Math 761 HW 2

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1. Show that a topological manifold is the same thing as a C^0 manifold.

Proof. From the definition, we only need to verify that the transition map is C^0. Let (U, ϕ) and (V, ψ) be two charts of M, which is a topological manifold. Also, we may as well assumen $U \cap V \neq \emptyset$. Then $\psi^{-1} \circ \phi$ and $\phi^{-1} \circ \psi$ are both continous, since $\phi, \phi^{-1}, \psi, \psi^{-1}$ are all continous map. So, a topological manifold is accutually a C^0 manifold. □

2. Check that the definition of a smooth function makes sense on a smooth manifold.

Proof. Assume this smooth function is f, and this smooth manifold is M. Then for every $p \in M$, there is a smooth chart (U_p, ϕ_p) such that $f \circ \phi^{-1}$ is smooth on $\phi_p(U_p)$. Let $\Gamma = \{U_p\}$, then Γ is countable. Since f is smooth on M, $\bigcup\limits_{p \in M} U_p = M$. Assume (V, ψ) is another smooth chart containing p, Then $V = \bigcup\limits_i U_i, U_i \in \Gamma$. Then in every open subset $\psi(U_i \cap V)$ of $\psi(V)$, $f \circ \psi^{-1} = f \circ \phi_i^{-1} \circ \phi_i \circ \psi^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ which is smooth by definition. So, we can see $f \circ \psi^{-1}$ is smooth on an open cover of $\psi(V)$, which indicates $f \circ \psi^{-1}$ is smooth on $\psi(V)$.

□

3. Write down (with proof) a smooth structure on the Möbius strip.

Proof. We define a mobius stirp as $M = \{[0,1] \times (0,1)\}/(0, \frac{1}{2} + x) = (1, \frac{1}{2} - x), x \in (-\frac{1}{2}, \frac{1}{2})$, and we denote this quotient map as p. And, we give it a smooth structure $\{(p((0,1) \times (0,1)), p^{-1}),\ (p(B((0, \frac{1}{2}), \frac{1}{2}) \cap (0,1) \times [0,1] \cup B(1, \frac{1}{2}) \cap (0,1) \times [0,1]), \tilde{\phi}\}$. We define $\tilde{\phi}$ as follows. If $p^{-1}(x) \in (0, y),\ y \in (0,1)$, we define $\tilde{\phi}(x) = (0, y)$. This is well defined. If $p^{-1}(x) \in B((0, \frac{1}{2}), \frac{1}{2})$, we define $\tilde{\phi}(x) = p^{-1}(x)$. If $p^{-1}(x) = (x^{'}, y') \in B((1, \frac{1}{2}), \frac{1}{2})$, We define $\tilde{\phi}(x) = (x^{'} - 1, 1 - y^{'})$. Now we need to check if the transition map is smooth or not. Fisrt, we can consider $p^{-1} \circ \tilde{\phi} : B((0, \frac{1}{2}), \frac{1}{2}) \backslash \{(0, y), y \in (0,1)\} \to (0,1) \times (0,1)$. It is easy to see that the domain of this map is two open sets, so we can check this map by restricting it to two half open disk without diameter. When restricting it to right half open ball, it is actually id, so it is smooth. When restricting it to left half open disk, it is actually a reflection compose a translation, so it is a smooth. On the other hand, we need to check if $p \circ \tilde{\phi}^{-1}$ is smooth. First, it is easy see its domain is two open half disk without diameter in $(0,1) \times (0,1)$. When we restrict to the left half open disk, $p \circ \tilde{\phi}^{-1} = id$, so it is smooth. When we restrict it to the right half open disk, we can view $p \circ \tilde{\phi}^{-1}$ as translation compose reflextion. So, it is also smooth. So, this is a smooth struture. □

4. The stereographic projection maps on S^n are given by

$$X \mapsto x = \frac{1}{1 - X^{n+1}}\big(X^1, \ldots, X^n\big), \qquad X \mapsto y = \frac{1}{1 + X^{n+1}}\big(X^1, \ldots, X^n\big)$$

from the north and south poles respectively. The inverse stereographic projection is given by

$$x \mapsto X = \frac{1}{1 + |x|^2}\big(2x^1, \ldots, 2x^n, |x|^2 - 1\big).$$

Derive the formula for the transition map

$$x \mapsto y = \frac{x}{|x|^2}.$$

Proof. We write stereographic projection map from north pole as $f(x_1, x_2, \cdots, x_{n+1}) = \frac{1}{1-X^{n+1}}(x_1, \cdots, x_n)$. And we write stereographic projection map from the south pole as $g(x_1, x_2, \cdots, x_{n+1}) = \frac{1}{1+X^{n+1}}(x_1, \cdots, x_{n+1})$. Then we can caculate that $f^{-1}(x_1, x_2, \cdots, x_n) = \frac{1}{1+|x|^2}(2x, |x|^2-1)$, and $g^{-1}(x_1, x_2, \cdots, x_n) = \frac{1}{1+|x|^2}(2x, 1-|x|^2)$. So, we can caculate that $g \circ f^{-1} = \frac{1}{1+\frac{|x|^2-1}{1+|x|^2}}(\frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}) = \frac{1+|x|^2}{2|x|^2}(\frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}) = \frac{x}{|x|^2}$, when $|x| \neq 0$. □

5. Check that the smooth structure on S^n induced by the stereographic charts is equivalent to the smooth structure induced by the hemisphere charts.

Proof. Let $U_{x_i^+}$ denote the open upper half sphere of the plane $x_i = 0$, and $\phi_{x_i^+}$ denote the projection according to it. Let $U_{x_i^-}$ denote the open lower half sphere of the plane $x_i = 0$, and $\phi_{x_i^-}$ denote the projection according to it. Let U_N denote the sphere removed the north pole, and let ϕ_N denote the according stereographic projection map. Let U_S denote the sphere removed the south pole, and ϕ_S denote according stereographic projection map. Then we need to check if $\{(U_{x_i^{+-}}, \phi_{x_i^{+-}}), (U_{N,S}, \phi_{N,S})\}$ is a smooth structure of the sphere. First, $\phi_{x_i^{+-}} \circ \phi^{-1}_{x_i^{+-}}$ is smooth, because it is just a composition of some rotations and reflections. And, from problem 4 we know $\phi_S \circ \phi_N^{-1}$ is smooth. Also, we have $\phi_N \circ \phi_S^{-1} = \frac{1}{|x|^2}x$, which is also smooth because $|x| \neq 0$. Then we verify if $\phi_N \circ \phi^{-1}_{x_i^+}$ smooth. And later we can do similar caculation to $\phi_N \circ \phi^{-1}_{x_i^-}$ and $\phi_S \circ \phi^{-1}_{x_i^{+,-}}$. $\phi_N \circ \phi^{-1}_{x_i^+} = \frac{1}{1-x_n}(x_1, \cdots, x_{i-1}, \sqrt{1-(x_1^2 + \cdots x_n^2)}, x_i, \cdots, x_{n-1})$. Since we have $1-(x_1^2 + \cdots x_n^2) > 0$, it is smooth. So, we can conclude $\phi_N \circ \phi^{-1}_{x_i^-}$ and $\phi_S \circ \phi^{-1}_{x_i^{+,-}}$ are also smooth by similar calculation. Now we need to verify the reverse composition. Like before, we acually only need to verify if $\phi_{x_i^+} \circ \phi_N^{-1}$ is smooth. We can calculate that $\phi_{x_i^+} \circ \phi_N^{-1} = \frac{1}{1+|x|^2}(2x_1, \cdots, 2x_{i-1}, 2x_{i+1}, \cdots, |x|^2 - 1)$, so $\phi_{x_i^+} \circ \phi_N^{-1}$ is indeed a smooth map. So, stereographic charts is equivalent to the smooth structure induced by the hemisphere charts. □

6. Show that $\mathbb{CP}^n$ is a compact smooth manifold.

Proof. We first prove $\mathbb{CP}^n$ is a $2n$ dimensional manifold. Here we define $\mathbb{CP}^n = \mathbb{C}^{n+1} - 0/(z_0, \cdots, z_{n+1}) = \lambda(z_0, \cdots, z_{n+1}), \lambda \in \mathbb{C}^*$. Let $\{U_i = \{z = (z_0, z_1, \cdots, z_{n+1}) \in \mathbb{CP}^n | z_i \neq 0\}\}$, then $\{U_i\}$ is an open cover of $\mathbb{CP}^n$. So, $\mathbb{CP}^n$ is second countable. We only need to prove each U_i is Hausdorff as a subspace to show $\mathbb{CP}^n$ is Hausdoff. Let $[z]$ and $[w]$ be two distinct points in U_j. Then the lines they represent are different, so $[z] = (z_1/z_j, \cdots, z_{j-1}/z_j, 1, z_{j+1}/z_j, \cdots, z_n/z_j) \neq [w] = (w_1/w_j, \cdots, w_{j-1}/w_j, 1, w_{j+1}/w_j, \cdots, w_n/w_j)$. On U_j we define a homeomorphism ϕ_j on U_j to $\mathbb{C}^n$

$$\varphi_j([u_0 : \cdots : u_n]) = \left(\frac{u_0}{u_j}, \ldots, \widehat{\frac{u_j}{u_j}}, \ldots, \frac{u_n}{u_j}\right) \in \mathbb{C}^n.$$

Since $[z] \neq [w]$, they map to distinct points in $\mathbb{C}^n$, and hence can be separated by disjoint open neighborhoods in $\mathbb{C}^n$. Pulling these neighborhoods back gives disjoint open sets in $\mathbb{CP}^n$ that separate $[z]$ and $[w]$. Therefore, $\mathbb{CP}^n$ is Hausdorff. Let ψ be the natrual homeomorphism from $\mathbb{C}^n$ to R^{2n}. We give $\mathbb{CP}^n$ a smooth structure $(U_j, \psi \circ \phi_j)$. It is easy to check $\psi \circ \phi_j$ is a homeomorphism from U_j to $\mathbb{R}^{2n}$. We only need to check $(\psi \circ \phi_j) \circ (\psi \circ \phi_i)^{-1} = (\psi \circ \phi_j) \circ (\phi_i^{-1} \circ \psi^{-1}) = \psi \circ (\phi_j \circ \phi_i^{-1}) \circ \psi^{-1}$. We can caculate that $\phi_j \circ \phi_i^{-1} : (z_1, \cdots, z_n) \to (\frac{1}{z_j})(z_1, z_2, \cdots, z_{i-1}, 1, z_i, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$. Since $z_j \neq 0$, $\phi_j \circ \phi_i^{-1}$ is holomorphic. So, $\psi \circ (\phi_j \circ \phi_i^{-1}) \circ \psi^{-1}$ is smooth in $\mathbb{R}^{2n}$. Therefore, $\mathbb{CP}^n$ is a $2n$ dimensional manifold. Now we prove $\mathbb{CP}^n$ is compact. We denote the quotient map $\mathbb{CP}^n \to \mathbb{C}^{n+1} - 0/(z_0, \cdots, z_{n+1}) = \lambda(z_0, \cdots, z_{n+1}), \lambda \in \mathbb{C}^*$ as p. Now we can consider a bounded closed set $K = \{z = (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} \ | \ \max_i |z_i| = 1\}$. Then $\mathbb{CP}^n = p(K)$. So, $\mathbb{CP}^n$ is compact, since the image of a compact set under a continous map is still compact. □