Problem 1

Let $S_n(x) = \sum_{k=1}^n f_k(x)$. For $m \geq n$,

$$\sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_k g_k = \sum_{k=n}^{m-1} S_k(g_k - g_{k+1}) + (S_m g_m - S_n g_n)$$

$|S_k| \leq C$ and $g_k - g_{k+1} \geq 0$ for any $k \in \mathbb{N}$. Since g_k converges to 0 and is monotonically decreasing, $g_m > 0, g_n > 0$. Therefore we have:

$$\left| \sum_{k=1}^m f_k g_k \right| \leq \sum_{k=n}^{m-1} C(g_k - g_{k+1}) + C(g_m + g_n) = 2Cg_n$$

Since g_n converges uniformly on E to 0, for any $\varepsilon > 0$, there exist $N(\varepsilon)$ such that for $n \geq N$, we have $g_n < \frac{\varepsilon}{2C}$. Thus we conclude that for $N \leq n \leq m$,

$$\left| \sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_k g_k \right| \leq 2Cg_n < \varepsilon$$

Since it holds for every $x \in E$, we have

$$\left\| \sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_k g_k \right\|_E \leq 2Cg_n < \varepsilon$$

By cauchy criterion, it converges uniformly.

Problem 2

Let $R_n(x) = \sum_{k=n}^\infty f_n(x)$, because $\sum_{k=1}^\infty f_n$ converges. For $m \geq n$,

$$\sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_k g_k = R_n g_n - R_{m+1} g_m + \sum_{k=n}^{m-1} R_{k+1}(g_{k+1} - g_k)$$

Since $g_k \geq g_{k+1}$, we have:

$$\left| \sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_k, g_k \right| \leq |R_n||g_n| + |R_{m+1}||g_m| + \sum_{k=n}^{m-1} |R_{k+1}|(g_k - g_{k+1})$$

Since $\sum_{k=1}^\infty f_n$ converges uniformly on E, for any $\varepsilon > 0$, there exist N such that for $n > N$, we have $|R_n| \leq \frac{\varepsilon}{4C}$. Also since g_k is bounded by C, we have

$$\begin{aligned}
\left| \sum_{k=1}^m f_k g_k - \sum_{k=1}^n f_j g_k \right| &\leq C \cdot \frac{\varepsilon}{4C} \cdot 2 + \frac{\varepsilon}{4C} \sum_{k=n}^{m-1} (g_k - g_{k+1}) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{4C}(g_n - g_m) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4C}(|g_n| + g_m|) \leq \varepsilon
\end{aligned}$$

By cauchy criterion, this implies uniform convergence.

Problem 3

(a) $\frac{1}{n} \geq \frac{1}{n+1}$ for any n, and $\frac{1}{n}$ converges uniformly to zero. For $\sin nx$, we have:

$$\begin{aligned}\left|\sum_{k=1}^{n} \sin kx\right| &= \left|\frac{\sin\frac{kx+x}{2}\sin\frac{kx}{2}}{\sin\frac{x}{2}}\right| \\ &\leq \frac{1}{\left|\sin\frac{x}{2}\right|}\end{aligned}$$

since $x \in [\varepsilon, 2\pi - \varepsilon]$ where $\varepsilon > 0$, $\sin\frac{x}{2} \geq \sin\frac{\varepsilon}{2} > 0$. Therefore $\frac{1}{|\sin\frac{x}{2}|}$ is bounded by $\frac{1}{|\sin\frac{\varepsilon}{2}|}$. By Dirichlet's test, it converges uniformly on $[\varepsilon, 2\pi - \varepsilon], \varepsilon > 0$.

(b) By alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Also $x^n \geq x^{n+1}$ for $x \in [0,1]$ and $|x^n| \leq 1$ for and $n \in \mathbb{N}$ and $x \in [0,1]$. By Abel's test, it converges uniformly on $[0,1]$.

(c) By integral test,

$$\begin{aligned}\int_2^{\infty} \frac{dx}{x(\log x)^2} &= -\frac{1}{\log(x)}\bigg|_2^{\infty} \\ &= \frac{1}{\log 2} < \infty\end{aligned}$$

Therefore $\sum_{k=2}^{\infty} \frac{1}{k(\log k)^2}$ converges. And since $\left|\frac{x^k}{k(\log k)^2}\right| \leq \frac{1}{k(\log k)^2}$ for $x \in [-1,1]$, by Weiestrass' M-test, it converges uniformly on $[-1,1]$.

Problem 4

(a)

$$\int_0^1 t^{x-1}e^{-t}dt = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+x-1}}{n!}dt$$

Because $x > 1$ and $t \in [0,1]$,

$$\frac{(-1)^n t^{n+x-1}}{n!} \leq \frac{t^{n+x-1}}{n!} \leq \frac{t^n}{n!} \leq t^n$$

and $\sum_{n=0}^{\infty} t^n$ converges. Therefore Weierstrass' M-test shows the series $\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+x-1}}{n!}$ converges uniformly on $t \in [0,1]$. So we can

$$\begin{aligned}\int_0^1 t^{x-1}e^{-t}dt &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{n+x-1}}{n!}dt \\ &= \sum_{n=0}^{n} \frac{(-1)^n}{n!(x+n)}\end{aligned}$$

(b) Let $D := R \backslash \bigcup_{n=0}^{\infty}(-n-\varepsilon, -n+\varepsilon)$ for any small $\varepsilon > 0$. Then for $x \in D$, $|x+n| \geq \varepsilon$ for any $n \in \mathbb{N}$.

$$\begin{aligned}\frac{(-1)^n}{n!(x+n)} &\leq \frac{1}{n!\,|x+n|} \\ &\leq \frac{1}{n!\varepsilon}\end{aligned}$$

By ratio test, $L = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, $\sum_{n=0}^{\infty} \frac{1}{n!\varepsilon}$ converges. Then $g(x)$ converges uniformly on D. Because $\sum_{n=0}^{N} \frac{(-1)^n}{n!(x+n)}$ is continuous for $N \in \mathbb{N}$, it defines a continuous function on D for any ε, hence it is also a continuous function on Ω as $\varepsilon \to 0$.

Problem 5

Since $f(t,x)$ is C^∞, $\frac{\partial}{\partial t}\frac{\partial^k f}{\partial x^k}(t,x)$ is continuous for any $(t,x) \in [0,T) \times [a,b]$ and any $k \in \mathbb{N} \cup \{0\}$. For $p < q \in [T-\varepsilon, T)$, $\forall \varepsilon > 0$,

$$\begin{aligned}\left|\frac{\partial^k}{\partial x^k} f(q,x) - \frac{\partial^k}{\partial x^k} f(p,x)\right| &= \left|\int_p^q \frac{d}{dt}\frac{\partial^k}{\partial x^k} f(t,x) dt\right| \\ &\leq |q-p|\, C_k \leq \varepsilon C_k\end{aligned}$$

By cauchy condition, $g(x) = \lim_{t \to T^-} f(t,x)$ exists for every x as it is the case for $k = 0$. Since f is continuous over all order derivatives, and x it independent of t, we also have

$$\begin{aligned}\frac{d^k}{dx^k} g(x) &= \frac{d^k}{dx^k} \lim_{t \to T^-} f(t,x) \\ &= \lim_{t \to T^-} \frac{d^k}{dx^k} f(t,x)\end{aligned}$$

Hence for a fixed k, $\forall \varepsilon > 0$, as $x \to x_0 \in [a,b], t \to T^-$,

$$\left|\frac{d^k}{dx^k} g(x) - \frac{d^k}{dx^k} f(t,x)\right| < \varepsilon$$

$$\left|\frac{d^k}{dx^k} g(x_0) - \frac{d^k}{dx^k} f(t,x_0)\right| < \varepsilon$$

$$\left|\frac{d^k}{dx^k} f(t,x) - \frac{d^k}{dx^k} f(t,x_0)\right| < \varepsilon$$

By triangular inequality, $\left|\frac{d^k}{dx^k} g(x) - \frac{d^k}{dx^k} g(x_0)\right| < 3\varepsilon$. Thus $g(x)$ is C^k. Since it is true for any k, $g(x)$ is C^∞.

Problem 6

(a) In order to show that $\bigcup_{n=1}^\infty S_n \supset C[0,1] - E$, it suffices to show that fix $f \in C[0,1] - E$, $f \in \bigcup_{n=1}^\infty S_n$. For $f \in C[0,1] - E$, there exist $x_0 \in [0,1]$ such that $f'(x_0)$ exists. Let $g(x) = \frac{f(x) - f(x_0)}{x - x_0}$, therefore

$$\left|\lim_{x \to x_0} g(x)\right| = L < +\infty$$

So $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. $|x - x_0| < \delta \implies L - \varepsilon < g(x) < L + \varepsilon$. So for $x \in [0,1], x \in (x_0 - \delta, x_0 + \delta) \backslash x_0$, there exist $n \in \mathbb{N}$ such that $g(x) < L + \varepsilon < n$. If $x = x_0$, then any $n > 0$ could satisfy.

Since f is continuous on a closed interval, it is bounded, therefore $|f(x) - f(x_0)| < M$ for some $M > 0$. So for $x \in [0,1], x \notin (x_0 - \delta, x_0 + \delta)$, there also exist n such that $g(x) < \frac{M}{\delta} < n$, where n depends only on δ. Thus $f \in \bigcup_{n=1}^\infty S_n$

(b) Let f_k be a sequence in S_n converging to some $f \in C[0,1]$. For each k, there exist x_k such that for any $x \in [0,1]$,

$$\left|\frac{f_k(x) - f_k(x_k)}{x - x_k}\right| \leq n$$

By Bolzano-Weiestrass theorem, $\{x_k\}$ has a subsequence converging to x_∞. extract the subsequence $\{x_{k_j}\}$ and $\{f_{k_j}\}$. we got

$$\left|\frac{f(x) - f(x_\infty)}{x - x_\infty}\right| = \lim_{j \to \infty} \left|\frac{f_{k_j}(x) - f_{k_j}(x_{k_j})}{x - x_{k_j}}\right| \leq n$$

Thus $f \in S_n$ and thus S_n is closed.

(c) To show $S_n^o = \phi$, it suffices to show that $C[0,1] - S_n$ is dense. Let $f \in C[0,1]$. Since $C[0,1]$ is compact, f is uniformly continuous on $C[0,1]$, thus $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in [0,1], |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Divide $[0,1]$ in to more than $\frac{1}{\delta}$ intervals n: $[0,x_1],[x_1,x_2],\dots[x_{n-1},x_n]$ such that $|x_i - x_{i+1}| < \delta$. Define $g(x)$ to be $g(x_i) = f(x_i)$, $|g'(x)| > n$ and $|g(x) - f(x)| < \varepsilon/2$. We can make g to be piecewise linear with absolute slope greater than n, and we can arbitrary change the slope sign so that the first and third condition can be satisfied. (like a chainsaw). Therefore $\|f-h\|_{C[0,1]} \leq \varepsilon$, and $g \in C[0,1] - S_n$, thus it is dense and $S_n^o = \phi$.

(d) According to Baire Category theorem, $\overline{C[0,1] - \bigcup_{n=1}^\infty S_n} = C[0,1]$, thus $C[0,1] - \bigcup_{n=1}^\infty S_n$ is dense. since $C[0,1] - \bigcup_{n=1}^\infty S_n \subset E$ by (a), E is also dense.

Problem 7

Let $k \leq \frac{nt}{\delta}$ be the biggest k. Since x_n is continuous in $[0,\delta]$,

$$\begin{aligned}
x_n(t) - x_0 &= \sum_{j=0}^{k-1} \int_{\frac{j\delta}{n}}^{\frac{(j+1)\delta}{n}} x_n'(s)ds + \int_{\frac{k\delta}{n}}^{t} x_n'(s)ds \\
&= \sum_{j=0}^{k-1} \int_{\frac{j\delta}{n}}^{\frac{(j+1)\delta}{n}} F(x_n(\frac{j\delta}{n}))ds + \int_{\frac{k\delta}{n}}^{t} F(x_n(\frac{k\delta}{n}))ds \\
&= \sum_{j=0}^{k-1} \int_{\frac{j\delta}{n}}^{\frac{(j+1)\delta}{n}} F(y_n(s))ds + \int_{\frac{k\delta}{n}}^{t} F(y_n(s))ds \\
&= \int_0^t F(y_n(s))ds
\end{aligned}$$

We prove by induction to show that $|x_n(t)|$ is uniformly bounded by $\max\{|x_0 + r|, |x_0 - r|\}$, which is equivalent to say $|x_n(t) - x_0| \leq r$ for any $n \in \mathbb{N}, t \in [0,\delta]$. Base case: $|x_n(0) - x_0| = 0 < r$. Suppose for any $n \in \mathbb{N}$, $t \leq \frac{j\delta}{n}$, we have $|x_n(t) - x_0| \leq tM$. When $t \in (\frac{j\delta}{n}, \frac{(j+1)\delta}{n}]$, we have

$$x_n(t) = x_n(\frac{j\delta}{n}) + (t - \frac{j\delta}{n})F(x_n(\frac{j\delta}{n}))$$

Since $M := \sup_{x\in[x_0-r,x_0+r]} |F(x)|$ and $\left|x_n(\frac{j\delta}{n}) - x_0\right| \leq \frac{j\delta}{n}M$ by hypothesis, $F(x_n(\frac{j\delta}{n})) \leq M$. Then

$$\begin{aligned}
|x_n(t) - x_0| &= \left|x_n(\frac{j\delta}{n}) + (t - \frac{j\delta}{n})F(x_n(\frac{j\delta}{n})) - x_0\right| \\
&\leq \left|x_n(\frac{j\delta}{n}) - x_0\right| + \left|F(x_n(\frac{j\delta}{n}))(t - \frac{j\delta}{n})\right| \\
&\leq \frac{j\delta}{n}M + M(t - \frac{j\delta}{n}) = tM
\end{aligned}$$

So $x_n(t)$ is uniformly bounded.

By the proof above, $F(x_n(\frac{j\delta}{n})) \leq M$ for any $j = 1,2,\dots,n-1$, therefore $|x_n'(t)|$ is bounded by M. Hence for any $\varepsilon > 0$, there exists $c = \frac{\varepsilon}{2M}$ such that whenever $t,s \in [0,\delta]$ and $|t-s| < c$, then

$$|x_n(t) - x_n(s)| \leq M|t-s| = \varepsilon/2 < \varepsilon$$

Therefore $x_n(t)$ is also equicontinuous on $[0,\delta]$. By Arzela-Ascoli theorem, $\{x_n(t)\}_{n=1}^\infty$ has a converging subsequence.

Problem 8

To prove that there exists a subsequence of f that converges uniformly on every compact sets $K \subset R$, it suffices to prove that this subsequence converges uniformly on $K_n = [-n, n]$ for any $n \in N$. Because any compact sets on R is close and bounded, so it will be a subset of K_n for some n.

For $n = 1$, since $H(x), K(x)$ are a continuous funtions on a closed interval, they are all bounded by $M_h(1), M_k(1)$. Thus $f_k(x), f_k'(x)$ is uniformly bounded on $[-1, 1]$. According to exercise 4.21, $f_k(x)$ is equicontinuous on $[-1, 1]$. By Arzela Ascoli, there exist a subsequence $f_k^1(x)$ that converges uniformly on $[-1, 1]$.

For $n = 2$, consider only $f_k^1(x)$, which is still uniformly bounded and equicontinuous on $[-2, 2]$. Apply A-A again, and we get $f_k^2(x)$ that converges uniformly on $[-2, 2]$.

We proceed with $n \to \infty$. Then we extract $f_1^1, f_2^2, f_3^3, \ldots$ and we claim that this sequence uniformly converges in $[-n, n]$ for any $n \in N$.

Explain: For a specific n, omitting the former of the sequence, starting from n : $f_n^n, f_{n+1}^{n+1} \ldots$. This forms a subsequence of f_k^n, thus it converges uniformly on $[-n, n]$. Hence this sequence uniformly converges on any compact set $K \subset R$.

For every $x \in R$, $x \in [-m, m]$ for some m, thus the sequence converges at x, so it converges pointwise.