

Homological Algebra

Based on lectures by Prof. Andrei Căldăraru
Notes taken by Lizi Guo¹

2025 MATH750
University of Wisconsin–Madison

¹These notes are not endorsed by the lecturer, and I have modified them significantly after lectures. They are neither fully faithful nor essentially surjective to the accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Contents

1	Introduction	2
1.1	History of Homological Algebra	2
1.2	Abelian Categories	2
1.3	Homology	4
1.4	Differential Graded Categories	6
2	Derived Functors	8
2.1	Exact Functors	8
2.2	Projective Objects	10
2.3	Projective Resolution	12
2.4	Motivations of Derived Functors	16
2.5	Left Derived Functors	18
2.6	Opposite Categories	23
2.7	Adjunction	24
2.8	Injective Resolution	27
2.9	Distinguished Triangles	30
2.10	Balanced Bifunctors	33
2.11	Acyclic Resolutions	34
3	Derived Categories	36
3.1	Triangulated Categories	36
3.2	Localization of a category	39
3.3	Derived Categories	40
3.4	Extension	41
3.5	Derived Functors in Derived Categories	42

1 Introduction

1.1 History of Homological Algebra

1940s Leray spectral sequence. The origins of homological methods trace back to Jean Leray’s work during World War II, where he introduced spectral sequences in the study of sheaf cohomology. This provided a systematic way to compute cohomology groups through filtrations.

1950s Cartan–Eilenberg developed the formal foundations of homological algebra in their book *Homological Algebra* (1956), introducing derived functors such as Ext and Tor.

Serre, *FAC: Faisceaux Algébriques Cohérents* (1955) introduced coherent sheaves on complex manifolds and algebraic varieties, linking algebraic geometry and sheaf theory.

Grothendieck’s Tôhoku paper (1957) unified homological algebra for abelian categories, generalizing the setting from modules to sheaves. This provided a categorical framework for derived functors and spectral sequences, combining Serre’s geometric perspective with algebraic formalism.

1960s Grothendieck and Verdier: derived categories. Alexander Grothendieck’s school, particularly Verdier’s thesis (1963), formalized the notion of the *derived category* $D(\mathcal{A})$, allowing one to work with complexes up to quasi-isomorphism. This revolutionized the formulation of cohomological theories in algebraic geometry.

1970s Triangulated categories. Verdier’s construction led to the concept of triangulated categories, providing the language for much of modern homological algebra, particularly in the study of sheaves and motives.

1980s Quillen’s model categories (1967, influential in the 1980s) introduced a homotopical framework generalizing chain complexes, forming the foundation of modern homotopical algebra.

Mukai’s equivalences (1981) established that for an abelian variety A , the derived categories of coherent sheaves satisfy

$$D^b(\mathrm{Coh}(A)) \simeq D^b(\mathrm{Coh}(\hat{A})),$$

opening the study of derived equivalences in algebraic geometry.

1990s Study of $D(X)$. Derived categories of coherent sheaves on varieties became a central object of study (Bondal–Orlov). Ideas from mirror symmetry connected derived categories with symplectic geometry.

Derived geometry. Spaces themselves began to be “derived” — with work by Jacob Lurie and Bertrand Toën–Gabriele Vezzosi developing derived and higher algebraic geometry. These ideas eventually linked to the *geometric Langlands program* and higher categorical structures.

2000s–Present Higher categories and derived geometry. Lurie’s *Higher Algebra* and *Spectral Algebraic Geometry* established ∞ -categorical approaches to homological algebra. Derived categories evolved into *stable ∞ -categories*, forming a modern unified language across topology, algebra, and geometry.

1.2 Abelian Categories

Definition 1.2.1 (Additive Categories). *A category \mathcal{C} is said to be **additive** if it satisfies the following properties.*

1. For each $A, B \in \mathcal{C}$, $\text{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition (enriched over the category of abelian groups).
2. \mathcal{C} has a zero object, denoted 0 . (This is an object that is simultaneously an initial object and a final object.)
3. It has products of two objects.

Remark 1.2.2. A functor between additive categories preserving the additive structure of Mor , is called an **additive functor**. For the following definitions, we assume the categories and functors are additive.

Definition 1.2.3 (Kernel). Let $f : B \rightarrow C$ be a morphism in an additive category. A **kernel** of f is a morphism $i : \ker f \rightarrow B$ such that:

$$f \circ i = 0,$$

and for any morphism $a : A \rightarrow B$ with $f \circ a = 0$, there exists a unique morphism $a' : A \rightarrow \ker f$ making the following diagram commute:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \text{ker } f & \xrightarrow{i} & B & \xrightarrow{f} & C \\
 & \uparrow a' & \nearrow a & & \nearrow & \\
 A & & & & & \\
 & \searrow & & & & \\
 & & 0 & & &
 \end{array}$$

Remark 1.2.4. The $\ker f$ is the largest object mapping to B that is sent to 0 under f . Equivalently, the kernel can be characterized as a *limit* of the following diagram:

$$\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
 B & \xrightarrow{f} & C
 \end{array}$$

That is, $\ker f$ together with $i : \ker f \rightarrow B$ satisfies the universal property of the limit:

$$\begin{array}{ccccc}
 A & & & & \\
 \searrow & & \searrow & & \\
 & \text{ker } f & \longrightarrow & 0 & \\
 & \downarrow i & & \downarrow & \\
 & B & \xrightarrow{f} & C &
 \end{array}$$

Definition 1.2.5 (Cokernel). Let $f : C \rightarrow B$ be a morphism in an additive category. A **cokernel** of f is a morphism $p : B \rightarrow \text{coker } f$ such that:

$$p \circ f = 0,$$

and for any morphism $a : B \rightarrow A$ with $a \circ f = 0$, there exists a unique morphism $a' : \text{coker } f \rightarrow A$ making the following diagram commute:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & C & \xrightarrow{f} & B & \xrightarrow{p} & \text{coker } f \\
 & \searrow & & \searrow a & \downarrow a' & \\
 & & & & & A \\
 & \nearrow & & & & \\
 & & 0 & & &
 \end{array}$$

Remark 1.2.6. The coker f is the smallest object mapping B to 0 under f . Equivalently, the cokernel can be characterized as a *colimit* of the following diagram:

$$\begin{array}{ccc}
 & & 0 \\
 & & \uparrow \\
 B & \xleftarrow{f} & C \\
 & & \\
 A & \xleftarrow{\quad} & \text{coker } f \longrightarrow 0 \\
 \uparrow & \nwarrow \exists! & \uparrow p \\
 & B & \xrightarrow{f} C \\
 & & \uparrow
 \end{array}$$

Remark 1.2.7. • $\text{im}(f) = \ker(\text{coker } f)$

- If $i : A \rightarrow B$ is a monomorphism, then we say that A is a subobject of B , where the map i is implicit. A quotient object is defined dually to subobject.
- The cokernel of a monomorphism is called the quotient.

Definition 1.2.8 (Abelian Categories). An **abelian category** is an additive category satisfying three additional properties.

1. Every map has a kernel and cokernel.
2. Every monomorphism is the kernel of its cokernel.
3. Every epimorphism is the cokernel of its kernel.

Theorem 1.2.9 (Mitchell Embedding Theorem). Every small abelian category admits a full, faithful and exact functor to the category \mathbf{RMod} for some ring R .

1.3 Homology

Definition 1.3.1 (Exactness). In an abelian category, we say a sequence

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

is **exact** at B if $\ker g = \text{im } f$. A sequence is an **exact sequence** if it is exact at each term. A **short exact sequence** is an exact sequence with five terms, the first and last of which are zeros.

Definition 1.3.2 (Chain Complexes). A **chain complex** is a sequence of objects and homomorphisms in an abelian category

$$\dots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_0} \dots$$

such that

$$d_i \circ d_{i+1} = 0$$

for all i .

Definition 1.3.3 (Chain Maps). If $(C., d^C)$ and $(D., d^D)$ are chain complexes, then a **chain map** $C. \rightarrow D.$ is a collection of homomorphisms $f_n : C_n \rightarrow D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$. In other words, the following diagram has to commute for all n :

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ d_n^C \downarrow & & \downarrow d_n^D \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

Theorem 1.3.4. Let \mathcal{A} be an Abelian category. The category of chain complexes in \mathcal{A} is an Abelian category.

Definition 1.3.5 (Homology). The **homology** of a chain complex $C.$ is

$$H_i(C.) = \frac{\ker(d_i : C_i \rightarrow C_{i-1})}{\operatorname{im}(d_{i+1} : C_{i+1} \rightarrow C_i)}.$$

Remark 1.3.6. By using increasing superscript indices, and set $C^i = C_{-i}$, we get the notion of **cochain complexes** and **cohomologies**. Notice the cohomological notation is dual to homological notation and we prefer to using homological one in this course.

Definition 1.3.7 (Quasi-isomorphism). A chain map

$$f_{\bullet} : C_{\bullet} \rightarrow D_{\bullet}$$

is called a quasi-isomorphism if

$$H_i(f_{\bullet}) : H_i(C_{\bullet}) \xrightarrow{\sim} H_i(D_{\bullet})$$

is an isomorphism for all i .

Lemma 1.3.8. $H_i : Ch(\mathcal{A}) \rightarrow \mathcal{A}$ is a functor.

Theorem 1.3.9 (Snake Lemma).

$$\begin{array}{ccccccc} \ker f & \longrightarrow & \ker g & \longrightarrow & \ker h & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ A' & \longrightarrow & B' & \xrightarrow{p} & C' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{coker } f & \longrightarrow & \text{coker } g & \longrightarrow & \text{coker } h & \longrightarrow & \end{array}$$

∂

Theorem 1.3.10 (Zig-Zag Lemma). *Suppose we have a short exact sequence of complexes*

$$0 \longrightarrow A. \xrightarrow{i} B. \xrightarrow{q} C. \longrightarrow 0.$$

Then there are maps

$$\partial : H_n(C.) \rightarrow H_{n-1}(A.)$$

such that there is a long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{q_*} & H_n(C) \\ & & & & \searrow \partial_* & & \\ & & H_{n-1}(A) & \xleftarrow{i_*} & H_{n-1}(B) & \xrightarrow{q_*} & H_{n-1}(C) \longrightarrow \cdots \end{array}$$

1.4 Differential Graded Categories

¹ A **differential graded (dg) object** C is a \mathbb{Z} -graded object

$$C = \bigoplus_{i \in \mathbb{Z}} C_i,$$

together with a differential operator

$$d : C \longrightarrow C[1],$$

such that $d^2 = 0$.

The notation $C[1]$ denotes the *shifted* graded object, whose underlying object is still C , but with components shifted by one degree:

$$(C[1])_n = C_{n-1}, \quad d_{C[1]} = (-1)^i d_C \text{ on } C^{i-1}.$$

Intuitively, “shifting by 1” means *moving the whole complex to the left by one*.

For cohomological notation:

$$C^i := C_{-i}, \quad C[1]^i = C^{i+1}.$$

Remark 1.4.1. A chain complex of R -module is nothing but a dg R -module. Philosophy: consider everything as a dg thing.

Definition 1.4.2 (Chain Homotopy). *Let \mathcal{A} be an additive category and let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes with values in \mathcal{A} . Let $f = \{f_n\}_{n \in \mathbb{Z}}$ and $f' = \{f'_n\}_{n \in \mathbb{Z}}$ be chain maps from C_* to D_* . A **chain homotopy** from f to f' is a collection of maps $h = \{h_n : C_n \rightarrow D_{n+1}\}$ which satisfy the identity*

$$f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C$$

for every integer n .

We say that f and f' are **chain homotopic** if there exists a chain homotopy from f to f' . We will say that f is a **chain homotopy equivalence** if there exists a chain map $g : D_* \rightarrow C_*$ such that $g \circ f$ and $f \circ g$ are chain homotopic to the identity morphisms id_{C_*} and id_{D_*} , respectively. Clearly, a chain homotopy equivalence induces an isomorphism on homology.

¹I have modified this section significantly, referring to Lurie’s Kerodon.

Definition 1.4.3 (Mapping Complexes). Let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes with values in an Abelian category \mathcal{A} . For each integer d , we let $[C, D]_d$ denote the abelian group $\prod_{n \in \mathbf{Z}} \text{Hom}_{\mathcal{A}}(C_n, D_{n+d})$ consisting of maps from C_* to D_* which are homogeneous of degree d . These abelian groups can be organized into a chain complex

$$\cdots \xrightarrow{\partial} [C, D]_2 \xrightarrow{\partial} [C, D]_1 \xrightarrow{\partial} [C, D]_0 \xrightarrow{\partial} [C, D]_{-1} \xrightarrow{\partial} [C, D]_{-2} \xrightarrow{\partial} \cdots,$$

whose boundary operator $\partial : [C, D]_d \rightarrow [C, D]_{d-1}$ is given by the formula $\partial \{f_n : C_n \rightarrow D_{n+d}\}_{n \in \mathbf{Z}} = \{\partial_D \circ f_n - (-1)^d f_{n-1} \circ \partial_C\}_{n \in \mathbf{Z}}$. We will refer to $[C, D]_*$ as the **mapping complex** associated to the chain complexes C_* and D_* .

Remark 1.4.4. • Chain maps from C_* to D_* can be identified with **0-cycles** of the chain complex $[C, D]_*$: that is, with elements $f = \{f_n\}_{n \in \mathbf{Z}} \in [C, D]_0$ satisfying $\partial(f) = 0$.

- Given a pair of chain maps $f, f' : C_* \rightarrow D_*$, a chain homotopy from f to f' is an element $h = \{h_n\}_{n \in \mathbf{Z}} \in [C, D]_1$ satisfying $\partial(h) = f' - f$. In particular, f and f' are **chain homotopic** if and only if they are **homologous** when viewed as 0-cycles of the complex $[C, D]_*$, so $\text{Hom}_{\text{hCh}(\mathcal{A})}(C_*, D_*)$ can be identified with the 0th homology group of $[C, D]_*$.

Definition 1.4.5 (Differential Graded Categories). A **differential graded category** \mathcal{C} consists of the following data:

1. A collection $\text{Ob}(\mathcal{C})$, whose elements we refer to as *objects* of \mathcal{C} .
2. For every pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a chain complex $(\text{Hom}_{\mathcal{C}}(X, Y)_*, \partial)$. For each integer n , we refer to the elements of $\text{Hom}_{\mathcal{C}}(X, Y)_n$ as *morphisms of degree n from X to Y* .
3. For every triple of objects $X, Y, Z \in \text{Ob}(\mathcal{C})$ and every pair of integers $m, n \in \mathbf{Z}$, a function

$$c_{Z,Y,X} : \text{Hom}_{\mathcal{C}}(Y, Z)_n \times \text{Hom}_{\mathcal{C}}(X, Y)_m \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)_{m+n}$$

which we will refer to as the *composition law*. Given a pair of morphisms $f \in \text{Hom}_{\mathcal{C}}(X, Y)_m$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)_n$, we will often denote the image $c_{Z,Y,X}(g, f) \in \text{Hom}_{\mathcal{C}}(X, Z)_{m+n}$ by $g \circ f$ or gf .

4. For every object $X \in \text{Ob}(\mathcal{C})$, a morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)_0$, which we will refer to as the *identity morphism*.

Remark 1.4.6. A dg category is a "category" enriched over $\text{Ch}(\mathcal{A})$ for some abelian category \mathcal{A} .

Example 1.4.7. Let \mathcal{A} be an abelian category. $\text{Ch}(\mathcal{A})$ is a dg category enriched over $\text{Ch}(\mathbf{Z})$.

Let \mathcal{A} be an abelian category.

Definition 1.4.8 (Homotopy Categories). The **homotopy category** of chain complexes in \mathcal{A} , denoted

$$\text{HoCh}(\mathcal{A}) = H^0(\text{Ch}(\mathcal{A})) = K(\mathcal{A}),$$

is the category defined as follows:

- Objects are chain complexes in \mathcal{A} ;
- Morphisms are chain maps, taken **modulo chain homotopy**.

That is,

$$\text{Hom}_{K(\mathcal{A})}(A, B) = \text{Hom}_{\text{Ch}(\mathcal{A})}(A, B) / (\text{chain homotopy}).$$

Remark 1.4.9.

$K(\mathcal{A})$ is not an abelian category,

but it carries a natural structure of a **triangulated category**.

2 Derived Functors

2.1 Exact Functors

Definition 2.1.1 (Exact Functors). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a (covariant) functor between abelian categories.

1. F is called **right exact** if for every short exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{C} , the induced sequence

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

is exact in \mathcal{D} .

2. F is called **left exact** if for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

in \mathcal{C} , the induced sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC$$

is exact in \mathcal{D} .

3. F is called **exact** if it is both left and right exact.

Remark 2.1.2. If F is *contravariant*, then F is left exact if and only if for every exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC$$

is exact. i.e. we get a left exact sequence after applying F .

Example 2.1.3 (Tensor is Left-exact). Let $M \in R\text{-Mod}$. Then the functor

$$- \otimes_R M : \text{Mod-}R \longrightarrow \text{Ab}$$

is **right exact**, where ${}_Z N_R \in \text{Mod-}R$ and ${}_R M_Z \in R\text{-Mod}$, then

$${}_Z N_R \otimes_R {}_R M_Z \in \text{Ab}.$$

Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$, we get

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \longrightarrow 0.$$

Hence the resulting sequence is *exact on the right* but not on the left — confirming that

$$- \otimes_R M \quad \text{is right exact (but not left exact).}$$

Example 2.1.4 (Covariant Hom is Left-exact). Let $M \in R\text{-Mod}$. Then the functor

$$\text{Hom}(M, -) : R\text{-Mod} \longrightarrow \text{Ab}$$

is **left exact**.

That is, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $R\text{-Mod}$, the induced sequence

$$0 \longrightarrow \text{Hom}(M, A) \longrightarrow \text{Hom}(M, B) \longrightarrow \text{Hom}(M, C)$$

is exact in Ab .

Consider again the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying the functor $\text{Hom}(\mathbb{Z}/2, -)$, we get:

$$0 \longrightarrow \text{Hom}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \text{Hom}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow 0.$$

Since any map $\mathbb{Z}/2 \rightarrow \mathbb{Z}$ is zero, we obtain:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2.$$

Thus the sequence is exact on the left but not on the right, confirming that

$$\text{Hom}(M, -) \text{ is left exact (but not right exact).}$$

Example 2.1.5 (Contravariant Hom is Left-exact). Let $M \in R\text{-Mod}$, the functor

$$\text{Hom}(-, M) : R\text{-Mod} \longrightarrow \text{Ab}$$

is **left exact**.

That is, for every short exact sequence in $R\text{-Mod}$,

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow \text{Hom}(C, M) \xrightarrow{v^*} \text{Hom}(B, M) \xrightarrow{u^*} \text{Hom}(A, M)$$

is exact in Ab .

(Note: this is the contravariant version of the Hom functor, hence the direction of arrows reverses.)

Example 2.1.6 (Quotient is right exact). Let $M \in R\text{-Mod}$ and let $I \triangleleft R$ be an ideal.

Consider the functor

$$M \longmapsto M/IM.$$

We have the natural isomorphism

$$M/IM \cong M \otimes_R R/I.$$

Hence this functor can be viewed as the tensor functor

$$- \otimes_R R/I : R\text{-Mod} \longrightarrow \text{Ab}.$$

Since the tensor functor $- \otimes_R N$ is **right exact** for any N , it follows that

$$M \longmapsto M/IM \text{ is right exact.}$$

Example 2.1.7. Consider the functor

$$(-)^G : \text{Rep}_G \longrightarrow \text{Vect},$$

which sends a representation V of G to its subspace of invariants

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}.$$

Note. This functor can be expressed as a Hom functor:

$$(-)^G \cong \text{Hom}_{G\text{-rep}}(\mathbf{1}_G, -),$$

where $\mathbf{1}_G$ denotes the trivial representation of G .

Since any Hom functor of the form $\text{Hom}(A, -)$ is **left exact**, it follows that

$$(-)^G : \text{Rep}_G \longrightarrow \text{Vect} \quad \text{is left exact.}$$

Example 2.1.8 (Pushforward is Left-exact). Let $f : X \rightarrow Y$ be a continuous map of topological spaces (or morphism of schemes). Given a sheaf \mathcal{F} of abelian groups on X , the **pushforward sheaf** (or **direct image sheaf**) of \mathcal{F} along f is the sheaf $f_*\mathcal{F}$ on Y defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)) \quad \text{for every open set } V \subseteq Y.$$

Pushforward is left exact. This gives a functor between categories of sheaves of abelian groups:

$$f_* : \text{Sh}(X) \longrightarrow \text{Sh}(Y),$$

For a short exact sequence of sheaves on X :

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

applying f_* gives

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

This sequence is exact on the left but not necessarily on the right. Thus f_* is a **left exact functor**.

Remark 2.1.9. An additive functor between abelian categories $F : \mathcal{C} \rightarrow \mathcal{D}$ can always be lifted to $F : \text{Ch}(\mathcal{C}) \rightarrow \text{Ch}(\mathcal{D})$. However, many functors we are interested in are not exact, but only left- or right-exact. Therefore, they cannot be lifted to $\mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{D})$ since $F(0) \not\cong 0$. The derived versions of these functors we will talk about later will fix this problem.

2.2 Projective Objects

Let \mathcal{C} be an abelian category.

Definition 2.2.1 (Projective Objects). *An object $P \in \mathcal{C}$ is called **projective** if for every epimorphism*

$$f : M' \twoheadrightarrow M$$

and every morphism

$$g : P \longrightarrow M,$$

there exists a morphism $h : P \rightarrow M'$ such that

$$f \circ h = g.$$

$$\begin{array}{ccccc} & & P & & \\ & \swarrow h & \downarrow g & & \\ M' & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

i.e. morphisms from P lift along surjections.

Example 2.2.2 (Free Modules are Projective). R is a projective R -module. More generally, every **free** R -module is projective.

Indeed, given a surjection of R -modules

$$M' \twoheadrightarrow M \longrightarrow 0,$$

and any map from a free module $R^{(I)} \rightarrow M$, each **basis element** of $R^{(I)}$ can be lifted to M' independently, so a lifting map $R^{(I)} \rightarrow M'$ exists.

$$\begin{array}{ccccc} & & R^{(I)} & & \\ & \swarrow h & \downarrow g & & \\ M' & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

Example 2.2.3. $\mathbb{Z}/2\mathbb{Z}$ is *not* a projective \mathbb{Z} -module.

Consider the surjection

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Suppose there were a lift of the identity map

$$\begin{array}{ccc} \mathbb{Z}/2 & \dashrightarrow & \mathbb{Z} \\ & \searrow \text{id} & \downarrow \times 2 \\ & & \mathbb{Z}/2 \end{array}$$

This would require a homomorphism $\mathbb{Z}/2 \rightarrow \mathbb{Z}$ such that composing with $\times 2$ gives the identity on $\mathbb{Z}/2$, which is impossible because \mathbb{Z} has no element of order 2.

Proposition 2.2.4 (Characterization of Projective Modules). *Let P be an R -module. The following statements are equivalent:*

(1) P is **projective**,

(2) The functor

$$\text{Hom}_R(P, -) : R\text{-Mod} \longrightarrow \text{Ab}$$

is **right-exact**, hence exact.

(3) P is a **direct summand of a free module**, i.e. there exists a free R -module F and a submodule $Q \subseteq F$ such that

$$F = P \oplus Q.$$

(4) Every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

Theorem 2.2.5. *Projective modules over local rings or PIDs are free.*

Remark 2.2.6 (Projective Modules = Vector Bundles). Geometrically, a vector bundle is locally free; algebraically, a projective module is locally a free module. These notions coincide on affine schemes.

Let $X = \text{Spec}(R)$ be an affine scheme. There is an equivalence of categories

$$\text{ProjMod}_R^{\text{fg}} \simeq \text{Vect}(X),$$

between finitely generated projective R -modules and vector bundles (locally free sheaves of finite rank or coherent sheaves with free stalks) on X .

2.3 Projective Resolution

Definition 2.3.1. Let M be an R -module (or an object in an abelian category). A projective resolution of M is a chain complex

$$P_{\bullet} = (\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0)$$

together with a chain map $q : P_{\bullet} \rightarrow M[0]$ such that:

1. $P_i = 0$ for all $i < 0$;
2. $H_i(P_{\bullet}) = 0$ for all $i \neq 0$;
3. $H_0(P_{\bullet}) \cong M$;
4. each P_i is projective.

Equivalently, this means that we have an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{q} M \longrightarrow 0$$

with each P_i projective.

Equivalently, the chain map between the projective chain complex P_{\bullet} to $M[0]$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow 0 & & \downarrow q & & \parallel & \parallel \\ \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \cdots \end{array}$$

is a quasi-isomorphism.

If the abelian category \mathcal{C} **has enough projectives** (every object is mapped surjectively by some projective object), then every object $M \in \mathcal{C}$ admits a (non-unique) projective resolution.

This can be done inductively as follows:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & P_1 & \twoheadrightarrow & K_0 \longrightarrow 0 \\
& & \uparrow & & \downarrow & & \downarrow \\
& & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\
& & & & \downarrow & & \downarrow \\
& & & & M & & \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

That is inductively finding a projective P_{i+1} mapping onto the kernel K_i of the last projection $P_i \rightarrow K_{i-1}$.

Remark 2.3.2. The category of R -modules $R\text{-Mod}$ has enough free objects, i.e. $R^{(M)} \twoheadrightarrow M$. Therefore, any R -module admits a free projective resolution.

Example 2.3.3 (Vector Spaces). Over a field $R = k$, every module is free. Given M ,

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$$

is already a projective resolution.

Example 2.3.4. Over \mathbb{Z} , take $M = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

Notice the kernel \mathbb{Z} is projective, so we are done.

Example 2.3.5. Over $k[x]$, let $M = k[x]/(x)$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & k[x] & \xrightarrow{x} & k[x] \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & k[x]/(x)
\end{array}$$

Example 2.3.6. Over $k[x, y]$, let $M = k[x, y]/(x, y)$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & k[x, y] & \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & k[x, y]^{\oplus 2} & \xrightarrow{(x \ y)} & k[x, y] \\
& & & & & & \downarrow \\
& & & & & & k[x, y]/(x, y)
\end{array}$$

Example 2.3.7. Over $R = k[x, y]/(xy)$, let $M = R/(x, y)$.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & R \oplus R & \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} & R \oplus R & \xrightarrow{\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}} & R \oplus R & \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} & R \oplus R & \xrightarrow{(x, y)} & R \\ & & & & & & & & & & \downarrow \\ & & & & & & & & & & \mathcal{M} \end{array}$$

Theorem 2.3.8 (Hilbert). *If R is a regular commutative local ring, then every minimal resolution of a module ends after $\dim R$ steps.*

Theorem 2.3.9 (Serre). *If R is not regular, there exists an R -module with an infinite minimal resolution.*

Theorem 2.3.10 (Eisenbud). *If R is a hypersurface ring, the infinite resolution is eventually 2-periodic.*

Question: How far from unique are projective resolution?

Proposition 2.3.11. *Let R be a ring (or work in an abelian category). Suppose*

$$f : M \longrightarrow N$$

is a morphism of R -modules, and let

$$P_\bullet \rightarrow M, \quad Q_\bullet \rightarrow N$$

be projective resolutions of M and N respectively.

Then there exists a chain map

$$\tilde{f} : P_\bullet \longrightarrow Q_\bullet$$

lifting f , i.e.

$$H_0(\tilde{f}) = f : M = H_0(P_\bullet) \longrightarrow H_0(Q_\bullet) = N.$$

Furthermore, \tilde{f} is not unique, but unique up to homotopy.

That is, \tilde{f} makes the following diagram commute up to homology:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\ & & \tilde{f}_1 \downarrow & & \tilde{f}_0 \downarrow & & \downarrow f \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

and if $\bar{f}, \bar{f}' : P_\bullet \rightarrow Q_\bullet$ are two such liftings, then they are homotopic:

$$\tilde{f} \simeq \tilde{f}'.$$

Equivalently, there is a natural isomorphism

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{\mathcal{K}(R\text{-mod})}(P_\bullet, Q_\bullet).$$

Remark 2.3.12. 1. Any functor F takes isomorphisms to isomorphisms.

2. Functors between abelian categories need *not* map quasi-isomorphisms to quasi-isomorphisms.

For example:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow 0$$

is a complex, and the natural map

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism, but there is no quasi-isomorphism back.

3. However, homotopic maps are always mapped to homotopic maps.

Theorem 2.3.13 (Horseshoe Lemma). *Let*

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$

be a short exact sequence of R -modules (or objects in an abelian category). Suppose we have projective resolutions

$$\cdots \rightarrow P'_1 \rightarrow P'_0 \rightarrow A' \rightarrow 0, \quad \cdots \rightarrow P''_1 \rightarrow P''_0 \rightarrow A'' \rightarrow 0.$$

Then there exists a projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

such that we have a short exact sequence of chain complexes

$$0 \longrightarrow P'_\bullet \longrightarrow P_\bullet \longrightarrow P''_\bullet \longrightarrow 0,$$

and each P_i fits into a short exact sequence

$$0 \longrightarrow P'_i \longrightarrow P_i \longrightarrow P''_i \longrightarrow 0.$$

We have the following diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P'_1 & \longrightarrow & P_1 & \longrightarrow & P''_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P'_0 & \longrightarrow & P_0 & \longrightarrow & P''_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Remark 2.3.14. This lemma shows that projective resolutions behave well with respect to short exact sequences.

2.4 Motivations of Derived Functors

Moral: If F behaves badly on X , replace X by something else on which F behaves well.

We will discuss two examples from homotopy theory.

Example 2.4.1 (Homotopy Fibre). Let

$$f^{-1}(y) = X_y \longrightarrow X$$

be the fiber of a map

$$f : X \longrightarrow Y$$

over a point $y \in Y$. That is, the pullback

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \{y\} & \xhookrightarrow{j} & Y \end{array}$$

However, the process of “taking fibers” is *not homotopy invariant* — this is bad. For instance, consider a map $\{x\} \rightarrow Y$, $x \mapsto y_0$, then

$$\begin{array}{c} x \\ \bullet \\ \downarrow \\ \text{---} \bullet \text{---} Y \\ y_0 \end{array}$$

But “taking fiber” is well-defined up to homotopy for certain maps — namely, for fibrations. To obtain a homotopy-invariant notion, *replace f by a homotopy equivalent fibration*.

$$\begin{array}{ccc} X & \xrightarrow{i} & X_f \\ & \searrow f & \downarrow p \\ & & Y \end{array}$$

Here $X_f = \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = f(x)\}$ is called the mapping path space, i is the inclusion and $p(x, \gamma) = \gamma(1)$. Then i is a homotopy equivalence and p is a fibration.

The *homotopy fiber* F_f of a map

$$f : X \rightarrow Y$$

is the ordinary fiber of the fibration p replacing f . By the way, the homotopy fiber is the homotopy pullback (limit) of the following diagram

$$\operatorname{holim}_{\leftarrow} \left(\begin{array}{ccc} & & X \\ & & \downarrow f \\ * & \longrightarrow & Y \end{array} \right) \simeq F_f$$

Back to the example $x \mapsto y_0$. The mapping path space X_f is the path space

$$P(Y, y_0) = \{\gamma \in Y^I \mid \gamma(0) = y_0\}$$

and the homotopy fibre is the loop space

$$\Omega(Y, y_0) = \{\gamma \in Y^I \mid \gamma(0) = \gamma(1) = y_0\}.$$

This is called a path fibration.

Example 2.4.2 (Homotopy Quotient). Let X be a topological space with a (left) action of a topological group G . Consider the functor that assigns to X the orbit space X/G . This functor is *not homotopy invariant*.

For example, let $X = \mathbb{R} \simeq *$, and let \mathbb{Z} act on \mathbb{R} by translation. Then

$$\mathbb{R}/\mathbb{Z} \cong S^1, \quad */\mathbb{Z} \cong *.$$

The quotient X/G is well-defined up to homotopy if the G -action on X is *free*. Moreover, any continuous G -action on a space can be replaced by a homotopy equivalent free G -action.

To make the quotient construction homotopy invariant, replace X by a free G -space $EG \times X$ where EG is a contractible free G -space (the universal G -bundle). Define the *homotopy quotient* as

$$X_{hG} := EG \times_G X.$$

When the action of G on X is free, there is a homotopy equivalence $X_{hG} \simeq X/G$.

If G acts on a single point $*$, we replace $*$ by EG (which has a free G -action). Then

$$*_{hG} = EG/G =: BG,$$

where BG is called the *classifying space* of G .

- To find $B\mathbb{Z}$, notice $E\mathbb{Z} = \mathbb{R}$ and we have seen $\mathbb{R}/\mathbb{Z} = S^1$. Therefore, $B\mathbb{Z} \simeq S^1$. On the other hand, $B\mathbb{Z} = K(\mathbb{Z}, 1)$ (true for all groups G) the Eilenberg–MacLane space, which is S^1 as well.
- To find BS^1 , first notice $S^1 \simeq \mathbb{C}^* \cong GL(1, \mathbb{C})$. Therefore, $BS^1 \simeq BGL(1, \mathbb{C})$, which is the classifying space for complex line bundles \mathbb{CP}^∞ .

There is a natural fibration

$$\begin{array}{ccc} EG \times_G X & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X/G & \longrightarrow & BG \end{array}$$

where the fiber of $EG \times_G X \rightarrow BG$ is homotopy equivalent to X .

For any coefficient ring R ,

$$H^*(G; R) \cong H^*(BG; R).$$

More generally, if G acts on a space X , the G -equivariant cohomology of X is defined as

$$H_G^*(X; R) := H^*(EG \times_G X; R),$$

which is invariant under G -equivariant homotopy equivalence.

2.5 Left Derived Functors

Definition 2.5.1 (Left Derived Functors). *Let \mathcal{C}, \mathcal{D} be abelian categories, and suppose \mathcal{C} has enough projectives. Let*

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

be a right-exact functor. For an object $M \in \text{Ob}(\mathcal{C})$, choose a projective resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Then the left derived functors of F are defined by

$$(L_i F)(M) := H_i(FP_\bullet),$$

that is, we apply F to the projective resolution and take homology.

Theorem 2.5.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a right-exact functor between abelian categories, and assume \mathcal{C} has enough projectives. Then:*

- (a) $L_i F$ is a well-defined functor, i.e. it does not depend on the chosen projective resolution.
- (b) $L_0 F \cong F$.
- (c) For any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in \mathcal{C} , there is a natural long exact sequence in \mathcal{D} :

$$\cdots \rightarrow (L_i F)(M') \rightarrow (L_i F)(M) \rightarrow (L_i F)(M'') \rightarrow (L_{i-1} F)(M') \rightarrow \cdots.$$

- (d) *This long exact sequence is functorial with respect to short exact sequences in \mathcal{C} .*

Remark 2.5.3. To define the left derived functors, we must first assign a projective resolution to every object in this abelian category a priori, using the axiom of choice. Later we will see that the derived functor does not depend on the choice of resolution.

Proposition 2.5.4. *Let P_\bullet and Q_\bullet be two projective resolutions of the same module (or object) M . Then there exists a canonical isomorphism*

$$(L_i^P F)(M) \cong (L_i^Q F)(M)$$

for all $i \geq 0$.

Let $\mathcal{C} = R\text{-Mod}$, the category of left R -modules. Consider the functor

$$F = M \otimes_R - : R\text{-Mod} \longrightarrow \mathbf{Ab},$$

which is *right exact*.

Definition 2.5.5 (Tor). *For $M \in R\text{-Mod}$, we define the left derived functors of F as*

$$\text{Tor}_i^R(M, -) := L_i F(-).$$

Remark 2.5.6. We could have derived in the first variable instead of the second:

$$\mathrm{Tor}_i^R(-, N) = L_i(- \otimes_R N),$$

and the result would be the same up to canonical isomorphism.

Example 2.5.7. Let us compute $\mathrm{Tor}_i^R(M, R)$.

(1) **Find a projective resolution of R :** Since R is a free (hence projective) module over itself,

$$0 \longrightarrow R \longrightarrow 0$$

is a projective resolution of R .

(2) **Apply the functor $M \otimes_R -$:**

$$0 \longrightarrow M \otimes_R R \longrightarrow 0$$

which simplifies to

$$0 \longrightarrow M \longrightarrow 0.$$

(3) **Take homology:**

$$H_0 = M, \quad H_1 = 0, \quad H_2 = 0, \dots$$

Hence,

$$\mathrm{Tor}_i^R(M, R) = \begin{cases} M & i = 0, \\ 0 & i > 0. \end{cases}$$

Example 2.5.8. Assume R is a commutative ring and $f \in R$ is *not* a zero-divisor. We want to compute

$$\mathrm{Tor}_i^R(M, R/(f)).$$

1. Find a projective resolution of $R/(f)$. Since f is not a zero-divisor, we have a short exact sequence

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0,$$

which is a projective (in fact free) resolution of $R/(f)$.

2. Apply the functor $M \otimes_R -$ Tensoring with M gives an exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow 0.$$

We denote this as a chain complex concentrated in degrees 1 and 0.

3. Compute homology

$$H_0 = M/fM = M \otimes_R R/(f),$$

$$H_1 = \ker(f : M \rightarrow M) = \{ m \in M \mid fm = 0 \}, \text{ the } f\text{-torsion submodule.}$$

Thus,

$$\mathrm{Tor}_i^R(M, R/(f)) = \begin{cases} M/fM & \text{if } i = 0, \\ \{ m \in M \mid fm = 0 \} & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

i.e. $\text{Tor}_1^R(M, R/(f))$ detects the f -torsion in M .

Example 2.5.9. Let $R = \mathbb{Z}$ and M be a \mathbb{Z} -module. Take $N = \mathbb{Z}/k\mathbb{Z}$.

$$\begin{aligned}\text{Tor}_0^{\mathbb{Z}}(M, N) &= M/kM, \\ \text{Tor}_1^{\mathbb{Z}}(M, N) &= \{m \in M \mid km = 0\},\end{aligned}$$

the k -torsion submodule of M .

Example 2.5.10. Let R be a principal ideal domain (PID). Then for any R -modules M, N ,

$$\text{Tor}_i^R(M, N) = 0 \quad \text{for all } i \geq 2.$$

The reason is every finitely generated R -module N has a projective (even free) resolution of length at most 1:

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where F is free. Fact: Over a PID, every submodule of a free module is free. Hence K is free, and thus the resolution stops after one step. Therefore all higher Tor-groups ($i \geq 2$) vanish.

Example 2.5.11. Consider the short exact sequence of \mathbb{Z} -modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Applying $\text{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2, -)$ yields

$$\cdots \longrightarrow \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow 0.$$

Since $\text{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = 0$ for $i > 0$, this simplifies to

$$0 \longrightarrow \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow 0,$$

so we again obtain

$$\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2, \quad \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2.$$

Example 2.5.12. Let

$$R = k[x, y], \quad M = R/(x, y), \quad N = R/(x).$$

1. Projective Resolution of $N = R/(x)$

A projective resolution of N is:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow N \longrightarrow 0.$$

2. Tensor with M Now apply $M \otimes_R -$:

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow 0.$$

But since x acts by 0 on $M = R/(x, y)$, the map $\cdot x$ is the zero map.

3. Compute Homology

We obtain

$$H_0 = M, \quad H_1 = M, \quad H_i = 0 \text{ for } i \geq 2.$$

Therefore,

$$\mathrm{Tor}_i^R(M, N) = \begin{cases} M & i = 0, 1, \\ 0 & i \geq 2. \end{cases}$$

Example 2.5.13. Let

$$R = k[x, y], \quad M = R/(x, y), \quad N = R/(x, y).$$

1. Koszul Resolution:

$$0 \longrightarrow R \xrightarrow{(-y, x)^T} R^{\oplus 2} \xrightarrow{(x, y)} R \longrightarrow R/(x, y) \longrightarrow 0.$$

2. Tensoring with $M = R/(x, y)$:

$$0 \longrightarrow M \xrightarrow{(-y, x)^T} M^{\oplus 2} \xrightarrow{(0, y)} M \longrightarrow 0.$$

Notice the maps are all zero.

3. Compute homology:

$$H_0 = H_1 = M, \quad H_2 = M^{\oplus 2}, \quad H_i = 0 \text{ for } i \geq 3.$$

Example 2.5.14. Let

$$R = \frac{k[x, y]}{(xy)}, \quad M = N = \frac{R}{(x, y)}.$$

1. Projective Resolution of N .

We use the periodic resolution

$$\cdots \longrightarrow R^{\oplus 2} \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R \longrightarrow N \longrightarrow 0,$$

2. Tensor with M .

$$M \otimes_R - : \quad \cdots \rightarrow M^{\oplus 2} \xrightarrow{0} M^{\oplus 2} \xrightarrow{0} M^{\oplus 2} \xrightarrow{0} M \rightarrow 0,$$

since x and y both act as 0 on $M = R/(x, y)$.

3. Compute Homology. We then obtain

$$H_0 = M, \quad H_i = M^{\oplus 2} \text{ for all } i > 0.$$

Hence,

$$\mathrm{Tor}_i^R(M, N) = \begin{cases} M, & i = 0, \\ M^{\oplus 2}, & i > 0. \end{cases}$$

The module $N = R/(x, y)$ does not admit a finite projective resolution. This shows that $R = k[x, y]/(xy)$ is *not regular* — its residue field has infinite projective dimension.

Let R be a ring, and let M be a left R -module. Recall the functor

$$\mathrm{Hom}_R(-, M) : R\text{-Mod} \longrightarrow \mathbf{Ab}.$$

is **contravariant** and **left exact**. We can also use projective resolution to derive it.

Definition 2.5.15 (Ext). *For $N \in R\text{-Mod}$, the **Ext functors** are defined as*

$$\mathrm{Ext}_R^i(N, M) := (L_{-i} \mathrm{Hom}_R(-, M))(N) = H^i(\mathrm{Hom}_R(P_\bullet, M)),$$

where $P_\bullet \rightarrow N \rightarrow 0$ is a projective resolution of N .

Example 2.5.16.

$$\mathrm{Ext}_i^R(N, M) = \begin{cases} M, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Example 2.5.17. Assume R is a commutative ring, M a R -module and $f \in R$ is not a zero divisor. We wish to compute

$$\mathrm{Ext}_R^i(R/(f), M).$$

1. Take a projective resolution of $R/(f)$:

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/(f) \longrightarrow 0.$$

2. Apply the functor $\mathrm{Hom}_R(-, M)$ to obtain

$$0 \longrightarrow \mathrm{Hom}_R(R, M) \xrightarrow{f} \mathrm{Hom}_R(R, M) \longrightarrow 0.$$

Since $\mathrm{Hom}_R(R, M) \cong M$, this becomes

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow 0.$$

3. Take cohomology:

$$H^0 = \ker(f : M \rightarrow M) = \{m \in M \mid fm = 0\},$$

$$H^1 = \mathrm{coker}(f : M \rightarrow M) = M/fM.$$

Hence

$$\mathrm{Ext}_R^i(R/(f), M) = \begin{cases} \{m \in M \mid fm = 0\}, & i = 0, \\ M/fM, & i = 1, \\ 0, & i > 1. \end{cases}$$

In particular:

$$\mathrm{Ext}_R^0(R/(f), M) = f\text{-torsion of } M, \quad \mathrm{Ext}_R^1(R/(f), M) = M/fM.$$

Example 2.5.18. Let $R = \mathbb{Z}$.

$$\mathrm{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/2, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/2, & i = 1. \end{cases}$$

2.6 Opposite Categories

Question: What to do with a left exact covariant functor?

Suppose we have a (covariant) functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}.$$

Then we can form the opposite functor

$$F^{\text{op}} : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{D}^{\text{op}},$$

which is still *covariant*. Recall that for morphisms in the opposite category,

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A).$$

Composition in \mathcal{C}^{op} is reversed:

$$f \circ_{\mathcal{C}^{\text{op}}} g = g \circ_{\mathcal{C}} f.$$

That is, in \mathcal{C} we have $f : A \rightarrow B$, while in \mathcal{C}^{op} the direction is reversed, but F^{op} still acts covariantly:

$$F^{\text{op}}(f^{\text{op}}) = (Ff)^{\text{op}}.$$

Hence the operation $(-)^{\text{op}}$ defines a functor

$$(-)^{\text{op}} : \mathbf{AbCat} \longrightarrow \mathbf{AbCat},$$

sending an abelian category to its opposite abelian category.

Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *left exact* functor. We claim that $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is *right exact*.

Given a short exact sequence in \mathcal{C}^{op} :

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

this corresponds to a short exact sequence in \mathcal{C} :

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Applying the left exact functor F gives:

$$0 \longrightarrow FC \longrightarrow FB \longrightarrow FA$$

exact in \mathcal{D} .

Reversing arrows back to the opposite category, we get:

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

exact in \mathcal{D}^{op} . Hence, F^{op} is right exact.

Definition 2.6.1 (Right Derived Functors). *Let \mathcal{C}, \mathcal{D} be abelian categories. Let*

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

be a left-exact functor. We define the right derived functors of F by

$$R^i F := L_i(F^{\text{op}}),$$

provided that \mathcal{C}^{op} has enough projectives, (which is equivalent to saying that \mathcal{C} has enough injectives).

2.7 Adjunction

Let \mathcal{C} and \mathcal{D} be categories, and let

$$F : \mathcal{C} \longrightarrow \mathcal{D}, \quad G : \mathcal{D} \longrightarrow \mathcal{C}$$

be functors.

Definition 2.7.1 (Adjunction). *We say that F is **left adjoint** to G (and that G is **right adjoint** to F), written*

$$F \dashv G,$$

if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Naturality means the following diagrams commute.

Natural in Y : For any morphism $f : Y_1 \rightarrow Y_2$ in \mathcal{D} , the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, Y_1) & \xrightarrow{\Phi_{X, Y_1}} & \mathrm{Hom}_{\mathcal{C}}(X, GY_1) \\ f \circ - \downarrow & & \downarrow Gf \circ - \\ \mathrm{Hom}_{\mathcal{D}}(FX, Y_2) & \xrightarrow{\Phi_{X, Y_2}} & \mathrm{Hom}_{\mathcal{C}}(X, GY_2) \end{array}$$

commutes for all $X \in \mathcal{C}$.

Natural in X : For any morphism $g : X_1 \rightarrow X_2$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX_2, Y) & \xrightarrow{\Phi_{X_2, Y}} & \mathrm{Hom}_{\mathcal{C}}(X_2, GY) \\ - \circ Fg \downarrow & & \downarrow - \circ g \\ \mathrm{Hom}_{\mathcal{D}}(FX_1, Y) & \xrightarrow{\Phi_{X_1, Y}} & \mathrm{Hom}_{\mathcal{C}}(X_1, GY) \end{array}$$

commutes for all $Y \in \mathcal{D}$.

Example 2.7.2 (Free-Forget Adjunction). Consider the categories

$$\mathcal{C} = \mathbb{Z}\text{-}\mathbf{Mod} \quad \text{and} \quad \mathcal{D} = \mathbf{Sets}.$$

There is a **forgetful functor**

$$\mathrm{Forget} : \mathbb{Z}\text{-}\mathbf{Mod} \longrightarrow \mathbf{Sets},$$

which sends an abelian group (i.e. a \mathbb{Z} -module) to its underlying set.

There is also a **free functor**

$$\mathrm{Free} : \mathbf{Sets} \longrightarrow \mathbb{Z}\text{-}\mathbf{Mod},$$

which sends a set S to the free abelian group

$$\mathrm{Free}(S) = \bigoplus_{s \in S} \mathbb{Z} e_s.$$

These two functors form an **adjoint pair**:

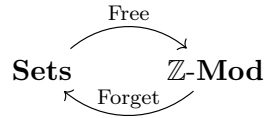
$$\mathbf{Free} \dashv \mathbf{Forget}.$$

That is, for all $S \in \mathbf{Sets}$ and $G \in \mathbb{Z}\text{-}\mathbf{Mod}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbf{Free}(S), G) \cong \mathrm{Hom}_{\mathbf{Sets}}(S, \mathbf{Forget}(G)),$$

natural in both S and G .

Concretely, a group homomorphism $\mathbf{Free}(S) \rightarrow G$ is completely determined by its values on the basis elements e_s , which correspond exactly to a function $S \rightarrow \mathbf{Forget}(G)$.



Example 2.7.3 (Tensor Algebra). Let

$$\mathcal{C} = k\text{-Alg}, \quad \mathcal{D} = k\text{-Vect}.$$

We have the **forgetful functor**

$$\mathbf{Forget} : k\text{-Alg} \longrightarrow k\text{-Vect},$$

which sends a (not necessarily commutative) k -algebra A to its underlying vector space. This functor has a **left adjoint**, the *free algebra functor*

$$T : k\text{-Vect} \longrightarrow k\text{-Alg}.$$

For any k -vector space V , the free k -algebra on V is given by the **tensor algebra**

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

with multiplication given by concatenation of tensors:

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

If $V = \langle x_1, \dots, x_n \rangle$, then

$$T(V) \cong k\langle x_1, \dots, x_n \rangle,$$

the noncommutative polynomial algebra in n variables.

Example 2.7.4 (Symmetric Algebra). Let

$$\mathcal{C} = k\text{-ComAlg}, \quad \mathcal{D} = k\text{-Vect}.$$

To make this Free-Forget adjunction work, the free object $\mathbf{Free}(V)$ must be a commutative k -algebra. We take

$$\mathbf{Free}(V) = S(V),$$

the **symmetric algebra** on V , defined as

$$S(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V),$$

which is the quotient of the tensor algebra $T(V)$ by the ideal generated by $v \otimes w - w \otimes v$.

If $V = \langle x_1, \dots, x_n \rangle$, then

$$S(V) = k[x_1, \dots, x_n].$$

Thus the polynomial algebra in n variables is the free commutative algebra on an n -dimensional vector space.

Example 2.7.5 (Extension-Restriction of scalars Adjunction). Let $R \rightarrow S$ be a ring homomorphism. On the module level, we have two categories:

$$S\text{-Mod} \quad \text{and} \quad R\text{-Mod}.$$

- The **restriction of scalars** (or *forgetful functor*)

$$\text{Forget} : S\text{-Mod} \longrightarrow R\text{-Mod}$$

regards an S -module as an R -module via the map $R \rightarrow S$.

- The **extension of scalars** functor

$$S \otimes_R - : R\text{-Mod} \longrightarrow S\text{-Mod}$$

sends $M \mapsto S \otimes_R M$.

These form an *adjoint pair*:

$$S \otimes_R - \dashv \text{Forget}.$$

That is, there is a natural isomorphism

$$\text{Hom}_{S\text{-Mod}}(S \otimes_R M, N) \cong \text{Hom}_{R\text{-Mod}}(M, \text{Forget}(N)).$$

On the scheme level, let $f : X = \text{Spec}(S) \rightarrow Y = \text{Spec}(R)$ be the induced morphism of affine schemes. Then on quasi-coherent sheaves, we have:

$$f^* : \text{QCoh}(Y) \longrightarrow \text{QCoh}(X), \quad f_* : \text{QCoh}(X) \longrightarrow \text{QCoh}(Y),$$

and they satisfy

$$f^* \dashv f_*.$$

Example 2.7.6. Let R, S be rings, M an (R, S) -bimodule, and N a right S -module. Then $\text{Hom}_S(M, N)$ is a right R -module, and the functor

$$\text{Hom}_S(M, -) : \text{mod-}S \longrightarrow \text{mod-}R$$

is **right adjoint** to

$$- \otimes_R M : \text{mod-}R \longrightarrow \text{mod-}S.$$

That is,

$$\text{Hom}_S(P, \text{Hom}_S(M, N)) \cong \text{Hom}_R(P \otimes_R M, N),$$

naturally in P and N .

The universal property of tensor product is as follow. Given right R -module P and left R -module M , we have:

$$\text{Maps}_{\mathbf{Ab}}(P \otimes_R M, N) = \{R\text{-balanced maps } f : P \times M \rightarrow N\},$$

that is, maps f satisfying:

1. f is a group homomorphism in each variable,
2. $f(pr, m) = f(p, rm)$ for all $p \in P, m \in M, r \in R$.

Each R -balanced map f gives rise to an element of

$$\text{Hom}_R(P, \text{Hom}_S(M, N)).$$

i.e.

$$f \mapsto (p \mapsto (m \mapsto f(p, m))).$$

Lemma 2.7.7. *Let \mathcal{C}, \mathcal{D} be abelian categories,*

$$F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$$

*such that $G \dashv F$. Then G is **right exact** and F is **left exact**.*

2.8 Injective Resolution

Definition 2.8.1. *Let \mathcal{A} be an abelian category. An object $I \in \mathcal{A}$ is called injective if for every diagram with exact row*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

there exists a morphism $B \rightarrow I$ making the diagram commute.

Equivalently, I is injective if and only if the functor

$$\text{Hom}_{\mathcal{A}}(-, I)$$

is *exact*.

Example 2.8.2. \mathbb{Z} is *not* an injective \mathbb{Z} -module.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}.$$

The following diagram cannot commute.

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ & & \downarrow 1 & \swarrow & \\ & & \mathbb{Z} & & \end{array}$$

Hence \mathbb{Z} is not injective.

However, if we replace \mathbb{Z} by \mathbb{Q} , we have:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ & & \downarrow 1 & \swarrow & \\ & & \mathbb{Q} & & \end{array}$$

where the extension exists, given by $1 \mapsto \frac{1}{2}$. Thus \mathbb{Q} is an injective \mathbb{Z} -module.

Theorem 2.8.3 (Baer's Criterion). *Let R be a ring and I an R -module. Then I is injective if and only if for every (left) ideal $J \subseteq R$, any map $J \rightarrow I$ extends to a map $R \rightarrow I$.*

That is, it suffices to check injectivity on diagrams of the form:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \longrightarrow & R \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

Corollary 2.8.4. *An abelian group G (i.e. a \mathbb{Z} -module) is injective if and only if it is divisible, that is,*

$$\forall x \in G, \forall n \in \mathbb{Z}_{>0}, \exists y \in G \text{ such that } ny = x.$$

i.e. check injectivity on diagrams of the form:

$$\begin{array}{ccccc} 0 & \longrightarrow & n\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & & \downarrow & \swarrow & \\ & & G & & \end{array}$$

Example 2.8.5. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & n\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & & \downarrow 1 & \swarrow 1/n & \\ & & \mathbb{Q} & & \end{array}$$

which shows that the map extends in \mathbb{Q} . Hence \mathbb{Q} is an injective \mathbb{Z} -module. Similarly, \mathbb{Q}/\mathbb{Z} is injective.

Corollary 2.8.6. *The category $\mathbb{Z}\text{-Mod}$ has enough injectives.*

Lemma 2.8.7. *Let \mathcal{C}, \mathcal{D} be abelian categories, and let*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

be an exact functor.

Assume that $F \dashv G$ for some functor

$$G : \mathcal{D} \longrightarrow \mathcal{C}.$$

Then for every injective object I in \mathcal{D} , the object $G(I)$ is injective in \mathcal{C} .

Theorem 2.8.8. *$\text{Mod-}R$ has enough injectives.*

Corollary 2.8.9. *For any left exact functor*

$$F : \text{Mod-}R \rightarrow \mathcal{C},$$

we can define the right derived functors

$$R^i F : \text{Mod-}R \rightarrow \mathcal{C}$$

by

$$(R^i F)(M) = H^i(FI^\bullet),$$

where I^\bullet is an injective resolution of M .

Example 2.8.10. We compute $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ from the definition, as a functor in the second variable.

$$\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i = 1. \end{cases}$$

1. Find an injective resolution of \mathbb{Z} .

Since divisible abelian groups are injective, we use the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where \mathbb{Q}/\mathbb{Z} is injective. Hence this is an injective resolution of \mathbb{Z} :

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

2. Apply the functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$

Applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to the above injective resolution, we obtain

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Since \mathbb{Q} is torsion-free,

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0.$$

On the other hand,

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \left\{ \frac{k}{n} \in \mathbb{Q}/\mathbb{Z} \mid k \in \mathbb{Z} \right\} \simeq \mathbb{Z}/n\mathbb{Z}.$$

3. Compute cohomology

The complex we obtained is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

so we get

$$\text{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0, \quad \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

$$\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \begin{cases} 0, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i = 1, \\ 0, & i > 1. \end{cases}$$

Example 2.8.11. Here is another way to compute

$$\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}).$$

Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ yields a long exact sequence:

$$0 \rightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \rightarrow \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \rightarrow 0.$$

Since $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ and $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, we again obtain:

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}.$$

2.9 Distinguished Triangles

Definition 2.9.1 (Mapping cone). *Let A^\bullet and B^\bullet be chain complexes, and let*

$$f : A^\bullet \longrightarrow B^\bullet$$

be a chain map. The mapping cone of f , denoted by $\text{Cone}(f)$, is the chain complex defined by

$$\text{Cone}(f) := A[1] \oplus B,$$

i.e.

$$\text{Cone}(f)^n = A^{n+1} \oplus B^n,$$

with differential

$$d_{\text{Cone}(f)}(a, b) = (-d_A(a), f(a) + d_B(b)).$$

In matrix form,

$$d_{\text{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

Explicitly, the complex is:

$$\cdots \longrightarrow A^i \oplus B^{i-1} \xrightarrow{d_{\text{Cone}(f)}} A^{i+1} \oplus B^i \xrightarrow{d_{\text{Cone}(f)}} A^{i+2} \oplus B^{i+1} \longrightarrow \cdots$$

Recall that for the mapping cone

$$d_{\text{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}.$$

The composition

$$d_{\text{Cone}(f)}^2 = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} = \begin{pmatrix} d_A^2 & 0 \\ -fd_A + d_Bf & d_B^2 \end{pmatrix},$$

is zero since f is a chain map.

Example 2.9.2 (Motivation from Topology). Let X, Y be topological spaces, and let

$$f : X \rightarrow Y$$

be a continuous map. The **mapping cone** of f is the quotient space

$$\text{Cone}(f) := (X \times [0, 1] \sqcup Y) / \sim$$

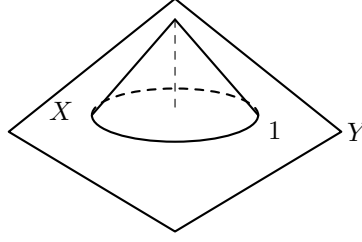
where the equivalence relation \sim is given by

$$(x, 0) \sim (x', 0) \quad \text{for all } x, x' \in X, \quad (x, 1) \sim f(x).$$

In words:

- all points $(x, 0)$ are identified together (forming the cone vertex),
- and each point $(x, 1)$ is glued to $f(x) \in Y$.

If f is an inclusion $f : X \hookrightarrow Y$, then geometrically: $\text{Cone}(f)$ is obtained by attaching a cone over X to Y along X .



There are canonical maps

$$X \xrightarrow{f} Y \longrightarrow \text{Cone}(f) \longrightarrow \Sigma X,$$

where $\Sigma X = X \times [0, 1]/(x, 0) \sim (x', 0), (x, 1) \sim (x', 1)$ is the suspension of X . The map $Y \rightarrow \text{Cone}(f)$ is the inclusion and the map $\text{Cone}(f) \rightarrow \Sigma X$ collapses Y to a point.

Definition 2.9.3. *There are canonical chain maps*

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \text{Cone}(f) \longrightarrow A^\bullet[1].$$

We call this a *distinguished triangle*, drawn as

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ & \nwarrow \quad \swarrow & \\ & \text{Cone}(f) & \end{array}$$

Notice the map $\text{Cone}(f) \rightarrow A^\bullet$ is shifting degrees by 1, which is not an honest chain map.

Algebraic Topology	Homological Algebra	Homotopical Algebra
Topological space	Chain complex	Object (0-morphism)
Continuous map	Chain map	Morphism (1-morphism)
Homotopy	Chain homotopy	Homotopy (2-morphism)
Homotopy equivalence	Chain homotopy equivalence	Homotopy equivalence
Homotopy group $\pi_n(X)$	Homology group $H_n(X)$	Homotopy group
Weak homotopy equivalence	Quasi-isomorphism	Weak equivalence
CW approximation	Projective/Injective resolution	Fibrant/Cofibrant replacement
Homotopy category hCW	Derived category $\mathcal{D}(R)$	Homotopy category
Suspension / Loop $\Sigma \dashv \Omega$	Shift $[1] \dashv [-1]$	Suspension object / Loop object
Mapping cone, mapping cocone	Chain mapping cone, cocone	Homotopy fiber, homotopy cofiber

Table 1: Analogy between Algebraic Topology, Homological Algebra, and Homotopical Algebra

Proposition 2.9.4. *For any chain map $f : A^\bullet \rightarrow B^\bullet$, we obtain a long exact sequence in cohomology:*

$$\cdots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(\text{Cone}(f)) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow H^{i+1}(B^\bullet) \longrightarrow \cdots$$

Proof.

$$0 \rightarrow B^\bullet \rightarrow \text{Cone}(f) \rightarrow A^\bullet[1] \rightarrow 0$$

is a short exact sequence. □

Corollary 2.9.5. *A chain map $f : A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism if and only if*

$$\text{Cone}(f) \simeq 0.$$

Fact: The homotopy category $\mathcal{K}(\mathcal{A})$ is *not* an abelian category.

However, *kernels* and *cokernels* are replaced by *cones*, which make sense in $\mathcal{K}(\mathcal{A})$. Intuitively, we roughly have

$$\text{Cone}(f) \sim (\ker f)[1] \oplus \text{coker}(f).$$

Let $f : A \rightarrow B$ with $A, B \in \mathcal{A}$.

We may view f as a morphism of complexes:

$$f : A[0] \rightarrow B[0].$$

Then the cone is:

$$\text{Cone}(f) : 0 \rightarrow A \xrightarrow{f} B \rightarrow 0 \rightarrow \dots$$

Cohomology of the cone is

$$H^{-1}(\text{Cone}(f)) = \ker f, \quad H^0(\text{Cone}(f)) = \text{coker } f.$$

Hence we get an exact sequence:

$$0 \rightarrow \ker f \rightarrow A \xrightarrow{f} B \rightarrow \text{coker } f \rightarrow 0.$$

Actually the cone $\text{Cone}(f)$ also encodes the $\text{Ext}^2(\text{coker } f, \ker f)$.

Example 2.9.6. If

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is short exact, then

$$\text{Cone}(f) \simeq C[0].$$

Thus, we obtain a distinguished triangle in $\mathcal{K}(\mathcal{A})$:

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ & \swarrow & \searrow \\ & \text{Cone}(f) \xrightarrow{\simeq} & C^\bullet \end{array}$$

What we want is a distinguished triangle

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ & \swarrow & \searrow \\ & C^\bullet & \end{array}$$

which can be found in derived category $\mathcal{D}(\mathcal{A})$.

2.10 Balanced Bifunctors

In this section we want to show the Tor and Ext functors are balanced.

Theorem 2.10.1 ($\text{Tor}^I \cong \text{Tor}^{II}$). *Let M, N be R -modules. Take projective resolutions*

$$P_\bullet \longrightarrow M, \quad Q_\bullet \longrightarrow N.$$

Then for every i there is a natural isomorphism

$$H_i(P_\bullet \otimes_R N) \cong H_i(M \otimes_R Q_\bullet).$$

Hence the two constructions of Tor agree:

$$\text{Tor}_i^R(M, N) := H_i(P_\bullet \otimes_R N) \cong H_i(M \otimes_R Q_\bullet) =: \text{Tor}_i^R(M, N).$$

Definition 2.10.2 (Bicomplex). *A bicomplex $(C_{\bullet, \bullet}, d^h, d^v)$ consists of modules $C_{i,j}$ with horizontal and vertical differentials*

$$d^h : C_{i,j} \rightarrow C_{i-1,j}, \quad d^v : C_{i,j} \rightarrow C_{i,j-1}$$

such that

$$(d^h)^2 = 0, \quad (d^v)^2 = 0, \quad d^h d^v = d^v d^h.$$

There are two standard ways to turn a bicomplex into a single complex.

Definition 2.10.3 (Direct-sum and product totalizations). *Given a bicomplex $C_{\bullet, \bullet}$, define complexes $\text{Tot}^\oplus(C)$ and $\text{Tot}^\Pi(C)$ by degree n :*

$$\text{Tot}^\oplus(C)_n = \bigoplus_{i+j=n} C_{i,j}, \quad \text{Tot}^\Pi(C)_n = \prod_{i+j=n} C_{i,j},$$

with differential $d = d^h + d^v$ (using the chosen sign convention). The direct-sum totalization takes finite sums along each diagonal; the product totalization allows infinite products.

Sketch proof of the theorem. Consider the bicomplex $C_{\bullet, \bullet} = P_\bullet \otimes_R Q_\bullet$ with

$$C_{i,j} = P_i \otimes_R Q_j, \quad d^h = d^P \otimes \text{id}, \quad d^v = (-1)^i \text{id} \otimes d^Q \text{ on } C_{i,j}.$$

(Any consistent sign convention is fine; we use the commuting convention $d^h d^v = d^v d^h$. Another possible sign convention is to require squares to be anti-commutative.)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_{i+1} \otimes Q_j & \xrightarrow{d^h} & P_i \otimes Q_j & \xrightarrow{d^h} & P_{i-1} \otimes Q_j \longrightarrow \cdots \\ & & \downarrow d^v & & \downarrow d^v & & \downarrow d^v \\ \cdots & \longrightarrow & P_{i+1} \otimes Q_{j-1} & \xrightarrow{d^h} & P_i \otimes Q_{j-1} & \xrightarrow{d^h} & P_{i-1} \otimes Q_{j-1} \longrightarrow \cdots \end{array}$$

Notice the bicomplex lives in the second quadrant. Consider the augmented bicomplex on the right:

$$\begin{array}{ccccccc} & & \cdots & & \cdots & & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & P_2 \otimes Q_1 & \xrightarrow{d^h} & P_1 \otimes Q_1 & \xrightarrow{d^h} & P_0 \otimes Q_1 \longrightarrow M \otimes Q_1 \longrightarrow 0 \\ & & \downarrow d^v & & \downarrow d^v & & \downarrow d^v \\ \cdots & \longrightarrow & P_2 \otimes Q_0 & \xrightarrow{d^h} & P_1 \otimes Q_0 & \xrightarrow{d^h} & P_0 \otimes Q_0 \longrightarrow M \otimes Q_0 \longrightarrow 0 \end{array}$$

denote the new bicomplex by $R_\bullet \otimes_R Q_\bullet$.

Check the following statements:

1. There is a natural map

$$\epsilon_n : \text{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet) \rightarrow M \otimes Q_\bullet.$$

2.

$$\text{Tot}^\oplus(R_\bullet \otimes_R Q_\bullet) = \text{Cone}(\epsilon_M)[-1]$$

3. The rows of $R_\bullet \otimes_R Q_\bullet$ are acyclic because R_\bullet is acyclic and Q_i are projective (hence flat).

4. $\text{Tot}^\oplus(R_\bullet \otimes_R Q_\bullet)$ is acyclic, by the acyclic assembly lemma or convergence theorem for spectral sequences.

Then ϵ_M is a quasi-isomorphism, i.e.

$$(P_\bullet \otimes_R Q_\bullet) \simeq M \otimes Q_\bullet.$$

Analogously,

$$(P_\bullet \otimes_R Q_\bullet) \simeq P_\bullet \otimes N.$$

□

Remark 2.10.4. One-sentence proof: The spectral sequences of the bicomplex $P_\bullet \otimes_R Q_\bullet$ under vertical and horizontal orientations converge to $H_\bullet(M \otimes_R Q_\bullet)$ and $H_\bullet(P_\bullet \otimes_R N)$ respectively.

Theorem 2.10.5 ($\text{Ext}^I \cong \text{Ext}^{II}$). *Ext is balanced. i.e. $M, N \in R\text{-Mod}$. Let $P_\bullet \twoheadrightarrow M$ be a projective resolution and $N \hookrightarrow I^\bullet$ an injective resolution.*

$$\text{Ext}_R^i(M, N) \cong H^i(\text{Hom}_R(P_\bullet, N)) \cong H^i(\text{Hom}_R(M, I^\bullet)).$$

2.11 Acyclic Resolutions

Let \mathcal{A}, \mathcal{B} be abelian categories and $F : \mathcal{A} \rightarrow \mathcal{B}$ a functor. Typically:

- If F is left exact, one defines its right derived functors $R^i F$ using injective resolutions.
- If F is right exact, one defines its left derived functors $L_i F$ using projective resolutions.

In many settings (e.g. sheaves), \mathcal{A} does *not* have enough projectives, so $L_* F$ cannot be defined by projective resolutions. The remedy is to resolve by objects that are *acyclic for F* .

First we need a new definition independent of resolution.

Definition 2.11.1 (Homological δ -functor). *A homological δ -functor from \mathcal{A} to \mathcal{B} is a family $(T_i)_{i \geq 0}$ of functors $T_i : \mathcal{A} \rightarrow \mathcal{B}$ together with, for every short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ in \mathcal{A} , connecting morphisms $\partial_i : T_i(X'') \rightarrow T_{i-1}(X')$ producing a natural long exact sequence*

$$\cdots \rightarrow T_i(X') \rightarrow T_i(X) \rightarrow T_i(X'') \xrightarrow{\partial_i} T_{i-1}(X') \rightarrow \cdots \rightarrow T_0(X'') \rightarrow 0,$$

natural in the short exact sequence and compatible with morphisms of short exact sequences.

Definition 2.11.2 (Left derived functors via universality). *A homological δ -functor $(L_i F)_{i \geq 0}$ is called the left derived functor of a right exact F if:*

- (i) $L_0 F \simeq F$ (naturally in X);

- (ii) (Universality) For any homological δ -functor (T_i) and any natural transformation $\eta_0 : T_0 \Rightarrow F$, there exists a unique family $\eta_i : T_i \Rightarrow L_i F$ of natural transformations extending η_0 and compatible with all connecting morphisms. Equivalently, $(L_i F)$ is initial in the category of homological δ -functors under F .

Now assume $F : \mathcal{A} \rightarrow \mathcal{B}$ is right exact and we want $L_* F$, but \mathcal{A} lacks enough projectives. We replace projectives by objects that are *left-acyclic* for F .

Definition 2.11.3 (F -adapted / F -left-acyclic objects). An object $A \in \mathcal{A}$ is F -left-acyclic if

$$L_i F(A) = 0 \quad \text{for all } i > 0.$$

Equivalently, for any (bounded above) resolution $\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A$ by such objects, the homology of F applied to the resolution vanishes in positive degrees.

Definition 2.11.4 (Left-acyclic resolution). A (bounded above) morphism $A_\bullet \rightarrow X$ is an F -left-acyclic resolution if it is a quasi-isomorphism and each A_n is F -acyclic.

Theorem 2.11.5 (Defining $L_* F$ via left-acyclic resolutions). If every $X \in \mathcal{A}$ admits a bounded-above F -left-acyclic resolution $A_\bullet \rightarrow X$, then

$$L_i F(X) := H_i(F(A_\bullet))$$

is well defined (independent of choices), gives a universal δ -functor, and agrees with the projective-resolution definition when \mathcal{A} has enough projectives.

Example 2.11.6 (Tensor and flat resolutions). Let $F = - \otimes_R N$ (right exact). Even if $R\text{-Mod}$ has few projectives, flat modules are \otimes -left-acyclic, and every module M admits a flat resolution $F_\bullet \rightarrow M$. Therefore

$$L_i(- \otimes_R N)(M) \cong H_i(F_\bullet \otimes_R N) = \text{Tor}_i^R(M, N).$$

Those constructions above work analogously for left exact functors.

Definition 2.11.7 (Right derived functors via universality). A homological δ -functor $(R^i F)_{i \geq 0}$ is called the right derived functor of a left exact F if:

- (i) $R^0 F \cong F$ (naturally in X);
- (ii) (Universality) For any cohomological δ -functor (T^i) and any natural transformation $\eta^0 : F \Rightarrow T^0$, there exists a unique family $\eta^i : R^i F \Rightarrow T^i$ compatible with the long exact sequences and extending η^0 .

Definition 2.11.8 (F -acyclic objects for left exact F). Assume F is left exact. An object $A \in \mathcal{A}$ is called F -acyclic if

$$R^i F(A) = 0 \quad \text{for all } i > 0.$$

Equivalently, for any injective resolution $A \rightarrow I^\bullet$, the cochain complex $F(I^\bullet)$ has vanishing cohomology in positive degrees.

Definition 2.11.9 (Acyclic resolution). A (bounded below) morphism $X \rightarrow A^\bullet$ is a F -(right) acyclic resolution of X if it is a quasi-isomorphism and each A^n is F -acyclic.

Theorem 2.11.10. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be left exact. If every $X \in \mathcal{A}$ admits a bounded-below F -acyclic resolution $X \rightarrow A^\bullet$, then*

$$R^i F(X) := H^i(F(A^\bullet))$$

is well defined (independent of the chosen resolution), yields a universal δ -functor, and agrees with the usual definition via injective resolutions whenever \mathcal{A} has enough injectives.

Example 2.11.11 (Sheaf cohomology). For the global sections functor $\Gamma(X, -)$ (left exact), soft/flabby/fine sheaves are Γ -acyclic. Hence $R^i \Gamma$ can be computed from soft (or flabby/fine) resolutions, without using injective sheaves.

3 Derived Categories

3.1 Triangulated Categories

Start with an abelian category \mathcal{A} . We want a category whose objects are chain complexes in \mathcal{A} :

$$\text{Ch}(\mathcal{A}).$$

Many constructions on complexes are not strictly unique, but are unique up to homotopy. This leads to the *homotopy category*

$$\mathcal{K}(\mathcal{A}) = \text{Ch}(\mathcal{A}) / (\text{chain homotopy}).$$

Problem: While \mathcal{A} and $\text{Ch}(\mathcal{A})$ are abelian (so we can speak of short exact sequences, kernels, cokernels), the homotopy category $\mathcal{K}(\mathcal{A})$ is *not* abelian.

Theorem 3.1.1. *The homotopy category $\mathcal{K}(\mathcal{A})$ is triangulated.*

Definition 3.1.2 (Triangulated category). *A triangulated category consists of:*

- *an additive category \mathcal{C} ;*
- *an auto-equivalence (the shift/suspension)*

$$T : \mathcal{C} \xrightarrow{\cong} \mathcal{C}, \quad X \mapsto TX;$$

- *a specified class of distinguished triangles*

$$(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA),$$

subject to the Verdier axioms TR1–TR4:

TR1

- (a) *For every object A , the triangle $A \xrightarrow{\text{id}} A \rightarrow 0 \rightarrow TA$ is distinguished.*
- (b) *Every morphism $f : A \rightarrow B$ sits in some distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$, where C is called a **cone** or **cofiber** of the morphism f .*
- (c) *Any triangle isomorphic to a distinguished triangle is distinguished.*

TR2 (Rotation) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ is distinguished, then so are

$$B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB \quad \text{and} \quad C \xrightarrow{h} TA \xrightarrow{-Tf} TB \xrightarrow{-Tg} TC.$$

TR3 (Functoriality up to homotopy) Given a morphism of the first two legs of distinguished triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ a \downarrow & & b \downarrow & & \downarrow c & & \downarrow Ta \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA' \end{array}$$

there exists $c : C \rightarrow C'$ making the diagram a morphism of triangles.

TR4 (Octahedral axiom) For composable maps $A \xrightarrow{f} B \xrightarrow{g} C$, the cones of f , g , and gf fit into an “octahedron” relating the associated distinguished triangles.

Lemma 3.1.3. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ is a distinguished triangle then $g \circ f = 0$.

Lemma 3.1.4 (Two-out-of-three for morphisms of triangles). Suppose we have a morphism of distinguished triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\ a \downarrow & & b \downarrow & & c \downarrow & & \downarrow Ta \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA' \end{array}$$

Then if any two of a, b, c are isomorphisms, so is the third.

Lemma 3.1.5 (Exactness of $\text{Hom}(D, -)$ on a distinguished triangle). Suppose

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

a distinguished triangle. For any object $D \in \text{Ob}(\mathcal{C})$, the sequence

$$\text{Hom}(D, A) \xrightarrow{f_*} \text{Hom}(D, B) \xrightarrow{g_*} \text{Hom}(D, C)$$

is exact at $\text{Hom}(D, B)$; i.e. $\ker(g_*) = \text{im}(f_*)$.

Corollary 3.1.6 (Long exact sequence from a triangle). Let \mathcal{C} be a triangulated category and

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

a distinguished triangle. For every object $D \in \mathcal{C}$ there is a natural long exact sequence

$$\cdots \longrightarrow \text{Hom}(D, A) \xrightarrow{f_*} \text{Hom}(D, B) \xrightarrow{g_*} \text{Hom}(D, C) \xrightarrow{h_*} \text{Hom}(D, TA) \xrightarrow{T(f)_*} \text{Hom}(D, TB) \xrightarrow{T(g)_*} \text{Hom}(D, TC) \xrightarrow{T(h)_*} \cdots$$

Definition 3.1.7 (Homological and cohomological functors). Let \mathcal{C} be a triangulated category and \mathcal{A} an abelian category.

- A functor $H : \mathcal{C} \rightarrow \mathcal{A}$ is called homological if for every distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA \quad \text{in } \mathcal{C},$$

there is a natural long exact sequence in \mathcal{A}

$$\cdots \longrightarrow H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C) \xrightarrow{H(h)} H(TA) \xrightarrow{H(Tf)} H(TB) \xrightarrow{H(Tg)} H(TC) \xrightarrow{H(Th)} \cdots$$

Equivalently, H sends distinguished triangles to exact triangles of long exact sequences.

- A functor $H : \mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$ is called cohomological if for every distinguished triangle as above there is a natural long exact sequence

$$\cdots \longrightarrow H(T^{-1}C) \xrightarrow{H(T^{-1}g)} H(T^{-1}B) \xrightarrow{H(T^{-1}f)} H(T^{-1}A) \longrightarrow H(C) \xrightarrow{H(g)} H(B) \xrightarrow{H(f)} H(A) \longrightarrow \cdots .$$

Example 3.1.8. For any $D \in \mathcal{C}$, the functors $\text{Hom}_{\mathcal{C}}(D, -) : \mathcal{C} \rightarrow \mathbf{Ab}$ and $\text{Hom}_{\mathcal{C}}(-, D) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$ are homological and cohomological, respectively.

Theorem 3.1.9. Let \mathcal{A} be an abelian category. The homotopy category $K(\mathcal{A})$ of chain complexes in \mathcal{A} carries a natural triangulated structure.

Definition 3.1.10 (Shift and distinguished triangles on $K(\mathcal{A})$). On $K(\mathcal{A})$ define the shift (suspension) by

$$X[1]^n := X^{n+1}, \quad d_{X[1]} := -d_X.$$

For a chain map $f : X \rightarrow Y$, let $\text{Cone}(f)$ be the mapping cone

$$\text{Cone}(f)^n := Y^n \oplus X^{n+1}, \quad d_{\text{Cone}(f)} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}.$$

Declare any triangle isomorphic in $K(\mathcal{A})$ to

$$X \xrightarrow{f} Y \longrightarrow \text{Cone}(f) \longrightarrow X[1]$$

to be a distinguished triangle. Equivalently, the class of distinguished triangles is the closure under isomorphism of these mapping-cone triangles.

Problem: Given a short exact sequence in an abelian category \mathcal{A}

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{p} C \longrightarrow 0,$$

we would like the degree-0 complexes $A[0] \rightarrow B[0] \rightarrow C[0] \rightarrow A[1]$ to be a distinguished triangle in the homotopy category $K(\mathcal{A})$. This is *false in general*. What we do have is a canonical map $g : \text{Cone}(f) \rightarrow C[0]$ that is a *quasi-isomorphism* (induces isomorphism on homology), but g need not be an isomorphism in $K(\mathcal{A})$.

$$\begin{array}{ccc} A[0] & \xrightarrow{f} & B[0] \\ & \searrow & \swarrow \\ & \text{Cone}(f) & \xrightarrow{g \simeq} C[0] \end{array}$$

Example 3.1.11. Take $\mathcal{A} = \mathbf{Ab}$ and the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Viewing $f : \mathbb{Z}[0] \rightarrow \mathbb{Z}[0]$ as a chain map, there is a quasi-isomorphism

$$g : \text{Cone}(f) \xrightarrow{\simeq} \mathbb{Z}/2[0].$$

But g is not necessarily an isomorphism in $K(\mathbf{Ab})$ (the cone and $\mathbb{Z}/2[0]$ need not be chain-homotopy equivalent). If we apply a degree-wise functor such as

$$F = (\mathbb{Z}/2) \otimes_{\mathbb{Z}} - : K(\mathbf{Ab}) \rightarrow K(\mathbf{Ab}),$$

then typically $F(\text{Cone}(f)) \not\simeq F(\mathbb{Z}/2[0])$ in $K(\mathbf{Ab})$ because F does not preserve all quasi-isomorphisms ($\mathbb{Z}/2$ is not flat). This shows why we must formally force quasi-isomorphisms to be invertible.

3.2 Localization of a category

Given an abelian category \mathcal{A} , we want a triangulated category $D(\mathcal{A})$ (the *derived category of \mathcal{A}*) and, for any (left/right) exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$, a *derived functor*

$$RF \text{ or } LF : D(\mathcal{A}) \longrightarrow D(\mathcal{B}).$$

The key construction: start with the homotopy category $K(\mathcal{A})$ and *invert all quasi-isomorphisms* to obtain $D(\mathcal{A})$.

Localizing a category works like localizing a ring R at a multiplicative set S : we formally add inverses of the chosen elements. For categories we replace elements by morphisms and use *roofs* to record “fractions.”

Definition 3.2.1 (Multiplicative system). *Let \mathcal{C} be a category and let S be a class of arrows of \mathcal{C} . We say S is a multiplicative system if:*

1. *For every object X of \mathcal{C} , the identity id_X lies in S , and the composite of two composable arrows of S lies in S .*
2. *Every solid diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ t \downarrow & & \downarrow s \\ Z & \xrightarrow{\quad f \quad} & W \end{array}$$

with $s \in S$ can be completed to a commutative square with $t \in S$.

3. *For every pair of morphisms $f, g : X \rightarrow Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.*

Definition 3.2.2 (Localization of a category). *Let \mathcal{C} be a category and let S be a multiplicative system in \mathcal{C} . The localization $S^{-1}\mathcal{C}$ is defined as follows.*

- (1) **Objects.** $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$.
- (2) **Morphisms.** A morphism $X \rightarrow Y$ in $S^{-1}\mathcal{C}$ is an equivalence class of pairs (a “roof”)

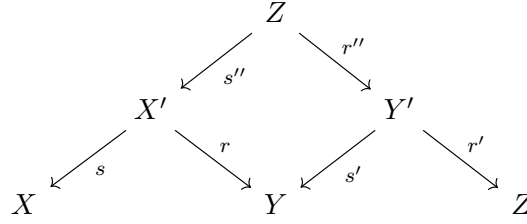
$$(r, s) : \quad X \xleftarrow{s} X' \xrightarrow{r} Y \quad (s \in S),$$

which we also denote by $s^{-1}r : X \rightarrow Y$.

- (3) **Equivalence of roofs.** Two roofs $(r_1, s_1) : X \xleftarrow{s_1} X_1 \xrightarrow{r_1} Y$ and $(r_2, s_2) : X \xleftarrow{s_2} X_2 \xrightarrow{r_2} Y$ are declared equivalent if there exist a third roof $(r_3, s_3) : X \xleftarrow{s_3} X_3 \xrightarrow{r_3} Y$ and morphisms $u : X_3 \rightarrow X_1$, $v : X_3 \rightarrow X_2$ in \mathcal{C} such that the following diagram commutes

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow s_1 & \uparrow u & \searrow r_1 & \\ X & \xleftarrow{s_3} & X_3 & \xrightarrow{r_3} & Y \\ & \swarrow s_2 & \downarrow v & \searrow r_2 & \\ & & X_2 & & \end{array}$$

(4) **Composition.** Given classes of roofs $(r, s) : X \xleftarrow{s} X' \xrightarrow{r} Y$ and $(r', s') : Y \xleftarrow{s'} Y' \xrightarrow{r'} Z$, The composition is defined as the big roof below



(5) **Identities.** The identity id_X in $S^{-1}\mathcal{C}$ is the class of the roof $(\text{id}_X, \text{id}_X) : X \xleftarrow{\text{id}_X} X \xrightarrow{\text{id}_X} X$.

There is a canonical functor

$$Q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}, \quad Q(X) = X, \quad Q(f : X \rightarrow Y) = (f, \text{id}_X),$$

which sends every $s \in S$ to an isomorphism.

Theorem 3.2.3 (Universal property). *Q inverts every $s \in S$ and is universal with this property: for any category \mathcal{D} and functor $F : \mathcal{C} \rightarrow \mathcal{D}$ sending all S -arrows to isomorphisms, there exists a unique $\bar{F} : S^{-1}\mathcal{C} \rightarrow \mathcal{D}$ with $F = \bar{F} \circ Q$.*

$$\begin{array}{ccc}
 & S^{-1}\mathcal{C} & \\
 Q \nearrow & & \downarrow \bar{F} \\
 \mathcal{C} & \xrightarrow{F} & \mathcal{D}
 \end{array}$$

Remark 3.2.4 (Ring localization as a special case). If \mathcal{C} is the one-object category of a ring R and $S \subset R$ multiplicative, then $S^{-1}\mathcal{C}$ is the one-object category of the localized ring $S^{-1}R$. Roofs (r, s) encode fractions r/s .

3.3 Derived Categories

Definition 3.3.1 (Derived categories). Let $K(\mathcal{A})$ be the homotopy category of complexes in an abelian category \mathcal{A} , W the class of quasi-isomorphisms. Then W is a multiplicative system (satisfies a calculus of fractions), and we define

$$D(\mathcal{A}) := W^{-1}K(\mathcal{A}).$$

Morphisms in $D(\mathcal{A})$ are roofs $X \xleftarrow{\cong} X' \rightarrow Y$ with the back arrow a quasi-isomorphism.

Theorem 3.3.2 ($D(\mathcal{A})$ is triangulated). Let \mathcal{A} be an abelian category. The derived category $D(\mathcal{A})$ inherits a natural triangulated structure from $K(\mathcal{A})$ as follows:

- The shift functor is the degree shift $[1]$.
- A triangle in $D(\mathcal{A})$ is distinguished iff it is isomorphic in $D(\mathcal{A})$ to the image under the localization $Q : K(\mathcal{A}) \rightarrow D(\mathcal{A})$ of a mapping-cone triangle in $K(\mathcal{A})$.

Lemma 3.3.3. If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence in \mathcal{A} , then in $D(\mathcal{A})$ there is a distinguished triangle

$$A[0] \longrightarrow B[0] \longrightarrow C[0] \longrightarrow A[1].$$

Proof. In $K(\mathcal{A})$ there is the cone triangle $A[0] \rightarrow B[0] \rightarrow \text{Cone}(f) \rightarrow A[1]$. The canonical map $g : \text{Cone}(f) \rightarrow C[0]$ is a quasi-isomorphism, so it becomes an isomorphism after localization to $D(\mathcal{A})$. \square

Question: What is the map $C[0] \rightarrow A[1]$?

In $K(\mathcal{A})$ the composite $C[0] \rightarrow A[1]$ coming from the short exact sequence would be zero. In $D(\mathcal{A})$ it is the composition

$$C[0] \xleftarrow[\sim]{g} \text{Cone}(f) \longrightarrow A[1],$$

where g is the quasi-isomorphism above and the second arrow is the cone boundary. Thus the connecting morphism is defined via the cone.

3.4 Extension

Fact: For $X, Y \in \mathcal{A}$ and $i \geq 0$,

$$\text{Ext}_{\mathcal{A}}^i(X, Y) \cong \text{Hom}_{D(\mathcal{A})}(X[0], Y[i]).$$

Definition 3.4.1. Two extensions of C by A (i.e. short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$) are equivalent if there exists a map between them (i.e. a commutative diagram)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

An extension of C : $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ gives a long exact sequence by applying $\text{Hom}(C, -)$:

$$0 \rightarrow \text{Hom}(C, A) \rightarrow \text{Hom}(C, B) \rightarrow \text{Hom}(C, C) \xrightarrow{\delta} \text{Ext}^1(C, A) \rightarrow \dots$$

Notice there is a canonical element $\text{id}_C \in \text{Hom}(C, C)$, giving rise to $\xi := \delta(\text{id}_C) \in \text{Ext}^1(C, A)$.

Theorem 3.4.2. The map $\Theta : (\text{extension}) \mapsto \xi(\text{extension})$ is an isomorphism

$$\{\text{extensions}\} / \sim \cong \text{Ext}^1(C, A).$$

Example 3.4.3. Work over a field k (let $\mathcal{A} = \text{Mod-}A$, where A is a k -algebra).

Assume $\text{Ext}^1(C, A) = k$.

For each $\alpha \in k$ we get an extension

$$0 \longrightarrow A \longrightarrow B_\alpha \longrightarrow C \longrightarrow 0.$$

For $\alpha_1 = \lambda \alpha_2$ with $\lambda \neq 0$, we have $B_{\alpha_1} \cong B_{\alpha_2}$.

Hence all B_α are isomorphic for $\alpha \neq 0$, but $B_0 \cong A \oplus C$.

Definition 3.4.4 (k -step extension). A k -step extension ($k \geq 1$) of C by A is an exact sequence

$$0 \longrightarrow A \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots \longrightarrow X_k \longrightarrow C \longrightarrow 0.$$

Two such extensions are equivalent if there is a morphism between them making the obvious diagram commute and which is the identity on the end terms A and C :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X_k & \longrightarrow & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & A & \longrightarrow & Y_1 & \longrightarrow & \dots & \longrightarrow & Y_k & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Theorem 3.4.5. *There is a bijection*

$$\{k\text{-step extensions of } C \text{ by } A\} / \sim \xrightarrow{\cong} \text{Ext}^k(C, A).$$

Example 3.4.6 (Construction of 2-step extension). Start with an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & X_1 & \xrightarrow{g} & X_2 \xrightarrow{h} C \longrightarrow 0 \\ & & & & \searrow & & \nearrow \\ & & & & \text{coker}(f) & & \\ & & & & \nearrow & & \searrow \\ & & 0 & & & & 0 \end{array}$$

and notice it splits into two short exact sequences. By applying $\text{Hom}(C, -)$ to the first short exact sequence, we obtain a class

$$\text{Ext}^1(C, \text{coker}(f)) \longrightarrow \text{Ext}^2(C, A)$$

sending the 1-step extension class to the 2-step extension class in $\text{Ext}^2(C, A)$.

Remark 3.4.7. Extensions can be composed. Concatenating a k -step extension of B by A and a l -step extension of C by B gives a $k + l$ -step extension of C by A . This is called the Yoneda product:

$$\text{Ext}^k(A, B) \times \text{Ext}^l(B, C) \rightarrow \text{Ext}^{k+l}(A, C)$$

Recall in $\mathcal{D}(\mathcal{A})$,

$$\text{Ext}_{\mathcal{A}}^l(A, B) \cong \text{Hom}_{\mathcal{D}(\mathcal{A})}(A[0], B[l]).$$

Then $A \rightarrow B[k]$ and $B \rightarrow C[l]$ gives $A \rightarrow B[k] \rightarrow C[k + l]$.

3.5 Derived Functors in Derived Categories

Reminder

Start with an abelian category \mathcal{A} . Recall

$$K(\mathcal{A}) = \text{Ch}(\mathcal{A}) / (\text{homotopies}).$$

There are full subcategories

$$K^-(\mathcal{A}), \quad K^+(\mathcal{A}), \quad K^b(\mathcal{A}),$$

which contain bounded-above, bounded-below and bounded complexes.

Problem: In $\mathcal{K}(\mathcal{A})$, there are complexes quasi-isomorphic to 0 but not isomorphic to 0.

Example 3.5.1. The complex

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

of free $\mathbb{Z}/4\mathbb{Z}$ -modules is qiso to 0 quasi-isomorphic to 0 but not isomorphic to 0 in $\mathcal{K}(\mathbb{Z}/4\mathbb{Z}\text{-Mod})$.

Theorem 3.5.2. *Let P^\bullet be a bounded-above complex of projectives in \mathcal{A} . If P^\bullet is quasi-isomorphic to 0, then P^\bullet is null-homotopic; in particular $P^\bullet \simeq 0$ in $K^-(\mathcal{A})$.*

Corollary 3.5.3. *If P^\bullet, Q^\bullet are bounded-above complexes of projectives and $f : P^\bullet \rightarrow Q^\bullet$ is a quasi-isomorphism, then f is an isomorphism in $K^-(\mathcal{A})$.*

Proof. $\text{Cone}(f)$ is a bounded-above complex of projectives and is acyclic. By the theorem $\text{Cone}(f) \simeq 0$ in $K^-(\mathcal{A})$, which implies f is an isomorphism in $K^-(\mathcal{A})$ by 5-lemma. \square

Write $\mathcal{P} \subset \mathcal{A}$ for projective objects and set

$$K^-(\mathcal{P}) := \{\text{bounded-above complexes with all terms projective}\} \subset K^-(\mathcal{A}).$$

Theorem 3.5.4. *The natural map*

$$(W \cap K^-(\mathcal{P}))^{-1} K^-(\mathcal{P}) \cong K^-(\mathcal{P})$$

is an equivalence.

Theorem 3.5.5 (Fully faithfulness). *The localization functor restricts to a fully faithful functor*

$$Q : K^-(\mathcal{P}) \longrightarrow D^-(\mathcal{A}).$$

Theorem 3.5.6 (Equivalence under enough projectives). *If \mathcal{A} has enough projectives, then*

$$Q : K^-(\mathcal{P}) \xrightarrow{\sim} D^-(\mathcal{A})$$

is an equivalence of categories.

Proof. Every bounded-above complex A^\bullet admits a quasi-isomorphism $P^\bullet \rightarrow A^\bullet$ with $P^\bullet \in K^-(\mathcal{P})$ (e.g. via Cartan–Eilenberg resolutions). \square

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be right exact between abelian categories.

Definition 3.5.7 (left derived functor via $K^-(\mathcal{P})$). *Assume \mathcal{A} has enough projectives. The derived functor LF is defined as*

$$\begin{array}{ccc} \mathcal{K}^-(\mathcal{P}_{\mathcal{A}}) & \xrightarrow{F} & \mathcal{K}^-(\mathcal{B}) \\ \exists \uparrow \wr \downarrow \cong & & \downarrow \\ \mathcal{D}^-(\mathcal{A}) & \xrightarrow{LF} & \mathcal{D}^-(\mathcal{B}) \end{array}$$