Math 750 HW 1

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1. Prove that an arbitrary direct sum of 'projective' objects in an abelian category is projective. (Hint: use the characterization of projectivity of an object P in terms of behavior of the functor $\mathrm{Hom}(P,-)$, and how Hom behaves with respect to direct sums.)

Proof. Since abelian category can be embedded into an $R-Mod$ category, we may talk about it in a $R-mod$ category. Let $p_1, \cdots, p_k$ be some projective objects. Let $P = P_1 \oplus \cdots \oplus P_k$. Let A, B be two objects, and $h : A \to B$ is surjective. Suppose we have $f : P \to B$. Let $\iota_i : P_i \to P$ be embedding. Then for $f \circ \iota_i : P_i \to B$, we can find $g_i : P_i \to A$ such that the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "h"] & B \\ & P_i \arrow[ul, "g_1"] \arrow[u, "f\circ\iota_i"'] \end{tikzcd}$$

. Since P is the direct sum of $P_1, \cdots, P_k$, there exist a $g(p_1, p_2, \cdots, p_k) = g_1(p_1) + \cdots + g_k(p_k) : P \to A$, which makes following diagram commute:

$$\begin{tikzcd} A \arrow[r, "h"] & B \\ & P. \arrow[ul, "g"] \arrow[u, "f"'] \end{tikzcd}$$

$\square$

2.Prove that the direct product of an infinite number of copies of $\mathbb{Z}$ (for example, indexed by $\mathbb{Z}$ itself) is not projective. So the above statement does not hold with "direct sum" replaced by "direct product".

Proof. Let A be an object of $\mathbb{Z} - mod$. We know A is projective iff A a direct summand of a free $\mathbb{Z} - Mod$. But $\mathbb{Z}$ is a PID, this indicates A is projective iff A is a free abelian group. So, we only need to show $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}_i$ is not a free abelian group. Because we can view any element of $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}_i$ as a sequence of integers, the cardinality of $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}_i$ should be $\aleph$. If $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}_i$ is a free abelian group, then it can be written as $\bigoplus\limits_{i\in I} \mathbb{Z}$. We claim that the cardinality of I can not be $\aleph_0$, since every element of a free abelian group is a finite sum. If I is countable, we can view the element of $\bigoplus\limits_{i\in I} \mathbb{Z}$ as some rational number in $[0,1]$, which make $\bigoplus\limits_{i\in I} \mathbb{Z}$ countable. But the cardinality of $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}_i$ is $\aleph$, so I is not countable. So, $Hom_{\mathbb{Z}}(\bigoplus\limits_{i\in I} \mathbb{Z}, \mathbb{Z}) = \prod\limits_{i\in I} \mathbb{Z}$ is not countable. But from Baer-Specker Theorem, $Hom_{\mathbb{Z}}(\bigoplus\limits_{i\in I} \mathbb{Z}, \mathbb{Z}) = Hom_{\mathbb{Z}}(\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}, \mathbb{Z})$ should be countable. This is a contradiction. So, $\prod\limits_{i\in\mathbb{Z}} \mathbb{Z}$ is not a free abelian group and a projective $\mathbb{Z}$ module. $\square$

3.Let R be a commutative ring. Prove that an R-module M is projective if and only if $M_{\mathfrak{p}}$ is a projective $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ in R. In other words, projectivity is a local property. (Hint: use the fact that localization is exact, and that a module is zero if and only if all its localizations are zero. You can use these two facts without proof, but look them up if you are not familiar with them.)

Proof. Let $S = R - \mathfrak{p} \subset R$. We know S is a multiplicative subset of R.
$'\Rightarrow'$: For every R- module M, we have $M_{\mathfrak{p}} = M/S = R/S \otimes_R M$. If $M \oplus M^{'} = \bigoplus\limits_{i\in I} R$, then we have $(M \otimes R/S) \oplus (M^{'} \otimes R/S) = \bigoplus\limits_{i\in I} R_{\mathfrak{p}} \Rightarrow M_{\mathfrak{p}} \oplus M^{'}_{\mathfrak{p}} = \bigoplus\limits_{i\in I} R_{\mathfrak{p}}$. So, $M_{\mathfrak{p}}$ is projective.
'$\Leftarrow$': It suffice to prove $Hom(P,-)$ is an exact functor. Let $0 \to M^{'} \to M^{''} \to M \to 0$ be an exact sequence. We can apply $Hom(P,-)$ to it to get another exact sequence: $0 \to Hom(P, M^{'}) \to Hom(P, M^{''}) \xrightarrow{f} Hom(P, M) \to Hom(P, M)/im(f) = C \to 0$. Now we can apply functor $1/S$ to this exact sequence. Then we get $0 \to Hom(P_{\mathfrak{p}}, M^{'}_{\mathfrak{p}}) \to Hom(P_{\mathfrak{p}}, M^{''}_{\mathfrak{p}}) \xrightarrow{f} Hom(P_{\mathfrak{p}}, M_{\mathfrak{p}}) \to C_{\mathfrak{p}} \to 0$. But we know $M_{\mathfrak{p}}$ is projective, so $C_{\mathfrak{p}} = 0 \Rightarrow C = 0$. So, $0 \to Hom(P, M^{'}) \to Hom(P, M^{''}) \xrightarrow{f} Hom(P, M) \to 0$ is exact. So, M is projective. $\square$

4.Let R be a commutative ring, and let f, g be elements in R such that g is not a zero-divisor in R and f is not a zero-divisor in $R/(g)$. Prove that the sequence

$$0 \to R \to R^{\oplus 2} \to R \to R/(f,g) \to 0$$

is exact, where the first and second differentials are given by $h \mapsto (hg, -hf)$ and $(h,k) \mapsto fh - gk$.

The sequence (f, g) is an example of a regular sequence in R, and the above is the Koszul resolution associated to it. A similar resolution exists for every regular sequence.

Proof. Let the first differential be d_1, the second differential be $d_2 = (h,k) \to (fh+gk)$. (i): d_1 is injective. If $(hg, -hf) = (0,0)$, then $hg = 0 \Rightarrow h = 0$. (ii): $d_2 \circ d_1(h) = fhg - ghf = 0$. And if $d_2\big((h,k)\big) = fh + gk = 0$, then $fh \in (g) \Rightarrow h \in (g) \Rightarrow h = h^{'}g$. So, $fh^{'}g + gk = 0 \Rightarrow g(fh^{'} + k) = 0 \Rightarrow k = -fh^{'}$. So, $(h,k) = d_1(h^{'})$. (iii): Let $\pi : R \to R/(f,g)$ be the quotient map. Then $\pi \circ d_2(h,k) = \pi(fh + gk) = 0$. Also, the kener of π is $(f,g) = Rf + Rg = im(d_2)$. Last, the quotient map is surjective. So, this sequence is exact. $\square$

5. Let A be an abelian group, and $a \in A$ an element. Prove that there exists a map $f : A \to \mathbb{Q}/\mathbb{Z}$ of abelian groups such that $f(a) \neq 0$. Conclude that the evaluation map

$$ev : A \to \prod_{f \in \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

is injective. (This will be used in class to prove that the category of abelian groups has enough injective modules.)

Proof. For every $a \in A, a \neq 0$, let $f_a = f_a(ka) = k\alpha : \mathbb{Z}a \to \mathbb{Q}/\mathbb{Z}, \alpha \neq 0, \alpha \in \mathbb{Q}/\mathbb{Z}$. If we can extend this group homomorphism to $f : A \to \mathbb{Q}/\mathbb{Z}, f|_{\mathbb{Z}a} = f_a$ then we can comclude ev is injective. Now Let $S = \{(M,f), \mathbb{Z}a \subset M \subset A, f|_{\mathbb{Z}a} = f_a\}$. We define a partial order on $S :$ $(M,f) \leq (M^{'}, f^{'})$, iff $M \subset M^{'},\ f^{'}|_M = f$. First, $S \neq \emptyset$. If there exist a chain (M_i, f_i) such that $(M_{i-1}, f_{i-1}) \leq (M_i, f_i)$, then we can let $M = \bigcup\limits_i M_i$. We can define $f : M \to \mathbb{Q}/\mathbb{Z}$ as follows: If $x \in M$, then $x \in M_i$. We define $f(x) = f_i(x)$, which is easy be checked as well defined. By Zorn's Lemma, we have a maximal element (M, f) of S. We claime $M = A$. Suppose not, we can find an element $e \in A \backslash M, e \neq 0$. We define an ideal I of $\mathbb{Z}$ by $I = \{k \in \mathbb{Z} | ke \in M\}$. So I can be written as $\mathbb{Z}n, n \in \mathbb{Z}$. Now we can define a group homomorphism $g(kn) = f(kne)$ from $\mathbb{Z}n \to \mathbb{Q}/Z$. Noticing that this homomorphism can be extended to $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ by defining $g^{'}(1) = \frac{g(n)}{n}$. Now we can extend f to $M + \mathbb{Z}e$ by defining $\tilde{f}(m + ke) = f(m) + g^{'}(k), k \in \mathbb{Z}$. We need to clarify that $\tilde{f}$ is well-defined. If $m_1 + k_1 e = m_2 + k_2 e$, then we have $f(m_1 - m_2) - g^{'}(k_2 - k_1) = f(m_1 - m_2) - \frac{k_2 - k_1}{n} g(n) = f(m_1 - m_2) - (k_2 - k_1) f(e) = f(m_1 + k_1 e - m_2 - k_2 e) = 0$. This is well defined witch indicates a contradiction. So, $M = A$, which means we can extend any group homomorphism of $\mathbb{Z}a$ to A. $\square$

6. Let R be a polynomial ring over a field k (in several variables), and let W be an element of R. A matrix factorization of W is a pair P, Q of matrices with elements in R (P of size $m \times n$, Q of size $n \times m$) such that

$$PQ = W \cdot I_m, \quad QP = W \cdot I_n.$$

(The idea is that instead of factoring the polynomial W as a product of two polynomials, we aim to factor it as a product of two matrices.)

(a) Find a non-trivial matrix factorization of the polynomial $xy - zw$ in $k[x, y, z, w]$. (Note that as a polynomial, $xy - zw$ is irreducible.)

Proof. Let $P =$

$$\begin{pmatrix} x & z \\ w & y \end{pmatrix}$$

, $Q =:$

$$\begin{pmatrix} x & -z \\ -w & y \end{pmatrix}$$

$\square$

(b) Prove that in any matrix factorization of a non-zero polynomial the matrices must be square, i.e., $m = n$.

Proof. Assume F is a field, $A \in M_{m\times n}(F), B \in M_{n\times m}(F)$. From the facts of linear algebra, we know $rank(AB) \leq \min\{rank(A), rank(B)\}, rank(BA) \leq \min\{rank(A), rank(B)\}$. So, if $AB = I_{m\times m}, BA = I_{n\times n}$, we must have $n \leq m, m \leq n \Rightarrow m = n$. For these problem, we can view R as a subring of its fraction field $F = R/(R-0)$. Now we know $P(Q/W) = I_m, (Q/W)P = I_n \Rightarrow m = n$. □

(These matrix factorizations appear as the 2-periodic part of resolutions of modules over hypersurface rings like R/W. For your own fun – not to be submitted – try to resolve over $R = k[x, y, z, w]/(xy - zw)$ the module $M = R/(x, y, z, w)$ and see how the matrix factorization in (a) appears.)