Math 761 HW 4

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1. Show that the map
$$\mathbb{R}^n \ni \vec{v} \longmapsto D\vec{v}\big|_p \in T_p\mathbb{R}^n$$
is surjective, i.e. that every derivation comes from a directional derivative. *Hint:* To find $\vec{v}$, apply $w \in T_p\mathbb{R}^n$ to the coordinate functions. Then for any smooth function f use Taylor's theorem on f and the Leibniz property.

Proof. We can prove this by showing $\{\frac{\partial}{\partial x_i}|_p, 1 \leq i \leq n\}$ is a basis of $T_p\mathbb{R}^n$. First, they are linear independent. If $D = \sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial x_i}|_p = 0$, then $D(x_j) = 0, 1 \leq j \leq n \Rightarrow a_i = 0$. So, they are linear independent. Now let $D \in T_p\mathbb{R}^n$, and $p = (p_1, \cdots, p_n)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then we can use Taylor Theorem: $f = f(p) + \sum\limits_{i=1}^{n}(x_i - p_i)\int_0^1 \frac{\partial f}{\partial x_i}\big(p + t(x-p)\big)dt \Rightarrow D(f) = \sum\limits_{i=1}^{n} D(x_i - p_i)\int_0^1 \frac{\partial f}{\partial x_i}(p)dt + (p_i - p_i)D\left(\int_0^1 \frac{\partial f}{\partial x_i}\big(p + t(x-p)\big)dt\right) = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}|_p(f)D(x_i)$. Since f is a random smooth function, we know $D = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}|_p D(x_i)$. So, $\{\frac{\partial}{\partial x_i}, 1 \leq i \leq n\}$ is a basis of $T_p\mathbb{R}^n$, which means this map is surjective. □

2. Prove Proposition 3.6 from Lee (*Properties of Differentials*). Let M, N, P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

Here we use the definition of Lee's book. We define the $T_p(M)$ as the collection of linear map $v : C^\infty(M) \to \mathbb{R}$ which satisfies $v(fg) = f(p)v(g) + g(p)v(f)\ \ f, g \in C^\infty(M)$.
And we define $dF_p : T_pM \to T_{F(p)}N$ by $dF_p(v)(f) = v(f \circ F)\quad f \in C^\infty(N), v \in T_pM$.

   (a) $dF_p : T_pM \to T_{F(p)}N$ is linear.

   Proof. $dF_p(v_1 + v_2)(f) = (v_1 + v_2)(f \circ F) = v_1(f \circ F) + v_2(f \circ F) = dF_p(v_1)(f) + dF_p(v_2)(f)\quad f \in C^\infty(N), v_1, v_2 \in T_pM \Rightarrow dF_p(v_1 + v_2) = dF_p(v_1) + dF_p(v_2)$. $dF_p(av)(f) = av(f \circ F) = a \cdot dF_p(v)(f)\quad f \in C^\infty(N), v \in T_pM,\quad a \in \mathbb{R} \Rightarrow dF_p(av) = a \cdot dF_p(v)$. So, $dF_p : T_pM \to T_{F(p)}N$ is linear. □

   (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}P$.

   Proof. $d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = v(f' \circ F) = dF_p(v)(f') = dF_p(v)(f \circ G) = dG_{F(p)} \circ dF_p(v)(f)\quad v \in T_p(M),\ f \in C^\infty(P) \Rightarrow d(G \circ F)_p = dG_{F(p)} \circ dF_p$

   □

   (c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

   Proof. $d(\mathrm{Id}_M)_p(v)(f) = v(f \circ Id_M) = v(f)\quad v \in T_p(M), f \in C^\infty(M) \Rightarrow d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$. □

   (d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and
$$(dF_p)^{-1} = d(F^{-1})_{F(p)}.$$

   Proof. $dF_p \circ d(F^{-1})_{F(p)} = d(F \circ F^{-1})_{F(p)} = d(\mathrm{Id}_N)_{F(p)} = \mathrm{Id}_{T_{F(p)}N}$. On the other direction, $d(F^{-1})_{F(p)} \circ dF_p = d(F^{-1} \circ F)_p = d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$. This indicates $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$. □

3. Note that if M is a smooth manifold and $U \subset M$ is open, then U is also a smooth manifold with smooth structure inherited from M (i.e. charts on U are given by intersecting U with charts on M). We say U is an *open submanifold* of M. Show that for $p \in U$ there is an isomorphism

$$T_pM \;\cong\; T_pU.$$

Proof. Let $\iota : U \to M$ be the inclusion map. We claim $d\iota_p$ is the isomorphism $T_pU \to T_pM$. Here we continue to use the definition of Lee.

First, we prove a lemma: Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

Proof. Let $h = f - g$, so that h is a smooth function that vanishes in a neighborhood U of p. Let $\psi \in C^\infty(M)$ be a smooth bump function that is identically equal to 1 on the support of h(This is true, because we can find a closed set $p \in A \subset U$(from class), which is disjoint with *Supp* (h). Then we can use HW3 problem 5) and is supported in $M \setminus \{p\}$. Because $\psi \equiv 1$ where h is nonzero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, $vh \;=\; v(\psi h) \;=\; 0$. By linearity, this implies $vf = vg$. □

Injectivity: suppose $v \in T_pU$ and $d\iota_p(v) = 0 \in T_pM$. Let B be a neighborhood of p such that $\overline{B} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, the extension lemma for smooth functions guarantees that there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} \equiv f$ on $\overline{B}$(application of partition of unity from class). Then since f and $\tilde{f}\,|_U$ are smooth functions on U that agree in a neighborhood of p, the lemma we proved implies $vf \;=\; v(\tilde{f}\,|_U) \;=\; v(\tilde{f} \circ \iota) \;=\; d\iota(v)_p\,\tilde{f} \;=\; 0$. Since this holds for every $f \in C^\infty(U)$, it follows that $v = 0$, so $d\iota_p$ is injective.

On the other hand, to prove surjectivity, suppose $w \in T_pM$ is arbitrary. Define an operator $v : C^\infty(U) \to \mathbb{R}$ by setting

$$vf \;=\; w\,\tilde{f},$$

where $\tilde{f}$ is any smooth function on all of M that agrees with f on $\overline{B}$(We have proved the existence of $\tilde{f}$). By the lemma we proved, vf is independent of the choice of $\tilde{f}$, so v is well defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$,

$$d\iota_p(v)\,g \;=\; v(g \circ \iota) \;=\; w(\,\tilde{g} \circ \iota\,) \;=\; wg,$$

where the last two equalities follow from the facts that $g \circ \iota$, $\tilde{g} \circ \iota$, and g all agree on $\overline{B}$. Therefore, $d\iota_p$ is also surjective. □

4. Show that for a vector space V (with its canonical smooth structure) and any point $p \in V$, the tangent space T_pV is canonically isomorphic to V.

Proof. We will construct a map $\Psi : V \to T_pV$ and prove that it is a linear isomorphism. This construction will be entirely independent of any choice of basis. We define the map $\Psi : V \to T_pV$ as follows: For any $v \in V$, $\Psi(v)$ is an element of T_pV (i.e., a derivation) whose action on any smooth function $f \in C^\infty(V)$ is given by:

$$[\Psi(v)](f) := \left.\frac{d}{dt}\right|_{t=0} f(p + tv)$$

(Noticing that $f(p + tv)$ is actually a smooth function on $\mathbb{R}$) This definition calculates the directional derivative of the function f at the point p in the direction v. We need to prove that for any $v \in V$, the operator $\Psi(v)$ we defined satisfies the two conditions for an element of T_pV (linearity and the Leibniz rule). (i): For any constants $a, b \in \mathbb{R}$ and functions $f, g \in C^\infty(V)$:

$$\begin{aligned}
[\Psi(v)](af + bg) &= \left.\frac{d}{dt}\right|_{t=0} (af + bg)(p + tv) \\
&= \left.\frac{d}{dt}\right|_{t=0} (af(p + tv) + bg(p + tv)) \\
&= a\left.\frac{d}{dt}\right|_{t=0} f(p + tv) + b\left.\frac{d}{dt}\right|_{t=0} g(p + tv) \quad \text{(by linearity of the derivative)} \\
&= a[\Psi(v)](f) + b[\Psi(v)](g)
\end{aligned}$$

Thus, $\Psi(v)$ is a linear operator.
(ii):

$$\begin{aligned}
[\Psi(v)](f \cdot g) &= \left.\frac{d}{dt}\right|_{t=0} (f \cdot g)(p + tv) = \left.\frac{d}{dt}\right|_{t=0} (f(p + tv) \cdot g(p + tv)) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} f(p + tv)\right) \cdot g(p + 0) + f(p + 0) \cdot \left(\left.\frac{d}{dt}\right|_{t=0} g(p + tv)\right) \quad \text{(by the product rule)} \\
&= ([\Psi(v)](f)) \cdot g(p) + f(p) \cdot ([\Psi(v)](g))
\end{aligned}$$

The Leibniz rule is also satisfied. Therefore, our defined $\Psi(v)$ is indeed a tangent vector in T_pV. Now we must prove that the map Ψ itself is a linear isomorphism.

(a) :We need to show that $\Psi(av_1 + bv_2) = a\Psi(v_1) + b\Psi(v_2)$. This means they are the same operator, i.e., they have the same effect on any function f.

$$\begin{aligned}
[\Psi(av_1 + bv_2)](f) &= \left.\frac{d}{dt}\right|_{t=0} f(p + t(av_1 + bv_2)) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(p + (at)v_1 + (bt)v_2) \\
&= a\left(\left.\frac{d}{ds}\right|_{s=0} f(p + sv_1)\right) + b\left(\left.\frac{d}{ds}\right|_{s=0} f(p + sv_2)\right) \quad \text{(by the chain rule)} \\
&= a[\Psi(v_1)](f) + b[\Psi(v_2)](f) = [a\Psi(v_1) + b\Psi(v_2)](f)
\end{aligned}$$

Since this holds for all f, we have $\Psi(av_1 + bv_2) = a\Psi(v_1) + b\Psi(v_2)$, so Ψ is linear.

(b) : To prove injectivity, we show its kernel is $\{0\}$. That is, if $\Psi(v) = 0$ (the zero derivation), then v must be the zero vector.
$\Psi(v) = 0$ means that for all smooth functions $f \in C^\infty(V)$, we have $[\Psi(v)](f) = 0$. Let $\lambda : V \to \mathbb{R}$ be any linear functional (i.e., $\lambda \in V^*$, the dual space of V). Linear functionals are smooth functions. Apply $\Psi(v)$ to $f = \lambda$:

$$[\Psi(v)](\lambda) = \left.\frac{d}{dt}\right|_{t=0} \lambda(p + tv)$$

Because λ is linear, $\lambda(p + tv) = \lambda(p) + t\lambda(v)$.

$$[\Psi(v)](\lambda) = \left.\frac{d}{dt}\right|_{t=0} (\lambda(p) + t\lambda(v)) = \lambda(v)$$

If $\Psi(v) = 0$, then $[\Psi(v)](\lambda) = \lambda(v) = 0$. This must hold for **all** linear functionals $\lambda \in V^*$. From a fundamental result in linear algebra, if the image of a vector v is zero under every linear functional, then the vector itself must be the zero vector. Thus, $v = 0$. This proves that Ψ is injective.

(c) : We already know that $\dim(T_pV) = \dim(V)$. Ψ is a linear map from V to T_pV. For two finite-dimensional vector spaces of the same dimension, an injective linear map is necessarily also surjective, and thus is an isomorphism.

In summary, Ψ is a linear isomorphism. Because we made no choice of basis during this entire construction, this isomorphism is canonical.

$\square$

5. Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

Let $p = (1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p .$$

Proof. Let $\psi(x, y) = (x, y - x^3) : \mathbb{R}^2 \to \mathbb{R}^2$. Then $\phi \circ \psi = \psi \circ \phi = \mathrm{Id}$. Also, ϕ, ψ are smooth, because $x, y + x^3, y - x^3$ are some polynomials of x, y. So, ϕ is a diffeomorphism of $\mathbb{R}^2$ and $\mathbb{R}^2$, which indicating ϕ is a global smooth coordinates. Let $f(x, y) = y : \mathbb{R}^2 \to \mathbb{R}$. Then $\frac{\partial}{\partial \tilde{x}}|_p(f) = d\psi|_{\phi(p)}(\frac{\partial}{\partial x})(f) = \frac{\partial}{\partial x}|_{\phi(p)}(f \circ \psi) = \frac{\partial}{\partial x}|_{(1,1)}(y - x^3) = -3$. But $\frac{\partial}{\partial x}|_p(f) = \frac{\partial}{\partial x}|_{(1,0)}(y) = 0 \neq \frac{\partial}{\partial \tilde{x}}|_p(f) = -3$.

$\square$