Problem 1

We define the sequence of partitions $\{P_k\}_{k=1}^{\infty}$ as following.

$$\delta := \frac{1}{2^{k+2}}, x_i = \frac{1}{i}, i \in N \tag{1}$$
$$P_k := \{x_1 = 1, x_1 - \delta, x_2 + \delta, x_2 - \delta, x_3 + \delta, x_3 - \delta, \ldots, x_k + \delta, x_k - \delta, 0\}, k \in \mathbb{N} \tag{2}$$

Then for a particular partition P_k, we have

$$\begin{aligned}
U(f,P) = &\sum_{i=1}^{k-1} \sup_{x\in[x_{i+1}+\delta, x_i-\delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + &(3)\\
&\sum_{i=2}^{k} \sup_{x\in[x_i-\delta, x_i+\delta]} 1/[1/x]2\delta &(4)\\
&+ \sup_{x\in[1-\delta,1]} 1/[1/x]\delta + \sup_{x\in[0,x_k-\delta]} 1/[1/x](x_k - \delta) &(5)\\
&= \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_{i-1} + \delta + x_{k+1}(x_k - \delta) &(6)\\
&= \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1}\right) + \delta + x_{k+1}(x_k - \delta) &(7)\\
&= \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + \delta + x_{k+1}(x_k - \delta) &(8)
\end{aligned}$$

and

$$\begin{aligned}
L(f,P) = &\sum_{i=1}^{k-1} \inf_{x\in[x_{i+1}+\delta, x_i-\delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + &(9)\\
&\sum_{i=2}^{k} \inf_{x\in[x_i-\delta, x_i+\delta]} 1/[1/x]2\delta &(10)\\
&+ \inf_{x\in[1-\delta,1]} 1/[1/x]\delta + \inf_{x\in[0,x_k-\delta]} 1/[1/x](x_k - \delta) &(11)\\
&= \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_i + \delta + 0 &(12)\\
&= \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1}\right) + 2\delta(x_k - 1) + \delta &(13)\\
&= \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + 2\delta(x_k - 1) + \delta &(14)
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, Letting $k \to \infty$ and we got

$$L(f,P_k) = \frac{\pi^2}{6} - 1 = U(f,P_k) \tag{15}$$

Problem 2

(a) Since f_n converges uniformly to f, $\forall \varepsilon > 0 \exists N > 0$ s.t. $n \geq N \implies \forall x \in [a,b], |f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)}$. So that given a partition $P = \{x_0 = a, x_1, \ldots, x_k = b\}$ we have

$$|U(f,P) - U(f_n,P)| = \sum_{i=1}^{k} \left| \sup_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x_{i-1},x_i]} f_n(x) \right| (x_i - x_{i-1}) \tag{16}$$

$$|L(f,P) - L(f_n,P)| = \sum_{i=1}^{k} \left| \inf_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f_n(x) \right| (x_i - x_{i-1}) \tag{17}$$

Since

$$\sup f(x) \leq \sup(f(x) - f_n(x)) + \sup f_n(x) < \frac{\varepsilon}{3(b-a)} + \sup f_n(x) \tag{18}$$

$$\inf f(x) \geq \inf(f(x) - f_n(x)) + \inf f_n(x) > -\frac{\varepsilon}{3(b-a)} + \inf f_n(x) \tag{19}$$

we have

$$|U(f,P) - U(f_n,P)| < \sum_{i=1}^{k} \frac{\varepsilon}{3(b-a)}(x_i - x_{i-1}) = \varepsilon/3 \tag{20}$$

$$|L(f,P) - L(f_n,P)| < \sum_{i=1}^{k} \frac{\varepsilon}{3(b-a)}(x_i - x_{i-1}) = \varepsilon/3 \tag{21}$$

Since f_n is Riemann integrable, by 5.6(iii), for any partition P we have $|U(f_n,P) - L(f_n,P)| < \varepsilon/3$. Combine it with (20), (21) and apply triangular inequality, we conclude that f is Riemann integrable by 5.6(iii).

(b) Suppose f_n is continuous at $x_0 \in [a,b]$ for any $n \in \mathbb{N}$. Then $\forall \varepsilon > 0, \exists \delta(n) > 0$ such that $|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \varepsilon/3$. Since f_n converges uniformly to f, if n is sufficiently large, we have $\forall x \in [a,b], |f(x) - f_n(x)| < \varepsilon/3$. Then $\forall \varepsilon > 0$, we could find $\delta(n)$ with sufficiently large n, such that if $|x - x_0| < \delta(n)$,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon \tag{22}$$

Then f is also continuous at x_0. Hence $\bigcap_{i=1}^{\infty} D_{f_i}^c \subset D_f^C$. Therefore $D_f \subset \bigcup_{i=1}^{\infty} D_{f_i}$, and

$$\mathcal{L}^*(D_f) \leq \mathcal{L}^*\left(\bigcup_{i=1}^{\infty} D_{f_i}\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^*(D_{f_i}) \tag{23}$$

By Lebesgue theorem, $\mathcal{L}^*(D_{f_i}) = 0$ for $i \in \mathbb{N}$, therefore $\mathcal{L}^*(D_f) = 0$ and hence f is Riemann integrable.

Problem 3

(a)