Math 751 HW 4

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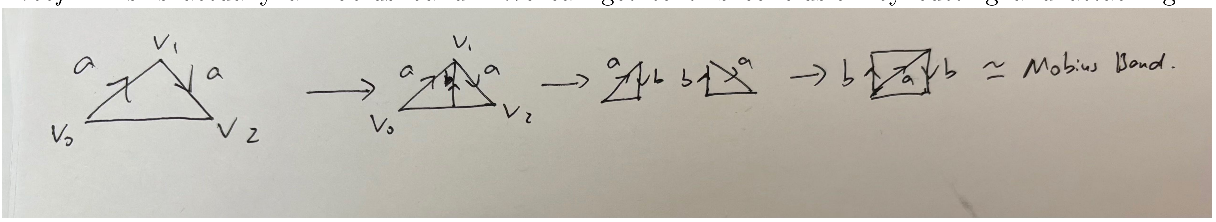
Due: Friday Night

1. What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

Proof. This is actually a mobius band. We can get to this conclusion by cutting and attaching:



□

2. Construct a Δ-complex structure on $\mathbb{RP}^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.

Proof. We view S^n as the unit sphere in $\mathbb{R}^{n+1}$ with standard basis $e_0, \ldots, e_n$. Consider the set of $2(n+1)$ points

$$\{\pm e_0, \ldots, \pm e_n\} \subset S^n.$$

Let

$$P = \operatorname{conv}\{\pm e_0, \ldots, \pm e_n\} \subset \mathbb{R}^{n+1}$$

be the $(n+1)$-dimensional cross–polytope. It is a convex polytope whose vertices are exactly the points $\pm e_i$. A standard fact is that each n-dimensional facet of P is of the form

$$\tau_\varepsilon = \operatorname{conv}(\varepsilon_0 e_0, \ldots, \varepsilon_n e_n), \qquad \varepsilon_i \in \{\pm 1\}.$$

All such τ_ε give the facets of P, and their faces give all lower-dimensional faces.

Thus the boundary ∂P becomes a finite Δ complex whose vertices are the $\pm e_i$, whose n-simplices are the τ_ε, and whose lower-dimensional simplices are faces of these.

Since P is a convex body containing the origin, the radial projection

$$r : \partial P \longrightarrow S^n, \qquad r(x) = \frac{x}{\|x\|}$$

is a homeomorphism. Composing each simplex embedding $\Delta^k \cong \sigma \subset \partial P$ with r yields a family of maps $\Delta^k \to S^n$ satisfying the axioms of a Δ-complex structure. Explicitly, for each $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ we define

$$\sigma_\varepsilon : \Delta^n \to S^n, \qquad \sigma_\varepsilon(t_0, \ldots, t_n) = \frac{\sum_{i=0}^n t_i\, \varepsilon_i e_i}{\|\sum_{i=0}^n t_i\, \varepsilon_i e_i\|}.$$

The restrictions of σ_ε to faces of Δ^n give all lower-dimensional simplices.

Therefore S^n is endowed with a Δ-complex structure whose vertices are precisely the $2(n+1)$ unit vectors $\pm e_i$.

Consider the antipodal map

$$a : S^n \to S^n, \qquad a(x) = -x.$$

On vertices we have $a(e_i) = -e_i$ and $a(-e_i) = e_i$; thus a simply permutes the vertices in pairs. On an n-simplex $\tau_\varepsilon = \operatorname{conv}(\varepsilon_0 e_0, \ldots, \varepsilon_n e_n)$ we obtain

$$a(\tau_\varepsilon) = \operatorname{conv}(-\varepsilon_0 e_0, \ldots, -\varepsilon_n e_n) = \tau_{-\varepsilon},$$

and it follows that

$$a \circ \sigma_\varepsilon = \sigma_{-\varepsilon}.$$

The same holds for all faces of τ_ε. Hence a maps each simplex of the Δ-complex structure on S^n to a simplex, and it maps faces to faces; a is a simplicial self-map of this Δ-complex.

Under the quotient map $q : S^n \to \mathbb{RP}^n$, the two vertices $\pm e_i$ are identified, so the quotient has $n+1$ vertices, which we may denote by $[e_i]$. An n-simplex of $\mathbb{RP}^n$ is the orbit of a pair $\{\sigma_\varepsilon, \sigma_{-\varepsilon}\}$. It is convenient to label such an orbit by the class $[\varepsilon] = \{\varepsilon, -\varepsilon\}$. The corresponding characteristic map can be written explicitly as

$$\bar{\sigma}_{[\varepsilon]} : \Delta^n \to \mathbb{RP}^n, \qquad \bar{\sigma}_{[\varepsilon]}(t_0, \dots, t_n) = \Big[\sum_{i=0}^{n} t_i \, \varepsilon_i e_i \Big],$$

where $[\cdot]$ denotes the line spanned by a nonzero vector in $\mathbb{R}^{n+1}$. This is well-defined because

$$\Big[\sum t_i \, \varepsilon_i e_i \Big] = \Big[-\sum t_i \, \varepsilon_i e_i \Big],$$

so replacing ε by $-\varepsilon$ does not change the image point in $\mathbb{RP}^n$.

Similarly, for a subset $I = \{i_0 < \cdots < i_k\} \subset \{0, \dots, n\}$ and a sign vector $\varepsilon = (\varepsilon_0, \dots, \varepsilon_k)$ we define a k-simplex

$$\bar{\sigma}_{I,[\varepsilon]} : \Delta^k \to \mathbb{RP}^n, \qquad (t_0, \dots, t_k) \longmapsto \Big[\sum_{j=0}^{k} t_j \, \varepsilon_j e_{i_j} \Big],$$

which is the image of the corresponding face of σ_ε.

These simplices satisfy the axioms of a Δ-complex and provide the desired Δ-complex structure on $\mathbb{RP}^n$ as the quotient of the Δ-complex on S^n whose vertices are the unit vectors along the coordinate axes. $\square$

3. Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. We consider the standard 2–simplex

$$\Delta^2 = [v_0, v_1, v_2]$$

and form the quotient space obtained by identifying the three vertices v_0, v_1, v_2 to a single point. This quotient admits a natural Δ-complex structure: it has one vertex v (the common image of v_0, v_1, v_2), three edges

$$e_1 = [v_0, v_1], \qquad e_2 = [v_1, v_2], \qquad e_3 = [v_0, v_2],$$

and one 2–simplex

$$\sigma = [v_0, v_1, v_2].$$

Thus the simplicial chain groups are

$$C_2 \cong \mathbb{Z}, \qquad C_1 \cong \mathbb{Z}^3, \qquad C_0 \cong \mathbb{Z},$$

and $C_n = 0$ for $n \geq 3$.

For an oriented edge $[a, b]$ we have $\partial_1[a, b] = b - a$. In the quotient space the vertices v_0, v_1, v_2 all correspond to the same vertex v, hence

$$\partial_1(e_1) = v - v = 0, \quad \partial_1(e_2) = v - v = 0, \quad \partial_1(e_3) = v - v = 0.$$

Therefore $\partial_1 = 0$, so

$$\ker \partial_1 = C_1 \cong \mathbb{Z}^3, \qquad \operatorname{im} \partial_1 = 0.$$

For the oriented 2–simplex $\sigma = [v_0, v_1, v_2]$ we have

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = e_2 - e_3 + e_1.$$

Identifying $C_1 \cong \mathbb{Z}^3$ via the basis (e_1, e_2, e_3), the map $\partial_2 : \mathbb{Z} \to \mathbb{Z}^3$ is given by

$$\partial_2(1) = (1, 1, -1).$$

This map is injective, hence

$$\ker \partial_2 = 0, \qquad \operatorname{im} \partial_2 = \langle (1, 1, -1) \rangle \cong \mathbb{Z}.$$

Thus the chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

In degree 2 we get

$$H_2 = \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2 = 0.$$

In degree 1 we obtain

$$H_1 = \ker \partial_1 / \operatorname{im} \partial_2 = C_1 / \operatorname{im} \partial_2 = \mathbb{Z}^3 / \langle (1, 1, -1) \rangle.$$

Consider the projection

$$\phi : \mathbb{Z}^3 \to \mathbb{Z}^2, \qquad \phi(x_1, x_2, x_3) = (x_1, x_2).$$

Its kernel is

$$\ker \phi = \{(t, t, -t) \mid t \in \mathbb{Z}\} = \langle (1, 1, -1) \rangle = \operatorname{im} \partial_2.$$

Hence ϕ induces an isomorphism

$$\mathbb{Z}^3 / \langle (1, 1, -1) \rangle \cong \mathbb{Z}^2,$$

so $H_1 \cong \mathbb{Z}^2$.

In degree 0, since $\partial_1 = 0$, we have

$$H_0 \cong \mathbb{Z}.$$

For $n \geq 2$ there are no nonzero chains, so $H_n = 0$ for $n \geq 2$.

Therefore the simplicial homology groups of the triangular parachute are

$$H_0 \cong \mathbb{Z}, \qquad H_1 \cong \mathbb{Z}^2, \qquad H_n = 0 \text{ for } n \geq 2.$$

□

4. Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.

Proof. Let X_n be the Δ-complex described in the statement, obtained from $(n + 1)$ two-simplices $\Delta_0^2, \ldots, \Delta_n^2$ with the prescribed identifications of edges.

First we determine the simplices of X_n.

For Δ_0^2, all three edges are identified to a single edge. In particular, the three vertices of Δ_0^2 are identified to one vertex.

For $i > 0$, in Δ_i^2 the edges $[v_0, v_1]$ and $[v_1, v_2]$ are identified to a single edge, so v_0, v_1, v_2 are again identified to a single vertex. The third edge $[v_0, v_2]$ is glued to the edge $[v_0, v_1]$ of Δ_{i-1}^2, whose endpoints are already identified with the vertex of Δ_{i-1}^2. Inductively, all vertices in the whole Δ-complex are identified to a single vertex.

Thus there is exactly one 0-simplex; we denote it by v.

Let e_0 be the single edge coming from the three edges of Δ_0^2. For each $i > 0$, the two edges $[v_0, v_1]$ and $[v_1, v_2]$ in Δ_i^2 are identified to a new edge; we call this edge e_i. The remaining edge $[v_0, v_2]$ of Δ_i^2 is glued to the edge $[v_0, v_1]$ of Δ_{i-1}^2, which is e_{i-1}. Hence there are exactly $n+1$ distinct 1-simplices

$$e_0, e_1, \ldots, e_n.$$

Finally, the $(n+1)$ triangles Δ_i^2 themselves give the $(n+1)$ 2-simplices of X_n; we denote the oriented 2-simplex in Δ_i^2 by σ_i.

Therefore the simplicial chain groups are

$$C_2(X_n) \cong \mathbb{Z}^{n+1} \quad \text{with basis } \{\sigma_0, \ldots, \sigma_n\},$$

$$C_1(X_n) \cong \mathbb{Z}^{n+1} \quad \text{with basis } \{e_0, \ldots, e_n\},$$

$$C_0(X_n) \cong \mathbb{Z} \quad \text{with basis } \{v\},$$

and $C_k(X_n) = 0$ for $k \geq 3$.

Since there is only one vertex, every edge starts and ends at the same vertex. Hence

$$\partial_1(e_i) = v - v = 0 \quad \text{for all } i,$$

so $\partial_1 = 0$.

To compute ∂_2, fix an ordering (v_0, v_1, v_2) of the vertices of each Δ_i^2, and denote the corresponding oriented 2-simplex by $\sigma_i = [v_0, v_1, v_2]$. In a Δ-complex each face is attached by an order-preserving map, so there is no extra sign coming from the gluings; the only signs come from the alternating sum in the boundary formula. For an oriented 2-simplex we have

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

For $i = 0$, all three edges of Δ_0^2 are identified with the same oriented edge e_0, hence

$$\partial_2(\sigma_0) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = e_0 - e_0 + e_0 = e_0.$$

For $i > 0$, the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 are glued to the new edge e_i, while $[v_0, v_2]$ is glued to the edge $[v_0, v_1]$ of Δ_{i-1}^2, which is e_{i-1}. Thus

$$\partial_2(\sigma_i) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = e_i - e_{i-1} + e_i = 2e_i - e_{i-1}, \qquad i = 1, \ldots, n.$$

Consequently

$$\operatorname{im} \partial_2 = \big\langle e_0,\ 2e_1 - e_0,\ 2e_2 - e_1, \ldots, 2e_n - e_{n-1} \big\rangle \subset C_1(X_n).$$

Since $\partial_1 = 0$, we have

$$H_0(X_n) \cong C_0(X_n)/\operatorname{im} \partial_1 \cong \mathbb{Z},$$

To compute $H_2(X_n)$, consider an arbitrary 2-chain

$$c = \sum_{i=0}^{n} c_i \sigma_i \in C_2(X_n).$$

Its boundary is

$$\partial_2(c) = c_0 e_0 + \sum_{i=1}^{n} c_i (2e_i - e_{i-1}).$$

Equating the coefficients of e_k to zero gives

$$\begin{cases} c_0 - c_1 = 0, & \text{(coefficient of } e_0\text{)} \\ 2c_k - c_{k+1} = 0, & \text{for } 1 \le k \le n-1, \\ 2c_n = 0, & \text{(coefficient of } e_n\text{)}. \end{cases}$$

From the last equation we obtain $c_n = 0$, then from $2c_{n-1} - c_n = 0$ we get $c_{n-1} = 0$, and inductively we find $c_{n-2} = 0, \dots, c_1 = 0$. Finally $c_0 - c_1 = 0$ implies $c_0 = 0$. Thus $\ker \partial_2 = 0$ and therefore

$$H_2(X_n) = \ker \partial_2 / \operatorname{im} \partial_3 = 0.$$

Because $\partial_1 = 0$, we have

$$H_1(X_n) = \ker \partial_1 / \operatorname{im} \partial_2 = C_1(X_n)/\operatorname{im} \partial_2 \cong \mathbb{Z}^{n+1}/\langle e_0,\ 2e_i - e_{i-1}\ (1 \le i \le n)\rangle.$$

In this quotient we can rewrite the relations as

$$e_0 = 0, \qquad e_{i-1} = 2e_i \quad (1 \le i \le n).$$

By successive substitution we obtain

$$e_{n-1} = 2e_n, \quad e_{n-2} = 2e_{n-1} = 2^2 e_n, \ \dots, \ e_0 = 2^n e_n.$$

Together with $e_0 = 0$ this gives $2^n e_n = 0$ in $H_1(X_n)$, so $H_1(X_n)$ is generated by the class of e_n and has exponent dividing 2^n.

To determine its order exactly, define a homomorphism

$$\phi : C_1(X_n) \longrightarrow \mathbb{Z}/2^n\mathbb{Z}, \qquad \phi(e_i) = 2^{\,n-i} \pmod{2^n}.$$

Then

$$\phi(e_0) = 2^n \equiv 0 \pmod{2^n},$$

and for $1 \le i \le n$,

$$\phi(2e_i - e_{i-1}) = 2 \cdot 2^{\,n-i} - 2^{\,n-(i-1)} = 2^{\,n-i+1} - 2^{\,n-i+1} = 0.$$

Hence $\operatorname{im} \partial_2 \subset \ker \phi$, so ϕ induces a surjective homomorphism

$$\bar{\phi} : H_1(X_n) \twoheadrightarrow \mathbb{Z}/2^n\mathbb{Z},$$

since $\bar{\phi}([e_n]) = \phi(e_n) = 1$.

Therefore $|H_1(X_n)| \ge 2^n$. On the other hand, every class in $H_1(X_n)$ is an integer multiple of $[e_n]$ and $2^n[e_n] = 0$, so $|H_1(X_n)| \le 2^n$. We conclude that $|H_1(X_n)| = 2^n$, and since $H_1(X_n)$ has an element of order 2^n (namely $[e_n]$), it is cyclic:

$$H_1(X_n) \cong \mathbb{Z}/2^n\mathbb{Z}.$$

Collecting the above computations, the simplicial homology groups of X_n are

$$H_k(X_n) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2^n\mathbb{Z}, & k = 1, \\ 0, & k \ge 2. \end{cases}$$

□

5. Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \dots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively. [The space X is an example of a lens space; see Example 2.43 for the general case.]

Proof. All vertices of the complex are identified to just two vertices: a central vertex x and an outer vertex y.

We take

$$a : y \to y, \quad b : y \to y, \quad c_i : y \to x \ (i = 1, \dots, n).$$

Thus there are $n + 2$ oriented 1–simplices.

For each i we have

- a horizontal face A_i with edges b, c_i, c_{i+1},
- a vertical face B_i with edges a, c_{i-1}, c_i.

So there are $2n$ 2–simplices.

The 3–simplices are the tetrahedra $T_1, \dots, T_n$.

Thus the simplicial chain groups are

$$C_3 \cong \mathbb{Z}^n, \qquad C_2 \cong \mathbb{Z}^{2n}, \qquad C_1 \cong \mathbb{Z}^{n+2}, \qquad C_0 \cong \mathbb{Z}^2$$

with bases $\{T_i\}$, $\{A_i, B_i\}$, $\{a, b, c_i\}$, $\{x, y\}$ respectively.

We choose orientations compatible with the picture.
For 1–simplices we have

$$\partial_1 a = 0, \quad \partial_1 b = 0, \quad \partial_1 c_i = x - y.$$

For 2–simplices we choose orientations so that

$$\partial_2 A_i = b - c_{i+1} + c_i, \qquad \partial_2 B_i = a + c_i - c_{i-1}.$$

Each tetrahedron T_i has faces $A_i, A_{i-1}, B_i, B_{i+1}$, and with suitable orientations we obtain

$$\partial_3 T_i = A_i - A_{i-1} + B_i - B_{i+1}.$$

Since $\partial_0 = 0$, we have $\ker \partial_0 = C_0 \cong \mathbb{Z}^2$ with basis $\{x, y\}$, and

$$\operatorname{im} \partial_1 = \langle x - y \rangle \cong \mathbb{Z},$$

because $\partial_1 c_i = x - y$ while $\partial_1 a = \partial_1 b = 0$. Hence

$$H_0(X) \cong C_0 / \operatorname{im} \partial_1 \cong \mathbb{Z}^2 / \langle x - y \rangle \cong \mathbb{Z}.$$

For group $H_1(X)$, we have $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2$.
A general 1–chain is

$$u = \alpha a + \beta b + \sum_{i=1}^{n} \gamma_i c_i,$$

and

$$\partial_1 u = \Big(\sum_{i=1}^{n} \gamma_i\Big)(x-y).$$

Thus $u \in \ker \partial_1$ iff $\sum_{i=1}^{n} \gamma_i = 0$.

Introduce

$$d_i := c_{i+1} - c_i, \qquad i = 1, \dots, n.$$

Then $\sum_{i=1}^{n} d_i = 0$, and any family (γ_i) with $\sum \gamma_i = 0$ can be expressed in terms of the differences d_i. Hence

$$\ker \partial_1 = \langle a, b, d_1, \dots, d_{n-1} \rangle \cong \mathbb{Z}^{n+1}.$$

Next, using the boundary formulas,

$$\partial_2 A_i = b - c_{i+1} + c_i = b - d_i, \qquad \partial_2 B_{i+1} = a + c_{i+1} - c_i = a + d_i.$$

For $i = 1, \dots, n$ set

$$g_i := a + d_i, \qquad h_i := b - d_i.$$

Then $\operatorname{im} \partial_2$ is generated by all g_i, h_i.

In the quotient $G := \ker \partial_1 / \operatorname{im} \partial_2$, from $g_i = 0$ for $i = 1, \dots, n-1$ we obtain

$$d_i \equiv -a, \qquad i = 1, \dots, n-1.$$

Since $d_n = -\sum_{i=1}^{n-1} d_i$, the relation $g_n = 0$ yields

$$0 \equiv a + d_n = a - \sum_{i=1}^{n-1} d_i \equiv a - (n-1)(-a) = na.$$

Thus $na \equiv 0$ in G. On the other hand, from $h_i = 0$ we get

$$b \equiv d_i \equiv -a,$$

so b and all d_i are multiples of a in G. Therefore G is generated by the class of a with the single relation $na = 0$, and hence

$$H_1(X) \cong G \cong \mathbb{Z}_n.$$

Now we prove $H_2(X) = 0$. It is equivalent to prove that for each $2-$cycle, we can find a chain in C_3, such that this two cycle equals the image of this chain.

Suppose we have $\sum \alpha_i A_i + \beta_i B_i \in C_2$ such that $\partial_2(\sum \alpha_i A_i + \beta_i B_i \in C_2) = 0$. And, we can calculate that

$$\partial_2(\sum \alpha_i A_i + \beta_i B_i \in C_2) = 0 \iff \sum \alpha_i = \sum \beta_i = 0, \alpha_i - \alpha_{i-1} = \beta_i - \beta_{i+1}$$

Now suppose we have $\sum \gamma_i T_i \in C_3$. Then we can calculate that

$$\partial(\sum \gamma_i (T_i) = \sum (\gamma_i - \gamma_{i+1}) A_i + (\gamma_i - \gamma_{i-1}) B_i$$

Then we need to solve the integer function

$$\gamma_i - \gamma_{i+1} = \alpha_i \ , \ \gamma_i - \gamma_{i-1} = \beta_i$$

Then we can let $\gamma_1 = 0$. Then we can solve

$$\gamma_i = -\alpha_1 - \dots - \alpha_{i-1} = \beta_2 + \dots + \beta_i, \ i \geq 2$$

.

The last equality can be easily seen from equalities $\alpha_i - \alpha_{i-1} = \beta_i - \beta_{i+1}$ and $\sum \alpha_i = \sum \beta_i = 0$. So, $H_2(X) = 0$

Now we compute $H_3(X)$.

$$H_3(X) = \ker \partial_3 / \operatorname{im} \partial_4 = \ker \partial_3,$$

since $C_4 = 0$.

Let $\sum t_i T_i$ be a 3–cycle. Then

$$0 = \partial_3\Big(\sum t_i T_i\Big) = \sum t_i (A_i - A_{i-1} + B_i - B_{i+1}).$$

The coefficient of each A_i gives $t_i - t_{i+1} = 0$, and the coefficient of each B_i gives $t_i - t_{i-1} = 0$. Hence all t_i are equal, say $t_i = t$ for all i, and

$$\ker \partial_3 = \langle T_1 + \cdots + T_n \rangle \cong \mathbb{Z}.$$

Thus

$$H_3(X) \cong \mathbb{Z}.$$

All in all, The simplicial homology groups of X are

$$H_k(X) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}_n & k = 1, \\ 0 & k = 2, \\ \mathbb{Z} & k = 3, \\ 0 & k \geq 4. \end{cases}$$

□

6. (a) Show the quotient space of a finite collection of disjoint 2-simplices obtained by identifying pairs of edges is always a surface, locally homeomorphic to $\mathbb{R}^2$.

Proof. Let

$$K = \bigsqcup_{i=1}^{m} \Delta_i^2$$

be a finite disjoint union of 2–simplices. Assume that every edge of K is contained in exactly one 2–simplex and is identified with exactly one other edge by a homeomorphism that sends endpoints to endpoints. Let $\sim$ be the equivalence relation generated by these identifications of edges and let

$$q : K \longrightarrow X = K/\!\sim$$

be the quotient map. We must show that every point $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^2$; K is compact; K is Hausdorff; K has countable open basis. Since K is compact, and the quotient map is continuous, K is compact. Next we proof K is local Euclidean.

Case 1. x comes from the interior of a 2–simplex.

Suppose $x = q(p)$ where p is in the interior of some 2–simplex Δ_i^2. Then p is not contained in any identified edge, hence its equivalence class is just $\{p\}$. Choose an open disk $D \subset \Delta_i^2$ with $p \in D$ and $\overline{D} \subset \operatorname{int} \Delta_i^2$. Since no other point of K is equivalent to a point of D, we have $q^{-1}(q(D)) = D$, which is open in K, so $q(D)$ is open in X. The restriction $q|_D : D \to q(D)$ is a homeomorphism, and D is homeomorphic to an open disk in $\mathbb{R}^2$, hence so is $q(D)$. Thus x has a neighborhood homeomorphic to $\mathbb{R}^2$.

Case 2. x comes from the interior of an edge.

Suppose $x = q(p)$ where p lies in the interior of an edge e of some 2–simplex Δ_i^2. Let e' be the edge paired with e, lying in a 2–simplex Δ_j^2, and let

$$\phi : e \longrightarrow e'$$

be the homeomorphism used to identify e with e'.

Choose small open neighborhoods $U_i \subset \Delta_i^2$ and $U_j \subset \Delta_j^2$ of e and e' such that

(a) U_i and U_j are both homeomorphic to an open half–disk, with e and e' corresponding to the diameter,

(b) $U_i \cap U_j = \emptyset$, and

(c) U_i and U_j contain no vertices of the simplices.

By composing the chosen homeomorphisms to half–disks with an appropriate homeomorphism of the boundary diameter, we may assume that, in these coordinates, the identification of e and e' is just the identity map on the diameter of the two half–disks.

Then $U_i \cup U_j$ is homeomorphic to the disjoint union of an upper and a lower open half–disk in $\mathbb{R}^2$, glued along their common diameter. The quotient of this space by identifying the two diameters by the identity map is clearly homeomorphic to an open disk in $\mathbb{R}^2$ (it is exactly the full open disk obtained by gluing two half–disks along the diameter).

Set $U = U_i \cup U_j \subset K$. By construction, every point of $q(U)$ has all of its $\sim$–equivalent points contained in U, so $q^{-1}(q(U)) = U$, which is open in K. Hence $q(U)$ is open in X, and the above description shows that $q(U)$ is homeomorphic to an open disk in $\mathbb{R}^2$. In particular, $x \in q(U)$ has a neighborhood homeomorphic to $\mathbb{R}^2$.

Case 3. x comes from vertices.

Let $x \in X$ be the image of a collection of vertices $v_1, \ldots, v_k$ in K, so that

$$q^{-1}(x) = \{v_1, \ldots, v_k\}.$$

For each v_ℓ choose a small closed disk $B_\ell \subset K$ centered at v_ℓ such that

(a) B_ℓ is contained in the union of the finitely many 2–simplices of K having v_ℓ as a vertex;

(b) for each such 2–simplex, the intersection with B_ℓ is a closed sector (a closed disk cut by two radii);

(c) the disks B_ℓ are pairwise disjoint; and

(d) B_ℓ contains no points of identified edges except those meeting at v_ℓ.

Let

$$B = \bigcup_{\ell=1}^{k} B_\ell, \qquad U = \operatorname{int} B \subset K.$$

As before, by choosing the B_ℓ sufficiently small we may ensure that every point of $q(U)$ has all of its $\sim$–equivalent points contained in U, hence $q^{-1}(q(U)) = U$, which is open in K, so $q(U)$ is open in X. We now analyze the local structure of $q(U)$.

Write $S = q(U)$ and let ∂B be the boundary of B in K. The intersection $\partial B \cap K$ is a union of finitely many arcs, each lying in the interior of an edge of some 2–simplex of K. When we pass to the quotient, these boundary arcs are glued in pairs according to the given identifications of edges. Let

$$L = q(\partial B) \subset S.$$

Then L is a finite 1–dimensional CW complex (a finite graph), which is the *link* of x.

We claim that L is homeomorphic to the circle S^1. First, by definition of the equivalence relation, all vertices $v_1, \ldots, v_k$ lie in a single equivalence class, so the union B is connected in K, hence $S = q(B)$ and $L = q(\partial B)$ are connected.

Next we look at the local degree in L. A point of L coming from the interior of one of the boundary arcs of ∂B has a neighborhood in L homeomorphic to an open interval, so it has degree 2. At the endpoints of the boundary arcs, the arcs meet in pairs coming from the two sides of an edge in K, because in the quotient X each edge is identified with exactly one other edge and thus has exactly two incident 2–simplices. Hence each vertex of L is also incident to exactly two edges. Therefore L is a connected finite graph in which every vertex has degree 2.

A standard graph–theoretic fact says that any finite connected graph in which every vertex has degree 2 is homeomorphic to the circle S^1. Thus $L \cong S^1$.

The set $S = q(U)$ is obtained from L by coning off: each radial segment in B_ℓ from v_ℓ to a point of ∂B_ℓ projects under q to a segment from x to a point of L, and these segments fit together to give a homeomorphism

$$S \;\cong\; \mathrm{Cone}(L)^\circ,$$

the interior of the cone on L. Since $L \cong S^1$, the cone on L is homeomorphic to a closed 2–disk, and its interior is homeomorphic to an open disk in $\mathbb{R}^2$. Thus S is an open neighborhood of x homeomorphic to $\mathbb{R}^2$.

Combining the three cases, we see that every point of X has a neighborhood homeomorphic to $\mathbb{R}^2$, so X is a 2–dimensional topological manifold, i.e. a surface.

In the end, we prove that K is local Euclidean and second-countable.

Each simplex Δ_i^2 is a compact metric space. A finite disjoint union of metric spaces is again metric, so K is a compact metric space; let d_0 be a metric on K.

For $x \in K$ denote its $\sim$–equivalence class by $[x]$. Because only edges are identified in pairs and there are only finitely many simplices and edges, each equivalence class is finite:

- if x lies in the interior of a simplex, then $[x] = \{x\}$;
- if x lies in the interior of an edge, then $[x]$ consists of x and its partner point in the paired edge;
- if x is a vertex, then $[x]$ is a finite set of vertices, since there are only finitely many simplices.

Finite subsets of a Hausdorff space are closed, so each $[x]$ is closed in K.

For distinct classes $[x] \neq [y]$ define

$$\delta([x],[y]) = \inf\{d_0(x',y') \mid x' \in [x],\, y' \in [y]\},$$

and set

$$d([x],[y]) = \begin{cases} 0, & [x] = [y], \\ \min\{1, \delta([x],[y])\}, & [x] \neq [y]. \end{cases}$$

Since K is compact metric and $[x],[y]$ are disjoint closed subsets, the infimum $\delta([x],[y])$ is a strictly positive minimum. Hence d is a well–defined metric on the set X of equivalence classes.

Let $\mathcal{T}_d$ be the topology induced by d and let $\mathcal{T}_q$ be the quotient topology on X. We claim that $\mathcal{T}_d = \mathcal{T}_q$.

First, q is continuous with respect to $\mathcal{T}_d$. Indeed, for any class $[x]$ and $\varepsilon > 0$,

$$q^{-1}\big(B_d([x],\varepsilon)\big) = \bigcup_{z \in [x]} B_{d_0}(z,\varepsilon),$$

which is open in K. By the universal property of the quotient topology, this implies $\mathcal{T}_q \subset \mathcal{T}_d$.

Conversely, let $U \in \mathcal{T}_q$ and pick a class $[x] \in U$. Then $q^{-1}(U)$ is open in K and contains the finite set $[x]$. For each $z \in [x]$ choose $\varepsilon_z > 0$ such that $B_{d_0}(z, \varepsilon_z) \subset q^{-1}(U)$. Let $\varepsilon = \min_{z \in [x]} \varepsilon_z > 0$. We claim that $B_d([x], \varepsilon) \subset U$.

Indeed, if $[y] \in B_d([x], \varepsilon)$, then by definition of d there exist $y' \in [y]$ and $z \in [x]$ with $d_0(y', z) < \varepsilon \leq \varepsilon_z$. Thus $y' \in B_{d_0}(z, \varepsilon_z) \subset q^{-1}(U)$, and hence $[y] = q(y') \in U$. So every point $[x] \in U$ has a d–ball contained in U, which shows that $U \in \mathcal{T}_d$. Therefore $\mathcal{T}_d \subset \mathcal{T}_q$, and we conclude that $\mathcal{T}_d = \mathcal{T}_q$.

Thus X is metrizable. Every metric space is Hausdorff and second countable, so X is a Hausdorff, second countable space that is locally homeomorphic to $\mathbb{R}^2$. Therefore X is a 2–dimensional topological manifold, i.e. a surface. $\square$

(b) Show the edges can always be oriented so as to define a Δ-complex structure on the quotient surface. [This is more difficult.]

Proof. Let $K = \bigsqcup_{i=1}^m \Delta_i^2$ be the finite disjoint union of 2–simplices and let $X = K/\sim$ be the quotient obtained by identifying the edges in pairs, as in (a). By part (a) the space X is a surface. We shall orient the edges of X and then use these orientations to construct a Δ–complex structure on X.

Step 1: Vertices and edges. Let V be the finite set of vertices of X, i.e. the images in X of the vertices of the simplices in K. Choose once and for all a total ordering of V,

$$v_1 < v_2 < \cdots < v_r.$$

Let E be the set of edges of X, i.e. the images of the edges of K under the quotient map $q\colon K \to X$. Each $e \in E$ is homeomorphic to a closed interval and has two (not necessarily distinct) endpoints in V.

For an edge $e \in E$ whose endpoints are distinct vertices $v_i \neq v_j$, we orient e from the smaller to the larger endpoint, i.e. from v_i to v_j if $i < j$. If e is a loop edge, so both endpoints equal some vertex v_i, we choose an arbitrary orientation on e.

For each oriented edge $e \in E$ with initial vertex a and terminal vertex b, choose a homeomorphism

$$\sigma_e\colon \Delta^1 \longrightarrow e$$

such that $\sigma_e(0) = a$ and $\sigma_e(1) = b$. Since distinct edges in E intersect only in their endpoints, the restrictions of the maps σ_e to the interiors of Δ^1 are embeddings with pairwise disjoint images.

Step 2: Triangles and their vertex orderings.

Let F be the set of 2–simplices of X, i.e. the images of the simplices Δ_i^2 in K. For each $f \in F$ its boundary in X is a union of three edges of E, and f is homeomorphic to a closed 2–disk.

Let $f \in F$ have vertices (in X) $w_0, w_1, w_2 \in V$, not necessarily distinct. After renumbering, we may assume

$$w_0 \leq w_1 \leq w_2$$

with respect to the fixed total order on V. We now describe a map from the standard 2–simplex to f.

Denote by Δ^2 the standard simplex with vertices e_0, e_1, e_2, and let

$$\delta^0, \delta^1, \delta^2\colon \Delta^1 \longrightarrow \Delta^2$$

be the three standard face inclusions, so that $\delta^0(\Delta^1)$ is the edge opposite e_0, etc. The three edges of f have endpoints $\{w_1, w_2\}$, $\{w_0, w_2\}$ and $\{w_0, w_1\}$ respectively. By Step 1 we have already chosen oriented edge maps

$$\sigma_{w_i w_j}\colon \Delta^1 \to e_{ij} \subset X$$

for each unordered pair $\{w_i, w_j\}$, where e_{ij} is the corresponding edge of f and $\sigma_{w_i w_j}(0)$, $\sigma_{w_i w_j}(1)$ are the ordered endpoints (from smaller to larger).

Define a map

$$\varphi_f\colon \partial\Delta^2 \longrightarrow \partial f \subset X$$

by requiring that on each side we have

$$\varphi_f\big|_{\delta^0(\Delta^1)} = \sigma_{w_1w_2}, \qquad \varphi_f\big|_{\delta^1(\Delta^1)} = \sigma_{w_0w_2}, \qquad \varphi_f\big|_{\delta^2(\Delta^1)} = \sigma_{w_0w_1}.$$

By construction the three edges e_{01}, e_{02}, e_{12} occur in ∂f in exactly the same cyclic order as the sides of $\partial\Delta^2$, and each is parametrized once, so $\varphi_f\colon \partial\Delta^2 \to \partial f$ is a homeomorphism.

Since f is a 2–disk and ∂f is its boundary circle, any homeomorphism of $\partial\Delta^2 \cong S^1$ to ∂f extends to a homeomorphism of the whole disk. Hence there exists a homeomorphism

$$\sigma_f\colon \Delta^2 \longrightarrow f$$

whose restriction to the boundary is exactly φ_f. In particular, for each $k = 0, 1, 2$ the composition

$$\sigma_f \circ \delta^k\colon \Delta^1 \longrightarrow X$$

is one of the previously chosen edge maps σ_e.

Step 3: Verification of the Δ–complex axioms. We claim that the collection of all maps

$$\{\sigma_f : \Delta^2 \to X \mid f \in F\} \cup \{\sigma_e : \Delta^1 \to X \mid e \in E\}$$

together with the inclusion maps of the vertices $v_i \in V$ defines a Δ–complex structure on X in the sense of Hatcher.

First, for each $f \in F$ the restriction of σ_f to the interior of Δ^2 is a homeomorphism onto the interior of f, and the interiors of distinct f's are disjoint, since the interiors of the original simplices in K were disjoint and the quotient map q is injective on these interiors. Similarly, each σ_e restricts to a homeomorphism from the interior of Δ^1 onto the interior of e, and the interiors of distinct edges are disjoint.

Second, by the boundary construction above, the faces of each 2–simplex are exactly the 1–simplices we have chosen: for every $f \in F$ and every face inclusion $\delta^k\colon \Delta^1 \to \Delta^2$ we have $\sigma_f \circ \delta^k = \sigma_e$ for some $e \in E$. The endpoints of each σ_e are vertices in V, so the faces of 1–simplices are 0–simplices.

Third, every point of X lies in the image of one of these simplices: the interiors of 2–simplices, the interiors of edges, and the vertices cover X.

Finally, the topology on X is the quotient topology from K. As in Hatcher, this coincides with the weak topology determined by the above maps σ_f and σ_e, since K is obtained by gluing standard simplices along their faces and q is obtained by identifying faces in pairs. Thus the axioms of a Δ–complex are satisfied.

Therefore the oriented edges, together with the maps σ_f, give X the structure of a Δ–complex whose 2–simplices are the given triangles. $\square$