Math 751 HW 5

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Due: Sunday night

13. Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

Proof. Recall that the reduced singular chain complex $\widetilde{C}_*(X)$ can be defined as the augmented chain complex

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon_X} \mathbb{Z} \to 0,$$

where $\varepsilon_X : C_0(X) \to \mathbb{Z}$ is the augmentation map, and we put $\widetilde{C}_n(X) = C_n(X)$ for $n \geq 0$, $\widetilde{C}_{-1}(X) = \mathbb{Z}$, with differentials

$$\widetilde{\partial}_n = \partial_n \ (n \geq 1), \qquad \widetilde{\partial}_0 = \varepsilon_X, \qquad \widetilde{\partial}_{-1} = 0.$$

Then by definition $\widetilde{H}_n(X) = H_n(\widetilde{C}_*(X))$ for $n \geq 0$.

Let $f, g : X \to Y$ be homotopic maps, and let $f_*, g_* : C_*(X) \to C_*(Y)$ be the induced chain maps on the usual singular chain complexes. It is well known that homotopic maps induce chain-homotopic chain maps, so there exist homomorphisms

$$P_n : C_n(X) \longrightarrow C_{n+1}(Y) \qquad (n \geq 0)$$

such that for all $n \geq 0$,

$$f_n - g_n = \partial^Y_{n+1} P_n + P_{n-1} \partial^X_n. \tag{$*_n$}$$

Step 1. Extend f_*, g_* to the augmented complexes. Define chain maps

$$\widetilde{f}_*, \widetilde{g}_* : \widetilde{C}_*(X) \longrightarrow \widetilde{C}_*(Y)$$

by setting

$$\widetilde{f}_n = f_n,\ \widetilde{g}_n = g_n \quad (n \geq 0), \qquad \widetilde{f}_{-1} = \widetilde{g}_{-1} = \mathrm{id}_{\mathbb{Z}}.$$

We must check that these are indeed chain maps. For $n \geq 1$ this is clear since the augmented differentials coincide with the usual ones. For $n = 0$ we need

$$\widetilde{\partial}^Y_0 \, \widetilde{f}_0 = \widetilde{f}_{-1} \, \widetilde{\partial}^X_0, \quad \text{i.e.} \quad \varepsilon_Y f_0 = \varepsilon_X.$$

This holds because the augmentation is natural in X: for every map $f : X \to Y$ we have $\varepsilon_Y \circ f_* = \varepsilon_X$ (one checks this on generators $\sigma : \Delta^0 \to X$). Thus $\widetilde{f}_*$ and $\widetilde{g}_*$ are chain maps between the augmented complexes.

Step 2. Extend the chain homotopy. Define maps

$$\widetilde{P}_n : \widetilde{C}_n(X) \longrightarrow \widetilde{C}_{n+1}(Y)$$

by

$$\widetilde{P}_n = P_n \quad (n \geq 0), \qquad \widetilde{P}_{-1} = 0.$$

For $n \geq 1$ the chain-homotopy identity

$$\widetilde{f}_n - \widetilde{g}_n = \widetilde{\partial}^Y_{n+1} \, \widetilde{P}_n + \widetilde{P}_{n-1} \, \widetilde{\partial}^X_n \tag{$\star_n$}$$

reduces to $(*_n)$ since all the relevant maps agree with the unaugmented ones.

For $n = 0$ we have

$$\widetilde{f}_0 - \widetilde{g}_0 = f_0 - g_0, \qquad \widetilde{\partial}^Y_1 \, \widetilde{P}_0 + \widetilde{P}_{-1} \, \widetilde{\partial}^X_0 = \partial^Y_1 P_0 + 0,$$

so $(\star_0)$ is exactly $(*_0)$, which we already know to hold. Thus $(\star_n)$ holds for all $n \geq 0$, i.e. $\widetilde{P}$ is a chain homotopy between $\widetilde{f}_*$ and $\widetilde{g}_*$ on the augmented complexes (we do not need to impose any condition in degree -1 for the purpose of computing homology in degrees ≥ 0).

Step 3. Pass to reduced homology. Since chain-homotopic chain maps induce the same maps on homology, we get

$$H_n(\widetilde{f}_*) = H_n(\widetilde{g}_*) : H_n(\widetilde{C}_*(X)) \longrightarrow H_n(\widetilde{C}_*(Y)) \quad \text{for all } n \geq 0.$$

By the definition of reduced homology, this means precisely

$$\widetilde{f}_* = \widetilde{g}_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y) \quad \text{for all } n \geq 0.$$

Hence homotopic maps induce the same homomorphisms on reduced homology.

□

16. (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

Proof. Consider the long exact sequence of the pair (X, A) in low degrees:

$$H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \to 0.$$

From exactness we have j_* surjective, so

$$H_0(X, A) = 0 \quad \iff \quad j_* = 0 \quad \iff \quad i_* \text{ is surjective.}$$

Thus it suffices to show that $i_* : H_0(A) \to H_0(X)$ is surjective iff A meets each path-component of X.

Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be the set of path-components of X. It is standard that

$$H_0(X) \;\cong\; \bigoplus_{\alpha \in \Lambda} \mathbb{Z}$$

with basis given by the classes $[x_\alpha]$ of 0-simplices x_α chosen in each C_α. Similarly, if $\{D_\beta\}_{\beta \in \Gamma}$ are the path-components of A, then

$$H_0(A) \;\cong\; \bigoplus_{\beta \in \Gamma} \mathbb{Z}$$

with basis given by $[a_\beta]$ for $a_\beta \in D_\beta$.

The inclusion $i : A \hookrightarrow X$ sends each component D_β into the unique component $C_{\varphi(\beta)}$ of X containing it. On homology this means

$$i_*([a_\beta]) = [x_{\varphi(\beta)}] \in H_0(X).$$

Hence the image of i_* is the subgroup of $H_0(X)$ generated by those $[x_\alpha]$ for which C_α meets A.

Therefore i_* is surjective if and only if every path-component C_α of X contains some point of A, i.e. A meets each path-component of X.

Combining this with the first paragraph, we conclude

$$H_0(X, A) = 0 \quad \iff \quad A \text{ meets each path-component of } X.$$

□

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof. Consider the long exact sequence of the pair (X, A):

$$H_1(A) \xrightarrow{i_1} H_1(X) \xrightarrow{j_1} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_0} H_0(X) \to H_0(X, A) \to 0.$$

First we show

$$H_1(X, A) = 0 \iff \big(i_1 : H_1(A) \to H_1(X) \text{ surjective and } i_0 : H_0(A) \to H_0(X) \text{ injective}\big). \tag{$*$}$$

Suppose $H_1(X, A) = 0$. Then j_1 has zero target, so $\ker j_1 = H_1(X)$. Exactness at $H_1(X)$ gives $\mathrm{im} i_1 = \ker j_1 = H_1(X)$, hence i_1 is surjective. Since $H_1(X, A) = 0$, the image of j_1 is 0, so by exactness at $H_1(X, A)$ we have $\ker \partial = 0$, that is, ∂ is injective. But the domain of ∂ is zero, hence $\partial = 0$. Exactness at $H_0(A)$ now gives $\ker i_0 = \mathrm{im} \partial = 0$, so i_0 is injective.

Conversely, assume i_1 is surjective and i_0 is injective. Surjectivity of i_1 implies $\mathrm{im} i_1 = H_1(X) = \ker j_1$, hence $j_1 = 0$. Exactness at $H_1(X, A)$ then gives $\ker \partial = \mathrm{im} j_1 = 0$, so ∂ is injective. Injectivity of i_0 says

$\ker i_0 = 0$, so exactness at $H_0(A)$ yields $\operatorname{im}\partial = \ker i_0 = 0$, hence $\partial = 0$. Thus ∂ is both injective and zero, which forces its domain $H_1(X, A)$ to be 0. This proves $(*)$.

Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be the set of path-components of X, and $\{D_\beta\}_{\beta\in\Gamma}$ the set of path-components of A. Choose points $x_\alpha \in C_\alpha$ and $a_\beta \in D_\beta$. Then

$$H_0(X) \cong \bigoplus_{\alpha\in\Lambda} \mathbb{Z}[x_\alpha], \qquad H_0(A) \cong \bigoplus_{\beta\in\Gamma} \mathbb{Z}[a_\beta],$$

where $[x_\alpha]$ and $[a_\beta]$ denote the homology classes of the corresponding 0-simplices.

The inclusion $i : A \hookrightarrow X$ sends each component D_β into the unique component $C_{\varphi(\beta)}$ containing it, and the induced map $i_0 : H_0(A) \to H_0(X)$ satisfies

$$i_0([a_\beta]) = [x_{\varphi(\beta)}].$$

If some path-component C_α of X contains two distinct path-components D_{β_1} and D_{β_2} of A, then

$$i_0([a_{\beta_1}] - [a_{\beta_2}]) = [x_{\varphi(\beta_1)}] - [x_{\varphi(\beta_2)}] = [x_\alpha] - [x_\alpha] = 0,$$

while $[a_{\beta_1}] - [a_{\beta_2}] \neq 0$ in $H_0(A)$, since $[a_{\beta_1}]$ and $[a_{\beta_2}]$ correspond to different direct-sum generators. Hence i_0 is not injective.

Conversely, if each path-component C_α of X contains *at most* one path-component of A, then distinct generators $[a_\beta]$ go to distinct generators $[x_{\varphi(\beta)}]$, so the homomorphism $i_0 : \bigoplus_\beta \mathbb{Z}[a_\beta] \to \bigoplus_\alpha \mathbb{Z}[x_\alpha]$ is injective. Thus

$$i_0 \text{ injective} \iff \text{each path-component of } X \text{ contains at most one path-component of } A.$$

Combining this characterization of injectivity of i_0 with $(*)$, we obtain $H_1(X, A) = 0 \iff H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A, as required. $\square$

17. (a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

Proof. Let $A = \{a_1, \dots, a_k\} \subset X$ be a finite nonempty set of points, so

$$H_i(A) \cong \begin{cases} \mathbb{Z}^k & i = 0, \\ 0 & i > 0. \end{cases}$$

We use the long exact sequence of the pair (X, A) and the computation of $H_0(X, A)$ from Problem 16(a): since X is path–connected and $A \neq \emptyset$, we have $H_0(X, A) = 0$.

Case 1: $X = S^2$.

The long exact sequence gives, in degrees ≥ 2,

$$0 = H_2(A) \longrightarrow H_2(S^2) \cong \mathbb{Z} \longrightarrow H_2(S^2, A) \longrightarrow H_1(A) = 0,$$

hence

$$H_2(S^2, A) \cong \mathbb{Z}.$$

For $n \geq 3$ we have $H_n(S^2) = H_n(A) = 0$, so $H_n(S^2, A) = 0$.

For $n = 1$ we look at

$$0 = H_1(S^2) \longrightarrow H_1(S^2, A) \longrightarrow H_0(A) \cong \mathbb{Z}^k \longrightarrow H_0(S^2) \cong \mathbb{Z} \longrightarrow H_0(S^2, A) = 0.$$

Thus $H_1(S^2, A) \cong \ker\big(\mathbb{Z}^k \to \mathbb{Z}\big)$, where the map $\mathbb{Z}^k \to \mathbb{Z}$ is induced by the inclusion $A \hookrightarrow S^2$ and is the "sum" map $(n_1, \dots, n_k) \mapsto n_1 + \cdots + n_k$. Its kernel is a free abelian group of rank $k - 1$, so

$$H_1(S^2, A) \cong \mathbb{Z}^{k-1}.$$

Summarizing,

$$H_n(S^2, A) \cong \begin{cases} \mathbb{Z} & n = 2, \\ \mathbb{Z}^{k-1} & n = 1, \\ 0 & n \neq 1, 2. \end{cases}$$

Case 2: $X = S^1 \times S^1 = T^2$.
We know

$$H_i(T^2) \cong \begin{cases} \mathbb{Z} & i = 0, 2, \\ \mathbb{Z}^2 & i = 1, \\ 0 & i > 2. \end{cases}$$

For $n \geq 3$ we again have $H_n(T^2) = H_n(A) = 0$, hence $H_n(T^2, A) = 0$.
For $n = 2$ the LES gives

$$0 = H_2(A) \longrightarrow H_2(T^2) \cong \mathbb{Z} \longrightarrow H_2(T^2, A) \longrightarrow H_1(A) = 0,$$

so

$$H_2(T^2, A) \cong \mathbb{Z}.$$

For $n = 1$ we have

$$0 \longrightarrow H_1(T^2) \cong \mathbb{Z}^2 \longrightarrow H_1(T^2, A) \longrightarrow H_0(A) \cong \mathbb{Z}^k \longrightarrow H_0(T^2) \cong \mathbb{Z} \longrightarrow H_0(T^2, A) = 0.$$

As before, $\mathbb{Z}^k \to \mathbb{Z}$ is surjective with kernel $\mathbb{Z}^{k-1}$, so we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow H_1(T^2, A) \longrightarrow \mathbb{Z}^{k-1} \longrightarrow 0.$$

All groups involved are free abelian, hence $H_1(T^2, A)$ is also free abelian, with rank

$$\mathrm{rk} H_1(T^2, A) = \mathrm{rk}\mathbb{Z}^2 + \mathrm{rk}\mathbb{Z}^{k-1} = 2 + (k-1) = k + 1,$$

and with no torsion (any torsion would inject into $\mathbb{Z}^2$). Therefore

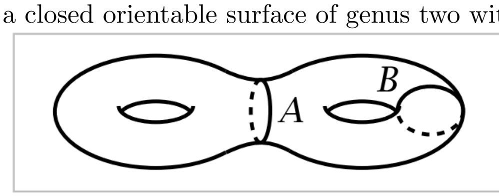
$$H_1(T^2, A) \cong \mathbb{Z}^{k+1}.$$

Thus

$$H_n(T^2, A) \cong \begin{cases} \mathbb{Z} & n = 2, \\ \mathbb{Z}^{k+1} & n = 1, \\ 0 & n \neq 1, 2. \end{cases}$$

□

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]



Proof. **Case A**
We can till use long exat sequence to solve this problem. So, the critical point is to identify

$$H_1(A) \to H_1(X), \; H_0(A) \to H_0(X)$$

which are induced by the inclusion map:

$$\iota : A \to X$$

Since, X is path-connected, we have known from before that $H_0(A) \to H_0(X)$ is isomorphism.
Now, we need to identify

$$H_1(A) = \mathbb{Z} \to H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

We know that ι has a homotopy to a continuous map f which is wraping $X's$ left torus's one-cell twice, but the second time traverse in opposite directions compared to the first time.
So, we have $H_1(A) \to H_1(X)$ is zero map.
Now consider the long exact sequence:

$$0 \to \mathbb{Z} \to H_2(X,A) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_1(X,A) \to \mathbb{Z} \to \mathbb{Z} \to H_0(X,A) \to 0$$

This long exact sequence can be taken apart into three short exact sequence:

$$0 \to \mathbb{Z} \to H_2(X,A) \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_1(X,A) \to 0$$

$$0 \to H_0(X,A) \to 0$$

$\Rightarrow$

$$H_2(X,A) = \mathbb{Z} \oplus \mathbb{Z},\ H_1(X,A) = \mathbb{Z}^4, H_0(X,A) = 0$$

Case B
It is easy to see that

$$H_1(B) = \mathbb{Z} \to H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

is sending 1 to $(0,0,0,1)$. Also, we know $H_0(B) \to H_0(X)$ is isomorphism, since X, B is path-connected.
Now, we can get three short exact sequence:

$$0 \to \mathbb{Z} \to H_2(X,B) \to 0$$

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_1(X,B) \to 0$$

$$0 \to H_0(X,B) \to 0$$

So, we have

$$H_2(X,B) = \mathbb{Z}$$

$$H_1(X,B) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$H_0(X,B) = 0$$

□

23. Show that the second barycentric subdivision of a Δ-complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a Δ-complex with the property that each simplex has all its vertices distinct, then show that for a Δ-complex with this property, barycentric subdivision produces a simplicial complex.

Proof. Let K be a Δ–complex. We denote by sd$K$ its barycentric subdivision (in the sense of Hatcher, obtained by subdividing each simplex barycentrically and gluing along faces).

Step 1. sd$K$ has the property that each simplex has all vertices distinct.
The vertices of sd$K$ are, by definition, the barycenters of all simplices of K. An n–simplex τ of sd$K$ is contained in a unique simplex σ of K (its interior lies in the interior of σ), and inside σ it is the convex hull of the barycenters of a chain of faces

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n = \sigma$$

of σ. The barycenter of σ_i lies in the interior of σ_i, and interiors of distinct simplices of K are disjoint, hence these barycenters are all distinct points of $|K|$. Thus every simplex of sd$K$ has $n+1$ distinct vertices.

Step 2. If K is a Δ–complex in which each simplex has all its vertices distinct, then sd$K$ is a simplicial complex.
Assume now K is a Δ–complex with the property:

$$(*) \text{ Each simplex of } K \text{ has all its vertices distinct.}$$

Step 2.1. In sd$K$ *every simplex is determined by its vertices.*
Let τ be an n–simplex of sd$K$ with vertices $v_0, \dots, v_n$. Each v_i is, by construction, the barycenter of a unique simplex σ_i of K (uniqueness because the barycenter lies in the interior of σ_i, and interiors of simplices of K are disjoint). The interior of τ lies in the interior of a unique simplex σ of K; this simplex σ is necessarily the largest among the σ_i's with respect to inclusion. Inside σ the simplex τ is one of the standard simplices of the barycentric subdivision sd$\Delta^{\dim \sigma}$, hence its vertices are the barycenters of a strictly increasing chain of faces of σ:

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n = \sigma.$$

Thus the data of the vertices of τ is equivalent to the data of this chain of faces.
Now suppose τ' is another simplex of sd$K$ with the same set of vertices as τ. Then τ' determines the same set of simplices $\{\sigma_0, \dots, \sigma_n\}$ in K. The interior of τ' also lies in the interior of some simplex of K, but this simplex must be σ since its barycenter is one of the vertices. Inside σ, both τ and τ' are simplices in the barycentric subdivision of the simplex σ. There, barycentric subdivision is an honest simplicial complex, so a simplex is uniquely determined by its set of vertices. Hence $\tau' = \tau$. We conclude that in sd$K$ each simplex is determined by its vertices.

Thus sd$K$ is a Δ–complex with the following, stronger property:

$$(**) \text{ No two distinct simplices of sd}K \text{ have the same set of vertices.}$$

Step 2.2. A Δ–complex with property $(**)$ is a simplicial complex.
Let L be a Δ–complex satisfying $(**)$. Let V be its set of vertices. Choose an injective map

$$\iota : V \longrightarrow \mathbb{R}^N$$

with image in general position (for instance, send the k–th vertex to the k–th standard basis vector e_k). For each n–simplex σ of L with vertices $v_0, \dots, v_n$, consider the geometric simplex

$$|\sigma|_{\text{geom}} := \big(\iota(v_0), \dots, \iota(v_n)\big) \subset \mathbb{R}^N.$$

Let K be the collection of all such geometric simplices. Since a simplex of L is completely determined by its vertex set, different simplices of L give different geometric simplices in K, and the intersection of two geometric simplices is the convex hull of their common vertices, hence a common face. Thus K is a simplicial complex in the usual sense.
There is an obvious map

$$h : |K| \longrightarrow |L|$$

defined simplexwise: on each geometric simplex $|\sigma|_{\text{geom}} \subset |K|$ we let h be the unique affine map sending $\iota(v_i)$ to the vertex v_i of L. These affine maps agree on common faces, so h is well-defined and continuous. It is bijective, since each point of $|L|$ lies in the interior of a unique simplex of L and the restriction of h to each simplex is a homeomorphism. As $|K|$ is compact and $|L|$ is Hausdorff, h is a homeomorphism. Hence L carries a simplicial complex structure (namely K).
Applying this to $L = \text{sd}K$, we conclude that sd$K$ is a simplicial complex whenever K satisfies $(*)$.

Step 3. Second barycentric subdivision of an arbitrary Δ–complex is simplicial.
Let K_0 be an arbitrary Δ–complex and set $K_1 = \text{sd}K_0$, $K_2 = \text{sd}K_1$. By Step 1, K_1 has the property that each simplex has all its vertices distinct, i.e. K_1 satisfies $(*)$. Therefore, by Step 2, $K_2 = \text{sd}K_1$ is a simplicial complex.
Thus the second barycentric subdivision of any Δ–complex is a simplicial complex. □

29. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Proof. Let

$$T^2 = S^1 \times S^1, \qquad X = S^1 \vee S^1 \vee S^2.$$

Step 1: Homology of T^2

We know from Hatcher's book:

$$H_0(T^2) \cong \mathbb{Z}, \qquad H_1(T^2) \cong \mathbb{Z}^2, \qquad H_2(T^2) \cong \mathbb{Z},$$

and $H_n(T^2) = 0$ for all $n \geq 3$.

Step 2: Homology of $X = S^1 \vee S^1 \vee S^2$

Recall that for a wedge of based spaces $\bigvee_i Y_i$ we have

$$\widetilde{H}_n\Big(\bigvee_i Y_i\Big) \cong \bigoplus_i \widetilde{H}_n(Y_i)$$

for all $n \geq 0$ ee.
We know the reduced homology of spheres:

$$\widetilde{H}_k(S^1) \cong \begin{cases} \mathbb{Z}, & k = 1, \\ 0, & k \neq 1, \end{cases} \qquad \widetilde{H}_k(S^2) \cong \begin{cases} \mathbb{Z}, & k = 2, \\ 0, & k \neq 2. \end{cases}$$

Hence

$$\widetilde{H}_1(X) \cong \widetilde{H}_1(S^1) \oplus \widetilde{H}_1(S^1) \oplus \widetilde{H}_1(S^2) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$\widetilde{H}_2(X) \cong \widetilde{H}_2(S^1) \oplus \widetilde{H}_2(S^1) \oplus \widetilde{H}_2(S^2) \cong \mathbb{Z},$$

and $\widetilde{H}_n(X) = 0$ for all $n \geq 3$.
Since X is path–connected, $H_0(X) \cong \mathbb{Z}$, and for $n \geq 1$, $H_n(X) \cong \widetilde{H}_n(X)$. Thus

$$H_0(X) \cong \mathbb{Z}, \qquad H_1(X) \cong \mathbb{Z}^2, \qquad H_2(X) \cong \mathbb{Z},$$

and $H_n(X) = 0$ for $n \geq 3$.
Comparing with Step 1, we see that T^2 and X have isomorphic homology groups in all dimensions.

Step 3: Universal covers

We now show that their universal covers are not homeomorphic.

3.1. Universal cover of T^2

The universal covering space of S^1 is $\mathbb{R}$ with the covering map $\mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. The product of covering spaces is again a covering space, hence the universal cover of

$$T^2 = S^1 \times S^1$$

is

$$\widetilde{T^2} \cong \mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2.$$

Thus every point of $\widetilde{T^2}$ has a neighborhood homeomorphic to an open disk in $\mathbb{R}^2$.

3.2. Universal cover of $X = S^1 \vee S^1 \vee S^2$

The 1–skeleton of X is $S^1 \vee S^1$, whose fundamental group is the free group F_2 on two generators. The universal cover of $S^1 \vee S^1$ is an infinite 4–valent tree T.
The space X is obtained from $S^1 \vee S^1$ by wedging a 2–sphere S^2 at the common basepoint. In the universal cover, the lift of this S^2 appears once at every vertex of the tree. More precisely, the universal cover $\widetilde{X}$ can be described as follows:

- Start with the universal covering tree T of $S^1 \vee S^1$.
- For each vertex v of T, attach a copy of S^2 by identifying its basepoint with v.

It is standard that the resulting CW–complex is simply connected and admits a free, properly discontinuous action of $\pi_1(X) \cong F_2$ with quotient X; hence this is the universal cover $\widetilde{X}$ of X.

3.3. A point of $\widetilde{X}$ with 1–dimensional local model

Consider an edge e in the tree $T \subset \widetilde{X}$ and let p be a point in the interior of e (so p is neither a vertex of the tree nor a point on any attached S^2). For sufficiently small $\varepsilon > 0$, the intersection of the open ε–ball around p with $\widetilde{X}$ is just a small open segment contained in the interior of e. Thus there exists an open neighborhood U of p in $\widetilde{X}$ that is homeomorphic to an open interval in $\mathbb{R}$.
In particular, U is homeomorphic to an open subset of $\mathbb{R}^1$.

Step 4: Using invariance of dimension

Assume for contradiction that the universal covers $\widetilde{T^2}$ and $\widetilde{X}$ are homeomorphic. Then there exists a homeomorphism

$$f : \widetilde{X} \longrightarrow \widetilde{T^2} \cong \mathbb{R}^2.$$

Let $p \in \widetilde{X}$ be the point in the interior of an edge as above, and let $U \subset \widetilde{X}$ be an open neighborhood of p that is homeomorphic to an open interval $(a, b) \subset \mathbb{R}$. Then $f(U)$ is an open subset of $\mathbb{R}^2$, and the restriction

$$f|_U : U \longrightarrow f(U)$$

is a homeomorphism.
Thus we obtain a homeomorphism between an open subset of $\mathbb{R}^1$ and an open subset of $\mathbb{R}^2$. This contradicts the *invariance of dimension* theorem, which states that if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are nonempty open sets and $U \cong V$, then $m = n$.
Therefore no such homeomorphism f can exist. Hence

$$\widetilde{X} \ncong \mathbb{R}^2,$$

so the universal covers of T^2 and X are not homeomorphic.

We have shown that

- T^2 and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but
- their universal covers are not homeomorphic.

This completes the proof. $\square$