrow, "\varepsilon"] & D^*(\mathcal{B}) \end{tikzcd}$$

*such that the following universal property holds:*

*for any other functor $G\colon \mathcal{D}^*(\mathcal{A}) \to \mathcal{D}^*(\mathcal{B})$ and any natural transformation*

$$\eta\colon G \circ Q_\mathcal{A} \Longrightarrow Q_\mathcal{B} \circ F,$$

*there exists a unique natural transformation*

$$\omega\colon G \circ Q_A \Longrightarrow LF \circ Q_A$$

*such that*

$$\eta = \varepsilon \circ \omega,$$

*i.e. the diagram*

$$\begin{tikzcd} G \circ Q \arrow[r, Rightarrow, "\omega"] \arrow[rr, Rightarrow, bend right, "\eta"'] & LF \circ Q \arrow[r, Rightarrow, "\varepsilon"] & F \circ Q \end{tikzcd}$$

*commutes up to the indicated natural transformations in the universal way.*

**Remark 3.6.2.** The definition of a *right* derived functor is completely analogous, except that all natural transformations go in the opposite direction.

**Theorem 3.6.3.** *Let $\mathcal{A}$ be an abelian category with enough projectives (/injectives) and let $F\colon \mathcal{A} \to \mathcal{B}$ be right exact. Extend $F$ to a functor*

$$F\colon \mathcal{K}^-(\mathcal{A}) \to \mathcal{K}^-(\mathcal{B})$$

*in the obvious way. Then the left derived functor of $F$ (in the above sense) exists and is given by a functor*

$$LF\colon \mathcal{D}^-(\mathcal{A}) \longrightarrow \mathcal{D}^-(\mathcal{B}).$$

*Dually, if $\mathcal{A}$ has enough injectives and $F\colon \mathcal{A} \to \mathcal{B}$ is left exact, then the right derived functor exists as a functor*

$$RF\colon \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B}).$$

*Proof.* For $C^\bullet \in \mathcal{K}^-(\mathcal{A})$, choose a fixed projective resolution $P^\bullet \to C^\bullet$ with $P^\bullet$ a bounded above complex of projectives. Define

$$LF(C^\bullet) := F(P^\bullet) \in \mathcal{D}^-(\mathcal{B}).$$

We need a natural transformation

$$\varepsilon_{C^\bullet}\colon LF(C^\bullet) = F(P^\bullet) \longrightarrow F(C^\bullet)$$

in $\mathcal{D}^-(\mathcal{B})$. This is induced by applying $F$ to the quasi-isomorphism $P^\bullet \to C^\bullet$.

One checks that this defines a functor $LF\colon \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$ together with $\varepsilon$ satisfying the required universal property. $\square$

### 3.7 Derived Functors in Derived Categories III

We want to understand, for objects $A, B \in \mathcal{A}$ and integers $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(A[0],\, B[i]\big) \cong \mathrm{Ext}^i_{\mathcal{A}}(A, B).$$

**Definition 3.7.1** (Hyper Ext)**.** *Let $A^\bullet, B^\bullet \in \mathcal{D}(\mathcal{A})$. Define the* hyper Ext *groups by*

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(A^\bullet,\, B^\bullet[i]\big), \qquad i \in \mathbb{Z}.$$

**Definition 3.7.2.** *Fix a bounded-above complex $C^\bullet \in \mathcal{K}^-(\mathcal{A})$. There is a functor*

$$\mathrm{Hom}(C^\bullet, -)\colon \mathcal{K}^+(\mathcal{A}) \longrightarrow \mathcal{K}^+(\mathrm{Ab})$$

*sending a complex $B^\bullet$ to the usual Hom complex* $\mathrm{Hom}(C^\bullet, B^\bullet)$*. If $\mathcal{A}$ has enough injectives, this functor admits a* right derived functor

$$\mathrm{RHom}(C^\bullet, -)\colon \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathrm{Ab}).$$

Explicitly, if $B^\bullet \in \mathcal{D}^+(\mathcal{A})$ and $B^\bullet \to I^\bullet$ is a bounded-below injective resolution, we set

$$\mathrm{RHom}(C^\bullet, B^\bullet) := \mathrm{Hom}(A^\bullet, I^\bullet)$$

viewed as an object of $\mathcal{D}^+(\mathrm{Ab})$.

The construction above extends (in the first variable as well) to a bifunctor

$$\mathrm{RHom}(-, -)\colon \mathcal{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathrm{Ab}).$$

**Theorem 3.7.3** (Yoneda-type description). *Let $A^\bullet, B^\bullet \in \mathcal{D}^+(\mathcal{A})$ and assume $\mathcal{A}$ has enough injectives. Then for all $i \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) \;\cong\; H^i\big(\mathrm{RHom}(A^\bullet, B^\bullet)\big).$$

**Corollary 3.7.4.** *For $A, B \in \mathcal{A}$ and $i \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}\big(A[0], B[i]\big) \;\cong\; \mathrm{Ext}^i_{\mathcal{A}}(A, B).$$

*Proof.* We only check the case $i = 0$; the general case is similar. For $A^\bullet, B^\bullet \in \mathcal{D}^+(\mathcal{A})$, want to show

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) \;\cong\; H^0\big(\mathrm{RHom}(A^\bullet, B^\bullet)\big).$$

Choose an injective resolution $B^\bullet \xrightarrow{\simeq} I^\bullet$ with $I^\bullet$ bounded below and consisting of injectives.

$$\begin{aligned} H^0\big(\mathrm{RHom}(A^\bullet, B^\bullet)\big) &= H^0\big(\mathrm{Hom}(A^\bullet, I^\bullet)\big) \\ &\cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, I^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet). \end{aligned}$$

The first equality is the definition of RHom. The second is a basic fact about the Hom complex: its 0-th cohomology is the group of chain maps modulo homotopy. The third is the key lemma below, and the last is induced by the quasi-isomorphism $B' \to I^\bullet$. □

**Lemma 3.7.5.** *Let $s\colon I^\bullet \to Y^\bullet$ be a chain map with the following properties:*

1. *$s$ is a quasi-isomorphism;*
2. *$I^\bullet$ is a bounded-below complex of injective objects.*

*Then $s$ has a homotopy inverse. In particular, for any $A^\bullet \in \mathcal{K}^+(\mathcal{A})$ the induced map*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, Y^\bullet)$$

*is an isomorphism, and localization $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ does not invert any additional maps with source $I^\bullet$. Hence $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, I^\bullet)$.*

We now state a general existence theorem for right derived functors.

**Theorem 3.7.6.** *Let $F\colon \mathcal{K}^*(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ be a functor of homotopy categories. Let $\mathcal{L} \subseteq \mathcal{K}^*(\mathcal{A})$ be a full subcategory such that:*

1. *$\mathcal{L}$ is a triangulated subcategory of $\mathcal{K}^+(\mathcal{A})$ (closed under shifts and cones);*
2. *$F$ sends acyclic complexes in $\mathcal{L}$ to acyclic complexes in $\mathcal{K}(\mathcal{B})$ (we say "$\mathcal{L}$ is adapted to $F$");*
3. *every object $C^\bullet \in \mathcal{K}^+(\mathcal{A})$ admits a quasi-isomorphism $C^\bullet \to L^\bullet$ with $L^\bullet \in \mathcal{L}$.*

*Then the right derived functor*

$$RF\colon \mathcal{D}^*(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$$

*exists and can be computed using representatives in $\mathcal{L}$. More precisely, if $X^\bullet \in \mathcal{D}^+(\mathcal{A})$ and $X^\bullet \simeq L^\bullet$ with $L^\bullet \in \mathcal{L}$, then*

$$RF(X^\bullet) \;\cong\; F(L^\bullet)$$

*in $\mathcal{D}(\mathcal{B})$, and this is independent (up to canonical isomorphism) of the choice of $L^\bullet$.*

## 3.8 Composition of Derived Functors

**Theorem 3.8.1.** *Let*

$$G\colon \mathcal{A} \longrightarrow \mathcal{B}, \qquad F\colon \mathcal{B} \longrightarrow \mathcal{C}$$

*be functors between abelian categories. Then*

$$R(F \circ G) \;\cong\; (RF) \circ (RG)$$

*if $G$ takes bounded below complexes of injectives to complexes adapted to $F$.*

$$\begin{tikzcd}
\mathcal{K}^+(\mathcal{A}) \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{K}^+(\mathcal{B}) \arrow[r, "F"] \arrow[d, "Q"] & \mathcal{K}^+(\mathcal{C}) \arrow[d, "Q"] \\
\mathcal{D}^+(\mathcal{A}) \arrow[r, "RG"'] & \mathcal{D}^+(\mathcal{B}) \arrow[r, "RF"'] & \mathcal{D}^+(\mathcal{C})
\end{tikzcd}$$

*Idea of proof.* Check that the pair

$$(RF \circ RG,\ \eta_F * \eta_G)$$

(where $\eta_F, \eta_G$ are the canonical natural transformations defining the derived functors) satisfies the same universal property as the right derived functor of $F \circ G$. Hence $R(F \circ G) \cong RF \circ RG$. $\square$

**Example 3.8.2.** Let $A, B$ be rings (or algebras), and let

$$N \text{ be an } (A,B)\text{–bimodule}, \qquad P \text{ be a } (B,C)\text{–bimodule}.$$

Consider functors

$$G = - \otimes_A N\colon \text{Mod-}A \longrightarrow \text{Mod-}B,$$

$$F = - \otimes_B P\colon \text{Mod-}B \longrightarrow \text{Mod-}C.$$

Then $G$ and $F$ are right exact functors, and their composition is

$$F \circ G = (- \otimes_A N) \otimes_B P \cong - \otimes_A (N \otimes_B P).$$

The left derived functors are

$$LG = - \otimes_A^L N, \qquad LF = - \otimes_B^L P,$$

and the theorem above implies

$$L(F \circ G) \;\simeq\; LF \circ LG \;\simeq\; \big(- \otimes_A^L N\big) \otimes_B^L P.$$

**Example 3.8.3** (Grothendieck spectral sequence)**.** Evaluating on an object $A \in \mathcal{A}$ and taking cohomology, one obtains the *Grothendieck spectral sequence* (under suitable hypotheses):

$$E_2^{p,q} = R^pF\big(R^qG(A)\big) \Longrightarrow R^{p+q}(F \circ G)(A),$$

or in homological indexing (for left derived functors),

$$E^2_{p,q} = L_pF\big(L_qG(A)\big) \Longrightarrow L_{p+q}(F \circ G)(A).$$

**Example 3.8.4** (Sheaf cohomology and Leray spectral sequence)**.** Let $f\colon X \to Y$ be a continuous map of topological spaces, and work with sheaves of abelian groups.

Take

$$G = f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y), \qquad F = \Gamma(Y,-)\colon \mathrm{Sh}(Y) \to \mathrm{Ab}\,.$$

Then $F$ and $G$ are left exact, so

$$R\Gamma(X,-) \;\cong\; R\Gamma\big(Y, Rf_*(-)\big).$$

If $\mathcal{F} = \underline{\mathbb{Z}}$ is the constant sheaf on $X$, we get

$$H^*(X,\mathbb{Z}) \;\cong\; H^*\big(Y, Rf_*\underline{\mathbb{Z}}\big),$$

and the associated Grothendieck spectral sequence is the Leray spectral sequence of $f$.

Consider the special case $X = Y \times Z$ and let

$$f\colon Y \times Z \longrightarrow Y$$

be the projection. Then the sheaf $R^q f_*\underline{\mathbb{Q}}$ is (under standard hypotheses) the constant sheaf on $Y$ with stalk $H^q(Z,\mathbb{Q})$. The Leray spectral sequence then yields

$$H^*(Y \times Z,\mathbb{Q}) \;\cong\; H^*\big(Y, H^*(Z,\mathbb{Q})\big) \;\cong\; H^*(Y,\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(Z,\mathbb{Q}),$$

which is a form of the Künneth formula.

# 4 Spectral Sequences

## 4.1 Spectral Sequences of Double Complexes

Let $(A^{p,q}, d_h, d_v)$ be a first–quadrant double complex of abelian groups (or objects in an abelian category), i.e.

$$d_h\colon A^{p,q} \to A^{p+1,q}, \qquad d_v\colon A^{p,q} \to A^{p,q+1},$$

with

$$d_h^2 = d_v^2 = 0, \qquad d_h d_v + d_v d_h = 0.$$

Define the total complex

$$C^\bullet = \mathrm{Tot}(A^{\bullet,\bullet}), \qquad C^n = \bigoplus_{p+q=n} A^{p,q},$$

with differential

$$d = d_h + d_v\colon C^n \longrightarrow C^{n+1}.$$

Then $d^2 = 0$ and we can consider the cohomology $H^*(C^\bullet)$.

Filter $C^\bullet$ by *columns*

$$F^p C^n = \bigoplus_{i \ge p,\; i+j=n} A^{i,j}.$$

The associated graded pieces are

$$\mathrm{gr}_F^p\, C^{p+q} \cong A^{p,q}.$$

This filtration gives a spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$E_0^{p,q} \cong A^{p,q}, \qquad d_0 = d_v\colon A^{p,q} \to A^{p,q+1}.$$

Thus the $E_1$–page is
$$E_1^{p,q} \cong H_v^q(A^{p,\bullet}),$$
vertical cohomology of each column, and the differential $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ is induced by $d_h$.

Taking cohomology again,
$$E_2^{p,q} \cong H_h^p\big(H_v^q(A^{\bullet,\bullet})\big).$$

**Theorem 4.1.1.** *With the filtration above there is a convergent spectral sequence*

$$E_2^{p,q} = H_h^p\big(H_v^q(A^{\bullet,\bullet})\big) \Longrightarrow H^{p+q}(C^\bullet),$$

*i.e. the spectral sequence converges the cohomology of the total complex.*

**Theorem 4.1.2.** *Let $f : X \to B$ be a fibration of topological spaces, and let $F$ be the (path-connected) fiber. The Serre cohomology spectral sequence is the following:*

$$E_2^{p,q} = H^p\left(B, H^q(F)\right) \Rightarrow H^{p+q}(X)$$

**Example 4.1.3.** We now use a Serre spectral sequence to compute the singular cohomology groups $H^*(\mathbb{CP}^n, \mathbb{Z})$.

Consider $\mathbb{C}^{n+1} \setminus \{0\}$ and the natural projection

$$\mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n, \qquad z \longmapsto [z].$$

The fiber over a point is $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Topologically, $\mathbb{C}^{n+1} \setminus \{0\} \simeq S^{2n+1}$ and $\mathbb{C}^* \simeq S^1$, so we have a fibration
$$S^1 \simeq \mathbb{C}^* \longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n.$$

Let $F \simeq S^1$ be the fiber and $B = \mathbb{CP}^n$ the base.

The cohomology of the fiber is

$$H^q(F, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, 1, \\ 0, & q \neq 0, 1. \end{cases}$$

The Serre cohomology spectral sequence of this fibration has

$$E_2^{p,q} = H^p(B; H^q(F, \mathbb{Z})) \Longrightarrow H^{p+q}(S^{2n+1}, \mathbb{Z}).$$

Since the local coefficient system is trivial here,

$$E_2^{p,0} = H^p(\mathbb{CP}^n, \mathbb{Z}), \qquad E_2^{p,1} = H^p(\mathbb{CP}^n, \mathbb{Z}),$$

and $E_2^{p,q} = 0$ for $q \neq 0, 1$.

On the other hand, we know

$$H^k(S^{2n+1}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 2n+1, \\ 0, & \text{otherwise}. \end{cases}$$

One now compares the $E_\infty$–page with these groups. The only nonzero cohomology of the total space in degrees 0 and $2n + 1$ forces almost all entries of the spectral sequence to eventually die, and one deduces by a standard argument that

$$H^k(\mathbb{CP}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise}. \end{cases}$$