Math 751 HW 2

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Chapter 1.1.2: Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. We may suppose X is a path-connected space, otherwise we can restrict this statement in one of X path-connected component. Let x, y be its two basepoints. Suppose we have two homotopic path $h_1, h_2 : [0,1] \to X,\ h_1(0) = h_2(0) = x,\ h_1(1) = h_2(1) = y$. Let $g^{'} : [0,1] \to X, g^{'}(0) = g^{'}(1) = y$. Then $\beta_{h_1}([g^{'}]) = [\overline{h_1} * g^{'} * h_1] = [\overline{h_1}][g^{'}][h_1] = [\overline{h_2}][g^{'}][h_2] = \beta_{h_2}([g^{'}])$. □

Chapter 1.1.5: Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. (i): $a \Rightarrow b$: Let $f : S^1 \to X$ is a continuous map. Noticing that D^2 is homeomorphic to $C = S^1 \times I / S^1 \times 1$. We actually need to extends f to C. Since f is homotopic to a constant map $g : S^1 \to x_0, x_0 \in X$, we have a homotopy $F : S^1 \times I \to X, F(S^1, 0) = f, F(S^1, 1) = g$. Now we define a continuous map $f^{'} : C \to X$ by defining $f^{'}(\overline{x}) = F(x), x \in S^1 \times I, \overline{x} \in C$. Then $f^{'}$ is well defined because $F(S^1, 1) = x_0$ and continuous.

(ii) : $b \Rightarrow c$: We may suppose X is a path-connected space, otherwise we can restrict this statement in one of X path-connected components. Let $p : I = [0,1] \to S^1 = [0,1]/0 \sim 1$ be the quotient map. Let $g : I = [0,1] \to X, g(0) = g(1) = x_0$. Then we can construct a continuous map $f : S^1 \to X$, such that $f \circ p = g$. In fact, we can let $f(\overline{x}) = g(x),\ x \in X,\ \overline{x} \in S^1$. Now we know we can extend f to $f^{'} : D^2 \to X$ such that $f^{'}|_{S^1} = f$. So we have following diagram which commutes:

$$\begin{tikzcd} D^2 \arrow[r, "f^{'}"] & X \\ S^1 \arrow[u, "\iota"] \arrow[ur, "f"] & \end{tikzcd}$$

. This actually induces a commutative diagram between the fundamental groups:

$$\begin{tikzcd} \pi_1(D^2, \iota \circ p(0)) \arrow[r, "f^{'}_*"] & \pi_1(X, x_0) \\ \pi_1(S^1, p(0)). \arrow[u, "\iota_*"] \arrow[ur, "f_*"] & \end{tikzcd}$$

. This indicates f_* is a zero map because D^2 is simply connected. So, $[f \circ p]$ is the identity element of $\pi_1(X, x_0)$, because $[p]$ is a loop of S^1. Since $g = f \circ p$, $[g]$ is also the identity element of $\pi_1(X, x_0)$. Since g is a random continuous map, $\pi_1(X, x_0)$ is trivial.

(iii): $c \Rightarrow a$: Let $f : S^1 \to X$ be a continuous map. Let $p : I = [0,1] \to I/0 \sim 1 = S^1$ be the quotient map. Then $[f \circ p]$ is an element of $\pi_1\big(X, x_0 = f \circ p(0)\big) = 0$. So, $f \circ p$ is homotopic to a constant map $\epsilon : I \to X, \epsilon(I) = x_0$. Let $\epsilon^{'} : S^1 \to X, \epsilon^{'}(S^1) = x_0$. Then we know $f \circ p \simeq \epsilon^{'} \circ p = \epsilon$. Suppose the homotopy rel $0, 1$ between $f \circ p, \epsilon$ is $H : I \times I \to X, H(t, 0) = f \circ p(t), H(t, 1) = \epsilon(t)$. We can built a homotopy $H^{'}(\overline{x}, t) = H(x, t), x \in I, \overline{x} \in S^1$ between f and $\epsilon^{'}$. This is well-defined, because H is a homotopy rel $\{0, 1\}$ □

Chapter: 1.1.6: We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map

$$\Phi : \pi_1(X, x_0) \to [S^1, X]$$

obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof. (i) : Suppose X is path-connected, we need to prove Φ is surjective. Let $h : S^1 \to X$. We may suppose $x_0 \notin h(S^1)$, otherwise we can end this proof. Let l be a path from x_0 to y, then we claim $\Phi(l * h * \bar{l}) \simeq h$. Let $S^1 = e^{2\pi i t}, t \in [0, 1)$. Then we construct a homotopy $H : S^1 \times I \to X$. Let $H(e^{2\pi i t_1}, t_2) = \begin{cases} l\big((1-3t_1)t_2 + 3t_1\big), \ t_1 \in [0, \frac{1}{3}] \\ h(e^{2\pi i(3t_1 - 1)}), \ t_1 \in [\frac{1}{3}, \frac{2}{3}] \\ l^{-1}\big((1-t_2)(3t_1 - 2)\big), \ t_1 \in [\frac{2}{3}, 1] \end{cases}$, them H is well-defined and continuous. Let $l' : S^1 \to y \in X$ be a constant map. Then we know $l * h * \bar{l} \simeq l' * h * \overline{l'} \simeq h$.

(ii) $\Rightarrow$: Suppose $\Phi[f] = \Phi[g]$. Then we know there is a homotopy $H : S^1 \times I \to X, H(s, 0) = f(s), H(s, 1) = g(s)$. Here we use $e^{2\pi i t}, t \in [0, 1)$ to express the circle. And, we may assume $e^{2\pi i 0} = s_0$. Let this homotopy restrict to $(s_0, t), t \in [0, 1]$, then we can get a loop $h(t) = H(s_0, t), t \in [0, 1]$ with basepoint on x_0. Then we can get a homotopy $H' : S^1 \times I \to X$ of f and $h * g * \overline{h}$, be defining $H'(e^{2\pi i t_1}, t_2) = \begin{cases} H(s_0, t_2 \cdot 3t_1), 0 \le t_1 \le \frac{1}{3} \\ H(e^{2\pi i(3t_1 - 1)}, t_2), \frac{1}{3} \le t_1 \le \frac{2}{3} \\ H(s_0, t_2(-3t_1 + 3)), \frac{2}{3} \le t_1 < 1 \end{cases}$

. Noticing that $H'(s_0, t) = x_0, H'(s, 0) = f(s), H'(s, 1) = h * g(s) * \overline{h}$.

$\Leftarrow$: Suppose $[f] = [h][g][h^{-1}]$, $f, g, h : S^1 \to X$, and we may assume $f(e^0) = g(e^0) = h(e^0) = x_0$, otherwise we can rotate the circle(Here we continue use $e^{2\pi i t}$ to express the circle). It suffice to prove $h * g * h^{-1} \simeq g$. We construct the homotopy H like what we did in (i) as follows: $H(e^{2\pi i t_1}, t_2) = \begin{cases} h\Big(e^{\big((1-3t_1)t_2 + 3t_1\big)2\pi i}\Big), \ t_1 \in [0, \frac{1}{3}] \\ h\big(e^{2\pi i(3t_1 - 1)}\big), \ t_1 \in [\frac{1}{3}, \frac{2}{3}] \\ h^{-1}\Big(e^{\big((1-t_2)(3t_1 - 2)\big)2\pi i}\Big), \ t_1 \in [\frac{2}{3}, 1] \end{cases}$ ☐

Chapter: 1.1.11: If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism

$$\pi_1(X_0, x_0) \to \pi_1(X, x_0).$$

Proof. First, we prove this is injective. Let $\iota : X_0 \to X$ be the injection. Let $[f] \in \pi_1(X_0, x_0)$, and suppose $\iota_*([f])$ is the identity element of $\pi_1(X, x_0)$. Then we have a homotopy $H : I \times I \to X$, such that $H(t, 0) = f(t),\ H(t, 1) = x_0,\ H(0, t) = H(1, t) = x_0$. Since $I \times I$ is path-connected and $x_0 \in H(I, I)$, $im(H) \subset X_0$. So, $[f]$ is the identity element of $\pi_1(X_0, x_0)$. Now, we prove this is surjective. For any $f : I \to X,\ f(0) = f(1) = x_0$, we have $im(f) \subset X_0$, since I is path-connected. So $\{f : I \to X, f(0) = f(1), f \text{ is continuous}\} = \{f : I \to X_0, f(0) = f(1), f \text{ is continuous}\}$, which indicates ι_* is surjective. ☐

chapter: 1.1.15: Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that $f_*\beta_h = \beta_{fh} f_*$ in the diagram at the right.

$$\begin{tikzcd} \pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"] & \pi_1(Y, f(x_0)) \end{tikzcd}$$

Proof. We may assume X, Y is path-connected, otherwise we can restrict the discussion to a path-connected component $X^{'} \subset X$ and $f(X^{'}) \subset Y$. Let $[g] \in \pi_1(X, x_0)$. $f_*\beta_h([g]) = f_*[h * g * \overline{h}] = [f \circ h] * [f \circ g] * [f \circ \overline{h}]$. $\beta_{fh} f_*([g]) = \beta_{fh}[f(g)] = [f \circ h][f \circ g][\overline{f \circ h}] = [f \circ h][f \circ g][f \circ \overline{h}] = f_*\beta_h([g])$. $\square$

Chapter 1.2.4: Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1\big(\mathbb{R}^3 - X\big)$.

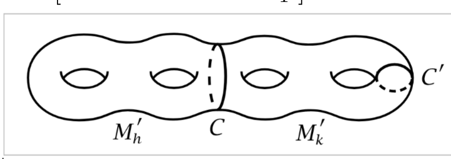
Proof. First we can deformation retract $\mathbb{R}^3 - X$ to S^2 without n pairs antipodal points by using $H(v,t) = \frac{v}{||tv||+(1-t)}, v \in \mathbb{R}^3, t \in [0,1], || \cdot ||$ is norm. So, we only need to compute the foundamental group of S^2_m, which is S^2 without $m > 1, m \in \mathbb{Z}$ points. Also, we know S^2 removed one point is homeomorphic to plane $\mathbb{R}^2$. So, we acutually need to compute the foundamental group of the plane removed $m-1$ points. Since this space is acutually homotopy equivelent to $\bigvee_{i=1}^{m-1} S^1$. From the class we know $\pi_1(\bigvee_{i=1}^{m-1} S^1) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{m-1}$. So, $\pi_1(\mathbb{R}^3 - X) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1}$.

$\square$

Chapter 1.2.7: Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Proof. Observing that we can also get X by collapsing one Longitude of $\mathbb{T}^2$. So, we may get a CW structure inherited from $\mathbb{T}^2$'s CW structure. We have one 0 cell, one 1 cell and one 2 cell. And its 1 skeleton is S^1. So, $\pi_1(X)$ has one generators a, b. Noticing that the attach map of the 2 cell is actually wraping S^1 and wraping it again in a reverse direction. So, it gives a relation that $aa^{-1} = 0$ which is a trivial relation. So, $\pi_1(X) = \mathbb{Z}$. $\square$

Chapter 1.2.9: In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M_h' and M_k' obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M_h' does not retract onto its boundary circle C, and hence M_g does not retract onto C. [*Hint:* abelianize π_1.] But show that M_g does retract onto the nonseparating circle C' in the figure.



Proof. We may suppose $M_h^{'}$ has genus h, and $M_k^{'}$ has genus k. From the class we know $M_h^{'}$ can be deformation retracted to $\bigvee_{i=1}^{2h} S^1$, so $\pi_1(M_h^{'}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2h}$. Lei $\iota : C \to M_h^{'}$ denote the inclusion, which also induce a homomorphism $\iota^* : \mathbb{Z} = \pi_1(C) \to \pi_1(M_h^{'})$. Suppose we have a retraction from $M_h^{'} \to C$, and denote the induced homomorphism by r^*. Then $r^* \circ \iota^* : \mathbb{Z} \to \mathbb{Z}$ should be identity. We first calculate that $\iota^*(\alpha) = [a_1, b_1][a_2, b_2] \cdots [a_h, b_h]$ a_i, b_i are generators of $\pi_1(M_h^{'})$, α is the generator of $\pi_1(C)$. Noticing that $\pi_1(M_h^{'})$ is not abelian, but $\pi_1(C))$ is abelian. So, $[\pi_1(M_h^{'}), \pi_1(M_h^{'}] \subset \ker(r^*)([G,G]$ denote the commutator subgroup.) So, $r^* \circ \iota^*$ is zero instead of identity. This is a contradiction.

Now we prove there is a retract from M_g to $C^{'}$. Take a small tubular neighborhood A of C'; it is an annulus $A \cong S^1 \times [-1,1]$ whose core $S^1 \times \{0\}$ is exactly C'. suppose $B = \overline{M_g \setminus \operatorname{int}(A)}$ is connected, and its boundary in M_g is $\partial B = (S^1 \times \{-1\}) \sqcup (S^1 \times \{+1\}) \subset A$.

First, we choose a basepoint $x_0 \in C'$ and define a constant map $r_B : B \to C'$ by $r_B \equiv x_0$.
Identify S^1 with the unit circle in $\mathbb{C}$ and write $x_0 = e^{i\theta_0}$. Define

$$r_A : S^1 \times [-1,1] \longrightarrow S^1, \qquad r_A\big(e^{i\theta}, t\big) = \frac{(1-|t|)\, e^{i\theta} + |t|\, e^{i\theta_0}}{\big|(1-|t|)\, e^{i\theta} + |t|\, e^{i\theta_0}\big|}.$$

Then $r_A(e^{i\theta},0) = e^{i\theta}$ (identity on the core C'), and $r_A(e^{i\theta},\pm 1) = e^{i\theta_0} = x_0$ (both boundary circles collapse to x_0). Thus r_A retracts A onto C' while fixing C' pointwise and sending ∂A to x_0.

Last, we define $r : M_g \to C'$ by $r\big|_A = r_A, \quad r\big|_B = r_B$. On the overlap $\partial B \subset A$ both pieces take the value x_0, so r is continuous. Moreover $r|_{C'} = \mathrm{id}_{C'}$. Hence r is a retraction of M_g onto C'.

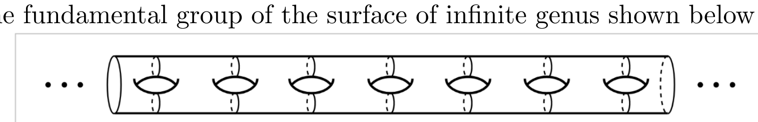
$\square$

Chapter 1.2.11: The *mapping torus* T_f of a map $f\colon X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x,0)$ with $(f(x),1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_*\colon \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. [One way to do this is to regard T_f as built from $X \vee S^1$ by attaching cells.]

Proof. FIrst suppose $X = S^1 \vee S^1$. Let f_* denoote the group homomorphism induced from f. Let s_0 denote the point where S^1 "wedge sum" with another S^1. Then we can cut through $s_0 \times I/(s_0,0) \sim (f(s_0),1)$. Then we can get the Cell complexes structure of $T_f(S^1 \vee S^1)$. First, we can observe that it only has one 0-cell s_0(since f is basepoint-preserving), three 1-cells d_1, d_2, d_3, two 2- cells B_1, B_2. And attaches two ends of the d_1, d_2, d_3 to x_0. So $T_f(S^1 \vee S^1)$'s 1-skeleton is homotopy equivalent to $S^1 \vee S^1 \vee S^1$, which indicates its foundamental group is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Let a, b, c denote its generators. Now we can consider the attach map Φ_1, Φ_2 of 2-cells. We may assume B_1 is a rectangle with four sides $r_1, r_2.r_3.r_4$. Then $\Phi(r_1)$ is wrapping around the d_1, and $\Phi(r_2)$ is wrapping d_3, and $\Phi(r_3) = f\big(\Phi(r_1)\big)$ in a reverse direction, and $\Phi(r_4)$ is wrapping d_3 in the reverse direction. So, once we give this rectangle an orientation, it will donate a relation $acf_*(a)^{-1}c^{-1}$. Similarly, another 2-cell will give a relation $bcf_*(b)^{-1}c^{-1}$. So, $\pi_1(T_f(S^1 \vee S^1)) =< a, b, c | acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1} >$

Now we may suppose $X = S^1 \times S^1 = \mathbb{T}^2$. We now give $T_f(X)$ a cell complexes structure. It suffice to construct the two-skeleton of $T_f(X)$. Now we have 1 0-cell s_0 (since f is a basepoint preserving map), still three 1-cells d_1, d_2, d_3, but three 2-cells B_1, B_2, B_3, one of which are inherited from the cell complexes structure of $\mathbb{T}^2$. So, $\pi_1(T_f(X))$ should have three generators called a, b, c(Here we may view a, b also as the generators of $\pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$, which is reasonable. Because two of the 1-cells actually comes from the 1-skeleton of $\mathbb{T}^2$). One 2-cell give a relation of $aba^{-1}b^{-1}$ just like $\mathbb{T}^2$(This is actually inherited from $\mathbb{T}^2$). Another two donate relations of $acf_*(a)^{-1}c^{-1}$ and $bcf_*(b)^{-1}c^{-1}$ like the situation of $X = S^1 \vee S^1$. So, $\pi_1(T_f(\mathbb{T}^2)) =< a, b, c | ab = ba, bcf_*(b)^{-1}c^{-1}, acf_*(a)^{-1}c^{-1} >$ $\square$

Chapter 1.2.16: Show that the fundamental group of the surface of infinite genus shown below is free on



an infinite number of generators.

Proof. Consider the connected sum X^n of n Torus $\mathbb{T}^2$. First, we remove an open disk on the left surface of the left end of X^n. We know it can be deformation retracted to $\underbrace{S^1 \vee \cdots \vee S^1}_{2n}$. Then we remove another open disk of X^n on the right of the surface of the right end of X^n. This action will add a new S^1 to its 1 skeleton. So, after these two actions, we can get a new topological space X^n_* which is compact and can be deformation retracted to $\underbrace{S^1 \vee \cdots \vee S^1}_{2n+1}$. So, its foundamental group is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n+1}$. Now, we prove the foundamental group of the surface of infinite genus X is free on an infinite generators. Let $\gamma : [0,1] \to X,\ \gamma(0) = \gamma(1) = x_0$. Since $[0,1]$ is compact, so $\gamma([0,1]) \subset X$ is compact. Since, X is Hausdorff, it is also closed. This indicates that we can find a X^n_* such that $x_0 \in X^n_*$ $, \gamma([0,1]) \subset X^n_*$. We know the foundamental group of X^n_* is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n+1}$(We may assume their generators are $G^n = \{a_i, b_i, c | 1 \le i \le n\}$), so there exists a homotopy $H : I \times I \to X^n_*$ between γ and a free product of some a_i, b_i, c, which preserve the basepoint x_0. This is actually also a homotopy in X. Now, we may view X as $\bigcup\limits_{n \in \mathbb{N}^*} X^n_*$, and Let $G = \bigcup\limits_n G^n$, so we just show G is the set of generators of $\pi_1(X)$. Now, we eed to prove there is no relation in $\pi_1(X)$ to show it is free. suppose $\gamma : I \to X,\ \gamma(0) = \gamma(1) = x_0$. Now suppose it can be written as some free product of some a_i, b_i, c. Now we suppose $[\gamma]$ is zero in $\pi_1(X)$, indicating there exist a homotopy $H : I \times I \to X$ between γ and the constant map, which preserves the basepoint x_0. Since $I \times I$ is a compact set, so $H(I \times I)$ is also a subset of X^m_* for a big enough $m \in N^*$, such that a_i, b_i, c is also generators of $\pi_1(X_*^m)$(This is true, because γ can be only written as a finitely product). But we have shown that for all $n \in \mathbb{N}^*$, we have $\pi_1(X_*^n)$ is a free group. This indicates γ must be an empty word, which means there is no relation in $\pi_1(X) \Rightarrow \pi_1(X)$ is a free group of infinitely generators. $\square$