Problem 1

We define the sequence of partitions $\{P_k\}_{k=1}^{\infty}$ as following.

$$\delta := \frac{1}{2^{k+2}}, x_i = \frac{1}{i}, i \in N \tag{1}$$

$$P_k := \{x_1 = 1, x_1 - \delta, x_2 + \delta, x_2 - \delta, x_3 + \delta, x_3 - \delta, \ldots, x_k + \delta, x_k - \delta, 0\}, k \in \mathbb{N} \tag{2}$$

Then for a particular partition P_k, we have

$$\begin{aligned}
U(f,P) = \sum_{i=1}^{k-1} \sup_{x \in [x_{i+1}+\delta, x_i - \delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + && (3)\\
\sum_{i=2}^{k} \sup_{x \in [x_i - \delta, x_i + \delta]} 1/[1/x] 2\delta && (4)\\
+ \sup_{x \in [1-\delta, 1]} 1/[1/x]\delta + \sup_{x \in [0, x_k - \delta]} 1/[1/x](x_k - \delta) && (5)\\
= \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_{i-1} + \delta + x_{k+1}(x_k - \delta) && (6)\\
= \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1} \right) + \delta + x_{k+1}(x_k - \delta) && (7)\\
= \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + \delta + x_{k+1}(x_k - \delta) && (8)
\end{aligned}$$

and

$$\begin{aligned}
L(f,P) = \sum_{i=1}^{k-1} \inf_{x \in [x_{i+1}+\delta, x_i - \delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + && (9)\\
\sum_{i=2}^{k} \inf_{x \in [x_i - \delta, x_i + \delta]} 1/[1/x] 2\delta && (10)\\
+ \inf_{x \in [1-\delta, 1]} 1/[1/x]\delta + \inf_{x \in [0, x_k - \delta]} 1/[1/x](x_k - \delta) && (11)\\
= \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_i + \delta + 0 && (12)\\
= \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1} \right) + 2\delta(x_k - 1) + \delta && (13)\\
= \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + 2\delta(x_k - 1) + \delta && (14)
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, Letting $k \to \infty$ and we got

$$L(f, P_k) = \frac{\pi^2}{6} - 1 = U(f, P_k) \tag{15}$$

Problem 2

(a) Since f_n converges uniformly to f, $\forall \varepsilon > 0 \exists N > 0$ s.t. $n \geq N \implies \forall x \in [a,b], |f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)}$. So that given a partition $P = \{x_0 = a, x_1, \ldots, x_k = b\}$ we have

$$|U(f,P) - U(f_n,P)| = \sum_{i=1}^{k} \left| \sup_{x \in [x_{i-1},x_i]} f(x) - \sup_{x \in [x_{i-1},x_i]} f_n(x) \right| (x_i - x_{i-1}) \tag{16}$$

$$|L(f,P) - L(f_n,P)| = \sum_{i=1}^{k} \left| \inf_{x \in [x_{i-1},x_i]} f(x) - \inf_{x \in [x_{i-1},x_i]} f_n(x) \right| (x_i - x_{i-1}) \tag{17}$$

Since

$$\sup f(x) \leq \sup(f(x) - f_n(x)) + \sup f_n(x) < \frac{\varepsilon}{3(b-a)} + \sup f_n(x) \tag{18}$$

$$\inf f(x) \geq \inf(f(x) - f_n(x)) + \inf f_n(x) > -\frac{\varepsilon}{3(b-a)} + \inf f_n(x) \tag{19}$$

we have

$$|U(f,P) - U(f_n,P)| < \sum_{i=1}^{k} \frac{\varepsilon}{3(b-a)}(x_i - x_{i-1}) = \varepsilon/3 \tag{20}$$

$$|L(f,P) - L(f_n,P)| < \sum_{i=1}^{k} \frac{\varepsilon}{3(b-a)}(x_i - x_{i-1}) = \varepsilon/3 \tag{21}$$

Since f_n is Riemann integrable, by 5.6(iii), for any partition P we have $|U(f_n,P) - L(f_n,P)| < \varepsilon/3$. Combine it with (20), (21) and apply triangular inequality, we conclude that f is Riemann integrable by 5.6(iii).

(b) Suppose f_n is continuous at $x_0 \in [a,b]$ for any $n \in \mathbb{N}$. Then $\forall \varepsilon > 0, \exists \delta(n) > 0$ such that $|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \varepsilon/3$. Since f_n converges uniformly to f, if n is sufficiently large, we have $\forall x \in [a,b], |f(x) - f_n(x)| < \varepsilon/3$. Then $\forall \varepsilon > 0$, we could find $\delta(n)$ with sufficiently large n, such that if $|x - x_0| < \delta(n)$,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon \tag{22}$$

Then f is also continuous at x_0. Hence $\bigcap_{i=1}^{\infty} D_{f_i}^c \subset D_f^C$. Therefore $D_f \subset \bigcup_{i=1}^{\infty} D_{f_i}$, and

$$\mathcal{L}^*(D_f) \leq \mathcal{L}^*\left(\bigcup_{i=1}^{\infty} D_{f_i}\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^*(D_{f_i}) \tag{23}$$

By Lebesgue theorem, $\mathcal{L}^*(D_{f_i}) = 0$ for $i \in \mathbb{N}$, therefore $\mathcal{L}^*(D_f) = 0$ and hence f is Riemann integrable.

Problem 3

(a) Since ϕ is continuous on $\mathbb{R}$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|p - q| < \delta \implies |\phi(p) - \phi(q)| < \varepsilon$. Since f is Riemann integrable, there exist a partition $P = \{x_i\}_{i=1}^n$ of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon\delta$. Thus

$$\sum_{i=1}^{n} \left((\sup - \inf)_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1}) < \varepsilon\delta \tag{24}$$

For $\phi \circ f$, we have

$$U(\phi \circ f,) - L(\phi \circ f, P) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_i]} \phi \circ f - \inf_{[x_{i-1}, x_i]} \phi \circ f \right) (x_i - x_{i-1}) \tag{25}$$

$$= \sum_{i:(\sup - \inf)_{[x_{i-1},x_i]} f < \delta/2} \tag{26}$$

$$\left(\sup_{[x_{i-1}, x_i]} \phi \circ f - \inf_{[x_{i-1}, x_i]} \phi \circ f \right) (x_i - x_{i-1}) \tag{27}$$

$$+ \sum_{i:(\sup - \inf)_{[x_{i-1},x_i]} f \geq \delta/2} \tag{28}$$

$$\left(\sup_{[x_{i-1}, x_i]} \phi \circ f - \inf_{[x_{i-1}, x_i]} \phi \circ f \right) (x_i - x_{i-1}) \tag{29}$$

For the first branch, we have $|f(x) - f(y)| < \delta/2$ for any $x, y \in [x_{i-1}, x_i]$. Since ϕ is continuous, we have $|\phi(f(x)) - \phi(f(y))| < \varepsilon$ and thus

$$\left(\sup_{[x_{i-1}, x_i]} \phi \circ f - \inf_{[x_{i-1}, x_i]} \phi \circ f \right) \leq \varepsilon \tag{30}$$

For the next branch, Since

$$\sum_{i \in Branch2} \frac{\delta}{2}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \left((\sup - \inf)_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1}) < \varepsilon\delta \tag{31}$$

we have $\sum_{i \in Branch2}(x_i - x_{i-1}) < \varepsilon\delta\frac{2}{\delta} = 2\varepsilon$ thus

$$\begin{aligned} \sum_{i \in Branch2} \left(\sup_{[x_{i-1}, x_i]} \phi \circ f - \inf_{[x_{i-1}, x_i]} \phi \circ f \right) (x_i - x_{i-1}) \\ < 2\varepsilon(\sup - \inf)_{[a,b]} \phi \circ f \end{aligned}$$

Since f is bounded, there exist M such that $f[a, b] \subset [-M, M]$. Since ϕ is continuous and the compactness of $[-M, M]$, $\phi \circ f$ is also bounded and therefore $(\sup - \inf)_{[a,b]} \phi \circ f$ is a constant. Adding up the two branches and we got $\phi \circ f$ is Riemann integrable by 5.6(iii).

(b) If f is continuous at $x_0 \in [a, b]$, then $\phi \circ f$ is alsoe continuous at x_0 thus

$$D_f^c \subset D_{\phi \circ f}^c \implies D_{\phi \circ f} \subset D_f \tag{32}$$

Since $\mathcal{L}^*(D_f) = 0$, we have $\mathcal{L}^*(D_{\phi \circ f}) = 0$ as well, so it is Riemann integrable.

Problem 4

(a) Let $E_n = E \cap [-n, n], n \in N$. Then $\mathcal{L}^*(E_n) = 0$ as well. Give $\varepsilon > 0$, Let $E_n \subset \bigcup_{i=1}^{\infty}(a_i, b_i), a_i \leq b_i$ and $\sum_{i=1}^{\infty} |a_i - b_i| \leq \frac{\varepsilon}{2^n(2n+1)}$. Let $c_i = \min(a_i, b_i), d_i = \max(a_i, b_i)$. So $E_n^2 =\subset \bigcup_{i=1}^{\infty}(c_i, d_i)$, and

$$\begin{aligned} \mathcal{L}^*(E_n^2) &= \sum_{i=1}^{\infty}(d_i - c_i) \\ &= \sum_{i=1}^{\infty}(a_i + b_i)(b_i - a_i) \\ &< (2n + 1)\sum_{i=1}^{\infty}(b_i - a_i) \leq \varepsilon/2^n \end{aligned}$$

We can verify that $E_n^2 = E^2 \cap [-n^2, n^2]$, therefore

$$\mathcal{L}^*(E^2) = \mathcal{L}^*\left(\bigcup_{n=1}^{\infty} E^2 \cap [-n^2, n^2]\right)$$
$$\leq \sum_{n=1}^{\infty} \mathcal{L}^*(E_n) < 2\varepsilon$$

Let $\varepsilon \to 0^+$ and done.

(b) If g is not continuous at x_0, then since $\sqrt{x}$ is continuous on $[0,1]$, f must be discontinuous at $\sqrt{x_0}$. Therefore $D_g \subset D_f^2$.By (a), we have

$$\mathcal{L}^*(D_g) \leq \mathcal{L}^*(D_f^2) = 0$$

By Lebesgue theorem, g is Riemann integrable on $[0,1]$.

Problem 5

Only if:

Let $\varepsilon > 0$. Since f is Riemann integrable, there is a partition P such that $U(f,P) - L(f,P) < \varepsilon/k$. Let $x_0, x_1, \ldots, x_n$ be the points of partition P. Define a set of indexes I such that for $i \in I, \Omega_k \cap [x_{i-1}, x_i] \neq \phi$. Then we have $\Omega_k \subset \bigcup_{i\in I}[x_{i-1}, x_i]$. Let $\delta = (x_i - x_{i-1})/2$, then we have

$$[x_{i-1}, x_i] \subset (x_{i-1} - \delta, x_i + \delta)$$
$$\Omega_k \subset \bigcup_{i\in I}(x_{i-1} - \delta, x_i + \delta)$$

And the length of this cover is $2\sum_{i\in I}(x_i - x_{i-1})$.

By the definition of $\Omega_k, \sup_{x,y\in[x_{i-1},x_i]} |f(x) - f(y)| \geq 1/k$, if $\Omega_k \cap [x_{i-1}, x_i] \neq \phi$. Thus

$$\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \geq 1/k$$

And

$$\frac{2}{k}\sum_{i\in I}(x_i - x_{i-1}) \leq 2\sum_{i\in I}\left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right)(x_i - x_{i-1})$$
$$\leq 2(U(f,P) - L(f,P)) < 2\varepsilon/k$$

Let $\varepsilon \to 0^+$ and the length of the cover towards zero. Hence $\mathcal{L}^*(\Omega) = 0$. By 5.1, $\mathcal{L}^*(D_f) = 0$.

If:

(b):

By prop 2.27, $C = \bigcup_{n=1}^{\infty}(\alpha_n, \beta_n)$ is open. Therefore $T = \mathbb{R} - \bigcup_{n=1}^{\infty}(\alpha_n, \beta_n)$ is closed. Hence $K = T \cap [a,b]$ is closed and bounded. Therefore it is compact, and f is uniform continuous on K if it is continuous.

(c): If D_f has lebesgue 0, then we cover every point in D_f by the small open cover. Since f is continuous on $R - D_f$, by (b), f is further uniformly continuous on K. Therefore for $\varepsilon > 0$, there exist δ such that $\forall x_0 \in K, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

Define $J_x = \{y : |x - y| < \delta$, obviously $\bigcup_{x\in K} J_x$ is an open cover of K. By compactness of K, we conclude that there is a finite subcover such that

$$K \subset \bigcup_{x=x_1}^{x_m} J_x$$

Sort all the endpoints of these finite subcovers and a, b, we shall form a partition P.

If $[x_{i-1}, x_i] \cap K = \phi$, then it is totally inside the C. The sum of its length is bounded by ε since the measure is zero. Since f is bounded, therefore $(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f) < M$ for some fied M.

If $[x_{i-1}, x_i] \cap K \neq \phi$, notice that no endpoints $x_j \in (x_{i-1}, x_i)$ for any $j \in [1, 2, \ldots, m]$. Thus there exist $j \in [1, 2, \ldots, m]$ such that $(x_{i-1}, x_i) \subset [x_j - \delta, x_j + \delta]$. then it is uniformly continuous and therefore $\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f < \varepsilon$.

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{case1} (\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f)(x_i - x_{i-1}) + \sum_{case2} (\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f)(x_i - x_{i-1}) \\ &\leq M\varepsilon + \varepsilon(b-a) \end{aligned}$$

Hence it is Riemann integrable.