Math 750 HW 2

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Due: Sunday night

The exercises in this homework concern the concept of projective dimension of a module and global dimension of a ring.

Let R be a ring, and M an R-module. We will say that M has *projective dimension* $\mathrm{pd}(M) = d$ if the length of the shortest projective resolution of M is d (i.e., it has $d+1$ non-zero projective modules in it). So a non-zero projective module has projective dimension 0. By convention we set $\mathrm{pd}(M) = -1$ for the zero module $M = 0$. If the module M does not admit a finite projective resolution we set $\mathrm{pd}(M) = \infty$.

One also defines an invariant of the ring itself: the *global dimension* of R (also called the homological dimension) is the supremum of $\mathrm{pd}(M)$ over all R-modules M. (If the ring is not commutative, there are left and right versions of global dimension.)

Our first goal is to prove that

$$\mathrm{pd}(M) \leq n \quad \text{if and only if} \quad \mathrm{Ext}^i_R(M, N) = 0$$

for all R-modules N and all $i > n$. In other words the projective dimension of M coincides with the largest non-zero i for which we can find a non-zero $\mathrm{Ext}^i(M, -)$.

1. Prove from the definition of Ext that if $\mathrm{pd}(M) = n$ then $\mathrm{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i > n$.

   Proof. From the problem we know we have a projective resolution:

   $$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

   And, we can apply $Hom(-, N)$ to get:

   $$0 \leftarrow Hom(P_n, N) \leftarrow \cdots \leftarrow Hom(P_0, N) \leftarrow Hom(M, N) \leftarrow 0$$

   So, it is easy to see the homology of this chain complex is zero when $i > n$. So, $\mathrm{Ext}^i_R(M, N) = 0$. $\square$

2. Prove that the reverse implication holds for $M = 0$ and for M projective.

   Proof. (i): Suppose $M = 0$. Then it is easy to calculate that $\mathrm{Ext}^i_R(M, N) = 0,\ i \geq 0$. And we know M has a projective resolution:

   $$0 \to \cdots \to 0 \to 0 \to 0 \to \cdots$$

   which indicates $\mathrm{pd}(M) = -1$.
   (ii): Suppose M is projective. Then we easy calculate that $\mathrm{Ext}^i_R(M) = 0,\ i \geq 1$. And, we have a projective resolution of M:

   $$0 \to \cdots \to 0 \to M \to M \to 0 \to \cdots$$

   which indicates that $\mathrm{pd}(M) = 0$. $\square$

3. Assume that M is not projective (hence $\mathrm{pd}(M) \geq 1$). Let

   $$0 \to K \to P \to M \to 0$$

   be a short exact sequence with P projective. Prove that if $\mathrm{Ext}^i_R(M, N) = 0$ for all N and all $i > n$, then $\mathrm{Ext}^i_R(K, N) = 0$ for all N and all $i > n - 1$.

   Proof. We have a long exact sequence :

   $$\cdots \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(P.N) \to \mathrm{Ext}^i_R(K, N) \to \mathrm{Ext}^{i+1}_R(M, N) \to \cdots$$

(i): Suppose $n \geq 1$. Then when $i > n-1 \geq 1$, this long exact sequence collapses

$$\cdots \to \mathrm{Ext}^i_R(M,N) \to 0 \to \mathrm{Ext}^i_R(K,N) \to 0 \to \cdots$$

$\Rightarrow \mathrm{Ext}^i_R(K,N) = 0$. (ii): We need to prove n cannot be 0. Suppose yes, Then we have $\mathrm{Ext}^i_R(M,N) = 0,\ \forall i \geq 1,\ \forall N$. Suppose we have a short exact sequence $0 \to M_1 \to M^{'} \to M_2 \to 0$. Then We get a long exact sequence

$$0 \to \mathrm{Hom}(M,M_1) \to \mathrm{Hom}(M,M^{'}) \to \mathrm{Hom}(M,M_2) \to \mathrm{Ext}^1_R(M,M_1) = 0 \to \mathrm{Ext}^1_R(M,M^{'}) = 0 \to 0$$

which collapses to short exact sequence

$$0 \to \mathrm{Hom}(M,M_1) \to \mathrm{Hom}(M,M^{'}) \to \mathrm{Hom}(M,M_2) \to 0$$

$\Rightarrow M$ is projective. ☐

4. Prove the second implication in the statement we want to prove (one implication was problem 1) by induction, using the fact that the category of R-modules has enough projectives.

Proof. Suppose this statement is right when $i > n-1 \geq -1$. Suppose $\mathrm{Ext}^i_R(M,N) = 0,\ i > n$. Since $R-$mod has enough projective objects, we can construct a short exact sequence :

$$0 \to K \to P \to M \to 0$$

Then we have $\mathrm{Ext}^I_R(K,N) = 0,\ i > n-1$. Then $\mathrm{pd}(K) \leq n-1$. So, we can find a projective resolution of K:

$$0 \to P_{n-1} \to \cdots \to P_0 \to K \to 0$$

So, we can get a projetive resolution of M:

$$0 \to P_{n-1} \to \cdots \to P_0 \to P \to M \to 0$$

So, $\mathrm{pd}(M) \leq n$. ☐

5. Prove that the global dimension of a PID is 1.

Proof. Since R is a PID, then for every $R-$mod M, we have a free resolution $0 \to F_1 \to F \to M \to 0$. So, $\mathrm{Ext}^i_R(M,N) = 0,\ i > 1$. So, the global dimension of R less than 1. Let $M = R/I$, and I is an ideal of R. Then M has at least length equaling 1 projective resolution. So, the global dimension of a PID is 1. ☐

6. Prove that the global dimension of $k[x]/(x^2)$ is ∞ (k a field).

Proof. Let R denote $k[x]/(x^2)$. Let I denote the ideal generated by $\overline{x}$. Then we have an infinite projective resolution of R/I:

$$\cdots \to R \xrightarrow{\cdot x} R \cdots \xrightarrow{\cdot x} R \to R/I$$

Now we apply $\mathrm{Hom}(-, R/I)$. And, we can get a chain complex of

$$\cdots \xleftarrow{0} R/I \xleftarrow{0} R/I \xleftarrow{\text{inclusion}} R/I$$

So, we can get $\mathrm{Ext}^i_R(R/I, R/I) = R/I$. This indicates $\mathrm{pd}(R/I) = \infty$. So, the global dimension of $k[x]/(x^2)$ is ∞. ☐

7. Let n be a non-zero integer. Find the global dimension of $\mathbb{Z}/n\mathbb{Z}$. (The answer will depend on n.)

Proof. We first prove: Let $\{R_i\}_{i\in I}$ be any family of rings and set $R = \prod_{i\in I} R_i$. Then

$$\operatorname{gldim}(R) = \sup_{i\in I} \operatorname{gldim}(R_i).$$

For each $i \in I$ let $e_i \in R$ be the idempotent with ith coordinate 1 and all other coordinates 0. For an R-module M put

$$M_i := R_i \otimes_R M \;\cong\; e_i M.$$

Define functors

$$\Phi : \operatorname{mod}(R) \longrightarrow \prod_{i\in I} \operatorname{mod}(R_i), \quad \Phi(M) := (M_i)_i,$$

and

$$\Psi : \prod_{i\in I} \operatorname{mod}(R_i) \longrightarrow \operatorname{mod}(R), \quad \Psi\big((N_i)_i\big) := \prod_{i\in I} N_i$$

with the coordinatewise R-action $(r_i)_i \cdot (n_i)_i = (r_i n_i)_i$. There are natural isomorphisms

$$\Phi\Psi\big((N_i)_i\big) \cong (N_i)_i \quad \text{and} \quad \Psi\Phi(M) \cong M,$$

hence Φ and Ψ yield an equivalence $\operatorname{mod}(R) \simeq \prod_i \operatorname{mod}(R_i)$.

Moreover, each R_i is (as a right R-module) isomorphic to $e_i R$, which is a direct summand of the free R-module R; hence R_i is projective (in particular flat) as a right R-module. It follows that the tensor functors $R_i \otimes_R (-)$ are exact, and Ψ is exact as well (being an inverse equivalence). Consequently, projective objects and exact sequences correspond under the equivalence.

Given $M \in \operatorname{mod}(R)$ and $M_i := R_i \otimes_R M$, apply Φ to any projective resolution $P_\bullet \to M$. Exactness of Φ shows that each $(P_\bullet)_i \to M_i$ is a projective resolution over R_i, hence

$$\operatorname{pd}_{R_i}(M_i) \le \operatorname{pd}_R(M) \qquad (\forall i \in I).$$

Conversely, suppose for each i we are given a projective resolution $Q_\bullet^{(i)} \to M_i$ of length at most d_i. Applying Ψ to the family $\big(Q_\bullet^{(i)}\big)_i$ and using that Ψ preserves projectives and exactness, we obtain a projective resolution of M of length $\sup_i d_i$. Therefore

$$\operatorname{pd}_R(M) = \sup_{i\in I} \operatorname{pd}_{R_i}(M_i)$$

Equivalently, one may express this via derived functors: for any $N \in \operatorname{mod}(R)$,

$$\operatorname{Ext}^n_R(M,N) \;\cong\; \prod_{i\in I} \operatorname{Ext}^n_{R_i}(M_i, N_i),$$

and the smallest n annihilating all these groups is exactly $\sup_i \operatorname{pd}_{R_i}(M_i)$.

Now compute

$$\operatorname{gldim}(R) = \sup_{M\in\operatorname{mod}(R)} \operatorname{pd}_R(M) = \sup_M \sup_{i\in I} \operatorname{pd}_{R_i}\big(R_i \otimes_R M\big) = \sup_{i\in I} \sup_{N\in\operatorname{mod}(R_i)} \operatorname{pd}_{R_i}(N) = \sup_{i\in I} \operatorname{gldim}(R_i).$$

In the penultimate equality we use that every R_i-module N occurs as the i-th component of some R-module under the equivalence (take N in the i-th slot and 0 elsewhere).

Now we can apply this result to this question.

Let $n = \prod_{i=1}^r p_i^{e_i}$ and $R = \mathbb{Z}/n\mathbb{Z}$. Then

$$\operatorname{gldim}(R) = \begin{cases} 0, & \text{if } e_i = 1 \text{ for all } i; \\ \infty, & \text{if } e_i \geq 2 \text{ for some } i. \end{cases}$$

By the Chinese remainder theorem,

$$R \;\cong\; \prod_{i=1}^r \mathbb{Z}/p_i^{e_i}\mathbb{Z}.$$

Then,

$$\operatorname{gldim}(R) = \sup_{1 \leq i \leq r} \operatorname{gldim}\bigl(\mathbb{Z}/p_i^{e_i}\mathbb{Z}\bigr),$$

so it suffices to compute the global dimension of $R_p := \mathbb{Z}/p^e\mathbb{Z}$.

Case $e = 1$. Then $R_p \cong \mathbb{F}_p$ is a field; every module is projective, so $\operatorname{gldim}(R_p) = 0$.

Case $e \geq 2$. Let $R = R_p = \mathbb{Z}/p^e\mathbb{Z}$, $\mathfrak{m} = (p)$, and $k = R/\mathfrak{m} \cong \mathbb{F}_p$. Consider the two–periodic complex of free R-modules

$$\cdots \xrightarrow{\times p} R \xrightarrow{\times p^{\,e-1}} R \xrightarrow{\times p} R \twoheadrightarrow k \to 0,$$

where the rightmost map is the natural quotient $R \to k$. We verify exactness:

$$\ker(\times p) = \{x \in R : px = 0\} = p^{e-1}R = \operatorname{im}(\times p^{e-1}),$$

and

$$\ker(\times p^{e-1}) = \{x \in R : p^{e-1}x = 0\} = pR = \operatorname{im}(\times p).$$

Moreover $(\times p) \circ (\times p^{e-1}) = \times p^e = 0$ in R. Hence this is an infinite projective (indeed free) resolution of k, so $\operatorname{pd}_R(k) = \infty$ and therefore $\operatorname{gldim}(R) = \infty$. $\square$

(One statement that I mentioned in class, but we will not prove, is that the global dimension of $k[x_1, \ldots, x_n]$ is n. More generally, a theorem of Auslander–Buchsbaum and/or Serre says that regular rings – i.e. those whose spectrum is smooth as an algebraic variety – are precisely those with finite global dimension.)