CHAPTER II

Schemes

This chapter and the next form the technical heart of this book. In this chapter we develop the basic theory of schemes, following Grothendieck [EGA]. Sections 1 to 5 are fundamental. They contain a review of sheaf theory (necessary even to define a scheme), then the basic definitions of schemes, morphisms, and coherent sheaves. This is the language that we use for the rest of the book.

Then in Sections 6, 7, 8, we treat some topics which could have been done in the language of varieties, but which are already more convenient to discuss using schemes. For example, the notion of Cartier divisor, and of an invertible sheaf, which belong to the new language, greatly clarify the discussion of Weil divisors and linear systems, which belong to the old language. Then in §8, the systematic use of nonclosed scheme points gives much more flexibility in the discussion of sheaves of differentials and nonsingular varieties, improving the treatment of (I, §5).

In §9 we give the definition of a formal scheme, which did not have an analogue in the theory of varieties. It was invented by Grothendieck as a good way of dealing with Zariski's theory of "holomorphic functions," which Zariski regarded as an analogue in abstract algebraic geometry of the holomorphic functions in a neighborhood of a subvariety in the classical case.

1 Sheaves

The concept of a sheaf provides a systematic way of keeping track of local algebraic data on a topological space. For example, the regular functions on open subsets of a variety, introduced in Chapter I, form a sheaf, as we will see shortly. Sheaves are essential in the study of schemes. In fact, we cannot even define a scheme without using sheaves. So we begin this chapter with sheaves. For additional information, see the book of Godement [1].

Definition. Let X be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups on X consists of the data

(a) for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and
(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV}: \mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

(0) $\mathscr{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,
(1) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and
(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

The reader who likes the language of categories may rephrase this definition as follows. For any topological space X, we define a category $\mathfrak{Top}(X)$, whose objects are the open subsets of X, and where the only morphisms are the inclusion maps. Thus $\operatorname{Hom}(V,U)$ is empty if $V \nsubseteq U$, and $\operatorname{Hom}(V,U)$ has just one element if $V \subseteq U$. Now a presheaf is just a contravariant functor from the category $\mathfrak{Top}(X)$ to the category $\mathfrak{Ab}$ of abelian groups.

We define a presheaf of rings, a presheaf of sets, or a presheaf with values in any fixed category $\mathfrak{C}$, by replacing the words "abelian group" in the definition by "ring", "set", or "object of $\mathfrak{C}$" respectively. We will stick to the case of abelian groups in this section, and let the reader make the necessary modifications for the case of rings, sets, etc.

As a matter of terminology, if $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the *sections* of the presheaf $\mathscr{F}$ over the open set U, and we sometimes use the notation $\Gamma(U,\mathscr{F})$ to denote the group $\mathscr{F}(U)$. We call the maps ρ_{UV} *restriction* maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$, if $s \in \mathscr{F}(U)$.

A sheaf is roughly speaking a presheaf whose sections are determined by local data. To be precise, we give the following definition.

Definition. A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions:

(3) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;
(4) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i,j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Note condition (3) implies that s is unique.)

Note. According to our definition, a sheaf is a presheaf satisfying certain extra conditions. This is equivalent to the definition found in some other books, of a sheaf as a topological space over X with certain properties (Ex. 1.13).

Example 1.0.1. Let X be a variety over the field k. For each open set $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let $\rho_{UV}: \mathcal{O}(U) \to \mathcal{O}(V)$ be the restriction map (in the usual sense). Then $\mathcal{O}$ is a sheaf of rings on X. It is clear that it is a presheaf of rings. To verify the conditions (3) and (4), we note that a function which is 0 locally is 0, and a function which is regular locally is regular, because of the definition of regular function (I, §3). We call $\mathcal{O}$ the *sheaf of regular functions* on X.

Example 1.0.2. In the same way, one can define the sheaf of continuous real-valued functions on any topological space, or the sheaf of differentiable functions on a differentiable manifold, or the sheaf of holomorphic functions on a complex manifold.

Example 1.0.3. Let X be a topological space, and A an abelian group. We define the *constant sheaf* $\mathscr{A}$ on X determined by A as follows. Give A the discrete topology, and for any open set $U \subseteq X$, let $\mathscr{A}(U)$ be the group of all continuous maps of U into A. Then with the usual restriction maps, we obtain a sheaf $\mathscr{A}$. Note that for every connected open set U, $\mathscr{A}(U) \cong A$, whence the name "constant sheaf." If U is an open set whose connected components are open (which is always true on a locally connected topological space), then $\mathscr{A}(U)$ is a direct product of copies of A, one for each connected component of U.

Definition. If $\mathscr{F}$ is a presheaf on X, and if P is a point of X, we define the *stalk* $\mathscr{F}_P$ of $\mathscr{F}$ at P to be the direct limit of the groups $\mathscr{F}(U)$ for all open sets U containing P, via the restriction maps ρ.

Thus an element of $\mathscr{F}_P$ is represented by a pair $\langle U,s \rangle$, where U is an open neighborhood of P, and s is an element of $\mathscr{F}(U)$. Two such pairs $\langle U,s \rangle$ and $\langle V,t \rangle$ define the same element of $\mathscr{F}_P$ if and only if there is an open neighborhood W of P with $W \subseteq U \cap V$, such that $s|_W = t|_W$. Thus we may speak of elements of the stalk $\mathscr{F}_P$ as *germs* of sections of $\mathscr{F}$ at the point P. In the case of a variety X and its sheaf of regular functions $\mathcal{O}$, the stalk $\mathcal{O}_P$ at a point P is just the local ring of P on X, which was defined in (I, §3).

Definition. If $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X, a *morphism* $\varphi: \mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\varphi(U)$: $\mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\
\downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\
\mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V)
\end{array}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, we use the same definition for a morphism of sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

Note that a morphism $\varphi:\mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism $\varphi_P:\mathscr{F}_P \to \mathscr{G}_P$ on the stalks, for any point $P \in X$. The following proposition (which would be false for presheaves) illustrates the local nature of a sheaf.

Proposition 1.1. *Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P:\mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for every $P \in X$.*

PROOF. If φ is an isomorphism it is clear that each φ_P is an isomorphism. Conversely, assume φ_P is an isomorphism for all $P \in X$. To show that φ is an isomorphism, it will be sufficient to show that $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for all U, because then we can define an inverse morphism ψ by $\psi(U) = \varphi(U)^{-1}$ for each U. First we show $\varphi(U)$ is injective. Let $s \in \mathscr{F}(U)$, and suppose $\varphi(s) \in \mathscr{G}(U)$ is 0. Then for every point $P \in U$, the image $\varphi(s)_P$ of $\varphi(s)$ in the stalk $\mathscr{G}_P$ is 0. Since φ_P is injective for each P, we deduce that $s_P = 0$ in $\mathscr{F}_P$ for each $P \in U$. To say that $s_P = 0$ means that s and 0 have the same image in $\mathscr{F}_P$, which means that there is an open neighborhood W_P of P, with $W_P \subseteq U$, such that $s|_{W_P} = 0$. Now U is covered by the neighborhoods W_P of all its points, so by the sheaf property (3), s is 0 on U. Thus $\varphi(U)$ is injective.

Next, we show that $\varphi(U)$ is surjective. Suppose we have a section $t \in \mathscr{G}(U)$. For each $P \in U$, let $t_P \in \mathscr{G}_P$ be its germ at P. Since φ_P is surjective, we can find $s_P \in \mathscr{F}_P$ such that $\varphi_P(s_P) = t_P$. Let s_P be represented by a section $s(P)$ on a neighborhood V_P of P. Then $\varphi(s(P))$ and $t|_{V_P}$ are two elements of $\mathscr{G}(V_P)$, whose germs at P are the same. Hence, replacing V_P by a smaller neighborhood of P if necessary, we may assume that $\varphi(s(P)) = t|_{V_P}$ in $\mathscr{G}(V_P)$. Now U is covered by the open sets V_P, and on each V_P we have a section $s(P) \in \mathscr{F}(V_P)$. If P,Q are two points, then $s(P)|_{V_P \cap V_Q}$ and $s(Q)|_{V_P \cap V_Q}$ are two sections of $\mathscr{F}(V_P \cap V_Q)$, which are both sent by φ to $t|_{V_P \cap V_Q}$. Hence by the injectivity of φ proved above, they are equal. Then by the sheaf property (4), there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_P} = s(P)$ for each P. Finally, we have to check that $\varphi(s) = t$. Indeed, $\varphi(s)$, t are two sections of $\mathscr{G}(U)$, and for each P, $\varphi(s)|_{V_P} = t|_{V_P}$, hence by the sheaf property (3) applied to $\varphi(s) - t$, we conclude that $\varphi(s) = t$.

Our next task is to define kernels, cokernels and images of morphisms of sheaves.

Definition. Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. We define the *presheaf kernel* of φ, *presheaf cokernel* of φ, and *presheaf image* of φ to be the presheaves given by $U \mapsto \ker(\varphi(U))$, $U \mapsto \operatorname{coker}(\varphi(U))$, and $U \mapsto \operatorname{im}(\varphi(U))$ respectively.

Note that if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the presheaf kernel of φ is a sheaf, but the presheaf cokernel and presheaf image of φ are in general not sheaves. This leads us to the notion of a sheaf associated to a presheaf.

Proposition-Definition 1.2. *Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$, with the property that for any sheaf $\mathscr{G}$, and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$. Furthermore the pair $(\mathscr{F}^+, \theta)$ is unique up to unique isomorphism. $\mathscr{F}^+$ is called the* sheaf associated *to the presheaf $\mathscr{F}$.*

Proof. We construct the sheaf $\mathscr{F}^+$ as follows. For any open set U, let $\mathscr{F}^+(U)$ be the set of functions s from U to the union $\bigcup_{P \in U} \mathscr{F}_P$ of the stalks of $\mathscr{F}$ over points of U, such that

(1) for each $P \in U$, $s(P) \in \mathscr{F}_P$, and
(2) for each $P \in U$, there is a neighborhood V of P, contained in U, and an element $t \in \mathscr{F}(V)$, such that for all $Q \in V$, the germ t_Q of t at Q is equal to $s(Q)$.

Now one can verify immediately (!) that $\mathscr{F}^+$ with the natural restriction maps is a sheaf, that there is a natural morphism $\theta : \mathscr{F} \to \mathscr{F}^+$, and that it has the universal property described. The uniqueness of $\mathscr{F}^+$ is a formal consequence of the universal property. Note that for any point P, $\mathscr{F}_P = \mathscr{F}^+_P$. Note also that if $\mathscr{F}$ itself was a sheaf, then $\mathscr{F}^+$ is isomorphic to $\mathscr{F}$ via θ.

Definition. A *subsheaf* of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for every open set $U \subseteq X$, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$, and the restriction maps of the sheaf $\mathscr{F}'$ are induced by those of $\mathscr{F}$. It follows that for any point P, the stalk $\mathscr{F}'_P$ is a subgroup of $\mathscr{F}_P$.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *kernel* of φ, denoted $\ker \varphi$, to be the presheaf kernel of φ (which is a sheaf). Thus $\ker \varphi$ is a subsheaf of $\mathscr{F}$.

We say that a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is *injective* if $\ker \varphi = 0$. Thus φ is injective if and only if the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open set of X.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *image* of φ, denoted $\operatorname{im} \varphi$, to be the sheaf associated to the presheaf image of φ. By the universal property of the sheaf associated to a presheaf, there is a natural map $\operatorname{im} \varphi \to \mathscr{G}$. In fact this map is injective (see Ex. 1.4), and thus $\operatorname{im} \varphi$ can be identified with a subsheaf of $\mathscr{G}$.

We say that a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves is *surjective* if $\operatorname{im} \varphi = \mathscr{G}$.

We say that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is *exact* if at each stage $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Thus a sequence

$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is exact if and only if φ is injective, and $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to 0$ is exact if and only if φ is surjective.

Now let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. We define the *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ to be the sheaf associated to the presheaf $U \to \mathscr{F}(U)/\mathscr{F}'(U)$. It follows that for any point P, the stalk $(\mathscr{F}/\mathscr{F}')_P$ is the quotient $\mathscr{F}_P/\mathscr{F}'_P$.

If $\varphi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *cokernel* of φ, denoted coker φ, to be the sheaf associated to the presheaf cokernel of φ.

Caution 1.2.1. We saw that a morphism $\varphi: \mathscr{F} \to \mathscr{G}$ of sheaves is injective if and only if the map on sections $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. The corresponding statement for surjective morphisms is not true: if $\varphi: \mathscr{F} \to \mathscr{G}$ is surjective, the maps $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ on sections need not be surjective. However, we can say that φ is surjective if and only if the maps $\varphi_P: \mathscr{F}_P \to \mathscr{G}_P$ on stalks are surjective for each P. More generally, a sequence of sheaves and morphisms is exact if and only if it is exact on stalks (Ex. 1.2). This again illustrates the local nature of sheaves.

So far we have talked only about sheaves on a single topological space. Now we define some operations on sheaves, associated with a continuous map from one topological space to another.

Definition. Let $f: X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathscr{F}$ on X, we define the *direct image* sheaf $f_*\mathscr{F}$ on Y by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. For any sheaf $\mathscr{G}$ on Y, we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathscr{G}(V)$, where U is any open set in X, and the limit is taken over all open sets V of Y containing $f(U)$. Do not confuse $f^{-1}\mathscr{G}$ with the sheaf $f^*\mathscr{G}$ which will be defined later for a morphism of ringed spaces (§5).

Note that f_* is a functor from the category $\mathfrak{Ab}(X)$ of sheaves on X to the category $\mathfrak{Ab}(Y)$ of sheaves on Y. Similarly, f^{-1} is a functor from $\mathfrak{Ab}(Y)$ to $\mathfrak{Ab}(X)$.

Definition. If Z is a subset of X, regarded as a topological subspace with the induced topology, if $i: Z \to X$ is the inclusion map, and if $\mathscr{F}$ is a sheaf on X, then we call $i^{-1}\mathscr{F}$ the *restriction* of $\mathscr{F}$ to Z, and we often denote it by $\mathscr{F}|_Z$. Note that the stalk of $\mathscr{F}|_Z$ at any point $P \in Z$ is just $\mathscr{F}_P$.

Exercises

1.1. Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

1.2. (a) For any morphism of sheaves $\varphi:\mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.
(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.
(c) Show that a sequence $\ldots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

1.3. (a) Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.
(b) Give an example of a surjective morphism of sheaves $\varphi:\mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

1.4. (a) Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+:\mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.
(b) Use part (a) to show that if $\varphi:\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{im} \varphi$ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

1.5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

1.6. (a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0.$$

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

1.7. Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.
(a) Show that $\operatorname{im} \varphi \cong \mathscr{F}/\ker \varphi$.
(b) Show that $\operatorname{coker} \varphi \cong \mathscr{G}/\operatorname{im} \varphi$.

1.8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U,\cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U,\mathscr{F}') \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U,\cdot)$ need not be exact; see (Ex. 1.21) below.

1.9. *Direct Sum.* Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

1.10. *Direct Limit.* Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

1.11. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

1.12. *Inverse Limit.* Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

1.13. *Espace Étalé of a Presheaf.* (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathscr{F}$ on X, we define a topological space $\text{Spé}(\mathscr{F})$, called the *espace étalé* of $\mathscr{F}$, as follows. As a set, $\text{Spé}(\mathscr{F}) = \bigcup_{P \in X} \mathscr{F}_P$. We define a projection map $\pi : \text{Spé}(\mathscr{F}) \to X$ by sending $s \in \mathscr{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map $\bar{s} : U \to \text{Spé}(\mathscr{F})$ by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = \text{id}_U$, in other words, it is a "section" of π over U. We now make $\text{Spé}(\mathscr{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to \text{Spé}(\mathscr{F})$ for all U, and all $s \in \mathscr{F}(U)$, are continuous. Now show that the sheaf $\mathscr{F}^+$ associated to $\mathscr{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of *continuous* sections of $\text{Spé}(\mathscr{F})$ over U. In particular, the original presheaf $\mathscr{F}$ was a sheaf if and only if for each U, $\mathscr{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathscr{F})$ over U.

1.14. *Support.* Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\text{Supp}\, s$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that $\text{Supp}\, s$ is a closed subset of U. We define the *support* of $\mathscr{F}$, $\text{Supp}\, \mathscr{F}$, to be $\{P \in X \mid \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

1.15. *Sheaf $\mathscr{H}om$.* Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

1.16. *Flasque Sheaves.* A sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque. See (I, §1) for irreducible topological spaces.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.

(c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.

(d) If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

(e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of *discontinuous sections* of $\mathscr{F}$ as follows. For each open set $U \subseteq X$, $\mathscr{G}(U)$ is the set of maps $s: U \to \bigcup_{P \in U} \mathscr{F}_P$ such that for each $P \in U$, $s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

1.17. *Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i: \{P\}^- \to X$ is the inclusion.

1.18. *Adjoint Property of f^{-1}.* Let $f: X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$\operatorname{Hom}_X(f^{-1}\mathscr{G},\mathscr{F}) = \operatorname{Hom}_Y(\mathscr{G},f_*\mathscr{F}).$$

Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

1.19. *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i: Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j: U \to X$ be its inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by *extending $\mathscr{F}$ by zero* outside U.

(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

1.20. *Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X,\mathscr{F})$ to be the subgroup of $\Gamma(X,\mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.

(a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V,\mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}^0_Z(\mathscr{F})$.

(b) Let $U = X - Z$, and let $j: U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U).$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

1.21. *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathscr{O}_X$ be the sheaf of regular functions on X (1.0.1).

(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathscr{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i: Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(c) Now let $X = \mathbf{P}^1$, and let Y be the union of two distinct points $P,Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathscr{F})$ is not surjective. This shows that the global section functor $\Gamma(X,\cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

(d) Again let $X = \mathbf{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P\in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X,\mathcal{O}) \to \Gamma(X,\mathscr{K}) \to \Gamma(X,\mathscr{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

1.22. *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i,j an isomorphism $\varphi_{ij}:\mathscr{F}_i|_{U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = \text{id}$, and (2) for each i,j,k, $\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i:\mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i,j, $\psi_j = \varphi_{ij}\circ\psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by *glueing* the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

2 Schemes

In this section we will define the notion of a scheme. First we define affine schemes: to any ring A (recall our conventions about rings made in the Introduction!) we associate a topological space together with a sheaf of rings on it, called Spec A. This construction parallels the construction of affine varieties (I, §1) except that the points of Spec A correspond to all prime ideals of A, not just the maximal ideals. Then we define an arbitrary scheme to be something which locally looks like an affine scheme. This definition has no parallel in Chapter I. An important class of schemes is given by the construction of the scheme Proj S associated to any graded ring S. This construction parallels the construction of projective varieties in (I, §2). Finally, we will show that the varieties of Chapter I, after a slight modification, can be regarded as schemes. Thus the category of schemes is an enlargement of the category of varieties.

Now we will construct the space Spec A associated to a ring A. As a set, we define Spec A to be the set of all prime ideals of A. If $\mathfrak{a}$ is any ideal of A, we define the subset $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ to be the set of all prime ideals which contain $\mathfrak{a}$.

Lemma 2.1.

(a) *If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

(b) *If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.*

(c) *If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.*

PROOF.

(a) Certainly if $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$, then $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Conversely, if $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$, and if $\mathfrak{p} \not\supseteq \mathfrak{b}$ for example, then there is a $b \in \mathfrak{b}$ such that $b \notin \mathfrak{p}$. Now for any $a \in \mathfrak{a}$, $ab \in \mathfrak{p}$, so we must have $a \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Thus $\mathfrak{p} \supseteq \mathfrak{a}$.

(b) $\mathfrak{p}$ contains $\sum \mathfrak{a}_i$ if and only if $\mathfrak{p}$ contains each $\mathfrak{a}_i$, simply because $\sum \mathfrak{a}_i$ is the smallest ideal containing all of the ideals $\mathfrak{a}_i$.

(c) The radical of $\mathfrak{a}$ is the intersection of the set of all prime ideals containing $\mathfrak{a}$. So $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$ if and only if $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$.

Now we define a topology on Spec A by taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets. Note that $V(A) = \varnothing$; $V((0)) = \operatorname{Spec} A$; and the lemma shows that finite unions and arbitrary intersections of sets of the form $V(\mathfrak{a})$ are again of that form. Hence they do form the set of closed sets for a topology on Spec A.

Next we will define a sheaf of rings $\mathcal{O}$ on Spec A. For each prime ideal $\mathfrak{p} \subseteq A$, let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathcal{O}(U)$ to be the set of functions $s: U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$, such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and such that s is locally a quotient of elements of A: to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{q}}$. (Note the similarity with the definition of the regular functions on a variety. The difference is that we consider functions into the various local rings, instead of to a field.)

Now it is clear that sums and products of such functions are again such, and that the element 1 which gives 1 in each $A_{\mathfrak{p}}$ is an identity. Thus $\mathcal{O}(U)$ is a commutative ring with identity. If $V \subseteq U$ are two open sets, the natural restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is a homomorphism of rings. It is then clear that $\mathcal{O}$ is a presheaf. Finally, it is clear from the local nature of the definition that $\mathcal{O}$ is a sheaf.

Definition. Let A be a ring. The *spectrum* of A is the pair consisting of the topological space Spec A together with the sheaf of rings $\mathcal{O}$ defined above.

Let us establish some basic properties of the sheaf $\mathcal{O}$ on Spec A. For any element $f \in A$, we denote by $D(f)$ the open complement of $V((f))$. Note that open sets of the form $D(f)$ form a base for the topology of Spec A. Indeed, if $V(\mathfrak{a})$ is a closed set, and $\mathfrak{p} \notin V(\mathfrak{a})$, then $\mathfrak{p} \not\supseteq \mathfrak{a}$, so there is an $f \in \mathfrak{a}$, $f \notin \mathfrak{p}$. Then $\mathfrak{p} \in D(f)$ and $D(f) \cap V(\mathfrak{a}) = \varnothing$.

Proposition 2.2. *Let A be a ring, and* (Spec A, $\mathcal{O}$) *its spectrum.*

(a) *For any $\mathfrak{p} \in$ Spec A, the stalk $\mathcal{O}_{\mathfrak{p}}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_{\mathfrak{p}}$.*

(b) *For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.*

(c) *In particular, Γ(Spec A, $\mathcal{O}$) $\cong A$.*

PROOF.

(a) First we define a homomorphism from $\mathcal{O}_{\mathfrak{p}}$ to $A_{\mathfrak{p}}$ by sending any local section s in a neighborhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in A_{\mathfrak{p}}$. This gives a well-defined homomorphism φ from $\mathcal{O}_{\mathfrak{p}}$ to $A_{\mathfrak{p}}$. The map φ is surjective, because any element of $A_{\mathfrak{p}}$ can be represented as a quotient a/f, with $a, f \in A$, $f \notin \mathfrak{p}$. Then $D(f)$ will be an open neighborhood of $\mathfrak{p}$, and a/f defines a section of $\mathcal{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is the given element. To show that φ is injective, let U be a neighborhood of $\mathfrak{p}$, and let $s, t \in \mathcal{O}(U)$ be elements having the same value $s(\mathfrak{p}) = t(\mathfrak{p})$ at $\mathfrak{p}$. By shrinking U if necessary, we may assume that $s = a/f$, and $t = b/g$ on U, where $a, b, f, g \in A$, and $f, g \notin \mathfrak{p}$. Since a/f and b/g have the same image in $A_{\mathfrak{p}}$, it follows from the definition of localization that there is an $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$ in A. Therefore $a/f = b/g$ in every local ring $A_{\mathfrak{q}}$ such that $f, g, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $D(f) \cap D(g) \cap D(h)$, which contains $\mathfrak{p}$. Hence $s = t$ in a whole neighborhood of $\mathfrak{p}$, so they have the same stalk at $\mathfrak{p}$. So φ is an isomorphism, which proves (a).

(b) and (c). Note that (c) is the special case of (b) when $f = 1$, and $D(f)$ is the whole space. So it is sufficient to prove (b). We define a homomorphism $\psi : A_f \to \mathcal{O}(D(f))$ by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which assigns to each $\mathfrak{p}$ the image of a/f^n in $A_{\mathfrak{p}}$.

First we show ψ is injective. If $\psi(a/f^n) = \psi(b/f^m)$, then for every $\mathfrak{p} \in D(f)$, a/f^n and b/f^m have the same image in $A_{\mathfrak{p}}$. Hence there is an element $h \notin \mathfrak{p}$ such that $h(f^m a - f^n b) = 0$ in A. Let $\mathfrak{a}$ be the annihilator of $f^m a - f^n b$. Then $h \in \mathfrak{a}$, and $h \notin \mathfrak{p}$, so $\mathfrak{a} \not\subseteq \mathfrak{p}$. This holds for any $\mathfrak{p} \in D(f)$, so we conclude that $V(\mathfrak{a}) \cap D(f) = \varnothing$. Therefore $f \in \sqrt{\mathfrak{a}}$, so some power $f^l \in \mathfrak{a}$, so $f^l(f^m a - f^n b) = 0$, which shows that $a/f^n = b/f^m$ in A_f. Hence ψ is injective.

The hard part is to show that ψ is surjective. So let $s \in \mathcal{O}(D(f))$. Then by definition of $\mathcal{O}$, we can cover $D(f)$ with open sets V_i, on which s is represented by a quotient a_i/g_i, with $g_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in V_i$, in other words, $V_i \subseteq D(g_i)$. Now the open sets of the form $D(h)$ form a base for the topology, so we may assume that $V_i = D(h_i)$ for some h_i. Since $D(h_i) \subseteq D(g_i)$, we have $V((h_i)) \supseteq V((g_i))$, hence by (2.1c), $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$, and in particular, $h_i^n \in (g_i)$ for some n. So $h_i^n = cg_i$, so $a_i/g_i = ca_i/h_i^n$. Replacing h_i by h_i^n (since $D(h_i) = D(h_i^n)$) and a_i by ca_i, we may assume that $D(f)$ is covered by the open subsets $D(h_i)$, and that s is represented by a_i/h_i on $D(h_i)$.

Next we observe that $D(f)$ can be covered by a finite number of the $D(h_i)$. Indeed, $D(f) \subseteq \bigcup D(h_i)$ if and only if $V((f)) \supseteq \bigcap V((h_i)) = V(\sum(h_i))$. By (2.1c) again, this is equivalent to saying $f \in \sqrt{\sum(h_i)}$, or $f^n \in \sum(h_i)$ for some n. This means that f^n can be expressed as a *finite* sum $f^n = \sum b_i h_i$, $b_i \in A$. Hence a finite subset of the h_i will do. So from now on we fix a finite set $h_1, \ldots, h_r$ such that $D(f) \subseteq D(h_1) \cup \ldots \cup D(h_r)$.

For the next step, note that on $D(h_i) \cap D(h_j) = D(h_i h_j)$ we have two elements of $A_{h_i h_j}$, namely a_i/h_i and a_j/h_j both of which represent s. Hence, according to the injectivity of ψ proved above, applied to $D(h_i h_j)$, we must have $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Hence for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

Since there are only finitely many indices involved, we may pick n so large that it works for all i,j at once. Rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$

Then replace each h_i by h_i^{n+1}, and a_i by $h_i^n a_i$. Then we still have s represented on $D(h_i)$ by a_i/h_i, and furthermore, we have $h_j a_i = h_i a_j$ for all i,j.

Now write $f^n = \sum b_i h_i$ as above, which is possible for some n since the $D(h_i)$ cover $D(f)$. Let $a = \sum b_i a_i$. Then for each j we have

$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i h_i a_j = f^n a_j.$$

This says that $a/f^n = a_j/h_j$ on $D(h_j)$. So $\psi(a/f^n) = s$ everywhere, which shows that ψ is surjective, hence an isomorphism.

To each ring A we have now associated its spectrum $(\operatorname{Spec} A, \mathcal{O})$. We would like to say that this correspondence is functorial. For that we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is the category of locally ringed spaces.

Definition. A *ringed space* is a pair $(X,\mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism* of ringed spaces from $(X,\mathcal{O}_X)$ to $(Y,\mathcal{O}_Y)$ is a pair $(f,f^\#)$ of a continuous map $f:X \to Y$ and a map $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y. The ringed space $(X,\mathcal{O}_X)$ is a *locally ringed space* if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring. A *morphism* of locally ringed spaces is a morphism $(f,f^\#)$ of ringed spaces, such that for each point $P \in X$, the induced map (see below) of local rings $f_P^\#:\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a *local homomorphism* of local rings. We explain this last condition. First of all, given a point $P \in X$, the morphism of sheaves $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$, for every open set V in Y. As V ranges over all open neighborhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of P. Taking direct limits, we obtain a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim_V \mathcal{O}_Y(V) \to \varinjlim_V \mathcal{O}_X(f^{-1}V),$$

and the latter limit maps to the stalk $\mathcal{O}_{X,P}$. Thus we have an induced homomorphism $f_P^{\#}:\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$. We require that this be a local homomorphism: If A and B are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively, a homomorphism $\varphi:A \to B$ is called a *local homomorphism* if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse. Thus a morphism $(f,f^{\#})$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces, and $f^{\#}$ is an isomorphism of sheaves.

Proposition 2.3.

(a) *If A is a ring, then* $(\operatorname{Spec} A, \mathcal{O})$ *is a locally ringed space.*

(b) *If $\varphi:A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces*

$$(f,f^{\#}):(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

(c) *If A and B are rings, then any morphism of locally ringed spaces from* Spec B *to* Spec A *is induced by a homomorphism of rings $\varphi:A \to B$ as in* (b).

PROOF.

(a) This follows from (2.2a).

(b) Given a homomorphism $\varphi:A \to B$, we define a map $f:\operatorname{Spec} B \to \operatorname{Spec} A$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Spec} B$. If $\mathfrak{a}$ is an ideal of A, then it is immediate that $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$, so f is continuous. For each $\mathfrak{p} \in \operatorname{Spec} B$, we can localize φ to obtain a local homomorphism of local rings $\varphi_{\mathfrak{p}}:A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. Now for any open set $V \subseteq \operatorname{Spec} A$ we obtain a homomorphism of rings $f^{\#}:\mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$ by the definition of $\mathcal{O}$, composing with the maps f and $\varphi_{\mathfrak{p}}$. This gives the morphism of sheaves $f^{\#}:\mathcal{O}_{\operatorname{Spec} A} \to f_*(\mathcal{O}_{\operatorname{Spec} B})$. The induced maps $f^{\#}$ on the stalks are just the local homomorphisms $\varphi_{\mathfrak{p}}$, so $(f,f^{\#})$ is a morphism of locally ringed spaces.

(c) Conversely, suppose given a morphism of locally ringed spaces $(f,f^{\#})$ from Spec B to Spec A. Taking global sections, $f^{\#}$ induces a homomorphism of rings $\varphi:\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$. By (2.2c), these rings are A and B, respectively, so we have a homomorphism $\varphi:A \to B$. For any $\mathfrak{p} \in \operatorname{Spec} B$, we have an induced local homomorphism on the stalks, $\mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B,\mathfrak{p}}$ or $A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$, which must be compatible with the map φ on global sections and the localization homomorphisms. In other words, we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\
A_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^{\#}"] & B_{\mathfrak{p}}.
\end{tikzcd}$$

Since $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, which shows that f coincides with the map Spec $B \to$ Spec A induced by φ. Now it is immediate that $f^{\#}$ also is induced by φ, so that the morphism $(f,f^{\#})$ of locally ringed spaces does indeed come from the homomorphism of rings φ.

Caution 2.3.0. Statement (c) of the proposition would be false, if in the definition of a morphism of locally ringed spaces, we did not insist that the induced maps on the stalks be *local* homomorphisms of local rings (see (2.3.2) below).

Now we come to the definition of a scheme.

Definition. An *affine scheme* is a locally ringed space $(X,\mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A *scheme* is a locally ringed space $(X,\mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme. We call X the *underlying topological space* of the scheme $(X,\mathcal{O}_X)$, and $\mathcal{O}_X$ its *structure sheaf*. By abuse of notation we will often write simply X for the scheme $(X,\mathcal{O}_X)$. If we wish to refer to the underlying topological space without its scheme structure, we write sp(X), read "space of X." A *morphism* of schemes is a morphism as locally ringed spaces. An *isomorphism* is a morphism with a two-sided inverse.

Example 2.3.1. If k is a field, Spec k is an affine scheme whose topological space consists of one point, and whose structure sheaf consists of the field k.

Example 2.3.2. If R is a discrete valuation ring, then $T =$ Spec R is an affine scheme whose topological space consists of two points. One point t_0 is closed, with local ring R; the other point t_1 is open and dense, with local ring equal to K, the quotient field of R. The inclusion map $R \to K$ corresponds to the morphism Spec $K \to T$ which sends the unique point of Spec K to t_1. There is another morphism of ringed spaces Spec $K \to T$ which sends the unique point of Spec K to t_0, and uses the inclusion $R \to K$ to define the associated map $f^{\#}$ on structure sheaves. This morphism is *not* induced by any homomorphism $R \to K$ as in (2.3b,c), since it is not a morphism of *locally ringed spaces*.

Example 2.3.3. If k is a field, we define the *affine line* over k, $\mathbf{A}_k^1$, to be Spec $k[x]$. It has a point ξ, corresponding to the zero ideal, whose closure is the whole space. This is called a *generic point*. The other points, which correspond to the maximal ideals in $k[x]$, are all closed points. They are in one-to-one correspondence with the nonconstant monic irreducible polynomials in x. In particular, if k is algebraically closed, the closed points of $\mathbf{A}_k^1$ are in one-to-one correspondence with elements of k.

Example 2.3.4. Let k be an algebraically closed field, and consider the *affine plane* over k, defined as $\mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$ (Fig. 6). The closed points of $\mathbf{A}_k^2$ are in one-to-one correspondence with ordered pairs of elements of k. Furthermore, the set of all closed points of $\mathbf{A}_k^2$, with the induced topology, is homeomorphic to the *variety* called $\mathbf{A}^2$ in Chapter I. In addition to the closed points, there is a *generic point* ξ, corresponding to the zero ideal of $k[x,y]$, whose closure is the whole space. Also, for each irreducible polynomial $f(x,y)$, there is a point η whose closure consists of η together with all closed points (a,b) for which $f(a,b) = 0$. We say that η is a *generic point* of the curve $f(x,y) = 0$.

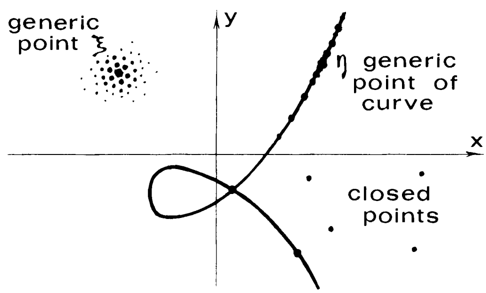

Figure 6. Spec $k[x, y]$.

Example 2.3.5. Let X_1 and X_2 be schemes, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let $\varphi:(U_1,\mathcal{O}_{X_1}|_{U_1}) \to (U_2,\mathcal{O}_{X_2}|_{U_2})$ be an isomorphism of locally ringed spaces. Then we can define a scheme X, obtained by *glueing* X_1 and X_2 along U_1 and U_2 via the isomorphism φ. The topological space of X is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology. Thus there are maps $i_1:X_1 \to X$ and $i_2:X_2 \to X$, and a subset $V \subseteq X$ is open if and only if $i_1^{-1}(V)$ is open in X_1 and $i_2^{-1}(V)$ is open in X_2. The structure sheaf $\mathcal{O}_X$ is defined as follows: for any open set $V \subseteq X$,

$$\mathcal{O}_X(V) = \{\langle s_1,s_2\rangle | s_1 \in \mathcal{O}_{X_1}(i_1^{-1}(V)) \text{ and } s_2 \in \mathcal{O}_{X_2}(i_2^{-1}(V)) \text{ and } \\ \varphi(s_1|_{i_1^{-1}(V)\cap U_1}) = s_2|_{i_2^{-1}(V)\cap U_2}\}.$$

Now it is clear that $\mathcal{O}_X$ is a sheaf, and that $(X,\mathcal{O}_X)$ is a locally ringed space. Furthermore, since X_1 and X_2 are schemes, it is clear that every point of X has a neighborhood which is affine, hence X is a scheme.

Example 2.3.6. As an example of glueing, let k be a field, let $X_1 = X_2 = \mathbf{A}_k^1$, let $U_1 = U_2 = \mathbf{A}_k^1 - \{P\}$, where P is the point corresponding to the maximal ideal (x), and let $\varphi: U_1 \to U_2$ be the identity map. Let X be obtained by glueing X_1 and X_2 along U_1 and U_2 via φ. We get an "affine line with the point P doubled."

———:———

This is an example of a scheme which is not an affine scheme (!). It is also an example of a nonseparated scheme, as we will see later (4.0.1).

Next we will define an important class of schemes, constructed from graded rings, which are analogous to projective varieties.

Let S be a graded ring. See (I, §2) for our conventions about graded rings. We denote by S_+ the ideal $\bigoplus_{d>0} S_d$.

We define the *set* Proj S to be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of S_+. If $\mathfrak{a}$ is a homogeneous ideal of S, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

Lemma 2.4.

(a) *If $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals in S, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

(b) *If $\{\mathfrak{a}_i\}$ is any family of homogeneous ideals of S, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.*

Proof. The proofs are the same as for (2.1a,b), taking into account the fact that a homogeneous ideal $\mathfrak{p}$ is prime if and only if for any two *homogeneous* elements $a,b \in S$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Because of the lemma we can define a topology on Proj S by taking the closed subsets to be the subsets of the form $V(\mathfrak{a})$.

Next we will define a sheaf of rings $\mathcal{O}$ on Proj S. For each $\mathfrak{p} \in \operatorname{Proj} S$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system consisting of all *homogeneous* elements of S which are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj} S$, we define $\mathcal{O}(U)$ to be the set of functions $s: U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that s is locally a quotient of elements of S: for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements a,f in S, of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Now it is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathcal{O}$ is a sheaf.

Definition. If S is any graded ring, we define (Proj $S,\mathcal{O}$) to be the topological space together with the sheaf of rings constructed above.

Proposition 2.5. *Let S be a graded ring.*

(a) *For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.*

(b) *For any homogeneous $f \in S_+$, let $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$.*

Then $D_+(f)$ *is open in* Proj S. *Furthermore, these open sets cover* Proj S, *and for each such open set, we have an isomorphism of locally ringed spaces*

$$(D_+(f),\mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)},$$

where $S_{(f)}$ *is the subring of elements of degree* 0 *in the localized ring* S_f.
(c) Proj S *is a scheme.*

Proof. Note first that (a) says that Proj S is a locally ringed space, and (b) tells us it is covered by open affine schemes, so (c) is a consequence of (a) and (b).

The proof of (a) is practically identical to the proof of (2.2a) above, so is left to the reader.

To prove (b), first note that $D_+(f) = \operatorname{Proj} S - V((f))$, so it is open. Since the elements of Proj S are those homogeneous prime ideals $\mathfrak{p}$ of S which do not contain all of S_+, it follows that the open sets $D_+(f)$ for homogeneous $f \in S_+$ cover Proj S. Now fix a homogeneous $f \in S_+$. We will define an isomorphism $(\varphi,\varphi^\#)$ of locally ringed spaces from $D_+(f)$ to Spec $S_{(f)}$. There is a natural homomorphism of rings $S \to S_f$, and $S_{(f)}$ is a subring of S_f. For any homogeneous ideal $\mathfrak{a} \subseteq S$, let $\varphi(\mathfrak{a}) = (\mathfrak{a}S_f) \cap S_{(f)}$. In particular, if $\mathfrak{p} \in D_+(f)$, then $\varphi(\mathfrak{p}) \in \operatorname{Spec} S_{(f)}$, so this gives the map φ as sets. The properties of localization show that φ is bijective as a map from $D_+(f)$ to Spec $S_{(f)}$. Furthermore, if $\mathfrak{a}$ is a homogeneous ideal of S, then $\mathfrak{p} \supseteq \mathfrak{a}$ if and only if $\varphi(\mathfrak{p}) \supseteq \varphi(\mathfrak{a})$. Hence φ is a homeomorphism. Note also if $\mathfrak{p} \in D_+(f)$, then the local rings $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\varphi(\mathfrak{p})}$ are naturally isomorphic. These isomorphisms and the homeomorphism φ induce a natural map of sheaves $\varphi^\# : \mathcal{O}_{\operatorname{Spec} S_{(f)}} \to \varphi_*(\mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)})$ which one recognizes immediately to be an isomorphism. Hence $(\varphi,\varphi^\#)$ is an isomorphism of locally ringed spaces, as required.

Example 2.5.1. If A is a ring, we define *projective n-space* over A to be the scheme $\mathbf{P}_A^n = \operatorname{Proj} A[x_0, \ldots ,x_n]$. In particular, if A is an algebraically closed field k, then $\mathbf{P}_k^n$ is a scheme whose subspace of closed points is naturally homeomorphic to the *variety* called projective n-space—see (Ex. 2.14d) below.

Next we will show that the notion of scheme does in fact generalize the notion of variety. It is not quite true that a variety is a scheme. As we have already seen in the examples above, the underlying topological space of a scheme such as $\mathbf{A}_k^1$ or $\mathbf{A}_k^2$ has more points than the corresponding variety. However, we will show that there is a natural way of adding generic points (Ex. 2.9) for every irreducible subset of a variety so that the variety becomes a scheme.

To state our result, we need a definition.

Definition. Let S be a fixed scheme. A *scheme over* S is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a morphism of X to Y as schemes over S, (also called an S-morphism) is a morphism $f:X \to Y$ which is compatible with the given morphisms to S. We denote by $\mathfrak{Sch}(S)$ the category of schemes over S. If A is a ring, then by abuse of notation we write $\mathfrak{Sch}(A)$ for the category of schemes over Spec A.

Proposition 2.6. *Let k be an algebraically closed field. There is a natural fully faithful functor $t:\mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ from the category of varieties over k to schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of* $\operatorname{sp}(t(V))$, *and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.*

PROOF. To begin with, let X be any topological space, and let $t(X)$ be the set of (nonempty) irreducible closed subsets of X. If Y is a closed subset of X, then $t(Y) \subseteq t(X)$. Furthermore, $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ and $t(\bigcap Y_i) = \bigcap t(Y_i)$. So we can define a topology on $t(X)$ by taking as closed sets the subsets of the form $t(Y)$, where Y is a closed subset of X. If $f:X_1 \to X_2$ is a continuous map, then we obtain a map $t(f):t(X_1) \to t(X_2)$ by sending an irreducible closed subset to the closure of its image. Thus t is a functor on topological spaces. Furthermore, one can define a continuous map $\alpha:X \to t(X)$ by $\alpha(P) = \{P\}^-$. Note that α induces a *bijection* between the set of open subsets of X and the set of open subsets of $t(X)$.

Now let k be an algebraically closed field. Let V be a variety over k, and let $\mathcal{O}_V$ be its sheaf of regular functions (1.0.1). We will show that $(t(V),\alpha_*(\mathcal{O}_V))$ is a scheme over k. Since any variety can be covered by open affine subvarieties (I, 4.3), it will be sufficient to show that if V is affine, then $(t(V),\alpha_*(\mathcal{O}_V))$ is a scheme. So let V be an affine variety with affine coordinate ring A. We define a morphism of locally ringed spaces

$$\beta:(V,\mathcal{O}_V) \to X = \operatorname{Spec} A$$

as follows. For each point $P \in V$, let $\beta(P) = \mathfrak{m}_P$, the ideal of A consisting of all regular functions which vanish at P. Then by (I, 3.2b), β is a bijection of V onto the set of closed points of X. It is easy to see that β is a homeomorphism onto its image. Now for any open set $U \subseteq X$, we will define a homomorphism of rings $\mathcal{O}_X(U) \to \beta_*(\mathcal{O}_V)(U) = \mathcal{O}_V(\beta^{-1}U)$. Given a section $s \in \mathcal{O}_X(U)$, and given a point $P \in \beta^{-1}(U)$, we define $s(P)$ by taking the image of s in the stalk $\mathcal{O}_{X,\beta(P)}$, which is isomorphic to the local ring $A_{\mathfrak{m}_P}$, and then passing to the quotient ring $A_{\mathfrak{m}_P}/\mathfrak{m}_P$ which is isomorphic to the field k. Thus s gives a function from $\beta^{-1}(U)$ to k. It is easy to see that this is a regular function, and that this map gives an isomorphism $\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}U)$. Finally, since the prime ideals of A are in 1-1 correspondence with the irreducible closed subsets of V (see (I, 1.4) and proof), these remarks show that $(X,\mathcal{O}_X)$ is isomorphic to $(t(V), \alpha_*\mathcal{O}_V)$, so the latter is indeed an affine scheme.

To give a morphism of $(t(V), \alpha_*\mathcal{O}_V)$ to Spec k, we have only to give a homomorphism of rings $k \to \Gamma(t(V), \alpha_*\mathcal{O}_V) = \Gamma(V, \mathcal{O}_V)$. We send $\lambda \in k$ to the constant function λ on V. Thus $t(V)$ becomes a scheme over k. Finally, if V and W are two varieties, then one can check (Ex. 2.15) that the natural map

$$\mathrm{Hom}_{\mathfrak{Var}(k)}(V, W) \to \mathrm{Hom}_{\mathfrak{Sch}(k)}(t(V), t(W))$$

is bijective. This shows that the functor $t: \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ is fully faithful. In particular it implies that $t(V)$ is isomorphic to $t(W)$ if and only if V is isomorphic to W.

It is clear from the construction that $\alpha: V \to t(V)$ induces a homeomorphism from V onto the set of closed points of $t(V)$, with the induced topology.

Note. We will see later (4.10) what the image of the functor t is.

Exercises

2.1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to Spec A_f.

2.2. Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

2.3. *Reduced Schemes*. A scheme $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\mathrm{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\mathrm{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{\mathrm{red}})$ is a scheme. We call it the *reduced scheme* associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\mathrm{red}} \to X$, which is a homeomorphism on the underlying topological spaces.

(c) Let $f: X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g: X \to Y_{\mathrm{red}}$ such that f is obtained by composing g with the natural map $Y_{\mathrm{red}} \to Y$.

2.4. Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f: X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^{\#}: \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha: \mathrm{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

2.5. Describe Spec $\mathbf{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to Spec $\mathbf{Z}$.

2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of Spec $k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

2.9. If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

2.10. Describe Spec $\mathbf{R}[x]$. How does its topological space compare to the set $\mathbf{R}$? To $\mathbf{C}$?

2.11. Let $k = \mathbf{F}_p$ be the finite field with p elements. Describe Spec $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

2.12. *Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij}: U_{ij} \to U_{ji}$ such that (1) for each i,j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i,j,k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i: X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by *glueing* the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod X_i$.

2.13. A topological space is *quasi-compact* if every open cover has a finite subcover.
(a) Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.
(b) If X is an affine scheme, show that sp(X) is quasi-compact, but not in general noetherian. We say a scheme X is *quasi-compact* if sp(X) is.
(c) If A is a noetherian ring, show that sp(Spec A) is a noetherian topological space.
(d) Give an example to show that sp(Spec A) can be noetherian even when A is not.

2.14. (a) Let S be a graded ring. Show that Proj $S = \varnothing$ if and only if every element of S_+ is nilpotent.
(b) Let $\varphi: S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \text{Proj } T \mid \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of Proj T, and show that φ determines a natural morphism $f: U \to$ Proj S.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d: S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f: \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(d) Let V be a projective variety with homogeneous coordinate ring S (I, §2). Show that $t(V) \cong \operatorname{Proj} S$.

2.15. (a) Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.

(b) If $f: X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

(c) Now show that if V,W are any two varieties over k, then the natural map

$$\operatorname{Hom}_{\mathfrak{Var}}(V,W) \to \operatorname{Hom}_{\mathfrak{Sch}/k}(t(V),t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

2.16. Let X be a scheme, let $f \in \Gamma(X,\mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

(a) If $U = \operatorname{Spec} B$ is an open *affine* subscheme of X, and if $\bar{f} \in B = \Gamma(U,\mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact. Let $A = \Gamma(X,\mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [*Hint*: Use an open affine cover of X.]

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if sp(X) is noetherian.) Let $b \in \Gamma(X_f,\mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f,\mathcal{O}_{X_f}) \cong A_f$.

2.17. *A Criterion for Affineness.*

(a) Let $f: X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X,\mathcal{O}_X)$, such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A. [*Hint*: Use (Ex. 2.4) and (Ex. 2.16d) above.]

2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi: A \to B$ be a homomorphism of rings, and let $f: Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective. [*Hint*: Consider $X' = \operatorname{Spec}(A/\ker \varphi)$ and use (b) and (c).]

2.19. Let A be a ring. Show that the following conditions are equivalent:

(i) Spec A is disconnected;
(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents*);
(iii) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

3 First Properties of Schemes

In this section we will give some of the first properties of schemes. In particular we will discuss open and closed subschemes, and products of schemes. In the exercises we introduce the notion of constructible subsets, and study the dimension of the fibres of a morphism.

Definition. A scheme is *connected* if its topological space is connected. A scheme is *irreducible* if its topological space is irreducible.

Definition. A scheme X is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently (Ex. 2.3), X is reduced if and only if the local rings $\mathcal{O}_P$, for all $P \in X$, have no nilpotent elements.

Definition. A scheme X is *integral* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Example 3.0.1. If $X = \operatorname{Spec} A$ is an affine scheme, then X is irreducible if and only if the nilradical nil A of A is prime; X is reduced if and only if nil $A = 0$; and X is integral if and only if A is an integral domain.

Proposition 3.1. *A scheme is integral if and only if it is both reduced and irreducible.*

Proof. Clearly an integral scheme is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2. Then $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ which is not an integral domain. Thus integral implies irreducible.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \mathcal{O}(U)$ with $fg = 0$. Let $Y = \{x \in U \mid f_x \in \mathfrak{m}_x\}$, and let $Z = \{x \in U \mid g_x \in \mathfrak{m}_x\}$. Then Y and Z are closed subsets (Ex. 2.16a), and $Y \cup Z = U$. But X is irreducible, so U is irreducible, so one of Y or Z is equal to U, say $Y = U$. But then the restriction of f to any open affine subset of U will be nilpotent (Ex. 2.18a), hence zero, so f is zero. This shows that X is integral.

Definition. A scheme X is *locally noetherian* if it can be covered by open affine subsets Spec A_i, where each A_i is a noetherian ring. X is *noetherian* if it is locally noetherian and quasi-compact. Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets Spec A_i, with each A_i a noetherian ring.

Caution 3.1.1. If X is a noetherian scheme, then sp(X) is a noetherian topological space, but not conversely (Ex. 2.13) and (Ex. 3.17).

Note that in this definition we do not require that every open affine subset be the spectrum of a noetherian ring. So while it is obvious from the definition that the spectrum of a noetherian ring is a noetherian scheme, the converse is not obvious. It is a question of showing that the noetherian property is a "local property". We will often encounter similar situations later in defining properties of a scheme or of a morphism of schemes, so we will give a careful statement and proof of the local nature of the noetherian property, to illustrate this type of situation.

Proposition 3.2. *A scheme X is locally noetherian if and only if for every open affine subset U = Spec A, A is a noetherian ring. In particular, an affine scheme X = Spec A is a noetherian scheme if and only if the ring A is a noetherian ring.*

PROOF. The "if" part follows from the definition, so we have to show if X is locally noetherian, and if U = Spec A is an open affine subset, then A is a noetherian ring. First note that if B is a noetherian ring, so is any localization B_f. The open subsets $D(f) \cong$ Spec B_f form a base for the topology of Spec B. Hence on a locally noetherian scheme X there is a base for the topology consisting of the spectra of noetherian rings. In particular, our open set U can be covered by spectra of noetherian rings.

So we have reduced to proving the following statement: let X = Spec A be an affine scheme, which can be covered by open subsets which are spectra of noetherian rings. Then A is noetherian. Let U = Spec B be an open subset of X, with B noetherian. Then for some $f \in A$, $D(f) \subseteq U$. Let $\bar{f}$ be the image of f in B. Then $A_f \cong B_{\bar{f}}$, hence A_f is noetherian. So we can cover X by open subsets $D(f) \cong$ Spec A_f with A_f noetherian. Since X is quasi-compact, a finite number will do.

So now we have reduced to a purely algebraic problem: A is a ring, $f_1, \ldots, f_r$ are a finite number of elements of A, which generate the unit ideal, and each localization A_{f_i} is noetherian. We have to show A is noetherian. First we establish a lemma. Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\varphi_i : A \rightarrow A_{f_i}$ be the localization map, $i = 1, \ldots, r$. Then

$$\mathfrak{a} = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i}).$$

The inclusion $\subseteq$ is obvious. Conversely, given an element $b \in A$ contained in this intersection, we can write $\varphi_i(b) = a_i/f_i^{n_i}$ in A_{f_i} for each i, where $a_i \in \mathfrak{a}$, and $n_i > 0$. Increasing the n_i if necessary, we can make them all equal to a fixed n. This means that in A we have

$$f_i^{m_i}(f_i^n b - a_i) = 0$$

for some m_i. And as before, we can make all the $m_i = m$. Thus $f_i^{m+n}b \in \mathfrak{a}$ for each i. Since $f_1, \ldots, f_r$ generate the unit ideal, the same is true of their Nth powers for any N. Take $N = n + m$. Then we have $1 = \sum c_i f_i^N$ for suitable $c_i \in A$. Hence

$$b = \sum c_i f_i^N b \in \mathfrak{a}$$

as required.

Now we can easily show that A is noetherian. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ be an ascending chain of ideals in A. Then for each i,

$$\varphi_i(\mathfrak{a}_1) \cdot A_{f_i} \subseteq \varphi_i(\mathfrak{a}_2) \cdot A_{f_i} \subseteq \ldots$$

is an ascending chain of ideals in A_{f_i}, which must become stationary because A_{f_i} is noetherian. There are only finitely many A_{f_i}, so from the lemma we conclude that the original chain is eventually stationary, and hence A is noetherian.

Definition. A morphism $f: X \to Y$ of schemes is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. The morphism f is *of finite type* if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}.

Definition. A morphism $f: X \to Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.

Note in each of these definitions that a property of a morphism $f: X \to Y$ is defined by the existence of an open affine cover of Y with certain properties. In fact in each case it is equivalent to require the given property for every open affine subset of Y (Ex. 3.1–3.4).

Example 3.2.1. If V is a variety over an algebraically closed field k, then the associated scheme $t(V)$ (see (2.6)) is an integral noetherian scheme of finite type over k. Indeed, V can be covered by a finite number of open affine subvarieties (I, 4.3), so $t(V)$ can be covered by a finite number of open affines of the form $\operatorname{Spec} A_i$, where each A_i is an integral domain which is a finitely generated k-algebra and hence noetherian.

Example 3.2.2. If P is a point of a variety V, with local ring $\mathcal{O}_P$, then Spec $\mathcal{O}_P$ is an integral noetherian scheme, which is not in general of finite type over k.

Next we come to open and closed subschemes.

Definition. An *open subscheme* of a scheme X is a scheme U, whose topological space is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X. An *open immersion* is a morphism $f: X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Note that every open subset of a scheme carries a unique structure of open subscheme (Ex. 2.2).

Definition. A *closed immersion* is a morphism $f: Y \to X$ of schemes such that f induces a homeomorphism of sp(Y) onto a closed subset of sp(X), and furthermore the induced map $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where we say $f: Y \to X$ and $f': Y' \to X$ are equivalent if there is an isomorphism $i: Y' \to Y$ such that $f' = f \circ i$.

Example 3.2.3. Let A be a ring, and let $\mathfrak{a}$ be an ideal of A. Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Spec} A/\mathfrak{a}$. Then the ring homomorphism $A \to A/\mathfrak{a}$ induces a morphism of schemes $f: Y \to X$ which is a closed immersion. The map f is a homeomorphism of Y onto the closed subset $V(\mathfrak{a})$ of X, and the map of structure sheaves $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective because it is surjective on the stalks, which are localizations of A and $A/\mathfrak{a}$, respectively (Ex. 2.18).

Thus for any ideal $\mathfrak{a} \subseteq A$ we obtain a structure of closed subscheme on the closed set $V(\mathfrak{a}) \subseteq X$. In particular, every closed subset Y of X has many closed subscheme structures, corresponding to all the ideals $\mathfrak{a}$ for which $V(\mathfrak{a}) = Y$. In fact, every closed subscheme structure on a closed subset Y of an affine scheme X arises from an ideal in this way (Ex. 3.11b) or (5.10).

Example 3.2.4. For some more specific examples, let $A = k[x,y]$, where k is a field. Then Spec $A = \mathbf{A}_k^2$ is the affine plane over k. The ideal $\mathfrak{a} = (xy)$ gives a reducible subscheme, consisting of the union of the x and y axes. The ideal $\mathfrak{a} = (x^2)$ gives a subscheme structure with nilpotents on the y-axis. The ideal $\mathfrak{a} = (x^2,xy)$ gives another subscheme structure on the y-axis, this one having nilpotents only in the local ring at the origin. We say the origin is an *embedded point* for this subscheme.

Example 3.2.5. Let V be an affine variety over the field k, and let W be a closed subvariety. Then W corresponds to a prime ideal $\mathfrak{p}$ in the affine coordinate ring A of V (I, §1). Let $X = t(V)$ and $Y = t(W)$ be the associated schemes. Then $X = \operatorname{Spec} A$ and Y is the closed subscheme defined by $\mathfrak{p}$. For each $n \geqslant 1$ let Y_n be the closed subscheme of X corresponding to the ideal $\mathfrak{p}^n$. Then $Y_1 = Y$, but for each $n > 1$, Y_n is a nonreduced scheme structure on the closed set Y, which does not correspond to any subvariety of V. We call Y_n the nth *infinitesimal neighborhood* of Y in X. The schemes Y_n reflect properties of the embedding of Y in X. Later (§9) we will study the "formal completion" of Y in X, which is roughly the limit of the schemes Y_n as $n \to \infty$.

Example 3.2.6. Let X be a scheme, and let Y be a closed subset. In general Y will have many possible closed subscheme structures. However, there is one which is "smaller" than any other, called the *reduced induced closed subscheme structure*, which we now describe.

First let $X = \operatorname{Spec} A$ be an affine scheme, and let Y be a closed subset. Let $\mathfrak{a} \subseteq A$ be the ideal obtained by intersecting all the prime ideals in Y. This is the largest ideal for which $V(\mathfrak{a}) = Y$. Then we take the reduced induced structure on Y to be the one defined by $\mathfrak{a}$.

Now let X be any scheme, and let Y be a closed subset. For each open affine subset $U_i \subseteq X$, consider the closed subset $Y_i = Y \cap U_i$ of U_i, and give it the reduced induced structure just defined for affines (which may depend on U_i). I claim that for any i,j, the restrictions to $Y_i \cap Y_j$ of the two structure sheaves just defined on Y_i and Y_j are isomorphic, and furthermore, that the three such isomorphisms on $Y_i \cap Y_j \cap Y_k$ are compatible for all i,j,k. One reduces easily to showing that if $U = \operatorname{Spec} A$ is an open affine, and if $f \in A$, and if $V = D(f) = \operatorname{Spec} A_f$, then the reduced induced structure on $Y \cap U$ obtained from A when restricted to $Y \cap V$ agrees with the one obtained from A_f. This corresponds to the algebraic fact that if $\mathfrak{a}$ is the intersection of those prime ideals of A which are in Y, then $\mathfrak{a}A_f$ is the intersection of those prime ideals of A_f which are in $Y \cap D(f)$.

So now we can glue the sheaves defined on the Y_i to obtain a sheaf on Y (Ex. 1.22), which gives us the desired reduced induced subscheme structure on Y. See (Ex. 3.11) below for a universal property of the reduced induced subscheme structure.

Definition. The *dimension* of a scheme X, denoted $\dim X$, is its dimension as a topological space (I, §1). If Z is an irreducible closed subset of X, then the *codimension* of Z in X, denoted $\operatorname{codim}(Z,X)$ is the supremum of integers n such that there exists a chain

$$Z = Z_0 < Z_1 < \ldots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y,X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z,X)$$

where the infimum is taken over all closed irreducible subsets of Y.

Example 3.2.7. If $X = \operatorname{Spec} A$ is an affine scheme, then the dimension of X is the same as the Krull dimension of A (I, §1).

Caution 3.2.8. Be careful in applying the concepts of dimension and codimension to arbitrary schemes. Our intuition is derived from working with schemes of finite type over a field, where these notions are well-behaved. For example, if X is an affine integral scheme of finite type over a field k, and if $Y \subseteq X$ is any closed irreducible subset, then (I, 1.8A) implies that $\dim Y + \operatorname{codim}(Y,X) = \dim X$. But on arbitrary (even noetherian) schemes, funny things can happen. See (Ex. 3.20–3.22), and also Nagata [7], and Grothendieck [EGA IV, §5].

Definition. Let S be a scheme, and let X, Y be schemes over S, i.e., schemes with morphisms to S. We define the *fibred product* of X and Y over S, denoted $X \times_S Y$, to be a scheme, together with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, such that given any scheme Z over S, and given morphisms $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$, and $g = p_2 \circ \theta$. The morphisms p_1 and p_2 are called the *projection morphisms* of the fibred product onto its factors.

$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[dr] \arrow[drr] & X \times_S Y \arrow[d] \arrow[dr] & \\
& X \arrow[dr] & Y \arrow[d] \\
& & S
\end{tikzcd}$$

If X and Y are schemes given without reference to any base scheme S, we take $S = \operatorname{Spec} \mathbf{Z}$ (Ex. 2.5) and define the *product* of X and Y, denoted $X \times Y$, to be $X \times_{\operatorname{Spec} \mathbf{Z}} Y$.

Theorem 3.3. *For any two schemes X and Y over a scheme S, the fibred product $X \times_S Y$ exists, and is unique up to unique isomorphism.*

PROOF. The idea is first to construct products for affine schemes and then glue. We proceed in seven steps.

Step 1. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$ all be affine. Then A and B are R-algebras, and I claim that $\operatorname{Spec}(A \otimes_R B)$ is a product for X and Y over S. Indeed, for any scheme Z, to give a morphism of Z to $\operatorname{Spec}(A \otimes_R B)$ is the same as to give a homomorphism of the ring $A \otimes_R B$ into the ring $\Gamma(Z, \mathcal{O}_Z)$, by (Ex. 2.4). But to give a homomorphism of $A \otimes_R B$ into any ring is the same as to give homomorphisms of A and B into that ring, inducing the same homomorphism on R. Applying (Ex. 2.4) again, we see that to give a morphism of Z into $\operatorname{Spec}(A \otimes_R B)$ is the same as giving morphisms of Z into X and into Y, which give rise to the same morphism of Z into S. Thus $\operatorname{Spec}(A \otimes_R B)$ is the desired product.

Step 2. It follows immediately from the universal property of the product that it is unique up to unique isomorphism, if it exists. We will need this uniqueness for those products already constructed, as we go along.

Step 3. Glueing morphisms. We have already seen how to glue sheaves (Ex. 1.22) and how to glue schemes (Ex. 2.12). Now we glue morphisms. If X and Y are schemes, then to give a morphism f from X to Y, it is equivalent to give an open cover $\{U_i\}$ of X, together with morphisms $f_i: U_i \to Y$, where U_i has the induced open subscheme structure, such that the restrictions of f_i and f_j to $U_i \cap U_j$ are the same, for each i,j. The proof is straightforward.

Step 4. If X,Y are schemes over a scheme S, if $U \subseteq X$ is an open subset, and if the product $X \times_S Y$ exists, then $p_1^{-1}(U) \subseteq X \times_S Y$ is a product for U and Y over S. Indeed, given a scheme Z, and morphisms $f:Z \to U$ and $g:Z \to Y$, f determines a map of Z to X by composing with the inclusion $U \subseteq X$. Hence there is a map $\theta:Z \to X \times_S Y$ compatible with f,g and the projections. But since $f(Z) \subseteq U$, we have $\theta(Z) \subseteq p_1^{-1}(U)$. So θ can be regarded as a morphism $Z \to p_1^{-1}(U)$. It is clearly unique, so $p_1^{-1}(U)$ is a product $U \times_S Y$.

Step 5. Suppose given X,Y schemes over S, suppose $\{X_i\}$ is an open covering of X, and suppose that for each i, $X_i \times_S Y$ exists. Then $X \times_S Y$ exists. Indeed, for each i,j, let $U_{ij} \subseteq X_i \times_S Y$ be $p_1^{-1}(X_{ij})$, where $X_{ij} = X_i \cap X_j$. Then by Step 4, U_{ij} is a product for X_{ij} and Y over S. Hence by the uniqueness of products there are (unique) isomorphisms $\varphi_{ij}: U_{ij} \to U_{ji}$ for each i,j compatible with all the projections. Furthermore, these isomorphisms are compatible with each other for each i,j,k, in the sense of (Ex. 2.12). Thus we are in a position to glue the schemes $X_i \times_S Y$ via the isomorphisms φ_{ij}. We obtain by (Ex. 2.12) a scheme $X \times_S Y$ which I claim is a product for X and Y over S. The projection morphisms p_1 and p_2 are defined by glueing the projections from the pieces $X_i \times_S Y$ (Step 3). Given a scheme Z and morphisms $f:Z \to X$, $g:Z \to Y$, let $Z_i = f^{-1}(X_i)$. Then we get maps $\theta_i:Z_i \to X_i \times_S Y$, hence by composition with the inclusions $X_i \times_S Y \subseteq X \times_S Y$ we get maps $\theta_i:Z_i \to X \times_S Y$. One verifies that these maps agree on $Z_i \cap Z_j$, so we can glue the morphisms (Step 3) to obtain a morphism $\theta:Z \to X \times_S Y$, compatible with the projections and f and g. The uniqueness of θ can be checked locally.

Step 6. We know from Step 1 that if X,Y,S are all affine, then $X \times_S Y$ exists. Thus using Step 5 we conclude that for any X, but Y, S affine, the product exists. Using Step 5 again, with X and Y interchanged, we find that the product exists for any X and any Y over an affine S.

Step 7. Given arbitrary X,Y,S, let $q:X \to S$ and $r: Y \to S$ be the given morphisms. Let S_i be an open affine cover of S. Let $X_i = q^{-1}(S_i)$ and let $Y_i = r^{-1}(S_i)$. Then by Step 6, $X_i \times_{S_i} Y_i$ exists. Note that this same scheme is a product for X_i and Y over S. Indeed, given morphisms $f:Z \to X_i$ and $g:Z \to Y$ over S, the image of g must land inside Y_i. Thus $X_i \times_S Y$ exists for each i, and one more application of Step 5 gives us $X \times_S Y$. This completes the proof.

Perhaps this is a good place to make some general remarks on the importance and uses of fibred products. To begin with, we can define the fibres of a morphism.

Definition. Let $f: X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y, and let $\operatorname{Spec} k(y) \to Y$ be the natural morphism (Ex. 2.7). Then we define the *fibre* of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

The fibre X_y is a scheme over $k(y)$, and one can show that its underlying topological space is homeomorphic to the subset $f^{-1}(y)$ of X (Ex. 3.10).

The notion of the fibre of a morphism allows us to regard a morphism as a family of schemes (namely its fibres) parametrized by the points of the image scheme. Conversely, this notion of family is a good way of making sense of the idea of a family of schemes varying algebraically. For example, given a scheme X_0 over a field k, we define a *family of deformations* of X_0 to be a morphism $f: X \to Y$ with Y connected, together with a point $y_0 \in Y$, such that $k(y_0) = k$, and $X_{y_0} \cong X_0$. The other fibres X_y of f are called *deformations* of X_0.

An interesting kind of family arises when we have a scheme X over $\operatorname{Spec} \mathbf{Z}$. In this case, taking the fibre over the generic point gives a scheme $X_{\mathbf{Q}}$ over $\mathbf{Q}$, while taking the fibre over a closed point, corresponding to a prime number p, gives a scheme X_p over the finite field $\mathbf{F}_p$. We say that X_p arises by *reduction* mod p of the scheme X.

Another important application of fibred products is to the notion of base extension. Let S be a fixed scheme which we think of as a *base scheme*, meaning that we are interested in the category of schemes over S. For example, think of $S = \operatorname{Spec} k$, where k is a field. If S' is another base scheme, and if $S' \to S$ is a morphism, then for any scheme X over S, we let $X' = X \times_S S'$, which will be a scheme over S'. We say that X' is obtained from X by making a *base extension* $S' \to S$. For example, think of $S' = \operatorname{Spec} k'$ where k' is an extension field of k. Note, by the way, that base extension is a transitive operation: if $S'' \to S' \to S$ are two morphisms, then $(X \times_S S') \times_{S'} S'' \cong X \times_S S''$.

This ties in with a general philosophy, emphasized by Grothendieck in his "Eléments de Géométrie Algébrique" ([EGA]), that one should try to develop all concepts of algebraic geometry in a relative context. Instead of always working over a fixed base field, and considering properties of one variety at a time, one should consider a morphism of schemes $f: X \to S$, and study properties of the morphism. It then becomes important to study the behavior of properties of f under base extension, and in particular, to relate properties of f to properties of the fibres of f. For example, if $f: X \to S$

is a morphism of finite type, and if $S' \to S$ is any base extension, then $f': X' \to S'$ is also a morphism of finite type, where $X' = X \times_S S'$. Hence we say the property of a morphism f being of finite type is *stable under base extension.* On the other hand, if for example $f: X \to S$ is a morphism of integral schemes, the fibres of f may be neither irreducible nor reduced. So the property of a scheme being integral is not stable under base extension.

Example 3.3.1. Let k be an algebraically closed field, let

$$X = \operatorname{Spec} k[x,y,t]/(ty - x^2),$$

let $Y = \operatorname{Spec} k[t]$, and let $f: X \to Y$ be the morphism determined by the natural homomorphism $k[t] \to k[x,y,t]/(ty - x^2)$. Then X and Y are integral schemes of finite type over k, and f is a surjective morphism. We identify the closed points of Y with elements of k. For $a \in k$, $a \neq 0$, the fibre X_a is the plane curve $ay = x^2$ in $\mathbf{A}_k^2$, which is an irreducible, reduced curve. But for $a = 0$, the fibre X_0 is the nonreduced scheme given by $x^2 = 0$ in $\mathbf{A}^2$. Thus we have a family (Fig. 7) in which most members are irreducible curves, but one is nonreduced. This shows how nonreduced schemes occur naturally even if one is primarily interested in varieties. We can say that the nonreduced scheme $x^2 = 0$ in $\mathbf{A}^2$ is a deformation of the irreducible parabola $ay = x^2$ as $a \to 0$.

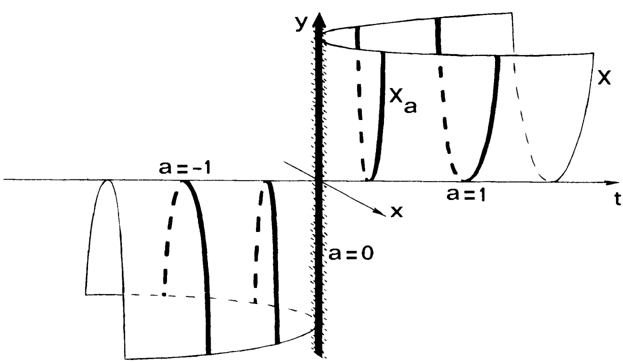

Figure 7. An algebraic family of schemes.

Example 3.3.2. Similarly, if $X = \operatorname{Spec} k[x,y,t]/(xy - t)$, we get a family whose general member X_a is an irreducible hyperbola $xy = a$, when $a \neq 0$, but whose special member X_0 is the reducible scheme $xy = 0$ consisting of two lines.

Exercises

3.1. Show that a morphism $f: X \to Y$ is locally of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

3.2. A morphism $f:X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

3.3. (a) Show that a morphism $f:X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

(c) Show also if f is of finite type, then for *every* open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for *every* open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

3.4. Show that a morphism $f:X \to Y$ is finite if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

3.5. A morphism $f:X \to Y$ is *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.

(b) Show that a finite morphism is *closed*, i.e., the image of any closed subset is closed.

(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

3.6. Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the *function field* of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

3.7. A morphism $f:X \to Y$, with Y irreducible, is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f:X \to Y$ is *dominant* if $f(X)$ is dense in Y. Now let $f:X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite. [*Hint*: First show that the function field of X is a finite field extension of the function field of Y.]

3.8. *Normalization.* A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of X. Show also that there is a morphism $\tilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f:Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

3.9. *The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

(a) Let k be a field, and let $\mathbf{A}_k^1 = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbf{A}_k^1 \times_{\operatorname{Spec} k} \mathbf{A}_k^1 \cong \mathbf{A}_k^2$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

(b) Let k be a field, let s and t be indeterminates over k. Then Spec $k(s)$, Spec $k(t)$, and Spec k are all one-point spaces. Describe the product scheme Spec $k(s) \times_{\text{Spec } k} \text{Spec } k(t)$.

3.10. *Fibres of a Morphism.*

(a) If $f:X \to Y$ is a morphism, and $y \in Y$ a point, show that $\text{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

(b) Let $X = \text{Spec } k[s,t]/(s - t^2)$, let $Y = \text{Spec } k[s]$, and let $f:X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

3.11. *Closed Subschemes.*

(a) Closed immersions are stable under base extension: if $f:Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f':Y \times_X X' \to X'$ is also a closed immersion.

*(b) If Y is a closed subscheme of an affine scheme $X = \text{Spec } A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion Spec $A/\mathfrak{a} \to \text{Spec } A$. [*Hints*: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \varnothing$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \ldots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subseteq A$.] *Note*: We will give another proof of this result using sheaves of ideals later (5.10).

(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(d) Let $f:Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the *scheme-theoretic image* of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

3.12. *Closed Subschemes of* Proj S.

(a) Let $\varphi:S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, and the morphism f:Proj $T \to$ Proj S is a closed immersion.

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \text{Proj } S$ defined as image of the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

3.13. *Properties of Morphisms of Finite Type.*
(a) A closed immersion is a morphism of finite type.
(b) A quasi-compact open immersion (Ex. 3.2) is of finite type.
(c) A composition of two morphisms of finite type is of finite type.
(d) Morphisms of finite type are stable under base extension.
(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.
(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.
(g) If $f:X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

3.14. If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

3.15. Let X be a scheme of finite type over a field k (not necessarily algebraically closed).
(a) Show that the following three conditions are equivalent (in which case we say that X is *geometrically irreducible*).

(i) $X \times_k \bar{k}$ is irreducible, where $\bar{k}$ denotes the algebraic closure of k. (By abuse of notation, we write $X \times_k \bar{k}$ to denote $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$.)
(ii) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.
(iii) $X \times_k K$ is irreducible for every extension field K of k.

(b) Show that the following three conditions are equivalent (in which case we say X is *geometrically reduced*).

(i) $X \times_k \bar{k}$ is reduced.
(ii) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.
(iii) $X \times_k K$ is reduced for all extension fields K of k.

(c) We say that X is *geometrically integral* if $X \times_k \bar{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

3.16. *Noetherian Induction.* Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

3.17. *Zariski Spaces.* A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9).

For example, let R be a discrete valuation ring, and let $T = \operatorname{sp}(\operatorname{Spec} R)$. Then T consists of two points t_0 = the maximal ideal, t_1 = the zero ideal. The open subsets are $\varnothing$, $\{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.
(a) Show that if X is a noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.
(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points*.
(c) Show that a Zariski space X satisfies the axiom T_0: given any two distinct points of X, there is an open set containing one but not the other.
(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.
(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 *specializes* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization*

of x_1, or that x_1 is a *generization* of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are *stable under specialization*.) Similarly, open subsets are *stable under generization*.

(f) Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha: X \to t(X)$ is a homeomorphism.

3.18. *Constructible Sets.* Let X be a Zariski topological space A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

(a) A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

(b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

(c) A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

(d) If $f: X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

3.19. The real importance of the notion of constructible subsets derives from the following theorem of Chevalley—see Cartan and Chevalley [1, exposé 7] and see also Matsumura [2, Ch. 2, §6]: let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y. Prove this theorem in the following steps.

(a) Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

*(b) In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi: A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$. [*Hint*: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.]

(c) Now use noetherian induction on Y to complete the proof.

(d) Give some examples of morphisms $f: X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

3.20. *Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

(a) For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.
(b) Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \text{tr.d. } K(X)/k$.
(c) If Y is a closed subset of X, then $\operatorname{codim}(Y,X) = \inf\{\dim \mathcal{O}_{P,X} | P \in Y\}$.
(d) If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y,X) = \dim X$.
(e) If U is a nonempty open subset of X, then $\dim U = \dim X$.
(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

3.21. Let R be a discrete valuation ring containing its residue field k. Let $X = \operatorname{Spec} R[t]$ be the affine line over $\operatorname{Spec} R$. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

***3.22.** *Dimension of the Fibres of a Morphism.* Let $f: X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.
(a) Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$, such that $\eta' \in f(Z)$, and show that $\operatorname{codim}(Z,X) \leqslant \operatorname{codim}(Y',Y)$.
(b) Let $e = \dim X - \dim Y$ be the *relative dimension* of X over Y. For any point $y \in f(X)$, show that every irreducible component of the fibre X_y has dimension $\geqslant e$. [*Hint*: Let $Y' = \{y\}^-$, and use (a) and (Ex. 3.20b).]
(c) Show that there is a dense open subset $U \subseteq X$, such that for any $y \in f(U)$, $\dim U_y = e$. [*Hint*: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is a finitely generated B-algebra. Take $t_1, \ldots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \ldots, t_e]$. Then X_1 is isomorphic to affine e-space over Y, and the morphism $X \to X_1$ is generically finite. Now use (Ex. 3.7) above.]
(d) Going back to our original morphism $f: X \to Y$, for any integer h, let E_h be the set of points $x \in X$ such that, letting $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geqslant h$. Show that (1) $E_e = X$ (use (b) above); (2) if $h > e$, then E_h is not dense in X (use (c) above); and (3) E_h is closed, for all h (use induction on $\dim X$).
(e) Prove the following theorem of Chevalley—see Cartan and Chevalley [1, exposé 8]. For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains an open dense subset of Y.

3.23. If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in (I, Ex. 3.15, 3.16), and if t is the functor of (2.6), then $t(V \times W) = t(V) \times_{\operatorname{Spec} k} t(W)$.

4 Separated and Proper Morphisms

We now come to two properties of schemes, or rather of morphisms between schemes, which correspond to well-known properties of ordinary topological spaces. Separatedness corresponds to the Hausdorff axiom for a topological space. Properness corresponds to the usual notion of properness, namely that the inverse image of a compact subset is compact. However, the usual definitions are not suitable in abstract algebraic geometry, because the Zariski topology is never Hausdorff, and the underlying topological space of a scheme does not accurately reflect all of its properties. So instead we will use definitions which reflect the functorial behavior of the morphism within the category of schemes. For schemes of finite type over $\mathbf{C}$, one can show that these notions, defined abstractly, are in fact the same as the usual notions if we consider those schemes as complex analytic spaces in the ordinary topology (Appendix B).

In this section we will define separated and proper morphisms. We will give criteria for a morphism to be separated or proper using valuation rings. Then we will show that projective space over any scheme is proper.

Definition. Let $f: X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique morphism $\Delta: X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2: X \times_Y X \to X$ is the identity map of $X \to X$. We say that the morphism f is *separated* if the diagonal morphism Δ is a closed immersion. In that case we also say X is *separated* over Y. A scheme X is *separated* if it is separated over Spec $\mathbf{Z}$.

Example 4.0.1. Let k be a field, and let X be the affine line with the origin doubled (2.3.6). Then X is not separated over k. Indeed, $X \times_k X$ is the affine plane with doubled axes and four origins. The image of Δ is the usual diagonal, with two of those origins. This is not closed, because all four origins are in the closure of $\Delta(X)$.

Example 4.0.2. We will see later (4.10) that if V is any variety over an algebraically closed field k, then the associated scheme $t(V)$ is separated over k.

Proposition 4.1. *If $f: X \to Y$ is any morphism of affine schemes, then f is separated.*

Proof. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then A is a B-algebra, and $X \times_Y X$ is also affine, given by Spec $A \otimes_B A$. The diagonal morphism Δ comes from the *diagonal homomorphism* $A \otimes_B A \to A$ defined by $a \otimes a' \to aa'$. This is a surjective homomorphism of rings, hence Δ is a closed immersion.

Corollary 4.2. *An arbitrary morphism $f: X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.*

Proof. One implication is obvious, so we have only to prove that if $\Delta(X)$ is a closed subset, then $\Delta: X \to X \times_Y X$ is a closed immersion. In other words, we have to check that $\Delta: X \to \Delta(X)$ is a homeomorphism, and that the morphism of sheaves $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective. Let $p_1: X \times_Y X \to X$ be the first projection. Since $p_1 \circ \Delta = \mathrm{id}_X$, it follows immediately that Δ gives a homeomorphism onto $\Delta(X)$. To see that the map of sheaves $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective is a local question. For any point $P \in X$, let U be an open affine neighborhood of P which is small enough so that $f(U)$ is contained in an open affine subset V of Y. Then $U \times_V U$ is an open affine neighborhood of $\Delta(P)$, and by the proposition, $\Delta: U \rightarrow U \times_V U$ is a closed immersion. So our map of sheaves is surjective in a neighborhood of P, which completes the proof.

Next we will discuss the valuative criterion of separatedness. The rough idea is that in order for a scheme X to be separated, it should not contain any subscheme which looks like a curve with a doubled point, as in the example above. Another way of saying this is that if C is a curve, and P a point of C, then given any morphism of $C - P$ into X, it should admit at most one extension to a morphism of all of C into X. (Compare (I, 6.8) where we showed that a projective variety has this property.)

In practice, this rough idea has to be modified. The question is local, so we replace the curve by its local ring at P, which is a discrete valuation ring. Then since our schemes may be quite general, we must consider arbitrary (not necessarily discrete) valuation rings. Finally, we make the criterion relative over the image scheme Y of a morphism.

See (I, §6) for the definition and basic properties of valuation rings.

Theorem 4.3 (Valuative Criterion of Separatedness). *Let $f: X \rightarrow Y$ be a morphism of schemes, and assume that X is noetherian. Then f is separated if and only if the following condition holds. For any field K, and for any valuation ring R with quotient field K, let $T =$ Spec R, let $U =$ Spec K, and let $i: U \rightarrow T$ be the morphism induced by the inclusion $R \subseteq K$. Given a morphism of T to Y, and given a morphism of U to X which makes a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is at most one morphism of T to X making the whole diagram commutative.

We will need two lemmas.

Lemma 4.4. *Let R be a valuation ring of a field K. Let $T =$ Spec R and let $U =$ Spec K. To give a morphism of U to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subseteq K$. To give a morphism of T to X is equivalent to giving two points x_0, x_1 in X, with x_0 a specialization (see* Ex. 3.17e) *of x_1, and an inclusion of fields $k(x_1) \subseteq K$,* *such that R dominates the local ring $\mathcal{O}$ of x_0 on the subscheme $Z = \{x_1\}^-$ of X with its reduced induced structure.*

PROOF. U is a one-point scheme, with structure sheaf K. To give a local homomorphism $\mathcal{O}_{x_1,X} \to K$ is the same as giving an inclusion of $k(x_1) \subseteq K$, so the first part is obvious. For the second part, let $t_0 = \mathfrak{m}_R$ be the closed point of T, and let $t_1 = (0)$ be the generic point of T. Given a morphism of T to X, let x_0 and x_1 be the images of t_0 and t_1. Since T is reduced, the morphism $T \to X$ factors through Z (Ex. 3.11). Furthermore, $k(x_1)$ is the function field of Z. So we have a local homomorphism of $\mathcal{O} = \mathcal{O}_{x_0,Z}$ to R compatible with the inclusion $k(x_1) \subseteq K$. In other words R dominates $\mathcal{O}$.

Conversely, given the data consisting of x_0, x_1, and the inclusion $k(x_1) \subseteq K$ such that R dominates $\mathcal{O}$, the inclusion $\mathcal{O} \to R$ gives a morphism $T \to \operatorname{Spec} \mathcal{O}$, which composed with the natural map $\operatorname{Spec} \mathcal{O} \to X$ gives the desired morphism $T \to X$.

Lemma 4.5. *Let $f: X \to Y$ be a quasi-compact morphism of schemes* (*see* Ex. 3.2). *Then the subset $f(X)$ of Y is closed if and only if it is stable under specialization* (Ex. 3.17e).

PROOF. One implication is obvious, so we have only to show that if $f(X)$ is stable under specialization, then it is closed. Clearly we may assume that X and Y are both reduced, and that $f(X)^- = Y$ (replace Y by the reduced induced structure on $f(X)^-$). So let $y \in Y$ be a point. We wish to show that $y \in f(X)$. Now we can replace Y by an affine neighborhood of y, and so assume that Y is affine. Then since f is quasi-compact, X will be a finite union of open affines X_i. We know that $y \in f(X)^-$. Hence $y \in f(X_i)^-$ for some i. Let $Y_i = f(X_i)^-$ with the reduced induced structure. Then Y_i is also affine, and we will consider the dominant morphism $X_i \to Y_i$ of reduced affine schemes. Let $X_i = \operatorname{Spec} A$ and $Y_i = \operatorname{Spec} B$. Then the corresponding ring homomorphism $B \to A$ is injective, because the morphism is dominant. The point $y \in Y_i$ corresponds to a prime ideal $\mathfrak{p} \subseteq B$. Let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a minimal prime ideal of B contained in $\mathfrak{p}$. (Minimal prime ideals exist, by Zorn's lemma, because the intersection of any family of prime ideals, totally ordered by inclusion, is again a prime ideal!) Then $\mathfrak{p}'$ corresponds to a point y' of Y_i which specializes to y. I claim $y' \in f(X_i)$. Indeed, let us localize A and B at $\mathfrak{p}'$. Localization is an exact functor, so $B_{\mathfrak{p}'} \subseteq A \otimes B_{\mathfrak{p}'}$. Now $B_{\mathfrak{p}'}$ is a field. Let $\mathfrak{q}_0'$ be any prime ideal of $A \otimes B_{\mathfrak{p}'}$. Then $\mathfrak{q}_0' \cap B_{\mathfrak{p}'} = (0)$. Let $\mathfrak{q}' \subseteq A$ be the inverse image of $\mathfrak{q}_0'$ under the localization map $A \to A \otimes B_{\mathfrak{p}'}$. Then $\mathfrak{q}' \cap B = \mathfrak{p}'$. So $\mathfrak{q}'$ corresponds to a point $x' \in X_i$ with $f(x') = y'$. Now go back to the morphism $f: X \to Y$. We have $x' \in X$, $f(x') = y'$, so $y' \in f(X)$. But $f(X)$ is stable under specialization by hypothesis, and $y' \rightsquigarrow y$, so $y \in f(X)$, which is what we wanted to prove.

PROOF OF THEOREM 4.3. First suppose f is separated, and suppose given a diagram as above where there are two morphisms h,h' of T to X making the whole diagram commutative.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, shift left, "h"] \arrow[ur, shift right, "h'"'] & Y
\end{tikzcd}$$

Then we obtain a morphism $h'' : T \to X \times_Y X$. Since the restrictions of h and h' to U are the same, the generic point t_1 of T has image in the diagonal $\Delta(X)$. Since $\Delta(X)$ is closed, the image of t_0 is also in the diagonal. Therefore h and h' both send the points t_0, t_1 to the same points x_0, x_1 of X. Since the inclusions of $k(x_1) \subseteq K$ induced by h and h' are also the same, it follows from (4.4) that h and h' are equal.

Conversely, let us suppose the condition of the theorem satisfied. To show that f is separated, it is sufficient by (4.2) to show that $\Delta(X)$ is a closed subset of $X \times_Y X$. And since we have assumed that X is noetherian, the morphism Δ is quasi-compact, so by (4.5) it will be sufficient to show that $\Delta(X)$ is stable under specialization. So let $\xi_1 \in \Delta(X)$ be a point, and let $\xi_1 \rightsquigarrow \xi_0$ be a specialization. Let $K = k(\xi_1)$ and let $\mathcal{O}$ be the local ring of ξ_0 on the subscheme $\{\xi_1\}^-$ with its reduced induced structure. Then $\mathcal{O}$ is a local ring contained in K, so by (I, 6.1A) there is a valuation ring R of K which dominates $\mathcal{O}$. Now by (4.4) we obtain a morphism of $T = \operatorname{Spec} R$ to $X \times_Y X$ sending t_0 and t_1 to ξ_0 and ξ_1. Composing with the projections p_1, p_2 gives two morphisms of T to X, which give the same morphism to Y, and whose restrictions to $U = \operatorname{Spec} K$ are the same, since $\xi_1 \in \Delta(X)$. So by the condition, these two morphisms of T to X must be the same. Therefore the morphism $T \to X \times_Y X$ factors through the diagonal morphism $\Delta : X \to X \times_Y X$, and so $\xi_0 \in \Delta(X)$. This completes the proof. Note in the last step it would not be sufficient to know only that $p_1(\xi_0) = p_2(\xi_0)$. For in general if $\xi \in X \times_Y X$ then $p_1(\xi) = p_2(\xi)$ does *not* imply $\xi \in \Delta(X)$.

Corollary 4.6. *Assume that all schemes are noetherian in the following statements.*

(a) *Open and closed immersions are separated.*
(b) *A composition of two separated morphisms is separated.*
(c) *Separated morphisms are stable under base extension.*
(d) *If $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of schemes over a base scheme S, then the product morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is also separated.*
(e) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms and if $g \circ f$ is separated, then f is separated.*
(f) *A morphism $f : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for each i.*

Proof. These statements all follow immediately from the condition of the theorem. We will give the proof of (c) to illustrate the method. Let $f : X \to Y$

be a separated morphism, let $Y' \to Y$ be any morphism, and let $X' = X \times_Y Y'$ be obtained by base extension. We must show that $f': X' \to Y'$ is separated. So suppose we are given morphisms of T to Y' and U to X' as in the theorem, and two morphisms of T to X' making the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, shift left] \arrow[ur, shift right] & Y' \arrow[r] & Y
\end{tikzcd}$$

commutative. Composing with the map $X' \to X$, we obtain two morphisms of T to X. Since f is separated, these are the same. But X' is the fibred product of X and Y' over Y, so by the universal property of the fibred product, the two maps of T to X' are the same. Hence f' is separated.

Note on Noetherian Hypotheses. You have probably noticed that in order to apply the theorem, it is not necessary to assume that all the schemes mentioned in the corollary are noetherian. In fact, even in the theorem itself, you can get by with assuming something less than X noetherian (see Grothendieck [EGA I, new ed., 5.5.4]). My feeling is that if a noetherian hypothesis will make statements and proofs substantially simpler, then I will make that hypothesis, even though it may not be necessary. My justification for this attitude is that most of the motivation and examples in algebraic geometry come from schemes of finite type over a field, and constructions made from them, and practically all the schemes encountered in this way are noetherian. This attitude will prevail in Chapter III, where noetherian hypotheses are built into the very foundations of our treatment of cohomology. The reader who wishes to avoid noetherian hypotheses is advised to read [EGA], especially [EGA IV, §8].

Definition. A morphism $f: X \to Y$ is *proper* if it is separated, of finite type, and universally closed. Here we say that a morphism is *closed* if the image of any closed subset is closed. A morphism $f: X \to Y$ is *universally closed* if it is closed, and for any morphism $Y' \to Y$, the corresponding morphism $f': X' \to Y'$ obtained by base extension is also closed.

Example 4.6.1. Let k be a field and let X be the affine line over k. Then X is separated and of finite type over k, but it is not proper over k. Indeed, take the base extension $X \to k$. The map $X \times_k X \to X$ we obtain is the projection map of the affine plane onto the affine line. This is not a closed map. For example, the hyperbola given by the equation $xy = 1$ is a closed subset of the plane, but its image under projection consists of the affine line minus the origin, which is not closed.

Of course it is clear that what is missing in this example is the point at infinity on the hyperbola. This suggests that the *projective* line would be proper over k. In fact, we will see later (4.9) that any projective variety over a field is proper.

Theorem 4.7 (Valuative Criterion of Properness). *Let $f: X \to Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if for every valuation ring R and for every morphism of U to X and T to Y forming a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(using the notation of (4.3)), there exists a unique morphism $T \to X$ making the whole diagram commutative.

PROOF. First assume that f is proper. Then by definition f is separated, so the uniqueness of the morphism $T \to X$ will follow from (4.3), once we know it exists. For the existence, we consider the base extension $T \to Y$, and let $X_T = X \times_Y T$. We get a map $U \to X_T$ from the given maps $U \to X$ and $U \to T$.

$$\begin{tikzcd} U \arrow[r] & X_T \arrow[r, no head] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & T \arrow[r] & Y \end{tikzcd}$$

Let $\xi_1 \in X_T$ be the image of the unique point t_1 of U. Let $Z = \{\xi_1\}^-$. Then Z is a closed subset of X_T. Since f is proper, it is universally closed, so the morphism $f': X_T \to T$ must be closed, so $f'(Z)$ is a closed subset of T. But $f'(\xi_1) = t_1$, which is the generic point of T, so in fact $f'(Z) = T$. Hence there is a point $\xi_0 \in Z$ with $f'(\xi_0) = t_0$. So we get a local homomorphism of local rings $R \to \mathcal{O}_{\xi_0,Z}$ corresponding to the morphism f'. Now the function field of Z is $k(\xi_1)$, which is contained in K, by construction of ξ_1. By (I, 6.1A), R is maximal for the relation of domination between local subrings of K. Hence R is isomorphic to $\mathcal{O}_{\xi_0,Z}$, and in particular R dominates it. Hence by (4.4) we obtain a morphism of T to X_T sending t_0,t_1 to ξ_0,ξ_1. Composing with the map $X_T \to X$ gives the desired morphism of T to X.

Conversely, suppose the condition of the theorem holds. To show f is proper, we have only to show that it is universally closed, since it is of finite type by hypothesis, and it is separated by (4.3). So let $Y' \to Y$ be any morphism, and let $f': X' \to Y'$ be the morphism obtained from f by base extension. Let Z be a closed subset of X', and give it the reduced induced structure.

$$\begin{tikzcd} Z \subseteq X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

We need to show that $f'(Z)$ is closed in Y'. Since f is of finite type, so is f' and so is the restriction of f' to Z (Ex. 3.13). In particular, the morphism $f':Z \to Y'$ is quasi-compact, so by (4.5) we have only to show that $f(Z)$ is stable under specialization. So let $z_1 \in Z$ be a point, let $y_1 = f'(z_1)$, and let $y_1 \rightsquigarrow y_0$ be a specialization. Let $\mathcal{O}$ be the local ring of y_0 on $\{y_1\}^-$ with its reduced induced structure. Then the quotient field of $\mathcal{O}$ is $k(y_1)$, which is a subfield of $k(z_1)$. Let $K = k(z_1)$, and let R be a valuation ring of K which dominates $\mathcal{O}$ (which exists by (I, 6.1A)).

From this data, by (4.4) we obtain morphisms $U \to Z$ and $T \to Y'$ forming a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & Z \arrow[d] \\ T \arrow[r] & Y'. \end{tikzcd}$$

Composing with the morphisms $Z \to X' \to X$ and $Y' \to Y$, we get morphisms $U \to X$ and $T \to Y$ to which we can apply the condition of the theorem. So there is a morphism of $T \to X$ making the diagram commute. Since X' is a fibred product, it lifts to give a morphism $T \to X'$. And since Z is closed, and the generic point of T goes to $z_1 \in Z$, this morphism factors to give a morphism $T \to Z$. Now let z_0 be the image of t_0. Then $f'(z_0) = y_0$, so $y_0 \in f'(Z)$. This completes the proof.

Corollary 4.8. *In the following statements, we take all schemes to be noetherian.*

(a) *A closed immersion is proper.*

(b) *A composition of proper morphisms is proper.*

(c) *Proper morphisms are stable under base extension.*

(d) *Products of proper morphisms are proper as in* (4.6d).

(e) *If $f:X \to Y$ and $g:Y \to Z$ are two morphisms, if $g \circ f$ is proper, and if g is separated, then f is proper.*

(f) *Properness is local on the base as in* (4.6f).

Proof. These results follow immediately from the condition of the theorem, taking into account (Ex. 3.13) which deals with the finite type property, and (4.6). We will give the proof of (e) to illustrate the method. Assume $g \circ f$ is proper and g is separated. Then f is of finite type by (Ex. 3.13). (We have assumed that X is noetherian, so f is automatically quasi-compact.) Also f is separated by (4.6). So we have to show that given a valuation ring R, and morphisms $U \to X$ and $T \to Y$ making a commutative diagram,

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[ur, dashed] \arrow[r, dashed, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}$$

then there exists a morphism of T to X making the diagram commutative.

Let $T \to Z$ be the composed map. Then since $g \circ f$ is proper, there is a map of T to X commuting with the map of $T \to Z$. By composing with f, we get a second map of T to Y. But now since g is separated, the two maps of T to Y are the same, so we are done.

Our next objective is to define projective morphisms and to show that any projective morphism is proper. Recall that in Section 2 we defined projective n-space $\mathbf{P}_A^n$ over any ring A to be Proj $A[x_0, \ldots, x_n]$. Note that if $A \to B$ is a homomorphism of rings, and Spec $B \to$ Spec A is the corresponding morphism of affine schemes, then $\mathbf{P}_B^n \cong \mathbf{P}_A^n \times_{\operatorname{Spec} A} \operatorname{Spec} B$. In particular, for any ring A, we have $\mathbf{P}_A^n \cong \mathbf{P}_{\mathbf{Z}}^n \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} A$. This motivates the following definition for any scheme Y.

Definition. If Y is any scheme, we define *projective n-space* over Y, denoted $\mathbf{P}_Y^n$, to be $\mathbf{P}_{\mathbf{Z}}^n \times_{\operatorname{Spec} \mathbf{Z}} Y$. A morphism $f: X \to Y$ of schemes is *projective* if it factors into a closed immersion $i: X \to \mathbf{P}_Y^n$ for some n, followed by the projection $\mathbf{P}_Y^n \to Y$. A morphism $f: X \to Y$ is *quasi-projective* if it factors into an open immersion $j: X \to X'$ followed by a projective morphism $g: X' \to Y$. (This definition of projective morphism is slightly different from the one in Grothendieck [EGA II, 5.5]. The two definitions are equivalent in case Y itself is quasi-projective over an affine scheme.)

Example 4.8.1. Let A be a ring, let S be a graded ring with $S_0 = A$, which is finitely generated as an A-algebra by S_1. Then the natural map Proj $S \to$ Spec A is a projective morphism. Indeed, by hypothesis S is a quotient of a polynomial ring $S' = A[x_0, \ldots, x_n]$. The surjective homomorphism of graded rings $S' \to S$ gives rise to a closed immersion Proj $S \to$ Proj $S' = \mathbf{P}_A^n$, which shows that Proj S is projective over A (Ex. 3.12).

Theorem 4.9. *A projective morphism of noetherian schemes is proper. A quasi-projective morphism of noetherian schemes is of finite type and separated.*

PROOF. Taking into account the results of (Ex. 3.13) and (4.6) and (4.8), it will be sufficient to show that $X = \mathbf{P}_{\mathbf{Z}}^n$ is proper over Spec $\mathbf{Z}$. Recall by (2.5) that X is a union of open affine subsets $V_i = D_+(x_i)$, and that V_i is isomorphic to Spec $\mathbf{Z}[x_0/x_i, \ldots, x_n/x_i]$. Thus X is of finite type. To show that X is proper, we will use the criterion of (4.7) and imitate the proof of (I, 6.8). So suppose given a valuation ring R and morphisms $U \to X$, $T \to \operatorname{Spec} \mathbf{Z}$ as shown:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbf{Z}. \end{tikzcd}$$

Let $\xi_1 \in X$ be the image of the unique point of U. Using induction on n, we may assume that ξ_1 is not contained in any of the hyperplanes $X - V_i$, which are each isomorphic to $\mathbf{P}^{n-1}$. In other words, we may assume that $\xi_1 \in \bigcap V_i$, and hence all of the functions x_i/x_j are invertible elements of the local ring $\mathcal{O}_{\xi_1}$.

We have an inclusion $k(\xi_1) \subseteq K$ given by the morphism $U \to X$. Let $f_{ij} \in K$ be the image of x_i/x_j. Then the f_{ij} are nonzero elements of K, and $f_{ik} = f_{ij} \cdot f_{jk}$ for all i,j,k. Let $v: K \to G$ be the valuation associated to the valuation ring R. Let $g_i = v(f_{i0})$ for $i = 0, \ldots, n$. Choose k such that g_k is minimal among the set $\{g_0, \ldots, g_n\}$, for the ordering of G. Then for each i we have

$$v(f_{ik}) = g_i - g_k \geqslant 0,$$

hence $f_{ik} \in R$ for $i = 0, \ldots, n$. Then we can define a homomorphism

$$\varphi: \mathbf{Z}[x_0/x_k, \ldots, x_n/x_k] \to R$$

by sending x_i/x_k to f_{ik}. It is compatible with the given field inclusion $k(\xi_1) \subseteq K$. This homomorphism φ gives a morphism $T \to V_k$, and hence a morphism of T to X which is the one required. The uniqueness of this morphism follows from the construction and the way the V_i patch together.

Proposition 4.10. *Let k be an algebraically closed field. The image of the functor $t: \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ of (2.6) is exactly the set of quasi-projective integral schemes over k. The image of the set of projective varieties is the set of projective integral schemes. In particular, for any variety V, $t(V)$ is an integral, separated scheme of finite type over k.*

PROOF. We have already seen in Section 3 that for any variety V, the associated scheme $t(V)$ is integral and of finite type over k. Since varieties were defined as locally closed subsets of projective space (I, §3), it is clear that $t(V)$ is also quasi-projective.

For the converse, it will be sufficient to show that any projective integral scheme Y over k is in the image of t. Let Y be a closed subscheme of $\mathbf{P}_k^n$, and let V be the set of closed points of Y. Then V is a closed subset of the *variety* $\mathbf{P}^n$. Since V is dense in Y (Ex. 3.14) we see that V is irreducible, so V is a projective variety, and we see also that $t(V)$ and Y have the same underlying topological space. But they are both reduced closed subschemes of $\mathbf{P}_k^n$, so they are isomorphic (Ex. 3.11).

Definition. An *abstract variety* is an integral separated scheme of finite type over an algebraically closed field k. If it is proper over k, we will also say it is *complete*.

Remark 4.10.1. From now on we will use the word "variety" to mean "abstract variety" in the sense just defined. We will identify the varieties of Chapter I with their associated schemes, and refer to them as quasi-projective varieties. We will use the words "curve," "surface," "three-fold," etc., to mean an abstract variety of dimension 1, 2, 3, etc.

Remark 4.10.2. The concept of an abstract variety was invented by Weil [1]. He needed it to provide a purely algebraic construction of the Jacobian variety of a curve, which at first appeared only as an abstract variety (Weil [2]). Then Chow [3] gave a different construction of the Jacobian variety showing that it was in fact a projective variety. Later Weil [6] himself showed that all abelian varieties were projective.

Meanwhile Nagata [1] found an example of a complete abstract non-projective variety, showing that in fact the new class of abstract varieties is larger than the class of projective varieties.

We can sum up the present state of knowledge of this subject as follows.

(a) Every complete curve is projective (III, Ex. 5.8).
(b) Every nonsingular complete surface is projective (Zariski [5]). See also Hartshorne [5, II.4.2].
(c) There exist singular nonprojective complete surfaces (Nagata [3]). See also (Ex. 7.13) and (III, Ex. 5.9).
(d) There exist nonsingular complete nonprojective three-folds (Nagata [4], Hironaka [2], and (Appendix B)).
(e) Every variety can be embedded as an open dense subset of a complete variety (Nagata [6]).

The following algebraic result will be used in (Ex. 4.6).

Theorem 4.11A. *If A is a subring of a field K, then the integral closure of A in K is the intersection of all valuation rings of K which contain A.*

Proof. Bourbaki [1, Ch. VI, §1, no. 3, Thm. 3, p. 92].

Exercises

4.1. Show that a finite morphism is proper.

4.2. Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated. [*Hint*: Consider the map $h: X \to Y \times_S Y$ obtained from f and g.]

4.3. Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

4.4. Let $f: X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S. We refer to this result by saying that "the image of a proper scheme is proper." [*Hint*: Factor f into the graph morphism $\Gamma_f: X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

4.5. Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has *center* x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.

(a) If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.

(b) If X is proper over k, then every valuation of K/k has a unique center on X.

*(c) Prove the converses of (a) and (b). [*Hint*: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.]

(d) If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a). [*Hint*: Let $a \in \Gamma(X, \mathcal{O}_X)$, with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in \mathfrak{m}_R$. Then use (b) to get a contradiction.]

Note. If X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

4.6. Let $f: X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism. [*Hint*: Use (4.11A).]

4.7. *Schemes Over* **R**. For any scheme X_0 over **R**, let $X = X_0 \times_{\mathbf{R}} \mathbf{C}$. Let $\alpha: \mathbf{C} \to \mathbf{C}$ be complex conjugation, and let $\sigma: X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to **C**. Then X is a scheme over **C**, and σ is a *semi-linear* automorphism, in the sense that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathbf{C} \arrow[r, "\alpha"] & \operatorname{Spec} \mathbf{C}. \end{tikzcd}$$

Since $\sigma^2 = \mathrm{id}$, we call σ an *involution*.

(a) Now let X be a separated scheme of finite type over **C**, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over **R**, such that $X_0 \times_{\mathbf{R}} \mathbf{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbf{R}} \mathbf{C}$ described above.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbf{R}$, and X,σ will denote the corresponding scheme with involution over $\mathbf{C}$.

(b) Show that X_0 is affine if and only if X is.

(c) If X_0, Y_0 are two such schemes over $\mathbf{R}$, then to give a morphism $f_0: X_0 \to Y_0$ is equivalent to giving a morphism $f: X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

(d) If $X \cong \mathbf{A}^1_{\mathbf{C}}$, then $X_0 \cong \mathbf{A}^1_{\mathbf{R}}$.

(e) If $X \cong \mathbf{P}^1_{\mathbf{C}}$, then either $X_0 \cong \mathbf{P}^1_{\mathbf{R}}$, or X_0 is isomorphic to the conic in $\mathbf{P}^2_{\mathbf{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

4.8. Let $\mathscr{P}$ be a property of morphisms of schemes such that:

(a) a closed immersion has $\mathscr{P}$;

(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(c) $\mathscr{P}$ is stable under base extension.

Then show that:

(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(e) if $f: X \to Y$ and $g: Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;

(f) If $f: X \to Y$ has $\mathscr{P}$, then $f_{\mathrm{red}}: X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$.

[*Hint*: For (e), consider the graph morphism $\Gamma_f: X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta: Y \to Y \times_Z Y$.]

4.9. Show that a composition of projective morphisms is projective. [*Hint*: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^{rs+r+s}$.] Conclude that projective morphisms have properties (a)–(f) of (Ex. 4.8) above.

***4.10.** *Chow's Lemma.* This result says that proper morphisms are fairly close to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g: X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.

(a) Reduce to the case X irreducible.

(b) Show that X can be covered by a finite number of open subsets U_i, $i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.

(c) Let $U = \bigcap U_i$, and consider the map

$$f: U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure (Ex. 3.11d) $f(U)^-$. Let $g: X' \to X$ be the projection onto the first factor, and let $h: X' \to P = P_1 \times_S \ldots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

(d) Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

4.11. If you are willing to do some harder commutative algebra, and stick to noetherian schemes, then we can express the valuative criteria of separatedness and properness using only *discrete* valuation rings.

(a) If $\mathcal{O}, \mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of

L dominating $\mathcal{O}$. Prove this in the following steps. By taking a polynomial ring over $\mathcal{O}$, reduce to the case where L is a *finite* extension field of K. Then show that for a suitable choice of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$, the ideal $\mathfrak{a} = (x_1)$ in $\mathcal{O}' = \mathcal{O}[x_2/x_1, \ldots, x_n/x_1]$ is not equal to the unit ideal. Then let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$, and let $\mathcal{O}'_{\mathfrak{p}}$ be the localization of $\mathcal{O}'$ at $\mathfrak{p}$. This is a noetherian local domain of dimension 1 dominating $\mathcal{O}$. Let $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ be the integral closure of $\mathcal{O}'_{\mathfrak{p}}$ in L. Use the theorem of Krull–Akizuki (see Nagata [7, p. 115]) to show that $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ is noetherian of dimension 1. Finally, take R to be a localization of $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ at one of its maximal ideals.

(b) Let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all *discrete* valuation rings.

4.12. *Examples of Valuation Rings.* Let k be an algebraically closed field.

(a) If K is a function field of dimension 1 over k (I, §6), then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of (I, §6).

(b) If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.

(1) If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.

(2) If $f: X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.

(3) Let $x_0 \in X$ be a closed point. Let $f: X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2: X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A). Show that R is a valuation ring of K/k, and that it has center x_0 on X. When is R a discrete valuation ring?

Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

5 Sheaves of Modules

So far we have discussed schemes and morphisms between them without mentioning any sheaves other than the structure sheaves. We can increase the flexibility of our technique enormously by considering sheaves of modules on a given scheme. Especially important are quasi-coherent and coherent sheaves, which play the role of modules (respectively, finitely generated modules) over a ring.

In this section we will develop the basic properties of quasi-coherent and coherent sheaves. In particular we will introduce the important "twisting sheaf" $\mathcal{O}(1)$ of Serre on a projective scheme.

We will start by defining sheaves of modules on a ringed space.

Definitions. Let $(X,\mathcal{O}_X)$ be a ringed space (see §2). A *sheaf of $\mathcal{O}_X$-modules* (or simply an *$\mathcal{O}_X$-module*) is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A *morphism* $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Note that the kernel, cokernel, and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. If $\mathcal{F}'$ is a subsheaf of $\mathcal{O}_X$-modules of an $\mathcal{O}_X$-module $\mathcal{F}$, then the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is an $\mathcal{O}_X$-module. Any direct sum, direct product, direct limit, or inverse limit of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, we denote the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$, or sometimes $\operatorname{Hom}_X(\mathcal{F},\mathcal{G})$ or $\operatorname{Hom}(\mathcal{F},\mathcal{G})$ if no confusion can arise. A sequence of $\mathcal{O}_X$-modules and morphisms is *exact* if it is exact as a sequence of sheaves of abelian groups.

If U is an open subset of X, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, the presheaf

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U,\mathcal{G}|_U)$$

is a sheaf, which we call the *sheaf $\mathcal{H}om$* (Ex. 1.15), and denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. It is also an $\mathcal{O}_X$-module.

We define the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. We will often write simply $\mathcal{F} \otimes \mathcal{G}$, with $\mathcal{O}_X$ understood.

An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the *rank* of $\mathcal{F}$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank 1 is also called an *invertible sheaf.*

A *sheaf of ideals* on X is a sheaf of modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Let $f:(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces (see §2). If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Since we have the morphism $f^\#:\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y, this gives $f_*\mathcal{F}$ a natural structure of $\mathcal{O}_Y$-module. We call it the *direct image* of $\mathcal{F}$ by the morphism f.

Now let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1} (Ex. 1.18) we have a morphism

$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X. We define $f^*\mathcal{G}$ to be the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module. We call it the *inverse image* of $\mathcal{G}$ by the morphism f.

As in (Ex. 1.18) one can show that f_* and f^* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules. To be precise, for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_Y$-module $\mathcal{G}$, there is a natural isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G},f_*\mathcal{F}).$$

Now that we have the general notion of a sheaf of modules on a ringed space, we specialize to the case of schemes. We start by defining the sheaf of modules $\tilde{M}$ on Spec A associated to a module M over a ring A.

Definition. Let A be a ring and let M be an A-module. We define the *sheaf associated* to M on Spec A, denoted by $\tilde{M}$, as follows. For each prime ideal $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s: U \to \coprod_{\mathfrak{p}\in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. We make $\tilde{M}$ into a sheaf by using the obvious restriction maps.

Proposition 5.1. *Let A be a ring, let M be an A-module, and let $\tilde{M}$ be the sheaf on $X =$ Spec A associated to M. Then:*

(a) *$\tilde{M}$ is an $\mathcal{O}_X$-module;*

(b) *for each $\mathfrak{p} \in X$, the stalk $(\tilde{M})_\mathfrak{p}$ of the sheaf $\tilde{M}$ at $\mathfrak{p}$ is isomorphic to the localized module $M_\mathfrak{p}$;*

(c) *for any $f \in A$, the A_f-module $\tilde{M}(D(f))$ is isomorphic to the localized module M_f;*

(d) *in particular, $\Gamma(X,\tilde{M}) = M$.*

PROOF. Recalling the construction of the structure sheaf $\mathcal{O}_X$ from §2, it is clear that $\tilde{M}$ is an $\mathcal{O}_X$-module. The proofs of (b), (c), (d) are identical to the proofs of (a), (b), (c) of (2.2), replacing A by M at appropriate places.

Proposition 5.2. *Let A be a ring and let $X =$ Spec A. Also let $A \to B$ be a ring homomorphism, and let f:Spec $B \to$ Spec A be the corresponding morphism of spectra. Then:*

(a) *the map $M \to \tilde{M}$ gives an exact, fully faithful functor from the category of A-modules to the category of $\mathcal{O}_X$-modules;*

(b) *if M and N are two A-modules, then $(M \otimes_A N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$;*

(c) *if $\{M_i\}$ is any family of A-modules, then $(\bigoplus M_i)^\sim \cong \bigoplus \tilde{M}_i$;*

(d) *for any B-module N we have $f_*(\tilde{N}) \cong ({}_AN)^\sim$, where ${}_AN$ means N considered as an A-module;*
(e) *for any A-module M we have $f^*(\tilde{M}) \cong (M \otimes_A B)^\sim$.*

PROOF. The map $M \to \tilde{M}$ is clearly functorial. It is exact, because localization is exact, and exactness of sheaves can be measured at the stalks (use (Ex. 1.2) and (5.1b)). It commutes with direct sum and tensor product, because these commute with localization. To say it is fully faithful means that for any A-modules M and N, we have $\operatorname{Hom}_A(M,N) = \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M},\tilde{N})$. The functor $\sim$ gives a natural map $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M},\tilde{N})$. Applying Γ and using (5.1d) gives a map the other way. These two maps are clearly inverse to each other, hence isomorphisms. The last statements about f_* and f^* follow directly from the definitions.

These sheaves of the form $\tilde{M}$ on affine schemes are our models for quasi-coherent sheaves. A quasi-coherent sheaf on a scheme X will be an $\mathcal{O}_X$-module which is locally of the form $\tilde{M}$. In the next few lemmas and propositions, we will show that this is a local property, and we will establish some facts about quasi-coherent and coherent sheaves.

Definition. Let $(X,\mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$, such that for each i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. We say that $\mathcal{F}$ is *coherent* if furthermore each M_i can be taken to be a finitely generated A_i-module.

Although we have just defined the notion of quasi-coherent and coherent sheaves on an arbitrary scheme, we will normally not mention coherent sheaves unless the scheme is noetherian. This is because the notion of coherence is not at all well-behaved on a nonnoetherian scheme.

Example 5.2.1. On any scheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent (and in fact coherent).

Example 5.2.2. If $X = \operatorname{Spec} A$ is an affine scheme, if $Y \subseteq X$ is the closed subscheme defined by an ideal $\mathfrak{a} \subseteq A$ (3.2.3), and if $i: Y \to X$ is the inclusion morphism, then $i_*\mathcal{O}_Y$ is a quasi-coherent (in fact coherent) $\mathcal{O}_X$-module. Indeed, it is isomorphic to $(A/\mathfrak{a})^\sim$.

Example 5.2.3. If U is an open subscheme of a scheme X, with inclusion map $j: U \to X$, then the sheaf $j_!(\mathcal{O}_U)$ obtained by extending $\mathcal{O}_U$ by zero outside of U (Ex. 1.19), is an $\mathcal{O}_X$-module, but it is not in general quasi-coherent. For example, suppose X is integral, and $V = \operatorname{Spec} A$ is any open affine subset of X, not contained in U. Then $j_!(\mathcal{O}_U)|_V$ has no global sections over V, and yet it is not the zero sheaf. Hence it cannot be of the form $\tilde{M}$ for any A-module M.

Example 5.2.4. If Y is a closed subscheme of a scheme X, then the sheaf $\mathcal{O}_X|_Y$ is not in general quasi-coherent on Y. In fact, it is not even an $\mathcal{O}_Y$-module in general.

Example 5.2.5. Let X be an integral noetherian scheme, and let $\mathscr{K}$ be the constant sheaf with group K equal to the function field of X (Ex. 3.6). Then $\mathscr{K}$ is a quasi-coherent $\mathcal{O}_X$-module, but it is not coherent unless X is reduced to a point.

Lemma 5.3. *Let $X = \operatorname{Spec} A$ be an affine scheme, let $f \in A$, let $D(f) \subseteq X$ be the corresponding open set, and let $\mathscr{F}$ be a quasi-coherent sheaf on X.*

(a) *If $s \in \Gamma(X,\mathscr{F})$ is a global section of $\mathscr{F}$ whose restriction to $D(f)$ is 0, then for some $n > 0$, $f^n s = 0$.*

(b) *Given a section $t \in \mathscr{F}(D(f))$ of $\mathscr{F}$ over the open set $D(f)$, then for some $n > 0$, $f^n t$ extends to a global section of $\mathscr{F}$ over X.*

PROOF. First we note that since $\mathscr{F}$ is quasi-coherent, X can be covered by open affine subsets of the form $V = \operatorname{Spec} B$, such that $\mathscr{F}|_V \cong \tilde{M}$ for some B-module M. Now the open sets of the form $D(g)$ form a base for the topology of X (see §2), so we can cover V by open sets of the form $D(g)$, for various $g \in A$. An inclusion $D(g) \subseteq V$ corresponds to a ring homomorphism $B \to A_g$ by (2.3). Hence $\mathscr{F}|_{D(g)} \cong (M \otimes_B A_g)^\sim$ by (5.2). Thus we have shown that if $\mathscr{F}$ is quasi-coherent on X, then X can be covered by open sets of the form $D(g_i)$ where for each i, $\mathscr{F}|_{D(g_i)} \cong \tilde{M}_i$ for some module M_i over the ring A_{g_i}. Since X is quasi-compact, a finite number of these open sets will do.

(a) Now suppose given $s \in \Gamma(X,\mathscr{F})$ with $s|_{D(f)} = 0$. For each i, s restricts to give a section s_i of $\mathscr{F}$ over $D(g_i)$, in other words, an element $s_i \in M_i$ (using (5.1d)). Now $D(f) \cap D(g_i) = D(fg_i)$, so $\mathscr{F}|_{D(fg_i)} = (M_i)_f^\sim$ using (5.1c). Thus the image of s_i in $(M_i)_f$ is zero, so by the definition of localization, $f^n s_i = 0$ for some n. This n may depend on i, but since there are only finitely many i, we can pick n large enough to work for them all. Then since the $D(g_i)$ cover X, we have $f^n s = 0$.

(b) Given an element $t \in \mathscr{F}(D(f))$, we restrict it for each i to get an element t of $\mathscr{F}(D(fg_i)) = (M_i)_f$. Then by the definition of localization, for some $n > 0$ there is an element $t_i \in M_i = \mathscr{F}(D(g_i))$ which restricts to $f^n t$ on $D(fg_i)$. The integer n may depend on i, but again we take one large enough to work for all i. Now on the intersection $D(g_i) \cap D(g_j) = D(g_i g_j)$ we have two sections t_i and t_j of $\mathscr{F}$, which agree on $D(fg_i g_j)$ where they are both equal to $f^n t$. Hence by part (a) above, there is an integer $m > 0$ such that $f^m(t_i - t_j) = 0$ on $D(g_i g_j)$. This m depends on i and j, but we take one m large enough for all. Now the local sections $f^m t_i$ of $\mathscr{F}$ on $D(g_i)$ glue together to give a global section s of $\mathscr{F}$, whose restriction to $D(f)$ is $f^{n+m} t$.

Proposition 5.4. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, there is an A-module M such that $\mathscr{F}|_U \cong \tilde{M}$. If X is noetherian, then $\mathscr{F}$ is coherent if and only if the same is true, with the extra condition that M be a finitely generated A-module.*

Proof. Let $\mathscr{F}$ be quasi-coherent on X, and let $U = \operatorname{Spec} A$ be an open affine. As in the proof of the lemma, there is a base for the topology consisting of open affines for which the restriction of $\mathscr{F}$ is the sheaf associated to a module. It follows that $\mathscr{F}|_U$ is quasi-coherent, so we can reduce to the case X affine $= \operatorname{Spec} A$. Let $M = \Gamma(X,\mathscr{F})$. Then in any case there is a natural map $\alpha: \tilde{M} \to \mathscr{F}$ (Ex. 5.3). Since $\mathscr{F}$ is quasi-coherent, X can be covered by open sets $D(g_i)$ with $\mathscr{F}|_{D(g_i)} \cong \tilde{M}_i$ for some A_{g_i}-module M_i. Now the lemma, applied to the open set $D(g_i)$, tells us exactly that $\mathscr{F}(D(g_i)) \cong M_{g_i}$, so $M_i = M_{g_i}$. It follows that the map α, restricted to $D(g_i)$, is an isomorphism. The $D(g_i)$ cover X, so α is an isomorphism.

Now suppose that X is noetherian, and $\mathscr{F}$ coherent. Then, using the above notation, we have the additional information that each M_{g_i} is a finitely generated A_{g_i}-module, and we want to prove that M is finitely generated. Since the rings A and A_{g_i} are noetherian, the modules M_{g_i} are noetherian, and we have to prove that M is noetherian. For this we just use the proof of (3.2) with A replaced by M in appropriate places.

Corollary 5.5. *Let A be a ring and let $X = \operatorname{Spec} A$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathscr{F} \mapsto \Gamma(X,\mathscr{F})$. If A is noetherian, the same functor also gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules.*

Proof. The only new information here is that $\mathscr{F}$ is quasi-coherent on X if and only if it is of the form $\tilde{M}$, and in that case $M = \Gamma(X,\mathscr{F})$. This follows from (5.4).

Proposition 5.6. *Let X be an affine scheme, let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathscr{F}'$ is quasi-coherent. Then the sequence*

$$0 \to \Gamma(X,\mathscr{F}') \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}'') \to 0$$

is exact.

Proof. We know already that Γ is a left-exact functor (Ex. 1.8) so we have only to show that the last map is surjective. Let $s \in \Gamma(X,\mathscr{F}'')$ be a global section of $\mathscr{F}''$. Since the map of sheaves $\mathscr{F} \to \mathscr{F}''$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x, such that $s|_{D(f)}$ lifts to a section $t \in \mathscr{F}(D(f))$ (Ex. 1.3). I claim that for some $n > 0$, $f^n s$ lifts to a global section of $\mathscr{F}$. Indeed, we can cover X with a finite number of open sets $D(g_i)$, such that for each i, $s|_{D(g_i)}$ lifts to a section $t_i \in \mathscr{F}(D(g_i))$. On $D(f) \cap D(g_i) = D(fg_i)$, we have two sections $t, t_i \in \mathscr{F}(D(fg_i))$ both lifting s. Therefore $t - t_i \in \mathscr{F}'(D(fg_i))$. Since $\mathscr{F}'$ is quasi-coherent, by (5.3b) for some $n > 0$, $f^n(t - t_i)$ extends to a section $u_i \in \mathscr{F}'(D(g_i))$. As usual, we pick one n to work for all i. Let $t_i' = f^n t_i + u_i$. Then t_i' is a lifting of $f^n s$ on $D(g_i)$, and furthermore t_i' and $f^n t$ agree on $D(fg_i)$. Now on $D(g_ig_j)$ we have two sections t_i' and t_j' of $\mathscr{F}$, both of which lift $f^n s$, so $t_i' - t_j' \in \mathscr{F}'(D(g_ig_j))$. Furthermore, t_i' and t_j' are equal on $D(fg_ig_j)$, so by (5.3a) we have $f^m(t_i' - t_j') = 0$ for some $m > 0$, which we may take independent of i and j. Now the sections $f^m t_i'$ of $\mathscr{F}$ glue to give a global section t'' of $\mathscr{F}$ over X, which lifts $f^{n+m}s$. This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$, $i = 1, \ldots, r$, such that $s|_{D(f_i)}$ lifts to a section of $\mathscr{F}$ over $D(f_i)$ for each i. Then by the claim, we can find an integer n (one for all i) and global sections $t_i \in \Gamma(X, \mathscr{F})$ such that t_i is a lifting of $f_i^n s$. Now the open sets $D(f_i)$ cover X, so the ideal $(f_1^n, \ldots, f_r^n)$ is the unit ideal of A, and we can write $1 = \sum_{i=1}^r a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathscr{F}$ whose image in $\Gamma(X, \mathscr{F}'')$ is $\sum a_i f_i^n s = s$. This completes the proof.

Remark 5.6.1. When we have developed the techniques of cohomology, we will see that this proposition is an immediate consequence of the fact that $H^1(X, \mathscr{F}') = 0$ for any quasi-coherent sheaf $\mathscr{F}'$ on an affine scheme X (III, 3.5).

Proposition 5.7. *Let X be a scheme. The kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent. If X is noetherian, the same is true for coherent sheaves.*

PROOF. The question is local, so we may assume X is affine. The statement about kernels, cokernels and images follows from the fact that the functor $M \mapsto \tilde{M}$ is exact and fully faithful from A-modules to quasi-coherent sheaves (5.2a and 5.5). The only nontrivial part is to show that an extension of quasi-coherent sheaves is quasi-coherent. So let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, with $\mathscr{F}'$ and $\mathscr{F}''$ quasi-coherent. By (5.6), the corresponding sequence of global sections over X is exact, say $0 \to M' \to M \to M'' \to 0$. Applying the functor $\sim$, we get an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \tilde{M}' & \to & \tilde{M} & \to & \tilde{M}'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{F}' & \to & \mathscr{F} & \to & \mathscr{F}'' & \to & 0. \end{array}$$

The two outside arrows are isomorphisms, since $\mathscr{F}'$ and $\mathscr{F}''$ are quasi-coherent. So by the 5-lemma, the middle one is also, showing that $\mathscr{F}$ is quasi-coherent.

In the noetherian case, if $\mathscr{F}'$ and $\mathscr{F}''$ are coherent, then M' and M'' are finitely generated, so M is also finitely generated, and hence $\mathscr{F}$ is coherent.

Proposition 5.8. *Let $f:X \to Y$ be a morphism of schemes.*

(a) *If $\mathscr{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, then $f^*\mathscr{G}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

(b) *If X and Y are noetherian, and if $\mathscr{G}$ is coherent, then $f^*\mathscr{G}$ is coherent.*

(c) *Assume that either X is noetherian, or f is quasi-compact* (Ex. 3.2) *and separated. Then if $\mathscr{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathscr{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules.*

PROOF.

(a) The question is local on both X and Y, so we can assume X and Y both affine. In this case the result follows from (5.5) and (5.2e).

(b) In the noetherian case, the same proof works for coherent sheaves.

(c) Here the question is local on Y only, so we may assume that Y is affine. Then X is quasi-compact (under either hypothesis) so we can cover X with a finite number of open affine subsets U_i. In the separated case, $U_i \cap U_j$ is again affine (Ex. 4.3). Call it U_{ijk}. In the noetherian case, $U_i \cap U_j$ is at least quasi-compact, so we can cover it with a finite number of open affine subsets U_{ijk}. Now for any open subset V of Y, giving a section s of $\mathscr{F}$ over $f^{-1}V$ is the same thing as giving a collection of sections s_i of $\mathscr{F}$ over $(f^{-1}V) \cap U_i$ whose restrictions to the open subsets $f^{-1}(V) \cap U_{ijk}$ are all equal. This is just the sheaf property (§1). Therefore, there is an exact sequence of sheaves on Y,

$$0 \to f_*\mathscr{F} \to \bigoplus_i f_*(\mathscr{F}|_{U_i}) \to \bigoplus_{i,j,k} f_*(\mathscr{F}|_{U_{ijk}}),$$

where by abuse of notation we denote also by f the induced morphisms $U_i \to Y$ and $U_{ijk} \to Y$. Now $f_*(\mathscr{F}|_{U_i})$ and $f_*(\mathscr{F}|_{U_{ijk}})$ are quasi-coherent by (5.2d). Thus $f_*\mathscr{F}$ is quasi-coherent by (5.7).

Caution 5.8.1. If X and Y are noetherian, it is *not* true in general that f_* of a coherent sheaf is coherent (Ex. 5.5). However, it is true if f is a finite morphism (Ex. 5.5) or a projective morphism (5.20) or (III, 8.8), or more generally, a proper morphism: see Grothendieck [EGA III, 3.2.1].

As a first application of these concepts, we will discuss the sheaf of ideals of a closed subscheme.

Definition. Let Y be a closed subscheme of a scheme X, and let $i: Y \to X$ be the inclusion morphism. We define the *ideal sheaf* of Y, denoted $\mathscr{I}_Y$, to be the kernel of the morphism $i^\#: \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Proposition 5.9. *Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathscr{I}_Y$ is a quasi-coherent sheaf of ideals on X. If X is noetherian, it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

PROOF. If Y is a closed subscheme of X, then the inclusion morphism $i: Y \rightarrow X$ is quasi-compact (obvious) and separated (4.6), so by (5.8), $i_*\mathcal{O}_Y$ is quasi-coherent on X. Hence $\mathscr{I}_Y$, being the kernel of a morphism of quasi-coherent sheaves, is also quasi-coherent. If X is noetherian, then for any open affine subset $U = \operatorname{Spec} A$ of X, the ring A is noetherian, so the ideal $I = \Gamma(U, \mathscr{I}_Y|_U)$, is finitely generated, so $\mathscr{I}_Y$ is coherent.

Conversely, given a scheme X and a quasi-coherent sheaf of ideals $\mathscr{J}$, let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathscr{J}$. Then Y is a subspace of X, and $(Y, \mathcal{O}_X/\mathscr{J})$ is the unique closed subscheme of X with ideal sheaf $\mathscr{J}$. The unicity is clear, so we have only to check that $(Y, \mathcal{O}_X/\mathscr{J})$ is a closed subscheme. This is a local question, so we may assume $X = \operatorname{Spec} A$ is affine. Since $\mathscr{J}$ is quasi-coherent, $\mathscr{J} = \tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subseteq A$. Then $(Y, \mathcal{O}_X/\mathscr{J})$ is just the closed subscheme of X determined by the ideal $\mathfrak{a}$ (3.2.3).

Corollary 5.10. *If $X = \operatorname{Spec} A$ is an affine scheme, there is a 1-1 correspondence between ideals $\mathfrak{a}$ in A and closed subschemes Y of X, given by $\mathfrak{a} \mapsto$ image of* Spec $A/\mathfrak{a}$ *in X* (3.2.3). *In particular, every closed subscheme of an affine scheme is affine.*

PROOF. By (5.5) the quasi-coherent sheaves of ideals on X are in 1-1 correspondence with the ideals of A.

Our next concern is to study quasi-coherent sheaves on the Proj of a graded ring. As in the case of Spec, there is a connection between modules over the ring and sheaves on the space, but it is more complicated.

Definition. Let S be a graded ring and let M be a graded S-module. (See (I, §7) for generalities on graded modules.) We define the *sheaf associated* to M on Proj S, denoted by $\tilde{M}$, as follows. For each $\mathfrak{p} \in \operatorname{Proj} S$, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in the localization $T^{-1}M$, where T is the multiplicative system of homogeneous elements of S not in $\mathfrak{p}$ (cf. definition of Proj in §2). For any open subset $U \subseteq \operatorname{Proj} S$ we define $\tilde{M}(U)$ to be the set of functions s from U to $\coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$ which are locally fractions. This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $m \in M$ and $f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_{(\mathfrak{q})}$. We make $\tilde{M}$ into a sheaf with the obvious restriction maps.

Proposition 5.11. *Let S be a graded ring, and M a graded S-module. Let $X = \operatorname{Proj} S$.*

(a) *For any* $\mathfrak{p} \in X$, *the stalk* $(\tilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$.

(b) *For any homogeneous* $f \in S_+$, *we have* $\tilde{M}|_{D_+(f)} \cong (M_{(f)})^{\sim}$ *via the isomorphism of* $D_+(f)$ *with* Spec $S_{(f)}$ *(see (2.5b)), where* $M_{(f)}$ *denotes the group of elements of degree* 0 *in the localized module* M_f.

(c) $\tilde{M}$ *is a quasi-coherent* $\mathcal{O}_X$*-module. If* S *is noetherian and* M *is finitely generated, then* $\tilde{M}$ *is coherent.*

Proof. For (a) and (b), just repeat the proof of (2.5), with M in place of S. Then (c) follows from (b).

Definition. Let S be a graded ring, and let $X = \operatorname{Proj} S$. For any $n \in \mathbf{Z}$, we define the sheaf $\mathcal{O}_X(n)$ to be $S(n)^{\sim}$. We call $\mathcal{O}_X(1)$ the *twisting sheaf* of Serre. For any sheaf of $\mathcal{O}_X$-modules, $\mathscr{F}$, we denote by $\mathscr{F}(n)$ the *twisted sheaf* $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 5.12. *Let* S *be a graded ring and let* $X = \operatorname{Proj} S$. *Assume that* S *is generated by* S_1 *as an* S_0*-algebra.*

(a) *The sheaf* $\mathcal{O}_X(n)$ *is an invertible sheaf on* X.

(b) *For any graded* S*-module* M, $\tilde{M}(n) \cong (M(n))^{\sim}$. *In particular,* $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n + m)$.

(c) *Let* T *be another graded ring, generated by* T_1 *as a* T_0*-algebra, let* $\varphi: S \to T$ *be a homomorphism preserving degrees, and let* $U \subseteq Y = \operatorname{Proj} T$ *and* $f: U \to X$ *be the morphism determined by* φ (Ex. 2.14). *Then* $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$ *and* $f_*(\mathcal{O}_Y(n)|_U) \cong (f_*\mathcal{O}_U)(n)$.

Proof.

(a) Recall that invertible means locally free of rank 1. Let $f \in S_1$, and consider the restriction $\mathcal{O}_X(n)|_{D_+(f)}$. By the previous proposition this is isomorphic to $S(n)^{\sim}_{(f)}$ on Spec $S_{(f)}$. We will show that this restriction is free of rank 1. Indeed, $S(n)_{(f)}$ is a free $S_{(f)}$-module of rank 1. For $S_{(f)}$ is the group of elements of degree 0 in S_f, and $S(n)_{(f)}$ is the group of elements of degree n in S_f. We obtain an isomorphism of one to the other by sending s to $f^n s$. This makes sense, for any $n \in \mathbf{Z}$, because f is invertible in S_f. Now since S is generated by S_1 as an S_0-algebra, X is covered by the open sets $D_+(f)$ for $f \in S_1$. Hence $\mathcal{O}(n)$ is invertible.

(b) This follows from the fact that $(M \otimes_S N)^{\sim} \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ for any two graded S-modules M and N, when S is generated by S_1. Indeed, for any $f \in S_1$ we have $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$.

(c) More generally, for any graded S-module M, $f^*(\tilde{M}) \cong (M \otimes_S T)^{\sim}|_U$ and for any graded T-module N, $f_*(\tilde{N}|_U) \cong ({}_S N)^{\sim}$. Furthermore, the sheaf $\tilde{T}$ on X is just $f_*(\mathcal{O}_U)$. The proofs are straightforward (cf. (5.2) for the affine case).

The twisting operation allows us to define a graded S-module associated to any sheaf of modules on $X = \operatorname{Proj} S$.

Definition. Let S be a graded ring, let $X = \operatorname{Proj} S$, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *graded S-module associated* to $\mathscr{F}$ as a group, to be $\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X,\mathscr{F}(n))$. We give it a structure of graded S-module as follows. If $s \in S_d$, then s determines in a natural way a global section $s \in \Gamma(X,\mathcal{O}_X(d))$. Then for any $t \in \Gamma(X,\mathscr{F}(n))$ we define the product $s \cdot t$ in $\Gamma(X,\mathscr{F}(n + d))$ by taking the tensor product $s \otimes t$ and using the natural map $\mathscr{F}(n) \otimes \mathcal{O}_X(d) \cong \mathscr{F}(n + d)$.

Proposition 5.13. *Let A be a ring, let $S = A[x_0,\ldots,x_r]$, $r \geqslant 1$, and let $X = \operatorname{Proj} S$. (This is just projective r-space over A.) Then $\Gamma_*(\mathcal{O}_X) \cong S$.*

PROOF. We cover X with the open sets $D_+(x_i)$. Then to give a section $t \in \Gamma(X,\mathcal{O}_X(n))$ is the same as giving sections $t_i \in \mathcal{O}_X(n)(D_+(x_i))$ for each i, which agree on the intersections $D_+(x_ix_j)$. Now t_i is just a homogeneous element of degree n in the localization S_{x_i}, and its restriction to $D_+(x_ix_j)$ is just the image of that element in $S_{x_ix_j}$. Summing over all n, we see that $\Gamma_*(\mathcal{O}_X)$ can be identified with the set of $(r + 1)$-tuples $(t_0,\ldots,t_r)$ where for each i, $t_i \in S_{x_i}$, and for each i,j, the images of t_i and t_j in $S_{x_ix_j}$ are the same.

Now the x_i are not zero divisors in S, so the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_ix_j}$ are all injective, and these rings are all subrings of $S' = S_{x_0\cdots x_r}$. Hence $\Gamma_*(\mathcal{O}_X)$ is the intersection $\bigcap S_{x_i}$ taken inside S'. Now any homogeneous element of S' can be written uniquely as a product $x_0^{i_0} \cdots x_r^{i_r}f(x_0,\ldots,x_r)$, where the $i_j \in \mathbf{Z}$, and f is a homogeneous polynomial not divisible by any x_i. This element will be in S_{x_i} if and only if $i_j \geqslant 0$ for $j \neq i$. It follows that the intersection of all the S_{x_i} (in fact the intersection of any two of them) is exactly S.

Caution 5.13.1. If S is a graded ring which is not a polynomial ring, then it is not true in general that $\Gamma_*(\mathcal{O}_X) = S$ (Ex. 5.14).

Lemma 5.14. *Let X be a scheme, let $\mathscr{L}$ be an invertible sheaf on X, let $f \in \Gamma(X,\mathscr{L})$, let X_f be the open set of points $x \in X$ where $f_x \notin \mathfrak{m}_x\mathscr{L}_x$, and let $\mathscr{F}$ be a quasi-coherent sheaf on X.*

(a) *Suppose that X is quasi-compact, and let $s \in \Gamma(X,\mathscr{F})$ be a global section of $\mathscr{F}$ whose restriction to X_f is 0. Then for some $n > 0$, we have $f^ns = 0$, where f^ns is considered as a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.*

(b) *Suppose furthermore that X has a finite covering by open affine subsets U_i, such that $\mathscr{L}|_{U_i}$ is free for each i, and such that $U_i \cap U_j$ is quasi-compact for each i,j. Given a section $t \in \Gamma(X_f,\mathscr{F})$, then for some $n > 0$, the section $f^nt \in \Gamma(X_f,\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ extends to a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.*

PROOF. This lemma is a direct generalization of (5.3), with an extra twist due to the presence of the invertible sheaf $\mathscr{L}$. It also generalizes (Ex. 2.16). To prove (a), we first cover X with a finite number (possible since X is quasi-compact) of open affines $U = \operatorname{Spec} A$ such that $\mathscr{L}|_U$ is free. Let $\psi: \mathscr{L}|_U \cong \mathcal{O}_U$ be an isomorphism expressing the freeness of $\mathscr{L}|_U$. Since $\mathscr{F}$ is quasi-coherent, by (5.4) there is an A-module M with $\mathscr{F}|_U \cong \tilde{M}$. Our section $s \in \Gamma(X,\mathscr{F})$ restricts to give an element $s \in M$. On the other hand, our section $f \in \Gamma(X,\mathscr{L})$ restricts to give a section of $\mathscr{L}|_U$, which in turn gives rise to an element $g = \psi(f) \in A$. Clearly $X_f \cap U = D(g)$. Now $s|_{X_f}$ is zero, so $g^n s = 0$ in M for some $n > 0$, just as in the proof of (5.3). Using the isomorphism

$$\mathrm{id} \times \psi^{\otimes n} \colon \mathscr{F} \otimes \mathscr{L}^n|_U \cong \mathscr{F}|_U,$$

we conclude that $f^n s \in \Gamma(U,\mathscr{F} \otimes \mathscr{L}^n)$ is zero. This statement is intrinsic (i.e., independent of ψ). So now we do this for each open set of the covering, pick one n large enough to work for all the sets of the covering, and we find $f^n s = 0$ on X.

To prove (b), we proceed as in the proof of (5.3), keeping track of the twist due to $\mathscr{L}$ as above. The hypothesis $U_i \cap U_j$ quasi-compact is used to be able to apply part (a) there.

Remark 5.14.1. The hypotheses on X made in the statements (a) and (b) above are satisfied either if X is noetherian (in which case every open set is quasi-compact) or if X is quasi-compact and separated (in which case the intersection of two open affine subsets is again affine, hence quasi-compact).

Proposition 5.15. *Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism $\beta \colon \Gamma_*(\mathscr{F})^\sim \to \mathscr{F}$.*

PROOF. First let us define the morphism β for any $\mathcal{O}_X$-module $\mathscr{F}$. Let $f \in S_1$. Since $\Gamma_*(\mathscr{F})^\sim$ is quasi-coherent in any case, to define β, it is enough to give the image of a section of $\Gamma_*(\mathscr{F})^\sim$ over $D_+(f)$ (see Ex. 5.3). Such a section is represented by a fraction m/f^d, where $m \in \Gamma(X,\mathscr{F}(d))$, for some $d \geqslant 0$. We can think of f^{-d} as a section of $\mathcal{O}_X(-d)$, defined over $D_+(f)$. Taking their tensor product, we obtain $m \otimes f^{-d}$ as a section of $\mathscr{F}$ over $D_+(f)$. This defines β.

Now let $\mathscr{F}$ be quasi-coherent. To show that β is an isomorphism we have to identify the module $\Gamma_*(\mathscr{F})_{(f)}$ with the sections of $\mathscr{F}$ over $D_+(f)$. We apply (5.14), considering f as a global section of the invertible sheaf $\mathscr{L} = \mathcal{O}(1)$. Since we have assumed that S is finitely generated by S_1 as an S_0-algebra, we can find finitely many elements $f_0, \ldots, f_r \in S_1$ such that X is covered by the open affine subsets $D_+(f_i)$. The intersections $D_+(f_i) \cap D_+(f_j)$ are also affine, and $\mathscr{L}|_{D_+(f_i)}$ is free for each i, so the hypotheses of (5.14) are satisfied. The conclusion of (5.14) tells us that $\mathscr{F}(D_+(f)) \cong \Gamma_*(\mathscr{F})_{(f)}$, which is just what we wanted.

Corollary 5.16. *Let A be a ring.*

(a) *If Y is a closed subscheme of $\mathbf{P}_A^r$, then there is a homogeneous ideal $I \subseteq S = A[x_0, \ldots, x_r]$ such that Y is the closed subscheme determined by I* (Ex. 3.12).

(b) *A scheme Y over* Spec A *is projective if and only if it is isomorphic to* Proj S *for some graded ring S, where $S_0 = A$, and S is finitely generated by S_1 as an S_0-algebra.*

PROOF.

(a) Let $\mathscr{I}_Y$ be the ideal sheaf of Y on $X = \mathbf{P}_A^r$. Now $\mathscr{I}_Y$ is a subsheaf of $\mathcal{O}_X$; the twisting functor is exact; the global section functor Γ is left exact; hence $\Gamma_*(\mathscr{I}_Y)$ is a submodule of $\Gamma_*(\mathcal{O}_X)$. But by (5.13), $\Gamma_*(\mathcal{O}_X) = S$. Hence $\Gamma_*(\mathscr{I}_Y)$ is a homogeneous ideal of S, which we will call I. Now I determines a closed subscheme of X (Ex. 3.12), whose sheaf of ideals will be $\tilde{I}$. Since $\mathscr{I}_Y$ is quasi-coherent by (5.9), we have $\mathscr{I}_Y \cong \tilde{I}$ by (5.15), and hence Y is the subscheme determined by I. In fact, $\Gamma_*(\mathscr{I}_Y)$ is the largest ideal in S defining Y (Ex. 5.10).

(b) Recall that by definition Y is projective over Spec A if it is isomorphic to a closed subscheme of $\mathbf{P}_A^r$ for some r (§4). By part (a), any such Y is isomorphic to Proj S/I, and we can take I to be contained in $S_+ = \bigoplus_{d>0} S_d$ (Ex. 3.12), so that $(S/I)_0 = A$. Conversely, any such graded ring S is a quotient of a polynomial ring, so Proj S is projective.

Definition. For any scheme Y, we define the *twisting sheaf* $\mathcal{O}(1)$ on $\mathbf{P}_Y^r$ to be $g^*(\mathcal{O}(1))$, where $g: \mathbf{P}_Y^r \to \mathbf{P}_{\mathbf{Z}}^r$ is the natural map (recall that $\mathbf{P}_Y^r$ was defined as $\mathbf{P}_{\mathbf{Z}}^r \times_{\mathbf{Z}} Y$).

Note that if $Y = \operatorname{Spec} A$, this is the same as the $\mathcal{O}(1)$ already defined on $\mathbf{P}_A^r = \operatorname{Proj} A[x_0, \ldots, x_r]$, by (5.12c).

Definition. If X is any scheme over Y, an invertible sheaf $\mathscr{L}$ on X is *very ample* relative to Y, if there is an immersion $i: X \to \mathbf{P}_Y^r$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathscr{L}$. We say that a morphism $i: X \to Z$ is an *immersion* if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z. (This definition of very ample differs slightly from the one in Grothendieck [EGA II, 4.4.2].)

Remark 5.16.1. Let Y be a noetherian scheme. Then a scheme X over Y is projective if and only if it is proper, and there exists a very ample sheaf on X relative to Y. Indeed, if X is projective over Y, then X is proper by (4.9). On the other hand, there is a closed immersion $i: X \to \mathbf{P}_Y^r$ for some r, so $i^*\mathcal{O}(1)$ is a very ample invertible sheaf on X. Conversely, if X is proper over Y, and $\mathscr{L}$ is a very ample invertible sheaf, then $\mathscr{L} \cong i^*(\mathcal{O}(1))$ for some immersion $i: X \to \mathbf{P}_Y^r$. But by (Ex. 4.4) the image of X is closed, so in fact i is a closed immersion, so X is projective over Y.

Note however that there may be several nonisomorphic very ample sheaves on a projective scheme X over Y. The sheaf $\mathscr{L}$ depends on the embedding of X into $\mathbf{P}_Y^r$ (Ex. 5.12). If $Y = \operatorname{Spec} A$, and if $X = \operatorname{Proj} S$, where S is a graded ring as in (5.16b), then the sheaf $\mathcal{O}(1)$ on X defined earlier is a very ample sheaf on X. However, there may be nonisomorphic graded rings having the same Proj and the same very ample sheaf $\mathcal{O}(1)$ (Ex. 2.14).

We end this section with some special results about sheaves on a projective scheme over a noetherian ring.

Definition. Let X be a scheme, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is *generated by global sections* if there is a family of global sections $\{s_i\}_{i \in I}$, $s_i \in \Gamma(X,\mathscr{F})$, such that for each $x \in X$, the images of s_i in the stalk $\mathscr{F}_x$ generate that stalk as an $\mathscr{O}_x$-module.

Note that $\mathscr{F}$ is generated by global sections if and only if $\mathscr{F}$ can be written as a quotient of a free sheaf. Indeed, the generating sections $\{s_i\}_{i \in I}$ define a surjective morphism of sheaves $\bigoplus_{i \in I} \mathscr{O}_X \rightarrow \mathscr{F}$, and conversely.

Example 5.16.2. Any quasi-coherent sheaf on an affine scheme is generated by global sections. Indeed, if $\mathscr{F} = \tilde{M}$ on Spec A, any set of generators for M as an A-module will do.

Example 5.16.3. Let $X = \operatorname{Proj} S$, where S is a graded ring which is generated by S_1 as an S_0-algebra. Then the elements of S_1 give global sections of $\mathscr{O}_X(1)$ which generate it.

Theorem 5.17 (Serre). *Let X be a projective scheme over a noetherian ring A, let $\mathscr{O}(1)$ be a very ample invertible sheaf on X, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then there is an integer n_0 such that for all $n \geqslant n_0$, the sheaf $\mathscr{F}(n)$ can be generated by a finite number of global sections.*

PROOF. Let $i: X \rightarrow \mathbf{P}_A^r$ be a closed immersion of X into a projective space over A, such that $i^*(\mathscr{O}(1)) = \mathscr{O}_X(1)$. Then $i_*\mathscr{F}$ is coherent on $\mathbf{P}_A^r$ (Ex. 5.5), and $i_*(\mathscr{F}(n)) = (i_*\mathscr{F})(n)$ (5.12) or (Ex. 5.1d), and $\mathscr{F}(n)$ is generated by global sections if and only if $i_*(\mathscr{F}(n))$ is (in fact, their global sections are the same), so we reduce to the case $X = \mathbf{P}_A^r = \operatorname{Proj} A[x_0, \ldots, x_r]$.

Now cover X with the open sets $D_+(x_i)$, $i = 0, \ldots, r$. Since $\mathscr{F}$ is coherent, for each i there is a finitely generated module M_i over $B_i = A[x_0/x_i, \ldots, x_n/x_i]$ such that $\mathscr{F}|_{D_+(x_i)} \cong \tilde{M}_i$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate this module. By (5.14) there is an integer n such that $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathscr{F}(n)$. As usual, we take one n to work for all i,j. Now $\mathscr{F}(n)$ corresponds to a B_i-module M_i' on $D_+(x_i)$, and the map $x_i^n: \mathscr{F} \rightarrow \mathscr{F}(n)$ induces an *isomorphism* of M_i to M_i'. So the sections $x_i^n s_{ij}$ generate M_i', and hence the global sections $t_{ij} \in \Gamma(X,\mathscr{F}(n))$ generate the sheaf $\mathscr{F}(n)$ everywhere.

Corollary 5.18. *Let X be projective over a noetherian ring A. Then any coherent sheaf $\mathscr{F}$ on X can be written as a quotient of a sheaf $\mathscr{E}$, where $\mathscr{E}$ is a finite direct sum of twisted structure sheaves $\mathscr{O}(n_i)$ for various integers n_i.*

PROOF. Let $\mathscr{F}(n)$ be generated by a finite number of global sections. Then we have a surjection $\bigoplus_{i=1}^{N} \mathcal{O}_X \to \mathscr{F}(n) \to 0$. Tensoring with $\mathcal{O}_X(-n)$ we obtain a surjection $\bigoplus_{i=1}^{N} \mathcal{O}_X(-n) \to \mathscr{F} \to 0$ as required.

Theorem 5.19. *Let k be a field, let A be a finitely generated k-algebra, let X be a projective scheme over A, and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then $\Gamma(X,\mathscr{F})$ is a finitely generated A-module. In particular, if $A = k$, $\Gamma(X,\mathscr{F})$ is a finite-dimensional k-vector space.*

PROOF. First we write $X = \operatorname{Proj} S$, where S is a graded ring with $S_0 = A$ which is finitely generated by S_1 as an S_0-algebra (5.16b). Let M be the graded S-module $\Gamma_*(\mathscr{F})$. Then by (5.15) we have $\tilde{M} \cong \mathscr{F}$. On the other hand, by (5.17), for n sufficiently large, $\mathscr{F}(n)$ is generated by a finite number of global sections in $\Gamma(X,\mathscr{F}(n))$. Let M' be the submodule of M generated by these sections. Then M' is a finitely generated S-module. Furthermore, the inclusion $M' \hookrightarrow M$ induces an inclusion of sheaves $\tilde{M}' \hookrightarrow \tilde{M} = \mathscr{F}$. Twisting by n we have an inclusion $\tilde{M}'(n) \hookrightarrow \mathscr{F}(n)$ which is actually an isomorphism, because $\mathscr{F}(n)$ is generated by global sections in M'. Twisting by $-n$ we find that $\tilde{M}' \cong \mathscr{F}$. Thus $\mathscr{F}$ is the sheaf associated to a finitely generated S-module, and so we have reduced to showing that if M is a finitely generated S-module, then $\Gamma(X,\tilde{M})$ is a finitely generated A-module.

Now by (I, 7.4), there is a finite filtration

$$0 = M^0 \subseteq M^1 \subseteq \ldots \subseteq M^r = M$$

of M by graded submodules, where for each i, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogeneous prime ideal $\mathfrak{p}_i \subseteq S$, and some integer n_i. This filtration gives a filtration of $\tilde{M}$, and the short exact sequences

$$0 \to \tilde{M}^{i-1} \to \tilde{M}^i \to \tilde{M}^i/\tilde{M}^{i-1} \to 0$$

give rise to left-exact sequences

$$0 \to \Gamma(X,\tilde{M}^{i-1}) \to \Gamma(X,\tilde{M}^i) \to \Gamma(X,\tilde{M}^i/\tilde{M}^{i-1}).$$

Thus to show that $\Gamma(X,\tilde{M})$ is finitely generated over A, it will be sufficient to show that $\Gamma(X,(S/\mathfrak{p})^{\sim}(n))$ is finitely generated, for each $\mathfrak{p}$ and n. Thus we have reduced to the following special case: Let S be a graded integral domain, finitely generated by S_1 as an S_0-algebra, where $S_0 = A$ is a finitely generated integral domain over k. Then $\Gamma(X,\mathcal{O}_X(n))$ is a finitely generated A-module, for any $n \in \mathbf{Z}$.

Let $x_0,\ldots,x_r \in S_1$ be a set of generators of S_1 as an A-module. Since S is an integral domain, multiplication by x_0 gives an injection $S(n) \to S(n+1)$ for any n. Hence there is an injection $\Gamma(X,\mathcal{O}_X(n)) \to \Gamma(X,\mathcal{O}_X(n+1))$ for any n. Thus it is sufficient to prove $\Gamma(X,\mathcal{O}_X(n))$ finitely generated for all sufficiently large n, say $n \geqslant 0$.

Let $S' = \bigoplus_{n \geqslant 0} \Gamma(X,\mathcal{O}_X(n))$. Then S' is a ring, containing S, and contained in the intersection $\bigcap S_{x_i}$ of the localizations of S at the elements $x_0,\ldots,x_r$.

(Use the same argument as in the proof of (5.13).) We will show that S' is integral over S.

Let $s' \in S'$ be homogeneous of degree $d \geqslant 0$. Since $s' \in S_{x_i}$ for each i, we can find an integer n such that $x_i^n s' \in S$. Choose one n that works for all i. Since the x_i generate S_1, the monomials in the x_i of degree m generate S_m for any m. So by taking a larger n, we may assume that $ys' \in S$ for all $y \in S_n$. In fact, since s' has positive degree, we can say that for any $y \in S_{\geqslant n} = \bigoplus_{e \geqslant n} S_e$, $ys' \in S_{\geqslant n}$. Now it follows inductively, for any $q \geqslant 1$ that $y \cdot (s')^q \in S_{\geqslant n}$ for any $y \in S_{\geqslant n}$. Take for example $y = x_0^n$. Then for every $q \geqslant 1$ we have $(s')^q \in (1/x_0^n)S$. This is a finitely generated sub-S-module of the quotient field of S'. It follows by a well-known criterion for integral dependence (Atiyah–Macdonald [1, p. 59]), that s' is integral over S. Thus S' is contained in the integral closure of S in its quotient field.

To complete the proof, we apply the theorem of finiteness of integral closure (I, 3.9A). Since S is a finitely generated k-algebra, S' will be a finitely generated S-module. It follows that for every n, S'_n is a finitely generated S_0-module, which is what we wanted to prove. In fact, our proof shows that $S'_n = S_n$ for all sufficiently large n (Ex. 5.9) and (Ex. 5.14).

Remark 5.19.1. This proof is a generalization of the proof of (I, 3.4a). We will give another proof of this theorem later, using cohomology (III, 5.2.1).

Remark 5.19.2. The hypothesis "A is a finitely generated k-algebra" is used only to be able to apply (I, 3.9A). Thus it would be sufficient to assume only that A is a "Nagata ring" in the sense of Matsumura [2, p. 231]—see also [loc. cit., Th. 72, p. 240].

Corollary 5.20. *Let $f: X \to Y$ be a projective morphism of schemes of finite type over a field k. Let $\mathscr{F}$ be a coherent sheaf on X. Then $f_*\mathscr{F}$ is coherent on Y.*

PROOF. The question is local on Y, so we may assume $Y = \operatorname{Spec} A$, where A is a finitely generated k-algebra. Then in any case, $f_*\mathscr{F}$ is quasi-coherent (5.8c), so $f_*\mathscr{F} = \Gamma(Y, f_*\mathscr{F})^\sim = \Gamma(X, \mathscr{F})^\sim$. But $\Gamma(X, \mathscr{F})$ is a finitely generated A-module by the theorem, so $f_*\mathscr{F}$ is coherent. See (III, 8.8) for another proof and generalization.

EXERCISES

5.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the *dual* of $\mathscr{E}$, denoted $\check{\mathscr{E}}$, to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.

(a) Show that $(\check{\mathscr{E}})^\vee \cong \mathscr{E}$.

(b) For any $\mathcal{O}_X$-module $\mathscr{F}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.

(c) For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.

(d) (*Projection Formula*). If $f:(X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

5.2. Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.

(a) To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho: M \otimes_R K \to L$.

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

5.3. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\tilde{}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism

$$\operatorname{Hom}_A(M, \Gamma(X,\mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M},\mathcal{F}).$$

5.4. Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

5.5. Let $f:X \to Y$ be a morphism of schemes.

(a) Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism (§3).

(c) If f is a finite morphism of noetherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

5.6. *Support*. Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

(a) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \tilde{M}$. For any $m \in M = \Gamma(X,\mathcal{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the *annihilator* of $m = \{a \in A | am = 0\}$.

(b) Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathcal{F} = V(\operatorname{Ann} M)$.

(c) The support of a coherent sheaf on a noetherian scheme is closed.

(d) For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathcal{H}^0_Z(\mathcal{F})$, where $Z = V(\mathfrak{a})$ and $\mathcal{F} = \tilde{M}$. [*Hint*: Use (Ex. 1.20) and (5.8) to show a priori that $\mathcal{H}^0_Z(\mathcal{F})$ is quasi-coherent. Then show that $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Z(\mathcal{F})$.]

(e) Let X be a noetherian scheme, and let Z be a closed subset. If $\mathcal{F}$ is a quasi-coherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathcal{H}^0_Z(\mathcal{F})$ is also quasi-coherent (respectively, coherent).

5.7. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.

(a) If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

(b) $\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.

(c) $\mathcal{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

5.8. Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x),$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is *upper semi-continuous*, i.e., for any $n \in \mathbf{Z}$, the set $\{x \in X | \varphi(x) \geqslant n\}$ is closed.

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

5.9. Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.

(a) Show that there is a natural homomorphism $\alpha: M \to \Gamma_*(\tilde{M})$.

(b) Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbf{Z}$ such that for all $d \geqslant d_0$, $\alpha_d: M_d \to \Gamma(X, \tilde{M}(d))$ is an isomorphism. [*Hint*: Use the methods of the proof of (5.19).]

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geqslant d} \cong M'_{\geqslant d}$. Here $M_{\geqslant d} = \bigoplus_{n \geqslant d} M_n$. We will say that a graded S-module M is *quasi-finitely generated* if it is equivalent to a finitely generated module. Now show that the functors $\tilde{\ }$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

5.10. Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

(a) For any homogeneous ideal $I \subseteq S$, we define the *saturation* $\bar{I}$ of I to be $\{s \in S | \text{for each } i = 0, \ldots, r, \text{ there is an } n \text{ such that } x_i^n s \in I\}$. We say that I is *saturated* if $I = \bar{I}$. Show that $\bar{I}$ is a homogeneous ideal of S.

(b) Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

(c) If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathscr{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.

(d) There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.

5.11. Let S and T be two graded rings with $S_0 = T_0 = A$. We define the *Cartesian product* $S \times_A T$ to be the graded ring $\bigoplus_{d \geqslant 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

The Cartesian product of rings is related to the *Segre embedding* of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \hookrightarrow \mathbf{P}_A^r$, and if $y_0, \ldots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \hookrightarrow \mathbf{P}_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and hence defines a projective embedding $\operatorname{Proj}(S \times_A T) \hookrightarrow \mathbf{P}_A^N$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbf{P}^r \times \mathbf{P}^s$ in its Segre embedding.

5.12. (a) Let X be a scheme over a scheme Y, and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L} \otimes \mathscr{M}$ is also very ample. [*Hint*: Use a Segre embedding.]

(b) Let $f: X \to Y$ and $g: Y \to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L} \otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

5.13. Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geqslant 0} S_n^{(d)}$ where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.

This construction is related to the d-uple *embedding* (I, Ex. 2.12) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \hookrightarrow \mathbf{P}_A^r$, then the set of monomials of degree d in the x_i is a set of generators for $S_1^{(d)} = S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbf{P}_A^r$.

5.14. Let A be a ring, and let X be a closed subscheme of $\mathbf{P}_A^r$. We define the *homogeneous coordinate ring* $S(X)$ of X for the given embedding to be $A[x_0, \ldots, x_r]/I$, where I is the ideal $\Gamma_*(\mathscr{I}_X)$ constructed in the proof of (5.16). (Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)!) Recall that a scheme X is *normal* if its local rings are integrally closed domains. A closed subscheme $X \subseteq \mathbf{P}_A^r$ is *projectively normal* for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbf{P}_k^r$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.

(a) Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure. [*Hint*: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathscr{S} = \bigoplus_{n \geqslant 0} \mathcal{O}_X(n)$ on X, and show that $\mathscr{S}$ is a sheaf of integrally closed domains.]

(b) Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

(c) Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.

(d) As a corollary of (a), show that a closed subscheme $X \subseteq \mathbf{P}_A^r$ is projectively normal if and only if it is normal, and for every $n \geqslant 0$ the natural map $\Gamma(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

5.15. *Extension of Coherent Sheaves*. We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathscr{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_U \cong \mathscr{F}$.

(a) On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathscr{F}$ is the *union* of its subsheaves $\mathscr{F}_\alpha$ if for every open set U, the group $\mathscr{F}(U)$ is the union of the subgroups $\mathscr{F}_\alpha(U)$.

(b) Let X be an affine noetherian scheme, U an open subset, and $\mathscr{F}$ coherent on U. Then there exists a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$. [*Hint*: Let $i: U \to X$ be the inclusion map. Show that $i_*\mathscr{F}$ is quasi-coherent, then use (a).]

(c) With $X,U,\mathscr{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that we can find $\mathscr{F}'$ a coherent subsheaf of $\mathscr{G}$, with $\mathscr{F}'|_U \cong \mathscr{F}$. [*Hint*: Use the same method, but replace $i_*\mathscr{F}$ by $\rho^{-1}(i_*\mathscr{F})$, where ρ is the natural map $\mathscr{G} \rightarrow i_*(\mathscr{G}|_U)$.]

(d) Now let X be any noetherian scheme, U an open subset, $\mathscr{F}$ a coherent sheaf on U, and $\mathscr{G}$ a quasi-coherent sheaf on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that there is a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{G}$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$. Taking $\mathscr{G} = i_*\mathscr{F}$ proves the result announced at the beginning. [*Hint*: Cover X with open affines, and extend over one of them at a time.]

(e) As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathscr{F}$ is the union of its coherent subsheaves. [*Hint*: If s is a section of $\mathscr{F}$ over an open set U, apply (d) to the subsheaf of $\mathscr{F}|_U$ generated by s.]

5.16. *Tensor Operations on Sheaves*. First we recall the definitions of various tensor operations on a module. Let A be a ring, and let M be an A-module. Let $T^n(M)$ be the tensor product $M \otimes \ldots \otimes M$ of M with itself n times, for $n \geqslant 1$. For $n = 0$ we put $T^0(M) = A$. Then $T(M) = \bigoplus_{n \geqslant 0} T^n(M)$ is a (noncommutative) A-algebra, which we call the *tensor algebra* of M. We define the *symmetric algebra* $S(M) = \bigoplus_{n \geqslant 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x,y \in M$. Then $S(M)$ is a commutative A-algebra. Its component $S^n(M)$ in degree n is called the nth *symmetric product* of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x,y \in M$. As an example, note that if M is a free A-module of rank r, then $S(M) \cong A[x_1,\ldots,x_r]$.

We define the *exterior algebra* $\bigwedge(M) = \bigoplus_{n \geqslant 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a *skew commutative* graded A-algebra. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$). The nth component $\bigwedge^n(M)$ is called the nth *exterior power* of M.

Now let $(X,\mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *tensor algebra, symmetric algebra*, and *exterior algebra* of $\mathscr{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathscr{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_X$-algebras, and their components in each degree are $\mathcal{O}_X$-modules.

(a) Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

(b) Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r\mathscr{F} \otimes \bigwedge^{n-r}\mathscr{F} \rightarrow \bigwedge^n\mathscr{F}$ is a perfect pairing for any r, i.e., it induces an isomorphism of $\bigwedge^r\mathscr{F}$ with $(\bigwedge^{n-r}\mathscr{F})^\vee \otimes \bigwedge^n\mathscr{F}$. As a special case, note if $\mathscr{F}$ has rank 2, then $\mathscr{F} \cong \mathscr{F}^\vee \otimes \bigwedge^2\mathscr{F}$.

(c) Let $0 \rightarrow \mathscr{F}' \rightarrow \mathscr{F} \rightarrow \mathscr{F}'' \rightarrow 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$,

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

(d) Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathscr{F}',\mathscr{F},\mathscr{F}''$ have ranks n',n,n'' respectively, there is an isomorphism $\bigwedge^n\mathscr{F} \cong \bigwedge^{n'}\mathscr{F}' \otimes \bigwedge^{n''}\mathscr{F}''$.

(e) Let $f:X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathscr{F}$, i.e., $f^*(S^n(\mathscr{F})) = S^n(f^*\mathscr{F})$ etc.

5.17. *Affine Morphisms.* A morphism $f:X \to Y$ of schemes is *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

(a) Show that $f:X \to Y$ is an affine morphism if and only if for *every* open affine $V \subseteq Y, f^{-1}(V)$ is affine. [*Hint*: Reduce to the case Y affine, and use (Ex. 2.17).]

(b) An affine morphism is quasi-compact and separated. Any finite morphism is affine.

(c) Let Y be a scheme, and let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there is a unique scheme X, and a morphism $f:X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathscr{A}(V)$, and for every inclusion $U \hookrightarrow V$ of open affines of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$. The scheme X is called $\mathbf{Spec}\, \mathscr{A}$. [*Hint*: Construct X by glueing together the schemes $\operatorname{Spec} \mathscr{A}(V)$, for V open affine in Y.]

(d) If $\mathscr{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f:X = \mathbf{Spec}\, \mathscr{A} \to Y$ is an affine morphism, and $\mathscr{A} \cong f_*\mathcal{O}_X$. Conversely, if $f:X \to Y$ is an affine morphism, then $\mathscr{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \mathbf{Spec}\, \mathscr{A}$.

(e) Let $f:X \to Y$ be an affine morphism, and let $\mathscr{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathscr{A}$-modules (i.e., quasi-coherent $\mathcal{O}_Y$-modules having a structure of $\mathscr{A}$-module). [*Hint*: For any quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, construct a quasi-coherent $\mathcal{O}_X$-module $\tilde{\mathscr{M}}$, and show that the functors f_* and $\tilde{\ }$ are inverse to each other.]

5.18. *Vector Bundles.* Let Y be a scheme. *A (geometric) vector bundle* of rank n over Y is a scheme X and a morphism $f:X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i:f^{-1}(U_i) \to \mathbf{A}^n_{U_i}$, such that for any i,j, and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbf{A}^n_V = \operatorname{Spec} A[x_1, \ldots, x_n]$ is given by a *linear* automorphism θ of $A[x_1, \ldots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij}x_j$ for suitable $a_{ij} \in A$.

An *isomorphism* $g:(X,f,\{U_i\},\{\psi_i\}) \to (X',f',\{U'_i\},\{\psi'_i\})$ of one vector bundle of rank n to another one is an isomorphism $g:X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X,f, together with the covering of Y consisting of all the U_i and U'_i, and the isomorphisms ψ_i and $\psi'_i \circ g$, is also a vector bundle structure on X.

(a) Let $\mathscr{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathscr{E})$ be the symmetric algebra on $\mathscr{E}$, and let $X = \mathbf{Spec}\, S(\mathscr{E})$, with projection morphism $f:X \to Y$. For each open affine subset $U \subseteq Y$ for which $\mathscr{E}|_U$ is free, choose a basis of $\mathscr{E}$, and let $\psi:f^{-1}(U) \to \mathbf{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathscr{E}(U))$ with $\mathcal{O}(U)[x_1, \ldots, x_n]$. Then $(X,f,\{U\},\{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathscr{E}_U$ chosen. We call it the *geometric vector bundle associated* to $\mathscr{E}$, and denote it by $\mathbf{V}(\mathscr{E})$.

(b) For any morphism $f : X \to Y$, a *section* of f over an open set $U \subseteq Y$ is a morphism $s : U \to X$ such that $f \circ s = \mathrm{id}_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{\text{set of sections of } f \text{ over } U\}$ is a sheaf of sets on Y, which we denote by $\mathscr{S}(X/Y)$. Show that if $f : X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathscr{S}(X/Y)$ has a natural structure of $\mathscr{O}_Y$-module, which makes it a locally free $\mathscr{O}_Y$-module of rank n. [*Hint*: It is enough to define the module structure locally, so we can assume $Y = \operatorname{Spec} A$ is affine, and $X = \mathbf{A}^n_Y$. Then a section $s : Y \to X$ comes from an A-algebra homomorphism $\theta : A[x_1, \ldots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

(c) Again let $\mathscr{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbf{V}(\mathscr{E})$, and let $\mathscr{S} = \mathscr{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathscr{S} \cong \mathscr{E}^\vee$, as follows. Given a section $s \in \Gamma(V, \mathscr{E}^\vee)$ over any open set V, we think of s as an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathscr{O}_V)$. So s determines an $\mathscr{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathscr{O}_V$. This determines a morphism of spectra $V = \mathbf{Spec}\, \mathscr{O}_V \to \mathbf{Spec}\, S(\mathscr{E}|_V) = f^{-1}(V)$, which is a section of X/Y. Show that this construction gives an isomorphism of $\mathscr{E}^\vee$ to $\mathscr{S}$.

(d) Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words "locally free sheaf" and "vector bundle" interchangeably, if no confusion seems likely to result.

6 Divisors

The notion of divisor forms an important tool for studying the intrinsic geometry on a variety or scheme. In this section we will introduce divisors, linear equivalence and the divisor class group. The divisor class group is an abelian group which is an interesting and subtle invariant of a variety. In §7 we will see that divisors are also important for studying maps from a given variety to a projective space.

There are several different ways of defining divisors, depending on the context. We will begin with Weil divisors, which are easiest to understand geometrically, but which are only defined on certain noetherian integral schemes. For more general schemes there is the notion of Cartier divisor which we treat next. Then we will explain the connection between Weil divisors, Cartier divisors, and invertible sheaves.

We start with an informal example. Let C be a nonsingular projective curve in $\mathbf{P}^2_k$, the projective plane over an algebraically closed field k. For each line L in $\mathbf{P}^2$, we consider $L \cap C$, which is a finite set of points on C. If C is a curve of degree d, and if we count the points with proper multiplicity, then $L \cap C$ will consist of exactly d points (I, Ex. 5.4). We write $L \cap C = \sum n_i P_i$, where $P_i \in C$ are the points, and n_i the multiplicities, and we call this formal sum a divisor on C. As L varies, we obtain a family of divisors on C, parametrized by the set of all lines in $\mathbf{P}^2$, which is the dual projective space $(\mathbf{P}^2)^*$. We call this set of divisors a linear system of divisors on C. Note that the embedding of C in $\mathbf{P}^2$ can be recovered just from knowing this linear system: if P is a point of C, we consider the set of divisors in the linear system which contain P. They correspond to the lines $L \in (\mathbf{P}^2)^*$ passing through P, and this set of lines determines P uniquely as a point of $\mathbf{P}^2$. This connection between linear systems and embeddings in projective space will be studied in detail in §7.

This example should already serve to illustrate the importance of divisors. To see the relation among the different divisors in the linear system, let L and L' be two lines in $\mathbf{P}^2$, and let $D = L \cap C$ and $D' = L' \cap C$ be the corresponding divisors. If L and L' are defined by linear homogeneous equations $f = 0$ and $f' = 0$ in $\mathbf{P}^2$, then f/f' gives a rational function on $\mathbf{P}^2$, which restricts to a rational function g on C. Now by construction, g has zeros at the points of D, and poles at the points of D', counted with multiplicities, in a sense which will be made precise below. We say that D and D' are linearly equivalent, and the existence of such a rational function can be taken as an intrinsic definition of the linear equivalence. We will make these concepts more precise in our formal discussion, starting now.

Weil Divisors

Definition. We say a scheme X is *regular in codimension one* (or sometimes *nonsingular in codimension one*) if every local ring $\mathcal{O}_x$ of X of dimension one is regular.

The most important examples of such schemes are nonsingular varieties over a field (I, §5) and noetherian normal schemes. On a nonsingular variety the local ring of every closed point is regular (I, 5.1), hence all the local rings are regular, since they are localizations of the local rings of closed points. On a noetherian normal scheme, any local ring of dimension one is an integrally closed domain, hence is regular (I, 6.2A).

In this section we will consider schemes satisfying the following condition:

(∗) X is a noetherian integral separated scheme which is regular in codimension one.

Definition. Let X satisfy (∗). A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* is an element of the free abelian group Div X generated by the prime divisors. We write a divisor as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the n_i are integers, and only finitely many n_i are different from zero. If all the $n_i \geqslant 0$, we say that D is *effective*.

If Y is a prime divisor on X, let $\eta \in Y$ be its generic point. Then the local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with quotient field K, the function field of X. We call the corresponding discrete valuation v_Y the *valuation of* Y. Note that since X is separated, Y is uniquely determined by its valuation (Ex. 4.5). Now let $f \in K^*$ be any nonzero rational function on X. Then $v_Y(f)$ is an integer. If it is positive, we say f has a *zero* along Y, of that order; if it is negative, we say f has a *pole* along Y, of order $-v_Y(f)$.

Lemma 6.1. *Let X satisfy $(*)$, and let $f \in K^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y.*

PROOF. Let $U = \operatorname{Spec} A$ be an open affine subset of X on which f is regular. Then $Z = X - U$ is a proper closed subset of X. Since X is noetherian, Z can contain at most finitely many prime divisors of X; all the others must meet U. Thus it will be sufficient to show that there are only finitely many prime divisors Y of U for which $v_Y(f) \neq 0$. Since f is regular on U, we have $v_Y(f) \geqslant 0$ in any case. And $v_Y(f) > 0$ if and only if Y is contained in the closed subset of U defined by the ideal Af in A. Since $f \neq 0$, this is a proper closed subset, hence contains only finitely many closed irreducible subsets of codimension one of U.

Definition. Let X satisfy $(*)$ and let $f \in K^*$. We define the *divisor* of f, denoted (f), by

$$(f) = \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisors of X. By the lemma, this is a finite sum, hence it is a divisor. Any divisor which is equal to the divisor of a function is called a *principal* divisor.

Note that if $f,g \in K^*$, then $(f/g) = (f) - (g)$ because of the properties of valuations. Therefore sending a function f to its divisor (f) gives a homomorphism of the multiplicative group K^* to the additive group $\operatorname{Div} X$, and the image, which consists of the principal divisors, is a subgroup of $\operatorname{Div} X$.

Definition. Let X satisfy $(*)$. Two divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor. The group $\operatorname{Div} X$ of all divisors divided by the subgroup of principal divisors is called the *divisor class group* of X, and is denoted by $\operatorname{Cl} X$.

The divisor class group of a scheme is a very interesting invariant. In general it is not easy to calculate. However, in the following propositions and examples we will calculate a number of special cases to give some idea of what it is like.

Proposition 6.2. *Let A be a noetherian domain. Then A is a unique factorization domain if and only if $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl} X = 0$.*

PROOF. (See also Bourbaki [1, Ch. 7, §3]). It is well-known that a UFD is integrally closed, so X will be normal. On the other hand, A is a UFD if and only if every prime ideal of height 1 is principal (I, 1.12A). So what we must show is that if A is an integrally closed domain, then every prime ideal of height 1 is principal if and only if $\operatorname{Cl}(\operatorname{Spec} A) = 0$.

One way is easy: if every prime ideal of height 1 is principal, consider a prime divisor $Y \subseteq X = \operatorname{Spec} A$. Y corresponds to a prime ideal $\mathfrak{p}$ of height 1. If $\mathfrak{p}$ is generated by an element $f \in A$, then clearly the divisor of f is $1 \cdot Y$. Thus every prime divisor is principal, so $\operatorname{Cl} X = 0$.

For the converse, suppose $\operatorname{Cl} X = 0$. Let $\mathfrak{p}$ be a prime ideal of height 1, and let Y be the corresponding prime divisor. Then there is an $f \in K$, the quotient field of A, with $(f) = Y$. We will show that in fact $f \in A$ and f generates $\mathfrak{p}$. Since $v_Y(f) = 1$, we have $f \in A_{\mathfrak{p}}$, and f generates $\mathfrak{p}A_{\mathfrak{p}}$. If $\mathfrak{p}' \subseteq A$ is any other prime ideal of height 1, then $\mathfrak{p}'$ corresponds to a prime divisor Y' of X, and $v_{Y'}(f) = 0$, so $f \in A_{\mathfrak{p}'}$. Now the algebraic result (6.3A) below implies that $f \in A$. In fact, $f \in A \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$. Now to show that f generates $\mathfrak{p}$, let g be any other element of $\mathfrak{p}$. Then $v_Y(g) \geqslant 1$ and $v_{Y'}(g) \geqslant 0$ for all $Y' \neq Y$. Hence $v_{Y'}(g/f) \geqslant 0$ for *all* prime divisors Y' (including Y). Thus $g/f \in A_{\mathfrak{p}'}$ for *all* $\mathfrak{p}'$ of height 1, so by (6.3A) again, $g/f \in A$. In other words, $g \in Af$, which shows that $\mathfrak{p}$ is a principal ideal, generated by f.

Proposition 6.3A. *Let A be an integrally closed noetherian domain. Then*

$$A = \bigcap_{\operatorname{ht} \mathfrak{p} = 1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1.

PROOF. Matsumura [2, Th. 38, p. 124].

Example 6.3.1. If X is affine n-space $\mathbf{A}_k^n$ over a field k, then $\operatorname{Cl} X = 0$. Indeed, $X = \operatorname{Spec} k[x_1, \ldots, x_n]$, and the polynomial ring is a UFD.

Example 6.3.2. If A is a Dedekind domain, then $\operatorname{Cl}(\operatorname{Spec} A)$ is just the ideal class group of A, as defined in algebraic number theory. Thus (6.2) generalizes the fact that A is a UFD if and only if its ideal class group is 0.

Proposition 6.4. *Let X be the projective space $\mathbf{P}_k^n$ over a field k. For any divisor $D = \sum n_i Y_i$, define the degree of D by* $\deg D = \sum n_i \deg Y_i$, *where* $\deg Y_i$ *is the degree of the hypersurface Y_i. Let H be the hyperplane $x_0 = 0$. Then*:

(a) *if D is any divisor of degree d, then $D \sim dH$*;
(b) *for any $f \in K^*$*, $\deg(f) = 0$;
(c) *the degree function gives an isomorphism* $\deg : \operatorname{Cl} X \to \mathbf{Z}$.

PROOF. Let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of X. If g is a homogeneous element of degree d, we can factor it into irreducible polynomials $g = g_1^{n_1} \cdots g_r^{n_r}$. Then g_i defines a hypersurface Y_i of degree $d_i = \deg g_i$, and we can define the divisor of g to be $(g) = \sum n_i Y_i$. Then $\deg(g) = d$. Now a rational function f on X is a quotient g/h of homogeneous polynomials of the same degree. Clearly $(f) = (g) - (h)$, so we see that $\deg(f) = 0$, which proves (b).

If D is any divisor of degree d, we can write it as a difference $D_1 - D_2$ of effective divisors of degrees d_1, d_2 with $d_1 - d_2 = d$. Let $D_1 = (g_1)$ and $D_2 = (g_2)$. This is possible, because an irreducible hypersurface in $\mathbf{P}^n$ corresponds to a homogeneous prime ideal of height 1 in S, which is principal. Taking power products we can get any effective divisor as (g) for some homogeneous g. Now $D - dH = (f)$ where $f = g_1/x_0^d g_2$ is a rational function on X. This proves (a). Statement (c) follows from (a), (b), and the fact that $\deg H = 1$.

Proposition 6.5. *Let X satisfy $(*)$, let Z be a proper closed subset of X, and let $U = X - Z$. Then:*

(a) *there is a surjective homomorphism* $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ *defined by* $D = \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$, *where we ignore those* $Y_i \cap U$ *which are empty;*

(b) *if* $\mathrm{codim}(Z,X) \geqslant 2$, *then* $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ *is an isomorphism;*

(c) *if Z is an irreducible subset of codimension* 1, *then there is an exact sequence*

$$\mathbf{Z} \to \mathrm{Cl}\, X \to \mathrm{Cl}\, U \to 0,$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

PROOF.

(a) If Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $f \in K^*$, and $(f) = \sum n_i Y_i$, then considering f as a rational function on U, we have $(f)_U = \sum n_i (Y_i \cap U)$, so indeed we have a homomorphism $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$. It is surjective because every prime divisor of U is the restriction of its closure in X.

(b) The groups $\mathrm{Div}\, X$ and $\mathrm{Cl}\, X$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension $\geqslant 2$ doesn't change anything.

(c) The kernel of $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{Cl}\, X$ generated by $1 \cdot Z$.

Example 6.5.1. Let Y be an irreducible curve of degree d in $\mathbf{P}_k^2$. Then $\mathrm{Cl}(\mathbf{P}^2 - Y) = \mathbf{Z}/d\mathbf{Z}$. This follows immediately from (6.4) and (6.5).

Example 6.5.2. Let k be a field, let $A = k[x,y,z]/(xy - z^2)$, and let $X = \mathrm{Spec}\, A$. Then X is an affine quadric cone in $\mathbf{A}_k^3$. We will show that $\mathrm{Cl}\, X = \mathbf{Z}/2\mathbf{Z}$, and that it is generated by a ruling of the cone, say $Y : y = z = 0$ (Fig. 8).

First note that Y is a prime divisor, so by (6.5) we have an exact sequence

$$\mathbf{Z} \to \mathrm{Cl}\, X \to \mathrm{Cl}(X - Y) \to 0,$$

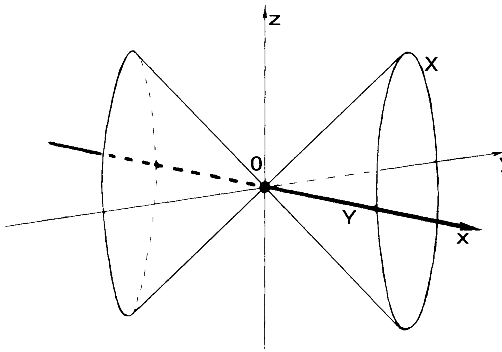

Figure 8. A ruling on the quadric cone.

where the first map sends $1 \mapsto 1 \cdot Y$. Now Y can be cut out set-theoretically by the function y. In fact, the divisor of y is $2 \cdot Y$, because $y = 0 \Rightarrow z^2 = 0$, and z generates the maximal ideal of the local ring at the generic point of Y. Hence $X - Y = \operatorname{Spec} A_y$. Now $A_y = k[x,y,y^{-1},z]/(xy - z^2)$. In this ring $x = y^{-1}z^2$, so we can eliminate x, and find $A_y \cong k[y,y^{-1},z]$. This is a UFD, so by (6.2), $\operatorname{Cl}(X - Y) = 0$.

Thus we see that $\operatorname{Cl} X$ is generated by Y, and that $2 \cdot Y = 0$. It remains to show that Y itself is not a principal divisor. Since A is integrally closed (Ex. 6.4), it is equivalent to show that the prime ideal of Y, namely $\mathfrak{p} = (y,z)$, is not principal (cf. proof of (6.2)). Let $\mathfrak{m} = (x,y,z)$, and note that $\mathfrak{m}/\mathfrak{m}^2$ is a 3-dimensional vector space over k generated by $\bar{x},\bar{y},\bar{z}$, the images of x,y,z. Now $\mathfrak{p} \subseteq \mathfrak{m}$, and the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ contains $\bar{y}$ and $\bar{z}$. Hence $\mathfrak{p}$ cannot be a principal ideal.

Proposition 6.6. *Let X satisfy* $(*)$. *Then* $X \times \mathbf{A}^1 (= X \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}[t])$ *also satisfies* $(*)$, *and* $\operatorname{Cl} X \cong \operatorname{Cl}(X \times \mathbf{A}^1)$.

PROOF. Clearly $X \times \mathbf{A}^1$ is noetherian, integral, and separated. To see that it is regular in codimension one, we note that there are two kinds of points of codimension one on $X \times \mathbf{A}^1$. Type 1 is a point x whose image in X is a point y of codimension one. In this case x is the generic point of $\pi^{-1}(y)$, where $\pi: X \times \mathbf{A}^1 \rightarrow X$ is the projection. Its local ring is $\mathcal{O}_x \cong \mathcal{O}_y[t]_{\mathfrak{m}_y}$, which is clearly a discrete valuation ring, since $\mathcal{O}_y$ is. The corresponding prime divisor $\{x\}^-$ is just $\pi^{-1}(\{y\}^-)$.

Type 2 is a point $x \in X \times \mathbf{A}^1$ of codimension one, whose image in X is the generic point of X. In this case $\mathcal{O}_x$ is a localization of $K[t]$ at some maximal ideal, where K is the function field of X. It is a discrete valuation ring because $K[t]$ is a principal ideal domain. Thus $X \times \mathbf{A}^1$ also satisfies $(*)$.

We define a map $\operatorname{Cl} X \rightarrow \operatorname{Cl}(X \times \mathbf{A}^1)$ by $D = \sum n_i Y_i \mapsto \pi^* D = \sum n_i \pi^{-1}(Y_i)$. If $f \in K^*$, then $\pi^*((f))$ is the divisor of f considered as an element of $K(t)$, the function field of $X \times \mathbf{A}^1$. Thus we have a homomorphism $\pi^*: \operatorname{Cl} X \rightarrow \operatorname{Cl}(X \times \mathbf{A}^1)$.

To show π^* is injective, suppose $D \in \operatorname{Div} X$, and $\pi^*D = (f)$ for some $f \in K(t)$. Since π^*D involves only prime divisors of type 1, f must be in K. For otherwise we could write $f = g/h$, with $g,h \in K[t]$, relatively prime. If g,h are not both in K, then (f) will involve some prime divisor of type 2 on $X \times \mathbf{A}^1$. Now if $f \in K$, it is clear that $D = (f)$, so π^* is injective.

To show that π^* is surjective, it will be sufficient to show that any prime divisor of type 2 on $X \times \mathbf{A}^1$ is linearly equivalent to a linear combination of prime divisors of type 1. So let $Z \subseteq X \times \mathbf{A}^1$ be a prime divisor of type 2. Localizing at the generic point of X, we get a prime divisor in $\operatorname{Spec} K[t]$, which corresponds to a prime ideal $\mathfrak{p} \subseteq K[t]$. This is principal, so let f be a generator. Then $f \in K(t)$, and the divisor of f consists of Z plus perhaps something purely of type 1. It cannot involve any other prime divisors of type 2. Thus Z is linearly equivalent to a divisor purely of type 1. This completes the proof.

Example 6.6.1. Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbf{P}_k^3$. We will show that $\operatorname{Cl} Q \cong \mathbf{Z} \oplus \mathbf{Z}$. We use the fact that Q is isomorphic to $\mathbf{P}_k^1 \times_k \mathbf{P}_k^1$ (I, Ex. 2.15). Let p_1 and p_2 be the projections of Q onto the two factors. Then as in the proof of (6.6) we obtain homomorphisms $p_1^*, p_2^* : \operatorname{Cl} \mathbf{P}^1 \to \operatorname{Cl} Q$. First we show that p_1^* and p_2^* are injective. Let $Y = pt \times \mathbf{P}^1$. Then $Q - Y = \mathbf{A}^1 \times \mathbf{P}^1$, and the composition

$$\operatorname{Cl} \mathbf{P}^1 \xrightarrow{p_2^*} \operatorname{Cl} Q \to \operatorname{Cl}(\mathbf{A}^1 \times \mathbf{P}^1)$$

is the isomorphism of (6.6). Hence p_2^* (and similarly p_1^*) is injective.

Now consider the exact sequence of (6.5) for Y:

$$\mathbf{Z} \to \operatorname{Cl} Q \to \operatorname{Cl}(\mathbf{A}^1 \times \mathbf{P}^1) \to 0.$$

In this sequence the first map sends 1 to Y. But if we identify $\operatorname{Cl} \mathbf{P}^1$ with $\mathbf{Z}$ by letting 1 be the class of a point, then this first map is just p_1^*, hence is injective. Since the image of p_2^* goes isomorphically to $\operatorname{Cl}(\mathbf{A}^1 \times \mathbf{P}^1)$ as we have just seen, we conclude that $\operatorname{Cl} Q \cong \operatorname{Im} p_1^* \oplus \operatorname{Im} p_2^* = \mathbf{Z} \oplus \mathbf{Z}$. If D is any divisor on Q, let (a,b) be the ordered pair of integers in $\mathbf{Z} \oplus \mathbf{Z}$ corresponding to the class of D under this isomorphism. Then we say D is of *type* (a,b) on Q.

Example 6.6.2. Continuing with the quadric surface $Q \subseteq \mathbf{P}^3$, we will show that the embedding induces a homomorphism $\operatorname{Cl} \mathbf{P}^3 \to \operatorname{Cl} Q$, and that the image of a hyperplane H, which generates $\operatorname{Cl} \mathbf{P}^3$, is the element $(1,1)$ in $\operatorname{Cl} Q = \mathbf{Z} \oplus \mathbf{Z}$. Let Y be any irreducible hypersurface of $\mathbf{P}^3$ which does not contain Q. Then we can assign multiplicities to the irreducible components of $Y \cap Q$ so as to obtain a *divisor* $Y \cdot Q$ on Q. Indeed, on each standard open set U_i of $\mathbf{P}^3$, Y is defined by a single function f; we can take the value of this function (restricted to Q) for each valuation of a prime divisor of Q to define the divisor $Y \cdot Q$. By linearity we extend this map to define a divisor $D \cdot Q$ on Q, for each divisor $D = \sum n_i Y_i$ on $\mathbf{P}^3$, such that no Y_i contains Q. Clearly linearly equivalent divisors restrict to linearly equivalent divisors. Since any divisor on $\mathbf{P}^3$ is linearly equivalent to one whose prime divisors don't contain Q by (6.4), we obtain a well-defined homomorphism $\mathrm{Cl}\, \mathbf{P}^3 \to \mathrm{Cl}\, Q$. Now if H is the hyperplane $w = 0$, then $H \cap Q$ is the divisor consisting of the two lines $x = w = 0$ and $y = w = 0$. One is in each family (I, Ex. 2.15) so $H \cap Q$ is of type $(1,1)$ in $\mathrm{Cl}\, Q = \mathbf{Z} \oplus \mathbf{Z}$. Note that the two families of lines correspond to $pt \times \mathbf{P}^1$ and $\mathbf{P}^1 \times pt$, so they are of type $(1,0)$ and $(0,1)$.

Example 6.6.3. Carrying this example one step further, let C be the twisted cubic curve $x = t^3$, $y = u^3$, $z = t^2u$, $w = tu^2$ which lies on Q. If Y is the quadric cone $yz = w^2$, then $Y \cap Q = C \cup L$ where L is the line $y = w = 0$. Since $Y \sim 2H$ on $\mathbf{P}^3$, $Y \cap Q$ is a divisor of type $(2,2)$. The line L has type $(1,0)$, so C is of type $(1,2)$. It follows that there does not exist any surface $Y \subseteq \mathbf{P}^3$, not containing Q, such that $Y \cap Q = C$, even set-theoretically! For in that case the divisor $Y \cap Q$ would be rC for some integer $r > 0$. This is a divisor of type $(r,2r)$ in $\mathrm{Cl}\, Q$. But if Y is a surface of degree d, then $Y \cap Q$ is of type (d,d), which can never equal $(r,2r)$. Thus Y does not exist.

Example 6.6.4. We will see later (V, 4.8) that if X is a nonsingular cubic surface in $\mathbf{P}^3$, then $\mathrm{Cl}\, X \cong \mathbf{Z}^7$.

Divisors on Curves

We will illustrate the notion of the divisor class group further by paying special attention to the case of divisors on curves. We will define the degree of a divisor on a curve, and we will show that on a complete nonsingular curve, the degree is stable under linear equivalence. Further study of divisors on curves will be found in Chapter IV.

To begin with, we need some preliminary information about curves and morphisms of curves. Recall our conventions about terminology from the end of Section 4:

Definition. Let k be an algebraically closed field. A *curve* over k is an integral separated scheme X of finite type over k, of dimension one. If X is proper over k, we say that X is *complete*. If all the local rings of X are regular local rings, we say that X is *nonsingular*.

Proposition 6.7. *Let X be a nonsingular curve over k with function field K. Then the following conditions are equivalent:*

(i) *X is projective;*
(ii) *X is complete;*
(iii) *$X \cong t(C_K)$, where C_K is the abstract nonsingular curve of* (I, §6), *and t is the functor from varieties to schemes of* (2.6).

PROOF.

(i) $\Rightarrow$ (ii) follows from (4.9).

(ii) $\Rightarrow$ (iii). If X is complete, then every discrete valuation ring of K/k has a unique center on X (Ex. 4.5). Since the local rings of X at the closed points are all discrete valuation rings, this implies that the closed points of X are in 1-1 correspondence with the discrete valuation rings of K/k, namely the points of C_K. Thus it is clear that $X \cong t(C_K)$.

(iii) $\Rightarrow$ (i) follows from (I, 6.9).

Proposition 6.8. *Let X be a complete nonsingular curve over k, let Y be any curve over k, and let $f: X \rightarrow Y$ be a morphism. Then either (1) $f(X) =$ a point, or (2) $f(X) = Y$. In case (2), $K(X)$ is a finite extension field of $K(Y)$, f is a finite morphism, and Y is also complete.*

PROOF. Since X is complete, $f(X)$ must be closed in Y, and proper over Spec k (Ex. 4.4). On the other hand, $f(X)$ is irreducible. Thus either (1) $f(X) = pt$, or (2) $f(X) = Y$, and in case (2), Y is also complete.

In case (2), f is dominant, so it induces an inclusion $K(Y) \subseteq K(X)$ of function fields. Since both fields are finitely generated extension fields of transcendence degree 1 of k, $K(X)$ must be a finite algebraic extension of $K(Y)$. To show that f is a finite morphism, let $V = \operatorname{Spec} B$ be any open affine subset of Y. Let A be the integral closure of B in $K(X)$. Then A is a finite B-module (I, 3.9A), and Spec A is isomorphic to an open subset U of X (I, 6.7). Clearly $U = f^{-1}V$, so this shows that f is a finite morphism.

Definition. If $f: X \rightarrow Y$ is a finite morphism of curves, we define the *degree* of f to be the degree of the field extension $[K(X): K(Y)]$.

Now we come to the study of divisors on curves. If X is a nonsingular curve, then X satisfies the condition $(*)$ used above, so we can talk about divisors on X. A prime divisor is just a closed point, so an arbitrary divisor can be written $D = \sum n_i P_i$, where the P_i are closed points, and $n_i \in \mathbf{Z}$. We define the *degree* of D to be $\sum n_i$.

Definition. If $f: X \rightarrow Y$ is a finite morphism of nonsingular curves, we define a homomorphism $f^*: \operatorname{Div} Y \rightarrow \operatorname{Div} X$ as follows. For any point $Q \in Y$, let $t \in \mathcal{O}_Q$ be a *local parameter* at Q, i.e., t is an element of $K(Y)$ with $v_Q(t) = 1$, where v_Q is the valuation corresponding to the discrete valuation ring $\mathcal{O}_Q$. We define $f^*Q = \sum_{f(P)=Q} v_P(t) \cdot P$. Since f is a finite morphism, this is a finite sum, so we get a divisor on X. Note that f^*Q is independent of the choice of the local parameter t. Indeed, if t' is another local parameter at Q, then $t' = ut$ where u is a unit in $\mathcal{O}_Q$. For any point $P \in X$ with $f(P) = Q$, u will be a unit in $\mathcal{O}_P$, so $v_P(t) = v_P(t')$. We extend the definition by linearity to all divisors on Y. One sees easily that f^* preserves linear equivalence, so it induces a homomorphism $f^*: \operatorname{Cl} Y \rightarrow \operatorname{Cl} X$.

Proposition 6.9. *Let $f: X \to Y$ be a finite morphism of nonsingular curves. Then for any divisor D on Y we have* $\deg f^*D = \deg f \cdot \deg D$.

PROOF. It will be sufficient to show that for any closed point $Q \in Y$ we have $\deg f^*Q = \deg f$. Let $V = \operatorname{Spec} B$ be an open affine subset of Y containing Q. Let A be the integral closure of B in $K(X)$. Then, as in the proof of (6.8), $U = \operatorname{Spec} A$ is the open subset $f^{-1}V$ of X. Let $\mathfrak{m}_Q$ be the maximal ideal of Q in B. We localize both B and A with respect to the multiplicative system $S = B - \mathfrak{m}_Q$, and we obtain a ring extension $\mathcal{O}_Q \hookrightarrow A'$, where A' is a finitely generated $\mathcal{O}_Q$-module. Now A' is torsion-free, and has rank equal to $r = [K(X):K(Y)]$, so A' is a free $\mathcal{O}_Q$-module of rank $r = \deg f$. If t is a local parameter at Q, it follows that A'/tA' is a k-vector space of dimension r.

On the other hand, the points P_i of X such that $f(P_i) = Q$ are in 1-1 correspondence with the maximal ideals $\mathfrak{m}_i$ of A', and for each i, $A'_{\mathfrak{m}_i} = \mathcal{O}_{P_i}$. Clearly $tA' = \bigcap_i (tA'_{\mathfrak{m}_i} \cap A')$, so by the Chinese remainder theorem,

$$\dim_k A'/tA' = \sum_i \dim_k A'/(tA'_{\mathfrak{m}_i} \cap A').$$

But

$$A'/(tA'_{\mathfrak{m}_i} \cap A') \cong A'_{\mathfrak{m}_i}/tA'_{\mathfrak{m}_i} = \mathcal{O}_{P_i}/t\mathcal{O}_{P_i},$$

so the dimensions in the sum above are just equal to $v_{P_i}(t)$. But $f^*Q = \sum v_{P_i}(t) \cdot P_i$, so we have shown that $\deg f^*Q = \deg f$ as required.

Corollary 6.10. *A principal divisor on a complete nonsingular curve X has degree zero. Consequently the degree function induces a surjective homomorphism* $\deg: \operatorname{Cl} X \to \mathbf{Z}$.

PROOF. Let $f \in K(X)^*$. If $f \in k$, then $(f) = 0$, so there is nothing to prove. If $f \notin k$, then the inclusion of fields $k(f) \subseteq K(X)$ induces a finite morphism $\varphi: X \to \mathbf{P}^1$. It is a morphism by (I, 6.12), and it is finite by (6.8). Now $(f) = \varphi^*(\{0\} - \{\infty\})$. Since $\{0\} - \{\infty\}$ is a divisor of degree 0 on $\mathbf{P}^1$, we conclude that (f) has degree 0 on X.

Thus the degree of a divisor on X depends only on its linear equivalence class, and we obtain a homomorphism $\operatorname{Cl} X \to \mathbf{Z}$ as stated. It is surjective, because the degree of a single point is 1.

Example 6.10.1. A complete nonsingular curve X is rational if and only if there exist two distinct points $P, Q \in X$ with $P \sim Q$. Recall that *rational* means birational to $\mathbf{P}^1$. If X is rational, then in fact it is isomorphic to $\mathbf{P}^1$ by (6.7). And on $\mathbf{P}^1$ we have already seen that any two points are linearly equivalent (6.4). Conversely, suppose X has two points $P \neq Q$ with $P \sim Q$. Then there is a rational function $f \in K(X)$ with $(f) = P - Q$. Consider the morphism $\varphi: X \to \mathbf{P}^1$ determined by f as in the proof of (6.10). We have $\varphi^*(\{0\}) = P$, so φ must be a morphism of degree 1. In other words, φ is birational, so X is rational.

Example 6.10.2. Let X be the nonsingular cubic curve $y^2z = x^3 - xz^2$ in $\mathbf{P}_k^2$, with char $k \neq 2$. We have already seen that X is not rational (I, Ex. 6.2). Let $\mathrm{Cl}^\circ X$ be the kernel of the degree map $\mathrm{Cl}\, X \to \mathbf{Z}$. Then from the previous example we know that $\mathrm{Cl}^\circ X \neq 0$. We will show in fact that there is a natural 1-1 correspondence between the set of closed points of X and the elements of the group $\mathrm{Cl}^\circ X$. On the one hand this elucidates the structure of the group $\mathrm{Cl}^\circ X$. On the other hand it gives us a group structure on the set of closed points of X, which makes X into a group variety (Fig. 9).

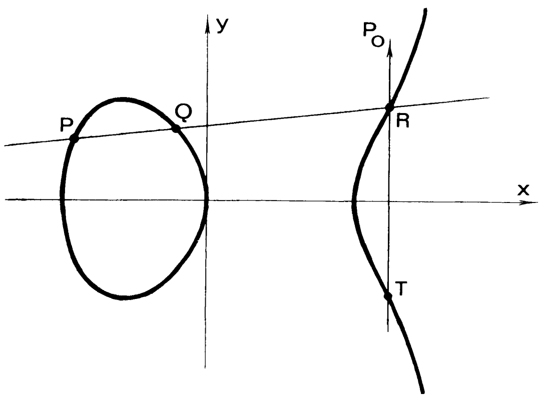

Figure 9. The group law on a cubic curve.

Let P_0 be the point $(0,1,0)$ on X. It is an inflection point, so the tangent line $z = 0$ at that point meets the curve in the divisor $3P_0$. If L is any other line in $\mathbf{P}^2$, meeting X in three points P,Q,R (which may coincide), then since L is linearly equivalent to the line $z = 0$ in $\mathbf{P}^2$, we have $P + Q + R \sim 3P_0$ on X, as in (6.6.2) above.

Now to any closed point $P \in X$, we associate the divisor $P - P_0 \in \mathrm{Cl}^\circ X$. This map is injective, because if $P - P_0 \sim Q - P_0$, then $P \sim Q$, and X would be rational by the previous example, which is impossible.

To show that the map from the closed points of X to $\mathrm{Cl}^\circ X$ is surjective, we proceed in several steps. Let $D \in \mathrm{Cl}^\circ X$. Then $D = \sum n_i P_i$, with $\sum n_i = 0$. Hence we can also write $D = \sum n_i(P_i - P_0)$. Now for any point R, let the line P_0R meet X further in the point T (always counting intersections with multiplicities—for example, if $R = P_0$, we take the line P_0R to be the tangent line at P_0, and then the third intersection T is also P_0). Then $P_0 + R + T \sim 3P_0$, so $R - P_0 \sim -(T - P_0)$. If i is an index such that $n_i < 0$ in D, we take $P_i = R$. Then replacing P_i by T, we get a linearly equivalent divisor with the ith coefficient $-n_i > 0$. Repeating this process, we may assume that $D = \sum n_i(P_i - P_0)$ with all $n_i > 0$. We now show by induction on $\sum n_i$ that $D \sim P - P_0$ for some point P. If $\sum n_i = 1$, there is nothing to prove. So suppose $\sum n_i \geqslant 2$, and let P,Q be two of the points P_i (maybe the same) which occur in D. Let the line PQ meet X in R, and let the line P_0R meet X in T.

Then we have

$$P + Q + R \sim 3P_0 \qquad \text{and} \qquad P_0 + R + T \sim 3P_0$$

so

$$(P - P_0) + (Q - P_0) \sim (T - P_0).$$

Replacing P and Q by T, we get D linearly equivalent to another divisor of the same form whose $\sum n_i$ is one less, so by induction $D \sim P - P_0$ for some P.

Thus we have shown that the group $\mathrm{Cl}\, X$ is in 1-1 correspondence with the set of closed points of X. One can show directly that the addition law determines a morphism of $X \times X \to X$, and the inverse law determines a morphism $X \to X$ (see for example Olson [1]). Thus X is a group variety in the sense of (I, Ex. 3.21). See (IV, 1.3.7) for a generalization.

Remark 6.10.3. This example of the cubic curve illustrates the general fact that the divisor class group of a variety has a discrete component (in this case $\mathbf{Z}$) and a continuous component (in this case $\mathrm{Cl}^\circ X$) which itself has the structure of an algebraic variety.

More specifically, if X is any complete nonsingular curve, then the group $\mathrm{Cl}^\circ X$ is isomorphic to the group of closed points of an abelian variety called the *Jacobian variety* of X. An *abelian variety* is a complete group variety over k. The dimension of the Jacobian variety is the *genus* of the curve. Thus the whole divisor class group of X is an extension of $\mathbf{Z}$ by the group of closed points of the Jacobian variety of X.

If X is a nonsingular projective variety of dimension $\geqslant 2$, then one can define a subgroup $\mathrm{Cl}^\circ X$ of $\mathrm{Cl}\, X$, namely the subgroup of divisor classes *algebraically equivalent* to zero, such that $\mathrm{Cl}\, X/\mathrm{Cl}^\circ X$ is a finitely generated abelian group, called the *Néron-Severi group* of X, and $\mathrm{Cl}^\circ X$ is isomorphic to the group of closed points of an abelian variety called the *Picard variety* of X.

Unfortunately we do not have space in this book to develop the theory of abelian varieties and to study the Jacobian and Picard varieties of a given variety. For more information and further references on this beautiful subject, see Lang [1], Mumford [2], Mumford [5], and Hartshorne [6]. See also (IV, §4), (V, Ex. 1.7), and Appendix B.

Cartier Divisors

Now we want to extend the notion of divisor to an arbitrary scheme. It turns out that using the irreducible subvarieties of codimension one doesn't work very well. So instead, we take as our point of departure the idea that a divisor should be something which locally looks like the divisor of a rational function. This is not exactly a generalization of the Weil divisors (as we will see), but it gives a good notion to use on arbitrary schemes.

Definition. Let X be a scheme. For each open affine subset $U = \operatorname{Spec} A$, let S be the set of elements of A which are not zero divisors, and let $K(U)$ be the localization of A by the multiplicative system S. We call $K(U)$ the *total quotient ring* of A. For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the *sheaf of total quotient rings* of $\mathcal{O}$. On an arbitrary scheme, the sheaf $\mathcal{K}$ replaces the concept of function field of an integral scheme. We denote by $\mathcal{K}^*$ the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings $\mathcal{K}$. Similarly $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$.

Definition. A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. Thinking of the properties of quotient sheaves, we see that a Cartier divisor on X can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each i,j, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. A Cartier divisor is *principal* if it is in the image of the natural map $\Gamma(X, \mathcal{K}^*) \rightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Two Cartier divisors are *linearly equivalent* if their difference is principal. (Although the group operation on $\mathcal{K}^*/\mathcal{O}^*$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.)

Proposition 6.11. *Let X be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains (in which case we say X is* locally factorial*). Then the group* Div X *of Weil divisors on X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

PROOF. First note that X is normal, hence satisfies (*), since a UFD is integrally closed. Thus it makes sense to talk about Weil divisors. Since X is integral, the sheaf $\mathcal{K}$ is just the constant sheaf corresponding to the function field K of X. Now let a Cartier divisor be given by $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open cover of X, and $f_i \in \Gamma(U_i, \mathcal{K}^*) = K^*$. We define the associated Weil divisor as follows. For each prime divisor Y, take the coefficient of Y to be $v_Y(f_i)$, where i is any index for which $Y \cap U_i \neq \varnothing$. If j is another such index, then f_i/f_j is invertible on $U_i \cap U_j$, so $v_Y(f_i/f_j) = 0$ and $v_Y(f_i) = v_Y(f_j)$. Thus we obtain a well-defined Weil divisor $D = \sum v_Y(f_i)Y$ on X. (The sum is finite because X is noetherian!)

Conversely, if D is a Weil divisor on X, let $x \in X$ be any point. Then D induces a Weil divisor D_x on the local scheme Spec $\mathcal{O}_x$. Since $\mathcal{O}_x$ is a UFD, D_x is a principal divisor, by (6.2), so let $D_x = (f_x)$ for some $f_x \in K$. Now the principal divisor (f_x) on X has the same restriction to Spec $\mathcal{O}_x$ as D, hence they differ only at prime divisors which do not pass through x. There are only finitely many of these which have a non-zero coefficient in D or (f_x), so there is an open neighborhood U_x of x such that D and (f_x) have the same restriction to U_x. Covering X with such open sets U_x, the functions f_x give a Cartier divisor on X. Note that if f, f' give the same Weil divisor on an open set U, then $f/f' \in \Gamma(U,\mathcal{O}^*)$, since X is normal (cf. proof of (6.2)). Thus we have a well-defined Cartier divisor.

These two constructions are inverse to each other, so we see that the groups of Weil divisors and Cartier divisors are isomorphic. Furthermore it is clear that the principal divisors correspond to each other.

Remark 6.11.1A. Since a regular local ring is UFD (Matsumura [2, Th. 48, p. 142]), this proposition applies in particular to any *regular* integral separated noetherian scheme. A scheme is *regular* if all of its local rings are regular local rings.

Remark 6.11.2. If X is a normal scheme, which is not necessarily locally factorial, we can define a subgroup of Div X consisting of the locally principal Weil divisors: D is *locally principal* if X can be covered by open sets U such that $D|_U$ is principal for each U. Then the above proof shows that the Cartier divisors are the same as the locally principal Weil divisors.

Example 6.11.3. Let X be the affine quadric cone Spec $k[x,y,z]/(xy - z^2)$ treated above (6.5.2). The ruling Y is a Weil divisor which is not locally principal in the neighborhood of the vertex of the cone. Indeed, our earlier proof shows that its prime ideal $\mathfrak{p}A_{\mathfrak{m}}$ is not a principal ideal even in the local ring $A_{\mathfrak{m}}$. Thus Y does not correspond to a Cartier divisor. On the other hand $2Y$ is locally principal, and in fact principal. So in this case the group of Cartier divisors modulo principal divisors is 0, whereas Cl $X \cong \mathbf{Z}/2\mathbf{Z}$.

Example 6.11.4. Let X be the cuspidal cubic curve $y^2z = x^3$ in $\mathbf{P}_k^2$, with char $k \neq 2$. In this case X does not satisfy (*), so we cannot talk about Weil divisors on X. However, we can talk about the group CaCl X of Cartier divisor classes modulo principal divisors. Imitating the case of the nonsingular cubic curve (6.10.2) we will show:

(a) there is a surjective degree homomorphism $\deg : \text{CaCl } X \to \mathbf{Z}$;

(b) there is a 1-1 correspondence between the set of nonsingular closed points of X and the kernel CaCl X of the degree map, which makes it into a group variety; and in fact

(c) there is a natural isomorphism of group varieties between $\text{CaCl}^\circ X$ and the additive group $\mathbf{G}_a$ of the field k (I, Ex. 3.21a).

To define the degree of a Cartier divisor on X, note that any Cartier divisor is linearly equivalent to one whose local function is invertible in some neighborhood of the singular point $Z = (0,0,1)$. Then this Cartier divisor corresponds to a Weil divisor $D = \sum n_i P_i$ on $X - Z$, and we define the degree of the original divisor to be $\deg D = \sum n_i$. The proof of (6.10) shows that if $f \in K$ is invertible at Z, then the principal divisor (f) on $X - Z$ has degree 0. Thus the degree of a Cartier divisor on X is well-defined, and it passes to linear equivalence classes to give a surjective homomorphism $\deg : \text{CaCl } X \to \mathbf{Z}$.

Now let P_0 be the point $(0,1,0)$ as in the case of the nonsingular cubic curve. To each closed point $P \in X - Z$, we associate the Cartier divisor D_P which is 1 in a neighborhood of Z, and which corresponds to the Weil divisor $P - P_0$ on $X - Z$. First note this map is injective: if $P \neq Q$ are two points in $X - Z$, and if $D_P \sim D_Q$, then there is an $f \in K^*$, which is invertible at Z, and such that $(f) = P - Q$ on $X - Z$. Then f gives a morphism of X to $\mathbf{P}^1$, which must be birational. But then the local ring of Z on X would dominate some discrete valuation ring of $\mathbf{P}^1$, and this is impossible, because Z is a singular point.

To show that every divisor in $\mathrm{CaCl}^\circ X$ is linearly equivalent to D_P for some closed point $P \in X - Z$, we proceed exactly as in the case of the nonsingular cubic curve above. The only difference is to note that the geometric constructions $R \mapsto T$ and $P,Q \mapsto R,T$ described above remain inside of $X - Z$. Thus the group $\mathrm{CaCl}^\circ X$ is in 1-1 correspondence with the set of closed points of $X - Z$, making it into a group variety.

In this case we are able to identify the group variety as $\mathbf{G}_a$. Of course, we know that X is a rational curve, and so $X - Z \cong \mathbf{A}_k^1$ (I, Ex. 3.2). But in fact, if we use the right parametrization, the group law corresponds. So define a morphism of $\mathbf{G}_a = \operatorname{Spec} k[t]$ to $X - Z$ by $t \mapsto (t,1,t^3)$. This is clearly an isomorphism of varieties. Using a little elementary analytic geometry (left to reader!) one shows that if $P = (t,1,t^3)$ and if $Q = (u,1,u^3)$, then the point T constructed above is just $(t + u,1,(t + u)^3)$. So we have an isomorphism of group varieties of $\mathbf{G}_a$ to $X - Z$ with the group structure of $\mathrm{CaCl}^\circ X$.

Invertible Sheaves

Recall that an *invertible sheaf* on a ringed space X is defined to be a locally free $\mathcal{O}_X$-module of rank 1. We will see now that invertible sheaves on a scheme are closely related to divisor classes modulo linear equivalence.

Proposition 6.12. *If $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves on a ringed space X, so is $\mathscr{L} \otimes \mathscr{M}$. If $\mathscr{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathscr{L}^{-1}$ on X such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathcal{O}_X$.*

PROOF. The first statement is clear, since $\mathscr{L}$ and $\mathscr{M}$ are both locally free of rank 1, and $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$. For the second statement, let $\mathscr{L}$ be any invertible sheaf, and take $\mathscr{L}^{-1}$ to be the dual sheaf $\mathscr{L}^\vee = \mathscr{H}om(\mathscr{L},\mathcal{O}_X)$. Then $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathscr{H}om(\mathscr{L},\mathscr{L}) = \mathcal{O}_X$ by (Ex. 5.1).

Definition. For any ringed space X, we define the *Picard group* of X, $\operatorname{Pic} X$, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$. The proposition shows that in fact it is a group.

Remark 6.12.1. We will see later (III, Ex. 4.5) that $\operatorname{Pic} X$ can be expressed as the cohomology group $H^1(X,\mathcal{O}_X^*)$.

Definition. Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$ as above. We define a subsheaf $\mathscr{L}(D)$ of the sheaf of total quotient rings $\mathscr{K}$ by taking $\mathscr{L}(D)$ to be the sub-$\mathscr{O}_X$-module of $\mathscr{K}$ generated by f_i^{-1} on U_i. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathscr{O}_X$-module. We call $\mathscr{L}(D)$ the *sheaf associated* to D.

Proposition 6.13. *Let X be a scheme. Then:*

(a) *for any Cartier divisor D, $\mathscr{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathscr{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathscr{K}$;*

(b) $\mathscr{L}(D_1 - D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$;

(c) *$D_1 \sim D_2$ if and only if $\mathscr{L}(D_1) \cong \mathscr{L}(D_2)$ as abstract invertible sheaves* (i.e., *disregarding the embedding in $\mathscr{K}$*).

PROOF.

(a) Since each $f_i \in \Gamma(U_i, \mathscr{K}^*)$, the map $\mathscr{O}_{U_i} \to \mathscr{L}(D)|_{U_i}$ defined by $1 \mapsto f_i^{-1}$ is an isomorphism. Thus $\mathscr{L}(D)$ is an invertible sheaf. The Cartier divisor D can be recovered from $\mathscr{L}(D)$ together with its embedding in $\mathscr{K}$, by taking f_i on U_i to be the inverse of a local generator of $\mathscr{L}(D)$. For any invertible subsheaf of $\mathscr{K}$, this construction gives a Cartier divisor, so we have a 1-1 correspondence as claimed.

(b) If D_1 is locally defined by f_i and D_2 is locally defined by g_i, then $\mathscr{L}(D_1 - D_2)$ is locally generated by $f_i^{-1}g_i$, so $\mathscr{L}(D_1 - D_2) = \mathscr{L}(D_1) \cdot \mathscr{L}(D_2)^{-1}$ as subsheaves of $\mathscr{K}$. This product is clearly isomorphic to the abstract tensor product $\mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$.

(c) Using (b), it will be sufficient to show that $D = D_1 - D_2$ is principal if and only if $\mathscr{L}(D) \cong \mathscr{O}_X$. If D is principal, defined by $f \in \Gamma(X, \mathscr{K}^*)$, then $\mathscr{L}(D)$ is globally generated by f^{-1}, so sending $1 \mapsto f^{-1}$ gives an isomorphism $\mathscr{O}_X \cong \mathscr{L}(D)$. Conversely, given such an isomorphism, the image of 1 gives an element of $\Gamma(X, \mathscr{K}^*)$ whose inverse will define D as a principal divisor.

Corollary 6.14. *On any scheme X, the map $D \mapsto \mathscr{L}(D)$ gives an injective homomorphism of the group* CaCl X *of Cartier divisors modulo linear equivalence to* Pic X.

Remark 6.14.1. The map CaCl $X \to$ Pic X may not be surjective, because there may be invertible sheaves on X which are not isomorphic to any invertible subsheaf of $\mathscr{K}$. For an example of Kleiman, see Hartshorne [5, I.1.3, p. 9]. On the other hand, this map is an isomorphism in most common situations. Nakai [2, p. 301] has shown that it is an isomorphism whenever X is projective over a field. We will show now that it is an isomorphism if X is integral.

Proposition 6.15. *If X is an integral scheme, the homomorphism* $\mathrm{CaCl}\, X \to \mathrm{Pic}\, X$ *of* (6.14) *is an isomorphism.*

PROOF. We have only to show that every invertible sheaf is isomorphic to a subsheaf of $\mathscr{K}$, which in this case is the constant sheaf K, where K is the function field of X. So let $\mathscr{L}$ be any invertible sheaf, and consider the sheaf $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{K}$. On any open set U where $\mathscr{L} \cong \mathscr{O}_X$, we have $\mathscr{L} \otimes \mathscr{K} \cong \mathscr{K}$, so it is a constant sheaf on U. Now because X is irreducible, it follows that any sheaf whose restriction to each open set of a covering of X is constant, is in fact a constant sheaf. Thus $\mathscr{L} \otimes \mathscr{K}$ is isomorphic to the constant sheaf $\mathscr{K}$, and the natural map $\mathscr{L} \to \mathscr{L} \otimes \mathscr{K} \cong \mathscr{K}$ expresses $\mathscr{L}$ as a subsheaf of $\mathscr{K}$.

Corollary 6.16. *If X is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism* $\mathrm{Cl}\, X \cong \mathrm{Pic}\, X$.

PROOF. This follows from (6.11) and (6.15).

Corollary 6.17. *If $X = \mathbf{P}_k^n$ for some field k, then every invertible sheaf on X is isomorphic to $\mathscr{O}(l)$ for some $l \in \mathbf{Z}$.*

PROOF. By (6.4), $\mathrm{Cl}\, X \cong \mathbf{Z}$, so by (6.16), $\mathrm{Pic}\, X \cong \mathbf{Z}$. Furthermore the generator of $\mathrm{Cl}\, X$ is a hyperplane, which corresponds to the invertible sheaf $\mathscr{O}(1)$. Hence $\mathrm{Pic}\, X$ is the free group generated by $\mathscr{O}(1)$, and any invertible sheaf $\mathscr{L}$ is isomorphic to $\mathscr{O}(l)$ for some $l \in \mathbf{Z}$.

We conclude this section with some remarks about closed subschemes of codimension one of a scheme X.

Definition. A Cartier divisor on a scheme X is *effective* if it can be represented by $\{(U_i, f_i)\}$ where all the $f_i \in \Gamma(U_i, \mathscr{O}_{U_i})$. In that case we define the *associated subscheme of codimension* 1, Y, to be the closed subscheme defined by the sheaf of ideals $\mathscr{I}$ which is locally generated by f_i.

Remark 6.17.1. Clearly this gives a 1-1 correspondence between effective Cartier divisors on X and *locally principal* closed subschemes Y, i.e., subschemes whose sheaf of ideals is locally generated by a single element. Note also that if X is an integral separated noetherian locally factorial scheme, so that the Cartier divisors correspond to Weil divisors by (6.11), then the effective Cartier divisors correspond exactly to the effective Weil divisors.

Proposition 6.18. *Let D be an effective Cartier divisor on a scheme X, and let Y be the associated locally principal closed subscheme. Then* $\mathscr{I}_Y \cong \mathscr{L}(-D)$.

PROOF. $\mathscr{L}(-D)$ is the subsheaf of $\mathscr{K}$ generated locally by f_i. Since D is effective, this is actually a subsheaf of $\mathcal{O}_X$, which is none other than the ideal sheaf $\mathscr{I}_Y$ of Y.

EXERCISES

6.1. Let X be a scheme satisfying (*). Then $X \times \mathbf{P}^n$ also satisfies (*), and $\mathrm{Cl}(X \times \mathbf{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbf{Z}$.

***6.2.** *Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbf{P}_k^n$ which is nonsingular in codimension one (hence satisfies (*)). For any divisor $D = \sum n_i Y_i$ on X, we define the *degree* of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself (I, §7).

(a) Let V be an irreducible hypersurface in $\mathbf{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbf{P}^n$ for which $Y_i \cap U_i \neq \varnothing$, and let $n_i = v_{Y_i}(\bar{f}_i)$, where $\bar{f}_i$ is the restriction of f_i to $U_i \cap X$. Then we define the *divisor* $V.X$ to be $\sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of $\mathrm{Div}\, \mathbf{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\, X$.

(b) If D is a principal divisor on $\mathbf{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus we get a homomorphism $\mathrm{Cl}\, \mathbf{P}^n \to \mathrm{Cl}\, X$.

(c) Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y_i)$ defined in (I, §7). Then use the generalized Bézout theorem (I, 7.7) to show that for any divisor D on $\mathbf{P}^n$, none of whose components contain X,

$$\deg(D.X) = (\deg D) \cdot (\deg X).$$

(d) If D is a principal divisor on X, show that there is a rational function f on $\mathbf{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism $\deg : \mathrm{Cl}\, X \to \mathbf{Z}$. (This gives another proof of (6.10), since any complete nonsingular curve is projective.) Finally, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Cl}\, \mathbf{P}^n \arrow[r] \arrow[d, "\cong"', "\deg"] & \mathrm{Cl}\, X \arrow[d, "\deg"] \\ \mathbf{Z} \arrow[r, "\cdot(\deg X)"] & \mathbf{Z} \end{tikzcd},$$

and in particular, we see that the map $\mathrm{Cl}\, \mathbf{P}^n \to \mathrm{Cl}\, X$ is injective.

***6.3.** *Cones.* In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex. 2.10). So let V be a projective variety in $\mathbf{P}^n$, which is of dimension $\geqslant 1$ and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in $\mathbf{A}^{n+1}$, and let $\bar{X}$ be its projective closure in $\mathbf{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi : \bar{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbf{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \mathrm{Cl}\, V \to \mathrm{Cl}(\bar{X} - P)$ is an isomorphism. Since $\mathrm{Cl}\, \bar{X} \cong \mathrm{Cl}(\bar{X} - P)$, we have also $\mathrm{Cl}\, V \cong \mathrm{Cl}\, \bar{X}$.

(b) We have $V \subseteq \bar{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\mathrm{Cl}\,\bar{X}$ is equal to π^* (class of $V.H$) where H is any hyperplane of $\mathbf{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence

$$0 \to \mathbf{Z} \to \mathrm{Cl}\,V \to \mathrm{Cl}\,X \to 0,$$

where the first arrow sends $1 \mapsto V.H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in X. (The injectivity of the first arrow follows from the previous exercise.)

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a unique factorization domain if and only if (1) V is projectively normal (Ex. 5.14), and (2) $\mathrm{Cl}\,V \cong \mathbf{Z}$ and is generated by the class of $V.H$.

(d) Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\mathrm{Cl}\,X \to \mathrm{Cl}(\mathrm{Spec}\,\mathcal{O}_P)$.

6.4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a *square-free* nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \ldots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring. [*Hint*: The quotient field K of A is just $k(x_1, \ldots, x_n)[z]/(z^2 - f)$. It is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbf{Z}/2\mathbf{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g,h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Now show that α is integral over $k[x_1, \ldots, x_n]$ if and only if $g,h \in k[x_1, \ldots, x_n]$. Conclude that A is the integral closure of $k[x_1, \ldots, x_n]$ in K.]

***6.5.** *Quadric Hypersurfaces.* Let char $k \neq 2$, and let X be the affine quadric hypersurface $\mathrm{Spec}\,k[x_0, \ldots, x_n]/(x_0^2 + x_1^2 + \ldots + x_r^2)$—cf. (I, Ex. 5.12).

(a) Show that X is normal if $r \geqslant 2$ (use (Ex. 6.4)).

(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \ldots + x_r^2$. Now imitate the method of (6.5.2) to show that:

(1) If $r = 2$, then $\mathrm{Cl}\,X \cong \mathbf{Z}/2\mathbf{Z}$;
(2) If $r = 3$, then $\mathrm{Cl}\,X \cong \mathbf{Z}$ (use (6.6.1) and (Ex. 6.3) above);
(3) If $r \geqslant 4$ then $\mathrm{Cl}\,X = 0$.

(c) Now let Q be the projective quadric hypersurface in $\mathbf{P}^n$ defined by the same equation. Show that:

(1) If $r = 2$, $\mathrm{Cl}\,Q \cong \mathbf{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;
(2) If $r = 3$, $\mathrm{Cl}\,Q \cong \mathbf{Z} \oplus \mathbf{Z}$;
(3) If $r \geqslant 4$, $\mathrm{Cl}\,Q \cong \mathbf{Z}$, generated by $Q.H$.

(d) Prove Klein's theorem, which says that if $r \geqslant 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbf{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection. (First show that for $r \geqslant 4$, the homogeneous coordinate ring $S(Q) = k[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_r^2)$ is a UFD.)

6.6. Let X be the nonsingular plane cubic curve $y^2 z = x^3 - xz^2$ of (6.10.2).

(a) Show that three points P,Q,R of X are collinear if and only if $P + Q + R = 0$ in the group law on X. (Note that the point $P_0 = (0,1,0)$ is the zero element in the group structure on X.)

(b) A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

(c) A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point. (An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity $\geqslant 3$ with the curve at P.)

(d) Let $k = \mathbf{C}$. Show that the points of X with coordinates in $\mathbf{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

***6.7.** Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbf{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, $\operatorname{CaCl}^\circ X$, is naturally isomorphic to the multiplicative group $\mathbf{G}_m$.

6.8. (a) Let $f\colon X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$.

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^*\colon \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphisms of (6.16).

(c) If X is a locally factorial integral closed subscheme of $\mathbf{P}_k^n$, and if $f\colon X \to \mathbf{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

***6.9.** *Singular Curves.* Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\tilde{X}$ be its normalization, and let $\pi\colon \tilde{X} \to X$ be the projection map (Ex. 3.8). For each point $P \in X$, let $\mathcal{O}_P$ be its local ring, and let $\tilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$. We use a * to denote the group of units in a ring.

(a) Show there is an exact sequence

$$0 \to \bigoplus_{P \in X} \tilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic} \tilde{X} \to 0.$$

[*Hint*: Represent $\operatorname{Pic} X$ and $\operatorname{Pic} \tilde{X}$ as the groups of Cartier divisors modulo principal divisors, and use the exact sequence of sheaves on X

$$0 \to \pi_*\mathcal{O}_{\tilde{X}}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\pi_*\mathcal{O}_{\tilde{X}}^* \to 0.]$$

(b) Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbf{G}_a \to \operatorname{Pic} X \to \mathbf{Z} \to 0,$$

and if X is a plane nodal cubic curve, there is an exact sequence

$$0 \to \mathbf{G}_m \to \operatorname{Pic} X \to \mathbf{Z} \to 0.$$

6.10. *The Grothendieck Group $K(X)$.* Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) If $X = \mathbf{A}_k^1$, then $K(X) \cong \mathbf{Z}$.

(b) If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the *rank* of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism $\mathrm{rank}: K(X) \to \mathbf{Z}$.

(c) If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0,$$

where the first map is extension by zero, and the second map is restriction. [*Hint*: For exactness in the middle, show that if $\mathscr{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \ldots \supseteq \mathscr{F}_n = 0$, such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use (Ex. 5.15).]

For further information about $K(X)$, and its applications to the generalized Riemann–Roch theorem, see Borel–Serre [1], Manin [1], and Appendix A.

***6.11.** *The Grothendieck Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that for any D, $\psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi: \mathrm{Cl}\, X \to K(X)$.

(b) For any coherent sheaf $\mathscr{F}$ on X, show that there exist locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\bigwedge^{r_0} \mathscr{E}_0) \otimes (\bigwedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det: K(X) \to \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathscr{L}(D)$.

(c) If $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank r, then $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$.

(d) Using the maps ψ, det, rank, and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbf{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

6.12. Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbf{Z}$, such that:

(1) If D is a divisor, $\deg \mathscr{L}(D) = \deg D$;

(2) If $\mathscr{F}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$; and

(3) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.

7 Projective Morphisms

In this section we gather together several topics concerned with morphisms of a given scheme to projective space. We will show how a morphism of a scheme X to a projective space is determined by giving an invertible sheaf

$\mathscr{L}$ on X and a set of its global sections. We will give some criteria for this morphism to be an immersion. Then we study the closely connected topic of ample invertible sheaves. We also introduce the more classical language of linear systems, which from the point of view of schemes is hardly more than another set of terminology for dealing with invertible sheaves and their global sections. However, the geometric understanding furnished by the concept of linear system is often very valuable. At the end of this section we define the **Proj** of a graded sheaf of algebras over a scheme X, and we give two important examples, namely the projective bundle $\mathbf{P}(\mathscr{E})$ associated with a locally free sheaf $\mathscr{E}$, and the definition of blowing up with respect to a coherent sheaf of ideals.

Morphisms to $\mathbf{P}^n$

Let A be a fixed ring, and consider the projective space $\mathbf{P}^n_A = \operatorname{Proj} A[x_0,\ldots,x_n]$ over A. On $\mathbf{P}^n_A$ we have the invertible sheaf $\mathcal{O}(1)$, and the homogeneous coordinates $x_0,\ldots,x_n$ give rise to global sections $x_0,\ldots,x_n \in \Gamma(\mathbf{P}^n_A,\mathcal{O}(1))$. One sees easily that the sheaf $\mathcal{O}(1)$ is generated by the global sections $x_0,\ldots,x_n$, i.e., the images of these sections generate the stalk $\mathcal{O}(1)_P$ of the sheaf $\mathcal{O}(1)$ as a module over the local ring $\mathcal{O}_P$, for each point $P \in \mathbf{P}^n_A$.

Now let X be any scheme over A, and let $\varphi: X \to \mathbf{P}^n_A$ be an A-morphism of X to $\mathbf{P}^n_A$. Then $\mathscr{L} = \varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, and the global sections $s_0,\ldots,s_n$, where $s_i = \varphi^*(x_i)$, $s_i \in \Gamma(X,\mathscr{L})$, generate the sheaf $\mathscr{L}$. Conversely, we will see that $\mathscr{L}$ and the sections s_i determine φ.

Theorem 7.1. *Let A be a ring, and let X be a scheme over A.*

(a) *If $\varphi: X \to \mathbf{P}^n_A$ is an A-morphism, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i)$, $i = 0,1,\ldots,n$.*

(b) *Conversely, if $\mathscr{L}$ is an invertible sheaf on X, and if $s_0,\ldots,s_n \in \Gamma(X,\mathscr{L})$ are global sections which generate $\mathscr{L}$, then there exists a unique A-morphism $\varphi: X \to \mathbf{P}^n_A$ such that $\mathscr{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.*

Proof. Part (a) is clear from the discussion above. To prove (b), suppose given $\mathscr{L}$ and the global sections $s_0,\ldots,s_n$ which generate it. For each i, let $X_i = \{P \in X \mid (s_i)_P \notin \mathfrak{m}_P\mathscr{L}_P\}$. Then (as we have seen before) X_i is an open subset of X, and since the s_i generate $\mathscr{L}$, the open sets X_i must cover X. We define a morphism from X_i to the standard open set $U_i = \{x_i \neq 0\}$ of $\mathbf{P}^n_A$ as follows. Recall that $U_i \cong \operatorname{Spec} A[y_0,\ldots,y_n]$ where $y_j = x_j/x_i$, with $y_i = 1$ omitted. We define a ring homomorphism $A[y_0,\ldots,y_n] \to \Gamma(X_i,\mathcal{O}_{X_i})$ by sending $y_j \to s_j/s_i$ and making it A-linear. This makes sense, because for each $P \in X_i$, $(s_i)_P \notin \mathfrak{m}_P\mathscr{L}_P$, and $\mathscr{L}$ is locally free of rank 1, so the quotient s_j/s_i is a well-defined element of $\Gamma(X_i,\mathcal{O}_{X_i})$. Now by (Ex. 2.4) this ring homomorphism determines a morphism of schemes (over A) $X_i \to U_i$. Clearly these morphisms glue (cf. Step 3 of proof of (3.3)), so we obtain a morphism $\varphi: X \to \mathbf{P}^n_A$. It is clear from the construction that φ is an A-

morphism, that $\mathscr{L} \cong \varphi^*(\mathcal{O}(1))$, and that the sections s_i correspond to $\varphi^*(x_i)$ under this isomorphism. It is clear that any morphism with these properties must be the one given by the construction, so φ is unique.

Example 7.1.1 (Automorphisms of $\mathbf{P}_k^n$). If $\|a_{ij}\|$ is an invertible $(n+1) \times (n+1)$ matrix of elements of a field k, then $x_i' = \sum a_{ij}x_j$ determines an automorphism of the polynomial ring $k[x_0, \ldots, x_n]$ and hence also an automorphism of $\mathbf{P}_k^n$. If $\lambda \in k$ is a nonzero element, then $\|\lambda a_{ij}\|$ determines the same automorphism of $\mathbf{P}_k^n$. So we are led to consider the group $\mathrm{PGL}(n,k) = \mathrm{GL}(n+1,k)/k^*$, which acts as a group of automorphisms of $\mathbf{P}_k^n$. By considering the points $(1,0, \ldots, 0)$, $(0,1,0, \ldots, 0), \ldots, (0,0, \ldots, 1)$, and $(1,1, \ldots, 1)$, one sees easily that this group acts faithfully, i.e., if $g \in \mathrm{PGL}(n,k)$ induces the trivial automorphism of $\mathbf{P}_k^n$, then g is the identity.

Now we will show conversely that every k-automorphism of $\mathbf{P}_k^n$ is an element of $\mathrm{PGL}(n,k)$. This generalizes an earlier result for $\mathbf{P}_k^1$ (I, Ex. 6.6). So let φ be a k-automorphism of $\mathbf{P}_k^n$. We have seen (6.17) that $\operatorname{Pic} \mathbf{P}_k^n \cong \mathbf{Z}$ and is generated by $\mathcal{O}(1)$. The automorphism φ induces an automorphism of $\operatorname{Pic} \mathbf{P}^n$, so $\varphi^*(\mathcal{O}(1))$ must be a generator of that group, hence isomorphic to either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$. But $\mathcal{O}(-1)$ has no global sections, so we conclude that $\varphi^*(\mathcal{O}(1)) \cong \mathcal{O}(1)$. Now $\Gamma(\mathbf{P}^n, \mathcal{O}(1))$ is a k-vector space with basis $x_0, \ldots, x_n$, by (5.13). Since φ is an automorphism, the $s_i = \varphi^*(x_i)$ must be another basis of this vector space, so we can write $s_i = \sum a_{ij}x_j$, where $\|a_{ij}\|$ is an invertible matrix of elements of k. Since φ is uniquely determined by the s_i according to the theorem, we see that φ coincides with the automorphism given by $\|a_{ij}\|$ as an element of $\mathrm{PGL}(n,k)$.

Example 7.1.2. If X is a scheme over A, $\mathscr{L}$ an invertible sheaf, and $s_0, \ldots, s_n$ any set of global sections, which do not necessarily generate $\mathscr{L}$, we can always consider the open set $U \subseteq X$ (possibly empty) over which the s_i do generate $\mathscr{L}$. Then $\mathscr{L}|_U$ and the $s_i|_U$ give a morphism $U \to \mathbf{P}_A^n$. Such is the case for example, if we take $X = \mathbf{P}_k^{n+1}$, $\mathscr{L} = \mathcal{O}(1)$, and $s_i = x_i$, $i = 0, \ldots, n$ (omitting x_{n+1}). These sections generate everywhere except at the point $(0,0, \ldots, 0,1) = P_0$. Thus $U = \mathbf{P}^{n+1} - P_0$, and the corresponding morphism $U \to \mathbf{P}^n$ is nothing other than the projection from the point P_0 to $\mathbf{P}^n$ (I, Ex. 3.14).

Next we give some criteria for a morphism to a projective space to be a closed immersion.

Proposition 7.2. *Let $\varphi: X \to \mathbf{P}_A^n$ be a morphism of schemes over A, corresponding to an invertible sheaf $\mathscr{L}$ on X and sections $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ as above. Then φ is a closed immersion if and only if*

(1) *each open set $X_i = X_{s_i}$ is affine, and*
(2) *for each i, the map of rings $A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ defined by $y_j \mapsto s_j/s_i$ is surjective.*

PROOF. First suppose φ is a closed immersion. Then $X_i = X \cap U_i$ is a closed subscheme of U_i. Therefore X_i is affine and the corresponding map of rings is surjective by (5.10). Conversely, suppose (1) and (2) satisfied. Then each X_i is a closed subscheme of U_i. Since in any case $X_i = \varphi^{-1}(U_i)$, and the X_i cover X, it is clear that X is a closed subscheme of $\mathbf{P}^n_A$.

With more hypotheses, we can give a more local criterion.

Proposition 7.3. *Let k be an algebraically closed field, let X be a projective scheme over k, and let $\varphi: X \to \mathbf{P}^n_k$ be a morphism (over k) corresponding to $\mathscr{L}$ and $s_0, \ldots, s_n \in \Gamma(X,\mathscr{L})$ as above. Let $V \subseteq \Gamma(X,\mathscr{L})$ be the subspace spanned by the s_i. Then φ is a closed immersion if and only if*

(1) *elements of V* separate points, i.e., *for any two distinct closed points $P,Q \in X$, there is an $s \in V$ such that $s \in \mathfrak{m}_P\mathscr{L}_P$ but $s \notin \mathfrak{m}_Q\mathscr{L}_Q$, or vice versa, and*
(2) *elements of V* separate tangent vectors, i.e., *for each closed point $P \in X$, the set $\{s \in V | s_P \in \mathfrak{m}_P\mathscr{L}_P\}$ spans the k-vector space $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$.*

PROOF. If φ is a closed immersion, we think of X as a closed subscheme of $\mathbf{P}^n_k$. In this case $\mathscr{L} = \mathscr{O}_X(1)$, and the vector space $V \subseteq \Gamma(X,\mathscr{O}_X(1))$ is just spanned by the images of $x_0, \ldots, x_n \in \Gamma(\mathbf{P}^n,\mathscr{O}(1))$. Given closed points $P \neq Q$ in X, there is a hyperplane containing P but not Q. If its equation is $\sum a_i x_i = 0$, $a_i \in k$, then $s = \sum a_i x_i$ restricted to X has the right property for (1). For (2), the hyperplanes passing through P give rise to sections which generate $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$. For simplicity suppose that P is the point $(1,0,0, \ldots ,0)$. Then on the open affine $U_0 \cong \operatorname{Spec} k[y_1, \ldots, y_n]$, $\mathscr{L}$ is trivial, P is the point $(0, \ldots ,0)$, and $\mathfrak{m}_P/\mathfrak{m}_P^2$ is exactly the vector space spanned by $y_1, \ldots, y_n$. We use the hypothesis k algebraically closed to ensure that every closed point of $\mathbf{P}^n_k$ is of the form $(a_0, \ldots ,a_n)$ for suitable $a_i \in k$, hence points can be separated by hyperplanes with coefficients in k.

For the converse, let $\varphi: X \to \mathbf{P}^n$ satisfy (1) and (2). Since the elements of V are pull-backs of sections of $\mathscr{O}(1)$ on $\mathbf{P}^n$, it is clear from (1) that the map φ is injective as a map of sets. Since X is projective over k, it is proper over k (4.9), so the image $\varphi(X)$ in $\mathbf{P}^n$ is closed (Ex. 4.4), and φ is a proper morphism (4.8e). In particular, φ is a closed map. But, being a morphism, it is also continuous, so we see that φ is a homeomorphism of X onto its image $\varphi(X)$ which is a closed subset of $\mathbf{P}^n$. To show that φ is a closed immersion, it remains only to show that the morphism of sheaves $\mathscr{O}_{\mathbf{P}^n} \to \varphi_*\mathscr{O}_X$ is surjective. This can be checked on the stalks. So it is sufficient to show, for each closed point P, that $\mathscr{O}_{\mathbf{P}^n,P} \to \mathscr{O}_{X,P}$ is surjective. Both local rings have the same residue field k, and our hypothesis (2) implies that the image of the maximal ideal $\mathfrak{m}_{\mathbf{P}^n,P}$ generates $\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$. We also need to use (5.20), which implies that $\varphi_*\mathscr{O}_X$ is a coherent sheaf on $\mathbf{P}^n$, and hence that $\mathscr{O}_{X,P}$ is a finitely generated $\mathscr{O}_{\mathbf{P}^n,P}$-module. Now our result is a consequence of the following lemma.

Lemma 7.4. *Let $f: A \to B$ be a local homomorphism of local noetherian rings, such that*

(1) *$A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism,*
(2) *$\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and*
(3) *B is a finitely generated A-module.*

Then f is surjective.

PROOF. Consider the ideal $\mathfrak{a} = \mathfrak{m}_A B$ of B. We have $\mathfrak{a} \subseteq \mathfrak{m}_B$, and by (2), $\mathfrak{a}$ contains a set of generators for $\mathfrak{m}_B/\mathfrak{m}_B^2$. Hence by Nakayama's lemma for the local ring B and the B-module $\mathfrak{m}_B$, we conclude that $\mathfrak{a} = \mathfrak{m}_B$. Now apply Nakayama's lemma to the A-module B. By (3), B is a finitely generated A-module. The element $1 \in B$ gives a generator for $B/\mathfrak{m}_A B = B/\mathfrak{m}_B = A/\mathfrak{m}_A$ by (1), so we conclude that 1 also generates B as an A-module, i.e., f is surjective.

Ample Invertible Sheaves

Now that we have seen that a morphism of a scheme X to a projective space can be characterized by giving an invertible sheaf on X and a suitable set of its global sections, we can reduce the study of varieties in projective space to the study of schemes with certain invertible sheaves and given global sections. Recall that in §5 we defined a sheaf $\mathscr{L}$ on X to be *very ample relative to* Y (where X is a scheme over Y) if there is an immersion $i: X \to \mathbf{P}_Y^n$ for some n such that $\mathscr{L} \cong i^*\mathcal{O}(1)$. In case $Y = \operatorname{Spec} A$, this is the same thing as saying that $\mathscr{L}$ admits a set of global sections $s_0, \ldots, s_n$ such that the corresponding morphism $X \to \mathbf{P}_A^n$ is an immersion. We have also seen (5.17) that if $\mathscr{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathscr{F}$ on X, there is an integer $n_0 > 0$ such that for all $n \geqslant n_0$, $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections. We will use this last property of being generated by global sections to define the notion of an ample invertible sheaf, which is more general, and in many ways is more convenient to work with than the notion of very ample sheaf.

Definition. An invertible sheaf $\mathscr{L}$ on a noetherian scheme X is said to be *ample* if for every coherent sheaf $\mathscr{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathscr{F}$) such that for every $n \geqslant n_0$, the sheaf $\mathscr{F} \otimes \mathscr{L}^n$ is generated by its global sections. (Here $\mathscr{L}^n = \mathscr{L}^{\otimes n}$ denotes the n-fold tensor power of $\mathscr{L}$ with itself.)

Remark 7.4.1. Note that "ample" is an absolute notion, i.e., it depends only on the scheme X, whereas "very ample" is a relative notion, depending on a morphism $X \to Y$.

Example 7.4.2. If X is affine, then any invertible sheaf is ample, because every coherent sheaf on an affine scheme is generated by its global sections (5.16.2).

Remark 7.4.3. Serre's theorem (5.17) asserts that a very ample sheaf $\mathscr{L}$ on a projective scheme X over a noetherian ring A is ample. The converse is false, but we will see below (7.6) that if $\mathscr{L}$ is ample, then some tensor power $\mathscr{L}^m$ of $\mathscr{L}$ is very ample. Thus "ample" can be viewed as a stable version of "very ample."

Remark 7.4.4. In Chapter III we will give a characterization of ample invertible sheaves in terms of the vanishing of certain cohomology groups (III, 5.3).

Proposition 7.5. *Let $\mathscr{L}$ be an invertible sheaf on a noetherian scheme X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*
(ii) *$\mathscr{L}^m$ is ample for all $m > 0$;*
(iii) *$\mathscr{L}^m$ is ample for some $m > 0$.*

PROOF. (i) $\Rightarrow$ (ii) is immediate from the definition of ample; (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i), assume that $\mathscr{L}^m$ is ample. Given a coherent sheaf $\mathscr{F}$ on X, there exists an $n_0 > 0$ such that $\mathscr{F} \otimes (\mathscr{L}^m)^n$ is generated by global sections for all $n \geqslant n_0$. Considering the coherent sheaf $\mathscr{F} \otimes \mathscr{L}$, there exists an $n_1 > 0$ such that $\mathscr{F} \otimes \mathscr{L} \otimes (\mathscr{L}^m)^n$ is generated by global sections for all $n \geqslant n_1$. Similarly, for each $k = 1,2, \ldots ,m - 1$, there is an $n_k > 0$ such that $\mathscr{F} \otimes \mathscr{L}^k \otimes (\mathscr{L}^m)^n$ is generated by global sections for all $n \geqslant n_k$. Now if we take $N = m \cdot \max\{n_i | i = 0,1, \ldots ,m - 1\}$, then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for all $n \geqslant N$. Hence $\mathscr{L}$ is ample.

Theorem 7.6. *Let X be a scheme of finite type over a noetherian ring A, and let $\mathscr{L}$ be an invertible sheaf on X. Then $\mathscr{L}$ is ample if and only if $\mathscr{L}^m$ is very ample over* Spec A *for some $m > 0$.*

PROOF. First suppose $\mathscr{L}^m$ is very ample for some $m > 0$. Then there is an immersion $i: X \to \mathbf{P}_A^n$ such that $\mathscr{L}^m \cong i^*(\mathcal{O}(1))$. Let $\bar{X}$ be the closure of X in $\mathbf{P}_A^n$. Then $\bar{X}$ is a projective scheme over A, so by (5.17), $\mathcal{O}_{\bar{X}}(1)$ is ample on $\bar{X}$. Now given any coherent sheaf $\mathscr{F}$ on X, it extends by (Ex. 5.15) to a coherent sheaf $\bar{\mathscr{F}}$ on $\bar{X}$. If $\bar{\mathscr{F}} \otimes \mathcal{O}_{\bar{X}}(l)$ is generated by global sections, then a fortiori $\mathscr{F} \otimes \mathcal{O}_X(l)$ is also generated by global sections. Thus we see that $\mathscr{L}^m$ is ample on X, and so by (7.5), $\mathscr{L}$ is also ample on X.

For the converse, suppose that $\mathscr{L}$ is ample on X. Given any $P \in X$, let U be an open affine neighborhood of P such that $\mathscr{L}|_U$ is free on U. Let Y be the closed set $X - U$, and let $\mathscr{I}_Y$ be its sheaf of ideals with the reduced induced scheme structure. Then $\mathscr{I}_Y$ is a coherent sheaf on X, so for some

$n > 0$, $\mathscr{I}_Y \otimes \mathscr{L}^n$ is generated by global sections. In particular, there is a section $s \in \Gamma(X, \mathscr{I}_Y \otimes \mathscr{L}^n)$ such that $s_P \notin \mathfrak{m}_P(\mathscr{I}_Y \otimes \mathscr{L}^n)_P$. Now $\mathscr{I}_Y \otimes \mathscr{L}^n$ is a subsheaf of $\mathscr{L}^n$, so we can think of s as an element of $\Gamma(X,\mathscr{L}^n)$. If X_s is the open set $\{Q \in X | s_Q \notin \mathfrak{m}_Q \mathscr{L}^n_Q\}$, then it follows from our choice of s that $P \in X_s$ and that $X_s \subseteq U$. Now U is affine, and $\mathscr{L}|_U$ is trivial, so s induces an element $f \in \Gamma(U,\mathcal{O}_U)$, and then $X_s = U_f$ is also affine.

Thus we have shown that for any point $P \in X$, there is an $n > 0$ and a section $s \in \Gamma(X,\mathscr{L}^n)$ such that $P \in X_s$ and X_s is affine. Since X is quasi-compact, we can cover X by a finite number of such open affines, corresponding to sections $s_i \in \Gamma(X,\mathscr{L}^{n_i})$. Replacing each s_i by a suitable power $s_i^k \in \Gamma(X,\mathscr{L}^{kn_i})$, which doesn't change X_{s_i}, we may assume that all n_i are equal to one n. Finally, since $\mathscr{L}^n$ is also ample, and since we are only trying to show that some power of $\mathscr{L}$ is very ample, we may replace $\mathscr{L}$ by $\mathscr{L}^n$. Thus we may assume now that we have global sections $s_1, \ldots, s_k \in \Gamma(X,\mathscr{L})$ such that each $X_i = X_{s_i}$ is affine, and the X_i cover X.

Now for each i, let $B_i = \Gamma(X_i,\mathcal{O}_{X_i})$. Since X is a scheme of finite type over A, each B_i is a finitely generated A-algebra (Ex. 3.3). So let $\{b_{ij} | j = 1, \ldots, k_i\}$ be a set of generators for B_i as an A-algebra. By (5.14), for each i,j, there is an integer n such that $s_i^n b_{ij}$ extends to a global section $c_{ij} \in \Gamma(X,\mathscr{L}^n)$. We can take one n large enough to work for all i,j. Now we take the invertible sheaf $\mathscr{L}^n$ on X, and the sections $\{s_i^n | i = 1, \ldots, k\}$ and $\{c_{ij} | i = 1, \ldots, k;$ $j = 1, \ldots, k_i\}$ and use all these sections to define a morphism (over A) $\varphi: X \to \mathbf{P}_A^N$ as in (7.1) above. Since X is covered by the X_i, the sections s_i^n already generate the sheaf $\mathscr{L}^n$, so this is indeed a morphism.

Let $\{x_i | i = 1, \ldots, k\}$ and $\{x_{ij} | i = 1, \ldots, k; j = 1, \ldots, k_i\}$ be the homogeneous coordinates of $\mathbf{P}_A^N$ corresponding to the sections of $\mathscr{L}^n$ mentioned above. For each $i = 1, \ldots, k$, let $U_i \subseteq \mathbf{P}_A^N$ be the open subset $x_i \neq 0$. Then $\varphi^{-1}(U_i) = X_i$, and the corresponding map of affine rings

$$A[\{y_i\}; \{y_{ij}\}] \to B_i$$

is surjective, because $y_{ij} \mapsto c_{ij}/s_i^n = b_{ij}$, and we chose the b_{ij} so as to generate B_i as an A-algebra. Thus X_i is mapped onto a closed subscheme of U_i. It follows that φ gives an isomorphism of X with a closed subscheme of $\bigcup_{i=1}^k U_i \subseteq \mathbf{P}_A^N$, so φ is an immersion. Hence $\mathscr{L}^n$ is very ample relative to Spec A, as required.

Example 7.6.1. Let $X = \mathbf{P}_k^n$, where k is a field. Then $\mathcal{O}(1)$ is very ample by definition. For any $d > 0$, $\mathcal{O}(d)$ corresponds to the d-uple embedding (Ex. 5.13), so $\mathcal{O}(d)$ is also very ample. Hence $\mathcal{O}(d)$ is ample for all $d > 0$. On the other hand, since the sheaf $\mathcal{O}(l)$ has no global sections for $l < 0$, one sees easily that the sheaves $\mathcal{O}(l)$ for $l \leqslant 0$ cannot be ample. So on $\mathbf{P}_k^n$, we have $\mathcal{O}(l)$ is ample $\Leftrightarrow$ very ample $\Leftrightarrow l > 0$.

Example 7.6.2. Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbf{P}_k^3$ over a field k. We have seen (6.6.1) that Pic $Q \cong \mathbf{Z} \oplus \mathbf{Z}$, and so we speak of the *type* (a,b), $a,b \in \mathbf{Z}$, of an invertible sheaf. Now $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$. If $a,b > 0$, then we consider an a-uple embedding $\mathbf{P}^1 \to \mathbf{P}^{n_1}$ and a b-uple embedding $\mathbf{P}^1 \to \mathbf{P}^{n_2}$. Taking their product, and following with a Segre embedding, we obtain a closed immersion

$$Q = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \to \mathbf{P}^n,$$

which corresponds to an invertible sheaf of type (a,b) on Q. Thus for any $a,b > 0$, the corresponding invertible sheaf is very ample, and hence ample. On the other hand, if $\mathscr{L}$ is of type (a,b) with either $a < 0$ or $b < 0$, then by restricting to a fibre of the product $\mathbf{P}^1 \times \mathbf{P}^1$, one sees that $\mathscr{L}$ is not generated by global sections. Hence if $a \leqslant 0$ or $b \leqslant 0$, $\mathscr{L}$ cannot be ample. So on Q, an invertible sheaf $\mathscr{L}$ of type (a,b) is ample $\Leftrightarrow$ very ample $\Leftrightarrow a,b > 0$.

Example 7.6.3. Let X be the nonsingular cubic curve $y^2z = x^3 - xz^2$ in $\mathbf{P}_k^2$, which was studied in (6.10.2). Let $\mathscr{L}$ be the invertible sheaf $\mathscr{L}(P_0)$. Then $\mathscr{L}$ is ample, because $\mathscr{L}(3P_0) \cong \mathscr{O}_X(1)$ is very ample. On the other hand, $\mathscr{L}$ is not very ample, because $\mathscr{L}(P_0)$ is not generated by global sections. If it were, then P_0 would be linearly equivalent to some other point $Q \in X$, which is impossible, since X is not rational (6.10.1). This shows that an ample sheaf need not be very ample.

Example 7.6.4. We will see later (IV, 3.3) that if D is a divisor on a complete nonsingular curve X, then $\mathscr{L}(D)$ is ample if and only if $\deg D > 0$. This is a consequence of the Riemann–Roch theorem.

Linear Systems

We will see in a minute how global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertible sheaf and a set of its global sections is the same as giving a certain set of effective divisors, all linearly equivalent to each other. This leads to the notion of linear system, which is the historically older notion. For simplicity, we will employ this terminology only when dealing with nonsingular projective varieties over an algebraically closed field. Over more general schemes the geometrical intuition associated with the concept of linear system may lead one astray, so it is safer to deal with invertible sheaves and their global sections in that case.

So let X be a nonsingular projective variety over an algebraically closed field k. In this case the notions of Weil divisor and Cartier divisor are equivalent (6.11). Furthermore, we have a one-to-one correspondence between linear equivalence classes of divisors and isomorphism classes of invertible sheaves (6.15). Another useful fact in this situation is that for any invertible sheaf $\mathscr{L}$ on X, the global sections $\Gamma(X,\mathscr{L})$ form a finite-dimensional k-vector space (5.19).

Let $\mathscr{L}$ be an invertible sheaf on X, and let $s \in \Gamma(X,\mathscr{L})$ be a nonzero section of $\mathscr{L}$. We define an effective divisor $D = (s)_0$, the *divisor of zeros* of s, as follows. Over any open set $U \subseteq X$ where $\mathscr{L}$ is trivial, let $\varphi: \mathscr{L}|_U \xrightarrow{\sim} \mathscr{O}_U$ be an isomorphism. Then $\varphi(s) \in \Gamma(U,\mathscr{O}_U)$. As U ranges over a covering of X, the collection $\{U,\varphi(s)\}$ determines an effective Cartier divisor D on X. Indeed, φ is determined up to multiplication by an element of $\Gamma(U,\mathscr{O}_U^*)$, so we get a well-defined Cartier divisor.

Proposition 7.7. *Let X be a nonsingular projective variety over the algebraically closed field k. Let D_0 be a divisor on X and let $\mathscr{L} \cong \mathscr{L}(D_0)$ be the corresponding invertible sheaf. Then:*

(a) *for each nonzero $s \in \Gamma(X,\mathscr{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0;*

(b) *every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X,\mathscr{L})$; and*

(c) *two sections $s,s' \in \Gamma(X,\mathscr{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.*

PROOF.

(a) We may identify $\mathscr{L}$ with the subsheaf $\mathscr{L}(D_0)$ of $\mathscr{K}$. Then s corresponds to a rational function $f \in K$. If D_0 is locally defined as a Cartier divisor by $\{U_i,f_i\}$ with $f_i \in K^*$, then $\mathscr{L}(D_0)$ is locally generated by f_i^{-1}, so we get a local isomorphism $\varphi: \mathscr{L}(D_0) \to \mathscr{O}$ by multiplying by f_i. So $D = (s)_0$ is locally defined by $f_i f$. Thus $D = D_0 + (f)$, showing that $D \sim D_0$.

(b) If $D > 0$ and $D = D_0 + (f)$, then $(f) \geqslant -D_0$. Thus f gives a global section of $\mathscr{L}(D_0)$ whose divisor of zeros is D.

(c) Still using the same construction, if $(s)_0 = (s')_0$, then s and s' correspond to rational functions $f,f' \in K$ such that $(f/f') = 0$. Therefore $f/f' \in \Gamma(X,\mathscr{O}_X^*)$. But since X is a projective variety over k algebraically closed, $\Gamma(X,\mathscr{O}_X) = k$, and so $f/f' \in k^*$ (I, 3.4).

Definition. A *complete linear system* on a nonsingular projective variety is defined as the set (maybe empty) of all effective divisors linearly equivalent to some given divisor D_0. It is denoted by $|D_0|$.

We see from the proposition that the set $|D_0|$ is in one-to-one correspondence with the set $(\Gamma(X,\mathscr{L}) - \{0\})/k^*$. This gives $|D_0|$ a structure of the set of closed points of a projective space over k.

Definition. A *linear system* $\mathfrak{d}$ on X is a subset of a complete linear system $|D_0|$ which is a linear subspace for the projective space structure of $|D_0|$. Thus $\mathfrak{d}$ corresponds to a sub-vector space $V \subseteq \Gamma(X,\mathscr{L})$, where $V = \{s \in \Gamma(X,\mathscr{L}) | (s)_0 \in \mathfrak{d}\} \cup \{0\}$. The *dimension* of the linear system $\mathfrak{d}$ is its dimension as a linear projective variety. Hence $\dim \mathfrak{d} = \dim V - 1$. (Note these dimensions are finite because $\Gamma(X,\mathscr{L})$ is a finite-dimensional vector space.)

Definition. A point $P \in X$ is a *base point* of a linear system $\mathfrak{d}$ if $P \in \operatorname{Supp} D$ for all $D \in \mathfrak{d}$. Here Supp D means the union of the prime divisors of D.

Lemma 7.8. *Let $\mathfrak{d}$ be a linear system on X corresponding to the subspace $V \subseteq \Gamma(X,\mathscr{L})$. Then a point $P \in X$ is a base point of $\mathfrak{d}$ if and only if $s_P \in \mathfrak{m}_P\mathscr{L}_P$ for all $s \in V$. In particular, $\mathfrak{d}$ is base-point-free if and only if $\mathscr{L}$ is generated by the global sections in V.*

PROOF. This follows immediately from the fact that for any $s \in \Gamma(X,\mathscr{L})$, the support of the divisor of zeros $(s)_0$ is the complement of the open set X_s.

Remark 7.8.1. We can rephrase (7.1) in terms of linear systems as follows: to give a morphism from X to $\mathbf{P}_k^n$ it is equivalent to give a linear system $\mathfrak{d}$ without base points on X, and a set of elements $s_0, \ldots, s_n \in V$, which span the vector space V. Often we will simply talk about the morphism to projective space determined by a linear system without base points $\mathfrak{d}$. In this case we understand that $s_0, \ldots, s_n$ should be chosen as a basis of V. If we chose a different basis, the corresponding morphism of $X \to \mathbf{P}^n$ would only differ by an automorphism of $\mathbf{P}^n$.

Remark 7.8.2. We can rephrase (7.3) in terms of linear systems as follows: Let $\varphi: X \to \mathbf{P}^n$ be a morphism corresponding to the linear system (without base points) $\mathfrak{d}$. Then φ is a closed immersion if and only if

(1) $\mathfrak{d}$ *separates points*, i.e., for any two distinct closed points $P,Q \in X$, there is a $D \in \mathfrak{d}$ such that $P \in \operatorname{Supp} D$ and $Q \notin \operatorname{Supp} D$, and
(2) $\mathfrak{d}$ *separates tangent vectors*, i.e., given a closed point $P \in X$ and a tangent vector $t \in T_P(X) = (\mathfrak{m}_P/\mathfrak{m}_P^2)'$, there is a $D \in \mathfrak{d}$ such that $P \in \operatorname{Supp} D$, but $t \notin T_P(D)$. Here we think of D as a locally principal closed subscheme, in which case the Zariski tangent space $T_P(D) = (\mathfrak{m}_{P,D}/\mathfrak{m}_{P,D}^2)'$ is naturally a subspace of $T_P(X)$.

The terminology of "separating points" and "separating tangent vectors" is perhaps somewhat explained by this geometrical interpretation.

Definition. Let $i: Y \hookrightarrow X$ be a closed immersion of nonsingular projective varieties over k. If $\mathfrak{d}$ is a linear system on X, we define the *trace* of $\mathfrak{d}$ on Y, denoted $\mathfrak{d}|_Y$, as follows. The linear system $\mathfrak{d}$ corresponds to an invertible sheaf $\mathscr{L}$ on X, and a sub-vector space $V \subseteq \Gamma(X,\mathscr{L})$. We take the invertible sheaf $i^*\mathscr{L} = \mathscr{L} \otimes \mathcal{O}_Y$ on Y, and we let $W \subseteq \Gamma(Y,i^*\mathscr{L})$ be the image of V under the natural map $\Gamma(X,\mathscr{L}) \to \Gamma(Y,i^*\mathscr{L})$. Then $i^*\mathscr{L}$ and W define the linear system $\mathfrak{d}|_Y$.

One can also describe $\mathfrak{d}|_Y$ geometrically as follows: it consists of all divisors $D.Y$ (defined as in (6.6.2)), where $D \in \mathfrak{d}$ is a divisor whose support does not contain Y.

Note that even if $\mathfrak{d}$ is a complete linear system, $\mathfrak{d}|_Y$ may not be complete.

Example 7.8.3. If $X = \mathbf{P}^n$, then the set of all effective divisors of degree $d > 0$ is a complete linear system of dimension $\binom{n+d}{n} - 1$. Indeed, it corresponds to the invertible sheaf $\mathcal{O}(d)$, whose global sections consist exactly of the space of all homogeneous polynomials in $x_0, \ldots, x_n$ of degree d. This is a vector space of dimension $\binom{n+d}{n}$, so the dimension of the complete linear system is one less.

Example 7.8.4. We can rephrase (Ex. 5.14d) in terms of linear systems as follows: a nonsingular projective variety $X \hookrightarrow \mathbf{P}_k^n$ is projectively normal if and only if for every $d > 0$, the trace on X of the linear system of all divisors of degree d on $\mathbf{P}^n$, is a complete linear system. By slight abuse of language, we say that "the linear system on X, cut out by the hypersurfaces of degree d in $\mathbf{P}^n$, is complete."

Example 7.8.5. Recall that the *twisted cubic curve* in $\mathbf{P}^3$ was defined by the parametric equations $x_0 = t^3$, $x_1 = t^2u$, $x_2 = tu^2$, $x_3 = u^3$. In other words, it is just the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$ (I, Ex. 2.9, Ex. 2.12). We will now show that any nonsingular curve X in $\mathbf{P}^3$, of degree 3, which is not contained in any $\mathbf{P}^2$, and which is abstractly isomorphic to $\mathbf{P}^1$, can be obtained from the given twisted cubic curve by an automorphism of $\mathbf{P}^3$. So we will refer to any such curve as a twisted cubic curve.

Let X be such a curve. The embedding of X in $\mathbf{P}^3$ is determined by the linear system $\mathfrak{d}$ of hyperplane sections of X (7.1). This is a linear system on X of dimension 3, because the planes in $\mathbf{P}^3$ form a linear system of dimension 3, and by hypothesis X is not contained in any plane, so the map $\Gamma(\mathbf{P}^3,\mathcal{O}(1)) \to \Gamma(X,i^*\mathcal{O}(1))$ is injective. On the other hand, $\mathfrak{d}$ is a linear system of degree 3, since X is a curve of degree 3. By the *degree* of a linear system on a complete nonsingular curve, we mean the degree of any of its divisors, which is independent of the divisor chosen (6.10). Now thinking of X as $\mathbf{P}^1$, the linear system $\mathfrak{d}$ must correspond to a 4-dimensional subspace $V \subseteq \Gamma(\mathbf{P}^1,\mathcal{O}(3))$. But $\Gamma(\mathbf{P}^1,\mathcal{O}(3))$ itself has dimension 4, so $V = \Gamma(\mathbf{P}^1,\mathcal{O}(3))$ and $\mathfrak{d}$ is a complete linear system. Since the embedding is determined by the linear system and the choice of basis of V by (7.1), we conclude that X is the same as the 3-uple embedding of $\mathbf{P}^1$, except for the choice of basis of V. This shows that there is an automorphism of $\mathbf{P}^3$ sending the given twisted cubic curve to X. (See (IV, Ex. 3.4) for generalization.)

Example 7.8.6. We define a *nonsingular rational quartic curve* in $\mathbf{P}^3$ to be a nonsingular curve X in $\mathbf{P}^3$, of degree 4, not contained in any $\mathbf{P}^2$, and which is abstractly isomorphic to $\mathbf{P}^1$. In this case we will see that two such curves need not be obtainable one from the other by an automorphism of $\mathbf{P}^3$. To give a morphism of $\mathbf{P}^1$ to $\mathbf{P}^3$ whose image has degree 4 and is not contained in any $\mathbf{P}^2$, we need a 4-dimensional subspace $V \subseteq \Gamma(\mathbf{P}^1,\mathcal{O}(4))$. This latter vector space has dimension 5. So if we choose two different subspaces V,V', the corresponding curves in $\mathbf{P}^3$ may not be related by an automorphism of $\mathbf{P}^3$. To be sure the image is nonsingular, we use the criterion of (7.3). Thus for example, one sees easily that the subspaces $V = (t^4, t^3u, tu^3, u^4)$ and $V' = (t^4, t^3u + at^2u^2, tu^3, u^4)$ for $a \in k^*$ give nonsingular rational quartic curves in $\mathbf{P}^3$ which are not equivalent by an automorphism of $\mathbf{P}^3$.

Proj, $\mathbf{P}(\mathscr{E})$, *and Blowing Up*

Earlier we have defined the Proj of a graded ring. Now we introduce a relative version of this construction, which is the **Proj** of a sheaf of graded algebras $\mathscr{S}$ over a scheme X. This construction is useful in particular because it allows us to construct the projective space bundle associated to a locally free sheaf $\mathscr{E}$, and it allows us to give a definition of blowing up with respect to an arbitrary sheaf of ideals. This generalizes the notion of blowing up a point introduced in (I, §4).

For simplicity, we will always impose the following conditions on a scheme X and a sheaf of graded algebras $\mathscr{S}$ before we define a **Proj**:

(†) X is a noetherian scheme, $\mathscr{S}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules, which has a structure of a sheaf of graded $\mathscr{O}_X$-algebras. Thus $\mathscr{S} \cong \bigoplus_{d \geqslant 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. We assume furthermore that $\mathscr{S}_0 = \mathscr{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathscr{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra. (It follows that $\mathscr{S}_d$ is coherent for all $d \geqslant 0$.)

Construction. Let X be a scheme and $\mathscr{S}$ a sheaf of graded $\mathscr{O}_X$-algebras satisfying (†). For each open affine subset $U = \operatorname{Spec} A$ of X, let $\mathscr{S}(U)$ be the graded A-algebra $\Gamma(U, \mathscr{S}|_U)$. Then we consider $\operatorname{Proj} \mathscr{S}(U)$ and its natural morphism $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$. If $f \in A$, and $U_f = \operatorname{Spec} A_f$, then since $\mathscr{S}$ is quasi-coherent, we see that $\operatorname{Proj} \mathscr{S}(U_f) \cong \pi_U^{-1}(U_f)$. It follows that if U, V are two open affine subsets of X, then $\pi_U^{-1}(U \cap V)$ is naturally isomorphic to $\pi_V^{-1}(U \cap V)$—here we leave some technical details to the reader. These isomorphisms allow us to glue the schemes $\operatorname{Proj} \mathscr{S}(U)$ together (Ex. 2.12). Thus we obtain a scheme $\mathbf{Proj}\, \mathscr{S}$ together with a morphism $\pi : \mathbf{Proj}\, \mathscr{S} \to X$ such that for each open affine $U \subseteq X$, $\pi^{-1}(U) \cong \operatorname{Proj} \mathscr{S}(U)$. Furthermore the invertible sheaves $\mathscr{O}(1)$ on each $\operatorname{Proj} \mathscr{S}(U)$ are compatible under this construction (5.12c), so they glue together to give an invertible sheaf $\mathscr{O}(1)$ on $\mathbf{Proj}\, \mathscr{S}$, canonically determined by this construction.

Thus to any X, $\mathscr{S}$ satisfying (†), we have constructed the scheme $\mathbf{Proj}\, \mathscr{S}$, the morphism $\pi : \mathbf{Proj}\, \mathscr{S} \to X$, and the invertible sheaf $\mathscr{O}(1)$ on $\mathbf{Proj}\, \mathscr{S}$. Everything we have said about the Proj of a graded ring S can be extended to this relative situation. We will not attempt to do this exhaustively, but will only mention certain aspects of the new situation.

Example 7.8.7. If $\mathscr{S}$ is the polynomial algebra $\mathscr{S} = \mathscr{O}_X[T_0, \ldots, T_n]$, then **Proj** $\mathscr{S}$ is just the relative projective space $\mathbf{P}_X^n$ with its twisting sheaf $\mathscr{O}(1)$ defined earlier (§5).

Caution 7.8.8. In general, $\mathscr{O}(1)$ may not be very ample on **Proj** $\mathscr{S}$ relative to X. See (7.10) and (Ex. 7.14).

Lemma 7.9. *Let $\mathscr{S}$ be a sheaf of graded algebras on a scheme X satisfying (†). Let $\mathscr{L}$ be an invertible sheaf on X, and define a new sheaf of graded algebras $\mathscr{S}' = \mathscr{S} * \mathscr{L}$ by $\mathscr{S}'_d = \mathscr{S}_d \otimes \mathscr{L}^d$ for each $d \geqslant 0$. Then $\mathscr{S}'$ also satisfies (†), and there is a natural isomorphism $\varphi : P' =$ **Proj** $\mathscr{S}' \xrightarrow{\sim} P =$ **Proj** $\mathscr{S}$, commuting with the projections π and π' to X, and having the property that*

$$\mathscr{O}_{P'}(1) \cong \varphi^* \mathscr{O}_P(1) \otimes \pi'^* \mathscr{L}.$$

PROOF. Let $\theta : \mathscr{O}_U \xrightarrow{\sim} \mathscr{L}|_U$ be a local isomorphism of $\mathscr{O}_U$ with $\mathscr{L}|_U$ over a small open affine subset U of X. Then θ induces an isomorphism of graded rings $\mathscr{S}(U) \cong \mathscr{S}'(U)$ and hence an isomorphism $\theta^* : \operatorname{Proj} \mathscr{S}'(U) \cong \operatorname{Proj} \mathscr{S}(U)$. If $\theta_1 : \mathscr{O}_U \cong \mathscr{L}|_U$ is a different local isomorphism, then θ and θ_1 differ by an element $f \in \Gamma(U, \mathscr{O}_U^*)$, and the corresponding isomorphism $\mathscr{S}(U) \cong \mathscr{S}'(U)$ differs by an automorphism ψ of $\mathscr{S}(U)$ which consists of multiplying by f^d in degree d. This does not affect the set of homogeneous prime ideals in $\mathscr{S}(U)$, and furthermore, since the structure sheaf of $\operatorname{Proj} \mathscr{S}(U)$ is formed by elements of degree *zero* in various localizations of $\mathscr{S}(U)$, the automorphism ψ of $\mathscr{S}(U)$ induces the *identity* automorphism of $\operatorname{Proj} \mathscr{S}(U)$. In other words, the isomorphism θ^* is independent of the choice of θ. So these local isomorphisms θ^* glue together to give a natural isomorphism $\varphi :$ **Proj** $\mathscr{S}' \xrightarrow{\sim}$ **Proj** $\mathscr{S}$, commuting with π and π'. When we form the sheaf $\mathscr{O}(1)$, however, the automorphism ψ of $\mathscr{S}(U)$ induces multiplication by f in $\mathscr{O}(1)$. Thus $\mathscr{O}_{P'}(1)$ looks like $\mathscr{O}_P(1)$ modified by the transition functions of $\mathscr{L}$. Stated precisely, this says $\mathscr{O}_{P'}(1) \cong \varphi^* \mathscr{O}_P(1) \otimes \pi'^* \mathscr{L}$.

Proposition 7.10. *Let $X, \mathscr{S}$ satisfy (†), let $P =$ **Proj** $\mathscr{S}$, with projection $\pi : P \to X$ and invertible sheaf $\mathscr{O}_P(1)$ constructed above. Then:*

(a) *π is a proper morphism. In particular, it is separated and of finite type;*

(b) *if X admits an ample invertible sheaf $\mathscr{L}$, then π is a projective morphism, and we can take $\mathscr{O}_P(1) \otimes \pi^* \mathscr{L}^n$ to be a very ample invertible sheaf on P over X, for suitable $n > 0$.*

PROOF.

(a) For each open affine $U \subseteq X$, the morphism $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$ is a projective morphism (4.8.1), hence proper (4.9). But the condition for a morphism to be proper is local on the base (4.8f), so π is proper.

(b) Let $\mathscr{L}$ be an ample invertible sheaf on X. Then for some $n > 0$, $\mathscr{S}_1 \otimes \mathscr{L}^n$ is generated by global sections. Since X is noetherian and $\mathscr{S}_1 \otimes \mathscr{L}^n$ is coherent, we can find a finite number of global sections which generate it, in other words we can find a surjective morphism of sheaves $\mathscr{O}_X^{N+1} \to \mathscr{S}_1 \otimes \mathscr{L}^n$ for some N. This allows us to define a surjective map of sheaves of graded $\mathscr{O}_X$-algebras $\mathscr{O}_X[T_0,\ldots,T_N] \to \mathscr{S} * \mathscr{L}^n$, which gives rise to a closed immersion $\mathbf{Proj}\ \mathscr{S} * \mathscr{L}^n \hookrightarrow \mathbf{Proj}\ \mathscr{O}_X[T_0,\ldots,T_N] = \mathbf{P}_X^N$ (Ex. 3.12). But $\mathbf{Proj}\ \mathscr{S} * \mathscr{L}^n \cong \mathbf{Proj}\ \mathscr{S}$ by (7.9), and the very ample invertible sheaf induced by this embedding is just $\mathscr{O}_P(1) \otimes \pi^* \mathscr{L}^n$.

Definition. Let X be a noetherian scheme, and let $\mathscr{E}$ be a locally free coherent sheaf on X. We define the associated *projective space bundle* $\mathbf{P}(\mathscr{E})$ as follows. Let $\mathscr{S} = S(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}$, $\mathscr{S} = \bigoplus_{d \geq 0} S^d(\mathscr{E})$ (Ex. 5.16). Then $\mathscr{S}$ is a sheaf of graded $\mathscr{O}_X$-algebras satisfying (†), and we define $\mathbf{P}(\mathscr{E}) = \mathbf{Proj}\ \mathscr{S}$. As such, it comes with a projection morphism $\pi: \mathbf{P}(\mathscr{E}) \to X$, and an invertible sheaf $\mathscr{O}(1)$.

Note that if $\mathscr{E}$ is free of rank $n + 1$ over an open set U, then $\pi^{-1}(U) \cong \mathbf{P}_U^n$, so $\mathbf{P}(\mathscr{E})$ is a "relative projective space" over X.

Proposition 7.11. *Let $X, \mathscr{E}, \mathbf{P}(\mathscr{E})$ be as in the definition. Then:*

(a) *if rank $\mathscr{E} \geq 2$, there is a canonical isomorphism of graded $\mathscr{O}_X$-algebras $\mathscr{S} \cong \bigoplus_{l \in \mathbf{Z}} \pi_*(\mathscr{O}(l))$, with the grading on the right hand side given by l. In particular, for $l < 0$, $\pi_*(\mathscr{O}(l)) = 0$; for $l = 0$, $\pi_*(\mathscr{O}_{\mathbf{P}(\mathscr{E})}) = \mathscr{O}_X$, and for $l = 1$, $\pi_*(\mathscr{O}(1)) = \mathscr{E}$;*

(b) *there is a natural surjective morphism $\pi^*\mathscr{E} \to \mathscr{O}(1)$.*

PROOF.

(a) is just a relative version of (5.13), and follows immediately from it.

(b) is a relative version of the fact that $\mathscr{O}(1)$ on $\mathbf{P}^n$ is generated by the global sections $x_0, \ldots, x_n$ (5.16.2).

Proposition 7.12. *Let $X, \mathscr{E}, \mathbf{P}(\mathscr{E})$ be as above. Let $g: Y \to X$ be any morphism. Then to give a morphism of Y to $\mathbf{P}(\mathscr{E})$ over X, it is equivalent to give an invertible sheaf $\mathscr{L}$ on Y and a surjective map of sheaves on Y, $g^*\mathscr{E} \to \mathscr{L}$.*

PROOF. This is a local version of (7.1). First note that if $f: Y \to \mathbf{P}(\mathscr{E})$ is a morphism over X, then the surjective map $\pi^*\mathscr{E} \to \mathscr{O}(1)$ on $\mathbf{P}(\mathscr{E})$ pulls back to give a surjective map $g^*\mathscr{E} = f^*\pi^*\mathscr{E} \to f^*\mathscr{O}(1)$, so we take $\mathscr{L} = f^*\mathscr{O}(1)$.

Conversely, given an invertible sheaf $\mathscr{L}$ on Y, and a surjective morphism $g^*\mathscr{E} \to \mathscr{L}$, I claim there is a unique morphism $f: Y \to \mathbf{P}(\mathscr{E})$ over X, such that $\mathscr{L} \cong f^*\mathscr{O}(1)$, and the map $g^*\mathscr{E} \to \mathscr{L}$ is obtained from $\pi^*\mathscr{E} \to \mathscr{O}(1)$ by applying f^*. In view of the claimed uniqueness of f, it is sufficient to verify this statement locally on X. Taking open affine subsets $U = \operatorname{Spec} A$ of X which are small enough so that $\mathscr{E}|_U$ is free, the statement reduces to (7.1). Indeed, if $\mathscr{E} \cong \mathscr{O}_X^{n+1}$, then to give a surjective morphism $g^*\mathscr{E} \to \mathscr{L}$ is the same as giving $n + 1$ global sections of $\mathscr{L}$ which generate.

Note. We refer to the exercises for further properties of $\mathbf{P}(\mathscr{E})$ and for the general notion of projective space bundle over a scheme X. Cf. (Ex. 5.18) for the notion of a vector bundle associated to a locally free sheaf.

Now we come to the generalized notion of blowing up. In (I, §4) we defined the blowing-up of a variety with respect to a point. Now we will define the blowing-up of a noetherian scheme with respect to any closed subscheme. Since a closed subscheme corresponds to a coherent sheaf of ideals (5.9), we may as well speak of blowing up a coherent sheaf of ideals.

Definition. Let X be a noetherian scheme, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. Consider the sheaf of graded algebras $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$, where $\mathscr{I}^d$ is the dth power of the ideal $\mathscr{I}$, and we set $\mathscr{I}^0 = \mathcal{O}_X$. Then $X,\mathscr{S}$ clearly satisfy (†), so we can consider $\tilde{X} = \mathbf{Proj}\, \mathscr{S}$. We define $\tilde{X}$ to be the *blowing-up* of X *with respect to the coherent sheaf of ideals* $\mathscr{I}$. If Y is the closed subscheme of X corresponding to $\mathscr{I}$, then we also call $\tilde{X}$ the *blowing-up* of X along Y, or *with center* Y.

Example 7.12.1. If X is $\mathbf{A}_k^n$ and $P \in X$ is the origin, then the blowing-up of P just defined is isomorphic to the one defined in (I, §4). Indeed, in this case $X = \operatorname{Spec} A$, where $A = k[x_1, \ldots ,x_n]$, and P corresponds to the ideal $I = (x_1, \ldots ,x_n)$. So $\tilde{X} = \operatorname{Proj} S$, where $S = \bigoplus_{d \geq 0} I^d$. We can define a surjective map of graded rings $\varphi : A[y_1, \ldots ,y_n] \rightarrow S$ by sending y_i to the element $x_i \in I$ considered as an element of S in degree 1. Thus $\tilde{X}$ is isomorphic to a closed subscheme of $\operatorname{Proj} A[y_1, \ldots ,y_n] = \mathbf{P}_A^{n-1}$. It is defined by the homogeneous polynomials in the y_i which generate the kernel of φ, and one sees easily that $\{x_i y_j - x_j y_i | i,j = 1, \ldots ,n\}$ will do.

Definition. Let $f : X \rightarrow Y$ be a morphism of schemes, and let $\mathscr{I} \subseteq \mathcal{O}_Y$ be a sheaf of ideals on Y. We define the *inverse image ideal sheaf* $\mathscr{I}' \subseteq \mathcal{O}_X$ as follows. First consider f as a continuous map of topological spaces $X \rightarrow Y$ and let $f^{-1}\mathscr{I}$ be the inverse image of the sheaf $\mathscr{I}$, as defined in §1. Then $f^{-1}\mathscr{I}$ is a sheaf of ideals in the sheaf of rings $f^{-1}\mathcal{O}_Y$ on the topological space X. Now there is a natural homomorphism of sheaves of rings on X, $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$, so we define $\mathscr{I}'$ to be the ideal sheaf in $\mathcal{O}_X$ generated by the image of $f^{-1}\mathscr{I}$. We will denote $\mathscr{I}'$ by $f^{-1}\mathscr{I} \cdot \mathcal{O}_X$ or simply $\mathscr{I} \cdot \mathcal{O}_X$, if no confusion seems likely to result.

Caution 7.12.2. If we consider $\mathscr{I}$ as a sheaf of $\mathcal{O}_Y$-modules, then in §5 we have defined the inverse image $f^*\mathscr{I}$ as a sheaf of $\mathcal{O}_X$-modules. It may happen that $f^*\mathscr{I} \neq f^{-1}\mathscr{I} \cdot \mathcal{O}_X$. The reason is that $f^*\mathscr{I}$ is defined as

$$f^{-1}\mathscr{I} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Since the tensor product functor is not in general left exact, $f^*\mathscr{I}$ may not be a subsheaf of $\mathcal{O}_X$. However, there is a natural map $f^*\mathscr{I} \rightarrow \mathcal{O}_X$ coming from the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_Y$, and $f^{-1}\mathscr{I} \cdot \mathscr{O}_X$ is just the image of $f^*\mathscr{I}$ under this map.

Proposition 7.13. *Let X be a noetherian scheme, $\mathscr{I}$ a coherent sheaf of ideals, and let $\pi: \tilde{X} \to X$ be the blowing-up of $\mathscr{I}$. Then:*

(a) *the inverse image ideal sheaf $\tilde{\mathscr{I}} = \pi^{-1}\mathscr{I} \cdot \mathscr{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$.*

(b) *if Y is the closed subscheme corresponding to $\mathscr{I}$, and if $U = X - Y$, then $\pi: \pi^{-1}(U) \to U$ is an isomorphism.*

PROOF.

(a) Since $\tilde{X}$ is defined as **Proj** $\mathscr{S}$, where $\mathscr{S} = \bigoplus_{d \geqslant 0} \mathscr{I}^d$, it comes equipped with a natural invertible sheaf $\mathscr{O}(1)$. For any open affine $U \subseteq X$, this sheaf $\mathscr{O}(1)$ on Proj $\mathscr{S}(U)$ is the sheaf associated to the graded $\mathscr{S}(U)$-module $\mathscr{S}(U)(1) = \bigoplus_{d \geqslant 0} \mathscr{I}^{d+1}(U)$. But this is clearly equal to the ideal $\mathscr{I} \cdot \mathscr{S}(U)$ generated by $\mathscr{I}$ in $\mathscr{S}(U)$, so we see that the inverse image ideal sheaf $\tilde{\mathscr{I}} = \pi^{-1}\mathscr{I} \cdot \mathscr{O}_{\tilde{X}}$ is in fact equal to $\mathscr{O}_{\tilde{X}}(1)$. Hence it is an invertible sheaf.

(b) If $U = X - Y$, then $\mathscr{I}|_U \cong \mathscr{O}_U$, so $\pi^{-1}U = \mathbf{Proj}\, \mathscr{O}_U[T] = U$.

Proposition 7.14 (Universal Property of Blowing Up). *Let X be a noetherian scheme, $\mathscr{I}$ a coherent sheaf of ideals, and $\pi: \tilde{X} \to X$ the blowing-up with respect to $\mathscr{I}$. If $f: Z \to X$ is any morphism such that $f^{-1}\mathscr{I} \cdot \mathscr{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g: Z \to \tilde{X}$ factoring f.*

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

PROOF. In view of the asserted uniqueness of g, the question is local on X. So we may assume that $X = \operatorname{Spec} A$ is affine, A is noetherian, and that $\mathscr{I}$ corresponds to an ideal $I \subseteq A$. Then $\tilde{X} = \operatorname{Proj} S$, where $S = \bigoplus_{d \geqslant 0} I^d$. Let $a_0, \ldots, a_n \in I$ be a set of generators for the ideal I. Then we can define a surjective map of graded rings $\varphi: A[x_0, \ldots, x_n] \to S$ by sending x_i to $a_i \in I$, considered as an element of degree one in S. This homomorphism gives rise to a closed immersion $\tilde{X} \hookrightarrow \mathbf{P}^n_A$. The kernel of φ is the homogeneous ideal in $A[x_0, \ldots, x_n]$ generated by all homogeneous polynomials $F(x_0, \ldots, x_n)$ such that $F(a_0, \ldots, a_n) = 0$ in A.

Now let $f: Z \to X$ be a morphism such that the inverse image ideal sheaf $f^{-1}\mathscr{I} \cdot \mathscr{O}_Z$ is an invertible sheaf $\mathscr{L}$ on Z. Since I is generated by $a_0, \ldots, a_n$, the inverse images of these elements, considered as global sections of $\mathscr{I}$, give global sections $s_0, \ldots, s_n$ of $\mathscr{L}$ which generate. Then by (7.1) there is a unique morphism $g: Z \to \mathbf{P}_A^n$ with the property that $\mathscr{L} \cong g^*\mathcal{O}(1)$ and that $s_i = g^{-1}x_i$ under this isomorphism. Now I claim that g factors through the closed subscheme $\tilde{X}$ of $\mathbf{P}_A^n$. This follows easily from the fact that if $F(x_0, \ldots, x_n)$ is a homogeneous element of degree d of $\ker \varphi$, where $\ker \varphi$ is the homogeneous ideal described above which determines $\tilde{X}$, then $F(a_0, \ldots, a_n) = 0$ in A and so $F(s_0, \ldots, s_n) = 0$ in $\Gamma(Z, \mathscr{L}^d)$.

Thus we have constructed a morphism $g: Z \to \tilde{X}$ factoring f. For any such morphism, we must necessarily have $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\tilde{X}}) \cdot \mathcal{O}_Z$ which is just $g^{-1}(\mathcal{O}_{\tilde{X}}(1)) \cdot \mathcal{O}_Z$. Therefore we have a surjective map $g^*\mathcal{O}_{\tilde{X}}(1) \to f^{-1}\mathscr{I} \cdot \mathcal{O}_Z = \mathscr{L}$. Now a surjective map of invertible sheaves on a locally ringed space is necessarily an isomorphism (Ex. 7.1), so we have $g^*\mathcal{O}_{\tilde{X}}(1) \cong \mathscr{L}$. Clearly the sections s_i of $\mathscr{L}$ must be the pull-backs of the sections x_i of $\mathcal{O}(1)$ on $\mathbf{P}_A^n$. Hence the uniqueness of g under our conditions follows from the uniqueness assertion of (7.1).

Corollary 7.15. *Let $f: Y \to X$ be a morphism of noetherian schemes, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. Let $\tilde{X}$ be the blowing-up of $\mathscr{I}$, and let $\tilde{Y}$ be the blowing-up of the inverse image ideal sheaf $\mathscr{J} = f^{-1}\mathscr{I} \cdot \mathcal{O}_Y$ on Y. Then there is a unique morphism $\tilde{f}: \tilde{Y} \to \tilde{X}$*

$$\begin{tikzcd} \tilde{Y} \arrow[r, dashed, "\tilde{f}"] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

making a commutative diagram as shown. Moreover, if f is a closed immersion, so is $\tilde{f}$.

Proof. The existence and uniqueness of $\tilde{f}$ follow immediately from the proposition. To show that $\tilde{f}$ is a closed immersion if f is, we go back to the definition of blowing up. $\tilde{X} = \mathbf{Proj}\, \mathscr{S}$ where $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$, and $\tilde{Y} = \mathbf{Proj}\, \mathscr{S}'$, where $\mathscr{S}' = \bigoplus_{d \geq 0} \mathscr{J}^d$. Since Y is a closed subscheme of X, we can consider $\mathscr{S}'$ as a sheaf of graded algebras on X. Then there is a natural surjective homomorphism of graded rings $\mathscr{S} \to \mathscr{S}'$, which gives rise to the closed immersion $\tilde{f}$.

Definition. In the situation of (7.15), if Y is a closed subscheme of X, we call the closed subscheme $\tilde{Y}$ of $\tilde{X}$ the *strict transform* of Y under the blowing-up $\pi: \tilde{X} \to X$.

Example 7.15.1. If Y is a closed subvariety of $X = \mathbf{A}_k^n$ passing through the origin P, then the strict transform $\tilde{Y}$ of Y in $\tilde{X}$ is a closed subvariety. Hence, provided Y is not just P itself, we can recover $\tilde{Y}$ as the closure of $\pi^{-1}(Y - P)$, where $\pi:\pi^{-1}(X - P) \to X - P$ is the isomorphism of (7.13b). This shows that our new definition of blowing up coincides with the one given in (I, §4) for any closed subvariety of $\mathbf{A}_k^n$. In particular, this shows that blowing up as defined earlier is intrinsic.

Now we will study blowing up in the special case that X is a variety. Recall (§4) that a variety is defined to be an integral separated scheme of finite type over an algebraically closed field k.

Proposition 7.16. *Let X be a variety over k, let $\mathscr{I} \subseteq \mathcal{O}_X$ be a nonzero coherent sheaf of ideals on X, and let $\pi:\tilde{X} \to X$ be the blowing-up with respect to $\mathscr{I}$. Then:*

(a) *$\tilde{X}$ is also a variety;*

(b) *π is a birational, proper, surjective morphism;*

(c) *if X is quasi-projective (respectively, projective) over k, then $\tilde{X}$ is also, and π is a projective morphism.*

Proof. First of all, since X is integral, the sheaf $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$ is a sheaf of integral domains on X, so $\tilde{X}$ is also integral. Next, we have already seen that π is proper (7.10). In particular, π is separated and of finite type, so it follows that $\tilde{X}$ is also separated and of finite type, i.e., $\tilde{X}$ is a variety. Now since $\mathscr{I} \neq 0$, the corresponding closed subscheme Y is not all of X, and so the open set $U = X - Y$ is nonempty. Since π induces an isomorphism from $\pi^{-1}U$ to U (7.13), we see that π is birational. Since π is proper, it is a closed map, so the image $\pi(\tilde{X})$ is a closed set containing U, which must be all of X since X is irreducible. Thus π is surjective. Finally, if X is quasi-projective (respectively, projective), then X admits an ample invertible sheaf (7.6), so by (7.10b) π is a projective morphism. It follows that $\tilde{X}$ is also quasi-projective (respectively, projective) (Ex. 4.9).

Theorem 7.17. *Let X be a quasi-projective variety over k. If Z is another variety and $f:Z \to X$ is any birational projective morphism, then there exists a coherent sheaf of ideals $\mathscr{I}$ on X such that Z is isomorphic to the blowing-up $\tilde{X}$ of X with respect to $\mathscr{I}$, and f corresponds to $\pi:\tilde{X} \to X$ under this isomorphism.*

Proof. The proof is somewhat difficult, so we divide it into steps.

Step 1. Since f is assumed to be a projective morphism, there exists a closed immersion $i:Z \to \mathbf{P}_X^n$ for some n.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbf{P}_X^n \arrow[d] \\ & X \end{tikzcd}$$

Let $\mathscr{L}$ be the invertible sheaf $i^*\mathscr{O}(1)$ on Z. Now we consider the sheaf of graded $\mathscr{O}_X$-algebras $\mathscr{S} = \mathscr{O}_X \oplus \bigoplus_{d \geqslant 1} f_*\mathscr{L}^d$. Each $f_*\mathscr{L}^d$ is a coherent sheaf on X, by (5.20), so $\mathscr{S}$ is quasi-coherent. However, $\mathscr{S}$ may not be generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra.

Step 2. For any integer $e > 0$, let $\mathscr{S}^{(e)} = \bigoplus_{d \geqslant 0} \mathscr{S}_d^{(e)}$, where $\mathscr{S}_d^{(e)} = \mathscr{S}_{de}$ (cf. Ex. 5.13). I claim that for e sufficiently large, $\mathscr{S}^{(e)}$ is generated as an $\mathscr{O}_X$-algebra by $\mathscr{S}_1^{(e)}$. Since X is quasi-compact, this question is local on X, so we may assume $X = \operatorname{Spec} A$ is affine, where A is a finitely generated k-algebra. Then Z is a closed subscheme of $\mathbf{P}_A^n$, and $\mathscr{S}$ corresponds to the graded A-algebra $S = A \oplus \bigoplus_{d \geqslant 1} \Gamma(Z, \mathscr{O}_Z(d))$. Let $T = A[x_0, \ldots, x_n]/I_Z$, where I_Z is a homogeneous ideal defining Z. Then, using the technique of (Ex. 5.9, Ex. 5.14), one can show that the A-algebras S,T agree in all large enough degrees (details left to reader). But T is generated as an A-algebra by T_1, so $T^{(e)}$ is generated by $T_1^{(e)}$, and this is the same as $S^{(e)}$ for e sufficiently large.

Step 3. Now let us replace our original embedding $i: Z \to \mathbf{P}_X^n$ by i followed by an e-uple embedding for e sufficiently large. This has the effect of replacing $\mathscr{L}$ by $\mathscr{L}^e$ and $\mathscr{S}$ by $\mathscr{S}^{(e)}$ (Ex. 5.13). Thus we may now assume that $\mathscr{S}$ is generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra. Note also by construction that $Z \cong$ **Proj** $\mathscr{S}$ (cf. (5.16)). So at least we have Z isomorphic to **Proj** of something. If $\mathscr{S}_1 = f_*\mathscr{L}$ were a sheaf of ideals in $\mathscr{O}_X$ we would be done. So in the next step, we try to make it into one.

Step 4. Now $\mathscr{L}$ is an invertible sheaf on the integral scheme Z, so we can find an embedding $\mathscr{L} \hookrightarrow \mathscr{K}_Z$ where $\mathscr{K}_Z$ is the constant sheaf of the function field of Z (proof of 6.15). Hence $f_*\mathscr{L} \subseteq f_*\mathscr{K}_Z$. But since f is assumed to be birational, we have $f_*\mathscr{K}_Z = \mathscr{K}_X$, and so $f_*\mathscr{L} \subseteq \mathscr{K}_X$. Now let $\mathscr{M}$ be an ample invertible sheaf on X, which exists because X is assumed to be quasi-projective. Then I claim that there is an $n > 0$ and an embedding $\mathscr{M}^{-n} \subseteq \mathscr{K}_X$ such that $\mathscr{M}^{-n} \cdot f_*\mathscr{L} \subseteq \mathscr{O}_X$. Indeed, let $\mathscr{J}$ be the *ideal sheaf of denominators* of $f_*\mathscr{L}$, defined locally as $\{a \in \mathscr{O}_X | a \cdot f_*\mathscr{L} \subseteq \mathscr{O}_X\}$. This is a nonzero coherent sheaf of ideals on X, because $f_*\mathscr{L}$ is a coherent subsheaf of $\mathscr{K}_X$, so locally one can just take common denominators for a set of generators of the corresponding finitely generated module. Since $\mathscr{M}$ is ample, $\mathscr{J} \otimes \mathscr{M}^n$ is generated by global sections for n sufficiently large. In particular, for suitable $n > 0$, there is a nonzero map $\mathscr{O}_X \to \mathscr{J} \otimes \mathscr{M}^n$, and hence a nonzero map $\mathscr{M}^{-n} \to \mathscr{J}$. Then by construction $\mathscr{M}^{-n} \cdot f_*\mathscr{L} \subseteq \mathscr{O}_X$.

Step 5. Since $\mathscr{M}^{-n} \cdot f_*\mathscr{L} \subseteq \mathscr{O}_X$, it is a coherent sheaf of ideals on X, which we call $\mathscr{I}$. This is the required ideal sheaf, as we will now show that Z is isomorphic to the blowing up of X with respect to $\mathscr{I}$. We already know that $Z \cong$ **Proj** $\mathscr{S}$. Therefore by (7.9) Z is also isomorphic to **Proj** $\mathscr{S} * \mathscr{M}^{-n}$. So to complete the proof, it will be sufficient to identify $(\mathscr{S} * \mathscr{M}^{-n})_d = \mathscr{M}^{-dn} \otimes f_*\mathscr{L}^d$ with $\mathscr{I}^d$ for any $d \geqslant 1$. First note that $f_*\mathscr{L}^d \subseteq \mathscr{K}_X$ for any d (same reason as above for $d = 1$), and since $\mathscr{M}$ is invertible, we can write $\mathscr{M}^{-dn} \cdot f_*\mathscr{L}^d$ instead of $\otimes$. Now since $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra, we have a natural surjective map $\mathscr{I}^d \to \mathscr{M}^{-dn} \cdot f_*\mathscr{L}^d$ for each

$d \geqslant 1$. It must also be injective, since both are subsheaves of $\mathscr{K}_X$, so it is an isomorphism. This shows finally that $Z \cong \mathbf{Proj} \bigoplus_{d \geqslant 0} \mathscr{I}^d$, which completes the proof.

Remark 7.17.1. Of course the sheaf of ideals $\mathscr{I}$ in the theorem is not unique. This is clear from the construction, but see also (Ex. 7.11).

Remark 7.17.2. We see from this theorem that blowing up arbitrary coherent sheaves of ideals is a very general process. Accordingly in most applications one learns more by blowing up only along some restricted class of subvarieties. For example, in his paper on resolution of singularities [4], Hironaka uses only blowing up along a nonsingular subvariety which is "normally flat" in its ambient space. In studying birational geometry of surfaces in Chapter V, we will use only blowing up at a point. In fact one of our main results there will be that any birational transformation of nonsingular projective surfaces can be factored into a finite number of blowings up (and blowings down) of points. One important application of the more general blowing-up we have been studying here is Nagata's theorem [6] that any (abstract) variety can be embedded as an open subset of a complete variety.

Example 7.17.3. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the points of indeterminacy of a rational map determined by an invertible sheaf. So let A be a ring, let X be a noetherian scheme over A, let $\mathscr{L}$ be an invertible sheaf on X, and let $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ be a set of global sections of $\mathscr{L}$. Let U be the open subset of X where the s_i generate the sheaf $\mathscr{L}$. Then according to (7.1) the invertible sheaf $\mathscr{L}|_U$ on U and the global sections $s_0, \ldots, s_n$ determine an A-morphism $\varphi: U \to \mathbf{P}^n_A$. We will now show how to blow up a certain sheaf of ideals $\mathscr{I}$ on X, whose corresponding closed subsheme Y has support equal to $X - U$ (i.e., the underlying topological space of Y is $X - U$), so that the morphism φ extends to a morphism $\tilde{\varphi}$ of $\tilde{X}$ to $\mathbf{P}^n_A$.

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\pi"] \arrow[drr, dashed, "\tilde{\varphi}"] & & \\
X & U \arrow[l, hook'] \arrow[r, "\varphi"] & \mathbf{P}^n_A
\end{tikzcd}$$

So let $\mathscr{F}$ be the coherent subsheaf of $\mathscr{L}$ generated by $s_0, \ldots, s_n$. We define a coherent sheaf of ideals $\mathscr{I}$ on X as follows: for any open set $V \subseteq X$, such that $\mathscr{L}|_V$ is free, let $\psi: \mathscr{L}|_V \xrightarrow{\sim} \mathscr{O}_V$ be an isomorphism, and take $\mathscr{I}|_V = \psi(\mathscr{F}|_V)$. Clearly the ideal sheaf $\mathscr{I}|_V$ is independent of the choice of ψ, so we get a well-defined coherent sheaf of ideals $\mathscr{I}$ on X. Note also that $\mathscr{I}_x = \mathscr{O}_x$ if and only if $x \in U$, so the corresponding closed subscheme Y has support equal to $X - U$. Let $\pi:\tilde{X} \to X$ be the blowing-up of $\mathscr{I}$. Then by (7.13a), $\pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf of ideals, so we see that the global sections π^*s_i of $\pi^*\mathscr{L}$ generate an *invertible* coherent subsheaf $\mathscr{L}'$ of $\pi^*\mathscr{L}$. Now $\mathscr{L}'$ and the sections π^*s_i define a morphism $\tilde{\varphi}:\tilde{X} \to \mathbf{P}^n_A$ whose restriction to $\pi^{-1}(U)$ corresponds to φ under the natural isomorphism $\pi:\pi^{-1}(U) \xrightarrow{\sim} U$ (7.13b).

In case X is a nonsingular projective variety over a field, we can rephrase this example in terms of linear systems. The given $\mathscr{L}$ and sections s_i determine a linear system $\mathfrak{d}$ on X. The base points of $\mathfrak{d}$ are just the points of the closed set $X - U$, and $\varphi:U \to \mathbf{P}^n_k$ is the morphism determined by the base-point-free linear system $\mathfrak{d}|_U$ on U. We call Y the *scheme* of base points of $\mathfrak{d}$. So our example shows that if we blow up Y, then $\mathfrak{d}$ extends to a base-point-free linear system $\tilde{\mathfrak{d}}$ on all of $\tilde{X}$.

Exercises

7.1. Let $(X,\mathcal{O}_X)$ be a locally ringed space, and let $f:\mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. [*Hint*: Reduce to a question of modules over a local ring by looking at the stalks.]

7.2. Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0,\ldots,s_n\}$ and $\{t_0,\ldots,t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X,\mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leqslant m$. Show that the corresponding morphisms $\varphi:X \to \mathbf{P}^n_k$ and $\psi:X \to \mathbf{P}^m_k$ differ by a suitable linear projection $\mathbf{P}^m - L \to \mathbf{P}^n$ and an automorphism of $\mathbf{P}^n$, where L is a linear subspace of $\mathbf{P}^m$ of dimension $m - n - 1$.

7.3. Let $\varphi:\mathbf{P}^n_k \to \mathbf{P}^m_k$ be a morphism. Then:
(a) either $\varphi(\mathbf{P}^n) = pt$ or $m \geqslant n$ and $\dim \varphi(\mathbf{P}^n) = n$;
(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbf{P}^n \to \mathbf{P}^N$ for a uniquely determined $d \geqslant 1$, (2) a linear projection $\mathbf{P}^N - \mathrm{L} \to \mathbf{P}^m$, and (3) an automorphism of $\mathbf{P}^m$. Also, φ has finite fibres.

7.4. (a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.
(b) Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate Pic X, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

7.5. Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L},\mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.
(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is ample.
(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.
(c) If $\mathscr{L},\mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.
(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.
(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geqslant n_0$.

7.6. *The Riemann–Roch Problem.* Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann–Roch problem is to determine $\dim|nD|$ as a function of n, and, in particular, its behavior for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim|nD| = \dim \Gamma(X,\mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X,\mathscr{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \hookrightarrow \mathbf{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim|nD| = P_X(n) - 1$, where P_X is the *Hilbert polynomial* of X (I, §7). Thus in this case $\dim|nD|$ is a polynomial function of n, for n large.

(b) If D corresponds to a torsion element of Pic X, of order r, then $\dim|nD| = 0$ if $r|n$, -1 otherwise. In this case the function is periodic of period r.

It follows from the general Riemann–Roch theorem that $\dim|nD|$ is a polynomial function for n large, whenever D is an *ample* divisor. See (IV, 1.3.2), (V, 1.6), and Appendix A. In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials $P_1,\ldots,P_r$, such that for all n sufficiently large, $\dim|nD| = P_{i(n)}(n)$, where $i(n) \in \{1,2,\ldots,r\}$ is a function of n.

7.7. *Some Rational Surfaces.* Let $X = \mathbf{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathscr{O}(2)$, whose space of global sections has a basis x^2,y^2,z^2,xy,xz,yz, where x,y,z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$, whose image is the Veronese surface (I, Ex. 2.13).

(b) Show that the subsystem defined by $x^2,y^2,z^2,y(x - z),(x - y)z$ gives a closed immersion of X into $\mathbf{P}^4$. The image is called the *Veronese surface* in $\mathbf{P}^4$. Cf. (IV, Ex. 3.11).

(c) Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbf{P}^4$. Furthermore, if we blow up P, to get a surface $\tilde{X}$, then this map extends to give a closed immersion of $\tilde{X}$ in $\mathbf{P}^4$. Show that $\tilde{X}$ is a surface of degree 3 in $\mathbf{P}^4$, and that the lines in X through P are transformed into straight lines in $\tilde{X}$ which do not meet. $\tilde{X}$ is the union of all these lines, so we say $\tilde{X}$ is a *ruled surface* (V, 2.19.1).

7.8. Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi:\mathbf{P}(\mathscr{E}) \rightarrow X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between *sections* of π (i.e., morphisms $\sigma:X \rightarrow \mathbf{P}(\mathscr{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathscr{E} \rightarrow \mathscr{L} \rightarrow 0$ of $\mathscr{E}$.

7.9. Let X be a regular noetherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank $\geqslant 2$ on X.

(a) Show that Pic $\mathbf{P}(\mathscr{E}) \cong$ Pic $X \times \mathbf{Z}$.

(b) If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbf{P}(\mathscr{E}) \cong \mathbf{P}(\mathscr{E}')$ (over X) if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.

7.10. *$\mathbf{P}^n$-Bundles Over a Scheme.* Let X be a noetherian scheme.

(a) By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a *projective n-space bundle* over X, as a scheme P with a morphism $\pi:P \rightarrow X$ such that P is locally isomorphic to $U \times \mathbf{P}^n$, $U \subseteq X$ open, and the transition automorphisms on Spec $A \times \mathbf{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0,\ldots,x_n]$ (e.g., $x_i' = \sum a_{ij}x_j$, $a_{ij} \in A$).

(b) If $\mathscr{E}$ is a locally free sheaf of rank $n + 1$ on X, then $\mathbf{P}(\mathscr{E})$ is a $\mathbf{P}^n$-bundle over X.

*(c) Assume that X is regular, and show that every $\mathbf{P}^n$-bundle P over X is isomorphic to $\mathbf{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X. [*Hint*: Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) \cong U \times \mathbf{P}^n$, and let $\mathscr{L}_0$ be the invertible sheaf $\mathscr{O}(1)$ on $U \times \mathbf{P}^n$. Show that $\mathscr{L}_0$ extends to an invertible sheaf $\mathscr{L}$ on P. Then show that $\pi_*\mathscr{L} = \mathscr{E}$ is a locally free sheaf on X and that $P \cong \mathbf{P}(\mathscr{E})$.] Can you weaken the hypothesis "X regular"?

(d) Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbf{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n + 1$ under the equivalence relation $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X.

7.11. On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

(a) If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $d \geqslant 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$ (cf. Ex. 5.13).

(b) If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I}$ and $\mathscr{I} \cdot \mathscr{J}$ give isomorphic blowings-up.

(c) If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1}U \rightarrow U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.

7.12. Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\tilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathscr{I}_Y + \mathscr{I}_Z$). Show that the strict transforms $\tilde{Y}$ and $\tilde{Z}$ of Y and Z in $\tilde{X}$ do not meet.

***7.13.** *A Complete Nonprojective Variety*. Let k be an algebraically closed field of char $\neq 2$. Let $C \subseteq \mathbf{P}_k^2$ be the nodal cubic curve $y^2z = x^3 + x^2z$. If $P_0 = (0,0,1)$ is the singular point, then $C - P_0$ is isomorphic to the multiplicative group $\mathbf{G}_m = \operatorname{Spec} k[t,t^{-1}]$ (Ex. 6.7). For each $a \in k$, $a \neq 0$, consider the translation of $\mathbf{G}_m$ given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a.

Now consider $C \times (\mathbf{P}^1 - \{0\})$ and $C \times (\mathbf{P}^1 - \{\infty\})$. We glue their open subsets $C \times (\mathbf{P}^1 - \{0, \infty\})$ by the isomorphism $\varphi\colon \langle P,u \rangle \mapsto \langle \varphi_u(P),u \rangle$ for $P \in C$, $u \in \mathbf{G}_m = \mathbf{P}^1 - \{0,\infty\}$. Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi\colon X \rightarrow \mathbf{P}^1$.

(a) Show that π is a proper morphism, and hence that X is a complete variety over k.

(b) Use the method of (Ex. 6.9) to show that $\operatorname{Pic}(C \times \mathbf{A}^1) \cong \mathbf{G}_m \times \mathbf{Z}$ and $\operatorname{Pic}(C \times (\mathbf{A}^1 - \{0\})) \cong \mathbf{G}_m \times \mathbf{Z} \times \mathbf{Z}$. [*Hint*: If A is a domain and if $*$ denotes the group of units, then $(A[u])^* \cong A^*$ and $(A[u,u^{-1}])^* \cong A^* \times \mathbf{Z}$.]

(c) Now show that the restriction map $\operatorname{Pic}(C \times \mathbf{A}^1) \rightarrow \operatorname{Pic}(C \times (\mathbf{A}^1 - \{0\}))$ is of the form $\langle t,n \rangle \mapsto \langle t,0,n \rangle$, and that the automorphism φ of $C \times (\mathbf{A}^1 - \{0\})$ induces a map of the form $\langle t,d,n \rangle \mapsto \langle t,d + n,n \rangle$ on its Picard group.

(d) Conclude that the image of the restriction map $\operatorname{Pic} X \rightarrow \operatorname{Pic}(C \times \{0\})$ consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

7.14. (a) Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathscr{E}$, such that the invertible sheaf $\mathscr{O}(1)$ on $\mathbf{P}(\mathscr{E})$ is *not* very ample relative to X.

(b) Let $f: X \to Y$ be a morphism of finite type, let $\mathscr{L}$ be an ample invertible sheaf on X, and let $\mathscr{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let $P = \mathbf{Proj}\, \mathscr{S}$, let $\pi: P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n \gg 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^n$ is very ample on P relative to Y. [*Hint*: Use (7.10) and (Ex. 5.12).]

8 Differentials

In this section we will define the sheaf of relative differential forms of one scheme over another. In the case of a nonsingular variety over $\mathbf{C}$, which is like a complex manifold, the sheaf of differential forms is essentially the same as the dual of the tangent bundle defined in differential geometry. However, in abstract algebraic geometry, we will define the sheaf of differentials first, by a purely algebraic method, and then define the tangent bundle as its dual. Hence we will begin this section with a review of the module of differentials of one ring over another. As applications of the sheaf of differentials, we will give a characterization of nonsingular varieties among schemes of finite type over a field. We will also use the sheaf of differentials on a nonsingular variety to define its tangent sheaf, its canonical sheaf, and its geometric genus. This latter is an important numerical invariant of a variety.

Kähler Differentials

Here we will review the algebraic theory of Kähler differentials. We will use Matsumura [2, Ch. 10] as our main reference, but proofs can also be found in the exposés of Cartier and Godement in Cartan and Chevalley [1, exposés 13, 17], or in Grothendieck [EGA 0_{IV}, §20.5].

Let A be a ring (commutative with identity as always), let B be an A-algebra, and let M be a B-module.

Definition. An A-*derivation* of B into M is a map $d: B \to M$ such that (1) d is additive, (2) $d(bb') = bdb' + b'db$, and (3) $da = 0$ for all $a \in A$.

Definition. We define the *module of relative differential forms* of B over A to be a B-module $\Omega_{B/A}$, together with an A-derivation $d: B \to \Omega_{B/A}$, which satisfies the following universal property: for any B-module M, and for any A-derivation $d': B \to M$, there exists a unique B-module homomorphism $f: \Omega_{B/A} \to M$ such that $d' = f \circ d$.

Clearly one way to construct such a module $\Omega_{B/A}$ is to take the free B-module F generated by the symbols $\{db | b \in B\}$, and to divide out by the submodule generated by all expressions of the form (1) $d(b + b') - db - db'$ for $b,b' \in B$, (2) $d(bb') - bdb' - b'db$ for $b,b' \in B$, and (3) da for $a \in A$. The derivation $d: B \to \Omega_{B/A}$ is defined by sending b to db. Thus we see that $\Omega_{B/A}$ exists. It follows from the definition that the pair $\langle \Omega_{B/A}, d \rangle$ is unique up to unique isomorphism. As a corollary of this construction, we see that $\Omega_{B/A}$ is generated as a B-module by $\{db \mid b \in B\}$.

Proposition 8.1A. *Let B be an A-algebra. Let $f: B \otimes_A B \to B$ be the "diagonal" homomorphism defined by $f(b \otimes b') = bb'$, and let $I = \ker f$. Consider $B \otimes_A B$ as a B-module by multiplication on the left. Then I/I^2 inherits a structure of B-module. Define a map $d: B \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$ (modulo I^2). Then $\langle I/I^2, d\rangle$ is a module of relative differentials for B/A.*

PROOF. Matsumura [2, p. 182].

Proposition 8.2A. *If A' and B are A-algebras, let $B' = B \otimes_A A'$. Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$. Furthermore, if S is a multiplicative system in B, then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

PROOF. Matsumura [2, p. 186].

Example 8.2.1 If $B = A[x_1, \ldots, x_n]$ is a polynomial ring over A, then $\Omega_{B/A}$ is the free B-module of rank n generated by $dx_1, \ldots, dx_n$ (Matsumura [2, p. 184]).

Proposition 8.3A (First Exact Sequence). *Let $A \to B \to C$ be rings and homomorphisms. Then there is a natural exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

PROOF. Matsumura [2, Th. 57 p. 186].

Proposition 8.4A (Second Exact Sequence). *Let B be an A-algebra, let I be an ideal of B, and let $C = B/I$. Then there is a natural exact sequence of C-modules*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where for any $b \in I$, if $\bar{b}$ is its image in I/I^2, then $\delta\bar{b} = db \otimes 1$. Note in particular that I/I^2 has a natural structure of C-module, and that δ is a C-linear map, even though it is defined via the derivation d.

PROOF. Matsumura [2, Th. 58, p. 187].

Corollary 8.5. *If B is a finitely generated A-algebra, or if B is a localization of a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module.*

PROOF. Indeed, B is a quotient of a polynomial ring (or its localization) so the result follows from (8.4A), (8.2A), and the example of the polynomial ring itself.

Now we will consider the module of differentials in the case of field extensions and local rings. Recall (I, §4) that an extension field K of a field k

is *separably generated* if there exists a transcendence base $\{x_\lambda\}$ for K/k such that K is a separable algebraic extension of $k(\{x_\lambda\})$.

Theorem 8.6A. *Let K be a finitely generated extension field of a field k. Then $\dim_K \Omega_{K/k} \geqslant$ tr.d. K/k, and equality holds if and only if K is separably generated over k. (Here $\dim_K$ denotes the dimension as a K-vector space.)*

PROOF. Matsumura [2, Th. 59, p. 191]. Note in particular that if K/k is a finite algebraic extension, then $\Omega_{K/k} = 0$ if and only if K/k is separable.

Proposition 8.7. *Let B be a local ring which contains a field k isomorphic to its residue field $B/\mathfrak{m}$. Then the map $\delta:\mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{B/k} \otimes_B k$ of (8.4A) is an isomorphism.*

PROOF. According to (8.4A), the cokernel of δ is $\Omega_{k/k} = 0$, so δ is surjective. To show that δ is injective, it will be sufficient to show that the map

$$\delta':\mathrm{Hom}_k(\Omega_{B/k} \otimes k, k) \rightarrow \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

of dual vector spaces is surjective. The term on the left is isomorphic to $\mathrm{Hom}_B(\Omega_{B/k},k)$, which by definition of the differentials, can be identified with the set $\mathrm{Der}_k(B,k)$ of k-derivations of B to k. If $d:B \rightarrow k$ is a derivation, then $\delta'(d)$ is obtained by restricting to $\mathfrak{m}$, and noting that $d(\mathfrak{m}^2) = 0$. Now, to show that δ' is surjective, let $h \in \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2,k)$. For any $b \in B$, we can write $b = \lambda + c$, $\lambda \in k$, $c \in \mathfrak{m}$, in a unique way. Define $db = h(\bar{c})$, where $\bar{c} \in \mathfrak{m}/\mathfrak{m}^2$ is the image of c. Then one verifies immediately that d is a k-derivation of B to k, and that $\delta'(d) = h$. Thus δ' is surjective, as required.

Theorem 8.8. *Let B be a local ring containing a field k isomorphic to its residue field. Assume furthermore that k is perfect, and that B is a localization of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to $\dim B$ if and only if B is a regular local ring.*

PROOF. First suppose $\Omega_{B/k}$ is free of rank $= \dim B$. Then by (8.7) we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim B$, which says by definition that B is a regular local ring (I, §5). Note in particular that this implies that B is an integral domain.

Now conversely, suppose that B is regular local of dimension r. Then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = r$, so by (8.7) we have $\dim_k \Omega_{B/k} \otimes k = r$. On the other hand, let K be the quotient field of B. Then by (8.2A) we have $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$. Now since k is perfect, K is a separably generated extension field of k (I, 4.8A), and so $\dim_K \Omega_{K/k} =$ tr.d. K/k by (8.6A). But we also have $\dim B =$ tr.d. K/k by (I, 1.8A). Finally, note that by (8.5), $\Omega_{B/k}$ is a finitely generated B-module. We conclude that $\Omega_{B/k}$ is a free B module of rank r by using the following well-known lemma.

Lemma 8.9. *Let A be a noetherian local integral domain, with residue field k and quotient field K. If M is a finitely generated A-module and if $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then M is free of rank r.*

PROOF. Since $\dim_k M \otimes k = r$, Nakayama's lemma tells us that M can be generated by r elements. So there is a surjective map $\varphi: A^r \to M \to 0$. Let R be its kernel. Then we obtain an exact sequence

$$0 \to R \otimes K \to K^r \to M \otimes K \to 0,$$

and since $\dim_K M \otimes K = r$, we have $R \otimes K = 0$. But R is torsion-free, so $R = 0$, and M is isomorphic to A^r.

Sheaves of Differentials

We now carry the definition of the module of differentials over to schemes. Let $f: X \to Y$ be a morphism of schemes. We consider the diagonal morphism $\Delta: X \to X \times_Y X$. It follows from the proof of (4.2) that Δ gives an isomorphism of X onto its image $\Delta(X)$, which is a *locally closed* subscheme of $X \times_Y X$, i.e., a closed subscheme of an open subset W of $X \times_Y X$.

Definition. Let $\mathscr{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Then we define the *sheaf of relative differentials* of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2)$ on X.

Remark 8.9.1. First note that $\mathscr{I}/\mathscr{I}^2$ has a natural structure of $\mathcal{O}_{\Delta(X)}$-module. Then since Δ induces an isomorphism of X to $\Delta(X)$, $\Omega_{X/Y}$ has a natural structure of $\mathcal{O}_X$-module. Furthermore, it follows from (5.9) that $\Omega_{X/Y}$ is quasi-coherent; if Y is noetherian and f is a morphism of finite type, then $X \times_Y X$ is also noetherian, and so $\Omega_{X/Y}$ is coherent.

Remark 8.9.2. Now if $U = \operatorname{Spec} A$ is an open affine subset of Y and $V = \operatorname{Spec} B$ is an open affine subset of X such that $f(V) \subseteq U$, then $V \times_U V$ is an open affine subset of $X \times_Y X$ isomorphic to $\operatorname{Spec}(B \otimes_A B)$, and $\Delta(X) \cap (V \times_U V)$ is the closed subscheme defined by the kernel of the diagonal homomorphism $B \otimes_A B \to B$. Thus $\mathscr{I}/\mathscr{I}^2$ is the sheaf associated to the module I/I^2 of (8.1A). It follows that $\Omega_{V/U} \cong (\Omega_{B/A})^{\sim}$. Thus our definition of the sheaf of differentials of X/Y is compatible, in the affine case, with the module of differentials defined above, via the functor $\sim$. This also shows that we could have defined $\Omega_{X/Y}$ by covering X and Y with open affine subsets V and U as above, and glueing the corresponding sheaves $(\Omega_{B/A})^{\sim}$. The derivations $d: B \to \Omega_{B/A}$ glue together to give a map $d: \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of abelian groups on X, which is a derivation of the local rings at each point.

Therefore, we can carry over our algebraic results to sheaves, and we obtain the following results.

Proposition 8.10. *Let $f: X \to Y$ be a morphism, let $g: Y' \to Y$ be another morphism, and let $f': X' = X \times_Y Y' \to Y'$ be obtained by base extension. Then $\Omega_{X'/Y'} \cong g'^*(\Omega_{X/Y})$ where $g': X' \to X$ is the first projection.*

PROOF. Follows from (8.2A).

Proposition 8.11. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of schemes. Then there is an exact sequence of sheaves on X,*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

PROOF. Follows from (8.3A).

Proposition 8.12. *Let $f: X \to Y$ be a morphism, and let Z be a closed subscheme of X, with ideal sheaf $\mathscr{I}$. Then there is an exact sequence of sheaves on Z,*

$$\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

PROOF. Follows from (8.4A).

Example 8.12.1. If $X = \mathbf{A}_Y^n$, then $\Omega_{X/Y}$ is a free $\mathcal{O}_X$-module of rank n, generated by the global sections $dx_1, \ldots, dx_n$, where $x_1, \ldots, x_n$ are affine coordinates for $\mathbf{A}^n$.

Next we will give an exact sequence relating the sheaf of differentials on a projective space to sheaves we already know. This is a fundamental result, upon which we will base all future calculations involving differentials on projective varieties.

Theorem 8.13. *Let A be a ring, let $Y = \operatorname{Spec} A$, and let $X = \mathbf{P}_A^n$. Then there is an exact sequence of sheaves on X,*

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

(The exponent $n + 1$ in the middle means a direct sum of $n + 1$ copies of $\mathcal{O}_X(-1)$.)

PROOF. Let $S = A[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{n+1}$, with basis $e_0, \ldots, e_n$ in degree 1. Define a (degree 0) homomorphism of graded S-modules $E \to S$ by sending $e_i \mapsto x_i$, and let M be the kernel. Then the exact sequence

$$0 \to M \to E \to S$$

of graded S-modules gives rise to an exact sequence of sheaves on X,

$$0 \to \tilde{M} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

Note that $E \to S$ is not surjective, but it is surjective in all degrees $\geqslant 1$, so the corresponding map of sheaves is surjective.

We will now proceed to show that $\tilde{M} \cong \Omega_{X/Y}$. First note that if we localize at x_i, then $E_{x_i} \to S_{x_i}$ is a surjective homomorphism of free S_{x_i}-modules, so M_{x_i} is free of rank n, generated by $\{e_j - (x_j/x_i)e_i \mid j \neq i\}$. It follows that if U_i is the standard open set of X defined by x_i, then $\tilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by the sections $(1/x_i)e_j - (x_j/x_i^2)e_i$ for $j \neq i$. (Here we need the additional factor $1/x_i$ to get elements of degree 0 in the module M_{x_i}.)

We define a map $\varphi_i:\Omega_{X/Y}|_{U_i} \to \tilde{M}|_{U_i}$ as follows. Recall that $U_i \cong \operatorname{Spec} A[x_0/x_i, \ldots, x_n/x_i]$, so $\Omega_{X/Y}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by $d(x_0/x_i), \ldots, d(x_n/x_i)$. So we define φ_i by

$$\varphi_i(d(x_j/x_i)) = (1/x_i^2)(x_ie_j - x_je_i).$$

Thus φ_i is an isomorphism. I claim now that the isomorphisms φ_i glue together to give an isomorphism $\varphi:\Omega_{X/Y} \to \tilde{M}$ on all of X. This is a simple calculation. On $U_i \cap U_j$, we have, for any k, $(x_k/x_i) = (x_k/x_j) \cdot (x_j/x_i)$. Hence in $\Omega|_{U_i \cap U_j}$ we have

$$d\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j}\, d\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}\, d\left(\frac{x_k}{x_j}\right).$$

Now applying φ_i to the left-hand side and φ_j to the right-hand side, we get the same thing both ways, namely $(1/x_ix_j)(x_je_k - x_ke_j)$. Thus the isomorphisms φ_i glue, which completes our proof.

Nonsingular Varieties

Our principal application of the sheaf of differentials is to nonsingular varieties. In (I, §5) we defined a nonsingular quasi-projective variety to be one whose local rings were all regular local rings. Here we extend that definition to abstract varieties.

Definition. An (abstract) variety X over an algebraically closed field k is *nonsingular* if all its local rings are regular local rings.

Note that we are apparently requiring more here, because in Chapter I we had only closed points, but now our varieties also have nonclosed points. However, the two definitions are equivalent, because every local ring at a nonclosed point is the localization of a local ring at a closed point, and we have the following algebraic result.

Theorem 8.14A. *Any localization of a regular local ring at a prime ideal is again a regular local ring.*

PROOF. Matsumura [2, p. 139].

The connection between nonsingularity and differentials is given by the following result.

Theorem 8.15. *Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ if and only if X is a nonsingular variety over k.*

PROOF. If $x \in X$ is a closed point, then the local ring $B = \mathcal{O}_{x,X}$ has dimension n, residue field k, and is a localization of a k-algebra of finite type. Furthermore the module $\Omega_{B/k}$ of differentials of B over k is equal to the stalk $(\Omega_{X/k})_x$ of the sheaf $\Omega_{X/k}$. Thus we can apply (8.8) and we see that $(\Omega_{X/k})_x$ is free of rank n if and only if B is a regular local ring. Now the theorem follows in view of (8.14A) and (Ex. 5.7).

Corollary 8.16. *If X is a variety over k, then there is an open dense subset U of X which is nonsingular.*

PROOF. (This gives a new proof of (I, 5.3).) If $n = \dim X$, then the function field K of X has transcendence degree n over k, and it is a finitely generated extension field, which is separably generated by (I, 4.8A). Therefore by (8.6A), $\Omega_{K/k}$ is a K-vector space of dimension n. Now $\Omega_{K/k}$ is just the stalk of the sheaf $\Omega_{X/k}$ at the generic point of X. Thus by (Ex. 5.7), $\Omega_{X/k}$ is locally free of rank n in some neighborhood of the generic point, i.e., on a nonempty open set U. Then U is nonsingular by the theorem.

Theorem 8.17. *Let X be a nonsingular variety over k. Let $Y \subseteq X$ be an irreducible closed subscheme defined by a sheaf of ideals $\mathscr{I}$. Then Y is nonsingular if and only if*

(1) *$\Omega_{Y/k}$ is locally free, and*
(2) *the sequence of (8.12) is exact on the left also:*

$$0 \rightarrow \mathscr{I}/\mathscr{I}^2 \rightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \rightarrow \Omega_{Y/k} \rightarrow 0.$$

Furthermore, in this case, $\mathscr{I}$ is locally generated by $r = \operatorname{codim}(Y,X)$ elements, and $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank r on Y.

PROOF. First suppose (1) and (2) hold. Then $\Omega_{Y/k}$ is locally free, so by (8.15) we have only to show that rank $\Omega_{Y/k} = \dim Y$. Let rank $\Omega_{Y/k} = q$. We know that $\Omega_{X/k}$ is locally free of rank n, so it follows from (2) that $\mathscr{I}/\mathscr{I}^2$ is locally free on Y of rank $n - q$. Hence by Nakayama's lemma, $\mathscr{I}$ can be locally generated by $n - q$ elements, and it follows that $\dim Y \geqslant n - (n - q) = q$ (I, Ex. 1.9). On the other hand, considering any closed point $y \in Y$, we have $q = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$ by (8.7), and so $q \geqslant \dim Y$ by (I, 5.2A). Thus $q = \dim Y$. This shows that Y is nonsingular, and at the same time establishes the statements at the end of the theorem, since we now have $n - q = \operatorname{codim}(Y,X)$.

Conversely, assume that Y is nonsingular. Then $\Omega_{Y/k}$ is locally free of rank $q = \dim Y$, so (1) is immediate. From (8.12) we have the exact sequence

$$\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega_{X/k} \otimes \mathcal{O}_Y \xrightarrow{\varphi} \Omega_{Y/k} \rightarrow 0.$$

We consider a closed point $y \in Y$. Then $\ker \varphi$ is locally free of rank $r = n - q$ at y, so it is possible to choose sections $x_1, \ldots, x_r \in \mathscr{I}$ in a suitable neighborhood of y, such that $dx_1, \ldots, dx_r$ generate $\ker \varphi$. Let $\mathscr{I}'$ be the ideal sheaf generated by $x_1, \ldots, x_r$, and let Y' be the corresponding closed subscheme. Then by construction, the $dx_1, \ldots, dx_r$ generate a free subsheaf of rank r of $\Omega_{X/k} \otimes \mathcal{O}_{Y'}$ in a neighborhood of y. It follows that in the exact sequence of (8.12) for Y',

$$\mathscr{I}'/\mathscr{I}'^2 \xrightarrow{\delta} \Omega_{X/k} \otimes \mathcal{O}_{Y'} \to \Omega_{Y'/k} \to 0,$$

we have δ injective (since its image is free of rank r), and $\Omega_{Y'/k}$ is locally free of rank $n - r$. The previous part of the proof now shows that Y' is irreducible and nonsingular of dimension $n - r$ (in a neighborhood of y). But $Y \subseteq Y'$, both are integral schemes of the same dimension, so we must have $Y = Y'$, $\mathscr{I} = \mathscr{I}'$, and this shows that $\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega_{X/k} \otimes \mathcal{O}_Y$ is injective, as required.

Next we include a result which tells us that under suitable conditions, a hyperplane section of a nonsingular variety in projective space is again nonsingular. There is actually a large class of such results, which say that if a projective variety has a certain property, then a sufficiently general hyperplane section has the same property. The result we give here is not the strongest, but it is sufficient for many applications. See also (III, 10.9) for another version in characteristic 0.

Theorem 8.18 (Bertini's Theorem). *Let X be a nonsingular closed subvariety of $\mathbf{P}_k^n$, where k is an algebraically closed field. Then there exists a hyperplane $H \subseteq \mathbf{P}_k^n$, not containing X, and such that the scheme $H \cap X$ is regular at every point. (In fact, we will see later* (III, 7.9.1) *that if* $\dim X \geqslant 2$, *then $H \cap X$ is connected, hence irreducible, and so $H \cap X$ is a nonsingular variety.) Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.*

Proof. For a closed point $x \in X$, let us consider the set $B_x = \{\text{hyperplanes } H | H \supseteq X \text{ or } H \not\supseteq X \text{ but } x \in H \cap X, \text{ and } x \text{ is not a regular point of } H \cap X.\}$ (Fig. 10). These are the bad hyperplanes with respect to the point x. Now a

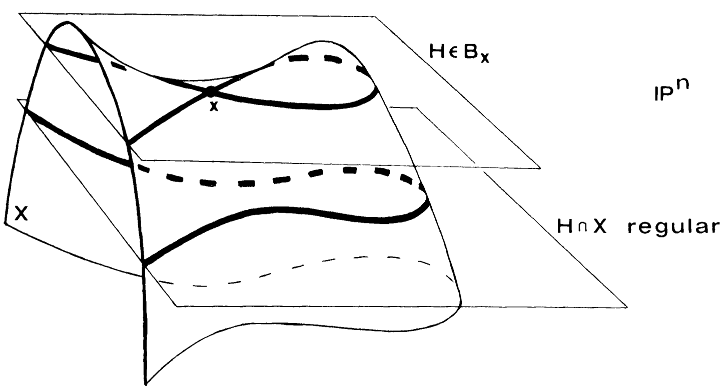

Figure 10. Hyperplane sections of a nonsingular variety.

hyperplane H is determined by a nonzero global section $f \in V = \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. Let us fix an $f_0 \in V$ such that $x \notin H_0$, the hyperplane defined by f_0. Then we can define a map of k-vector spaces

$$\varphi_x : V \to \mathcal{O}_{x,X}/\mathfrak{m}_x^2$$

as follows. Given $f \in V$, then f/f_0 is a regular function on $\mathbf{P}^n - H_0$, which induces a regular function on $X - X \cap H_0$. We take $\varphi_x(f)$ to be the image of f/f_0 in the local ring $\mathcal{O}_{x,X}$ modulo $\mathfrak{m}_x^2$. Now the scheme $H \cap X$ is defined at x by the ideal generated by f/f_0 in $\mathcal{O}_x$. So $x \in H \cap X$ if and only if $\varphi_x(f) \in \mathfrak{m}_x$, and x is nonregular on $H \cap X$ if and only if $\varphi_x(f) \in \mathfrak{m}_x^2$, because in that case, the local ring $\mathcal{O}_x/(\varphi(f))$ will not be regular. Thus we see that the hyperplanes $H \in B_x$ correspond exactly to those $f \in \ker \varphi_x$ (note also that $\varphi_x(f) = 0 \Leftrightarrow H \supseteq X$.)

Since x is a closed point and k is algebraically closed, $\mathfrak{m}_x$ is generated by linear forms in the coordinates, so we see that φ_x is surjective. If $\dim X = r$, then $\dim_k \mathcal{O}_x/\mathfrak{m}_x^2 = r + 1$. We have $\dim V = n + 1$, so $\dim \ker \varphi_x = n - r$. This shows that B_x is a linear system of hyperplanes (in the sense of §7) of dimension $n - r - 1$.

Now, considering the complete linear system $|H|$ as a projective space, consider the subset $B \subseteq X \times |H|$ consisting of all pairs $\langle x,H \rangle$ such that $x \in X$ is a closed point and $H \in B_x$. Clearly B is the set of closed points of a closed subset of $X \times |H|$, which we denote also by B, and which we give a reduced induced scheme structure. We have just seen that the first projection $p_1 : B \to X$ is surjective, with fibre a projective space of dimension $n - r - 1$. Hence B is irreducible, and has dimension $(n - r - 1) + r = n - 1$. Therefore, considering the second projection $p_2 : B \to |H|$, we have $\dim p_2(B) \leqslant n - 1$. Since $\dim|H| = n$, we conclude that $p_2(B) < |H|$. If $H \in |H| - p_2(B)$, then $H \not\supseteq X$ and every point of $H \cap X$ is regular, so that H satisfies the requirements of the theorem. Finally note that since X is projective, $p_2 : X \times |H| \to |H|$ is a proper morphism; B is closed in $X \times |H|$, so $p_2(B)$ is closed in $|H|$. Thus $|H| - p_2(B)$ is an open dense subset of $|H|$, which proves the last statement of the theorem.

Remark 8.18.1. This result continues to hold even if X has a finite number of singular points, because the set of hyperplanes containing any one of them is a proper closed subset of $|H|$.

Applications

Now we will apply the preceding ideas to define some invariants of nonsingular varieties over a field.

Definition. Let X be a nonsingular variety over k. We define the *tangent sheaf* of X to be $\mathscr{T}_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$. It is a locally free sheaf of rank $n = \dim X$. We define the *canonical sheaf* of X to be $\omega_X = \bigwedge^n \Omega_{X/k}$, the nth exterior power of the sheaf of differentials, where $n = \dim X$. It is an invertible sheaf on X. If X is projective and nonsingular, we define the *geometric genus* of X to be $p_g = \dim_k \Gamma(X,\omega_X)$. It is a nonnegative integer.

Remark 8.18.2. Earlier (I, Ex. 7.2) we defined the arithmetic genus p_a of a variety in projective space. In the case of a projective nonsingular curve, the arithmetic genus and the geometric genus coincide. This is a consequence of the Serre duality theorem which we will prove later (III, 7.12.2). For varieties of dimension $\geqslant 2$, however, p_a and p_g need not be equal (Ex. 8.3). See also (III, 7.12.3).

Remark 8.18.3. Since the sheaf of differentials, the tangent sheaf, and the canonical sheaf are all defined intrinsically, any numbers which we can define from them, such as the geometric genus, are invariants of X up to isomorphism. In fact, we will now show that the geometric genus is a *birational invariant* of a nonsingular projective variety. This makes it extremely important for the classification problem.

Theorem 8.19. *Let X and X' be two birationally equivalent nonsingular projective varieties over k. Then $p_g(X) = p_g(X')$.*

PROOF. Recall from (I, §4) that for X and X' to be birationally equivalent means that there are rational maps from X to X' and from X' to X which are inverses to each other. Considering the rational map from X to X', let $V \subseteq X$ be the largest open set for which there is a morphism $f: V \rightarrow X'$ representing this rational map. Then from (8.11) we have a map $f^*\Omega_{X'/k} \rightarrow \Omega_{V/k}$. These are locally free sheaves of the same rank $n = \dim X$, so we get an induced map on the exterior powers: $f^*\omega_{X'} \rightarrow \omega_V$. This map in turn induces a map on the space of global sections $f^*: \Gamma(X',\omega_{X'}) \rightarrow \Gamma(V,\omega_V)$. Now since f is birational, by (I, 4.5), there is an open set $U \subseteq V$ such that $f(U)$ is open in X', and f induces an isomorphism from U to $f(U)$. Thus $\omega_V|_U \cong \omega_{X'}|_{f(U)}$ via f. Since a nonzero global section of an invertible sheaf cannot vanish on a dense open set, we conclude that the map of vector spaces $f^*: \Gamma(X',\omega_{X'}) \rightarrow \Gamma(V,\omega_V)$ must be injective.

Next we will compare $\Gamma(V,\omega_V)$ with $\Gamma(X,\omega_X)$. First I claim that $X - V$ has codimension $\geqslant 2$ in X. Indeed, this follows from the valuative criterion of properness (4.7). If $P \in X$ is a point of codimension 1, then $\mathcal{O}_{P,X}$ is a discrete valuation ring (because X is nonsingular). We already have a map of the generic point of X to X'; and X' is projective, hence proper over k, so there exists a unique morphism $\operatorname{Spec} \mathcal{O}_{P,X} \rightarrow X'$ compatible with the given birational map. This extends to a morphism of some neighborhood of P to X', so we must have $P \in V$ by definition of V.

Now we can show that the natural restriction map $\Gamma(X,\omega_X) \rightarrow \Gamma(V,\omega_V)$ is bijective. It is enough to show, for any open affine subset $U \subseteq X$ such that

$\omega_X|_U \cong \mathcal{O}_U$, that $\Gamma(U,\mathcal{O}_U) \to \Gamma(U \cap V,\mathcal{O}_{U \cap V})$ is bijective. Since X is nonsingular, hence normal, and since $U - U \cap V$ has codimension $\geqslant 2$ in U, this is an immediate consequence of (6.3A).

Combining our results, we see that $p_g(X') \leqslant p_g(X)$. We obtain the reverse inequality by symmetry, and thus conclude that $p_g(X) = p_g(X')$.

Next we study the behavior of the tangent sheaf and the canonical sheaf for a nonsingular subvariety of a variety X.

Definition. Let Y be a nonsingular subvariety of a nonsingular variety X over k. The locally free sheaf $\mathscr{I}/\mathscr{I}^2$ of (8.17) we call the *conormal* sheaf of Y in X. Its dual $\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}/\mathscr{I}^2,\mathcal{O}_Y)$ is called the *normal sheaf* of Y in X. It is locally free of rank $r = \operatorname{codim}(Y,X)$.

Note that if we take the dual on Y of the exact sequence of locally free sheaves on Y given in (8.17), then we obtain an exact sequence

$$0 \to \mathscr{T}_Y \to \mathscr{T}_X \otimes \mathcal{O}_Y \to \mathscr{N}_{Y/X} \to 0.$$

This shows that the normal sheaf we have just defined corresponds to the usual geometric notion of normal vectors being tangent vectors of the ambient space modulo tangent vectors of the subspace.

Proposition 8.20. *Let Y be a nonsingular subvariety of codimension r in a nonsingular variety X over k. Then $\omega_Y \cong \omega_X \otimes \bigwedge^r \mathscr{N}_{Y/X}$. In case $r = 1$, consider Y as a divisor, and let $\mathscr{L}$ be the associated invertible sheaf on X. Then $\omega_Y \cong \omega_X \otimes \mathscr{L} \otimes \mathcal{O}_Y$.*

PROOF. We take the highest exterior powers of the locally free sheaves in the exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0$$

(Ex. 5.16d). Thus we find that $\omega_X \otimes \mathcal{O}_Y \cong \omega_Y \otimes \bigwedge^r(\mathscr{I}/\mathscr{I}^2)$. Since formation of the highest exterior power commutes with taking the dual sheaf, we find $\omega_Y \cong \omega_X \otimes \bigwedge^r \mathscr{N}_{Y/X}$. In the special case $r = 1$, we have $\mathscr{I}_Y \cong \mathscr{L}^{-1}$ by (6.18). Thus $\mathscr{I}/\mathscr{I}^2 \cong \mathscr{L}^{-1} \otimes \mathcal{O}_Y$, and $\mathscr{N}_{Y/X} \cong \mathscr{L} \otimes \mathcal{O}_Y$. So applying the previous result with $r = 1$, we obtain $\omega_Y \cong \omega_X \otimes \mathscr{L} \otimes \mathcal{O}_Y$.

Example 8.20.1. Let $X = \mathbf{P}_k^n$. Taking the dual of the exact sequence of (8.13) gives us this exact sequence involving the tangent sheaf of $\mathbf{P}^n$:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathscr{T}_X \to 0.$$

To obtain the canonical sheaf of $\mathbf{P}^n$, we take the highest exterior powers of the exact sequence of (8.13) and we find $\omega_X \cong \mathcal{O}_X(-n-1)$. Since $\mathcal{O}(l)$ has no global sections for $l < 0$, we find that $p_g(\mathbf{P}^n) = 0$ for any $n \geqslant 1$. Recall that a *rational variety* is defined as a variety birational to $\mathbf{P}^n$ for some n

(I, Ex. 4.4). We conclude from (8.19) that if X is any nonsingular projective rational variety, then $p_g(X) = 0$. This fact will enable us to demonstrate the existence of nonrational varieties in all dimensions.

Example 8.20.2. Let $X = \mathbf{P}_k^n$, with $n \geqslant 2$. For any integer $d \geqslant 1$, the divisor dH, where H is a hyperplane, is a very ample divisor (7.6.1). Thus dH becomes a hyperplane section of X in a suitable projective embedding (the d-uple embedding), and we can apply Bertini's theorem (8.18). We find that there is a subscheme $Y \in |dH|$ which is regular at every one of its points. If Y had at least two irreducible components, say Y_1 and Y_2, then since $n \geqslant 2$, their intersection $Y_1 \cap Y_2$ would be nonempty (I, 7.2). But this cannot happen because Y would be singular at any point of $Y_1 \cap Y_2$, so we conclude in fact that Y is irreducible, hence a nonsingular variety. Thus we see for any $d \geqslant 1$ that there are nonsingular hypersurfaces of degree d in $\mathbf{P}^n$. In fact, they form a dense open subset of the complete linear system $|dH|$. (This generalizes (I, Ex. 5.5).)

Example 8.20.3. Let Y be a nonsingular hypersurface of degree d in $\mathbf{P}^n$, $n \geqslant 2$. Then from (8.20) and the first example above, we conclude that $\omega_Y \cong \mathcal{O}_Y(d - n - 1)$. Let's look at some particular cases.

$n = 2$, $d = 1$. Y is a line in $\mathbf{P}^2$, so $Y \cong \mathbf{P}^1$, and we have $\omega_Y \cong \mathcal{O}_Y(-2)$ which we already knew.

$n = 2$, $d = 2$. Y is a conic in $\mathbf{P}^2$, and $\omega_Y \cong \mathcal{O}_Y(-1)$. In this case Y is the 2-uple embedding of $\mathbf{P}^1$, so pulling ω_Y back to $\mathbf{P}^1$ gives $\omega_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(-2)$, which is again what we already knew.

$n = 2, d = 3$. Y is a nonsingular plane cubic curve, and $\omega_Y \cong \mathcal{O}_Y$. Therefore $p_g(Y) = \dim \Gamma(Y,\mathcal{O}_Y) = 1$, and we see that Y is not rational! This generalizes (I, Ex. 6.2), where we gave just one example of a nonsingular cubic curve, and showed by a different method that it was not rational.

$n = 2, d \geqslant 4$. Y is a nonsingular plane curve of degree d, $\omega_Y \cong \mathcal{O}_Y(d - 3)$, and $d - 3 > 0$. Hence $p_g > 0$, and Y is not rational. In fact, $p_g = \frac{1}{2}(d - 1)(d - 2)$ (Ex. 8.4f), so we see that plane curves of different degrees $d,d' \geqslant 3$ are not birational to each other. Another way of seeing this is as follows. For any nonsingular projective curve, we can consider the *degree* of the canonical sheaf. Since a nonsingular projective curve is unique in its birational equivalence class (I, §6), this number is in fact a birational invariant. In the present case its value is $d(d - 3)$, since $\mathcal{O}(1)$ has degree d on Y. These numbers are also distinct for different $d,d' \geqslant 3$. This shows the existence of infinitely many mutually nonbirational curves.

$n = 3, d = 1$. This gives $Y \cong \mathbf{P}^2$, $\omega_Y \cong \mathcal{O}_Y(-3)$ which we knew.

$n = 3, d = 2$. Here Y is a nonsingular quadric surface, and $\omega_Y \cong \mathcal{O}_Y(-2)$. We have $p_g(Y) = 0$, which is consistent with the fact that Y is rational (I, Ex. 4.5). In terms of the isomorphism $Y \cong \mathbf{P}^1 \times \mathbf{P}^1$, ω_Y corresponds to a divisor class of type $(-2,-2)$—see (6.6.1). This illustrates the general fact

(Ex. 8.3) that the canonical sheaf on a direct product of nonsingular varieties is the tensor product of the pull-backs of the canonical sheaves on the two factors.

$n = 3$, $d = 3$. Y is a nonsingular cubic surface in $\mathbf{P}^3$, $\omega_Y \cong \mathcal{O}_Y(-1)$ and so $p_g(Y) = 0$. In this case also, Y is a rational surface, as we will see later (Chapter V).

$n = 3$, $d = 4$. In this case $\omega_Y \cong \mathcal{O}_Y$. The canonical sheaf is trivial so $p_g = 1$. This is a nonrational surface which belongs to the class of "K3 surfaces."

$n = 3$, $d \geqslant 5$. Here $\omega_Y \cong \mathcal{O}_Y(d - 4)$ with $d - 4 > 0$. Hence $p_g > 0$, and Y is not rational. Surfaces such as these on which the canonical sheaf is very ample belong to the class of "surfaces of general type."

$n = 4$, $d = 3,4$. The cubic and the quartic threefold in $\mathbf{P}^4$ both have $p_g = 0$, but it has recently been shown (by different methods) that they are not in general rational varieties. For the cubic threefold, see Clemens and Griffiths [1]. For the quartic threefold, see Iskovskih and Manin [1].

n arbitrary, $d \geqslant n + 1$. In this case we obtain a nonsingular hypersurface Y in $\mathbf{P}^n$, with $\omega_Y \cong \mathcal{O}_Y(d - n - 1)$ and $d - n - 1 \geqslant 0$. Hence $p_g(Y) \geqslant 1$, and so Y is not rational. This shows the existence of nonrational varieties in all dimensions.

Some Local Algebra

Here we will gather some results from local algebra, mainly concerning depth and Cohen–Macaulay rings, which are useful in algebraic geometry. Then we relate them to the geometric notion of local complete intersection, and give an application to blowing up. We refer to Matsumura [2, Ch. 6] for proofs.

If A is a ring, and M is an A-module, recall that a sequence $x_1, \ldots, x_r$ of elements of A is called a *regular sequence* for M if x_1 is not a zero divisor in M, and for all $i = 2, \ldots, r$, x_i is not a zero divisor in $M/(x_1, \ldots, x_{i-1})M$. If A is a local ring with maximal ideal $\mathfrak{m}$, then the *depth* of M is the maximum length of a regular sequence $x_1, \ldots, x_r$ for M with all $x_i \in \mathfrak{m}$. These definitions apply to the ring A itself, and we say that a local noetherian ring A is *Cohen-Macaulay* if depth A = dim A. Now we list some properties of Cohen-Macaulay rings.

Theorem 8.21A. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$.*

(a) *If A is regular, then it is Cohen–Macaulay.*

(b) *If A is Cohen–Macaulay, then any localization of A at a prime ideal is also Cohen–Macaulay.*

(c) *If A is Cohen–Macaulay, then a set of elements $x_1, \ldots, x_r \in \mathfrak{m}$ forms a regular sequence for A if and only if* $\dim A/(x_1, \ldots, x_r) = \dim A - r$.

(d) *If A is Cohen–Macaulay, and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence for A, then $A/(x_1, \ldots, x_r)$ is also Cohen–Macaulay.*

(e) *If A is Cohen–Macaulay, and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence, let I be the ideal $(x_1, \ldots, x_r)$. Then the natural map $(A/I)[t_1, \ldots, t_r] \to \operatorname{gr}_I A = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$, defined by sending $t_i \mapsto x_i$, is an isomorphism. In other words, I/I^2 is a free A/I-module of rank r, and for each $n \geqslant 1$, the natural map $S^n(I/I^2) \to I^n/I^{n+1}$ is an isomorphism, where S^n denotes the* nth *symmetric power.*

PROOFS. Matsumura [2: (a) p. 121; (b) p. 104; (c) p. 105; (d) p. 104; (e) p. 110].

In keeping with the terminology for schemes (Ex. 3.8), we will say that a noetherian ring A is *normal* if for every prime ideal $\mathfrak{p}$, the localization $A_{\mathfrak{p}}$ is an integrally closed domain. A normal ring is a finite direct product of integrally closed domains.

Theorem 8.22A (Serre). *A noetherian ring A is normal if and only if it satisfies the following two conditions:*

(1) *for every prime ideal $\mathfrak{p} \subseteq A$ of height $\leqslant 1$, $A_{\mathfrak{p}}$ is regular (hence a field or a discrete valuation ring); and*
(2) *for every prime ideal $\mathfrak{p} \subseteq A$ of height $\geqslant 2$, we have* depth $A_{\mathfrak{p}} \geqslant 2$.

PROOF. Matsumura [2, Th. 39, p. 125]. Condition (1) is sometimes called "R_1", or "regular in codimension 1". Condition (2), supplemented by the requirement that for ht $\mathfrak{p} = 1$, depth $A_{\mathfrak{p}} = 1$, which is a consequence of (1) in our case, is called the "condition S_2 of Serre".

Now we apply these results to algebraic geometry. We will say that a scheme is *Cohen–Macaulay* if all of its local rings are Cohen–Macaulay.

Definition. Let Y be a closed subscheme of a nonsingular variety X over k. We say that Y is a *local complete intersection* in X if the ideal sheaf $\mathscr{I}_Y$ of Y in X can be locally generated by $r = \operatorname{codim}(Y,X)$ elements at every point.

Example 8.22.1. If Y itself is nonsingular, then by (8.17) it is a local complete intersection inside any nonsingular X which contains it.

Remark 8.22.2. In fact, the notion of being a local complete intersection is an intrinsic property of the scheme Y, i.e., independent of the nonsingular variety containing it. This is proved using the cotangent complex of a morphism, which extends the concept of relative differentials introduced above—see Lichtenbaum and Schlessinger [1]. We will not use this fact in the sequel.

Proposition 8.23. *Let Y be a locally complete intersection subscheme of a nonsingular variety X over k. Then:*

(a) *Y is Cohen–Macaulay;*

(b) *Y is normal if and only if it is regular in codimension 1.*

PROOF.

(a) Since X is nonsingular, it is Cohen–Macaulay by (8.21Aa). Since $\mathscr{I}_Y$ is locally generated by $r = \operatorname{codim}(Y,X)$ elements, those elements locally form a regular sequence in $\mathcal{O}_X$, by (8.21Ac), and so Y is Cohen–Macaulay by (8.21Ad).

(b) We already know that normal implies regular in codimension 1 (I, 6.2A). For the converse, we use (8.22A) applied to the local rings of Y. Condition (1) is our hypothesis, and condition (2) holds automatically because Y is Cohen–Macaulay.

As our last application, we consider the blowing-up of a nonsingular variety along a nonsingular subvariety (cf. §7 for definition of blowing-up). The following theorem will be useful in comparing invariants of X and $\tilde{X}$ (Ex. 8.5).

Theorem 8.24. *Let X be a nonsingular variety over k, and let $Y \subseteq X$ be a nonsingular closed subvariety, with ideal sheaf $\mathscr{I}$. Let $\pi: \tilde{X} \to X$ be the blowing-up of $\mathscr{I}$, and let $Y' \subseteq \tilde{X}$ be the subscheme defined by the inverse image ideal sheaf $\mathscr{I}' = \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\tilde{X}}$. Then:*

(a) *$\tilde{X}$ is also nonsingular;*

(b) *Y', together with the induced projection map $\pi: Y' \to Y$, is isomorphic to $\mathbf{P}(\mathscr{I}/\mathscr{I}^2)$, the projective space bundle associated to the (locally free) sheaf $\mathscr{I}/\mathscr{I}^2$ on Y;*

(c) *under this isomorphism, the normal sheaf $\mathscr{N}_{Y'/\tilde{X}}$ corresponds to $\mathcal{O}_{\mathbf{P}(\mathscr{I}/\mathscr{I}^2)}(-1)$.*

PROOF. We prove (b) first. Since $\tilde{X} = \mathbf{Proj} \bigoplus \mathscr{I}^d$, we have

$$Y' \cong \mathbf{Proj} \bigoplus (\mathscr{I}^d \otimes \mathcal{O}_X/\mathscr{I}) = \mathbf{Proj} \bigoplus \mathscr{I}^d/\mathscr{I}^{d+1}.$$

But Y is nonsingular, so $\mathscr{I}$ is locally generated by a regular sequence in $\mathcal{O}_X$, and we can apply (8.21Ae). This implies that $\mathscr{I}/\mathscr{I}^2$ is locally free and that for each $n \geqslant 1$, $\mathscr{I}^n/\mathscr{I}^{n+1} \cong S^n(\mathscr{I}/\mathscr{I}^2)$. Thus $Y' \cong \mathbf{Proj} \bigoplus S^d(\mathscr{I}/\mathscr{I}^2)$, which by definition is $\mathbf{P}(\mathscr{I}/\mathscr{I}^2)$.

In particular, Y' is locally isomorphic to $Y \times \mathbf{P}^{r-1}$, where $r = \operatorname{codim}(Y,X)$, so Y' is also nonsingular. Since Y' is locally principal in $\tilde{X}$ (7.13a), it follows that $\tilde{X}$ is also nonsingular: if a quotient of a noetherian local ring by an element which is not a zero divisor is regular, then the local ring itself is regular.

To prove (c), we recall from the proof of (7.13) that $\mathscr{I}' = \pi^{-1}(\mathscr{I}) \cdot \mathcal{O}_{\tilde{X}}$ is isomorphic to $\mathcal{O}_{\tilde{X}}(1)$. It follows that $\mathscr{I}'/\mathscr{I}'^2 \cong \mathcal{O}_{Y'}(1)$, and hence $\mathscr{N}_{Y'/\tilde{X}} \cong \mathcal{O}_{Y'}(-1)$.

We will use the following algebraic result in the exercises.

Theorem 8.25A (I. S. Cohen). *Let A be a complete local ring containing a field k. Assume that the residue field $k(A) = A/\mathfrak{m}$ is a separably generated extension of k. Then there is a subfield $K \subseteq A$, containing k, such that $K \to A/\mathfrak{m}$ is an isomorphism. (The subfield K is called a* field of representatives *for A.)*

PROOF. Matsumura [2, p. 205].

EXERCISES

8.1 Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also. [*Hint*: In copying the proof of (8.7), first pass to $B/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $B/\mathfrak{m}^2$.]

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B +$ tr.d. $k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathcal{O}_x \text{ is a regular local ring}\}$ is an open dense subset of X.

8.2. Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X,\mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x\mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is also locally free. [*Hint*: Use a method similar to the proof of Bertini's theorem (8.18).]

8.3. *Product Schemes.*

(a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X \times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b) If X and Y are nonsingular varieties over a field k, show that $\omega_{X \times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

8.4. *Complete Intersections in* $\mathbf{P}^n$. A closed subscheme Y of $\mathbf{P}^n_k$ is called a (*strict, global*) *complete intersection* if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbf{P}^n)$ elements (I, Ex. 2.17).

(a) Let Y be a closed subscheme of codimension r in $\mathbf{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \ldots, H_r$, such that $Y = H_1 \cap \ldots \cap H_r$ *as schemes*, i.e., $\mathscr{I}_Y = \mathscr{I}_{H_1} + \ldots + \mathscr{I}_{H_r}$. [*Hint*: Use the fact that the unmixedness theorem holds in S (Matsumura [2, p. 107]).]

(b) If Y is a complete intersection of dimension $\geqslant 1$ in $\mathbf{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14). [*Hint*: Apply (8.23) to the affine cone over Y.]

(c) With the same hypotheses as (b), conclude that for all $l \geqslant 0$, the natural map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l)) \rightarrow \Gamma(Y, \mathcal{O}_Y(l))$ is surjective. In particular, taking $l = 0$, show that Y is connected.

(d) Now suppose given integers $d_1, \ldots, d_r \geqslant 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \ldots, H_r$ in $\mathbf{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \ldots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbf{P}^n$.

(e) If Y is a nonsingular complete intersection as in (d), show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

(f) If Y is a nonsingular hypersurface of degree d in $\mathbf{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

(g) If Y is a nonsingular curve in $\mathbf{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d,e, then $p_g(Y) = \frac{1}{2}de(d + e - 4) + 1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

8.5. *Blowing up a Nonsingular Subvariety*. As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geqslant 2$, let $\pi : \tilde{X} \rightarrow X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

(a) Show that the maps $\pi^* : \operatorname{Pic} X \rightarrow \operatorname{Pic} \tilde{X}$, and $\mathbf{Z} \rightarrow \operatorname{Pic} X$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

(b) Show that $\omega_{\tilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$. [*Hint*: By (a) we can write in any case $\omega_{\tilde{X}} \cong f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M}$ on X, and some integer q. By restricting to $\tilde{X} - Y' \cong X - Y$, show that $\mathscr{M} \cong \omega_X$. To determine q, proceed as follows. First show that $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y. Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbf{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r - 1$.]

8.6. *The Infinitesimal Lifting Property*. The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that Spec A is a nonsingular variety over k. Let $0 \rightarrow I \rightarrow B' \rightarrow B \rightarrow 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \rightarrow B$. Then there exists a k-algebra homomorphism $g : A \rightarrow B'$ making a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
 & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[ur, dashed, "g"] & B \arrow[d] \\
 & 0
\end{tikzcd}$$

We call this result the *infinitesimal lifting property* for A. We prove this result in several steps.

(a) First suppose that $g: A \to B'$ is a given homomorphism lifting f. If $g': A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about Spec A being nonsingular.)

(b) Now let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h: P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h induces an A-linear map $\bar{h}: J/J^2 \to I$.

(c) Now use the hypothesis Spec A nonsingular and (8.17) to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0.$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\bar{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g: A \to B'$.

8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathscr{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathscr{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathscr{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathscr{F}$. Such a pair $X', \mathscr{I}$ we call an *infinitesimal extension* of the scheme X by the sheaf $\mathscr{F}$. One such extension, the *trivial* one, is obtained as follows. Take $\mathscr{O}_{X'} = \mathscr{O}_X \oplus \mathscr{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathscr{O}_{X'}$ is an infinitesimal extension of X by $\mathscr{F}$.

The general problem of classifying extensions of X by $\mathscr{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

8.8. Let X be a projective nonsingular variety over k. For any $n > 0$ we define the nth *plurigenus* of X to be $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$. Thus in particular $P_1 = p_g$. Also, for any q, $0 \leqslant q \leqslant \dim X$ we define an integer $h^{q,0} = \dim_k \Gamma(X, \Omega_{X/k}^q)$ where $\Omega_{X/k}^q = \bigwedge^q \Omega_{X/k}$ is the sheaf of regular q-forms on X. In particular, for $q = \dim X$, we recover the geometric genus again. The integers $h^{q,0}$ are called *Hodge numbers*.

Using the method of (8.19), show that P_n and $h^{q,0}$ are *birational* invariants of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

9 Formal Schemes

One feature which clearly distinguishes the theory of schemes from the older theory of varieties is the possibility of having nilpotent elements in the structure sheaf of a scheme. In particular, if Y is a closed subvariety of a variety X, defined by a sheaf of ideals $\mathscr{I}$, then for any $n \geqslant 1$ we can consider the closed subscheme Y_n defined by the nth power $\mathscr{I}^n$ of the sheaf of ideals $\mathscr{I}$. For $n \geqslant 2$, this is a scheme with nilpotent elements. It carries information about Y together with the infinitesimal properties of the embedding of Y in X.

The formal completion of Y in X, which we will define precisely below, is an object which carries information about all the infinitesimal neighborhoods Y_n of Y at once. Thus it is thicker than any Y_n, but it is contained inside any actual open neighborhood of Y in X. We might call it the formal neighborhood of Y in X.

The idea of considering these formal completions is already implicit in the memoir of Zariski [3], where he uses the "holomorphic functions along a subvariety" for his proof of the connectedness principle. We will give different proofs of some of Zariski's results, using cohomology, in (III, §11). A striking application of formal schemes as something in between a subvariety and an ambient variety is in Grothendieck's proof of the Lefschetz theorems on Pic and π_1 [SGA 2]. This material is also explained in Hartshorne [5, Ch. IV].

We will define an arbitrary formal scheme as something which looks locally like the completion of a usual scheme along a closed subscheme.

Inverse Limits of Abelian Groups

First we recall the notion of inverse limit. An *inverse system* of abelian groups is a collection of abelian groups A_n, for each $n \geqslant 1$, together with homomorphisms $\varphi_{n'n}: A_{n'} \to A_n$ for each $n' \geqslant n$, such that for each $n'' \geqslant n' \geqslant n$ we have $\varphi_{n''n} = \varphi_{n'n} \circ \varphi_{n''n'}$. We will denote the inverse system by $(A_n, \varphi_{n'n})$, or simply (A_n), with the φ being understood. If (A_n) is an inverse system of abelian groups, we define the *inverse limit* $A = \varprojlim A_n$ to be the set of sequences $\{a_n\} \in \prod A_n$ such that $\varphi_{n'n}(a_{n'}) = a_n$ for all $n' \geqslant n$. Clearly A is a group. The inverse limit A can be characterized by the following universal property: given a group B, and homomorphisms $\psi_n: B \to A_n$ for each n, such that for any $n' \geqslant n$, $\psi_n = \varphi_{n'n} \circ \psi_{n'}$, then there exists a unique homomorphism $\psi: B \to A$ such that $\psi_n = p_n \circ \psi$ for each n, where $p_n: A \to A_n$ is the restriction of the nth projection map $\prod A_n \to A_n$.

If the groups A_n have the additional structure of vector spaces over a field k, or modules over a ring R, then the above discussion makes sense in the category of k-vector spaces or R-modules.

Next we study exactness properties of the inverse limit (cf. Atiyah–Macdonald [1, Ch.10]). A *homomorphism* $(A_n) \to (B_n)$ of inverse systems of abelian groups is a collection of homomorphisms $f_n: A_n \to B_n$ for each n, which are compatible with the maps of the inverse system, i.e., for each $n' \geqslant n$, we have a commutative diagram

$$\begin{tikzcd} A_{n'} \arrow[r, "f_{n'}"] \arrow[d, "\varphi_{n'n}"] & B_{n'} \arrow[d, "\psi_{n'n}"] \\ A_n \arrow[r, "f_n"] & B_n. \end{tikzcd}$$

A sequence

$$0 \to (A_n) \to (B_n) \to (C_n) \to 0$$

of homomorphisms of inverse systems is *exact* if the corresponding sequence of groups is exact for each n. Given such a short exact sequence of inverse systems, one sees easily that the sequence of inverse limits

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$$

is also exact. However, the last map need not be surjective. So we say that $\varprojlim$ is a left exact functor.

To give a criterion for exactness of $\varprojlim$ on the right, we make the following definition: an inverse system $(A_n, \varphi_{n'n})$ satisfies the *Mittag–Leffler condition* (ML) if for each n, the decreasing family $\{\varphi_{n'n}(A_{n'}) \subseteq A_n | n' \geqslant n\}$ of subgroups of A_n is stationary. In other words, for each n, there is an $n_0 \geqslant n$, such that for all $n', n'' \geqslant n_0$, $\varphi_{n'n}(A_{n'}) = \varphi_{n''n}(A_{n''})$ as subgroups of A_n.

Suppose an inverse system (A_n) satisfies (ML). Then for each n, we let $A'_n \subseteq A_n$ be the *stable image* $\varphi_{n'n}(A_{n'})$ for any $n' \geqslant n_0$, which exists by the definition. Then one sees easily that (A'_n) is also an inverse system, with the induced maps, and that the maps of the new system (A'_n) are all surjective. Furthermore, it is clear that $\varprojlim A'_n = \varprojlim A_n$. So we see that $A = \varprojlim A_n$ maps surjectively to each A'_n.

Proposition 9.1. *Let*

$$0 \to (A_n) \xrightarrow{f} (B_n) \xrightarrow{g} (C_n) \to 0$$

be a short exact sequence of inverse systems of abelian groups. Then:

(a) *if* (B_n) *satisfies* (ML), *so does* (C_n).

(b) *if* (A_n) *satisfies* (ML), *then the sequence of inverse limits*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Proof. (See also Grothendieck [EGA 0_{III}, 13.2].)

(a) For each $n' \geqslant n$, the image of $B_{n'}$ in B_n maps surjectively to the image of $C_{n'}$ in C_n, so (ML) for (B_n) implies (ML) for (C_n) immediately.

(b) The only nonobvious part is to show that the last map is surjective. So let $\{c_n\} \in \varprojlim C_n$. For each n, let $E_n = g^{-1}(c_n)$. Then E_n is a subset of B_n, and (E_n) is an inverse system of sets. Furthermore, each E_n is bijective, in a noncanonical way, with A_n, because of the exactness of the sequence $0 \to A_n \to B_n \to C_n \to 0$. Thus since (A_n) satisfies (ML), one sees easily that (E_n) satisfies the Mittag–Leffler condition as an inverse system of sets (same definition). Since each E_n is nonempty, it follows from considering the inverse system of stable images as above, that $\varprojlim E_n$ is also nonempty. Taking any element of this set gives an element of $\varprojlim B_n$ which maps to $\{c_n\}$.

Example 9.1.1. If all the maps $\varphi_{n'n}: A_{n'} \to A_n$ are surjective, then (A_n) satisfies (ML), so (9.1b) applies.

Example 9.1.2. If (A_n) is an inverse system of finite-dimensional vector spaces over a field, or more generally, an inverse system of modules with descending chain condition over a ring, then (A_n) satisfies (ML).

Inverse Limits of Sheaves

In any category $\mathfrak{C}$, we define the notion of inverse limit by analogy with the universal property of the inverse limit of abelian groups above. Thus if $(A_n, \varphi_{n'n})$ is an inverse system of objects of $\mathfrak{C}$ (same definition as above), then an *inverse limit* $A = \varprojlim A_n$ is an object A of $\mathfrak{C}$, together with morphisms $p_n: A \to A_n$ for each n, such that for each $n' \geqslant n$, $p_n = \varphi_{n'n} \circ p_{n'}$, satisfying the following universal property: given any object B of $\mathfrak{C}$, together with morphisms $\psi_n: B \to A_n$ for each n, such that for each $n' \geqslant n$, $\psi_n = \varphi_{n'n} \circ \psi_{n'}$, there exists a unique morphism $\psi: B \to A$ such that for each n, $\psi_n = p_n \circ \psi$. Clearly the inverse limit is unique if it exists. But the question of existence depends on the particular category considered.

Proposition 9.2. *Let X be a topological space, and let $\mathfrak{C}$ be the category of sheaves of abelian groups on X. Then inverse limits exist in $\mathfrak{C}$. Furthermore, if $(\mathscr{F}_n)$ is an inverse system of sheaves on X, and $\mathscr{F} = \varprojlim \mathscr{F}_n$ is its inverse limit, then for any open set U, we have $\Gamma(U, \mathscr{F}) = \varprojlim \Gamma(U, \mathscr{F}_n)$ in the category of abelian groups.*

PROOF. Given an inverse system of sheaves $(\mathscr{F}_n)$ on X, we consider the presheaf $U \to \varprojlim \Gamma(U,\mathscr{F}_n)$, where this inverse limit is taken in the category of abelian groups. Now using the sheaf property for each $\mathscr{F}_n$, one verifies immediately that this presheaf is a sheaf. Call it $\mathscr{F}$. Now given any other sheaf $\mathscr{G}$, and a system of compatible maps $\psi_n : \mathscr{G} \to \mathscr{F}_n$ for each n, it follows from the universal property of an inverse limit of abelian groups that we obtain unique maps, for each U, $\Gamma(U,\mathscr{G}) \to \Gamma(U,\mathscr{F})$. These give a sheaf map $\mathscr{G} \to \mathscr{F}$, thus verifying that $\mathscr{F}$ is the inverse limit of the $\mathscr{F}_n$ in $\mathfrak{C}$.

Caution 9.2.1. Even though inverse limits exist in the category $\mathfrak{C}$ of abelian sheaves on a topological space, one must beware of using intuition derived from the category of abelian groups. In particular, the statement of (9.1b) is *false* in $\mathfrak{C}$, even if all maps in the inverse system (A_n) are surjective. So in studying exactness questions, we will always pass to sections over an open set, and thus reduce to questions about abelian groups. For more details about exactness of $\varprojlim$ in $\mathfrak{C}$, see Hartshorne [7, I, §4].

Completion of a Ring

One important application of inverse limits is to define the completion of a ring with respect to an ideal. This generalizes the notion of completion of a local ring which was discussed in (I, §5). It also forms the algebraic model for the completion of a scheme along a closed subscheme which will come next.

So let A be a commutative ring with identity (as always), and let I be an ideal of A. We denote by I^n the nth power of the ideal I. Then we have natural homomorphisms

$$\ldots \to A/I^3 \to A/I^2 \to A/I,$$

which make (A/I^n) into an inverse system of rings. The inverse limit ring $\varprojlim A/I^n$ is denoted by $\hat{A}$ and is called the *completion of A with respect to I* or the *I-adic completion of A*. For each n we have a natural map $A \to A/I^n$, so by the universal property we obtain a homomorphism $A \to \hat{A}$.

Similarly, if M is any A-module, we define $\hat{M} = \varprojlim M/I^nM$, and call it the *$I$-adic completion of M*. It has a natural structure of $\hat{A}$-module.

Theorem 9.3A. *Let A be a noetherian ring, and I an ideal of A. We denote by* ^ *the I-adic completion as above. Then:*

(a) *$\hat{I} = \varprojlim I/I^n$ is an ideal of $\hat{A}$. For any n, $\hat{I}^n = I^n\hat{A}$, and $\hat{A}/\hat{I}^n \cong A/I^n$;*

(b) *if M is a finitely generated A-module, then $\hat{M} \cong M \otimes_A \hat{A}$;*

(c) *the functor $M \mapsto \hat{M}$ is an exact functor on the category of finitely generated A-modules;*

(d) *$\hat{A}$ is a noetherian ring;*

(e) *if (M_n) is an inverse system, where each M_n is a finitely generated A/I^n-module, each $\varphi_{n'n} : M_{n'} \mapsto M_n$ is surjective, and* $\ker \varphi_{n'n} = I^nM_{n'}$, *then $M = \varprojlim M_n$ is a finitely generated $\hat{A}$-module, and for each n, $M_n \cong M/I^nM$.*

PROOFS.

(a) Atiyah–Macdonald [1, p. 109].
(b) [Ibid., p. 108].
(c) [Ibid., p. 108].
(d) [Ibid., p. 113].
(e) Bourbaki [1, Ch. III, §2, no. 11, Prop. & Cor. 14].

Formal Schemes

We begin by defining the completion of a scheme along a closed subscheme. For technical reasons we will limit our discussion to noetherian schemes.

Definition. Let X be a noetherian scheme, and let Y be a closed subscheme, defined by a sheaf of ideals $\mathscr{I}$. Then we define the *formal completion of X along Y*, denoted $(\hat{X},\mathcal{O}_{\hat{X}})$, to be the following ringed space. We take the topological space Y, and on it the sheaf of rings $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X/\mathscr{I}^n$. Here we consider each $\mathcal{O}_X/\mathscr{I}^n$ as a sheaf of rings on Y, and make them into an inverse system in the natural way.

Remark 9.3.1. The structure sheaf $\mathcal{O}_{\hat{X}}$ of $\hat{X}$ actually depends only on the closed subset Y, and not on the particular scheme structure on Y. For if $\mathscr{J}$ is another sheaf of ideals defining a closed subscheme structure on Y, then since X is a noetherian scheme, there are integers m,n such that $\mathscr{I} \supseteq \mathscr{J}^m$ and $\mathscr{J} \supseteq \mathscr{I}^n$. Thus the inverse systems $(\mathcal{O}_X/\mathscr{J}^n)$ and $(\mathcal{O}_X/\mathscr{I}^m)$ are cofinal with each other, and hence have the same inverse limit.

One sees easily that the stalks of the sheaf $\mathcal{O}_{\hat{X}}$ are local rings, so in fact $(\hat{X},\mathcal{O}_{\hat{X}})$ is a locally ringed space. If $U = \operatorname{Spec} A$ is an open affine subset of X, and if $I \subseteq A$ is the ideal $\Gamma(U,\mathscr{I})$, then from (9.2) we see that $\Gamma(\hat{X} \cap U,\mathcal{O}_{\hat{X}}) = \hat{A}$, the I-adic completion of A. Thus the process of completing X along Y is analogous to the I-adic completion of a ring discussed above. However, one should note that the local rings of $\hat{X}$ are in general *not* complete, and their dimension ($=\dim X$) is *not* equal to the dimension of the underlying topological space Y.

Definition. With $X,Y,\mathscr{I}$ as in the previous definition, let $\mathscr{F}$ be a coherent sheaf on X. We define the *completion of $\mathscr{F}$ along Y*, denoted $\hat{\mathscr{F}}$, to be the sheaf $\varprojlim \mathscr{F}/\mathscr{I}^n\mathscr{F}$ on Y. It has a natural structure of $\mathcal{O}_{\hat{X}}$-module.

Definition. A *noetherian formal scheme* is a locally ringed space $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ which has a finite open cover $\{\mathfrak{U}_i\}$ such that for each i, the pair $(\mathfrak{U}_i,\mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic, as a locally ringed space, to the completion of some noetherian scheme X_i along a closed subscheme Y_i. A *morphism* of noetherian formal schemes is a morphism as locally ringed spaces. A sheaf $\mathfrak{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules is said to be *coherent* if there is a finite open cover $\mathfrak{U}_i$ as above, with $\mathfrak{U}_i \cong \hat{X}_i$, and for each i there is a coherent sheaf $\mathscr{F}_i$ on X_i such that $\mathfrak{F}|_{\mathfrak{U}_i} \cong \hat{\mathscr{F}}_i$ as $\mathcal{O}_{\hat{X}_i}$-modules via the given isomorphism $\mathfrak{U}_i \cong \hat{X}_i$.

Examples 9.3.2. If X is any noetherian scheme, and Y a closed subscheme, then its completion $\hat{X}$ is a formal scheme. Such a formal scheme, which can be obtained by completing a *single* noetherian scheme along a closed subscheme, is called *algebraizable*. It is not so easy to give examples, but there are nonalgebraizable noetherian formal schemes—see Hironaka and Matsumura [1, §5] or Hartshorne [5, V, 3.3, p. 205].

Example 9.3.3. If X is a noetherian scheme, and we take $Y = X$, then $\hat{X} = X$. Thus the category of noetherian formal schemes includes all noetherian schemes.

Example 9.3.4. If X is a noetherian scheme, and Y is a closed point P, then $\hat{X}$ is a one point space $\{P\}$ with the completion $\hat{\mathcal{O}}_P$ of the local ring at P as its structure sheaf. An $\hat{\mathcal{O}}_P$-module M, considered as a sheaf on $\hat{X}$, is coherent if and only if M is a finitely generated module. Indeed, clearly coherent implies finitely generated. But the converse is also true since we can obtain $\hat{X}$ by completing the scheme Spec $\hat{\mathcal{O}}_P$ at its closed point, and any finitely generated $\hat{\mathcal{O}}_P$-module M corresponds to a coherent sheaf on Spec $\hat{\mathcal{O}}_P$.

Next we will study the structure of coherent sheaves on a formal scheme. As in the study of coherent sheaves on usual schemes in §5, we first analyze what happens in the affine case.

Definition. An *affine (noetherian) formal scheme* is a formal scheme obtained by completing a single affine noetherian scheme along a closed subscheme. If $X = \operatorname{Spec} A$, $Y = V(I)$, and $\mathfrak{X} = \hat{X}$, then for any finitely generated A-module M, we define the sheaf $M^\triangle$ on $\mathfrak{X}$ to be the completion of the coherent sheaf $\tilde{M}$ on X. Thus by definition, $M^\triangle$ is a coherent sheaf on $\mathfrak{X}$.

Proposition 9.4. *Let A be a noetherian ring, I an ideal of A, let $X = \operatorname{Spec} A$, $Y = V(I)$, and let $\mathfrak{X} = \hat{X}$. Then:*

(a) *$\mathfrak{I} = I^\triangle$ is a sheaf of ideals in $\mathcal{O}_\mathfrak{X}$, and for any n, $\mathcal{O}_\mathfrak{X}/\mathfrak{I}^n \cong (A/I^n)^\sim$ as sheaves on Y;*

(b) *if M is a finitely generated A-module, then $M^\triangle = \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_\mathfrak{X}$.*

(c) *The functor $M \mapsto M^\triangle$ is an exact functor from the category of finitely generated A-modules to the category of coherent $\mathcal{O}_\mathfrak{X}$-modules.*

Proof. In each case we have a statement about sheaves on $\mathfrak{X}$. Since the open affine subsets of X form a base for the topology of X, and their intersections with Y a base for the topology of Y, it will be sufficient to establish the corresponding property of the sections over any such open set. So let $U = \operatorname{Spec} B$ be an open affine subset of X, let $J = \Gamma(U, \tilde{I})$, and for any finitely generated A-module M, let $N = \Gamma(U, \tilde{M})$. Then B is a noetherian ring (3.2), N is a finitely generated B-module (5.4), and the functor $M \mapsto N$ is an exact functor from A-modules to B-modules (5.5).

We prove (c) first. So let M be a finitely generated A-module. Then $M^{\triangle} = \varprojlim \tilde{M}/\tilde{I}^n\tilde{M}$ by definition, so by (9.2), $\Gamma(U,M^{\triangle}) = \varprojlim \Gamma(U,\tilde{M}/\tilde{I}^n\tilde{M})$. But this is equal to $\varprojlim N/J^nN = \hat{N}$, where $\hat{}$ now denotes the J-adic completion of a B-module. Now $M \mapsto N$ is exact as we saw above, and $N \mapsto \hat{N}$ is exact by (9.3A). Thus $M \mapsto \Gamma(U,M^{\triangle})$ is exact for each U, and so $M \mapsto M^{\triangle}$ is exact.

(a) For any U as above, $\Gamma(U,I^{\triangle}) = \varprojlim \Gamma(U,\tilde{I}/\tilde{I}^n) = \hat{J}$. Furthermore $\Gamma(U,\mathcal{O}_{\mathfrak{X}}) = \hat{B}$ similarly. But by (9.3A), $\hat{J}$ is an ideal of $\hat{B}$, so this shows that $\mathfrak{I} = I^{\triangle}$ is a sheaf of ideals in $\mathcal{O}_{\mathfrak{X}}$.

Now we consider the exact sequence of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

According to (c) which we have already proved, this gives an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules

$$0 \to \mathfrak{I}^n \to \mathcal{O}_{\mathfrak{X}} \to (A/I^n)^{\triangle} \to 0.$$

Observe that the inverse system which defines $(A/I^n)^{\triangle}$ as the completion of $(A/I^n)^{\sim}$ is eventually stationary, since this sheaf is annihilated by $\tilde{I}^n$. Hence $(A/I^n)^{\triangle} = (A/I^n)^{\sim}$, and we conclude that $\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n \cong (A/I^n)^{\sim}$ as required.

(b) We have a slight abuse of notation in our statement: since $\tilde{M}$ and $\mathcal{O}_X$ are sheaves on X, we should actually write $M^{\triangle} \cong \tilde{M}|_Y \otimes_{\mathcal{O}_X|_Y} \mathcal{O}_{\mathfrak{X}}$. But we will simply regard $M^{\triangle}$ and $\mathcal{O}_{\mathfrak{X}}$ as sheaves on X, by extending by zero outside of Y (Ex. 1.19). For any finitely generated A module M, and for U an open set as above, we have $\Gamma(U,M^{\triangle}) = \hat{N}$ as before. On the other hand, $\tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U,\tilde{M}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,\mathcal{O}_{\mathfrak{X}}) = N \otimes_B \hat{B}.$$

Since $\hat{N} \cong N \otimes_B \hat{B}$ by (9.3A), we conclude that the corresponding sheaves are isomorphic too: $M^{\triangle} \cong \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$.

Definition. Let $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme. A sheaf of ideals $\mathfrak{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ is called an *ideal of definition* for $\mathfrak{X}$ if Supp $\mathcal{O}_{\mathfrak{X}}/\mathfrak{I} = \mathfrak{X}$ and the locally ringed space $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a noetherian scheme.

Proposition 9.5. *Let $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme.*

(a) *If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two ideals of definition, then there are integers $m,n > 0$ such that $\mathfrak{I}_1 \supseteq \mathfrak{I}_2^m$ and $\mathfrak{I}_2 \supseteq \mathfrak{I}_1^n$.*

(b) *There is a unique largest ideal of definition $\mathfrak{I}$, characterized by the fact that $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a reduced scheme. In particular, ideals of definition exist.*

(c) *If $\mathfrak{I}$ is an ideal of definition, so is $\mathfrak{I}^n$, for any $n > 0$.*

PROOF.

(a) Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two ideals of definition. Then on the topological space $\mathfrak{X}$, we have surjective maps of sheaves of rings $f_1:\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}_1$ and $f_2:\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}_2$. For any point $P \in \mathfrak{X}$, the stalk $(\mathfrak{I}_2)_P$ of $\mathfrak{I}_2$ at P is contained in $\mathfrak{m}_P$, the maximal ideal of the local ring $\mathcal{O}_{\mathfrak{X},P}$. Indeed, $\mathcal{O}_{\mathfrak{X},P}/(\mathfrak{J}_2)_P$ is the local ring of P on the scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_2)$. In particular, it is nonzero, so $(\mathfrak{J}_2)_P \subseteq \mathfrak{m}_P$. Now we consider the sheaf of ideals $f_1(\mathfrak{J}_2)$ on the scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_1)$. For each point P, its stalk is contained inside the maximal ideal of the local ring. Hence every local section of $f_1(\mathfrak{J}_2)$ is nilpotent (Ex. 2.18), and since $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_1)$ is a noetherian scheme, $f_1(\mathfrak{J}_2)$ itself is nilpotent. This shows that for some $m > 0$, $\mathfrak{J}_1 \supseteq \mathfrak{J}_2^m$. The other way follows by symmetry.

(b) Suppose $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_1)$ is a reduced scheme. Then in the proof of (a), we find $f_1(\mathfrak{J}_2) = 0$, so $\mathfrak{J}_1 \supseteq \mathfrak{J}_2$. Thus such an $\mathfrak{J}_1$ is largest, if it exists. Since it is unique, the existence becomes a local question. Thus we may assume that $\mathfrak{X}$ is the completion of an affine noetherian scheme X along a closed subscheme Y. By (9.3.1) we may assume that Y has the reduced induced structure. Let $X = \operatorname{Spec} A$, $Y = V(I)$. Then by (9.4), $\mathfrak{J} = I^{\triangle}$ is an ideal in $\mathcal{O}_{\mathfrak{X}}$, and $\mathcal{O}_{\mathfrak{X}}/\mathfrak{J} \cong (A/I)^{\sim} = \mathcal{O}_Y$. Thus $\mathfrak{J}$ is an ideal of definition for which $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J})$ is reduced. This shows the existence of the largest ideal of definition.

(c) Let $\mathfrak{J}$ be any ideal of definition, and suppose given $n > 0$. Let $\mathfrak{J}_0$ be the unique largest ideal of definition. Then by (a), there is an integer r such that $\mathfrak{J} \supseteq \mathfrak{J}_0^r$, and hence $\mathfrak{J}^n \supseteq \mathfrak{J}_0^{nr}$. First note that $\mathfrak{J}_0^{nr}$ is an ideal of definition. Indeed, this can be checked locally. If $\mathfrak{J}_0 = I^{\triangle}$ on an affine, using the notation of (b), then $\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_0^{nr} \cong (A/I^{nr})^{\sim}$ by (9.4), so $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}_0^{nr})$ is a scheme with support Y. Let's call this scheme Y', and let $f:\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{Y'}$ be the corresponding map of sheaves. Then $(Y',\mathcal{O}_{Y'}/f(\mathfrak{J})) = (\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J})$ is a noetherian scheme, by hypothesis, so $f(\mathfrak{J})$ is a coherent sheaf. Therefore $f(\mathfrak{J}^n) = f(\mathfrak{J})^n$ is also coherent, and we conclude that $(Y',\mathcal{O}_{Y'}/f(\mathfrak{J}^n)) = (\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^n)$ is also a noetherian scheme.

Proposition 9.6. *Let $\mathfrak{X}$ be a noetherian formal scheme and let $\mathfrak{J}$ be an ideal of definition. For each $n > 0$ we denote by Y_n the scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^n)$.*

(a) *If $\mathfrak{F}$ is a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, then for each n, $\mathscr{F}_n = \mathfrak{F}/\mathfrak{J}^n\mathfrak{F}$ is a coherent sheaf of $\mathcal{O}_{Y_n}$-modules, and $\mathfrak{F} \cong \varprojlim \mathscr{F}_n$.*

(b) *Conversely, suppose given for each n a coherent $\mathcal{O}_{Y_n}$-module $\mathscr{F}_n$, together with surjective maps $\varphi_{n'n}:\mathscr{F}_{n'} \to \mathscr{F}_n$ for each $n' \geqslant n$, making $\{\mathscr{F}_n\}$ into an inverse system of sheaves. Assume furthermore that for each $n' \geqslant n$, $\ker \varphi_{n'n} = \mathfrak{J}^n\mathscr{F}_{n'}$. Then $\mathfrak{F} = \varprojlim \mathscr{F}_n$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, and for each n, $\mathscr{F}_n \cong \mathfrak{F}/\mathfrak{J}^n\mathfrak{F}$.*

PROOF.

(a) The question is local, so we may assume that $\mathfrak{X}$ is affine, equal to the completion of $X = \operatorname{Spec} A$ along $Y = V(I)$, and that $\mathfrak{F} = M^{\triangle}$ for some finitely generated A-module M. Then as in the proof of (9.4a) we see that $\mathfrak{F}/\mathfrak{J}^n\mathfrak{F} \cong (M/I^nM)^{\sim}$ for each n. Thus $\mathscr{F}_n$ is coherent on $Y_n = \operatorname{Spec}(A/I^n)$, and $\mathfrak{F} \cong \varprojlim \mathscr{F}_n$.

(b) Again the question is local, so we may assume that $\mathfrak{X}$ is affine as above. Furthermore, we may assume that A is I-adically complete, because replacing A by $\hat{A}$ does not change $\hat{X}$. For each n, let $M_n = \Gamma(Y_n, \mathscr{F}_n)$. Then

(M_n) is an inverse system of modules satisfying the hypotheses of (9.3Ae). Therefore we conclude that $M = \varprojlim M_n$ is a finitely generated A-module (since A is complete), and that for each n, $M_n \cong M/I^nM$. But then $\mathfrak{F} = \varprojlim \mathscr{F}_n$ is just $M^\triangle$, hence it is a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Furthermore $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong (M/I^nM)^\sim$ as in (a), so $\mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathscr{F}_n$.

Theorem 9.7. *Let A be a noetherian ring, I an ideal, and assume that A is I-adically complete. Let $X = \operatorname{Spec} A$, $Y = V(I)$, and $\mathfrak{X} = \hat{X}$. Then the functors $M \mapsto M^\triangle$ and $\mathfrak{F} \mapsto \Gamma(\mathfrak{X},\mathfrak{F})$ are exact, and inverse to each other, on the categories of finitely generated A-modules and coherent $\mathcal{O}_{\mathfrak{X}}$-modules respectively. Thus they establish an equivalence of categories. In particular, every coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathfrak{F}$ is of the form $M^\triangle$ for some M.*

PROOF. We have already seen that $M \mapsto M^\triangle$ is exact (9.4). If M is an A-module of finite type, then $\Gamma(\mathfrak{X},M^\triangle) = \varprojlim M/I^nM = \hat{M}$, and $\hat{M} = M$ because A is complete (9.3Ab). Thus one composition of our two functors is the identity.

Conversely, let $\mathfrak{F}$ be a coherent $\mathcal{O}_{\mathfrak{X}}$-module, and let $\mathfrak{I} = I^\triangle$. Then by (9.6a), $\mathfrak{F} \cong \varprojlim \mathscr{F}_n$, where for each $n > 0$, $\mathscr{F}_n = \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}$. Now the inverse system of sheaves $(\mathscr{F}_n)$ satisfies the hypotheses of (9.6b), and the proof of (9.6b) shows in fact that $\mathfrak{F} \cong M^\triangle$, for some finitely generated A-module M. Furthermore, by (9.2), $\Gamma(\mathfrak{X},\mathfrak{F}) = \varprojlim \Gamma(Y,(M/I^nM)^\sim) = \varprojlim M/I^nM = \hat{M}$, which is equal to M since A is complete. This shows that $\Gamma(\mathfrak{X},\mathfrak{F})$ is a finitely generated A-module, and $\mathfrak{F} \cong \Gamma(\mathfrak{X},\mathfrak{F})^\triangle$. Thus the other composition of our two functors is the identity.

It remains to show that the functor $\Gamma(\mathfrak{X},\cdot)$ is exact on the category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. So let

$$0 \to \mathfrak{F}_1 \to \mathfrak{F}_2 \to \mathfrak{F}_3 \to 0$$

be an exact sequence of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. For each i, let $M_i = \Gamma(\mathfrak{X},\mathfrak{F}_i)$. Then the M_i are finitely generated A-modules, and we have at least a left-exact sequence

$$0 \to M_1 \to M_2 \to M_3.$$

Let R be the cokernel on the right. Then applying the functor $^\triangle$ we obtain an exact sequence

$$0 \to M_1^\triangle \to M_2^\triangle \to M_3^\triangle \to R^\triangle \to 0$$

on $\mathfrak{X}$. But for each i, $M_i^\triangle \cong \mathfrak{F}_i$ as we saw above, so we conclude that $R^\triangle = 0$. But also by the above, $R = \Gamma(\mathfrak{X},R^\triangle)$, so $R = 0$. This shows that $\Gamma(\mathfrak{X},\cdot)$ is exact, which concludes the proof.

Corollary 9.8. *If X is any noetherian scheme, Y a closed subscheme, and $\mathfrak{X} = \hat{X}$ the completion along Y, then the functor $\mathscr{F} \mapsto \hat{\mathscr{F}}$ is an exact functor from coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Furthermore, if $\mathscr{I}$ is the sheaf of ideals of Y, and $\hat{\mathscr{I}}$ its completion, then we have $\hat{\mathscr{F}}/\hat{\mathscr{I}}^n\hat{\mathscr{F}} \cong \mathscr{F}/\mathscr{I}^n\mathscr{F}$ for each n, and $\hat{\mathscr{F}} \cong \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$.*

PROOF. These questions are all local, in which case they reduce to (9.4).

Corollary 9.9. *Any kernel, cokernel, or image of a morphism of coherent sheaves on a noetherian formal scheme is again coherent.*

PROOF. These questions are also local, in which case they follow from (9.7).

Remark 9.9.1. It is also true that an extension of coherent sheaves on a noetherian formal scheme is coherent (Ex. 9.4). On the other hand, some properties of coherent sheaves on usual schemes do not carry over to formal schemes. For example, if $\mathfrak{X}$ is the completion of a projective variety $X \subseteq \mathbf{P}_k^n$ along a closed subvariety Y, and if $\mathcal{O}_{\mathfrak{X}}(1) = \mathcal{O}_X(1)^{\wedge}$, then there may be nonzero coherent sheaves $\mathfrak{F}$ on $\mathfrak{X}$ such that $\Gamma(\mathfrak{X}, \mathfrak{F}(\nu)) = 0$ for all $\nu \in \mathbf{Z}$. In particular, no twist of $\mathfrak{F}$ is generated by global sections (III, Ex. 11.7).

EXERCISES

9.1. Let X be a noetherian scheme, Y a closed subscheme, and $\hat{X}$ the completion of X along Y. We call the ring $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the ring of *formal-regular functions* on X along Y. In this exercise we show that if Y is a connected, nonsingular, positive-dimensional subvariety of $X = \mathbf{P}_k^n$ over an algebraically closed field k, then $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

(a) Let $\mathscr{I}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on Y, $\mathscr{I}/\mathscr{I}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.

(b) Show that for any $r \geqslant 1$, $\Gamma(Y, \mathscr{I}^r/\mathscr{I}^{r+1}) = 0$.

(c) Use the exact sequences

$$0 \to \mathscr{I}^r/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^r \to 0$$

and induction on r to show that $\Gamma(Y, \mathcal{O}_X/\mathscr{I}^r) = k$ for all $r \geqslant 1$. (Use (8.21Ae).)

(d) Conclude that $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$. (Actually, the same result holds without the hypothesis Y nonsingular, but the proof is more difficult—see Hartshorne [3, (7.3)].)

9.2. Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subseteq X = \mathbf{P}_k^n$ be as above, and let $f: X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

9.3. Prove the analogue of (5.6) for formal schemes, which says, if $\mathfrak{X}$ is an affine formal scheme, and if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, and if $\mathfrak{F}'$ is coherent, then the sequence of global sections

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is exact. For the proof, proceed in the following steps.

(a) Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$, and for each $n > 0$ consider the exact sequence

$$0 \to \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'' \to 0.$$

Use (5.6), slightly modified, to show that for every open affine subset $\mathfrak{U} \subseteq \mathfrak{X}$, the sequence

$$0 \to \Gamma(\mathfrak{U},\mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U},\mathfrak{F}/\mathfrak{I}^n\mathfrak{F}) \to \Gamma(\mathfrak{U},\mathfrak{F}'') \to 0$$

is exact.

(b) Now pass to the limit, using (9.1), (9.2), and (9.6). Conclude that $\mathfrak{F} \cong \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}$ and that the sequence of global sections above is exact.

9.4. Use (Ex. 9.3) to prove that if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $\mathfrak{X}$, and if $\mathfrak{F}',\mathfrak{F}''$ are coherent, then $\mathfrak{F}$ is coherent also.

9.5. If $\mathfrak{F}$ is a coherent sheaf on a noetherian formal scheme $\mathfrak{X}$, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

9.6. Let $\mathfrak{X}$ be a noetherian formal scheme, let $\mathfrak{I}$ be an ideal of definition, and for each n, let Y_n be the scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n)$. Assume that the inverse system of groups $(\Gamma(Y_n,\mathcal{O}_{Y_n}))$ satisfies the Mittag–Leffler condition. Then prove that $\operatorname{Pic} \mathfrak{X} = \varprojlim \operatorname{Pic} Y_n$. As in the case of a scheme, we define $\operatorname{Pic} \mathfrak{X}$ to be the group of locally free $\mathcal{O}_{\mathfrak{X}}$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

(a) Use the fact that $\ker(\Gamma(Y_{n+1},\mathcal{O}_{Y_{n+1}}) \to \Gamma(Y_n,\mathcal{O}_{Y_n}))$ is a nilpotent ideal to show that the inverse system $(\Gamma(Y_n,\mathcal{O}_{Y_n}^*))$ of units in the respective rings also satisfies (ML).

(b) Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, and assume that for each n, there is some isomorphism $\varphi_n\colon \mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathcal{O}_{Y_n}$. Then show that there is an isomorphism $\mathfrak{F} \cong \mathcal{O}_{\mathfrak{X}}$. Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})$ and $(\mathcal{O}_{Y_n})$! Conclude that the natural map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is injective.

(c) Given an invertible sheaf $\mathscr{L}_n$ on Y_n for each n, and given isomorphisms $\mathscr{L}_{n+1} \otimes \mathcal{O}_{Y_n} \cong \mathscr{L}_n$, construct maps $\mathscr{L}_{n'} \to \mathscr{L}_n$ for each $n' \geqslant n$ so as to make an inverse system, and show that $\mathfrak{L} = \varprojlim \mathscr{L}_n$ is a coherent sheaf on $\mathfrak{X}$. Then show that $\mathfrak{L}$ is locally free of rank 1, and thus conclude that the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective. Again be careful, because even though each $\mathscr{L}_n$ is locally free of rank 1, the open sets needed to make them free might get smaller and smaller with n.

(d) Show that the hypothesis "$(\Gamma(Y_n,\mathcal{O}_{Y_n}))$ satisfies (ML)" is satisfied if either $\mathfrak{X}$ is affine, or each Y_n is projective over a field k.

Note: See (III, Ex. 11.5–11.7) for further examples and applications.

CHAPTER III

Cohomology

In this chapter we define the general notion of cohomology of a sheaf of abelian groups on a topological space, and then study in detail the cohomology of coherent and quasi-coherent sheaves on a noetherian scheme.

Although the end result is usually the same, there are many different ways of introducing cohomology. There are the fine resolutions often used in several complex variables—see Gunning and Rossi [1]; the Čech cohomology used by Serre [3], who first introduced cohomology into abstract algebraic geometry; the canonical flasque resolutions of Godement [1]; and the derived functor approach of Grothendieck [1]. Each is important in its own way.

We will take as our basic definition the derived functors of the global section functor (§1, 2). This definition is the most general, and also best suited for theoretical questions, such as the proof of Serre duality in §7. However, it is practically impossible to calculate, so we introduce Čech cohomology in §4, and use it in §5 to compute explicitly the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space $\mathbf{P}^r$. This calculation is the basis of many later results on projective varieties.

In order to prove that the Čech cohomology agrees with the derived functor cohomology, we need to know that the higher cohomology of a quasi-coherent sheaf on an affine scheme is zero. We prove this in §3 in the noetherian case only, because it is technically much simpler than the case of an arbitrary affine scheme ([EGA III, §1]). Hence we are bound to include noetherian hypotheses in all theorems involving cohomology.

As applications, we show for example that the arithmetic genus of a projective variety X, whose definition in (I, §7) depended on a projective embedding of X, can be computed in terms of the cohomology groups $H^i(X,\mathcal{O}_X)$, and hence is intrinsic (Ex. 5.3). We also show that the arithmetic genus is constant in a family of normal projective varieties (9.13).

Another application is Zariski's main theorem (11.4) which is important in the birational study of varieties.

The latter part of the chapter (§8–12) is devoted to families of schemes, i.e., the study of the fibres of a morphism. In particular, we include a section on flat morphisms and a section on smooth morphisms. While these can be treated without cohomology, it seems to be an appropriate place to include them, because flatness can be understood better using cohomology (9.9).

1 Derived Functors

In this chapter we will assume familiarity with the basic techniques of homological algebra. Since notation and terminology vary from one source to another, we will assemble in this section (without proofs) the basic definitions and results we will need. More details can be found in the following sources: Godement [1, esp. Ch. I, §1.1–1.8, 2.1–2.4, 5.1–5.3], Hilton and Stammbach [1, Ch. II,IV,IX], Grothendieck [1, Ch. II, §1,2,3], Cartan and Eilenberg [1, Ch. III,V], Rotman [1, §6].

Definition. An *abelian category* is a category $\mathfrak{A}$, such that: for each $A,B \in \operatorname{Ob} \mathfrak{A}$, $\operatorname{Hom}(A,B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. (Hilton and Stammbach [1, p. 78].)

The following are all abelian categories.

Example 1.0.1. $\mathfrak{Ab}$, the category of abelian groups.

Example 1.0.2. $\mathfrak{Mod}(A)$, the category of modules over a ring A (commutative with identity as always).

Example 1.0.3. $\mathfrak{Ab}(X)$, the category of sheaves of abelian groups on a topological space X.

Example 1.0.4. $\mathfrak{Mod}(X)$, the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X,\mathcal{O}_X)$.

Example 1.0.5. $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X (II, 5.7).

Example 1.0.6. $\mathfrak{Coh}(X)$, the category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X (II, 5.7).

Example 1.0.7. $\mathfrak{Coh}(\mathfrak{X})$, the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ (II, 9.9).

In the rest of this section, we will be stating some basic results of homological algebra in the context of an arbitrary abelian category. However, in most books, these results are proved only for the category of modules over a ring, and proofs are often done by "diagram-chasing": you pick an element and chase its images and pre-images through a diagram. Since diagram-chasing doesn't make sense in an arbitrary abelian category, the conscientious reader may be disturbed. There are at least three ways to handle this difficulty. (1) Provide intrinsic proofs for all the results, starting from the axioms of an abelian category, and without even mentioning an element. This is cumbersome, but can be done—see, e.g., Freyd [1]. Or (2), note that in each of the categories we use (most of which are in the above list of examples), one can in fact carry out proofs by diagram-chasing. Or (3), accept the "full embedding theorem" (Freyd [1, Ch. 7]), which states roughly that any abelian category is equivalent to a subcategory of $\mathfrak{Ab}$. This implies that any category-theoretic statement (e.g., the 5-lemma) which can be proved in $\mathfrak{Ab}$ (e.g., by diagram-chasing) also holds in any abelian category.

Now we begin our review of homological algebra. A *complex* $A^{\cdot}$ in an abelian category $\mathfrak{A}$ is a collection of objects A^i, $i \in \mathbf{Z}$, and morphisms $d^i: A^i \to A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i. If the objects A^i are specified only in a certain range, e.g., $i \geqslant 0$, then we set $A^i = 0$ for all other i. A *morphism* of complexes, $f: A^{\cdot} \to B^{\cdot}$ is a set of morphisms $f^i: A^i \to B^i$ for each i, which commute with the coboundary maps d^i.

The ith *cohomology object* $h^i(A^{\cdot})$ of the complex $A^{\cdot}$ is defined to be $\ker d^i/\operatorname{im} d^{i-1}$. If $f: A^{\cdot} \to B^{\cdot}$ is a morphism of complexes, then f induces a natural map $h^i(f): h^i(A^{\cdot}) \to h^i(B^{\cdot})$. If $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ is a short exact sequence of complexes, then there are natural maps $\delta^i: h^i(C^{\cdot}) \to h^{i+1}(A^{\cdot})$ giving rise to a long exact sequence

$$\ldots \to h^i(A^{\cdot}) \to h^i(B^{\cdot}) \to h^i(C^{\cdot}) \xrightarrow{\delta^i} h^{i+1}(A^{\cdot}) \to \ldots$$

Two morphisms of complexes $f,g: A^{\cdot} \to B^{\cdot}$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i: A^i \to B^{i-1}$ for each i (which need not commute with the d^i) such that $f - g = dk + kd$. The collection of morphisms, $k = (k^i)$ is called a *homotopy operator*. If $f \sim g$, then f and g induce the *same* morphism $h^i(A^{\cdot}) \to h^i(B^{\cdot})$ on the cohomology objects, for each i.

A covariant functor $F: \mathfrak{A} \to \mathfrak{B}$ from one abelian category to another is *additive* if for any two objects A,A' in $\mathfrak{A}$, the induced map $\operatorname{Hom}(A,A') \to \operatorname{Hom}(FA,FA')$ is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathfrak{A}$, the sequence

$$0 \to FA' \to FA \to FA''$$

is exact in $\mathfrak{B}$. If we can write a 0 on the right instead of the left, we say F is *right exact*. If it is both left and right exact, we say it is *exact*. If only the middle part $FA' \to FA \to FA''$ is exact, we say F is *exact in the middle*.

For a contravariant functor we make analogous definitions. For example, $F:\mathfrak{A} \to \mathfrak{B}$ is *left exact* if it is additive, and for every short exact sequence as above, the sequence

$$0 \to FA'' \to FA \to FA'$$

is exact in $\mathfrak{B}$.

Example 1.0.8. If $\mathfrak{A}$ is an abelian category, and A is a fixed object, then the functor $B \to \operatorname{Hom}(A,B)$, usually denoted $\operatorname{Hom}(A,\cdot)$, is a covariant left exact functor from $\mathfrak{A}$ to $\mathfrak{A}\mathrm{b}$. The functor $\operatorname{Hom}(\cdot,A)$ is a contravariant left exact functor from $\mathfrak{A}$ to $\mathfrak{A}\mathrm{b}$.

Next we come to resolutions and derived functors. An object I of $\mathfrak{A}$ is *injective* if the functor $\operatorname{Hom}(\cdot,I)$ is exact. An *injective resolution* of an object A of $\mathfrak{A}$ is a complex $I^{\cdot}$, defined in degrees $i \geqslant 0$, together with a morphism $\varepsilon:A \to I^0$, such that I^i is an injective object of $\mathfrak{A}$ for each $i \geqslant 0$, and such that the sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \dots$$

is exact.

If every object of $\mathfrak{A}$ is isomorphic to a subobject of an injective object of $\mathfrak{A}$, then we say $\mathfrak{A}$ *has enough injectives*. If $\mathfrak{A}$ has enough injectives, then every object has an injective resolution. Furthermore, a well-known lemma states that any two injective resolutions are homotopy equivalent.

Now let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F:\mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor. Then we construct the *right derived functors* R^iF, $i \geqslant 0$, of F as follows. For each object A of $\mathfrak{A}$, choose once and for all an injective resolution $I^{\cdot}$ of A. Then we define $R^iF(A) = h^i(F(I^{\cdot}))$.

Theorem 1.1A. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F:\mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then*

(a) *For each $i \geqslant 0$, R^iF as defined above is an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.*

(b) *There is a natural isomorphism $F \cong R^0F$.*

(c) *For each short exact sequence $0 \to A' \to A \to A'' \to 0$ and for each $i \geqslant 0$ there is a natural morphism $\delta^i:R^iF(A'') \to R^{i+1}F(A')$, such that we obtain a long exact sequence*

$$\dots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \dots.$$

(d) *Given a morphism of the exact sequence of* (c) *to another* $0 \to B' \to B \to B'' \to 0$, *the* δ*'s give a commutative diagram*

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B'). \end{array}$$

(e) *For each injective object* I *of* $\mathfrak{A}$, *and for each* $i > 0$, *we have* $R^iF(I) = 0$.

Definition. With $F:\mathfrak{A} \to \mathfrak{B}$ as in the theorem, an object J of $\mathfrak{A}$ is *acyclic* for F if $R^iF(J) = 0$ for all $i > 0$.

Proposition 1.2A. *With* $F:\mathfrak{A} \to \mathfrak{B}$ *as in* (1.1A), *suppose there is an exact sequence*

$$0 \to A \to J^0 \to J^1 \to \dots$$

where each J^i *is acyclic for* F, $i \geqslant 0$. (*We say* $J^{\cdot}$ *is an* F-acyclic resolution *of* A.) *Then for each* $i \geqslant 0$ *there is a natural isomorphism* $R^iF(A) \cong h^i(F(J^{\cdot}))$.

We leave to the reader the analogous definitions of projective objects, projective resolutions, an abelian category having enough projectives, and the left derived functors of a covariant right exact functor. Also, the right derived functors of a left exact contravariant functor (use projective resolutions) and the left derived functors of a right exact contravariant functor (use injective resolutions).

Next we will give a universal property of derived functors. For this purpose, we generalize slightly with the following definition.

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories. A (*covariant*) δ-*functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i \geqslant 0}$, together with a morphism $\delta^i: T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$, and each $i \geqslant 0$, such that:

(1) For each short exact sequence as above, there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to \dots$$
$$\dots \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to T^{i+1}(A) \to \dots;$$

(2) for each morphism of one short exact sequence (as above) into another $0 \to B' \to B \to B'' \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B'). \end{array}$$

Definition. The δ-functor $T = (T^i):\mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, given any other δ-functor $T' = (T'^i):\mathfrak{A} \to \mathfrak{B}$, and given any morphism of functors $f^0 : T^0 \to T'^0$, there exists a unique sequence of morphisms $f^i : T^i \to T'^i$ for each $i \geqslant 0$, starting with the given f^0, which commute with the δ^i for each short exact sequence.

Remark 1.2.1. If $F : \mathfrak{A} \to \mathfrak{B}$ is a covariant additive functor, then by definition there can exist at most one (up to unique isomorphism) universal δ-functor T with $T^0 = F$. If T exists, the T^i are sometimes called the *right satellite functors* of F.

Definition. An additive functor $F : \mathfrak{A} \to \mathfrak{B}$ is *effaceable* if for each object A of $\mathfrak{A}$, there is a monomorphism $u : A \to M$, for some M, such that $F(u) = 0$. It is *coeffaceable* if for each A there exists an epimorphism $u : P \to A$ such that $F(u) = 0$.

Theorem 1.3A. *Let $T = (T^i)_{i \geqslant 0}$ be a covariant δ-functor from $\mathfrak{A}$ to $\mathfrak{B}$. If T^i is effaceable for each $i > 0$, then T is universal.*

PROOF. Grothendieck [1, II, 2.2.1]

Corollary 1.4. *Assume that $\mathfrak{A}$ has enough injectives. Then for any left exact functor $F : \mathfrak{A} \to \mathfrak{B}$, the derived functors $(R^iF)_{i \geqslant 0}$ form a universal δ-functor with $F \cong R^0F$. Conversely, if $T = (T^i)_{i \geqslant 0}$ is any universal δ-functor, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for each $i \geqslant 0$.*

PROOF. If F is a left exact functor, then the $(R^iF)_{i \geqslant 0}$ form a δ-functor by (1.1A). Furthermore, for any object A, let $u : A \to I$ be a monomorphism of A into an injective. Then $R^iF(I) = 0$ for $i > 0$ by (1.1A), so $R^iF(u) = 0$. Thus R^iF is effaceable for each $i > 0$. It follows from the theorem that (R^iF) is universal.

On the other hand, given a universal δ-functor T, we have T^0 left exact by the definition of δ-functor. Since $\mathfrak{A}$ has enough injectives, the derived functors R^iT^0 exist. We have just seen that (R^iT^0) is another universal δ-functor. Since $R^0T^0 = T^0$, we find $R^iT^0 \cong T^i$ for each i, by (1.2.1).

2 Cohomology of Sheaves

In this section we define cohomology of sheaves by taking the derived functors of the global section functor. Then as an application of general techniques of cohomology we prove Grothendieck's theorem about the vanishing of cohomology on a noetherian topological space. To begin with, we must verify that the categories we use have enough injectives.

Proposition 2.1A. *If A is a ring, then every A-module is isomorphic to a submodule of an injective A-module.*

PROOF. Godement [1, I, 1.2.2] or Hilton and Stammbach [1, I, 8.3].

Proposition 2.2. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

PROOF. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. For each point $x \in X$, the stalk $\mathscr{F}_x$ is an $\mathcal{O}_{x,X}$-module. Therefore there is an injection $\mathscr{F}_x \to I_x$, where I_x is an injective $\mathcal{O}_{x,X}$-module (2.1A). For each point x, let j denote the inclusion of the one-point space $\{x\}$ into X, and consider the sheaf $\mathscr{I} = \prod_{x \in X} j_*(I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$, and j_* is the direct image functor (II, §1).

Now for any sheaf $\mathscr{G}$ of $\mathcal{O}_X$-modules, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G},\mathscr{I}) = \prod \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G},j_*(I_x))$ by definition of the direct product. On the other hand, for each point $x \in X$, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G},j_*(I_x)) \cong \operatorname{Hom}_{\mathcal{O}_{x,X}}(\mathscr{G}_x,I_x)$ as one sees easily. Thus we conclude first that there is a natural morphism of sheaves of $\mathcal{O}_X$-modules $\mathscr{F} \to \mathscr{I}$ obtained from the local maps $\mathscr{F}_x \to I_x$. It is clearly injective. Second, the functor $\operatorname{Hom}_{\mathcal{O}_X}(\cdot,\mathscr{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathscr{G} \mapsto \mathscr{G}_x$, which is exact, followed by $\operatorname{Hom}_{\mathcal{O}_{x,X}}(\cdot,I_x)$, which is exact, since I_x is an injective $\mathcal{O}_{x,X}$-module. Hence $\operatorname{Hom}(\cdot,\mathscr{I})$ is an exact functor, and therefore $\mathscr{I}$ is an injective $\mathcal{O}_X$-module.

Corollary 2.3. *If X is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

PROOF. Indeed, if we let $\mathcal{O}_X$ be the constant sheaf of rings $\mathbf{Z}$, then $(X,\mathcal{O}_X)$ is a ringed space, and $\mathfrak{Mod}(X) = \mathfrak{Ab}(X)$.

Definition. Let X be a topological space. Let $\Gamma(X,\cdot)$ be the global section functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We define the *cohomology functors* $H^i(X,\cdot)$ to be the right derived functors of $\Gamma(X,\cdot)$. For any sheaf $\mathscr{F}$, the groups $H^i(X,\mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$. Note that even if X and $\mathscr{F}$ have some additional structure, e.g., X a scheme and $\mathscr{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathscr{F}$ simply as a sheaf of abelian groups on the underlying topological space X.

We let the reader write out the long exact sequences which follow from the general properties of derived functors (1.1A).

Recall (II, Ex. 1.16) that a sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Lemma 2.4. *If $(X,\mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

PROOF. For any open subset $U \subseteq X$, let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$, which is the restriction of $\mathcal{O}_X$ to U, extended by zero outside U (II, Ex. 1.19). Now let $\mathscr{I}$ be an injective $\mathcal{O}_X$-module, and let $V \subseteq U$ be open sets. Then we have an inclusion $0 \to \mathcal{O}_V \to \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathscr{I}$ is injective, we get a surjection $\operatorname{Hom}(\mathcal{O}_U,\mathscr{I}) \to \operatorname{Hom}(\mathcal{O}_V,\mathscr{I}) \to 0$. But $\operatorname{Hom}(\mathcal{O}_U,\mathscr{I}) = \mathscr{I}(U)$ and $\operatorname{Hom}(\mathcal{O}_V,\mathscr{I}) = \mathscr{I}(V)$, so $\mathscr{I}$ is flasque.

Proposition 2.5. *If $\mathscr{F}$ is a flasque sheaf on a topological space X, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$.*

PROOF. Embed $\mathscr{F}$ in an injective object $\mathscr{I}$ of $\mathfrak{Ab}(X)$ and let $\mathscr{G}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0.$$

Then $\mathscr{F}$ is flasque by hypothesis, $\mathscr{I}$ is flasque by (2.4), and so $\mathscr{G}$ is flasque by (II, Ex. 1.16c). Now since $\mathscr{F}$ is flasque, we have an exact sequence (II, Ex. 1.16b)

$$0 \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{I}) \to \Gamma(X,\mathscr{G}) \to 0.$$

On the other hand, since $\mathscr{I}$ is injective, we have $H^i(X,\mathscr{I}) = 0$ for $i > 0$ (1.1Ae). Thus from the long exact sequence of cohomology, we get $H^1(X,\mathscr{F}) = 0$ and $H^i(X,\mathscr{F}) \cong H^{i-1}(X,\mathscr{G})$ for each $i \geqslant 2$. But $\mathscr{G}$ is also flasque, so by induction on i we get the result.

Remark 2.5.1. This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X,\cdot)$. Hence we can calculate cohomology using flasque resolutions (1.2A). In particular, we have the following result.

Proposition 2.6. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X,\cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ coincide with the cohomology functors $H^i(X,\cdot)$.*

PROOF. Considering $\Gamma(X,\cdot)$ as a functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, we calculate its derived functors by taking injective resolutions in the category $\mathfrak{Mod}(X)$. But any injective is flasque (2.4), and flasques are acyclic (2.5) so this resolution gives the usual cohomology functors (1.2A).

Remark 2.6.1. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $A = \Gamma(X,\mathcal{O}_X)$. Then for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$, $\Gamma(X,\mathscr{F})$ has a natural structure of A-module. In particular, since we can calculate cohomology using resolutions in the category $\mathfrak{Mod}(X)$, all the cohomology groups of $\mathscr{F}$ have a natural structure of A-module; the associated exact sequences are sequences of A-modules, and so forth. Thus for example, if X is a scheme over Spec B for some ring B, the cohomology groups of any $\mathcal{O}_X$-module $\mathscr{F}$ have a natural structure of B-module.

A Vanishing Theorem of Grothendieck

Theorem 2.7 (Grothendieck [1]). *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathscr{F}$ on X, we have $H^i(X,\mathscr{F}) = 0$.*

Before proving the theorem, we need some preliminary results, mainly concerning direct limits. If $(\mathscr{F}_\alpha)$ is a direct system of sheaves on X, indexed by a directed set A, then we have defined the direct limit $\varinjlim \mathscr{F}_\alpha$ (II, Ex. 1.10).

Lemma 2.8. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

PROOF. Let $(\mathscr{F}_\alpha)$ be a directed system of flasque sheaves. Then for any inclusion of open sets $V \subseteq U$, and for each α, we have $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$ is surjective. Since $\varinjlim$ is an exact functor, we get

$$\varinjlim \mathscr{F}_\alpha(U) \to \varinjlim \mathscr{F}_\alpha(V)$$

is also surjective. But on a noetherian topological space, $\varinjlim \mathscr{F}_\alpha(U) = (\varinjlim \mathscr{F}_\alpha)(U)$ for any open set (II, Ex. 1.11). So we have

$$(\varinjlim \mathscr{F}_\alpha)(U) \to (\varinjlim \mathscr{F}_\alpha)(V)$$

is surjective, and so $\varinjlim \mathscr{F}_\alpha$ is flasque.

Proposition 2.9. *Let X be a noetherian topological space, and let $(\mathscr{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geqslant 0$*

$$\varinjlim H^i(X,\mathscr{F}_\alpha) \to H^i(X,\varinjlim \mathscr{F}_\alpha).$$

PROOF. For each α we have a natural map $\mathscr{F}_\alpha \to \varinjlim \mathscr{F}_\alpha$. This induces a map on cohomology, and then we take the direct limit of these maps. For $i = 0$, the result is already known (II, Ex. 1.11). For the general case, we consider the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ consisting of all directed systems of objects of $\mathfrak{Ab}(X)$, indexed by A. This is an abelian category. Furthermore, since $\varinjlim$ is an exact functor, we have a natural transformation of δ-functors

$$\varinjlim H^i(X,\cdot) \to H^i(X,\varinjlim \cdot)$$

from $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ to $\mathfrak{Ab}$. They agree for $i = 0$, so to prove they are the same, it will be sufficient to show they are both effaceable for $i > 0$. For in that case, they are both universal by (1.3A), and so must be isomorphic.

So let $(\mathscr{F}_\alpha) \in \mathfrak{ind}_A(\mathfrak{Ab}(X))$. For each α, let $\mathscr{G}_\alpha$ be the sheaf of discontinuous sections of $\mathscr{F}_\alpha$ (II, Ex. 1.16e). Then $\mathscr{G}_\alpha$ is flasque, and there is a natural inclusion $\mathscr{F}_\alpha \to \mathscr{G}_\alpha$. Furthermore, the construction of $\mathscr{G}_\alpha$ is functorial, so the $\mathscr{G}_\alpha$ also form a direct system, and we obtain a monomorphism $u:(\mathscr{F}_\alpha) \to (\mathscr{G}_\alpha)$ in the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$. Now the $\mathscr{G}_\alpha$ are all flasque, so $H^i(X,\mathscr{G}_\alpha) = 0$ for $i > 0$ (2.5). Thus $\varinjlim H^i(X,\mathscr{G}_\alpha) = 0$, and the functor on the left-hand side is effaceable for $i > 0$. On the other hand, $\varinjlim \mathscr{G}_\alpha$ is also flasque by (2.8). So $H^i(X,\varinjlim \mathscr{G}_\alpha) = 0$ for $i > 0$, and we see that the functor on the right-hand side is also effaceable. This completes the proof.

Remark 2.9.1. As a special case we see that cohomology commutes with infinite direct sums.

Lemma 2.10. *Let Y be a closed subset of X, let $\mathscr{F}$ be a sheaf of abelian groups on Y, and let $j: Y \to X$ be the inclusion. Then $H^i(Y,\mathscr{F}) = H^i(X,j_*\mathscr{F})$, where $j_*\mathscr{F}$ is the extension of $\mathscr{F}$ by zero outside Y* (II, Ex. 1.19).

Proof. If $\mathscr{J}^{\cdot}$ is a flasque resolution of $\mathscr{F}$ on Y, then $j_*\mathscr{J}^{\cdot}$ is a flasque resolution of $j_*\mathscr{F}$ on X, and for each i, $\Gamma(Y,\mathscr{J}^i) = \Gamma(X,j_*\mathscr{J}^i)$. So we get the same cohomology groups.

Remark 2.10.1. Continuing our earlier abuse of notation (II, Ex. 1.19), we often write $\mathscr{F}$ instead of $j_*\mathscr{F}$. This lemma shows there will be no ambiguity about the cohomology groups.

Proof of (2.7). First we fix some notation. If Y is a closed subset of X, then for any sheaf $\mathscr{F}$ on X we let $\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$, where $j: Y \to X$ is the inclusion. If U is an open subset of X, we let $\mathscr{F}_U = i_!(\mathscr{F}|_U)$, where $i: U \to X$ is the inclusion. In particular, if $U = X - Y$, we have an exact sequence (II, Ex. 1.19)

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

We will prove the theorem by induction on $n = \dim X$, in several steps.

Step 1. Reduction to the case X irreducible. If X is reducible, let Y be one of its irreducible components, and let $U = X - Y$. Then for any $\mathscr{F}$ we have an exact sequence

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X,\mathscr{F}_Y) = 0$ and $H^i(X,\mathscr{F}_U) = 0$ for $i > n$. But Y is closed and irreducible, and $\mathscr{F}_U$ can be regarded as a sheaf on the closed subset $\bar{U}$, which has one fewer irreducible components than X. Thus using (2.10) and induction on the number of irreducible components, we reduce to the case X irreducible.

Step . Suppose X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. For otherwise, X would have a proper irreducible closed subset, and $\dim X$ would be $\geqslant 1$. Thus $\Gamma(X,\cdot)$ induces an equivalence of categories $\mathfrak{Ab}(X) \to \mathfrak{Ab}$. In particular, $\Gamma(X,\cdot)$ is an exact functor, so $H^i(X,\mathscr{F}) = 0$ for $i > 0$, and for all $\mathscr{F}$.

Step 3. Now let X be irreducible of dimension n, and let $\mathscr{F} \in \mathfrak{Ab}(X)$. Let $B = \bigcup_{U \subseteq X} \mathscr{F}(U)$, and let A be the set of all finite subsets of B. For each $\alpha \in A$, let $\mathscr{F}_\alpha$ be the subsheaf of $\mathscr{F}$ generated by the sections in α (over various open sets). Then A is a directed set, and $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$. So by (2.9), it will be sufficient to prove vanishing of cohomology for each $\mathscr{F}_\alpha$. If α' is a subset of α, then we have an exact sequence

$$0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a sheaf generated by $\#(\alpha - \alpha')$ sections over suitable open sets. Thus, using the long exact sequence of cohomology, and induction on $\#(\alpha)$, we reduce to the case that $\mathscr{F}$ is generated by a single section over some open set U. In that case $\mathscr{F}$ is a quotient of the sheaf $\mathbf{Z}_U$ (where $\mathbf{Z}$ denotes the constant sheaf $\mathbf{Z}$ on X). Letting $\mathscr{R}$ be the kernel, we have an exact sequence

$$0 \to \mathscr{R} \to \mathbf{Z}_U \to \mathscr{F} \to 0.$$

Again using the long exact sequence of cohomology, it will be sufficient to prove vanishing for $\mathscr{R}$ and for $\mathbf{Z}_U$.

Step 4. Let U be an open subset of X and let $\mathscr{R}$ be a subsheaf of $\mathbf{Z}_U$. For each $x \in U$, the stalk $\mathscr{R}_x$ is a subgroup of $\mathbf{Z}$. If $\mathscr{R} = 0$, skip to Step 5. If not, let d be the least positive integer which occurs in any of the groups $\mathscr{R}_x$. Then there is a nonempty open subset $V \subseteq U$ such that $\mathscr{R}|_V \cong d \cdot \mathbf{Z}|_V$ as a subsheaf of $\mathbf{Z}|_V$. Thus $\mathscr{R}_V \cong \mathbf{Z}_V$ and we have an exact sequence

$$0 \to \mathbf{Z}_V \to \mathscr{R} \to \mathscr{R}/\mathbf{Z}_V \to 0.$$

Now the sheaf $\mathscr{R}/\mathbf{Z}_V$ is supported on the closed subset $(U - V)^-$ of X, which has dimension $<n$, since X is irreducible. So using (2.10) and the induction hypothesis, we know $H^i(X,\mathscr{R}/\mathbf{Z}_V) = 0$ for $i \geqslant n$. So by the long exact sequence of cohomology, we need only show vanishing for $\mathbf{Z}_V$.

Step 5. To complete the proof, we need only show that for any open subset $U \subseteq X$, we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_Y \to 0.$$

Now $\dim Y < \dim X$ since X is irreducible, so using (2.10) and the induction hypothesis, we have $H^i(X,\mathbf{Z}_Y) = 0$ for $i \geqslant n$. On the other hand, $\mathbf{Z}$ is flasque, since it is a constant sheaf on an irreducible space (II, Ex. 1.16a). Hence $H^i(X,\mathbf{Z}) = 0$ for $i > 0$ by (2.5). So from the long exact sequence of cohomology we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. q.e.d.

Historical Note: The derived functor cohomology which we defined in this section was introduced by Grothendieck [1]. It is the theory which is used in [EGA]. The use of sheaf cohomology in algebraic geometry started with Serre [3]. In that paper, and in the later paper [4], Serre used Čech cohomology for coherent sheaves on an algebraic variety with its Zariski topology. The equivalence of this theory with the derived functor theory follows from the "theorem of Leray" (Ex. 4.11). The same argument, using Cartan's "Theorem B" shows that the Čech cohomology of a coherent analytic sheaf on a complex analytic space is equal to the derived functor cohomology. Gunning and Rossi [1] use a cohomology theory computed by fine resolutions of a sheaf on a paracompact Hausdorff space. The equivalence of this theory with ours is shown by Godement [1, Thm. 4.7.1, p. 181 and Ex. 7.2.1, p. 263], who shows at the same time that both theories coincide with his theory which is defined by a canonical flasque resolution. Godement also shows [1, Thm. 5.10.1, p. 228] that on a paracompact Hausdorff space, his theory coincides with Čech cohomology. This provides a bridge to the standard topological theories with constant coefficients, as developed in the book of Spanier [1]. He shows that on a paracompact Hausdorff space, Čech cohomology and Alexander cohomology and singular cohomology all agree (see Spanier [1, pp. 314, 327, 334]).

The vanishing theorem (2.7) was proved by Serre [3] for coherent sheaves on algebraic curves and projective algebraic varieties, and later [5] for abstract algebraic varieties. It is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$.

EXERCISES

2.1. (a) Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P,Q be distinct closed points of X, and let $U = X - \{P,Q\}$. Show that $H^1(X,\mathbf{Z}_U) \neq 0$.

*(b) More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n + 1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X,\mathbf{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

2.2. Let $X = \mathbf{P}_k^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X,\mathcal{O}) = 0$ for all $i > 0$.

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H_Y^i(X,\cdot)$. They are the *cohomology groups* of X *with supports* in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathcal{F}') \to \Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{F}'') \to 0$$

is exact.

(c) Show that if $\mathcal{F}$ is flasque, then $H_Y^i(X,\mathcal{F}) = 0$ for all $i > 0$.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(X - Y,\mathcal{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H_Y^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F}|_U) \to \\ &\to H_Y^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(U,\mathcal{F}|_U) \to \\ &\to H_Y^2(X,\mathcal{F}) \to \ldots . \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H_Y^i(X,\mathcal{F}) \cong H_Y^i(V,\mathcal{F}|_V).$$

2.4. *Mayer Vietoris Sequence.* Let Y_1,Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\begin{aligned} \ldots &\to H^i_{Y_1 \cap Y_2}(X,\mathcal{F}) \to H^i_{Y_1}(X,\mathcal{F}) \oplus H^i_{Y_2}(X,\mathcal{F}) \to H^i_{Y_1 \cup Y_2}(X,\mathcal{F}) \to \\ &\to H^{i+1}_{Y_1 \cap Y_2}(X,\mathcal{F}) \to \ldots . \end{aligned}$$

2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j: X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all i, $\mathscr{F}$, we have
$$H^i_P(X,\mathscr{F}) = H^i_P(X_P,\mathscr{F}_P).$$

2.6. Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathscr{R} \subseteq \mathbf{Z}_U$, and for every map $f:\mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.
(a) Show that $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.
(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1,\mathscr{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \operatorname{Spec} A$ is a noetherian affine scheme, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathscr{F}$ of $\mathscr{O}_X$-modules. The key point is to show that if I is an injective A-module, then the sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is flasque. We begin with some algebraic preliminaries.

Proposition 3.1A (Krull's Theorem). *Let A be a noetherian ring, let $M \subseteq N$ be finitely generated A-modules, and let $\mathfrak{a}$ be an ideal of A. Then the $\mathfrak{a}$-adic topology on M is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n > 0$, there exists an $n' \geqslant n$ such that $\mathfrak{a}^n M \supseteq M \cap \mathfrak{a}^{n'} N$.*

PROOF. Atiyah–Macdonald [1, 10.11] or Zariski–Samuel [1, vol. II, Ch. VIII, Th. 4].

Recall (II, Ex. 5.6) that for any ring A, and any ideal $\mathfrak{a} \subseteq A$, and any A-module M, we have defined the submodule $\Gamma_\mathfrak{a}(M)$ to be $\{m \in M \mid \mathfrak{a}^n m = 0$ for some $n > 0\}$.

Lemma 3.2. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_\mathfrak{a}(I)$ is also an injective A-module.*

PROOF. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi:\mathfrak{b} \to J$, there exists a homomorphism $\psi: A \to J$ extending φ. (This is a well-known criterion for an injective module—Godement [1, I, 1.4.1]). Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some power of $\mathfrak{a}$, so there exists an $n > 0$ such that $\mathfrak{a}^n\varphi(\mathfrak{b}) = 0$, or equivalently, $\varphi(\mathfrak{a}^n\mathfrak{b}) = 0$. Now applying (3.1A) to the inclusion $\mathfrak{b} \subseteq A$, we find that there is an $n' \geqslant n$ such that $\mathfrak{a}^n\mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^{n'}$. Hence $\varphi(\mathfrak{b} \cap \mathfrak{a}^{n'}) = 0$, and so the map $\varphi:\mathfrak{b} \to J$ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$. Now we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] & A/\mathfrak{a}^{n'} \arrow[dr, dashed] \arrow[drr, dashed, "\psi'"] & & \\
\mathfrak{b} \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "\varphi"'] & \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'}) \arrow[u, hook] \arrow[r] & J \arrow[r] & I
\end{tikzcd}$$

Since I is injective, the composed map of $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$ to I extends to a map $\psi':A/\mathfrak{a}^{n'} \to I$. But the image of ψ' is annihilated by $\mathfrak{a}^{n'}$, so it is contained in J. Composing with the natural map $A \to A/\mathfrak{a}^{n'}$, we obtain the required map $\psi:A \to J$ extending φ.

Lemma 3.3. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

Proof. For each $i > 0$, let $\mathfrak{b}_i$ be the annihilator of f^i in A. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \ldots$, and since A is noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \ldots$. Now let $\theta:I \to I_f$ be the natural map, and let $x \in I_f$ be any element. Then by definition of localization, there is a $y \in I$ and an $n \geqslant 0$ such that $x = \theta(y)/f^n$. We define a map φ from the ideal (f^{n+r}) of A to I by sending f^{n+r} to $f^r y$. This is possible, because the annihilator of f^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$, and $\mathfrak{b}_r$ annihilates $f^r y$. Since I is injective, φ extends to a map $\psi:A \to I$. Let $\psi(1) = z$. Then $f^{n+r}z = f^r y$. But this implies that $\theta(z) = \theta(y)/f^n = x$. Hence θ is surjective.

Proposition 3.4. *Let I be an injective module over a noetherian ring A. Then the sheaf $\tilde{I}$ on $X =$ Spec A is flasque.*

Proof. We will use noetherian induction on $Y = (\operatorname{Supp} \tilde{I})^-$. See (II, Ex. 1.14) for the notion of support. If Y consists of a single closed point of X, then $\tilde{I}$ is a skyscraper sheaf (II, Ex. 1.17) which is obviously flasque.

In the general case, to show that $\tilde{I}$ is flasque, it will be sufficient to show, for any open set $U \subseteq X$, that $\Gamma(X,\tilde{I}) \to \Gamma(U,\tilde{I})$ is surjective. If $Y \cap U = \varnothing$, there is nothing to prove. If $Y \cap U \neq \varnothing$, we can find an $f \in A$ such that the open set $X_f = D(f)$ (II, §2) is contained in U and $X_f \cap Y \neq \varnothing$. Let $Z = X - X_f$, and consider the following diagram:

$$\begin{tikzcd}
\Gamma(X,\tilde{I}) \arrow[r] & \Gamma(U,\tilde{I}) \arrow[r] & \Gamma(X_f,\tilde{I}) \\
\Gamma_Z(X,\tilde{I}) \arrow[u, hook] \arrow[r] & \Gamma_Z(U,\tilde{I}), \arrow[u, hook] &
\end{tikzcd}$$

where Γ_Z denotes sections with support in Z (II, Ex. 1.20). Now given a section $s \in \Gamma(U,\tilde{I})$, we consider its image s' in $\Gamma(X_f,\tilde{I})$. But $\Gamma(X_f,\tilde{I}) = I_f$ (II, 5.1), so by (3.3), there is a $t \in I = \Gamma(X,\tilde{I})$ restricting to s'. Let t' be the restriction of t to $\Gamma(U,\tilde{I})$. Then $s - t'$ goes to 0 in $\Gamma(X_f,\tilde{I})$, so it has support in Z. Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X,\tilde{I}) \rightarrow \Gamma_Z(U,\tilde{I})$ is surjective.

Let $J = \Gamma_Z(X,\tilde{I})$. If $\mathfrak{a}$ is the ideal generated by f, then $J = \Gamma_{\mathfrak{a}}(I)$ (II, Ex. 5.6), so by (3.2), J is also an injective A-module. Furthermore, the support of $\tilde{J}$ is contained in $Y \cap Z$, which is strictly smaller than Y. Hence by our induction hypothesis, $\tilde{J}$ is flasque. Since $\Gamma(U,\tilde{J}) = \Gamma_Z(U,\tilde{I})$ (II, Ex. 5.6), we conclude that $\Gamma_Z(X,\tilde{I}) \rightarrow \Gamma_Z(U,\tilde{I})$ is surjective, as required.

Theorem 3.5. *Let $X = \operatorname{Spec} A$ be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathscr{F}$ on X, and for all $i > 0$, we have $H^i(X,\mathscr{F}) = 0$.*

PROOF. Given $\mathscr{F}$, let $M = \Gamma(X,\mathscr{F})$, and take an injective resolution $0 \rightarrow M \rightarrow I^{\cdot}$ of M in the category of A-modules. Then we obtain an exact sequence of sheaves $0 \rightarrow \tilde{M} \rightarrow \tilde{I}^{\cdot}$ on X. Now $\mathscr{F} = \tilde{M}$ (II, 5.5) and each $\tilde{I}^i$ is flasque by (3.4), so we can use this resolution of $\mathscr{F}$ to calculate cohomology (2.5.1). Applying the functor Γ, we recover the exact sequence of A-modules $0 \rightarrow M \rightarrow I^{\cdot}$. Hence $H^0(X,\mathscr{F}) = M$, and $H^i(X,\mathscr{F}) = 0$ for $i > 0$.

Remark 3.5.1. This result is also true without the noetherian hypothesis, but the proof is more difficult [EGA III, 1.3.1].

Corollary 3.6. *Let X be a noetherian scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $\mathscr{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathscr{G}$.*

PROOF. Cover X with a finite number of open affines $U_i = \operatorname{Spec} A_i$, and let $\mathscr{F}|_{U_i} = \tilde{M}_i$ for each i. Embed M_i in an injective A_i-module I_i. For each i, let $f: U_i \rightarrow X$ be the inclusion, and let $\mathscr{G} = \bigoplus f_*(\tilde{I}_i)$. For each i we have an injective map of sheaves $\mathscr{F}|_{U_i} \rightarrow \tilde{I}_i$. Hence we obtain a map $\mathscr{F} \rightarrow f_*(\tilde{I}_i)$. Taking the direct sum over i gives a map $\mathscr{F} \rightarrow \mathscr{G}$ which is clearly injective. On the other hand, for each i, $\tilde{I}_i$ is flasque (3.4) and quasi-coherent on U_i. Hence $f_*(\tilde{I}_i)$ is also flasque (II, Ex. 1.16d) and quasi-coherent (II, 5.8). Taking the direct sum of these, we see that $\mathscr{G}$ is flasque and quasi-coherent.

Theorem 3.7 (Serre [5]). *Let X be a noetherian scheme. Then the following conditions are equivalent:*

(i) *X is affine;*
(ii) *$H^i(X,\mathscr{F}) = 0$ for all $\mathscr{F}$ quasi-coherent and all $i > 0$;*
(iii) *$H^1(X,\mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.*

PROOF. (i) $\Rightarrow$ (ii) is (3.5). (ii) $\Rightarrow$ (iii) is trivial, so we have only to prove (iii) $\Rightarrow$ (i). We use the criterion of (II, Ex. 2.17). First we show that X can be covered by open affine subsets of the form X_f, with $f \in A = \Gamma(X,\mathcal{O}_X)$. Let P be a closed point of X, let U be an open affine neighborhood of P, and let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathscr{I}_{Y \cup \{P\}} \to \mathscr{I}_Y \to k(P) \to 0,$$

where $\mathscr{I}_Y$ and $\mathscr{I}_{Y \cup \{P\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{P\}$, respectively. The quotient is the skyscraper sheaf $k(P) = \mathcal{O}_P/\mathfrak{m}_P$ at P. Now from the exact sequence of cohomology, and hypothesis (iii), we get an exact sequence

$$\Gamma(X,\mathscr{I}_Y) \to \Gamma(X,k(P)) \to H^1(X,\mathscr{I}_{Y \cup \{P\}}) = 0.$$

So there is an element $f \in \Gamma(X,\mathscr{I}_Y)$ which goes to 1 in $k(P)$, i.e., $f_P \equiv 1 \pmod{\mathfrak{m}_P}$. Since $\mathscr{I}_Y \subseteq \mathcal{O}_X$, we can consider f as an element of A. Then by construction, we have $P \in X_f \subseteq U$. Furthermore, $X_f = U_{\bar{f}}$, where $\bar{f}$ is the image of f in $\Gamma(U,\mathcal{O}_U)$, so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f. By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \ldots, f_r \in A$.

Now by (II, Ex. 2.17), to show that X is affine, we need only verify that $f_1, \ldots, f_r$ generate the unit ideal in A. We use $f_1, \ldots, f_r$ to define a map $\alpha: \mathcal{O}_X^r \to \mathcal{O}_X$ by sending $\langle a_1, \ldots, a_r \rangle$ to $\sum f_i a_i$. Since the X_{f_i} cover X, this is a surjective map of sheaves. Let $\mathscr{F}$ be the kernel:

$$0 \to \mathscr{F} \to \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \to 0.$$

We filter $\mathscr{F}$ as follows:

$$\mathscr{F} = \mathscr{F} \cap \mathcal{O}_X^r \supseteq \mathscr{F} \cap \mathcal{O}_X^{r-1} \supseteq \ldots \supseteq \mathscr{F} \cap \mathcal{O}_X$$

for a suitable ordering of the factors of $\mathcal{O}_X^r$. Each of the quotients of this filtration is a coherent sheaf of ideals in $\mathcal{O}_X$. Thus using our hypothesis (iii) and the long exact sequence of cohomology, we climb up the filtration and deduce that $H^1(X,\mathscr{F}) = 0$. But then $\Gamma(X,\mathcal{O}_X^r) \xrightarrow{\alpha} \Gamma(X,\mathcal{O}_X)$ is surjective, which tells us that $f_1, \ldots, f_r$ generate the unit ideal in A. q.e.d.

Remark 3.7.1. This result is analogous to another theorem of Serre in complex analytic geometry, which characterizes Stein spaces by the vanishing of coherent analytic sheaf cohomology.

Exercises

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [*Hint*: Use (3.7), and for any coherent sheaf $\mathscr{F}$ on X, consider the filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \ldots$, where $\mathscr{N}$ is the sheaf of nilpotent elements on X.]

3.2. Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

(a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdot)$.

(b) Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X,\tilde{M}),$$

where $H^i_Y(X,\cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

(c) For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

3.4. *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A-module, then $\mathit{depth}_{\mathfrak{a}}\, M$ is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geqslant 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

(b) Show inductively, for M finitely generated, that for any $n \geqslant 0$, the following conditions are equivalent:

(i) $\operatorname{depth}_{\mathfrak{a}} M \geqslant n$;
(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i) $\operatorname{depth} \mathcal{O}_P \geqslant 2$;
(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

3.6. Let X be a noetherian scheme.

(a) Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

*(b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines . . .]

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X,\cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

3.7. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

(a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U,\tilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, x_2, \ldots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \ldots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

4 Čech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space X, with respect to a given open covering of X. We will prove that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in §2. The value of this result is that it gives a practical method for computing cohomology of quasi-coherent sheaves on a scheme.

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$ by $U_{i_0, \ldots, i_p}$.

Now let $\mathscr{F}$ be a sheaf of abelian groups on X. We define a complex $C^{\cdot}(\mathfrak{U}, \mathscr{F})$ of abelian groups as follows. For each $p \geqslant 0$, let

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \ldots < i_p} \mathscr{F}(U_{i_0, \ldots, i_p}).$$

Thus an element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is determined by giving an element

$$\alpha_{i_0, \ldots, i_p} \in \mathscr{F}(U_{i_0, \ldots, i_p}),$$

for each $(p + 1)$-tuple $i_0 < \ldots < i_p$ of elements of I. We define the coboundary map $d: C^p \to C^{p+1}$ by setting

$$(d\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}} \big|_{U_{i_0, \ldots, i_{p+1}}}.$$

Here the notation $\hat{i}_k$ means omit i_k. Then since $\alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}}$ is an element of $\mathscr{F}(U_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}})$, we restrict to $U_{i_0, \ldots, i_{p+1}}$ to get an element of $\mathscr{F}(U_{i_0, \ldots, i_{p+1}})$. One checks easily that $d^2 = 0$, so we have indeed defined a complex of abelian groups.

Remark 4.0.1. If $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$, it is sometimes convenient to have the symbol $\alpha_{i_0, \ldots, i_p}$ defined for *all* $(p + 1)$-tuples of elements of I. If there is a repeated index in the set $\{i_0, \ldots, i_p\}$, we define $\alpha_{i_0, \ldots, i_p} = 0$. If the indices are all distinct, we define $\alpha_{i_0, \ldots, i_p} = (-1)^\sigma \alpha_{\sigma i_0, \ldots, \sigma i_p}$, where σ is the permutation for which $\sigma i_0 < \ldots < \sigma i_p$. With these conventions, one can check that the formula given above for $d\alpha$ remains correct for any $(p + 2)$-tuple $i_0, \ldots, i_{p+1}$ of elements of I.

Definition. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathscr{F}$ on X, we define the pth *Čech cohomology group of* $\mathscr{F}$, *with respect to the covering* $\mathfrak{U}$, to be

$$\check{H}^p(\mathfrak{U},\mathscr{F}) = h^p(C^{\cdot}(\mathfrak{U},\mathscr{F})).$$

Caution 4.0.2. Keeping X and $\mathfrak{U}$ fixed, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves of abelian groups on X, we do *not* in general get a long exact sequence of Čech cohomology groups. In other words, the functors $\check{H}^p(\mathfrak{U},\cdot)$ do not form a δ-functor (§1). For example, if $\mathfrak{U}$ consists of the single open set X, then this results from the fact that the global section functor $\Gamma(X,\cdot)$ is not exact.

Example 4.0.3. To illustrate how well suited Čech cohomology is for computations, we will compute some examples. Let $X = \mathbf{P}^1_k$, let $\mathscr{F}$ be the sheaf of differentials Ω (II, §8), and let $\mathfrak{U}$ be the open covering by the two open sets $U = \mathbf{A}^1$ with affine coordinate x, and $V = \mathbf{A}^1$ with affine coordinate $y = 1/x$. Then the Čech complex has only two terms:

$$C^0 = \Gamma(U,\Omega) \times \Gamma(V,\Omega)$$
$$C^1 = \Gamma(U \cap V,\Omega).$$

Now

$$\Gamma(U,\Omega) = k[x]\,dx$$
$$\Gamma(V,\Omega) = k[y]\,dy$$
$$\Gamma(U \cap V,\Omega) = k\left[x,\frac{1}{x}\right] dx,$$

and the map $d: C^0 \to C^1$ is given by

$$x \mapsto x$$
$$y \mapsto \frac{1}{x}$$
$$dy \mapsto -\frac{1}{x^2}\,dx.$$

So ker d is the set of pairs $\langle f(x)\,dx, g(y)\,dy\rangle$ such that

$$f(x) = -\frac{1}{x^2}\,g\left(\frac{1}{x}\right).$$

This can happen only if $f = g = 0$, since one side is a polynomial in x and the other side is a polynomial in $1/x$ with no constant term. So $\check{H}^0(\mathfrak{U},\Omega) = 0$.

To compute H^1, note that the image of d is the set of all expressions

$$\left(f(x) + \frac{1}{x^2}\,g\left(\frac{1}{x}\right)\right) dx,$$

where f and g are polynomials. This gives the subvector space of $k[x,1/x]\,dx$ generated by all $x^n\,dx$, $n \in \mathbf{Z}$, $n \neq -1$. Therefore $\check{H}^1(\mathfrak{U},\Omega) \cong k$, generated by the image of $x^{-1}\,dx$.

Example 4.0.4. Let S^1 be the circle (in its usual topology), let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$, and let $\mathfrak{U}$ be the open covering by two connected open semi-circles U, V, which overlap at each end, so that $U \cap V$ consists of two small intervals. Then

$$C^0 = \Gamma(U,\mathbf{Z}) \times \Gamma(V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$
$$C^1 = \Gamma(U \cap V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$

and the map $d: C^0 \to C^1$ takes $\langle a,b\rangle$ to $\langle b-a, b-a\rangle$. Thus $\check{H}^0(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$ and $\check{H}^1(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$. Since we know this is the right answer (Ex. 2.7), this illustrates the general principle that Čech cohomology agrees with the usual cohomology provided the open covering is taken fine enough so that there is no cohomology on any of the open sets (Ex. 4.11).

Now we will study some properties of the Čech cohomology groups.

Lemma 4.1. *For any $X,\mathfrak{U},\mathscr{F}$ as above, we have $\check{H}^0(\mathfrak{U},\mathscr{F}) \cong \Gamma(X,\mathscr{F})$.*

PROOF. $\check{H}^0(\mathfrak{U},\mathscr{F}) = \ker(d: C^0(\mathfrak{U},\mathscr{F}) \to C^1(\mathfrak{U},\mathscr{F}))$. If $\alpha \in C^0$ is given by $\{\alpha_i \in \mathscr{F}(U_i)\}$, then for each $i < j$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. So $d\alpha = 0$ says the sections α_i and α_j agree on $U_i \cap U_j$. Thus it follows from the sheaf axioms that $\ker d = \Gamma(X,\mathscr{F})$.

Next we define a "sheafified" version of the Čech complex. For any open set $V \subseteq X$, let $f: V \to X$ denote the inclusion map. Now given $X,\mathfrak{U},\mathscr{F}$ as above, we construct a complex $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ of sheaves on X as follows. For each $p \geqslant 0$, let

$$\mathscr{C}^p(\mathfrak{U},\mathscr{F}) = \prod_{i_0 < \ldots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}}),$$

and define

$$d: \mathscr{C}^p \to \mathscr{C}^{p+1}$$

by the same formula as above. Note by construction that for each p we have $\Gamma(X,\mathscr{C}^p(\mathfrak{U},\mathscr{F})) = C^p(\mathfrak{U},\mathscr{F})$.

Lemma 4.2. *For any sheaf of abelian groups $\mathscr{F}$ on X, the complex $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ is a resolution of $\mathscr{F}$,* i.e., *there is a natural map $\varepsilon: \mathscr{F} \to \mathscr{C}^0$ such that the sequence of sheaves*

$$0 \to \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \to \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \to \ldots$$

is exact.

PROOF. We define $\varepsilon:\mathscr{F} \rightarrow \mathscr{C}^0$ by taking the product of the natural maps $\mathscr{F} \rightarrow f_*(\mathscr{F}|_{U_i})$ for $i \in I$. Then the exactness at the first step follows from the sheaf axioms for $\mathscr{F}$.

To show the exactness of the complex $\mathscr{C}^\cdot$ for $p \geqslant 1$, it is enough to check exactness on the stalks. So let $x \in X$, and suppose $x \in U_j$. For each $p \geqslant 1$, we define a map

$$k:\mathscr{C}^p(\mathfrak{U},\mathscr{F})_x \rightarrow \mathscr{C}^{p-1}(\mathfrak{U},\mathscr{F})_x$$

as follows. Given $\alpha_x \in \mathscr{C}^p(\mathfrak{U},\mathscr{F})_x$, it is represented by a section $\alpha \in \Gamma(V,\mathscr{C}^p(\mathfrak{U},\mathscr{F}))$ over a neighborhood V of x, which we may choose so small that $V \subseteq U_j$. Now for any p-tuple $i_0 < \ldots < i_{p-1}$, we set

$$(k\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}},$$

using the notational convention of (4.0.1). This makes sense because $V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{j,i_0,\ldots,i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the required map k. Now one checks that for any $p \geqslant 1$, $\alpha \in \mathscr{C}_x^p$,

$$(dk + kd)(\alpha) = \alpha.$$

Thus k is a homotopy operator for the complex $\mathscr{C}_x^\cdot$, showing that the identity map is homotopic to the zero map. It follows (§1) that the cohomology groups $h^p(\mathscr{C}_x^\cdot)$ of this complex are 0 for $p \geqslant 1$.

Proposition 4.3. *Let X be a topological space, let $\mathfrak{U}$ be an open covering, and let $\mathscr{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$.*

PROOF. Consider the resolution $0 \rightarrow \mathscr{F} \rightarrow \mathscr{C}^\cdot(\mathfrak{U},\mathscr{F})$ given by (4.2). Since $\mathscr{F}$ is flasque, the sheaves $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are flasque for each $p \geqslant 0$. Indeed, for any $i_0,\ldots,i_p$, $\mathscr{F}|_{U_{i_0,\ldots,i_p}}$ is a flasque sheaf on $U_{i_0,\ldots,i_p}$; f_* preserves flasque sheaves (II, Ex. 1.16d), and a product of flasque sheaves is flasque. So by (2.5.1) we can use this resolution to compute the usual cohomology groups of $\mathscr{F}$. But $\mathscr{F}$ is flasque, so $H^p(X,\mathscr{F}) = 0$ for $p > 0$ by (2.5). On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X,\mathscr{C}^\cdot(\mathfrak{U},\mathscr{F}))) = \check{H}^p(\mathfrak{U},\mathscr{F}).$$

So we conclude that $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$ for $p > 0$.

Lemma 4.4. *Let X be a topological space, and $\mathfrak{U}$ an open covering. Then for each $p \geqslant 0$ there is a natural map, functorial in $\mathscr{F}$,*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \rightarrow H^p(X,\mathscr{F}).$$

PROOF. Let $0 \rightarrow \mathscr{F} \rightarrow \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{F}$ in $\mathfrak{Ab}(X)$. Comparing with the resolution $0 \rightarrow \mathscr{F} \rightarrow \mathscr{C}^\cdot(\mathfrak{U},\mathscr{F})$ of (4.2), it follows from a general result on complexes (Hilton and Stammbach [1, IV, 4.4]) that there is a morphism of complexes $\mathscr{C}^\cdot(\mathfrak{U},\mathscr{F}) \rightarrow \mathscr{I}^\cdot$, inducing the identity map on $\mathscr{F}$, and unique up to homotopy. Applying the functors $\Gamma(X,\cdot)$ and h^p, we get the required map.

Theorem 4.5. *Let X be a noetherian separated scheme, let $\mathfrak{U}$ be an open affine cover of X, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for all $p \geqslant 0$, the natural maps of* (4.4) *give isomorphisms*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^p(X,\mathscr{F}).$$

PROOF. For $p = 0$ we have an isomorphism by (4.1). For the general case, embed $\mathscr{F}$ in a flasque, quasi-coherent sheaf $\mathscr{G}$ (3.6), and let $\mathscr{R}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

For each $i_0 < \ldots < i_p$, the open set $U_{i_0,\ldots,i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme (II, Ex. 4.3). Since $\mathscr{F}$ is quasi-coherent, we therefore have an exact sequence

$$0 \to \mathscr{F}(U_{i_0,\ldots,i_p}) \to \mathscr{G}(U_{i_0,\ldots,i_p}) \to \mathscr{R}(U_{i_0,\ldots,i_p}) \to 0$$

of abelian groups, by (3.5) or (II, 5.6). Taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^{\cdot}(\mathfrak{U},\mathscr{F}) \to C^{\cdot}(\mathfrak{U},\mathscr{G}) \to C^{\cdot}(\mathfrak{U},\mathscr{R}) \to 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. Since $\mathscr{G}$ is flasque, its Čech cohomology vanishes for $p > 0$ by (4.3), so we have an exact sequence

$$0 \to \check{H}^0(\mathfrak{U},\mathscr{F}) \to \check{H}^0(\mathfrak{U},\mathscr{G}) \to \check{H}^0(\mathfrak{U},\mathscr{R}) \to \check{H}^1(\mathfrak{U},\mathscr{F}) \to 0$$

and isomorphisms

$$\check{H}^p(\mathfrak{U},\mathscr{R}) \xrightarrow{\sim} \check{H}^{p+1}(\mathfrak{U},\mathscr{F})$$

for each $p \geqslant 1$. Now comparing with the long exact sequence of usual cohomology for the above short exact sequence, using the case $p = 0$, and (2.5), we conclude that the natural map

$$\check{H}^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is an isomorphism. But $\mathscr{R}$ is also quasi-coherent (II, 5.7), so we obtain the result for all p by induction.

EXERCISES

4.1. Let $f:X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,

$$H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F}).$$

[*Hint*: Use (II, 5.8).]

4.2. Prove Chevalley's theorem: Let $f:X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f:X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha:\mathscr{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta: f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y. [*Hint*: Apply $\mathscr{H}om(\cdot,\mathscr{F})$ to α and use (II, Ex. 5.17e).]

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider ker β and coker β, and use noetherian induction on Y.

4.3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U,\mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j | i,j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j\in J}$, together with a map $\lambda: J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U},\mathscr{F}) \to \check{H}^i(\mathfrak{V},\mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U},\mathscr{F}).$$

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U},\mathscr{F}) \to H^i(X,\mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is an isomorphism. [*Hint*: Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$, so that we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

Define a complex $D^{\cdot}(\mathfrak{U})$ by

$$0 \to C^{\cdot}(\mathfrak{U},\mathscr{F}) \to C^{\cdot}(\mathfrak{U},\mathscr{G}) \to D^{\cdot}(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D^{\cdot}(\mathfrak{U}) \to C^{\cdot}(\mathfrak{U},\mathscr{R}),$$

and see what happens under refinement.]

4.5. For any ringed space $(X,\mathcal{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves (II, §6). Show that Pic $X \cong H^1(X,\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U,\mathcal{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathcal{L}$ on X, cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i:\mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U},\mathcal{O}_X^*)$. Now use (Ex. 4.4).]

4.6. Let $(X,\mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X,\mathcal{O}_X/\mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathcal{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X,\mathcal{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X,\mathcal{I}) \to \ldots .$$

4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0,x_1,x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1,0,0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U,\mathcal{O}_X) \oplus \Gamma(V,\mathcal{O}_X) \to \Gamma(U \cap V,\mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X,\mathcal{O}_X) = 1,$$

$$\dim H^1(X,\mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\operatorname{cd}(X)$, to be the least integer n such that $H^i(X,\mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\operatorname{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\operatorname{cd}(X) \leqslant \dim X$.

(a) In the definition of $\operatorname{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

(c) Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\operatorname{cd}(X) \leqslant r$.

*(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude (independently of (2.7)) that $\operatorname{cd}(X) \leqslant \dim X$.

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\operatorname{cd}(X - Y) \leqslant r - 1$.

4.9. Let $X = \operatorname{Spec} k[x_1,x_2,x_3,x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure

$\bar{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.]

***4.10.** Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II, §8). [*Hint*: Use (II, Ex. 8.6) and (4.5).]

4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathcal{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

5 The Cohomology of Projective Space

In this section we make explicit calculations of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine covering. These explicit calculations form the basis for various general results about cohomology of coherent sheaves on projective varieties.

Let A be a noetherian ring, let $S = A[x_0, \ldots, x_r]$, and let $X = \operatorname{Proj} S$ be the projective space $\mathbf{P}_A^r$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (II, §5). For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ (see II, §5).

Theorem 5.1. *Let A be a noetherian ring, and let $X = \mathbf{P}_A^r$, with $r \geqslant 1$. Then:*

(a) *the natural map $S \rightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;*

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ *for* $0 < i < r$ *and all* $n \in \mathbf{Z}$;

(c) $H^r(X, \mathcal{O}_X(-r - 1)) \cong A$;

(d) *The natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n - r - 1)) \rightarrow H^r(X, \mathcal{O}_X(-r - 1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbf{Z}$.

Proof. Let $\mathcal{F}$ be the quasi-coherent sheaf $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)$. Since cohomology commutes with arbitrary direct sums on a noetherian topological space (2.9.1), the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology of the sheaves $\mathcal{O}(n)$. So we will compute the cohomology of $\mathcal{F}$, and keep track of the grading by n, so that we can sort out the pieces at the end. Note that all the cohomology groups in question have a natural structure of A-module (2.6.1).

For each $i = 0, \ldots, r$, let U_i be the open set $D_+(x_i)$. Then each U_i is an open affine subset of X, and the U_i cover X, so we can compute the cohomology of $\mathscr{F}$ by using Čech cohomology for the covering $\mathfrak{U} = (U_i)$, by (4.5). For any set of indices $i_0, \ldots, i_p$, the open set $U_{i_0, \ldots, i_p}$ is just $D_+(x_{i_0} \cdots x_{i_p})$, so by (II, 5.11) we have

$$\mathscr{F}(U_{i_0, \ldots, i_p}) \cong S_{x_{i_0} \cdots x_{i_p}},$$

the localization of S with respect to the element $x_{i_0} \cdots x_{i_p}$. Furthermore, the grading on $\mathscr{F}$ corresponds to the natural grading of $S_{x_{i_0} \cdots x_{i_p}}$ under this isomorphism. Thus the Čech complex of $\mathscr{F}$ is given by

$$C^{\cdot}(\mathfrak{U},\mathscr{F}): \prod S_{x_{i_0}} \to \prod S_{x_{i_0} x_{i_1}} \to \ldots \to S_{x_0 \cdots x_r},$$

and the modules all have a natural grading compatible with the grading on $\mathscr{F}$.

Now $H^0(X,\mathscr{F})$ is the kernel of the first map, which is just S, as we have seen earlier (II, 5.13). This proves (a).

Next we consider $H^r(X,\mathscr{F})$. It is the cokernel of the last map in the Čech complex, which is

$$d^{r-1}: \prod_k S_{x_0 \cdots \hat{x}_k \cdots x_r} \to S_{x_0 \cdots x_r}.$$

We think of $S_{x_0 \cdots x_r}$ as a free A-module with basis $x_0^{l_0} \cdots x_r^{l_r}$, with $l_i \in \mathbf{Z}$. The image of d^{r-1} is the free submodule generated by those basis elements for which at least one $l_i \geqslant 0$. Thus $H^r(X,\mathscr{F})$ is a free A-module with basis consisting of the "negative" monomials

$$\{x_0^{l_0} \cdots x_r^{l_r} | l_i < 0 \text{ for each } i\}.$$

Furthermore the grading is given by $\sum l_i$. There is only one such monomial of degree $-r-1$, namely $x_0^{-1} \cdots x_r^{-1}$, so we see that $H^r(X,\mathcal{O}_X(-r-1))$ is a free A-module of rank 1. This proves (c).

To prove (d), first note that if $n < 0$, then $H^0(X,\mathcal{O}_X(n)) = 0$ by (a), and $H^r(X,\mathcal{O}_X(-n-r-1)) = 0$ by what we have just seen, since in that case $-n-r-1 > -r-1$, and there are no negative monomials of that degree. So the statement is trivial for $n < 0$. For $n \geqslant 0$, $H^0(X,\mathcal{O}_X(n))$ has a basis consisting of the usual monomials of degree n, i.e., $\{x_0^{m_0} \cdots x_r^{m_r} | m_i \geqslant 0$ and $\sum m_i = n\}$. The natural pairing with $H^r(X,\mathcal{O}_X(-n-r-1))$ into $H^r(X,\mathcal{O}_X(-r-1))$ is determined by

$$(x_0^{m_0} \cdots x_r^{m_r}) \cdot (x_0^{l_0} \cdots x_r^{l_r}) = x_0^{m_0+l_0} \cdots x_r^{m_r+l_r},$$

where $\sum l_i = -n-r-1$, and the object on the right becomes 0 if any $m_i + l_i \geqslant 0$. So it is clear that we have a perfect pairing, under which $x_0^{-m_0-1} \cdots x_r^{-m_r-1}$ is the dual basis element of $x_0^{m_0} \cdots x_r^{m_r}$.

It remains to prove statement (b), which we will do by induction on r. If $r = 1$ there is nothing to prove, so let $r > 1$. If we localize the complex $C^{\cdot}(\mathfrak{U},\mathscr{F})$ with respect to x_r, as graded S-modules, we get the Čech complex for the sheaf $\mathscr{F}|_{U_r}$ on the space U_r, with respect to the open affine covering $\{U_i \cap U_r | i = 0,\dots,r\}$. By (4.5), this complex gives the cohomology of $\mathscr{F}|_{U_r}$ on U_r, which is 0 for $i > 0$ by (3.5). Since localization is an exact functor, we conclude that $H^i(X,\mathscr{F})_{x_r} = 0$ for $i > 0$. In other words, every element of $H^i(X,\mathscr{F})$, for $i > 0$, is annihilated by some power of x_r.

To complete the proof of (b), we will show that for $0 < i < r$, multiplication by x_r induces a bijective map of $H^i(X,\mathscr{F})$ into itself. Then it will follow that this module is 0.

Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0.$$

This gives the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

on X, where H is the hyperplane $x_r = 0$. Twisting by all $n \in \mathbf{Z}$ and taking the direct sum, we have

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0,$$

where $\mathscr{F}_H = \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_H(n)$. Taking cohomology, we get a long exact sequence

$$\dots \to H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F}) \to H^i(X,\mathscr{F}_H) \to \dots.$$

Considered as graded S-modules, $H^i(X,\mathscr{F}(-1))$ is just $H^i(X,\mathscr{F})$ shifted one place, and the map $H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ of the exact sequence is multiplication by x_r.

Now H is isomorphic to $\mathbf{P}_A^{r-1}$, and $H^i(X,\mathscr{F}_H) = H^i(H,\bigoplus \mathcal{O}_H(n))$ by (2.10). So we can apply our induction hypothesis to $\mathscr{F}_H$, and find that $H^i(X,\mathscr{F}_H) = 0$ for $0 < i < r - 1$. Furthermore, for $i = 0$ we have an exact sequence

$$0 \to H^0(X,\mathscr{F}(-1)) \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{F}_H) \to 0$$

by (a), since $H^0(X,\mathscr{F}_H)$ is just $S/(x_r)$. At the other end of the exact sequence we have

$$0 \to H^{r-1}(X,\mathscr{F}_H) \xrightarrow{\delta} H^r(X,\mathscr{F}(-1)) \xrightarrow{x_r} H^r(X,\mathscr{F}) \to 0.$$

Indeed, we have described $H^r(X,\mathscr{F})$ above as the free A-module with basis formed by the negative monomials in $x_0,\dots,x_r$. So it is clear that x_r is surjective. On the other hand, the kernel of x_r is the free submodule generated by those negative monomials $x_0^{l_0} \cdots x_r^{l_r}$ with $l_r = -1$. Since $H^{r-1}(X,\mathscr{F}_H)$ is the free A-module with basis consisting of the negative monomials in $x_0,\dots,x_{r-1}$, and δ is division by x_r, the sequence is exact. In particular, δ is injective.

Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $x_r: H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ is bijective for $0 < i < r$, as required. q.e.d.

Theorem 5.2 (Serre [3]). *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over* Spec A. *Let $\mathscr{F}$ be a coherent sheaf on X. Then:*

(a) *for each $i \geqslant 0$, $H^i(X,\mathscr{F})$ is a finitely generated A-module;*

(b) *there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X,\mathscr{F}(n)) = 0$.*

PROOF. Since $\mathcal{O}_X(1)$ is a very ample sheaf on X over Spec A, there is a closed immersion $i: X \to \mathbf{P}_A^r$ of schemes over A, for some r, such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^r}(1)$—cf. (II, 5.16.1). If $\mathscr{F}$ is coherent on X, then $i_*\mathscr{F}$ is coherent on $\mathbf{P}_A^r$ (II, Ex. 5.5), and the cohomology is the same (2.10). Thus we reduce to the case $X = \mathbf{P}_A^r$.

For $X = \mathbf{P}_A^r$, we observe that (a) and (b) are true for any sheaf of the form $\mathcal{O}_X(q)$, $q \in \mathbf{Z}$. This follows immediately from the explicit calculations (5.1). Hence the same is true for any finite direct sum of such sheaves.

To prove the theorem for arbitrary coherent sheaves, we use descending induction on i. For $i > r$, we have $H^i(X,\mathscr{F}) = 0$, since X can be covered by $r + 1$ open affines (Ex. 4.8), so the result is trivial in this case.

In general, given a coherent sheaf $\mathscr{F}$ on X, we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E}$, which is a finite direct sum of sheaves $\mathcal{O}(q_i)$, for various integers q_i (II, 5.18). Let $\mathscr{R}$ be the kernel,

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Then $\mathscr{R}$ is also coherent. We get an exact sequence of A-modules

$$\ldots \to H^i(X,\mathscr{E}) \to H^i(X,\mathscr{F}) \to H^{i+1}(X,\mathscr{R}) \to \ldots.$$

Now the module on the left is finitely generated because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$, as remarked above. The module on the right is finitely generated by the induction hypothesis. Since A is a noetherian ring, we conclude that the one in the middle is also finitely generated. This proves (a).

To prove (b), we twist and again write down a piece of the long exact sequence

$$\ldots \to H^i(X,\mathscr{E}(n)) \to H^i(X,\mathscr{F}(n)) \to H^{i+1}(X,\mathscr{R}(n)) \to \ldots.$$

Now for $n \gg 0$, the module on the left vanishes because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$. The module on the right also vanishes for $n \gg 0$ because of the induction hypothesis. Hence $H^i(X,\mathscr{F}(n)) = 0$ for $n \gg 0$. Note since there are only finitely many i involved in statement (b), namely $0 < i \leqslant r$, it is sufficient to determine n_0 separately for each i. This proves (b).

Remark 5.2.1. As a special case of (a), we see that for any coherent sheaf $\mathscr{F}$ on X, $\Gamma(X,\mathscr{F})$ is a finitely generated A-module. This generalizes, and gives another proof of (II, 5.19).

As an application of these results, we give a cohomological criterion for an invertible sheaf to be ample (II, §7).

Proposition 5.3. *Let A be a noetherian ring, and let X be a proper scheme over* Spec A. *Let $\mathscr{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*
(ii) *For each coherent sheaf $\mathscr{F}$ on X, there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X,\mathscr{F} \otimes \mathscr{L}^n) = 0$.*

PROOF. (i) $\Rightarrow$ (ii). If $\mathscr{L}$ is ample on X, then for some $m > 0$, $\mathscr{L}^m$ is very ample on X over Spec A, by (II, 7.6). Since X is proper over Spec A, it is necessarily projective (II, 5.16.1). Now applying (5.2) to each of the sheaves $\mathscr{F},\mathscr{F} \otimes \mathscr{L},\mathscr{F} \otimes \mathscr{L}^2,\ldots,\mathscr{F} \otimes \mathscr{L}^{m-1}$ gives (ii). Cf. (II, 7.5) for a similar technique of proof.

(ii) $\Rightarrow$ (i). To show that $\mathscr{L}$ is ample, we will show that for any coherent sheaf $\mathscr{F}$ on X, there is an integer n_0 such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for all $n \geqslant n_0$. This is the definition of ampleness (II, §7).

Given $\mathscr{F}$, let P be a closed point of X, and let $\mathscr{I}_P$ be the ideal sheaf of the closed subset $\{P\}$. Then there is an exact sequence

$$0 \rightarrow \mathscr{I}_P\mathscr{F} \rightarrow \mathscr{F} \rightarrow \mathscr{F} \otimes k(P) \rightarrow 0,$$

where $k(P)$ is the skyscraper sheaf $\mathscr{O}_X/\mathscr{I}_P$. Tensoring with $\mathscr{L}^n$, we get

$$0 \rightarrow \mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n \rightarrow \mathscr{F} \otimes \mathscr{L}^n \rightarrow \mathscr{F} \otimes \mathscr{L}^n \otimes k(P) \rightarrow 0.$$

Now by our hypothesis (ii), there is an n_0 such that $H^1(X,\mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n \geqslant n_0$. Therefore

$$\Gamma(X,\mathscr{F} \otimes \mathscr{L}^n) \rightarrow \Gamma(X,\mathscr{F} \otimes \mathscr{L}^n \otimes k(P))$$

is surjective for all $n \geqslant n_0$. It follows from Nakayama's lemma over the local ring $\mathscr{O}_P$, that the stalk of $\mathscr{F} \otimes \mathscr{L}^n$ at P is generated by global sections. Since it is a coherent sheaf, we conclude that for each $n \geqslant n_0$, there is an open neighborhood U of P, depending on n, such that the global sections of $\mathscr{F} \otimes \mathscr{L}^n$ generate the sheaf at every point of U.

In particular, taking $\mathscr{F} = \mathscr{O}_X$, we find there is an integer $n_1 > 0$ and an open neighborhood V of P such that $\mathscr{L}^{n_1}$ is generated by global sections over V. On the other hand, for each $r = 0,1,\ldots,n_1 - 1$, the above argument gives a neighborhood U_r of P such that $\mathscr{F} \otimes \mathscr{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_P = V \cap U_0 \cap \ldots \cap U_{n_1-1}.$$

Then over U_P, all of the sheaves $\mathscr{F} \otimes \mathscr{L}^n$, for $n \geqslant n_0$, are generated by global sections. Indeed, any such sheaf can be written as a tensor product

$$(\mathscr{F} \otimes \mathscr{L}^{n_0+r}) \otimes (\mathscr{L}^{n_1})^m$$

for suitable $0 \leqslant r < n_1$ and $m \geqslant 0$.

Now cover X by a finite number of the open sets U_P, for various closed points P, and let the new n_0 be the maximum of the n_0 corresponding to those points P. Then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections over all of X, for all $n \geqslant n_0$. q.e.d.

Exercises

5.1. Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X,\mathscr{F}).$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

5.2. (a) Let X be a projective scheme over a field k, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbf{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathscr{O}_X(1)$. [*Hints*: Use induction on dim Supp $\mathscr{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.]$$

(b) Now let $X = \mathbf{P}_k^r$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0,\ldots,x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

5.3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathscr{O}_X) - 1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X,\mathscr{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X,\mathscr{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X,\mathscr{O}_X).$$

[*Hint*: Use (I, 3.4).]

(b) If X is a closed subvariety of $\mathbf{P}_k^r$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P: K(X) \to \mathbf{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

(b) Now let $X = \mathbf{P}_k^r$. For each $i = 0,1,\ldots,r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0,\ldots,r\}$, and
(2) the map $P: K(X) \to \mathbf{Q}[z]$ is injective.

[*Hint*: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

5.5. Let k be a field, let $X = \mathbf{P}_k^r$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then:

(a) for all $n \in \mathbf{Z}$, the natural map

$$H^0(X,\mathcal{O}_X(n)) \to H^0(Y,\mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y,\mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y,\mathcal{O}_Y)$.

[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

5.6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}_k^3$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that Pic $Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.

(a) Use the special cases $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

(1) if $|a - b| \leqslant 1$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(2) if $a,b < 0$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(3) If $a \leqslant -2$, then $H^1(Q,\mathcal{O}_Q(a,0)) \neq 0$.

(b) Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a,b), with $a,b > 0$, then Y is connected;
(2) now assume k is algebraically closed. Then for any $a,b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).
(3) an irreducible nonsingular curve Y of type (a,b), $a,b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1,3)$, which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab - a - b + 1$. [*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.]

5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i: Y \to X$ is the inclusion.

(b) $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathscr{O}_{X_{\text{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathscr{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f: X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X. [*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f: \tilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

5.9. *A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbf{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since $\dim X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\ldots \to H^1(X, \omega) \to \operatorname{Pic} X' \to \operatorname{Pic} X \xrightarrow{\delta} H^2(X, \omega) \to \ldots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathscr{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since $\operatorname{char} k = 0$, you need only show that $\delta(\mathscr{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X, \omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k.

Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X,\Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathscr{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathscr{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

5.10. Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \ldots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\Gamma(X,\mathscr{F}^1(n)) \to \Gamma(X,\mathscr{F}^2(n)) \to \ldots \to \Gamma(X,\mathscr{F}^r(n))$$

is exact.

6 Ext Groups and Sheaves

In this section we develop the properties of Ext groups and sheaves, which we will need for the duality theorem. We work on a ringed space $(X,\mathcal{O}_X)$, and all sheaves will be sheaves of $\mathcal{O}_X$-modules.

If $\mathscr{F}$ and $\mathscr{G}$ are $\mathcal{O}_X$-modules, we denote by $\operatorname{Hom}(\mathscr{F},\mathscr{G})$ the group of $\mathcal{O}_X$-module homomorphisms, and by $\mathscr{H}om(\mathscr{F},\mathscr{G})$ the sheaf Hom (II, §5). If necessary, we put a subscript X to indicate which space we are on: $\operatorname{Hom}_X(\mathscr{F},\mathscr{G})$. For fixed $\mathscr{F}$, $\operatorname{Hom}(\mathscr{F},\cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, and $\mathscr{H}om(\mathscr{F},\cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Since $\mathfrak{Mod}(X)$ has enough injectives (2.2) we can make the following definition.

Definition. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We define the functors $\operatorname{Ext}^i(\mathscr{F},\cdot)$ as the right derived functors of $\operatorname{Hom}(\mathscr{F},\cdot)$, and $\mathscr{E}xt^i(\mathscr{F},\cdot)$ as the right derived functors of $\mathscr{H}om(\mathscr{F},\cdot)$.

Consequently, according to the general properties of derived functors (1.1A) we have $\operatorname{Ext}^0 = \operatorname{Hom}$, a long exact sequence for a short exact sequence in the second variable, $\operatorname{Ext}^i(\mathscr{F},\mathscr{G}) = 0$ for $i > 0$, $\mathscr{G}$ injective in $\mathfrak{Mod}(X)$, and ditto for the $\mathscr{E}xt$ sheaves.

Lemma 6.1. *If $\mathscr{I}$ is an injective object of $\mathfrak{Mod}(X)$, then for any open subset $U \subseteq X$, $\mathscr{I}|_U$ is an injective object of $\mathfrak{Mod}(U)$.*

Proof. Let $j: U \to X$ be the inclusion map. Then given an inclusion $\mathscr{F} \subseteq \mathscr{G}$ in $\mathfrak{Mod}(U)$, and given a map $\mathscr{F} \to \mathscr{I}|_U$, we get an inclusion $j_!\mathscr{F} \subseteq j_!\mathscr{G}$, and a map $j_!\mathscr{F} \to j_!(\mathscr{I}|_U)$, where $j_!$ is extension by zero (II, Ex. 1.19). But $j_!(\mathscr{I}|_U)$ is a subsheaf of $\mathscr{I}$, so we have a map $j_!\mathscr{F} \to \mathscr{I}$. Since $\mathscr{I}$ is injective in $\mathfrak{Mod}(X)$, this extends to a map of $j_!\mathscr{G}$ to $\mathscr{I}$. Restricting to U gives the required map of $\mathscr{G}$ to $\mathscr{I}|_U$.

Proposition 6.2. *For any open subset $U \subseteq X$ we have*

$$\mathscr{E}xt^i_X(\mathscr{F},\mathscr{G})|_U \cong \mathscr{E}xt^i_U(\mathscr{F}|_U,\mathscr{G}|_U).$$

PROOF. We use (1.3A). Both sides give δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(U)$. They agree for $i = 0$, both sides vanish for $i > 0$ and $\mathscr{G}$ injective, by (6.1), so they are equal.

Proposition 6.3. *For any $\mathscr{G} \in \mathfrak{Mod}(X)$, we have:*
(a) $\mathscr{E}xt^0(\mathscr{O}_X,\mathscr{G}) = \mathscr{G}$;
(b) $\mathscr{E}xt^i(\mathscr{O}_X,\mathscr{G}) = 0$ *for* $i > 0$;
(c) $\operatorname{Ext}^i(\mathscr{O}_X,\mathscr{G}) \cong H^i(X,\mathscr{G})$ *for all* $i \geqslant 0$.

PROOF. The functor $\mathscr{H}om(\mathscr{O}_X,\cdot)$ is the identity functor, so its derived functors are 0 for $i > 0$. This proves (a) and (b). The functors $\operatorname{Hom}(\mathscr{O}_X,\cdot)$ and $\Gamma(X,\cdot)$ are equal, so their derived functors (as functors from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$) are the same. Then use (2.6).

Proposition 6.4. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence in $\mathfrak{Mod}(X)$, then for any $\mathscr{G}$ we have a long exact sequence*

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathscr{F}'',\mathscr{G}) \to \operatorname{Hom}(\mathscr{F},\mathscr{G}) \to \operatorname{Hom}(\mathscr{F}',\mathscr{G}) \to \\ \to \operatorname{Ext}^1(\mathscr{F}'',\mathscr{G}) \to \operatorname{Ext}^1(\mathscr{F},\mathscr{G}) \to \ldots, \end{aligned}$$

and similarly for the $\mathscr{E}xt$ sheaves.

PROOF. Let $0 \to \mathscr{G} \to \mathscr{I}^{\cdot}$ be an injective resolution of $\mathscr{G}$. For any injective sheaf $\mathscr{I}$, the functor $\operatorname{Hom}(\cdot,\mathscr{I})$ is exact, so we get a short exact sequence of complexes

$$0 \to \operatorname{Hom}(\mathscr{F}'',\mathscr{I}^{\cdot}) \to \operatorname{Hom}(\mathscr{F},\mathscr{I}^{\cdot}) \to \operatorname{Hom}(\mathscr{F}',\mathscr{I}^{\cdot}) \to 0.$$

Taking the associated long exact sequence of cohomology groups h^i gives the sequence of Ext^i.

Similarly, using (6.1) we see that $\mathscr{H}om(\cdot,\mathscr{I})$ is an exact functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Thus the same argument gives the exact sequence of $\mathscr{E}xt^i$.

Proposition 6.5. *Suppose there is an exact sequence*

$$\ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

in $\mathfrak{Mod}(X)$, where the $\mathscr{L}_i$ are locally free sheaves of finite rank (in this case we say $\mathscr{L}_{\cdot}$ is a locally free resolution of $\mathscr{F}$). Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$ we have

$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \cong h^i(\mathscr{H}om(\mathscr{L}_{\cdot},\mathscr{G})).$$

PROOF. Both sides are δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. For $i = 0$ they are equal, because then $\mathscr{H}om(\cdot,\mathscr{G})$ is contravariant and left exact. Both sides vanish for $i > 0$ and $\mathscr{G}$ injective, because then $\mathscr{H}om(\cdot,\mathscr{G})$ is exact. So by (1.3A) they are equal.

Example 6.5.1. If X is a scheme, which is quasi-projective over Spec A, where A is a noetherian ring, then by (II, 5.18), any coherent sheaf $\mathscr{F}$ on X is a quotient of a locally free sheaf. Thus any coherent sheaf on X has a locally free resolution $\mathscr{L}. \to \mathscr{F} \to 0$. So (6.5) tells us that we can calculate $\mathscr{E}xt$ by taking locally free resolutions in the first variable.

Caution 6.5.2. The results (6.4) and (6.5) do *not* imply that $\mathscr{E}xt$ can be construed as a derived functor in its first variable. In fact, we cannot even define the right derived functors of Hom or $\mathscr{H}om$ in the first variable because the category $\mathfrak{Mod}(X)$ does not have enough projectives (Ex. 6.2). However, see (Ex. 6.4) for a universal property.

Lemma 6.6. *If $\mathscr{L} \in \mathfrak{Mod}(X)$ is locally free of finite rank, and $\mathscr{I} \in \mathfrak{Mod}(X)$ is injective, then $\mathscr{L} \otimes \mathscr{I}$ is also injective.*

PROOF. We must show that the functor $\operatorname{Hom}(\cdot,\mathscr{L} \otimes \mathscr{I})$ is exact. But it is the same as the functor $\operatorname{Hom}(\cdot \otimes \mathscr{L}^{\vee},\mathscr{I})$ (II, Ex. 5.1), which is exact because $\cdot \otimes \mathscr{L}^{\vee}$ is exact, and $\mathscr{I}$ is injective.

Proposition 6.7. *Let $\mathscr{L}$ be a locally free sheaf of finite rank, and let $\mathscr{L}^{\vee} = \mathscr{H}om(\mathscr{L},\mathscr{O}_X)$ be its dual. Then for any $\mathscr{F},\mathscr{G} \in \mathfrak{Mod}(X)$ we have*

$$\operatorname{Ext}^i(\mathscr{F} \otimes \mathscr{L},\mathscr{G}) \cong \operatorname{Ext}^i(\mathscr{F},\mathscr{L}^{\vee} \otimes \mathscr{G}),$$

and for the sheaf $\mathscr{E}xt$ we have

$$\mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{L},\mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F},\mathscr{L}^{\vee} \otimes \mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \otimes \mathscr{L}^{\vee}.$$

PROOF. The case $i = 0$ follows from (II, Ex. 5.1). For the general case, note that all of them are δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ (respectively, $\mathfrak{Mod}(X)$), since tensoring with $\mathscr{L}^{\vee}$ is an exact functor. For $i > 0$ and $\mathscr{G}$ injective they all vanish, by (6.6), so by (1.3A) they are equal.

Next we will give some properties which are more particular to the case of schemes.

Proposition 6.8. *Let X be a noetherian scheme, let $\mathscr{F}$ be a coherent sheaf on X, let $\mathscr{G}$ be any $\mathscr{O}_X$-module, and let $x \in X$ be a point. Then we have*

$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G})_x \cong \operatorname{Ext}^i_{\mathscr{O}_x}(\mathscr{F}_x,\mathscr{G}_x)$$

for any $i \geqslant 0$, where the right-hand side is Ext *over the local ring $\mathscr{O}_x$.*

PROOF. Of course, Ext over a ring A is defined as the right derived functor of $\operatorname{Hom}_A(M,\cdot)$ for any A-module M, considered as a functor from $\mathfrak{Mod}(A)$ to $\mathfrak{Mod}(A)$. Or, by considering a one-point space with the ring A attached, it becomes a special case of the Ext of a ringed space defined above.

Our question is local, by (6.2), so we may assume that X is affine. Then $\mathscr{F}$ has a locally free (or even a free) resolution $\mathscr{L}. \rightarrow \mathscr{F} \rightarrow 0$, which on the stalks at x gives a free resolution $(\mathscr{L}.)_x \rightarrow \mathscr{F}_x \rightarrow 0$. So by (6.5) we can calculate both sides by these resolutions. Since $\mathscr{H}om(\mathscr{L},\mathscr{G})_x = \operatorname{Hom}_{\mathscr{O}_x}(\mathscr{L}_x,\mathscr{G}_x)$ for a locally free sheaf $\mathscr{L}$, and since the stalk functor is exact, we get the equality of Ext's.

Note that even the case $i = 0$ is not true without some special hypothesis on $\mathscr{F}$, such as $\mathscr{F}$ coherent.

Proposition 6.9. *Let X be a projective scheme over a noetherian ring A, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf, and let $\mathscr{F},\mathscr{G}$ be coherent sheaves on X. Then there is an integer $n_0 > 0$, depending on $\mathscr{F}$, $\mathscr{G}$, and i, such that for every $n \geqslant n_0$ we have*

$$\operatorname{Ext}^i(\mathscr{F},\mathscr{G}(n)) \cong \Gamma(X,\mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n))).$$

PROOF. If $i = 0$, this is true for any $\mathscr{F},\mathscr{G},n$. If $\mathscr{F} = \mathscr{O}_X$, then the left-hand side is $H^i(X,\mathscr{G}(n))$ by (6.3). So for $n \gg 0$ and $i > 0$ it is 0 by (5.2). On the other hand, the right-hand side is always 0 for $i > 0$ by (6.3), so we have the result for $\mathscr{F} = \mathscr{O}_X$.

If $\mathscr{F}$ is a locally free sheaf, we reduce to the case $\mathscr{F} = \mathscr{O}_X$ by (6.7).

Finally, if $\mathscr{F}$ is an arbitrary coherent sheaf, write it as a quotient of a locally free sheaf $\mathscr{E}$ (II, 5.18), and let $\mathscr{R}$ be the kernel:

$$0 \rightarrow \mathscr{R} \rightarrow \mathscr{E} \rightarrow \mathscr{F} \rightarrow 0.$$

Since $\mathscr{E}$ is locally free, by the earlier results, for $n \gg 0$, we have an exact sequence

$$0 \rightarrow \operatorname{Hom}(\mathscr{F},\mathscr{G}(n)) \rightarrow \operatorname{Hom}(\mathscr{E},\mathscr{G}(n)) \rightarrow \operatorname{Hom}(\mathscr{R},\mathscr{G}(n)) \rightarrow \operatorname{Ext}^1(\mathscr{F},\mathscr{G}(n)) \rightarrow 0$$

and isomorphisms, for all $i > 0$

$$\operatorname{Ext}^i(\mathscr{R},\mathscr{G}(n)) \xrightarrow{\sim} \operatorname{Ext}^{i+1}(\mathscr{F},\mathscr{G}(n)),$$

and similarly for the sheaf $\mathscr{H}om$ and $\mathscr{E}xt$. Now by (Ex. 5.10), the sequence of global sections of the sheaf sequence is exact after twisting a little more, so from the case $i = 0$, using (6.7), we get the case $i = 1$ for $\mathscr{F}$. But $\mathscr{R}$ is also coherent, so by induction we get the general result.

Remark 6.9.1. More generally, on any ringed space X, the relation between the global Ext and the sheaf $\mathscr{E}xt$ can be expressed by a spectral sequence (see Grothendieck [1] or Godement [1, II, 7.3.3]).

Now, for future reference, we recall the notion of projective dimension of a module over a ring. Let A be a ring, and let M be an A-module. A *projective resolution* of M is a complex $L.$ of projective A-modules, such that

$$\ldots \rightarrow L_2 \rightarrow L_1 \rightarrow L_0 \rightarrow M \rightarrow 0$$

is exact. If $L_i = 0$ for $i > n$, and $L_n \neq 0$, we say it has *length* n. Then we define the *projective dimension* of M, denoted $\mathrm{pd}(M)$, to be the least length of a projective resolution of M (or $+\infty$ if there is no finite projective resolution).

Proposition 6.10A. *Let A be a ring, and M an A-module. Then:*
(a) *M is projective if and only if $\mathrm{Ext}^1(M,N) = 0$ for all A-modules N;*
(b) *$\mathrm{pd}(M) \leqslant n$ if and only if $\mathrm{Ext}^i(M,N) = 0$ for all $i > n$ and all A-modules N.*

PROOF. Matsumura [2, pp. 127–128].

Proposition 6.11A. *If A is a regular local ring, then:*
(a) *for every M, $\mathrm{pd}(M) \leqslant \dim A$;*
(b) *If $k = A/\mathfrak{m}$, then $\mathrm{pd}(k) = \dim A$.*

PROOF. Matsumura [2, Th. 42, p. 131].

Proposition 6.12A. *Let A be a regular local ring of dimension n, and let M be a finitely generated A-module. Then we have*

$$\mathrm{pd}\, M + \mathrm{depth}\, M = n.$$

PROOF. Matsumura [2, p. 113, Ex. 4] or Serre [11, IVD, Prop. 21].

EXERCISES

6.1. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}',\mathscr{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}'$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\mathrm{Hom}(\mathscr{F}'',\cdot)$, in particular the map

$$\delta : \mathrm{Hom}(\mathscr{F}'',\mathscr{F}'') \to \mathrm{Ext}^1(\mathscr{F}'',\mathscr{F}'),$$

and let $\xi \in \mathrm{Ext}^1(\mathscr{F}'',\mathscr{F}')$ be $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and elements of the group $\mathrm{Ext}^1(\mathscr{F}'',\mathscr{F}')$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

6.2. Let $X = \mathbf{P}^1_k$, with k an infinite field.
(a) Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j : V \to X$ is the inclusion.]
(b) Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathscr{L} \to \mathscr{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathscr{L}$ is an invertible sheaf on X.]

6.3. Let X be a noetherian scheme, and let $\mathscr{F},\mathscr{G} \in \mathfrak{Mod}(X)$.
(a) If $\mathscr{F},\mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is coherent, for all $i \geqslant 0$.
(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

6.4. Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees*. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathscr{E}xt^i(\cdot,\mathscr{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is coeffaceable (§1) for $i > 0$.]

6.5. Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\operatorname{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:
(a) $\mathscr{F}$ is locally free $\Leftrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;
(b) $\operatorname{hd}(\mathscr{F}) \leqslant n \Leftrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;
(c) $\operatorname{hd}(\mathscr{F}) = \sup_x \operatorname{pd}_{\mathscr{O}_x} \mathscr{F}_x$.

6.6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.
(a) M is projective if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > 0$. [*Hint*: Use (6.11A) and descending induction on i to show that $\operatorname{Ext}^i(M,N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p. 129]).]
(b) Use (a) to show that for any n, $\operatorname{pd} M \leqslant n$ if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > n$.

6.7. Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\tilde{M},\tilde{N}) \cong \operatorname{Ext}^i_A(M,N)$$

and

$$\mathscr{E}xt^i_X(\tilde{M},\tilde{N}) \cong \operatorname{Ext}^i_A(M,N)^{\sim}.$$

6.8. Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).
(a) First show that open sets of the form X_s, for various $s \in \Gamma(X,\mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X. [*Hint*: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathscr{L},s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathscr{O}_x$, $f \notin \mathscr{O}_\zeta$. Let $D = (f)_\infty$, and let $\mathscr{L} = \mathscr{L}(D)$, $s \in \Gamma(X,\mathscr{L}(D))$ correspond to D (II, §6).]
(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

6.9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is *regular* if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution
$$0 \to \mathscr{E}_n \to \ldots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum(-1)^i\gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

6.10. *Duality for a Finite Flat Morphism.*

(a) Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathscr{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathscr{O}_X$-module, hence corresponds to a quasi-coherent $\mathscr{O}_X$-module, which we call $f^!\mathscr{G}$ (II, Ex. 5.17e).

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism
$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

(c) For each $i \geqslant 0$, there is a natural map
$$\varphi_i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$
[*Hint*: First construct a map
$$\operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}).$$
Then compose with a suitable map from $f_*f^!\mathscr{G}$ to $\mathscr{G}$.]

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathscr{O}_X$ is locally free on Y (this is equivalent to saying f flat—see §9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y. [*Hints*: First do $i = 0$. Then do $\mathscr{F} = \mathscr{O}_X$, using (Ex. 4.1). Then do $\mathscr{F}$ locally free. Do the general case by induction on i, writing $\mathscr{F}$ as a quotient of a locally free sheaf.]

7 The Serre Duality Theorem

In this section we prove the Serre duality theorem for the cohomology of coherent sheaves on a projective scheme. First we do the case of projective space itself, which follows easily from the explicit calculations of §5. Then on an arbitrary projective scheme X, we show that there is a coherent sheaf ω_X, which plays a role in duality theory similar to the canonical sheaf of a nonsingular variety. In particular, if X is Cohen–Macaulay, it gives a duality theorem just like the one on projective space. Finally, if X is a nonsingular variety over an algebraically closed field, we show that the dualizing sheaf ω_X° coincides with the canonical sheaf ω_X. At the end of the section, we mention the connection between duality and residues of differential forms.

Let k be a field, let $X = \mathbf{P}_k^n$ be the n-dimensional projective space over k, and let $\omega_X = \wedge^n \Omega_{X/k}$ be the canonical sheaf on X (II, §8).

Theorem 7.1 (Duality for $\mathbf{P}_k^n$). *Let $X = \mathbf{P}_k^n$ over a field k. Then:*

(a) $H^n(X,\omega_X) \cong k$. *Fix one such isomorphism;*

(b) *for any coherent sheaf $\mathscr{F}$ on X, the natural pairing*

$$\operatorname{Hom}(\mathscr{F},\omega) \times H^n(X,\mathscr{F}) \to H^n(X,\omega) \cong k$$

is a perfect pairing of finite-dimensional vector spaces over k;

(c) *for every $i \geqslant 0$ there is a natural functorial isomorphism*

$$\operatorname{Ext}^i(\mathscr{F},\omega) \xrightarrow{\sim} H^{n-i}(X,\mathscr{F})',$$

where $'$ denotes the dual vector space, which for $i = 0$ is the one induced by the pairing of (b).

PROOF.

(a) It follows from (II, 8.13) that $\omega_X \cong \mathcal{O}_X(-n-1)$ (see II, 8.20.1). Thus (a) follows from (5.1c).

(b) A homomorphism of $\mathscr{F}$ to ω induces a map of cohomology groups $H^n(X,\mathscr{F}) \to H^n(X,\omega)$. This gives the natural pairing. If $\mathscr{F} \cong \mathcal{O}(q)$ for some $q \in \mathbf{Z}$, then $\operatorname{Hom}(\mathscr{F},\omega) \cong H^0(X,\omega(-q))$, so the result follows from (5.1d). Hence (b) holds also for a finite direct sum of sheaves of the form $\mathcal{O}(q_i)$. If $\mathscr{F}$ is an arbitrary coherent sheaf, we can write it as a cokernel $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ of a map of sheaves $\mathscr{E}_i$, each $\mathscr{E}_i$ being a direct sum of sheaves $\mathcal{O}(q_i)$. Now $\operatorname{Hom}(\cdot,\omega)$ and $H^n(X,\cdot)'$ are both left-exact contravariant functors, so by the 5-lemma we get an isomorphism $\operatorname{Hom}(\mathscr{F},\omega) \xrightarrow{\sim} H^n(X,\mathscr{F})'$.

(c) Both sides are contravariant δ-functors, for $\mathscr{F} \in \mathfrak{Coh}(X)$, indexed by $i \geqslant 0$. For $i = 0$ we have an isomorphism by (b). Thus to show they are isomorphic, by (1.3A), it will be sufficient to show both sides are coeffaceable for $i > 0$. Given $\mathscr{F}$ coherent, it follows from (II, 5.18) and its proof that we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E} = \bigoplus_{i=1}^N \mathcal{O}(-q)$, with $q \gg 0$. Then $\operatorname{Ext}^i(\mathscr{E},\omega) = \bigoplus H^i(X,\omega(q)) = 0$ for $i > 0$ by (5.1). On the other hand, $H^{n-i}(X,\mathscr{E})' = \bigoplus H^{n-i}(X,\mathcal{O}(-q))'$, which is 0 for $i > 0$, $q > 0$, as we see again from (5.1) by inspection. Thus both sides are coeffaceable for $i > 0$, so the δ-functors are universal, hence isomorphic.

Remark 7.1.1. One may ask, why bother phrasing (7.1) with the sheaf ω_X, rather than simply writing $\mathcal{O}_X(-n-1)$, which is what we use in the proof? One reason is that this is the form of the theorem which generalizes well. But a more intrinsic reason is that when written this way, the isomorphism of (a) can be made independent of the choice of basis of $\mathbf{P}^n$, hence stable under automorphisms of $\mathbf{P}^n$. Thus it is truly a *natural* isomorphism. To do this, consider the Čech cocycle

$$\alpha = \frac{x_0^n}{x_1 \cdots x_n}\, d\left(\frac{x_1}{x_0}\right) \wedge \ldots \wedge d\left(\frac{x_n}{x_0}\right)$$

in $C^n(\mathfrak{U},\omega)$, where $\mathfrak{U}$ is the standard open covering. Then one can show that α determines a generator of $H^n(X,\omega)$, which is stable under change of variables.

To generalize (7.1) to other schemes, we take properties (a) and (b) as our guide, and make the following definition.

Definition. Let X be a proper scheme of dimension n over a field k. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace* morphism $t:H^n(X,\omega_X^\circ) \to k$, such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing

$$\operatorname{Hom}(\mathscr{F},\omega_X^\circ) \times H^n(X,\mathscr{F}) \to H^n(X,\omega_X^\circ)$$

followed by t gives an *isomorphism*

$$\operatorname{Hom}(\mathscr{F},\omega_X^\circ) \xrightarrow{\sim} H^n(X,\mathscr{F})'.$$

Proposition 7.2. *Let X be a proper scheme over k. Then a dualizing sheaf for X, if it exists, is unique. More precisely, if ω° is one, with its trace map t, and if ω',t' is another, then there is a unique isomorphism $\varphi:\omega^\circ \xrightarrow{\sim} \omega'$ such that $t = t' \circ H^n(\varphi)$.*

PROOF. Since ω' is dualizing, we get an isomorphism $\operatorname{Hom}(\omega^\circ,\omega') \cong H^n(\omega^\circ)'$. So there is a unique morphism $\varphi:\omega^\circ \to \omega'$ corresponding to the element $t \in H^n(\omega^\circ)'$, i.e., such that $t' \circ H^n(\varphi) = t$. Similarly, using the fact that ω° is dualizing, there is a unique morphism $\psi:\omega' \to \omega^\circ$ such that $t \circ H^n(\psi) = t'$. It follows that $t \circ H^n(\psi \circ \varphi) = t$. But again since ω° is dualizing, this implies that $\psi \circ \varphi$ is the identity map of ω°. Similarly $\varphi \circ \psi$ is the identity map of ω', so φ is an isomorphism. (This proof is a special case of the uniqueness of an object representing a functor (see Grothendieck [EGA I, new ed., Ch. 0, §1]). For by definition (ω°,t) represents the functor $\mathscr{F} \mapsto H^n(X,\mathscr{F})'$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(k)$.)

The question of existence of dualizing sheaves is more difficult. In fact they exist for any X proper over k, but we will prove the existence here only for projective schemes. First we need some preliminary results.

Lemma 7.3. *Let X be a closed subscheme of codimension r of $P = \mathbf{P}_k^N$. Then $\mathscr{E}xt_P^i(\mathscr{O}_X,\omega_P) = 0$ for all $i < r$.*

PROOF. For any i, the sheaf $\mathscr{F}^i = \mathscr{E}xt_P^i(\mathscr{O}_X,\omega_P)$ is a coherent sheaf on P (Ex. 6.3), so after twisting by a suitably large integer q, it will be generated by global sections (II, 5.17). Thus to show $\mathscr{F}^i$ is zero, it will be sufficient to show that $\Gamma(P,\mathscr{F}^i(q)) = 0$ for all $q \gg 0$. But by (6.7) and (6.9) we have

$$\Gamma(P,\mathscr{F}^i(q)) \cong \operatorname{Ext}_P^i(\mathscr{O}_X,\omega_P(q))$$

for $q \gg 0$. On the other hand, by (7.1) this last Ext group is dual to $H^{N-i}(P,\mathcal{O}_X(-q))$. For $i < r$, $N - i > \dim X$, so this group is 0 by (2.7) or (Ex. 4.8d).

Lemma 7.4. *With the same hypotheses as* (7.3), *let* $\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X,\omega_P)$. *Then for any* $\mathcal{O}_X$*-module* $\mathcal{F}$, *there is a functorial isomorphism*

$$\operatorname{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \operatorname{Ext}_P^r(\mathcal{F},\omega_P).$$

PROOF. Let $0 \to \omega_P \to \mathcal{I}^\cdot$ be an injective resolution of ω_P in $\mathfrak{Mod}(P)$. Then we calculate $\operatorname{Ext}_P^i(\mathcal{F},\omega_P)$ as the cohomology groups h^i of the complex $\operatorname{Hom}_P(\mathcal{F},\mathcal{I}^\cdot)$. But since $\mathcal{F}$ is an $\mathcal{O}_X$-module, any morphism $\mathcal{F} \to \mathcal{I}^i$ factors through $\mathcal{J}^i = \mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^i)$. Thus we have

$$\operatorname{Ext}_P^i(\mathcal{F},\omega_P) = h^i(\operatorname{Hom}_X(\mathcal{F},\mathcal{J}^\cdot)).$$

Now each $\mathcal{J}^i$ is an *injective* $\mathcal{O}_X$-module. Indeed, for $\mathcal{F} \in \mathfrak{Mod}(X)$, $\operatorname{Hom}_X(\mathcal{F},\mathcal{J}^i) = \operatorname{Hom}_P(\mathcal{F},\mathcal{I}^i)$, so $\operatorname{Hom}_X(\cdot,\mathcal{J}^i)$ is an exact functor. Furthermore, by (7.3) we have $h^i(\mathcal{J}^\cdot) = 0$ for $i < r$, so the complex $\mathcal{J}^\cdot$ is exact up to the rth step. Since the $\mathcal{J}^i$ are injective, it is actually split exact up to the rth step. This implies that we can write the complex as a direct sum of two injective complexes, $\mathcal{J}^\cdot = \mathcal{J}_1^\cdot \oplus \mathcal{J}_2^\cdot$, where $\mathcal{J}_1^\cdot$ is in degrees $0 \leqslant i \leqslant r$ and is exact, and $\mathcal{J}_2^\cdot$ is in degrees $i \geqslant r$. It follows that $\omega_X^\circ = \ker(d^r:\mathcal{J}_2^r \to \mathcal{J}_2^{r+1})$, and that for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\operatorname{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \operatorname{Ext}_P^r(\mathcal{F},\omega_P).$$

(It also follows that $\operatorname{Ext}_P^i(\mathcal{F},\omega_P) = 0$ for $i < r$, which we won't need.)

Proposition 7.5. *Let X be a projective scheme over a field k. Then X has a dualizing sheaf.*

PROOF. Embed X as a closed subscheme of $P = \mathbf{P}_k^N$ for some N, let r be its codimension, and let $\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X,\omega_P)$. Then by (7.4) we have an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\operatorname{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \operatorname{Ext}_P^r(\mathcal{F},\omega_P).$$

On the other hand, when $\mathcal{F}$ is coherent, the duality theorem for P (7.1) gives an isomorphism

$$\operatorname{Ext}_P^r(\mathcal{F},\omega_P) \cong H^{N-r}(P,\mathcal{F})'.$$

But $N - r = n$, the dimension of X, and $\mathcal{F}$ is a sheaf on X, so we obtain a functorial isomorphism, for $\mathcal{F} \in \mathfrak{Coh}(X)$,

$$\operatorname{Hom}_X(\mathcal{F},\omega_X) \cong H^n(X,\mathcal{F})'.$$

In particular, taking $\mathcal{F} = \omega_X^\circ$, the element $1 \in \operatorname{Hom}(\omega_X^\circ,\omega_X^\circ)$ gives us a homomorphism $t:H^n(X,\omega_X^\circ) \to k$, which we take as our trace map. Then it is clear by functoriality that (ω_X°,t) is a dualizing sheaf for X.

Now we can prove the duality theorem for a projective scheme X. Recall that a scheme is *Cohen–Macaulay* if all of its local rings are Cohen–Macaulay rings (II, §8).

Theorem 7.6 (Duality for a Projective Scheme). *Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω_X° be a dualizing sheaf on X, and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:*

(a) *for all $i \geqslant 0$ and $\mathscr{F}$ coherent on X, there are natural functorial maps*

$$\theta^i : \operatorname{Ext}^i(\mathscr{F},\omega_X^\circ) \rightarrow H^{n-i}(X,\mathscr{F})',$$

such that θ^0 is the map given in the definition of dualizing sheaf above;

(b) *the following conditions are equivalent:*

(i) *X is Cohen–Macaulay and equidimensional* (i.e., *all irreducible components have the same dimension*);

(ii) *for any $\mathscr{F}$ locally free on X, we have $H^i(X,\mathscr{F}(-q)) = 0$ for $i < n$ and $q \gg 0$;*

(iii) *the maps θ^i of* (a) *are isomorphisms for all $i \geqslant 0$ and all $\mathscr{F}$ coherent on X.*

PROOF.

(a) As in the proof of (7.1c), we can write any coherent sheaf $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$, with $q \gg 0$. Then $\operatorname{Ext}^i(\mathscr{E},\omega_X^\circ) \cong \bigoplus H^i(X,\omega_X^\circ(q))$, which is 0 for $i > 0$ and $q \gg 0$ by (5.2). Thus the functor $\operatorname{Ext}^i(\cdot,\omega_X^\circ)$ is coeffaceable for $i > 0$, so we have a universal contravariant δ-functor by (1.3A). On the right-hand side we have a contravariant δ-functor, indexed by $i \geqslant 0$, so there is a unique morphism of δ-functors (θ^i) reducing to the given θ^0 for $i = 0$.

(b) (i) $\Rightarrow$ (ii). Embed X as a closed subscheme of $P = \mathbf{P}_k^N$. Then for any $\mathscr{F}$ locally free on X, and any closed point $x \in X$, we have depth $\mathscr{F}_x = n$, since X is Cohen–Macaulay and equidimensional of dimension n. Let $A = \mathcal{O}_{P,x}$ be the local ring of x on P. Then A is a regular local ring of dimension N. (Since k is algebraically closed, x is rational over k, so the fact that A is regular can be seen directly. Or it follows from the fact that P is a nonsingular variety over k (II, §8).) Now depth $\mathscr{F}_x$ is the same, whether calculated over $\mathcal{O}_{X,x}$ or over A. Thus we conclude from (6.12A) that $\operatorname{pd}_A \mathscr{F}_x = N - n$. Therefore by (6.8) and (6.10A) we have

$$\mathscr{E}xt_P^i(\mathscr{F},\cdot) = 0$$

for $i > N - n$.

On the other hand, using (7.1), we find that $H^i(X,\mathscr{F}(-q))$ is dual to $\operatorname{Ext}_P^{N-i}(\mathscr{F},\omega_P(q))$. For $q \gg 0$, this Ext is isomorphic to $\Gamma(P,\mathscr{E}xt_P^{N-i}(\mathscr{F},\omega_P(q)))$ by (6.9). But this is 0 for $N - i > N - n$, as we have just seen. In other words, $H^i(X,\mathscr{F}(-q)) = 0$ for $i < n$ and $q \gg 0$.

(ii) $\Rightarrow$ (i). Running the above argument backwards, using condition (ii) with $\mathscr{F} = \mathcal{O}_X$, we find that

$$\mathscr{E}xt_P^i(\mathcal{O}_X,\omega_P) = 0$$

for $i > N - n$. This implies that over a local ring $A = \mathcal{O}_{P,x}$ as above, we have $\operatorname{Ext}^i_A(\mathcal{O}_{X,x},A) = 0$ for all $i > N - n$. Therefore by (Ex. 6.6) we have $\operatorname{pd}_A \mathcal{O}_{X,x} \leqslant N - n$, and so by (6.12A), depth $\mathcal{O}_{X,x} \geqslant n$. But since $\dim X = n$, we must have equality for every closed point of X. This shows, using (II, 8.21Ab), that X is Cohen–Macaulay and equidimensional.

(ii) $\Rightarrow$ (iii). Since we have already seen that $\operatorname{Ext}^i(\cdot,\omega_X))$ is a universal contravariant δ-functor, to show that the θ^i are isomorphisms, it will be sufficient to show that the δ-functor $(H^{n-i}(X,\cdot)')$ is universal also. For this it suffices by (1.3A) to show that $H^{n-i}(X,\cdot)'$ is coeffaceable for $i > 0$. So given a coherent sheaf $\mathscr{F}$, write $\mathscr{F}$ as a quotient of $\mathscr{E} = \bigoplus \mathcal{O}(-q)$ with $q \gg 0$. Then $H^{n-i}(X,\mathscr{E})' = 0$ for $i > 0$ by (ii), so the functor is coeffaceable.

(iii) $\Rightarrow$ (ii). If θ^i is an isomorphism, then for any $\mathscr{F}$ locally free, we have

$$H^i(X,\mathscr{F}(-q)) \cong \operatorname{Ext}^{n-i}(\mathscr{F}(-q),\omega_X)'.$$

But this Ext is isomorphic to $H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X(q))$ by (6.3) and (6.7), so it is 0 for $n - i > 0$ and $q \gg 0$ by (5.2). q.e.d.

Remark 7.6.1. In particular, if X is nonsingular over k, or more generally a local complete intersection, then X is Cohen–Macaulay (II, 8.21A) and (II, 8.23), so the θ^i are isomorphisms. In these two cases, one can show directly (cf. proof of (7.11) below) that $\operatorname{pd}_P \mathcal{O}_X = N - n$, and thus avoid use of the algebraic results (6.12A) and (Ex. 6.6).

Corollary 7.7. *Let X be a projective Cohen–Macaulay scheme of equidimension n over k. Then for any locally free sheaf $\mathscr{F}$ on X there are natural isomorphisms*

$$H^i(X,\mathscr{F}) \cong H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X)'.$$

PROOF. Use (6.3) and (6.7).

Corollary 7.8 (Lemma of Enriques–Severi–Zariski (Zariski [4])). *Let X be a normal projective scheme of dimension $\geqslant 2$. Then for any locally free sheaf $\mathscr{F}$ on X,*

$$H^1(X,\mathscr{F}(-q)) = 0$$

for $q \gg 0$.

PROOF. Since X is normal of dimension $\geqslant 2$, we have depth $\mathscr{F}_x \geqslant 2$ for every closed point $x \in X$ by (II, 8.22A). So the result follows by the same method as the proof of (i) $\Rightarrow$ (ii) in (7.6b).

Corollary 7.9. *Let X be an integral, normal projective variety of dimension $\geqslant 2$ over an algebraically closed field k. Let Y be a closed subset of codimension 1 which is the support of an effective ample divisor. Then Y is connected.*

PROOF. By (II, 7.6) we may assume that Y is the support of a very ample divisor D. Let $\mathcal{O}(1)$ be the corresponding very ample invertible sheaf. For each $q > 0$, let Y_q be the closed subscheme supported on Y corresponding to the divisor qD (II, 6.17.1). Then we have an exact sequence (II, 6.18)

$$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to \mathcal{O}_{Y_q} \to 0.$$

Taking cohomology and applying (7.8), we find that for $q \gg 0$,

$$H^0(X,\mathcal{O}_X) \to H^0(Y,\mathcal{O}_{Y_q}) \to 0$$

is surjective. But $H^0(X,\mathcal{O}_X) = k$ (I, 3.4a), and $H^0(Y,\mathcal{O}_{Y_q})$ contains k, so we conclude that $H^0(Y,\mathcal{O}_{Y_q}) = k$. Hence Y is connected. (If not, there would be at least one copy of k for each connected component.)

Remark 7.9.1. This implies that the schemes $H \cap X$ mentioned in Bertini's theorem (II, 8.18) are in fact *irreducible* and nonsingular when $\dim X \geqslant 2$. Indeed, they are connected by (7.9). On the other hand, they are regular by (II, 8.18). Hence the local rings are all integral domains, so we could not have two irreducible components meeting at a point.

Now that we have proved the duality theorem (7.6), our next task is to give more information about the dualizing sheaf ω_X° in some special cases. Again we need some algebraic preliminaries.

Let A be a ring, and let $f_1,\ldots,f_r \in A$. We define the *Koszul complex* $K.(f_1,\ldots,f_r)$ as follows: K_1 is a free A-module of rank r with basis $e_1,\ldots,e_r$. For each $p = 0,\ldots,r$, $K_p = \wedge^p K_1$. We define the boundary map $d: K_p \to K_{p-1}$ by its action on the basis vectors:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum (-1)^{j-1} f_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_p}.$$

Thus $K.(f_1,\ldots,f_r)$ is a (homological) complex of A-modules. If M is any A-module, we set $K.(f_1,\ldots,f_r; M) = K.(f_1,\ldots,f_r) \otimes_A M$.

Proposition 7.10A. *Let A be a ring, $f_1,\ldots,f_r \in A$, and let M be an A-module. If the f_i form a regular sequence for M, then*

$$h_i(K.(f_1,\ldots,f_r; M)) = 0 \qquad \text{for } i > 0$$

and

$$h_0(K.(f_1,\ldots,f_r; M)) \cong M/(f_1,\ldots,f_r)M.$$

PROOF. Matsumura [2, Th. 43, p. 135] or Serre [11, IV.A].

Theorem 7.11. *Let X be a closed subscheme of $P = \mathbf{P}_k^N$ which is a local complete intersection of codimension r. Let $\mathscr{I}$ be the ideal sheaf of X. Then $\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^{\vee}$. In particular, ω_X° is an invertible sheaf on X.*

PROOF. We have to calculate $\omega_X^\circ = \mathscr{E}xt_P^r(\mathscr{O}_X,\omega_P)$. Let U be an open affine subset over which $\mathscr{I}$ can be generated by r elements $f_1,\dots,f_r \in A = \Gamma(U,\mathscr{O}_U)$ and let $x \in X \cap U$ be a point corresponding to an ideal $\mathfrak{m} \subseteq A$. Because X has codimension r and $A_{\mathfrak{m}}$ is Cohen-Macaulay, $f_1,\dots,f_r$ form a regular sequence for $A_{\mathfrak{m}}$ (II, 8.21A). Therefore the localized Koszul complex $K.(f_1,\dots,f_r; A_{\mathfrak{m}})$ gives a free resolution of $A_{\mathfrak{m}}/(f_1,\dots,f_r)A_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$, so replacing U by a smaller neighborhood of x if necessary, $K.(f_1,\dots,f_r)$ gives a free resolution of $A/(f_1,\dots,f_r)$ over A. Sheafifying gives a free resolution $K.(f_1,\dots,f_r;\mathscr{O}_P)$ of $\mathscr{O}_X$ over U with which we can calculate $\mathscr{E}xt_P^r(\mathscr{O}_X,\omega_P)$ (6.5). We get

$$h^r(\mathscr{H}om(K.(f_1,\dots,f_r;\mathscr{O}_P),\omega_P)) \cong \omega_P/(f_1,\dots,f_r)\omega_P.$$

In other words,

$$\mathscr{E}xt_P^r(\mathscr{O}_X,\omega_P) \cong \omega_P \otimes \mathscr{O}_X$$

over U. However, this isomorphism depends on the choice of basis $f_1,\dots,f_r$ for $\mathscr{I}$. If $g_i = \sum c_{ij}f_j$, $i = 1,\dots,r$, is another basis, then the exterior powers of the matrix $\|c_{ij}\|$ give an isomorphism of Koszul complexes. In particular, we have a factor of $\det|c_{ij}|$ on K_r, so our isomorphism of $\mathscr{E}xt^r$ changes by $\det|c_{ij}|$.

To remedy this situation, we consider the sheaf $\mathscr{I}/\mathscr{I}^2$ on X, which is locally free of rank r (II, 8.21A). In particular, it is free over U, with basis $f_1,\dots,f_r$. Therefore $\bigwedge^r(\mathscr{I}/\mathscr{I}^2)$ is free of rank 1, with basis $f_1 \wedge \dots \wedge f_r$. If we change to the basis $g_1,\dots,g_r$, this element changes by det $|c_{ij}|$. Therefore, we can obtain an intrinsic isomorphism above by tensoring with this free sheaf of rank 1 (check variance!)

$$\mathscr{E}xt_P^r(\mathscr{O}_X,\omega_P) \cong \omega_P \otimes \mathscr{O}_X \otimes \textstyle\bigwedge^r(\mathscr{I}/\mathscr{I}^2)^\vee.$$

This isomorphism, defined over U, is independent of the choice of basis. Therefore when we cover P with such open sets, these isomorphisms glue together, and we obtain the required isomorphism $\omega_X^\circ \cong \omega_P \otimes \bigwedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$.

Corollary 7.12. *If X is a projective nonsingular variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.*

PROOF. Embed X in $P = \mathbf{P}_k^N$. Then X is a local complete intersection in P (II, 8.17), and $\omega_X \cong \omega_P \otimes \bigwedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$ by (II, 8.20).

Remark 7.12.1. Thus for a projective nonsingular variety X, the duality theorem (7.6) and its corollary (7.7) hold with ω_X in place of ω_X°. In particular, we obtain an isomorphism $H^n(X,\omega_X) \cong k$, whose existence is by no means obvious a priori.

Remark 7.12.2. If X is a projective nonsingular curve, we find that $H^1(X,\mathscr{O}_X)$ and $H^0(X,\omega_X)$ are dual vector spaces. Hence the arithmetic genus $p_a = \dim H^1(X,\mathscr{O}_X)$ and the geometric genus $p_g = \dim \Gamma(X,\omega_X)$ are equal—cf. (Ex. 5.3a) and (II. 8.18.2).

Remark 7.12.3. If X is a projective nonsingular surface, then $H^0(X,\omega)$ is dual to $H^2(X,\mathcal{O}_X)$, so $p_g = \dim H^2(X,\mathcal{O}_X)$. On the other hand $p_a = \dim H^2(X,\mathcal{O}_X) - \dim H^1(X,\mathcal{O}_X)$ by (Ex. 5.3a). Thus $p_g \geqslant p_a$. The difference, $p_g - p_a = \dim H^1(X,\mathcal{O}_X)$ is usually denoted by q, and is called the *irregularity* of X. For example, the surface of (II, Ex. 8.3c) has irregularity 2.

Corollary 7.13. *Let X be a nonsingular projective variety of dimension n. For any $p = 0,1,\ldots,n$, let $\Omega^p = \bigwedge^p \Omega_{X/k}$ be the sheaf of differential p-forms. Then for each $p,q = 0,1,\ldots,n$, we have a natural isomorphism*

$$H^q(X,\Omega^p) \cong H^{n-q}(X,\Omega^{n-p})'.$$

PROOF. Indeed, for any p, $\Omega^{n-p} \cong (\Omega^p)^\check{} \otimes \omega$ (II, Ex. 5.16b). Then use (7.7).

Remark 7.13.1. The numbers $h^{p,q} = \dim H^q(X,\Omega^p)$ are important biregular invariants of the variety X.

Remark 7.14 (Residues of Differentials on Curves). A weakness of the duality theorem as we have proved it is that even for a nonsingular projective variety X, we don't have much information about the trace map $t: H^n(X,\omega) \to k$. We know only that it exists. In the case of curves, there is another way of proving the duality theorem, using residues, which improves this situation.

Let X be a complete nonsingular curve over an algebraically closed field k, and let K be the function field of X. Let Ω_X be the sheaf of differentials of X over k, and for a closed point $P \in X$, let Ω_P be its stalk at P. Let Ω_K be the module of differentials of K over k. Then one first proves:

Theorem 7.14.1 (Existence of Residues). *For each closed point $P \in X$, there is a unique k-linear map* $\operatorname{res}_P: \Omega_K \to k$ *with the following properties*:

(a) $\operatorname{res}_P(\tau) = 0$ *for all* $\tau \in \Omega_P$;

(b) $\operatorname{res}_P(f^n df) = 0$ *for all* $f \in K^*$, *all* $n \neq -1$;

(c) $\operatorname{res}_P(f^{-1}df) = v_P(f)\cdot 1$, *where v_P is the valuation associated to P.*

From these properties we see immediately how to calculate the residue of any differential. Indeed, let $t \in \mathcal{O}_P$ be a uniformizing parameter. Then dt is a generator for Ω_K as a K-vector space, so we can write any $\tau \in \Omega_K$ as $g\,dt$ for some $g \in K$. Furthermore, since $\mathcal{O}_P$ is a valuation ring, we can write $g = \sum_{i<0} a_i t^i + h$ with $a_i \in k$, $h \in \mathcal{O}_P$, and the sum finite. Thus $\tau = \sum a_i t^i dt + h\,dt$. Now from linearity and (a), (b), (c) we find

(d) $\operatorname{res}_P \tau = a_{-1}$.

Thus the uniqueness of res_P is clear.

The existence is more difficult. One approach by Serre [7, Ch. II] is to take (d) as the definition of the residue. Then one has an awkward time proving that it is independent of the choice of the uniformizing parameter t, especially in the case of characteristic $p > 0$. Another approach by Tate [2] gives an intrinsic construction of the residue map by a clever use of certain k-linear transformations of K.

The basic result about residues is:

Theorem 7.14.2 (Residue Theorem). *For any* $\tau \in \Omega_K$, *we have* $\sum_{P \in X} \operatorname{res}_P \tau = 0$.

In Serre's approach this theorem is first proved on $\mathbf{P}^1$, by explicit calculation. Then the general case is obtained by using a finite morphism $X \to \mathbf{P}^1$ and studying the relationship between the residues in both places. In Tate's approach the residue theorem follows directly from the construction of the residue map.

Once one has the theory of residues, the duality theorem for X can be proved by a method of Weil using repartitions. We refer to the lucid expositions of Serre and Tate mentioned above for the details of this classic story.

The connection with our approach can be explained as follows. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0,$$

where $\mathcal{K}_X$ is the constant sheaf K_X, is a flasque resolution of $\mathcal{O}_X$ (cf. Ex. 2.2). Furthermore,

$$\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{P \in X} i_*(K_X/\mathcal{O}_P)$$

where we consider $K_X/\mathcal{O}_P$ as an $\mathcal{O}_P$-module, and $i:\{P\} \to X$ is the inclusion map. Tensoring with Ω_X, we get a flasque resolution of Ω_X:

$$0 \to \Omega_X \to \Omega_X \otimes \mathcal{K}_X \to \bigoplus_{P \in X} i_*(\Omega_K/\Omega_P) \to 0.$$

Taking cohomology, we get an exact sequence

$$\Omega_K \to \bigoplus_{P \in X} \Omega_K/\Omega_P \to H^1(X,\Omega_X) \to 0.$$

We define a map

$$\bigoplus_{P \in X} \Omega_K/\Omega_P \to k$$

by taking the sum of all the maps $\operatorname{res}_P:\Omega_K/\Omega_P \to k$. Then by (7.14.2) this map vanishes on the image of Ω_K, hence it passes to the quotient and gives a map $t:H^1(X,\Omega_X) \to k$. This is the trace map of our duality theorem, which appears now in a much more explicit form.

Remark 7.15 (The Kodaira Vanishing Theorem). Our discussion of the cohomology of projective varieties would not be complete without mentioning the Kodaira vanishing theorem. It says if X is a projective nonsingular variety of dimension n over $\mathbf{C}$, and if $\mathcal{L}$ is an ample invertible sheaf on X, then:

(a) $H^i(X,\mathcal{L} \otimes \omega) = 0$ for $i > 0$;
(b) $H^i(X,\mathcal{L}^{-1}) = 0$ for $i < n$.

Of course (a) and (b) are equivalent to each other by Serre duality. The theorem is proved using methods of complex analytic differential geometry. At present there is no purely algebraic proof. On the other hand, Raynaud has recently shown that this result does not hold over fields of characteristic $p > 0$.

The first proof was given by Kodaira [1]. For other proofs, including the generalization by Nakano, see Wells [1, Ch. VI, §2], Mumford [3], and Ramanujam [1]. For a relative version of the theorem, see Grauert and Riemenschneider [1].

References for the Duality Theorem. The duality theorem was first proved by Serre [2] (in the form of (7.7)) for locally free sheaves on a compact complex manifold, and in the case of abstract algebraic geometry by Serre [1]. Our proof follows Grothendieck [5] and Grothendieck [SGA 2, exp. XII], with some improvements suggested by Lipman. The duality theorem and the theory of residues have been generalized to the case of an arbitrary proper morphism by Grothendieck—see Grothendieck [4] and Hartshorne [2]. Deligne has given another proof of the existence of a dualizing sheaf, and Verdier [1] has shown that this one agrees with the sheaf ω for a nonsingular variety. Kunz [1] gives another construction, using differentials, of the dualizing sheaf ω_X for an integral projective scheme X over k.

The duality theorem has also been generalized to the case of a proper morphism of complex analytic spaces—see Ramis and Ruget [1] and Ramis, Ruget, and Verdier [1]. For a generalization to noncompact complex manifolds, see Suominen [1].

In the case of curves, the duality theorem is the most important ingredient in the proof of the Riemann–Roch theorem (IV, §1). See Serre [7, Ch. II] for the history of this approach, and also Gunning [1] for a proof in the language of compact Riemann surfaces.

Exercises

7.1. Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X,\mathscr{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

7.2. Let $f: X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f^!\omega_Y^\circ$ is a dualizing sheaf for X, where $f^!$ is defined as in (Ex. 6.10).

(b) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t: f_*\omega_X \to \omega_Y$.

7.3. Let $X = \mathbf{P}_k^n$. Show that $H^q(X,\Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leqslant p, q \leqslant n$.

***7.4.** *The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n - p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology $H^{n-p}(X,\Omega_X^{n-p}) \to H^{n-p}(Y,\Omega_Y^{n-p})$. Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y: H^{n-p}(Y, \Omega_Y^{n-p}) \to k$. Composing, we obtain a linear map $H^{n-p}(X, \Omega_X^{n-p}) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X, \Omega_X^p)$, which we call the *cohomology class* of Y.

(a) If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X, \Omega^n)$ and t_X is the trace map.

(b) If $X = \mathbf{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its *degree* as a projective variety (I, §7). [*Hint*: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

(c) For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log: \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X)$ which we denote by c—see (Ex. 4.5).

(d) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathcal{L}(Y))$, where $\mathcal{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

8 Higher Direct Images of Sheaves

For the remainder of this chapter we will be studying families of schemes. Recall (II, §3) that a family of schemes is simply a morphism $f: X \to Y$, and the members of the family are the fibres $X_y = X \times_Y \operatorname{Spec} k(y)$ for various points $y \in Y$. To study a family, we need some form of "relative cohomology of X over Y," or "cohomology along the fibres of X over Y." This notion is provided by the higher direct image functors R^if_* which we define below. The precise relationship between these functors and the cohomology of the fibres X_y will be studied in §11, 12.

Definition. Let $f: X \to Y$ be a continuous map of topological spaces. Then we define the *higher direct image* functors $R^if_*: \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_* (II, §1).

This makes sense because f_* is obviously left exact, and $\mathfrak{Ab}(X)$ has enough injectives (2.3).

Proposition 8.1. *For each $i \geq 0$ and each $\mathcal{F} \in \mathfrak{Ab}(X)$, $R^if_*(\mathcal{F})$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

on Y.

PROOF. Let us denote the sheaf associated to the above presheaf by $\mathcal{H}^i(X, \mathcal{F})$. Then, since the operation of taking the sheaf associated to a presheaf is exact, the functors $\mathcal{H}^i(X, \cdot)$ form a δ-functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}(Y)$. For $i = 0$ we have $f_* \mathcal{F} = \mathcal{H}^0(X, \mathcal{F})$ by definition of f_*. For an injective object $\mathcal{I} \in \mathfrak{Ab}(X)$ we have $R^if_*(\mathcal{I}) = 0$ for $i > 0$ because R^if_* is a derived functor.

On the other hand, for each V, $\mathscr{I}|_{f^{-1}(V)}$ is injective in $\mathfrak{Ab}(f^{-1}(V))$ by (6.1) (think of X as a ringed space with the constant sheaf $\mathbf{Z}$), so $\mathscr{H}^i(X,\mathscr{I}) = 0$ for $i > 0$ also. Hence there is a unique isomorphism of δ-functors $R^if_*(\cdot) \cong \mathscr{H}^i(X,\cdot)$ by (1.3A).

Corollary 8.2. *If $V \subseteq Y$ is any open subset, then*

$$R^if_*(\mathscr{F})|_V = R^if'_*(\mathscr{F}|_{f^{-1}(V)})$$

where $f': f^{-1}(V) \to V$ is the restricted map.

PROOF. Obvious.

Corollary 8.3. *If $\mathscr{F}$ is a flasque sheaf on X, then $R^if_*(\mathscr{F}) = 0$ for all $i > 0$.*

PROOF. Since the restriction of a flasque sheaf to an open subset is flasque, this follows from (2.5).

Proposition 8.4. *Let $f: X \to Y$ be a morphism of ringed spaces. Then the functors R^if_* can be calculated on $\mathfrak{Mod}(X)$ as the derived functors of $f_*: \mathfrak{Mod}(X) \to \mathfrak{Mod}(Y)$.*

PROOF. To calculate the derived functors of f_* on $\mathfrak{Mod}(X)$, we use resolutions by injective objects of $\mathfrak{Mod}(X)$. Any injective of $\mathfrak{Mod}(X)$ is flasque by (2.4), hence acyclic for f_* on $\mathfrak{Ab}(X)$ by (8.3), so they can be used to calculate R^if_* by (1.2A).

Proposition 8.5. *Let X be a noetherian scheme, and let $f: X \to Y$ be a morphism of X to an affine scheme $Y = \operatorname{Spec} A$. Then for any quasi-coherent sheaf $\mathscr{F}$ on X, we have*

$$R^if_*(\mathscr{F}) \cong H^i(X,\mathscr{F})^\sim.$$

PROOF. By (II, 5.8), $f_*\mathscr{F}$ is a quasi-coherent sheaf on Y. Hence $f_*\mathscr{F} \cong \Gamma(Y,f_*\mathscr{F})^\sim$. But $\Gamma(Y,f_*\mathscr{F}) = \Gamma(X,\mathscr{F})$. So we have an isomorphism for $i = 0$.

Since $^\sim$ is an exact functor from $\mathfrak{Mod}(A)$ to $\mathfrak{Mod}(Y)$, both sides are δ-functors from $\mathfrak{Qco}(X)$ to $\mathfrak{Mod}(Y)$. Furthermore, by (3.6), any quasi-coherent sheaf $\mathscr{F}$ on X can be embedded in a flasque, quasi-coherent sheaf. Hence both sides are effaceable for $i > 0$. We conclude from (1.3A) that there is a unique isomorphism of δ-functors as above, reducing to the given one for $i = 0$.

Note that we must work in the category $\mathfrak{Qco}(X)$, because already the case $i = 0$ fails if $\mathscr{F}$ is not quasi-coherent.

Corollary 8.6. *Let $f: X \to Y$ be a morphism of schemes, with X noetherian. Then for any quasi-coherent sheaf $\mathscr{F}$ on X, the sheaves $R^if_*(\mathscr{F})$ are quasi-coherent on Y.*

Proof. The question is local on Y, so we may use (8.5).

Proposition 8.7. *Let $f: X \to Y$ be a morphism of separated noetherian schemes. Let $\mathscr{F}$ be a quasi-coherent sheaf on X, let $\mathfrak{U} = (U_i)$ be an open affine cover of X, and let $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ be the Čech resolution of $\mathscr{F}$ given by (4.2). Then for each $p \geqslant 0$,*

$$R^p f_*(\mathscr{F}) \cong h^p(f_*\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})).$$

Proof. For any open affine subset $V \subseteq Y$, the open subsets $U_i \cap f^{-1}(V)$ of X are all affine (check!—cf. (II, Ex. 4.3)). Hence we may reduce to the case Y affine. The sheaves $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are all quasi-coherent, so we have

$$f_*\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F}) \cong C^{\cdot}(\mathfrak{U},\mathscr{F})^{\sim}$$

by (II, 5.8). Now the result follows from (4.5) and (8.5).

Theorem 8.8. *Let $f: X \to Y$ be a projective morphism of noetherian schemes, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X over Y, and let $\mathscr{F}$ be a coherent sheaf on X. Then:*

(a) *for all $n \gg 0$, the natural map $f^*f_*(\mathscr{F}(n)) \to \mathscr{F}(n)$ is surjective;*

(b) *for all $i \geqslant 0$, $R^if_*(\mathscr{F})$ is a coherent sheaf on Y;*

(c) *for $i > 0$* and *$n \gg 0$, $R^if_*(\mathscr{F}(n)) = 0$.*

Proof. Since Y is quasi-compact, the question is local on Y, so we may assume Y is affine, say $Y = \operatorname{Spec} A$. Then, using (8.5), (a) says that $\mathscr{F}(n)$ is generated by global sections, which is (II, 5.17). (b) says that $H^i(X,\mathscr{F})$ is a finitely generated A-module, which is (5.2a). Finally, (c) says that $H^i(X,\mathscr{F}(n)) = 0$, which is (5.2b).

Remark 8.8.1. Part (b) of this theorem is true more generally for a proper morphism of noetherian schemes—see Grothendieck [EGA III, 3.2.1]. The analogous theorem for a proper morphism of complex analytic spaces was proved by Grauert [1].

Exercises

8.1. Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^if_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F}).$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

8.2. Let $f: X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F})$ for each $i \geqslant 0$. (This gives another proof of (Ex. 4.1).)

8.3. Let $f:X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Prove the *projection formula* (cf. (II, Ex. 5.1))

$$R^if_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^if_*(\mathscr{F}) \otimes \mathscr{E}.$$

8.4. Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $n + 1$, $n \geqslant 1$. Let $X = \mathbf{P}(\mathscr{E})$ (II, §7), with the invertible sheaf $\mathscr{O}_X(1)$ and the projection morphism $\pi:X \to Y$.

(a) Then $\pi_*(\mathscr{O}(l)) \cong S^l(\mathscr{E})$ for $l \geqslant 0$, $\pi_*(\mathscr{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathscr{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbf{Z}$; and $R^n\pi_*(\mathscr{O}(l)) = 0$ for $l > -n - 1$.

(b) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathscr{O} \to 0,$$

cf. (II, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \wedge^n\Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1}\mathscr{E})(-n - 1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathscr{O}_Y$ (cf. (7.1.1)).

(c) Now show, for any $l \in \mathbf{Z}$, that

$$R^n\pi_*(\mathscr{O}(l)) \cong \pi_*(\mathscr{O}(-l - n - 1))\check{} \otimes (\wedge^{n+1}\mathscr{E})\check{}.$$

(d) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

(e) In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).

9 Flat Morphisms

In this section we introduce the notion of a flat morphism of schemes. By taking the fibres of a flat morphism, we get the notion of a flat family of schemes. This provides a concise formulation of the intuitive idea of a "continuous family of schemes." We will show, through various results and examples, why flatness is a natural as well as a convenient condition to put on a family of schemes.

First we recall the algebraic notion of a flat module. Let A be a ring, and let M be an A-module. We say that M is *flat* over A if the functor $N \mapsto M \otimes_A N$ is an exact functor for $N \in \mathfrak{Mod}(A)$. If $A \to B$ is a ring homomorphism, we say that B is *flat* over A if it is flat as a module.

Proposition 9.1A.

(a) *An A-module M is flat if and only if for every finitely generated ideal $\mathfrak{a} \subseteq A$, the map $\mathfrak{a} \otimes M \to M$ is injective.*

(b) Base extension: *If M is a flat A-module, and $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.*

(c) Transitivity: *If B is a flat A-algebra, and N is a flat B-module, then N is also flat as an A-module.*

(d) Localization: *M is flat over A if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

(e) *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* A*-modules. If* M' *and* M'' *are both flat then* M *is flat; if* M *and* M'' *are both flat, then* M' *is flat.*

(f) *A finitely generated module* M *over a local noetherian ring* A *is flat if and only if it is free.*

PROOFS. Matsumura [2, Ch. 2, §3] or Bourbaki [1, Ch. I.].

Example 9.1.1. If A is a ring and $S \subseteq A$ is a multiplicative system, then the localization $S^{-1}A$ is a flat A-algebra. If $A \to B$ is a ring homomorphism, if M is a B-module which is flat over A, and if S is a multiplicative system in B, then $S^{-1}M$ is flat over A.

Example 9.1.2. If A is a noetherian ring and $\mathfrak{a} \subseteq A$ an ideal, then the $\mathfrak{a}$-adic completion $\hat{A}$ is a flat A-algebra (II, 9.3A).

Example 9.1.3. Let A be a principal ideal domain. Then an A-module M is flat if and only if it is torsion-free. Indeed, by (9.1Aa) we must check that for every ideal $\mathfrak{a} \subseteq A$, $\mathfrak{a} \otimes M \to M$ is injective. But $\mathfrak{a}$ is principal, say generated by t, so this just says that t is not a zero divisor in M, i.e., M is torsion-free.

Definition. Let $f: X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *flat over* Y *at a point* $x \in X$, if the stalk $\mathscr{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module, where $y = f(x)$ and we consider $\mathscr{F}_x$ as an $\mathcal{O}_{y,Y}$-module via the natural map $f^{\#}: \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. We say simply $\mathscr{F}$ is *flat over* Y if it is flat at every point of X. We say X is *flat* over Y if $\mathcal{O}_X$ is.

Proposition 9.2.

(a) *An open immersion is flat.*

(b) Base change: *let* $f: X \to Y$ *be a morphism, let* $\mathscr{F}$ *be an* $\mathcal{O}_X$*-module which is flat over* Y*, and let* $g: Y' \to Y$ *be any morphism. Let* $X' = X \times_Y Y'$*, let* $f': X' \to Y'$ *be the second projection, and let* $\mathscr{F}' = p_1^*(\mathscr{F})$*. Then* $\mathscr{F}'$ *is flat over* Y'.

(c) Transitivity: let $f: X \to Y$ *and* $g: Y \to Z$ *be morphisms. Let* $\mathscr{F}$ *be an* $\mathcal{O}_X$*-module which is flat over* Y*, and assume also that* Y *is flat over* Z*. Then* $\mathscr{F}$ *is flat over* Z.

(d) *Let* $A \to B$ *be a ring homomorphism, and let* M *be a* B*-module. Let* $f: X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ *be the corresponding morphism of affine schemes, and let* $\mathscr{F} = \tilde{M}$*. Then* $\mathscr{F}$ *is flat over* Y *if and only if* M *is flat over* A.

(e) *Let* X *be a noetherian scheme, and* $\mathscr{F}$ *a coherent* $\mathcal{O}_X$*-module. Then* $\mathscr{F}$ *is flat over* X *if and only if it is locally free.*

PROOF. These properties all follow from the corresponding properties of modules, taking into account that the functor $\sim$ is compatible with $\otimes$ (II, 5.2).

Next, as an illustration of the convenience of flat morphisms, we show that "cohomology commutes with flat base extension":

Proposition 9.3. *Let $f: X \to Y$ be a separated morphism of finite type of noetherian schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $u: Y' \to Y$ be a flat morphism of noetherian schemes.*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

Then for all $i \geqslant 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^* \mathscr{F}).$$

Proof. The question is local on Y and on Y', so we may assume they are both affine, say $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Then by (8.5) what we have to show is that

$$H^i(X,\mathscr{F}) \otimes_A A' \cong H^i(X',\mathscr{F}').$$

Since X is separated and noetherian, and $\mathscr{F}$ is quasi-coherent, we can calculate $H^i(X,\mathscr{F})$ by Čech cohomology with respect to an open affine cover $\mathfrak{U}$ of X (4.5). On the other hand, $\{v^{-1}(U) | U \in \mathfrak{U}\}$ forms an open affine cover $\mathfrak{U}'$ of X', and clearly the Čech complex $C^{\cdot}(\mathfrak{U}',v^*\mathscr{F})$ is just $C^{\cdot}(\mathfrak{U},\mathscr{F}) \otimes_A A'$. Since A' is flat over A, the functor $\cdot \otimes_A A'$ commutes with taking cohomology groups of the Čech complex, so we get our result. Note that g is also separated and of finite type by base extension, so X' is also noetherian and separated, allowing us to apply (4.5) on X'.

Remark 9.3.1. Even if u is not flat, this proof shows that there is a natural map $u^*R^if_*(\mathscr{F}) \to R^ig_*(v^*\mathscr{F})$.

Corollary 9.4. *Let $f: X \to Y$ and $\mathscr{F}$ be as in (9.3), and assume Y affine. For any point $y \in Y$, let X_y be the fibre over y, and let $\mathscr{F}_y$ be the induced sheaf. On the other hand, let $k(y)$ denote the constant sheaf $k(y)$ on the closed subset $\{y\}^-$ of Y. Then for all $i \geqslant 0$ there are natural isomorphisms*

$$H^i(X_y,\mathscr{F}_y) \cong H^i(X,\mathscr{F} \otimes k(y)).$$

Proof. First let $Y' \subseteq Y$ be the reduced induced subscheme structure on $\{y\}^-$, and let $X' = X \times_Y Y'$, which is a closed subscheme of X. Then both sides of our desired isomorphism depend only on the sheaf $\mathscr{F}' = \mathscr{F} \otimes k(y)$ on X'. Thus we can replace $X, Y, \mathscr{F}$ by $X', Y', \mathscr{F}'$, i.e., we can assume that Y is an integral affine scheme and that $y \in Y$ is its generic point. In that case, $\operatorname{Spec} k(y) \to Y$ is a flat morphism, so we can apply (9.3) and conclude that

$$H^i(X_y,\mathscr{F}_y) \cong H^i(X,\mathscr{F}) \otimes k(y).$$

But after our reduction, $H^i(X,\mathscr{F})$ is already a $k(y)$-module, so tensoring with $k(y)$ has no effect, and we obtain the desired result. (This result is used in §12.)

Flat Families

For many reasons it is important to have a good notion of an algebraic family of varieties or schemes. The most naive definition would be just to take the fibres of a morphism. To get a good notion, however, we should require that certain numerical invariants remain constant in a family, such as the dimension of the fibres. It turns out that if we are dealing with nonsingular (or even normal) varieties over a field, then the naive definition is already a good one. Evidence for this is the theorem (9.13) that in such a family, the arithmetic genus is constant.

On the other hand, if we deal with nonnormal varieties, or more general schemes, the naive definition will not do. So we consider a flat family of schemes, which means the fibres of a flat morphism, and this is a very good notion. Why the algebraic condition of flatness on the structure sheaves should give a good definition of a family is something of a mystery. But at least we will justify this choice by showing that flat families have many good properties, and by giving necessary and sufficient conditions for flatness in some special cases. In particular, we will show that a family of closed subschemes of projective space (over an integral scheme) is flat if and only if the Hilbert polynomials of the fibres are the same.

Proposition 9.5. *Let $f:X \to Y$ be a flat morphism of schemes of finite type over a field k. For any point $x \in X$, let $y = f(x)$. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

Here for any scheme X and any point $x \in X$, by $\dim_x X$ we mean the dimension of the local ring $\mathcal{O}_{x,X}$.

PROOF. First we make a base change $Y' \to Y$ where $Y' = \operatorname{Spec} \mathcal{O}_{y,Y}$, and consider the new morphism $f':X' \to Y'$ where $X' = X \times_Y Y'$. Then f' is also flat by (9.2), x lifts to X', and the three numbers in question are the same. Thus we may assume that y is a closed point of Y, and $\dim_y Y = \dim Y$.

Now we use induction on $\dim Y$. If $\dim Y = 0$, then X_y is defined by a nilpotent ideal in X, so we have $\dim_x(X_y) = \dim_x X$, and $\dim_y Y = 0$.

If $\dim Y > 0$, we make a base extension to Y_{red}. Nothing changes, so we may assume that Y is reduced. Then we can find an element $t \in \mathfrak{m}_y \subseteq \mathcal{O}_{y,Y}$ such that t is not a zero divisor. Let $Y' = \operatorname{Spec} \mathcal{O}_{y,Y}/(t)$, and make the base extension $Y' \to Y$. Then $\dim Y' = \dim Y - 1$ by (I, 1.8A) and (I, 1.11A). Since f is flat, $f^{\#}t \in \mathfrak{m}_x$ is also not a zero divisor. So for the same reason, $\dim_x X' = \dim_x X - 1$. Of course the fibre X_y does not change under base extension, so we have only to prove our formula for $f':X' \to Y'$. But this follows from the induction hypothesis, so we are done.

Corollary 9.6. *Let $f: X \rightarrow Y$ be a flat morphism of schemes of finite type over a field k, and assume that Y is irreducible. Then the following conditions are equivalent:*

(i) *every irreducible component of X has dimension equal to* $\dim Y + n$;
(ii) *for any point $y \in Y$ (closed or not), every irreducible component of the fibre X_y has dimension n.*

PROOF.
(i) $\Rightarrow$ (ii). Given $y \in Y$, let $Z \subseteq X_y$ be an irreducible component, and let $x \in Z$ be a closed point, which is not in any other irreducible component of X_y. Applying (9.5) we have

$$\dim_x Z = \dim_x X - \dim_y Y.$$

Now $\dim_x Z = \dim Z$ since x is a closed point (II, Ex. 3.20). On the other hand, since Y is irreducible and X is equidimensional, and both are of finite type over k, we have (II, Ex. 3.20)

$$\dim_x X = \dim X - \dim\{x\}^-$$
$$\dim_y Y = \dim Y - \dim\{y\}^-.$$

Finally, since x is a closed point of the fibre X_y, $k(x)$ is a finite algebraic extension of $k(y)$ and so

$$\dim\{x\}^- = \dim\{y\}^-.$$

Combining all these, and using (i) we find $\dim Z = n$.

(ii) $\Rightarrow$ (i). This time let Z be an irreducible component of X, and let $x \in Z$ be a closed point which is not contained in any other irreducible component of X. Then applying (9.5), we have

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

But $\dim_x(X_y) = n$ by (ii), $\dim_x X = \dim Z$, and $\dim_y Y = \dim Y$, since $y = f(x)$ must be a closed point of Y. Thus

$$\dim Z = \dim Y + n$$

as required.

Definition. A point x of a scheme X is an *associated point* of X if the maximal ideal $\mathfrak{m}_x$ is an associated prime of 0 in the local ring $\mathcal{O}_{x,X}$, or in other words, if every element of $\mathfrak{m}_x$ is a zero divisor.

Proposition 9.7. *Let $f: X \rightarrow Y$ be a morphism of schemes, with Y integral and regular of dimension 1. Then f is flat if and only if every associated point $x \in X$ maps to the generic point of Y. In particular, if X is reduced, this says that every irreducible component of X dominates Y.*

Proof. First suppose that f is flat, and let $x \in X$ be a point whose image $y = f(x)$ is a closed point of Y. Then $\mathcal{O}_{y,Y}$ is a discrete valuation ring. Let $t \in \mathfrak{m}_y - \mathfrak{m}_y^2$ be a uniformizing parameter. Then t is not a zero divisor in $\mathcal{O}_{y,Y}$. Since f is flat, $f^{\#}t \in \mathfrak{m}_x$ is not a zero divisor, so x is not an associated point of X.

Conversely, suppose that every associated point of X maps to the generic point of Y. To show f is flat, we must show that for any $x \in X$, letting $y = f(x)$, the local ring $\mathcal{O}_{x,X}$ is flat over $\mathcal{O}_{y,Y}$. If y is the generic point, $\mathcal{O}_{y,Y}$ is a field, so there is nothing to prove. If y is a closed point, $\mathcal{O}_{y,Y}$ is a discrete valuation ring, so by (9.1.3) we must show that $\mathcal{O}_{x,X}$ is a torsion-free module. If it is not, then $f^{\#}t$ must be a zero divisor in $\mathfrak{m}_x$, where t is a uniformizing parameter of $\mathcal{O}_{y,Y}$. Therefore $f^{\#}t$ is contained in some associated prime ideal $\mathfrak{p}$ of (0) in $\mathcal{O}_x$ (Matsumura [2, Cor. 2, p. 50]). Then $\mathfrak{p}$ determines a point $x' \in X$, which is an associated point of X, and whose image by f is y, which is a contradiction.

Finally, note that if X is reduced, its associated points are just the generic points of its irreducible components, so our condition says that each irreducible component of X dominates Y.

Example 9.7.1. Let Y be a curve with a node, and let $f: X \to Y$ be the map of its normalization to it. Then f is not flat. For if it were, then $f_*\mathcal{O}_X$ would be a flat sheaf of $\mathcal{O}_Y$-modules. Since it is coherent, it would be locally free by (9.2e). And finally, since its rank is 1, it would be an invertible sheaf on Y. But there are two points P_1, P_2 of X going to the node Q of Y, so $(f_*\mathcal{O}_X)_Q$ needs two generators as an $\mathcal{O}_Y$-module, hence it cannot be locally free.

Example 9.7.2. The result of (9.7) also fails if Y is regular of dimension >1. For example, let $Y = \mathbf{A}^2$, and let X be obtained by blowing up a point. Then X and Y are both nonsingular, and X dominates Y, but f is not flat, because the dimension of the fibre over the blown-up point is too big (9.5).

Proposition 9.8. *Let Y be a regular, integral scheme of dimension 1, let $P \in Y$ be a closed point, and let $X \subseteq \mathbf{P}^n_{Y-P}$ be a closed subscheme which is flat over $Y - P$. Then there exists a unique closed subscheme $\bar{X} \subseteq \mathbf{P}^n_Y$, flat over Y, whose restriction to $\mathbf{P}^n_{Y-P}$ is X.*

Proof. Take $\bar{X}$ to be the scheme-theoretic closure of X in $\mathbf{P}^n_Y$ (II, Ex. 3.11d). Then the associated points of $\bar{X}$ are just those of X, so by (9.7), $\bar{X}$ is flat over Y. Furthermore, $\bar{X}$ is unique, because any other extension of X to $\mathbf{P}^n_Y$ would have some associated points mapping to P.

Remark 9.8.1. This proposition says that we can "pass to the limit," when we have a flat family of closed subschemes of $\mathbf{P}^n$ over a punctured curve. Hence it implies that "the Hilbert scheme is proper." The Hilbert scheme is a scheme H which parametrizes all closed subschemes of $\mathbf{P}_k^n$. It has the property that to give a closed subscheme $X \subseteq \mathbf{P}_T^n$, flat over T, for any scheme T, is equivalent to giving a morphism $\varphi: T \to H$. Here, naturally, for any $t \in T$, $\varphi(t)$ is the point of H corresponding to the fibre $X_t \subseteq \mathbf{P}_{k(t)}^n$.

Now once one knows that the Hilbert scheme exists (see Grothendieck [5, exp. 221]) then the question of its properness can be decided using the valuative criterion of properness (II, 4.7). And the result just proved is the essential point needed to show that each connected component of H is proper over k.

Example 9.8.2. Even though the dimension of the fibres is constant in a flat family, we cannot expect properties such as "irreducible" or "reduced" to be preserved in a flat family. Take for example, the families given in (II, 3.3.1) and (II, 3.3.2). In each case the total space X is integral, the base Y is a nonsingular curve and the morphism $f: X \to Y$ is surjective, so the family is flat. Also most fibres are integral in both families. However, the special fibre in one is a doubled line (not reduced), and the special fibre in the other is two lines (not irreducible).

Example 9.8.3 (Projection from a Point). We get some new insight into the geometric process of projection from a point (I, Ex. 3.14) using (9.8). Let $P = (0,0,\dots,0,1) \in \mathbf{P}^{n+1}$, and consider the projection $\varphi: \mathbf{P}^{n+1} - \{P\} \to \mathbf{P}^n$, which is defined by $(x_0,\dots,x_{n+1}) \mapsto (x_0,\dots,x_n)$. For each $a \in k$, $a \neq 0$, consider the automorphism σ_a of $\mathbf{P}^{n+1}$ defined by $(x_0,\dots,x_{n+1}) \mapsto (x_0,\dots,x_n,ax_{n+1})$. Now let X_1 be a closed subscheme of $\mathbf{P}^{n+1}$, not containing P. For each $a \neq 0$, let $X_a = \sigma_a(X_1)$. Then the X_a form a flat family parametrized by $\mathbf{A}^1 - \{0\}$. It is flat, because the X_a are all isomorphic as abstract schemes, and in fact, the whole family is isomorphic to $X_1 \times (\mathbf{A}^1 - \{0\})$ if we forget the embedding in $\mathbf{P}^{n+1}$.

Now according to (9.8) this family extends uniquely to a flat family defined over all of $\mathbf{A}^1$, and clearly the fibre X_0 over 0 agrees, at least set-theoretically, with the projection $\varphi(X_1)$ of X_1. Thus we see that there is a flat family over $\mathbf{A}^1$, whose fibres for all $a \neq 0$ are isomorphic to X_1, and whose fibre at 0 is some scheme with the same underlying space as $\varphi(X_1)$.

Example 9.8.4. We will now calculate the flat family just described in the special case where X_1 is a twisted cubic curve in $\mathbf{P}^3$, φ is a projection to $\mathbf{P}^2$, and $\varphi(X_1)$ is a nodal cubic curve in $\mathbf{P}^2$. The remarkable result of this calculation is that the special fibre X_0 of our flat family consists of the curve $\varphi(X_1)$ together with some nilpotent elements at the double point! We say that X_0 is a scheme with an embedded point. It seems as if the scheme X_0 is retaining the information that it is the limit of a family of space curves, by having these nilpotent elements which point out of the plane. In particular, X_0 is not a closed subscheme of $\mathbf{P}^2$ (Fig. 11).

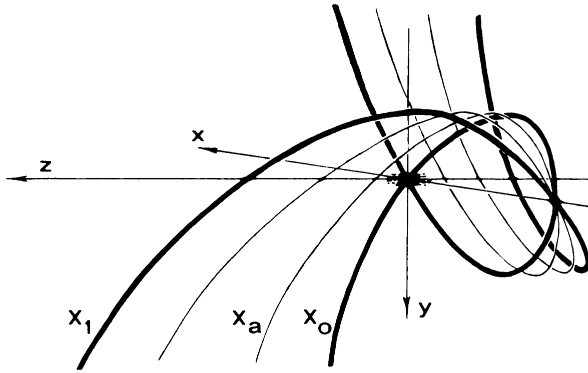

Figure 11. A flat family of subschemes of $\mathbf{P}^3$.

Now for the calculation. We are just interested in what happens near the double point, so we will use affine coordinates x,y in $\mathbf{A}^2$ and x,y,z in $\mathbf{A}^3$. Let X_1 be given by the parametric equations

$$\begin{cases} x = t^2 - 1 \\ y = t^3 - t \\ z = t. \end{cases}$$

Then since $t = z$, $t^2 = x + 1$, $t^3 = y + z$, we recognize this as a twisted cubic curve in $\mathbf{A}^3$ (I, Ex. 1.2).

Now for any $a \neq 0$, the scheme X_a is given by

$$\begin{cases} x = t^2 - 1 \\ y = t^3 - t \\ z = at. \end{cases}$$

To get the ideal $I \subseteq k[a,x,y,z]$ of the total family $\bar{X}$ extended over all of $\mathbf{A}^1$, we eliminate t from the parametric equations, and make sure a is not a zero divisor in $k[a,x,y,z]/I$, so that $\bar{X}$ will be flat. We find

$$I = (a^2(x + 1) - z^2, ax(x + 1) - yz, xz - ay, y^2 - x^2(x + 1)).$$

From this, setting $a = 0$, we obtain the ideal $I_0 \subseteq k[x,y,z]$ of X_0, which is

$$I_0 = (z^2,yz,xz,y^2 - x^2(x + 1)).$$

So we see that X_0 is a scheme with support equal to the nodal cubic curve $y^2 = x^2(x + 1)$. At any point where $x \neq 0$, we get z in the ideal, so X_0 is reduced there. But in the local ring at the node $(0,0,0)$, we have the element z with $z^2 = 0$, a nonzero nilpotent element.

So here we have an example of a flat family of curves, whose general member is nonsingular, but whose special member is singular, with an embedded point. See also (9.10.1) and (IV, Ex. 3.5).

Example 9.8.5 (Algebraic Families of Divisors). Let X be a scheme of finite type over an algebraically closed field k, let T be a nonsingular curve over k, and let D be an effective Cartier divisor on $X \times T$ (II, §6). Then we can think of D as a closed subscheme of $X \times T$, which is locally described on a small open set U as the zeros of a single element $f \in \Gamma(U,\mathcal{O}_U)$ such that f is not a zero divisor. For any closed point $t \in T$, let $X_t(\cong X)$ be the fibre of $X \times T$ over t. We say that the intersection divisor $D_t = D.X_t$ *is defined* if at every point of X_t, the image $\bar{f} \in \Gamma(U \cap X_t,\mathcal{O}_{X_t})$ of a local equation f of D is not a zero divisor. In that case the covering $\{U \cap X_t\}$ and the elements $\bar{f}$ define a Cartier divisor D_t on X_t. If D_t is defined for all t, we say that the divisors $\{D_t | t \in T\}$ form an *algebraic family of divisors on X parametrized by T*.

This definition, which is natural in the context of Cartier divisors, is connected with flatness in the following way: the original Cartier divisor D, considered as a scheme over T, is flat over T if and only if $D_t = D.X_t$ is defined for each $t \in T$. Indeed, let $x \in D$ be any point, let $A = \mathcal{O}_{x,X \times T}$ be the local ring of x on $X \times T$, let $f \in A$ be a local equation for D, let $p_2(x) = t$, and let $u \in \mathcal{O}_{t,T}$ be a uniformizing parameter. Then $D.X_t$ is defined at x if and only if $\bar{f} \in A/uA$ is not a zero divisor. Since u is automatically not a zero divisor in A, this is equivalent to saying that (u,f) is a regular sequence (II, §8) in A. On the other hand, D is flat over T at x if and only if $\mathcal{O}_{x,D}$ is flat over $\mathcal{O}_{t,T}$. By (9.1.3) this is equivalent to $\mathcal{O}_{x,D}$ being torsion-free, i.e., u not being a zero divisor in $\mathcal{O}_{x,D}$. But $\mathcal{O}_{x,D} \cong A/fA$, so this says that (f,u) is a regular sequence in A. Since the property of being a regular sequence is independent of the order of the sequence (Matsumura [2, Th. 28, p. 102]), the two conditions are equivalent.

Theorem 9.9. *Let T be an integral noetherian scheme. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme. For each point $t \in T$, we consider the Hilbert polynomial $P_t \in \mathbf{Q}[z]$ of the fibre X_t considered as a closed subscheme of $\mathbf{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

Proof. Recall that the Hilbert polynomial was defined in (I, §7), and computed another way in (Ex. 5.2). We will use the defining property that

$$P_t(m) = \dim_{k(t)} H^0(X_t,\mathcal{O}_{X_t}(m))$$

for all $m \gg 0$.

First we generalize, replacing $\mathcal{O}_X$ by any coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_T$, and using the Hilbert polynomial of $\mathcal{F}_t$. Thus we may assume $X = \mathbf{P}^n_T$. Second, the question is local on T. In fact, by comparing any point to the generic point, we see that it is sufficient to consider the case $T = \operatorname{Spec} A$, with A a *local* noetherian ring.

So now let $T = \operatorname{Spec} A$ with A a local noetherian domain, let $X = \mathbf{P}^n_T$, and let $\mathcal{F}$ be a coherent sheaf on X. We will show that the following conditions are equivalent:

(i) $\mathcal{F}$ is flat over T;
(ii) $H^0(X,\mathcal{F}(m))$ is a free A-module of finite rank, for all $m \gg 0$;

(iii) the Hilbert polynomial P_t of $\mathscr{F}_t$ on $X_t = \mathbf{P}^n_{k(t)}$ is independent of t, for any $t \in T$.

(i) $\Rightarrow$ (ii). We compute $H^i(X,\mathscr{F}(m))$ by Čech cohomology using the standard open affine cover $\mathfrak{U}$ of X. Then

$$H^i(X,\mathscr{F}(m)) = h^i(C^{\cdot}(\mathfrak{U},\mathscr{F}(m))).$$

Since $\mathscr{F}$ is flat, each term $C^i(\mathfrak{U},\mathscr{F}(m))$ of the Čech complex is a flat A-module. On the other hand, if $m \gg 0$, then $H^i(X,\mathscr{F}(m)) = 0$ for $i > 0$, by (5.2). Thus the complex $C^{\cdot}(\mathfrak{U},\mathscr{F}(m))$ is a resolution of the A-module $H^0(X,\mathscr{F}(m))$: we have an exact sequence

$$0 \rightarrow H^0(X,\mathscr{F}(m)) \rightarrow C^0(\mathfrak{U},\mathscr{F}(m)) \rightarrow C^1(\mathfrak{U},\mathscr{F}(m)) \rightarrow \ldots \rightarrow C^n(\mathfrak{U},\mathscr{F}(m)) \rightarrow 0.$$

Splitting this into short exact sequences, using (9.1Ae) and the fact that the C^i are all flat, we conclude that $H^0(X,\mathscr{F}(m))$ is a flat A-module. But it is also finitely generated (5.2), and hence free of finite rank by (9.1Af).

(ii) $\Rightarrow$ (i). Let $S = A[x_0, \ldots ,x_n]$, and let M be the graded S-module

$$M = \bigoplus_{m \geqslant m_0} H^0(X,\mathscr{F}(m)),$$

where m_0 is chosen large enough so that the $H^0(X,\mathscr{F}(m))$ are all free for $m \geqslant m_0$. Then $\mathscr{F} = \tilde{M}$ by (II, 5.15). Note that M is the same as $\Gamma_*(\mathscr{F})$ in degrees $m \geqslant m_0$, so $\tilde{M} = \Gamma_*(\mathscr{F})^{\sim}$. Since M is a free (and hence flat) A-module, we see that $\mathscr{F}$ is flat over A (9.1.1).

(ii) $\Rightarrow$ (iii). It will be enough to show that

$$P_t(m) = \operatorname{rank}_A H^0(X,\mathscr{F}(m))$$

for $m \gg 0$. To prove this we will show, for any $t \in T$, that

$$H^0(X_t,\mathscr{F}_t(m)) \cong H^0(X,\mathscr{F}(m)) \otimes_A k(t)$$

for all $m \gg 0$.

First we let $T' = \operatorname{Spec} A_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal corresponding to t, and we make the flat base extension $T' \rightarrow T$. Thus by (9.3) we reduce to the case where t is the closed point of T. Denote the closed fibre X_t by X_0, $\mathscr{F}_t$ by $\mathscr{F}_0$, and $k(t)$ by k. Take a presentation of k over A,

$$A^q \rightarrow A \rightarrow k \rightarrow 0.$$

Then we get an exact sequence of sheaves on X,

$$\mathscr{F}^q \rightarrow \mathscr{F} \rightarrow \mathscr{F}_0 \rightarrow 0.$$

Now by (Ex. 5.10) for $m \gg 0$ we get an exact sequence

$$H^0(X,\mathscr{F}(m)^q) \rightarrow H^0(X,\mathscr{F}(m)) \rightarrow H^0(X_0,\mathscr{F}_0(m)) \rightarrow 0.$$

On the other hand, we can tensor the sequence $A^q \rightarrow A \rightarrow k \rightarrow 0$ with $H^0(X,\mathscr{F}(m))$. Comparing, we deduce that

$$H^0(X_0,\mathscr{F}_0(m)) \cong H^0(X,\mathscr{F}(m)) \otimes_A k$$

for all $m \gg 0$, as required.

(iii) $\Rightarrow$ (ii). According to (II, 8.9) we can check the freeness of $H^0(X,\mathscr{F}(m))$ by comparing its rank at the generic point and the closed point of T. Hence the argument of (ii) $\Rightarrow$ (iii) above is reversible.

Corollary 9.10. *Let T be a connected noetherian scheme, and let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme which is flat over T. For any $t \in T$, let X_t be the fibre, considered as a closed subscheme of $\mathbf{P}^n_{k(t)}$. Then the dimension of X_t, the degree of X_t, and the arithmetic genus of X_t are all independent of t.*

PROOF. By base extension to the irreducible components of T with their reduced induced structure, we reduce to the case T integral. Now the result follows from the theorem and the facts (I, §7) and (Ex. 5.3) that

$$\dim X_t = \deg P_t,$$
$$\deg X_t = (r!)\cdot(\text{leading coefficient of } P_t),$$

where $r = \dim X$, and

$$p_a(X_t) = (-1)^r(P_t(0) - 1).$$

Definition. Let k be an algebraically closed field, let $f:X \rightarrow T$ be a surjective map of varieties over k, and assume that for each closed point $t \in T$, we have

(1) $f^{-1}(t)$ is irreducible of dimension equal to $\dim X - \dim T$, and
(2) if $\mathfrak{m}_t \subseteq \mathscr{O}_{t,T}$ is the maximal ideal, and if $\zeta \in f^{-1}(t)$ is the generic point, then $f^{\#}\mathfrak{m}_t$ generates the maximal ideal $\mathfrak{m}_\zeta \subseteq \mathscr{O}_{\zeta,X}$.

Under these circumstances, we let $X_{(t)}$ be the variety $f^{-1}(t)$ (with the reduced induced structure) and we say that the $X_{(t)}$ form an *algebraic family of varieties*, parametrized by T. The second condition is necessary to be sure that $X_{(t)}$ occurs with "multiplicity one" in the family. It is equivalent to saying that the scheme-theoretic fibre X_t is reduced at its generic point.

Example 9.10.1. In the flat family of (9.8.4), if we take the fibres with their reduced induced structures, we get an algebraic family of varieties $X_{(t)}$ parametrized by $\mathbf{A}^1$. For $t \neq 0$ it is a nonsingular rational curve, and for $t = 0$ it is the plane nodal cubic curve. Note that the arithmetic genus is not constant in this family: $p_a(X_{(t)}) = 0$ for $t \neq 0$ and $p_a(X_{(0)}) = 1$. This accounts for the appearance of nilpotent elements in the scheme-theoretic fibre X_0, since in a flat family of schemes p_a is constant by (9.10). The embedded point at 0 alters the constant term of the Hilbert polynomial so that we get $p_a(X_0) = 0$.

Theorem 9.11. *Let $X_{(t)}$ be an algebraic family of normal varieties parametrized by a nonsingular curve T over an algebraically closed field k. Then $X_{(t)}$ is a flat family of schemes.*

Proof. Let $f: X \to T$ be the defining morphism of the family. Then f is a flat morphism by (9.7). So we have only to show that for each closed point $t \in T$, the scheme-theoretic fibre X_t coincides with the variety $X_{(t)}$. In other words, we must show that X_t is reduced. For any point $x \in X$, let $A = \mathcal{O}_{x,X}$ be its local ring, let $f(x) = t$, and denote also by t a uniformizing parameter in the local ring $\mathcal{O}_{t,T}$. Then A/tA is the local ring of x on X_t. By hypothesis X_t is irreducible, so t has a unique minimal prime ideal $\mathfrak{p}$ in A. Furthermore, t generates the maximal ideal of the local ring of the generic point of X_t on X, which says that t generates the maximal ideal of $A_{\mathfrak{p}}$. Finally, the local ring of x on $X_{(t)}$ is $A/\mathfrak{p}$, so our hypothesis says that $A/\mathfrak{p}$ is normal. Now our result is a consequence of the following lemma, which tells us that $\mathfrak{p} = tA$, so $X_{(t)} = X_t$.

Lemma 9.12 (Lemma of Hironaka [1]). *Let A be a local noetherian domain, which is a localization of an algebra of finite type over a field k. Let $t \in A$, and assume*

(1) *tA has only one minimal associated prime ideal $\mathfrak{p}$,*
(2) *t generates the maximal ideal of $A_{\mathfrak{p}}$,*
(3) *$A/\mathfrak{p}$ is normal.*

Then $\mathfrak{p} = tA$ and A is normal.

Proof. Let $\tilde{A}$ be the normalization of A. Then $\tilde{A}$ is a finitely generated A-module by (I, 3.9A). We will show that the maps

$$\varphi: A/tA \to \tilde{A}/t\tilde{A}$$

and

$$\psi: A/tA \to A/\mathfrak{p}$$

are both isomorphisms.

First we localize at $\mathfrak{p}$. Then ψ is an isomorphism by hypothesis. Therefore $A_{\mathfrak{p}}$ is a discrete valuation ring, hence normal. So $\tilde{A}_{\mathfrak{p}} = A_{\mathfrak{p}}$ and φ is also an isomorphism.

Now suppose that at least one of φ,ψ is not an isomorphism. Then, after localizing A at a suitable prime ideal, we may assume that φ and ψ are isomorphisms at every localization $A_{\mathfrak{q}}$ with $\mathfrak{q} \neq \mathfrak{m}$, but that at least one of φ,ψ is not an isomorphism at $\mathfrak{m}$. By the previous step, we have $\mathfrak{p} < \mathfrak{m}$, so $\dim A \geqslant 2$. Now $\tilde{A}$ is normal of dimension $\geqslant 2$, so it has depth $\geqslant 2$ (II, 8.22A), so $\tilde{A}/t\tilde{A}$ has depth $\geqslant 1$. Therefore it does not have $\mathfrak{m}$ as an associated prime. On the other hand, $\tilde{A}/t\tilde{A}$ agrees with $A/\mathfrak{p}$ outside of $\mathfrak{m}$, so we conclude that $\tilde{A}/t\tilde{A}$ is an integral domain. Thus we have a natural map $(A/tA)_{\mathrm{red}} \to \tilde{A}/t\tilde{A}$. But $(A/tA)_{\mathrm{red}} \cong A/\mathfrak{p}$ since ψ is an isomorphism outside $\mathfrak{m}$. Thus $\tilde{A}/t\tilde{A}$ is a finitely generated $(A/\mathfrak{p})$-module with the same quotient field. Since $A/\mathfrak{p}$ is normal by hypothesis, we conclude that $\tilde{A}/t\tilde{A} \cong A/\mathfrak{p}$. Therefore φ is surjective, so we can write $\tilde{A} = A + t\tilde{A}$. By Nakayama's lemma, this implies that $A = \tilde{A}$. Thus $A/tA \cong A/\mathfrak{p}$ and both φ and ψ are isomorphisms. But this is a contradiction, so we conclude that φ and ψ were already isomorphisms on the original ring A before localization.

To conclude, we find that $\mathfrak{p} = tA$ because ψ is an isomorphism. Since φ is an isomorphism, we have $\tilde{A} = A + t\tilde{A}$, so by Nakayama's lemma as before, we find that $A = \tilde{A}$, so A is normal.

Corollary 9.13 (Igusa [1]). *Let $X_{(t)}$ be an algebraic family of normal varieties in $\mathbf{P}_k^n$, parametrized by a variety T. Then the Hilbert polynomial of $X_{(t)}$, and hence also the arithmetic genus $p_a(X_{(t)})$, are independent of t.*

PROOF. Any two closed points of T lie in the image of a morphism $g: T' \to T$, where T' is a nonsingular curve, or can be connected by a finite number of such curves, so by base extension, we reduce to the case where T is a nonsingular curve. Then the result follows from (9.10) and (9.11).

Example 9.13.1 (Infinitesimal Deformations). Now that we have seen that flatness is a natural condition for algebraic families of varieties, we come to an important nonclassical example of flatness in the category of schemes. Let X_0 be a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the *ring of dual numbers* over k. An *infinitesimal deformation* of X_0 is a scheme X', flat over D, and such that $X' \otimes_D k \cong X_0$.

These arise geometrically in the following way. If $f: X \to T$ is any flat family, having a point $t \in T$ with $X_t \cong X_0$, we say that X is a (*global*) *deformation* of X_0. Now given an element of the Zariski tangent space of T at t, we obtain a morphism Spec $D \to T$ (II, Ex. 2.8). Then by base extension we obtain an X' flat over Spec D with closed fibre X_0. Thus the study of the infinitesimal deformations of X_0 ultimately will help in the study of global deformations.

Example 9.13.2. Continuing the same ideas, it is often possible to classify the infinitesimal deformations of a scheme X. In particular, if X is nonsingular over an algebraically closed field k, we will show that the set of infinitesimal deformations of X, up to isomorphism, is in one-to-one correspondence with the elements of the cohomology group $H^1(X, \mathscr{T}_X)$, where $\mathscr{T}_X$ is the tangent sheaf.

Indeed, given X' flat over D, we consider the exact sequence

$$0 \to k \xrightarrow{t} D \to k \to 0$$

of D-modules. By flatness, we obtain an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. Thus X' is an infinitesimal extension of the scheme X by the sheaf $\mathcal{O}_X$, in the sense of (II, Ex. 8.7). Conversely, such an extension gives X' flat over D. Now these extensions are classified by $H^1(X, \mathscr{T}_X)$ by (Ex. 4.10).

Remark 9.13.3. There is a whole subject called *deformation theory* devoted to the study of deformations of a given scheme (or variety) X_0 over a field k.

It is closely related to the moduli problem. There one attempts to classify all varieties, and put them into algebraic families. Here we study only those that are close to a given one X_0.

Deformation theory is one area of algebraic geometry where the influence of schemes has been enormous. Because even if one's primary interest is in a variety X_0 over k, by working in the category of schemes, one can consider flat families over arbitrary Artin rings with residue field k, whose closed fibre is X_0. Taking the limit of Artin rings, one can study flat families over a complete local ring. Both of these types of families are intermediate between X_0 itself and a global deformation $f: X \to T$ where T is another variety. Thus they form a powerful tool for studying all deformations of X_0. For some references on deformation theory see Schlessinger [1], or Morrow and Kodaira [1, Ch. 4].

Exercises

9.1. A flat morphism $f: X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X$, $f(U)$ is open in Y. [*Hint*: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

9.2. Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

9.3. Some examples of flatness and nonflatness.

(a) If $f: X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

(b) Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x,y]$ and $X = \operatorname{Spec} k[x,y,z,w]/(z,w) \cap (x+z, y+w)$.

(c) Again let $Y = \operatorname{Spec} k[x,y]$, but take $X = \operatorname{Spec} k[x,y,z,w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{\mathrm{red}} \cong Y$, X has no embedded points, but that f is not flat.

9.4. *Open Nature of Flatness.* Let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X \mid f \text{ is flat at } x\}$ is an open subset of X (possibly empty)—see Grothendieck [EGA IV_3, 11.1.1].

9.5. *Very Flat Families.* For any closed subscheme $X \subseteq \mathbf{P}^n$, we denote by $C(X) \subseteq \mathbf{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k[x_0, \ldots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k[x_0, \ldots, x_{n+1}]$.

(a) Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbf{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbf{P}^{n+1}$.

(b) To remedy this situation, we make the following definition. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subseteq S_t = k(t)[x_0, \ldots, x_n]$ be the homogeneous ideal of X_t in $\mathbf{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is *very flat* if for all $d \geqslant 0$,

$$\dim_{k(t)} (S_t/I_t)_d$$

is independent of t. Here $(\quad)_d$ means the homogeneous part of degree d.

(c) If $\{X_t\}$ is a very flat family in $\mathbf{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbf{P}^{n+1}$, and hence flat.

(d) If $\{X_{(t)}\}$ is an algebraic family of projectively normal varieties in $\mathbf{P}_k^n$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_{(t)}\}$ is a very flat family of schemes.

9.6. Let $Y \subseteq \mathbf{P}^n$ be a nonsingular variety of dimension $\geqslant 2$ over an algebraically closed field k. Suppose $\mathbf{P}^{n-1}$ is a hyperplane in $\mathbf{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbf{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbf{P}^n$ if and only if Y' is a complete intersection in $\mathbf{P}^{n-1}$. [*Hint*: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

9.7. Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an *infinitesimal deformation of Y as a closed subscheme* of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y,\mathcal{N}_{Y/X})$, where

$$\mathcal{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2,\mathcal{O}_Y).$$

***9.8.** Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\operatorname{Hom}_A(\cdot,A)$, and let $T^1(A)$ be the cokernel:

$$\operatorname{Hom}_A(\Omega_{P/k} \otimes A,A) \to \operatorname{Hom}_A(J/J^2,A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

9.9. A k-algebra A is said to be *rigid* if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x,y,z,w]/(x,y) \cap (z,w)$, and show that A is rigid. This corresponds to two planes in $\mathbf{A}^4$ which meet at a point.

9.10. A scheme X_0 over a field k is *rigid* if it has no infinitesimal deformations.

(a) Show that $\mathbf{P}_k^1$ is rigid, using (9.13.2).

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f: X \to \mathbf{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbf{P}_k^1$, but there is no open neighborhood U of 0 in $\mathbf{A}^2$ for which $f^{-1}(U) \cong U \times \mathbf{P}^1$.

*(c) Show, however, that one can trivialize a global deformation of $\mathbf{P}^1$ after a flat base extension, in the following sense: let $f: X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbf{P}_k^1$. Then there exists a nonsingular curve T', and a flat morphism $g: T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f': X' \to T'$ is isomorphic to $\mathbf{P}_{T'}^1 \to T'$.

9.11. Let Y be a nonsingular curve of degree d in $\mathbf{P}_k^n$, over an algebraically closed field k. Show that
$$0 \leqslant p_a(Y) \leqslant \tfrac{1}{2}(d-1)(d-2).$$
[*Hint*: Compare Y to a suitable projection of Y into $\mathbf{P}^2$, as in (9.8.3) and (9.8.4).]

10 Smooth Morphisms

The notion of smooth morphism is a relative version of the notion of nonsingular variety over a field. In this section we will give some basic results about smooth morphisms. As an application, we give Kleiman's elegant proof of the characteristic 0 Bertini theorem. For further information about smooth and étale morphisms, see Altman and Kleiman [1, Ch. VI, VII], Matsumura [2, Ch. 11], and Grothendieck [SGA 1, exp. I, II, III].

For simplicity, we assume that all schemes in this section are of finite type over a field k.

Definition. A morphism $f: X \to Y$ of schemes of finite type over k is *smooth of relative dimension n* if:

(1) f is flat;
(2) if $X' \subseteq X$ and $Y' \subseteq Y$ are irreducible components such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y' + n$;
(3) for each point $x \in X$ (closed or not),
$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n.$$

Example 10.0.1. For any Y, $\mathbf{A}_Y^n$ and $\mathbf{P}_Y^n$ are smooth of relative dimension n over Y.

Example 10.0.2. If X is integral, then condition (3) is equivalent to saying $\Omega_{X/Y}$ is locally free on X of rank n (II, 8.9).

Example 10.0.3. If $Y = \operatorname{Spec} k$ and k is algebraically closed, then X is smooth over k if and only if X is regular of dimension n. In particular, if X is irreducible and separated over k, then it is smooth if and only if it is a nonsingular variety. Cf. (II, 8.8) and (II, 8.15).

Proposition 10.1.
(a) *An open immersion is smooth of relative dimension* 0.
(b) Base change. *If $f: X \to Y$ is smooth of relative dimension n, and $g: Y' \to Y$ is any morphism, then the morphism $f': X' \to Y'$ obtained by base extension is also smooth of relative dimension n.*
(c) Composition. *If $f: X \to Y$ is smooth of relative dimension n, and $g: Y \to Z$ is smooth of relative dimension m, then $g \circ f: X \to Z$ is smooth of relative dimension $n + m$.*

(d) Product. *If X and Y are smooth over Z, of relative dimensions n and m, respectively, then $X \times_Z Y$ is smooth over Z of relative dimension $n + m$.*

PROOFS.
(a) is trivial.
(b) f' is flat by (9.2). According to (9.6), the condition (2) in the definition of smoothness is equivalent to saying that every irreducible component of every fibre $X_{y'}$ of f has dimension n. This condition is preserved under base extension (II, Ex. 3.20). Finally, $\Omega_{X/Y}$ is stable under base extension (II, 8.10), so the number $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x))$ is also. Hence f' is smooth.
(c) $g \circ f$ is flat by (9.2). If $X' \subseteq X$, $Y' \subseteq Y$, and $Z' \subseteq Z$ are irreducible components such that $f(X') \subseteq Y'$ and $g(Y') \subseteq Z'$, then clearly $\dim X' = \dim Z' + n + m$ by hypothesis. For the last condition, we use the exact sequence of (II, 8.11)

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Tensoring with $k(x)$ we have

$$f^*\Omega_{Y/Z} \otimes k(x) \to \Omega_{X/Z} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0.$$

Now the first has dimension m, and the last has dimension n, by hypothesis. So the middle one has dimension $\leqslant n + m$.
On the other hand, let $z = g(f(x))$. Then

$$\Omega_{X/Z} \otimes k(x) = \Omega_{X_z/k(z)} \otimes k(x),$$

since relative differentials commute with base extension. Let X' be an irreducible component of X_z containing x, with its reduced induced structure. Then we have a surjective map

$$\Omega_{X_z/k(z)} \otimes k(x) \to \Omega_{X'/k(z)} \otimes k(x) \to 0$$

by (II, 8.12). But X' is an integral scheme of finite type over $k(z)$, of dimension $n + m$, by (9.6), so $\Omega_{X'/k(z)}$ is a coherent sheaf of rank $\geqslant n + m$ by (II, 8.6A). Hence it requires at least $n + m$ generators at every point, so

$$\dim_{k(x)}(\Omega_{X'/k(z)} \otimes k(x)) \geqslant n + m.$$

Combining our inequalities, we find that

$$\dim_{k(x)}(\Omega_{X/Z} \otimes k(x)) = n + m$$

as required.
(d) This statement is a consequence of (b) and (c) since we can factor into $X \times_Z Y \xrightarrow{p_2} Y \to Z$.

Theorem 10.2. *Let $f: X \to Y$ be a morphism of schemes of finite type over k. Then f is smooth of relative dimension n if and only if:*

(1) *f is flat; and*
(2) *for each point $y \in Y$, let $X_{\bar{y}} = X_y \otimes_{k(y)} k(y)^-$, where $k(y)^-$ is the algebraic closure of $k(y)$. Then $X_{\bar{y}}$ is equidimensional of dimension n and*

regular. (*We say "the fibres of f are* geometrically regular of equidimension n.")

PROOF. If f is smooth of relative dimension n, so is any base extension. In particular, $X_{\bar{y}}$ is smooth of relative dimension n over $k(y)^{-}$, so is regular (10.0.3).

Conversely, suppose (1) and (2) satisfied. Then f is flat by (1). From (2) we conclude that every irreducible component of X_y has dimension n, which gives condition (2) of the definition of smoothness by (9.6). Finally, since $k(y)^{-}$ is algebraically closed, regularity of $X_{\bar{y}}$ implies that $\Omega_{X_{\bar{y}}/k(y)^{-}}$ is locally free of rank n (10.0.3). This in turn implies that $\Omega_{X_y/k(y)}$ is locally free of rank n (see e.g. Matsumura [2, (4.E), p. 29]), and so for any $x \in X$,

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = \dim_{k(x)}(\Omega_{X_y/k(y)} \otimes k(x)) = n$$

as required.

Next we will study when a morphism of nonsingular varieties is smooth. Recall (II, Ex. 2.8) that for a point x in a scheme X we define the *Zariski tangent space* T_x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. If $f:X \to Y$ is a morphism, and $y = f(x)$, then there is a natural induced mapping on the tangent spaces

$$T_f : T_x \to T_y \otimes_{k(y)} k(x).$$

Before stating our criterion, we recall an algebraic fact.

Lemma 10.3.A. *Let $A \to B$ be a local homomorphism of local noetherian rings. Let M be a finitely generated B-module, and let $t \in A$ be a nonunit that is not a zero divisor. Then M is flat over A if and only if:*

(1) *t is not a zero divisor in M; and*
(2) *M/tM is flat over A/tA.*

PROOF. This is a special case of the "Local criterion of flatness." See Bourbaki [1, III, §5] or Altman and Kleiman [1, V, §3].

Proposition 10.4. *Let $f:X \to Y$ be a morphism of nonsingular varieties over an algebraically closed field k. Let $n = \dim X - \dim Y$. Then the following conditions are equivalent:*

(i) *f is smooth of relative dimension n;*
(ii) *$\Omega_{X/Y}$ is locally free of rank n on X;*
(iii) *for every closed point $x \in X$, the induced map on the Zariski tangent spaces $T_f : T_x \to T_y$ is surjective.*

PROOF.
(i) $\Rightarrow$ (ii) follows from the definition of smoothness, since X is integral (10.0.2).

(ii) $\Rightarrow$ (iii). From the exact sequence of (II, 8.11), tensoring with $k(x)$, we have

$$f^*\Omega_{Y/k} \otimes k(x) \to \Omega_{X/k} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0.$$

Now X and Y are both smooth over k, so the dimensions of these vector spaces are equal to dim Y, dim X, and n respectively. Therefore the map on the left is injective. But for a closed point x, $k(x) \cong k$, so using (II, 8.7) we see that this map is just the natural map

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$$

induced by f. Taking dual vector spaces over k, we find that T_f is surjective.

(iii) $\Rightarrow$ (i). First we show f is flat. For this, it is enough to show that $\mathcal{O}_x$ is flat over $\mathcal{O}_y$ for every *closed* point $x \in X$, where $y = f(x)$, by localization of flatness. Since X and Y are nonsingular, these are both regular local rings. Furthermore, since T_f is surjective, we have $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ injective as above. So let $t_1, \ldots, t_r$ be a regular system of parameters for $\mathcal{O}_y$. Then their images in $\mathcal{O}_x$ form part of a regular system of parameters of $\mathcal{O}_x$. Since $\mathcal{O}_x/(t_1, \ldots, t_r)$ is automatically flat over $\mathcal{O}_y/(t_1, \ldots, t_r) = k$, we can use (10.3A) to show by descending induction on i that $\mathcal{O}_x/(t_1, \ldots, t_i)$ is flat over $\mathcal{O}_y/(t_1, \ldots, t_i)$ for each i. In particular, for $i = 0$, $\mathcal{O}_x$ is flat over $\mathcal{O}_y$. Thus f is flat.

Now we can read the argument of (ii) $\Rightarrow$ (iii) backwards to conclude that

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$$

for each closed point $x \in X$. On the other hand since f is flat, it is dominant, so for the generic point $\zeta \in X$, we have

$$\dim_{k(\zeta)}(\Omega_{X/Y} \otimes k(\zeta)) \geqslant n$$

by (II, 8.6A). We conclude that $\Omega_{X/Y}$ is coherent of rank $\geqslant n$, so it must be locally free of rank $= n$ by (II, 8.9). Therefore $\Omega_{X/Y} \otimes k(x)$ has dimension n at every point of X, so f is smooth of relative dimension n.

Next we will give some special results about smoothness which hold only in characteristic zero.

Lemma 10.5. *Let $f: X \to Y$ be a dominant morphism of integral schemes of finite type over an algebraically closed field k of characteristic* 0. *Then there is a nonempty open set $U \subseteq X$ such that $f: U \to Y$ is smooth.*

Proof. Replacing X and Y by suitable open subsets, we may assume that they are both nonsingular varieties over k (II, 8.16). Next, since we are in characteristic 0, $K(X)$ is a separably generated field extension of $K(Y)$ (I, 4.8A). So by (II, 8.6A), $\Omega_{X/Y}$ is free of rank $n = \dim X - \dim Y$ at the generic point of X. Therefore it is locally free of rank n on some nonempty open set $U \subseteq X$. We conclude that $f: U \to Y$ is smooth by (10.4).

Example 10.5.1. Let k be an algebraically closed field of characteristic p, let $X = Y = \mathbf{P}_k^1$, and let $f:X \to Y$ be the Frobenius morphism (I, Ex. 3.2). Then f is not smooth on any open set. Indeed, since $d(t^p) = 0$, the natural map $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is the zero map, and so $\Omega_{X/Y} \cong \Omega_{X/k}$ is locally free of rank 1. But f has relative dimension 0, so it is nowhere smooth.

Proposition 10.6. *Let $f:X \to Y$ be a morphism of schemes of finite type over an algebraically closed field k of characteristic 0. For any r, let*

$$X_r = \{\text{closed points } x \in X \mid \text{rank } T_{f,x} \leqslant r\}.$$

Then

$$\dim \overline{f(X_r)} \leqslant r.$$

PROOF. Let Y' be any irreducible component of $\overline{f(X_r)}$, and let X' be an irreducible component of $\bar{X}_r$ which dominates Y'. We give X' and Y' their reduced induced structures, and consider the induced dominant morphism $f':X' \to Y'$. Then by (10.5) there is a nonempty open subset $U' \subseteq X'$ such that $f':U' \to Y'$ is smooth. Now let $x \in U' \cap X_r$, and consider the commutative diagram of maps of Zariski tangent spaces

$$\begin{array}{ccc} T_{x,U'} & \longrightarrow & T_{x,X} \\ \Big\downarrow{\scriptstyle T_{f',x}} & & \Big\downarrow{\scriptstyle T_{f,x}} \\ T_{y,Y'} & \longrightarrow & T_{y,Y} \end{array}$$

The horizontal arrows are injective, because U' and Y' are locally closed subschemes of X and Y, respectively. On the other hand, rank $T_{f,x} \leqslant r$ since $x \in X_r$, and $T_{f',x}$ is surjective because f' is smooth (10.4). We conclude that $\dim T_{y,Y'} \leqslant r$, and therefore $\dim Y' \leqslant r$.

Corollary 10.7 (Generic Smoothness). *Let $f:X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic* 0, *and assume that X is nonsingular. Then there is a nonempty open subset $V \subseteq Y$ such that $f:f^{-1}V \to V$ is smooth.*

PROOF. We may assume Y is nonsingular by (II, 8.16). Let $r = \dim Y$. Let $X_{r-1} \subseteq X$ be the subset defined in (10.6). Then $\dim \overline{f(X_{r-1})} \leqslant r - 1$ by (10.6), so removing it from Y, we may assume that rank $T_f \geqslant r$ for every closed point of X. But since Y is nonsingular of dimension r, this implies that T_f is surjective for every closed point of X. Hence f is smooth by (10.4).

Note that if the original f was not dominant, then $V \subseteq Y - \overline{f(X)}$, and $f^{-1}V$ will be empty.

For the next results, we recall the notion of a group variety (I, Ex. 3.21). A *group variety* G over an algebraically closed field k is a variety G, together with morphisms $\mu: G \times G \to G$ and $\rho: G \to G$, such that the set $G(k)$ of k-rational points of G (which is just the set of all closed points of G, since k is algebraically closed) becomes a group under the operation induced by μ, with ρ giving the inverses.

We say that a group variety G *acts* on a variety X if we have a morphism $\theta: G \times X \to X$ which induces a homomorphism $G(k) \to \operatorname{Aut} X$ of groups.

A *homogeneous space* is a variety X, together with a group variety G acting on it, such that the group $G(k)$ acts transitively on the set $X(k)$ of k-rational points of X.

Remark 10.7.1. Any group variety is a homogeneous space if we let it act on itself by left multiplication.

Example 10.7.2. The projective space $\mathbf{P}_k^n$ is a homogeneous space for the action of $G = \mathrm{PGL}(n)$—cf. (II, 7.1.1).

Example 10.7.3. A homogeneous space is necessarily a nonsingular variety. Indeed, it has an open subset which is nonsingular by (II, 8.16). But we have a transitive group of automorphisms acting, so it is nonsingular everywhere.

Theorem 10.8 (Kleiman [3]). *Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic 0. Let $f: Y \to X$ and $g: Z \to X$ be morphisms of nonsingular varieties Y, Z to X. For any $\sigma \in G(k)$, let Y^σ be Y with the morphism $\sigma \circ f$ to X. Then there is a nonempty open subset $V \subseteq G$ such that for every $\sigma \in V(k)$, $Y^\sigma \times_X Z$ is nonsingular and either empty or of dimension exactly*

$$\dim Y + \dim Z - \dim X.$$

PROOF. First we consider the morphism

$$h: G \times Y \to X$$

defined by composing f with the group action $\theta: G \times X \to X$. Now G is nonsingular since it is a group variety (10.7.3), and Y is nonsingular by hypothesis, so $G \times Y$ is nonsingular by (10.1). Since char $k = 0$, we can apply generic smoothness (10.7) to h, and conclude that there is a nonempty open subset $U \subseteq X$ such that $h: h^{-1}(U) \to U$ is smooth. Now G acts on $G \times Y$ by left multiplication on G; G acts on X by θ, and these two actions are compatible with the morphism h, by construction. Therefore, for any $\sigma \in G(k)$, $h: h^{-1}(U^\sigma) \to U^\sigma$ is also smooth. Since the U^σ cover X, we conclude that h is smooth everywhere.

Next, we consider the fibred product

$$W = (G \times Y) \times_X Z,$$

with maps g' and h' to $G \times Y$ and Z as shown.

$$\begin{tikzcd} W \arrow[r, "h'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\ G \times Y \arrow[r, "h"] \arrow[d, "p_1"] & X \\ G & \end{tikzcd}$$

Since h is smooth, h' is also smooth by base extension (10.1). Since Z is nonsingular, it is smooth over k, so by composition (10.1), W is also smooth over k, so W is nonsingular.

Now we consider the morphism

$$q = p_1 \circ g' : W \to G.$$

Applying generic smoothness (10.7) again, we find there is a nonempty open subset $V \subseteq G$ such that $q : q^{-1}(V) \to V$ is smooth. Therefore, if $\sigma \in V(k)$ is any closed point, the fibre W_σ will be nonsingular. But W_σ is just $Y^\sigma \times_X Z$, so this is what we wanted to show. Note that W_σ may not be irreducible, but our result shows that each connected component is a nonsingular variety.

To find the dimension of W_σ, we first note that h is smooth of relative dimension

$$\dim G + \dim Y - \dim X.$$

Hence h' has the same relative dimension, and we see that

$$\dim W = \dim G + \dim Y - \dim X + \dim Z.$$

If W is nonempty, then q on $q^{-1}(V)$ has relative dimension equal to $\dim W - \dim G$, so for each σ,

$$\dim W_\sigma = \dim Y + \dim Z - \dim X.$$

Corollary 10.9 (Bertini). *Let X be a nonsingular projective variety over an algebraically closed field k of characteristic 0. Let $\mathfrak{d}$ be a linear system without base points. Then almost every element of $\mathfrak{d}$, considered as a closed subscheme of X, is nonsingular (but maybe reducible).*

PROOF. Let $f : X \to \mathbf{P}^n$ be the morphism to $\mathbf{P}^n$ determined by $\mathfrak{d}$ (II, 7.8.1). We consider $\mathbf{P}^n$ as a homogeneous space under the action of $G = \mathrm{PGL}(n)$ (10.7.2). We apply the theorem taking $g : H \to \mathbf{P}^n$ to be the inclusion map of a hyperplane $H \cong \mathbf{P}^{n-1}$. We conclude that for almost all $\sigma \in G(k)$, $X \times_{\mathbf{P}^n} H^\sigma = f^{-1}(H^\sigma)$ is nonsingular. But the divisors $f^{-1}(H^\sigma)$ are just the elements of the linear system $\mathfrak{d}$, by construction of f. Thus almost all elements of $\mathfrak{d}$ are nonsingular.

Remark 10.9.1. We will see later (Ex. 11.3) that if $\dim f(X) \geqslant 2$, then all the divisors in $\mathfrak{d}$ are connected. Hence almost all of them are irreducible and nonsingular.

Remark 10.9.2. The hypothesis "X projective" is not necessary if we talk about a finite-dimensional linear system $\mathfrak{d}$. In particular, if X was projective, and $\mathfrak{d}$ was a linear system with base points Σ, then by considering the base-point-free linear system $\mathfrak{d}$ on $X - \Sigma$ we obtain the more general statement that "a general member of $\mathfrak{d}$ can have singularities only at the base points."

Remark 10.9.3. This result fails in characteristic $p > 0$. For example, in (10.5.1) the morphism f corresponds to the one-dimensional linear system $\{pP | P \in \mathbf{P}^1\}$. Thus every divisor in $\mathfrak{d}$ is a point with multiplicity p.

Remark 10.9.4. Compare this result to the earlier Bertini theorem (II, 8.18).

Exercises

10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq \mathbf{A}_k^2$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

10.2. Let $f: X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f: f^{-1}(U) \to U$ is smooth.

10.3. A morphism $f: X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;
(ii) f is flat, and $\Omega_{X/Y} = 0$;
(iii) f is flat and unramified.

10.4. Show that a morphism $f: X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x \in X$, let $y = f(x)$. Let $\hat{\mathcal{O}}_x$ and $\hat{\mathcal{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x) \subseteq \hat{\mathcal{O}}_x$ and $k(y) \subseteq \hat{\mathcal{O}}_y$ so that $k(y) \subseteq k(x)$ via the natural map $\hat{\mathcal{O}}_y \to \hat{\mathcal{O}}_x$. Then our condition is that for every $x \in X$, $k(x)$ is a separable algebraic extension of $k(y)$, and the natural map

$$\hat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \hat{\mathcal{O}}_x$$

is an isomorphism.

10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f: U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathscr{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f: U \to X$ for which $f^*\mathscr{F}$ is a free $\mathcal{O}_U$-module, then $\mathscr{F}$ is locally free on X.