Advanced Linear Algebra Notes

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Chapter 0

Preliminaries

0.1 Preliminaries

Multisets

A **Multiset** M with **underlying set** S is a set of ordered pairs

$$M = \{(s_i, n_i) | s_i \in S, n_i \in \mathbb{Z}^+, s_i \neq s_j \forall i \neq j\}$$

Matrices

$\mathcal{M}_{m,n}(F)$: the set of $m \times n$ matrices.

Properties of transpose:

1. $\left(A^T\right)^T = A$
2. $\left(A + B\right)^T = A^T + B^T$
3. $\left(rA\right)^T = rA^T \ \forall r \in F$
4. $\left(AB\right)^T = B^T A^T$ provided that AB is defined.
5. $\det\left(A^T\right) = \det\left(A\right)$

Partitions/Block Matrices

Actually there is another expression:

$$\begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix} = \sum_{i=1}^{k} A_i B_i$$

provided that $A_i B_i$ is defined.

Determinants

1. $\det\bigl(AB\bigr) = \det\bigl(A\bigr)\det\bigl(B\bigr)$
2. If

$$M = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_n \end{bmatrix}$$

   then $\det\bigl(M\bigr) = \prod \det(B_i)$.

Polynomials

More on it later. It's basically drawing analogies between polynomials and integers.

Function

Definition. *Let $f : S \to T$ be a function from a set S to T. Assume that $0 \in T$, the* **support** *is*

$$\mathrm{supp}(f) = s \in S | f(s) \neq 0$$

If $X \subseteq S$ and $Y \subseteq T$,

$$f(X) = \{f(x) | x \in X\}$$
$$f^{-1}(Y) = s \in S | f(s) \in Y$$

where f doese not need to be injective.

$$f|_A(a) = f(a)$$
$$\bar{f} : U \to T, U \subseteq S \text{ for } \bar{f}|_S = f$$

Equivalence Relations

Example

Let $p(x) \sim q(x) \overset{def}{\iff} p(x) = aq(x)$ for some nonzero constant $a \in F$. Trivially this is an equivalence relation. Since

$$p(x) \sim q(x) \implies \deg(p(x)) = \deg(p(x)),$$

the function of retrieving degrees is an invariant function of $\sim$, not a complete invariant. The set of all monic polynomials is a set of canonical forms.

Example

Row equivalence on $\mathcal{M}_{m,n}(F)$. The subset of reduced row echelon form matrices is a set of canonical form for row equivalence.

Example

$A, B \in \mathcal{M}_n(F)$ are row equivalent $\overset{def}{\iff}$ $\exists$ invertible P $s.t.$ $A = PB$.
$A, B \in \mathcal{M}_n(F)$ are column equivalent $\overset{def}{\iff}$ $\exists$ invertible Q $s.t.$ $A = BQ$.
A, B are equivalent $\overset{def}{\iff}$ $\exists$ invertible P, Q $s.t.$ $A = PBQ$.
After elementary operations

$$J_k = \begin{bmatrix} I_k & O \\ O & O \end{bmatrix} \in \mathcal{M}_{m,n}$$

So $\{J_k\}$ is a canonical form and rank is a complete invariant.

Example

$A, B \in \mathcal{M}_n(F)$ is similar $\overset{def}{\iff}$ $\exists$ invertible P $s.t.$ $A = PBP^{-1}$. To be developed.

Example

$A, B \in \mathcal{M}_n(F)$ is congruent $\overset{def}{\iff}$ $\exists$ invertible P $s.t.$ $A = PBP^T$.
Anyway, this book aims to find their canonical form.

Zorns's Lemma

posets: partially ordered set (not necessarily comparable for all elements)
Thus total/linear ordered set.
chain: total ordered subset.

Zorn's Lemma. *If P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.*

equivalent to axiom of choice, well-ordering principle.

Cardinality

$|S| = |T|$ $\overset{def}{\iff}$ there is a bijective function.

Part I

Chapter 1

Vector Spaces

1.1 Vector Spaces