Problem 1

We define the sequence of partitions $\{P_k\}_{k=1}^{\infty}$ as following.

$$\delta := \frac{1}{2^{k+2}}, x_i = \frac{1}{i}, i \in N$$
$$P_k := \{x_1 = 1, x_1 - \delta, x_2 + \delta, x_2 - \delta, x_3 + \delta, x_3 - \delta, \ldots, x_k + \delta, x_k - \delta, 0\}, k \in \mathbb{N}$$

Then for a particular partition P_k, we have

$$\begin{aligned}
U(f,P) = \sum_{i=1}^{k-1} & \sup_{x\in[x_{i+1}+\delta, x_i-\delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + \\
& \sum_{i=2}^{k} \sup_{x\in[x_i-\delta, x_i+\delta]} 1/[1/x]2\delta \\
& + \sup_{x\in[1-\delta,1]} 1/[1/x]\delta + \sup_{x\in[0,x_k-\delta]} 1/[1/x](x_k - \delta) \\
& = \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_{i-1} + \delta + x_{k+1}(x_k - \delta) \\
& = \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1} \right) + \delta + x_{k+1}(x_k - \delta) \\
& = \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + \delta + x_{k+1}(x_k - \delta)
\end{aligned}$$

and

$$\begin{aligned}
L(f,P) = \sum_{i=1}^{k-1} & \inf_{x\in[x_{i+1}+\delta, x_i-\delta]} 1/[1/x](x_i - x_{i+1} - 2\delta) + \\
& \sum_{i=2}^{k} \inf_{x\in[x_i-\delta, x_i+\delta]} 1/[1/x]2\delta \\
& + \inf_{x\in[1-\delta,1]} 1/[1/x]\delta + \inf_{x\in[0,x_k-\delta]} 1/[1/x](x_k - \delta) \\
& = \sum_{i=1}^{k-1} x_i(x_i - x_{i+1} - 2\delta) + 2\delta \sum_{i=2}^{k} x_i + \delta + 0 \\
& = \sum_{i=1}^{k-1} \left(\frac{1}{i^2} - \frac{1}{i} + \frac{1}{i+1} \right) + 2\delta(x_k - 1) + \delta \\
& = \sum_{i=1}^{k-1} \frac{1}{i^2} - 1 + \frac{1}{k} + 2\delta(x_k - 1) + \delta
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, Letting $k \to \infty$ and we got

$$L(f,P_k) = \frac{\pi^2}{6} - 1 = U(f,P_k)$$