Math 761 HW 3

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1. Let $M = \mathbb{RP}^1$. Define $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ by

$$f([X, Y]) = [X^{1/3}, Y^{1/3}].$$

Show that $f : M \to M$ is a homeomorphism but not a diffeomorphism.

Proof. First we prove f is a bijection. Let $g([X, Y]) = [X^3, Y^3] : \mathbb{RP}^1 \to \mathbb{RP}^1$, then we have $f \circ g = [X, Y] = g \circ f$. So, f is indeed a bijection. Next, we show f is homeomorphism. (i): f is continuous. Noticing that the smooth charts $\{(U_X, \phi_X), (U_Y, \phi_Y)\}$ of $\mathbb{RP}^1$ also makes $\mathbb{RP}^1$ a C^0 manifold. So, if f is continuous make senses as C^0 manifold meaning(Lee's definition), we can show it is continuous. For $p \in U_X$, we can choose U_X, U_X to make sure $f(U_X) \subset U_X$, and $\phi_X^{-1} \circ f \circ \phi_X = r \to r^{\frac{1}{3}},\ r \in \mathbb{R}$, which is continuous. Similarly, for $p \in U_Y$, we can choose U_Y, U_Y to make sure $f(U_Y) \subset U_Y$, and $\phi_Y^{-1} \circ f \circ \phi_Y = r \to r^{\frac{1}{3}},\ r \in \mathbb{R}$, which is continuous. So, f is continuous. (ii) g is continuous. Like what we did in (i), we verify this by viewing $\mathbb{RP}^1$ as C^0 manifold. For $p \in U_X$, we can choose U_X, U_X to make sure $g(U_X) \subset U_X$, and $\phi_X^{-1} \circ g \circ \phi_X = r \to r^3,\ r \in \mathbb{R}$, which is continuous. Similarly, for $p \in U_Y$, we can choose U_Y, U_Y to make sure $fg(U_Y) \subset U_Y$, and $\phi_Y^{-1} \circ g \circ \phi_Y = r \to r^3,\ r \in \mathbb{R}$, which is continuous. So, g is continuous, which means f is homeomorphism. So now we prove this is a homeomorphism. We claim f is not smooth at $[0, 1]$. Since f is continuous, then by the definition from the class, we need to find two charts $(U, \phi), (V, \psi)$ containing $[0, 1]$ and $f\big([0, 1]\big)$, and $\psi \circ f \circ \phi^{-1}$ should be smooth on some resonable domain of $\phi(U)$. Noticing that the only charts containing $[0, 1], f\big([0, 1]\big)$ is $(U_Y, \phi_Y), (U_Y, \phi_Y)$. But $\phi_Y \circ f \circ \phi_Y^{-1} = x^{\frac{1}{3}}$ is not smooth at $0 = \phi_Y\big([0, 1]\big)$. So, f is not smooth, therefore not a diffeomorphism. □

2. (Lee 2.7) Let M be a nonempty smooth n-manifold with $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

Proof. Let $f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0, & t \leq 0 \end{cases}$ From the class we know this is a smooth function. Let $h(t) = f(1 - t^2)$. Then by leibnize law, it is also a smooth function. Let $h_{a,b}(t) = h(\frac{2}{b-a}t - \frac{a+b}{b-a})$. Then we know $h_{a,b}(t)$ is also smooth. And, its support is $[a, b]$. We fix a point $p \in M$. Let (U, ϕ) be its chart, and we may assume $\phi(p) = 0$. We can find an open ball $B_r \subset \phi(U)$ with radius $r > 0$. Then we may as well assume $(U = \phi^{-1}(B_r), \phi = \phi|_{\phi^{-1}(B_r)})$. For every $n > 2, n \in \mathbb{Z}$. We can define some 'annuli' in $\phi(U)$, which are $\Gamma_{n,k} = \{x \in \phi(U) | \frac{r}{n} \cdot k < |x|^2 < \frac{r}{n} \cdot (k+1)\}, 0 \leq k \leq n-1$. For each these annulus we can define relevant smooth function $H_{n,k}(x) = h_{\frac{r}{n}(k+\frac{1}{3}), \frac{r}{n}(k+\frac{2}{3})}(|x|^2)$. And we can define some smooth function on M through $H_{n,k}(x)$. We define $F_{n,k}(x) = \begin{cases} H_{n,k}(\phi(x)), x \in U \\ 0 \ \ \textbf{elsewhere} \end{cases}$. These are well-defined smooth maps on M. Each of their support $\phi^{-1}\{|x| \in [\sqrt{\frac{r}{n}(k+\frac{1}{3})}, \sqrt{\frac{r}{n}(k+\frac{2}{3})}]\}$ is disjoint. Now we claim for fixed n, $F_{n,k}(x)$ are linear independent. If $\sum\limits_i a_i F_{n,i} = 0$, then $\sum\limits_i a_i F_{n,i} = 0$ in $\{|x| \in [\sqrt{\frac{r}{n}(k+\frac{1}{3})}, \sqrt{\frac{r}{n}(k+\frac{2}{3})}]\}$ for a fixed k, which actually means $a_k = 0$. So, $F_{n,k}$ are linear independtly. Noticing n is a random positive integral number, $C^\infty(M)$ is infinite-dimensional. □

3. (Lee 2.9) Let $p(z)$ be a degree d polynomial in one complex variable. Show that the map $p : \mathbb{C} \to \mathbb{C}$ extends to a smooth map from $\mathbb{CP}^1 \to \mathbb{CP}^1$, where we take $\mathbb{C} \subset \mathbb{CP}^1$ to be a standard coordinate chart.

Proof. Let $U_{z_1} = \{[Z_1, Z_2] \in \mathbb{CP}^1 | Z_1 \neq 0\}, U_{z_2} = \{[Z_1, Z_2] \in \mathbb{CP}^1 | Z_2 \neq 0\}$. Let $\phi_{z_1} = [Z_1, Z_2] \to \frac{Z_2}{Z_1}, \phi_{z_2} = [Z_1, Z_2] \to \frac{Z_1}{Z_2}$. And, let ϕ be the natural homeomorphism from $\mathbb{C} \to \mathbb{R}^2$. Then we know $(U_{z_1}, \phi \circ \phi_{z_1}), (U_{z_2}, \phi \circ \phi_{z_2})$ have already made $\mathbb{CP}^1$ a smooth manifold. But here we add another smooth chart in order to extend this smooth map. Let $p(z) = a_d z^d + \cdots + a_0, p^{'}(z) = a_d + a_{d-1}z + \cdots + a_0 z^d$. Let $U \subset \mathbb{C}$ be an open set containing 0, which makes $p^{'}(U) \neq 0$. Then we know $U^{'} = \{[1, z] | z \in U\}$ is an open subset of $\mathbb{CP}^1$, since it is the image of U under $\phi_{z_2}^{-1}$. So, we give $\mathbb{CP}^1$ an atlas $\{(U_{z_1}, \phi \circ \phi_{z_1}), (U_{z_2}, \phi \circ \phi_{z_2}), (U^{'}, \phi \circ \phi_{z_1})\}$. Now we define the extension of $p(z)$. Let $P([Z_1, Z_2]) = [a_d Z_1^d + a_{d-1} Z_1^{d-1} Z_2 + \cdots + a_0 Z_2^d, Z_2^d]$, which is well defined. If we view $\mathbb{C}$ as $\phi_{z_2}(U_{z_2})$, then $\phi_{z_2} \circ P \circ \phi_{z_2}^{-1}(z) = \phi_{z_2}([p(z), 1]) = p(z)$, which means P is actually an extension. Now, we prove smoothness. For every $[Z_1, Z_2] \in U_{z_2}$, we cam choose U_{z_2}, U_{z_2} to make sure $P(U_{z_2}) \subset U_{z_2}$. And $\phi_{z_2} \circ P \circ \phi_{z_2}^{-1} = p(z)$, which is holomorphic, so $\phi \circ \phi_{z_2} \circ P \circ \phi_{z_2}^{-1} \circ \phi^{-1}$ is smooth. Now we need to consider $[1, 0]$, which is the only element of $\mathbb{CP}^1$ but $\notin U_{z_2}$. Now we pick $U^{'}, U_{z_1}$, since $P(U^{'}) \subset U_{z_1}$. And $\phi_{z_1} \circ P \circ \phi_{z_1}^{-1} = z^d / p^{'}(z)$. Since $p^{'}(z) \neq 0, z \in U$, we know $z^d / p^{'}(z)$ is holomorphic on U, which means $\phi \circ \phi_{z_1} \circ P \circ \phi_{z_1}^{-1} \circ \phi^{-1}$ is smooth on $\phi(U) \subset \mathbb{R}^2$. Now, we prove P is indeed a smooth from $\mathbb{CP}^1 \to \mathbb{CP}^1$. □

4.(Lee 2.10) Let M and N be smooth manifolds. Given a continuous map $F : M \to N$, consider the map

$$F^* : C^0(N) \to C^0(M), \quad f \mapsto f \circ F.$$

(a) Show that F^* is a linear map.

Proof. $F^*(f_1 + f_2) = (f_1 + f_2) \circ F = f_1 \circ F + f_2 \circ F = F^*(f_1) + F^*(f_2)$. Also, let $a \in \mathbb{R}$, then we have $F^*(af) = af \circ F = a(f \circ F) = aF^*(f)$. So, F^* is a linear map. □

(b) Show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

Proof. $\Rightarrow$: From the lemma of the class, we may assume M, N both have maximal atlas. Let $p \in M, F(p) \in N$. Since f is smooth, we can find a chart $(V^{'}, \psi^{'}), F(p) \in V^{'}$ such that $f \circ (\psi^{'})^{-1}$ is smooth on $\psi^{'}(V^{'})$. Since F is smooth, we can find two charts $(U, \phi), (V, \psi)$, $p \in U, F(p) \in V, F(U) \subset V$, such that $\psi \circ F \circ \phi$ is smooth. Now let $(F^{-1}(V^{'} \cap V) \cap U, \phi)$ be the new chart of p, and $(V^{'} \cap V, \psi^{'})$ be the new chart of $F(p)$. Then $f \circ F \circ \phi^{-1} = (f \circ (\psi^{'})^{-1}) \circ (\psi^{'} \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1})$ is smooth on $\phi(F^{-1}(V^{'} \cap V) \cap U)$. Because p is a random point of M, $f \circ F$ is smooth on M.

$\Leftarrow$: We still assume M, N have maximal atlas, and the dimension of N is $n > 0$. Let (U, ϕ) be the chart containing p, and (V, ψ) the chart containing $F(p)$. We may as well assume $\psi(F(p)) = 0$. Let $B_r \subset \psi(V)$ be an open ball of radius r, such that $\overline{B_r} \subset V$. Then $(F^{-1}(B_r) \cap U, \phi)$ is a new chart containing p. We may assume $U = F^{-1}(B_r) \cap U$. Now we have partition of unity f_1, f_2 of $V, N \backslash \overline{B_r}$ and $f_1|_{\bar{B_r}} = 1$. Let y_i be the projection of $\mathbb{R}^n$ to its $i-$th coordinate, so y_i is smooth. Now we define a smooth function f on N : $f_i = f_1 \cdot y_i \circ \psi$. From the assumption we know $f_i \circ F$ is smooth. Then we can find a chart (U_i, ϕ_i) of p such that $f_i \circ \phi_i^{-1}$ is smooth. Now let $U^{'} = U \bigcap_{i=1}^{n} U_i$. Let $(U^{'}, \phi)$ be p's new chart. Then $f_i \circ F \circ \phi^{-1} = f_i \circ F \circ \phi_i^{-1} \circ \phi_i \circ \phi^{-1}$ is smooth on $\phi(U^{'})$. Noticing that $f_i \circ F \circ \phi^{-1} = y_i \circ \psi \circ F \circ \phi^{-1}$ on $\phi(U^{'})$. So, $f_i \circ F \circ \phi^{-1}$ is smooth acutually means $\psi \circ F \circ \phi^{-1}$ is smooth on $\phi(U^{'})$. So, we find two smooth charts $(U^{'}, \phi), (V, \psi), p \in U^{'}, F(p) \in V, F(U^{'}) \subset V$, and $\psi \circ F \circ \phi^{-1}$ is smooth. This proves F is smooth. □

(c) Suppose that F is a homeomorphism. Prove that F is a diffeomorphism if and only if F^* induces an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof. $\Rightarrow$: From (a),(b) we know F^* is a linear map from $C^\infty(N)$ to $C^\infty(M)$. We only need to prove F^* is an isomorphism. Since F is a diffeomorphism, we have $F^{'} : N \to M$ which is smooth, and satisfying $F \circ F^{'} = id_N, F^{'} \circ F = id_M$. Then from (a),(b) we know $(F^{'})^*$ is a linear map from $C^\infty(M)$ to $C^\infty(N)$, and $(F^{'})^* \circ F^*(f) = (F^{'})^*(f \circ F) = f \circ F \circ F^{'} = f, f : N \to \mathbb{R}, f$ is smooth. Also, $F^* \circ (F^{'})^*(f) = f \circ F^{'} \circ F = f, f : M \to \mathbb{R}, f$ is smooth. So, F^* is a isomorphism.

$\Leftarrow$: Let F^{-1} be F's continuous inverse. Since F^* is an iosmorphism from $C^\infty(N)$ to $C^\infty(M)$, for every $f \in C^\infty(M)$, we have $g \in C^\infty(N)$, such that $f = g \circ F$, which indicates $f \circ F^{-1} = g$. So, $(F^{-1})^*(C^\infty(M)) \subset C^\infty(N)$, then F^{-1} is smooth by (b). In consequence, F is a diffeomorphism. □

5. (Lee 2-14) Suppose that A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Proof. From Lee's Level sets theorem of smooth function, we know there are two smooth function $g_A, g_B : M \to [0, +\infty)$, satisfying $g_A^{-1}(0) = A, g_B^{-1}(0) = B$, then clearly we can let $f = \frac{g_A}{g_A + g_B}$, because $g_A + g_B \neq 0, \forall x \in M$, since $A \cap B = \emptyset$. First we know f is smooth, and $0 \leq f \leq 1$. And, $f = 0$ iff $g_A = 0$, so $f^{-1}(0) = g_A^{-1}(0) = A$. Also, $f = 1$ iff $g_B = 0$. So, $f^{-1}(1) = g_B^{-1}(0) = B$. $\square$

6. Construct a diffeomorphism between $\mathrm{Gr}_k(\mathbb{R}^n)$ and $\mathrm{Gr}_{n-k}(\mathbb{R}^n)$. (Hint: use an inner product on $\mathbb{R}^n$.)

Proof. Let Q be a $(n-k)$ dimensional subspace of $\mathbb{R}^n$, and let P be a k dimensional subspace of $\mathbb{R}^n$. $P^\perp$ is the the orthogonal complement of P, and $Q^\perp$ is the orthogonal complement of Q. $U_Q = \{$ all k dimensional subspace which intersects $Q = 0\}$, $U_{Q^\perp} = \{$ all $n-k$ dimensional subspace which intersects $Q^\perp = 0\}$. Let $F(P) = P^\perp : Gr_k(\mathbb{R}^n) \to Gr_{n-k}(\mathbb{R}^n)$, then F is a bijection, since $(p^\perp)^\perp = P$. Noticing that $P \cap Q = 0$, and $dim(P) + dim(Q) = n$, we know $P + Q = \mathbb{R}^n$. From the knowledge of linear algebra we know $(P+Q)^\perp = P^\perp \cap Q^\perp$, so $P^\perp \cap Q^\perp = 0$. Then we know $F(U_Q) \subset U_{Q^\perp}$. We now have found two charts $(U_Q, \phi_Q), (U_{Q^\perp}, \psi_{Q^\perp})$(Here we use the smooth charts definition of Grassmann manifold from class). Let $v_1, \cdots, v_k$ be orthonormal basis of $Q^\perp$, and $v_{k+1}, \cdots, v_n$ be orthonormal basis of Q. Then we have an orthonormal basis $v_1, \cdots, v_n$ of $\mathbb{R}^n$. Let $A = (a_{ij}), 1 \leq i \leq n-k;\ 1 \leq j \leq k$ be a $n-k \times k$ matrix such that $\epsilon_i = v_i + \sum\limits_{1 \leq j \leq n-k} a_{ji} v_{k+j},\ 1 \leq i \leq k$ form a basis of P. Then we need to find a basis of $P^\perp$. Let $v = \sum\limits_{1 \leq i \leq n} x_i v_i$. Then $v \in P^\perp$ iff $\begin{cases} x_1 + x_{k+1}a_{11} + x_{k+2}a_{21} + \cdots + x_n a_{n-k,1} = 0 \\ x_2 + x_{k+1}a_{12} + x_{k+2}a_{22} + \cdots + x_n a_{n-k,2} = 0 \\ \vdots \\ x_k + x_{k+1}a_{1k} + x_{k+2}a_{2k} + \cdots + x_n a_{n-k,k} = 0 \end{cases}$. We know the solution space of this system of this linear equations is $P^\perp$. Now we can find its basis. Let $x_{k+1} = \cdots = x_{k+i-1} = 0,\ x_{k+i} = 1,\ x_{k+i+1} = \cdots = x_{k+n-k} = 0$. We can get a solution $v^i = -a_{i1}v_1 - a_{i2}v_2 + \cdots - a_{ik}v_k + v_{k+i}$ of this system. It is easy to see the rank of $\{v^i\}$ is $n-k$. Also, this basis indicates that $P^\perp$ can be viewed as a linear map from Q to $Q^\perp$, whose matrix under $v_{k+1}, \cdots, v_n$ is

$$\begin{pmatrix} -a_{11} & -a_{21} & \cdots & -a_{n-k,1} \\ -a_{12} & -a_{22} & \cdots & -a_{n-k,2} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{1k} & -a_{2k} & \cdots & -a_{n-k,k} \end{pmatrix}$$

, which is exactly $-A^\top$ equal to $\psi_{Q^\perp} \circ F \circ \phi_Q^{-1}(A)$. So, clearly $\psi_{Q^\perp} \circ F \circ \phi_Q^{-1}$ is smooth on $\phi_Q(U_Q)$. For a $n-k$ dimensional subspace P', from before we know $F^{-1}(P') = (P')^\perp$. So, symmetrically, F^{-1} is also smooth from $Gr_{n-k}(\mathbb{R}^n)$ to $Gr_k(\mathbb{R})^n$, indicating that F is a diffeomorphism. $\square$