Math 751 HW 3

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Due: Sunday night

3. Let $p : \widetilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff iff X is compact Hausdorff.

Proof. ($\Leftarrow$) Suppose X is compact Hausdorff.

Hausdorffness. Since p is a local homeomorphism, take two distinct points $\tilde{x}, \tilde{y} \in \widetilde{X}$.

- If $p(\tilde{x}) \neq p(\tilde{y})$, choose disjoint neighborhoods U, V of $p(\tilde{x}), p(\tilde{y})$ in X. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint neighborhoods of $\tilde{x}$ and $\tilde{y}$.
- If $p(\tilde{x}) = p(\tilde{y}) = x$, choose a *uniformizing neighborhood* U of x so that $p^{-1}(U) = \bigsqcup_i U_i$, where each U_i is mapped homeomorphically onto U. The points $\tilde{x}$ and $\tilde{y}$ lie in distinct U_i, which are disjoint open sets.

Hence $\widetilde{X}$ is Hausdorff.

Compactness. For each $x \in X$, let U_x be a uniformizing neighborhood of x. We know X is T_2 and compact, so X is T_4 i.e normal. Also, T_2 implies T_1, so $\{x\}$ is a closed set in X. Since X is normal, there is another open neighborhood $U_x^{'} \ni x$ such that $\overline{U_x^{'}} \subset U_x$. Then we know $\{U_x^{'}\}$ is an open cover of X. Then we can find finite open cover $U_{x_1}^{'}, \cdots, U_{x_n}^{'}$. Let $V_{x_i} = p^{-1}(U_{x_i}^{'})$. We claim that $\{V_{x_i}\}$ is an open cover of $\tilde{X}$. This is because for each $\tilde{x} \in \tilde{X}$, we have $p(\tilde{x}) \in U_{x_i}^{'}$ for some i. Then $\tilde{x} \in V_{x_i}$. Now let $V_{x_i}^{'} = p^{-1}(U_{x_i})$. We know $V_{x_i} \subset V_{x_i}^{'}$, and $V_{x_i}^{'}$ is the union of finite disjoint union open set $W_1^{'}, \cdots, W_n^{'}$ which are homeomorphic to U_{x_i}. And V_{x_i} is also a disjoint union of some open set $W_1, \cdots, W_n$ which are subset of $W_1^{'}, \cdots, W_n^{'}$. Then based on homeomorphism, $\overline{W_i} \subset W_i^{'}$ and $\overline{W_i}$ is homeomorphic to $\overline{U_x^{'}}$. We know X is compact, so every closed set of X is compact. So, $\overline{U_x^{'}}$ is compact. $\Rightarrow \overline{W_i}$. Since $p^{-1}(x)$ is finite for all $x \in X$, we know $\tilde{X}$ can be covered by finite compact closed set, which indicates $\tilde{X}$ is compact.

($\Rightarrow$) Suppose $\widetilde{X}$ is compact Hausdorff.

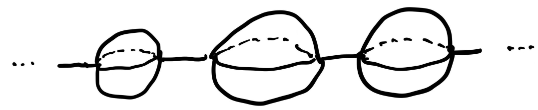
Compactness. Since p is continuous, $X = p(\widetilde{X})$ is compact as the continuous image of a compact space.

Hausdorffness. Let $x \neq y$ be points in X. Their fibers $F_x = p^{-1}(x), \quad F_y = p^{-1}(y)$ are finite and disjoint compact subsets of the Hausdorff space $\widetilde{X}$. Hence we can choose disjoint open sets $A, B \subset \widetilde{X}$ with $F_x \subset A$ and $F_y \subset B$ and $\overline{A} \cap \overline{B} = \varnothing$.

For each $\tilde{x} \in F_x$, take a uniformizing neighborhood $U_{\tilde{x}}$ of x such that the sheet $S_{\tilde{x}}$ containing $\tilde{x}$ is contained in A. Similarly, for each $\tilde{y} \in F_y$, take $V_{\tilde{y}}$ such that the corresponding sheet $T_{\tilde{y}}$ lies in B. Let $U = \bigcap_{\tilde{x} \in F_x} U_{\tilde{x}}, \qquad V = \bigcap_{\tilde{y} \in F_y} V_{\tilde{y}}$. Then U and V are open neighborhoods of x and y, respectively. If $z \in U \cap V$, some lift of z would belong to both A and B, which is impossible since $A \cap B = \varnothing$. Thus $U \cap V = \varnothing$, and X is Hausdorff. $\square$

4. Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof. (i): First, we need to calculate the fundamental group of X. Observing that X has a cell complexes structure with two 0 cells; three 1 cells; two 2 cells. And its 1 skeleton is homotopy equivalent to $S^1 \vee S^1$. Also, these two 2 cells actually contribute the same relation to the fundamental group making its fundamental group being $\mathbb{Z}$. If we make one end of the diameter being the basepoint, we know the fundamental group can be represented by a circle which go through an longitude and the diameter and wrap it again and again. So, base on the construction from the existence of universal covering space, we can get a space like this :



This is a simply-connected space, which can be seen by easily analyzing its cell complexes structure. And the covering map is sending the sphere to sphere using identity; sending the straight line to the diameter. So, we find a universal covering.

(ii): We still first compute the fundamental group of X. By analyzing its cell complexes, we still get $\pi_1(X) = \mathbb{Z}$. From its cell complexes we know its generator is the circle which is connected to this sphere. So, we can get a cover just like (i), but we need to fix some labeling. First label the connecting segments a, b consecutively, then p' sends the spheres to the sphere, sends the connecting segments labeled as a to the half-circle inside the sphere and the connecting segments labeled as b to the half-circle outside the sphere, and sends the north/south pole to the points where the circle intersects the sphere. □

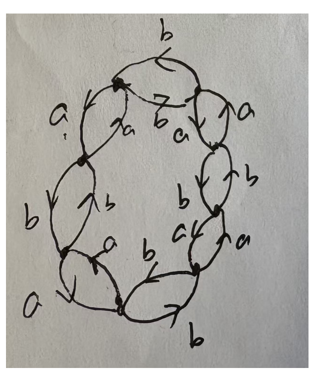
8. Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$. [Exercise 11 in Chapter 0 may be helpful.]

   Proof. Suppose $p_1 : \tilde{X} \to X$ and $p_2 : \tilde{Y} \to Y$ are covering map. $f : X \to Y$, $g : Y \to X$ and $f \circ g \simeq id_X$; $g \circ f \simeq id_Y$. Consider $f \circ p_1 : \tilde{X} \to Y$, since $\tilde{X}$ is simply connected we know this can be lifted to $\tilde{Y}$ as $\tilde{f'}$. Also, we can do the same to $g \circ p_2 : \tilde{Y} \to X$. And, we call this lift $\tilde{g'}$. Now we get a continuous map $F = \tilde{g'} \circ \tilde{f'}$ from $\tilde{X}$ to $\tilde{X}$. Since $p_1 \circ F = g \circ f \circ p_1 \simeq p_1$, we can lift this homotopy to $H : \tilde{X} \times I \to \tilde{X}$ such that $H(\tilde{x}, 0) = F$, and $p_1 \circ H(\tilde{x}, 1) = p_1$. So, $H(\tilde{x}, 1)$ must be a isomorphism $\tilde{p_1}$, because any lift of p_1 between universal covering must be a homeomorphism. So, we have $\tilde{p_1} \simeq F$ $\Rightarrow \tilde{p_1}^{-1} \circ F \simeq id_{\tilde{X}}; F \circ \tilde{p_1}^{-1} \simeq id_{\tilde{X}} \Rightarrow \tilde{f'}$ is a homotopy equivalent. Similarly, $\tilde{g'}$ is also a homotopy equivalent.

   □

12. Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

    Proof. Let F_2 denote the free group generated by a, b. Let H denote the normal subgroup generated by $a^2, b^2, (ab)^4$. Let G denote F_2/H. Then all we need to do is draw a cayley graph of G. Then we can let G acts naturally on the cayley graph of itself we can get a normal covering, which corresponding to the covering space this question requested. First, we pick the generators S of G. $S = \{a, b, a^{-1}, b^{-1}\}$. And, the vetex are $V = \{e, a, ab, aba, abab, ababa, ababab, abababa\}$. So, $|V| = 8$. Then we can draw its cayley graph like following:



$\square$

17. Given a group G and a normal subgroup N, show that there exists a normal covering space $\widetilde{X} \to X$ with $\pi_1(X) \approx G$, $\pi_1(\widetilde{X}) \approx N$, and deck transformation group $G(\widetilde{X}) \approx G/N$.

Proof. From chapter 1.2, we know we can construct a connected cell complexes X such that $\pi_1(X) = G$. From class we know the cayley complex for group G is the universal covering of X. Then by Galois corresponding we can find a normal covering $\tilde{X} \to X$ such that $\pi_1(\tilde{X})$ is N. So, the deck group is G/N. $\square$

18. For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\widetilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a "universal" abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.

Proof. Let X be path-connected, locally path-connected, and semilocally simply-connected, fix a basepoint x_0, and write $\pi = \pi_1(X, x_0)$. By the classification of covering spaces, a connected covering $p : \widetilde{X} \to X$ corresponds to a subgroup $H = p_*\big(\pi_1(\widetilde{X}, \tilde{x}_0)\big) \leq \pi$ (up to conjugacy). If p is normal then $H \trianglelefteq \pi$ and $\mathrm{Deck}(\widetilde{X}/X) \cong \pi/H$. If $\mathrm{Deck}(\widetilde{X}/X)$ is abelian, then π/H is abelian, hence $[\pi, \pi] \leq H$. Therefore, in every abelian covering the corresponding subgroup contains the commutator subgroup, and the smallest such normal subgroup is $[\pi, \pi]$ itself. Let $p_{ab} : \widetilde{X}_{ab} \to X$ be the covering corresponding to $H_0 = [\pi, \pi]$. Then p_{ab} is normal and $\mathrm{Deck}(\widetilde{X}_{ab}/X) \cong \pi/H_0 \cong \pi^{ab} \cong H_1(X; \mathbb{Z})$, so p_{ab} is an abelian covering. If $q : Y \to X$ is any other abelian covering with corresponding normal subgroup H', then $[\pi, \pi] \leq H'$, and the classification theorem produces a covering map $r : \widetilde{X}_{ab} \longrightarrow Y$ with $q \circ r = p_{ab}$. Hence $\widetilde{X}_{ab}$ covers every abelian covering of X; this is the *universal abelian covering* of X.

If $Z \to X$ is another abelian covering that covers every abelian covering of X, then there are covering maps both ways between Z and $\widetilde{X}_{ab}$. For connected coverings, mutual coverings imply an isomorphism (the two lifts of the identity compose to a deck transformation, hence a homeomorphism). Therefore the universal abelian covering is unique up to isomorphism.

(i): $X = S^1 \vee S^1$. Let $\tilde{X} = \{(x, y) \in \mathbb{R}^2 | x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$. Now we let $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\tilde{X}$ by $(m, n)(x, y) = (x + m, y + n)$. It is easy to check this is an covering action. So, we know $\tilde{X} \to \tilde{X}/\mathbb{Z} \oplus \mathbb{Z} = S^1 \vee S^1$ is a normal covering, whose covering map is the quotient map p. And, $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(S^1 \vee S^1)/p^*(\pi_1(\tilde{X}))$. Since $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, then $\pi_1(S^1 \vee S^1)/[\pi_1(S^1 \vee S^1), \pi_1(S^1 \vee S^1)] = \mathbb{Z} \oplus \mathbb{Z}$.(Here $[G, G]$ denote the commutator subgroup) But we know $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(S^1 \vee S^1)/p^*(\pi_1(\tilde{X}))$, which denote $p^*(\pi_1(\tilde{X})) = [\pi_1(S^1 \vee S^1), \pi_1(S^1 \vee S^1)]$. So, this is the universal abelian covering.

(ii): $X = S^1 \vee S^1 \vee S^1$. Similarly, Let $\tilde{X} = \{(x, y, z) \in R^3 |$ at least two of them $\in \mathbb{Z}\}$. Let $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ acts on $\tilde{X}$ by $(r, s, t)(x, y, z) = (x + r, y + s, z + t)$. This is a covering action. So, we have a normal covering $\tilde{X} \to \tilde{X}/\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = S^1 \vee S^1 \vee S^1$ whose covering map is the quotient map p. So, $\pi_1(S^1 \vee S^1 \vee S^1)/p_*(\pi_1(\tilde{X})) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \pi_1(S^1 \vee S^1 \vee S^1)/[\pi_1(S^1 \vee S^1 \vee S^1), \pi_1(S^1 \vee S^1 \vee S^1)]$, which indicates $\tilde{X}$ is the universal abelian covering.

$\square$

19. Use the preceding problem to show that a closed orientable surface M_g of genus g has a connected normal covering space with deck transformation group isomorphic to $\mathbb{Z}^n$ (the product of n copies of $\mathbb{Z}$) if and only if $n \leq 2g$. For $n = 3$ and $g \geq 3$, describe such a covering space explicitly as a subspace of $\mathbb{R}^3$ with translations of $\mathbb{R}^3$ as deck transformations. Show that such a covering space in $\mathbb{R}^3$ exists if and only if there is an embedding of M_g in the 3–torus $T^3 = S^1 \times S^1 \times S^1$ such that the induced map $\pi_1(M_g) \longrightarrow \pi_1(T^3)$ is surjective.

Proof. We first prove the formal half satement of this question.
"$\Rightarrow$": Suppose we have a normal covering space with deck group isomorphic to $\mathbb{Z}^n$. Then we know $\pi_1(M_g)/H$ is isomorphic to $\mathbb{Z}^n$, and H is normal subgroup of $\pi_1(M_g)$. We know $\pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)] = \mathbb{Z}^{2g}$($[\cdot, \cdot]$ still indicates the commutator group), so by the conclusion of last problem we know $2g \geq n$.
" $\Leftarrow$ ": By the conclusion of the previous problem, we know there is an universal abelian covering $\tilde{X}$ with the covering map p such that $\pi_1(M_g)/p_*(\pi_1(\tilde{X})) = \pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)] = \mathbb{Z}^{2g}$. Let $q : \pi_1(M_g) \to \pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)]$ denote the quotient group homomorphism. Since $\mathbb{Z}^{2g-n}$ is normal subgroup of $\mathbb{Z}^{2g}$, $q^{-1}(\mathbb{Z}^{2g-n})$ is a normal subgroup of $\pi_1(M_g)$. Then by classification of covering space, we have a normal covering $\tilde{X}'$ with covering map p' such that $p'_*(\pi_1(\tilde{X}')) = q^{-1}(\mathbb{Z}^{2g-n})$. So, the deck group of $\tilde{X}'$ is $\pi_1(M_g)/q^{-1}(\mathbb{Z}^{2g-n}) = \mathbb{Z}^n$.

Start with the unit cube $[0, 1]^3$ whose opposite faces are identified to form the three–torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Take its 1-skeleton L(the union of all edges of the cube), which represents the coordinate loops of the torus. Let N be a small tubular neighborhood of this 1-skeleton in T^3. Then the boundary surface $\Sigma_3 = \partial N \subset T^3$ is an embedded surface of genus 3 (the standard genus–3 Heegaard surface of T^3). The inclusion $i : \Sigma_3 \hookrightarrow T^3$ induces a surjection $i_* : \pi_1(\Sigma_3) \twoheadrightarrow \pi_1(T^3) = \mathbb{Z}^3$, since the three coordinate circles of T^3 are represented by loops on Σ_3. Now lift Σ_3 to the universal covering $p : \mathbb{R}^3 \to T^3$, and let $\widetilde{\Sigma}_3 = p^{-1}(\Sigma_3) \subset \mathbb{R}^3$. This is a triply periodic surface in $\mathbb{R}^3$, invariant under the translation group $\mathbb{Z}^3$. Each connected component $\widehat{\Sigma}_3$ projects onto Σ_3 as a connected normal covering with deck transformation group $\mathbb{Z}^3$. Let $Q = [0, 1]^3$ and $Q^\circ = (0, 1)^3$. Choose pairwise disjoint small closed disks $\{D_i^+, D_i^-\} \subset M_3 \cap Q^\circ \quad (i = 1, \ldots, k), \quad k := g - 3$, such that all their $\mathbb{Z}^3$–translates remain pairwise disjoint. For each i pick an embedded arc $\sigma_i : [0, 1] \to Q^\circ, \quad \sigma_i(0) \in \partial D_i^+, \ \sigma_i(1) \in \partial D_i^-$, whose interior is disjoint from M_3 and from the other arcs; moreover all $\mathbb{Z}^3$–translates of all σ_i are mutually disjoint. Take $\delta \ll \varepsilon$ and let $T_i = N_\delta(\sigma_i)$ be a thin tube. Attach T_i along its annular ends to $\partial D_i^\pm$. Performing this for $i = 1, \ldots, k$ adds k 1–handles to M_3 in Q, hence increases the genus by k (a standard fact: connecting two small disks by a tube raises the genus by 1).

Repeat the same modification in every translate $v + Q$, $v \in \mathbb{Z}^3$. Define the $\mathbb{Z}^3$–periodic 1–complex $K_g := L \cup \bigcup_{i=1}^k \bigcup_{v\in\mathbb{Z}^3} (\sigma_i + v) \subset \mathbb{R}^3, \qquad \widetilde{M}_g := \partial N_\varepsilon(K_g) \subset \mathbb{R}^3$. Then $\mathbb{Z}^3$ preserves $\widetilde{M}_g$ and acts freely and properly discontinuously, so the restriction of the universal covering $\mathbb{R}^3 \to \mathbb{T}^3$ yields a covering $p|_{\widetilde{M}_g} : \widetilde{M}_g \longrightarrow \widetilde{M}_g/\mathbb{Z}^3$.

Let $\Gamma_g = p(K_g) \subset \mathbb{T}^3$ be the projected graph. Inside one fundamental domain it is a wedge of $3+k = g$ loops, therefore $\beta_1(\Gamma_g) = g$. Its regular neighborhood $N_\varepsilon(\Gamma_g)$ is a handlebody; hence the boundary $M_g := \partial N_\varepsilon(\Gamma_g) \subset \mathbb{T}^3$ is a closed orientable surface of genus $\beta_1(\Gamma_g) = g$ (standard lemma: the boundary of a regular neighborhood of a connected graph has genus equal to the first Betti number of the graph). Clearly, $\widetilde{M}_g/\mathbb{Z}^3 \cong M_g$, and the covering $p|_{\widetilde{M}_g} : \widetilde{M}_g \to M_g$ is connected and *normal* with deck group $\mathbb{Z}^3$.

$\Rightarrow$ Suppose $\widehat{M} \subset \mathbb{R}^3$ is a closed surface such that the translation group $\mathbb{Z}^3$ acts freely and properly discontinuously by deck transformations and $\widehat{M}/\mathbb{Z}^3 \cong M_g$. Then the quotient $\mathbb{R}^3/\mathbb{Z}^3$ is T^3, and M_g embeds naturally in T^3. The covering projection diagram implies that the inclusion induces a surjective map $\pi_1(M_g) \twoheadrightarrow \pi_1(T^3) = \mathbb{Z}^3$.

$\Leftarrow$ Conversely, if there exists an embedding $i : M_g \hookrightarrow T^3$ such that i_* is surjective, then lifting $i(M_g)$ to the universal cover $p : \mathbb{R}^3 \to T^3$ yields $\widetilde{M} = p^{-1}(i(M_g)) \subset \mathbb{R}^3$. Each connected component $\widehat{M}$ projects onto M_g as a connected normal covering whose deck transformation group is $\mathbb{Z}^3$. Hence such a covering in $\mathbb{R}^3$ exists if and only if M_g admits an embedding in T^3 inducing a surjection on π_1.

$\square$

23. Show that if a group G acts freely and properly discontinuously on a Hausdorff space X, then the action is a covering space action. (Here "properly discontinuously" means that each $x \in X$ has a

neighborhood U such that $\{g \in G \mid U \cap g(U) \neq \varnothing\}$ is finite.) In particular, a free action of a finite group on a Hausdorff space is a covering space action.

Proof. We need to prove for each $x \in X$, there exists a neighborhood $x \in U$ such that $g(U) \cap U = \emptyset, g \neq e$. First, we know there exist a $x \in U^{'}$ such that there are only finite $g_1, \cdots, g_n; g_i \neq e$ such that $g_i(U) \cap U \neq \emptyset$. Consider the set $\{g_i(x) = x_i\}$. We know x, x_i are different points since this is a free action. Since X is hausdorff, we can find some open neighborhood $U^{''} \ni x, U_i \ni x_i$ such that $U \cap U_i, U_i \cap U_j = \emptyset$. Let $U = U^{'} \cap U^{''} \cap g_1^{-1}(g_1(U^{''}) \cap U_1) \cap \cdots \cap g_n^{-1}(g_n(U^{''}) \cap U_n)$. Then we claim $g(U) \cap U = \emptyset, g \neq e$. Suppose not. Since U is a subset of $U^{'}$, we know that g must be a g_i. Without lost of generality, we assume this is g_1. So, there is a $y \in U$ such that $g_1(y) \in U$. But noticing that $y \in g_1^{-1}(g_1(U^{''}) \cap U_1)$, $g_1(y) \in g_1(U^{''}) \cap U_1 \Rightarrow g_1(y) \in U_1$. But we know $U_1 \cap U^{''} = \emptyset \Rightarrow U_1 \cap U = \emptyset$. So, we get a contradiction. So, G is a covering action. □

27. For a universal cover $p : \widetilde{X} \to X$ we have two actions of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$, namely the action given by lifting loops at x_0 and the action given by restricting deck transformations to the fiber. Are these two actions the same when $X = S^1 \vee S^1$ or $X = S^1 \times S^1$? Do the actions always agree when $\pi_1(X, x_0)$ is abelian?

Proof. Let $G = \pi_1(X, x_0)$ be the fundamental group and $F = p^{-1}(x_0)$ be the fiber over the basepoint x_0. The universal cover $\tilde{X}$ is simply connected.

To compare these two actions, we first need a way to label the points in the fiber F. Let us choose a basepoint $\tilde{x}_0 \in F$ such that $p(\tilde{x}_0) = x_0$.

Since $\tilde{X}$ is path-connected and simply connected, for any other point $\tilde{x} \in F$, there exists a unique path homotopy class of paths from $\tilde{x}_0$ to $\tilde{x}$. More importantly, this establishes a bijection between G and F: $\beta : G \to F$ This map is defined as follows: for an element $g = [\gamma] \in G$ (where γ is a loop in X based at x_0), $\beta(g)$ is the endpoint of the unique lift $\tilde{\gamma}$ of γ that starts at $\tilde{x}_0$. That is, $\beta(g) = \tilde{\gamma}(1)$.

- $\beta(e) = \tilde{x}_0$, where e is the identity element of G.
- This is a bijection because $\tilde{X}$ is a universal cover.

We now use this bijection β to analyze the two actions. Let us take an arbitrary point $\tilde{x} \in F$. By the bijection, there exists a unique $h \in G$ such that $\tilde{x} = \beta(h)$. We will examine how each action, when applied by an element $g \in G$, acts on $\beta(h)$.

Action 1: Lifting Loops (denoted Φ)

This action is defined as: for $g = [\gamma] \in G$ and $\tilde{x} \in F$, $\Phi(g)(\tilde{x})$ is the endpoint of the unique lift of the loop γ starting at $\tilde{x}$.

Let's compute $\Phi(g)(\beta(h))$: $\beta(h)$ is the endpoint of the lift of h (as a loop) starting at $\tilde{x}_0$. Let this lifted path be $\tilde{h}$. $\Phi(g)(\beta(h))$ is the endpoint of the lift of g (as a loop) starting at $\beta(h) = \tilde{h}(1)$. Let this lifted path be $\tilde{g}_{\beta(h)}$. Consider the concatenated path $\tilde{h} * \tilde{g}_{\beta(h)}$. This path starts at $\tilde{x}_0$. Its projection is $p \circ (\tilde{h} * \tilde{g}_{\beta(h)}) = (p \circ \tilde{h}) * (p \circ \tilde{g}_{\beta(h)}) = h * g$ (the product of loops). Therefore, the endpoint of this concatenated path, which is $\tilde{g}_{\beta(h)}(1)$, is precisely the endpoint of the lift of the loop hg starting from $\tilde{x}_0$. By the definition of β, this endpoint is $\beta(hg)$.

So, we have: $\Phi(g)(\beta(h)) = \beta(hg)$ Note: $\Phi(g_1 g_2)(\beta(h)) = \beta(h(g_1 g_2))$. And $\Phi(g_2)(\Phi(g_1)(\beta(h))) = \Phi(g_2)(\beta(hg_1)) = \beta((hg_1)g_2)$. By associativity, $\Phi(g_1 g_2) = \Phi(g_2) \circ \Phi(g_1)$. This is a right action.

2. Action 2: Deck Transformations (denoted Ψ)

This action is defined as: for $g = [\gamma] \in G$, $\Psi(g)$ is the restriction of the corresponding element in the deck transformation group $Deck(\tilde{X}/X)$. There is an isomorphism (dependent on the choice of $\tilde{x}_0$) $\theta : G \to Deck(\tilde{X}/X)$. $\theta(g) = \tau_g$, where τ_g is the unique deck transformation that maps the basepoint $\tilde{x}_0$ to $\beta(g)$ (the endpoint of the lift of g from $\tilde{x}_0$). That is, $\tau_g(\tilde{x}_0) = \beta(g)$. The action $\Psi(g)$ is simply

the action of τ_g. Let's compute $\Psi(g)(\tilde{x}) = \tau_g(\beta(h))$. $\beta(h)$ is the endpoint of the lift of h starting at $\tilde{x}_0$. Let this lift be $\tilde{h}$. Consider the path $\tau_g \circ \tilde{h}$. It starts at $\tau_g(\tilde{h}(0)) = \tau_g(\tilde{x}_0) = \beta(g)$. Its projection is $p \circ (\tau_g \circ \tilde{h}) = (p \circ \tau_g) \circ \tilde{h} = p \circ \tilde{h} = h$. Thus, $\tau_g \circ \tilde{h}$ is the unique lift of the loop h starting at $\beta(g)$. $\tau_g(\beta(h)) = \tau_g(\tilde{h}(1))$ is the endpoint of this lifted path. Now consider another path: (the lift of g from $\tilde{x}_0$) * (the lift of h from $\beta(g)$). This path starts at $\tilde{x}_0$. Its projection is $g * h$. Therefore, the endpoint of this path (which is $\tau_g(\beta(h))$) is precisely the endpoint of the lift of the loop gh starting from $\tilde{x}_0$. By the definition of β, this endpoint is $\beta(gh)$.
So, we have: $\Psi(g)(\beta(h)) = \beta(gh)$ Note: This is a left action.

The two actions Φ and Ψ are considered "the same" if and only if they induce the same permutation on the fiber F for every $g \in G$. That is, for all $g \in G$ and all $\tilde{x} \in F$, we must have: $\Phi(g)(\tilde{x}) = \Psi(g)(\tilde{x})$ Using our bijection β, this is equivalent to: For all $g \in G$ and all $h \in G$, we must have: $\Phi(g)(\beta(h)) = \Psi(g)(\beta(h))$ $\beta(hg) = \beta(gh)$ Since β is a bijection, this is equivalent to: $hg = gh \quad \text{for all } g, h \in G$ This is precisely the definition of G being an Abelian group.

So, when $X = S^1 \vee S^1$, $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ which is nonabelian. These two actions is not equivalent. When $X = S^1 \times S^1$, $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$ which is abelian. So, these two actions is equivalent. □

30. Draw the Cayley graph of the group $\mathbb{Z} * \mathbb{Z}_2 = \langle a, b \mid b^2 \rangle$.

Proof. In the free product $\mathbb{Z} * \mathbb{Z}_2$, every element has a unique *reduced word* over the alphabet $\{a^{\pm 1}, b\}$ in which no subword aa^{-1}, $a^{-1}a$, or bb occurs; here $b = b^{-1}$. We take vertices of $\mathrm{Cay}(G, S)$ to be these reduced words, and we join g to gs by an edge for each $s \in S$.

From any vertex g there are exactly three adjacent vertices ga, ga^{-1}, gb. They are distinct: for instance, if $ga = gb$ then left-multiplying by g^{-1} gives $a = b$, which would force $a^2 = 1$ and contradict the defining relations of the free product; the other equalities are similar.

To see that the graph has no nontrivial cycles, consider a closed edge-path based at the identity. Its label is a word w in $\{a^{\pm 1}, b\}$ that represents $1 \in G$. By the normal form theorem for free products, the only way w can equal 1 is by free reduction to the empty word, i.e., the path immediately backtracks at each cancellation. Thus the underlying graph is a tree. Since each vertex has degree 3, the Cayley graph is the infinite 3-regular tree. Edges may be labeled by a, a^{-1}, b (with the b-edges undirected because $b = b^{-1}$). □