Gale-Shapley Algorithm

Shusen Wang

Example 1

Inputs

Ada > Becky > Cindy		Alex		Ada	Bob > Alex > Chris	
Becky > Ada > Cindy		Bob		Becky	Alex > Bob > Chris	
Ada > Becky > Cindy		Chris		Cindy	Alex > Bob > Chris	

Iteration 1

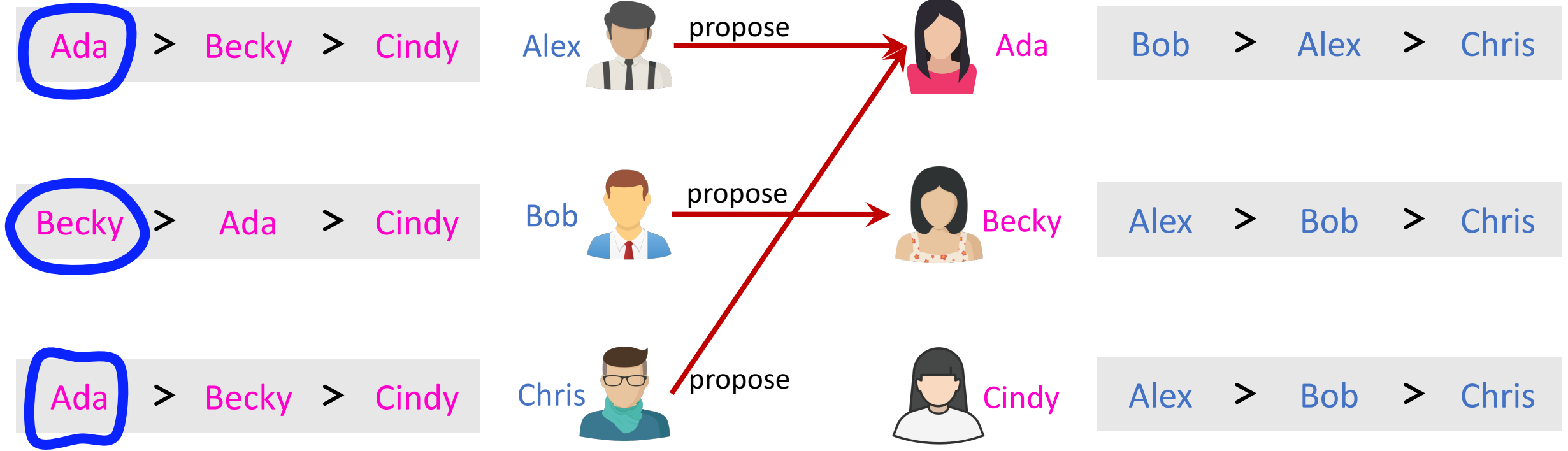| Ada > Becky > Cindy | Alex | Ada | Bob > Alex > Chris |
| Becky > Ada > Cindy | Bob | Becky | Alex > Bob > Chris |
| Ada > Becky > Cindy | Chris | Cindy | Alex > Bob > Chris |

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Iteration 1

Ada > Becky > Cindy   Alex —propose→ Ada   Bob > Alex > Chris

Becky > Ada > Cindy   Bob —propose→ Becky   Alex > Bob > Chris

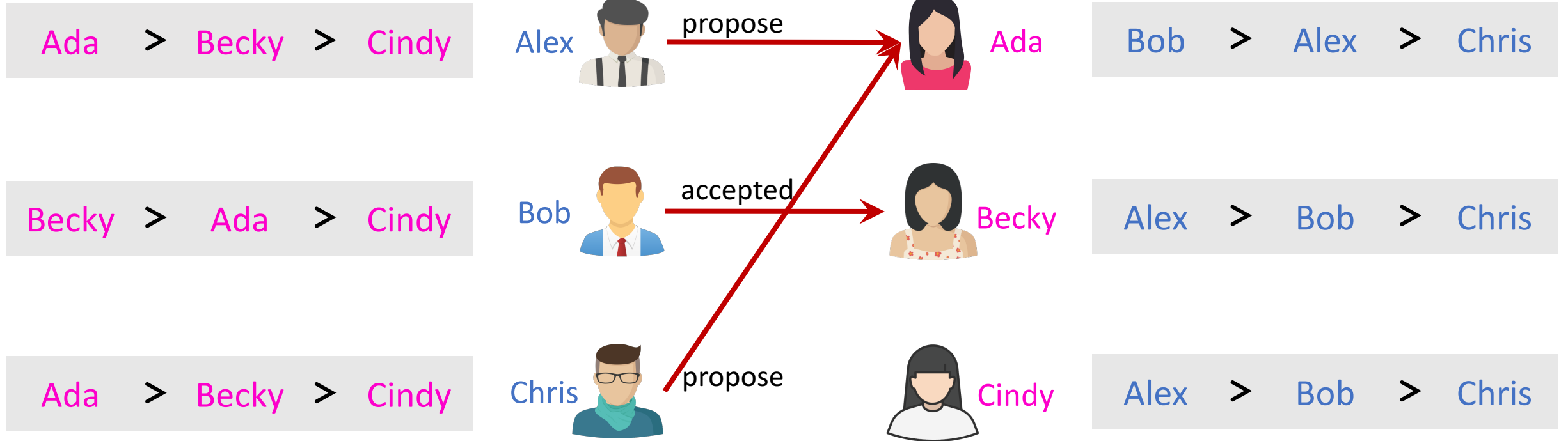Ada > Becky > Cindy   Chris —propose→   Cindy   Alex > Bob > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Iteration 1

Ada	>	Becky	>	Cindy

Alex → propose → Ada

Bob	>	Alex	>	Chris

Becky	>	Ada	>	Cindy

Bob → accepted → Becky

Alex	>	Bob	>	Chris

Ada	>	Becky	>	Cindy

Chris → propose → Ada

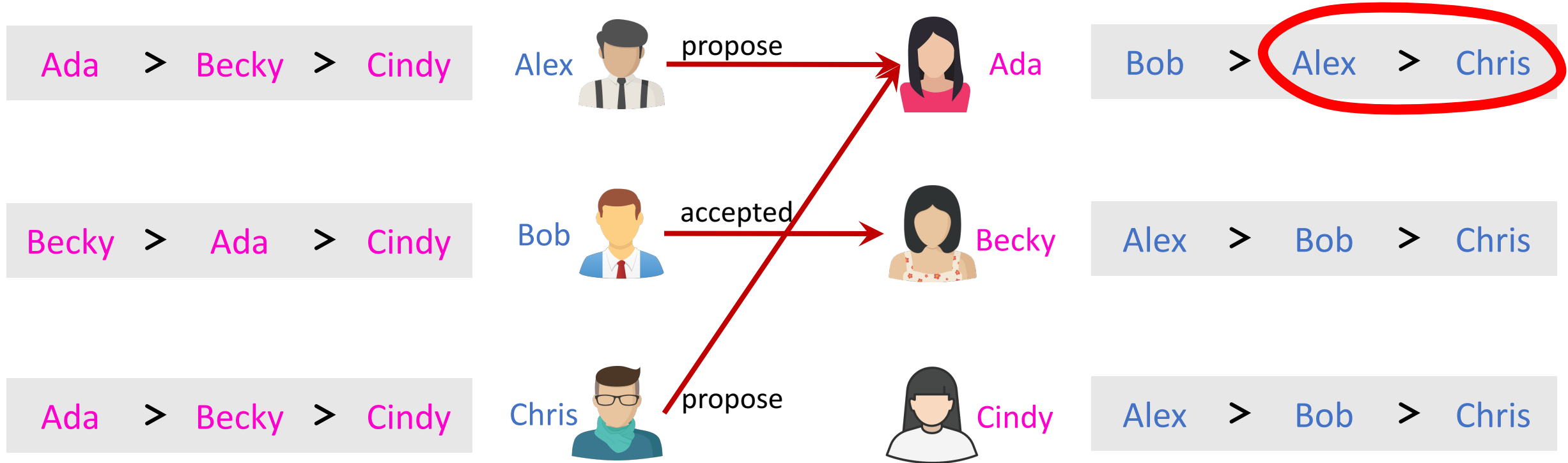Cindy

Alex	>	Bob	>	Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 1

| Ada | > | Becky | > | Cindy | | Alex | propose ⟶ | Ada | | Bob | > | Alex | > | Chris |

Ada > Becky > Cindy   Alex [propose] ⟶ Ada   Bob > Alex > Chris

Becky > Ada > Cindy   Bob [accepted] ⟶ Becky   Alex > Bob > Chris

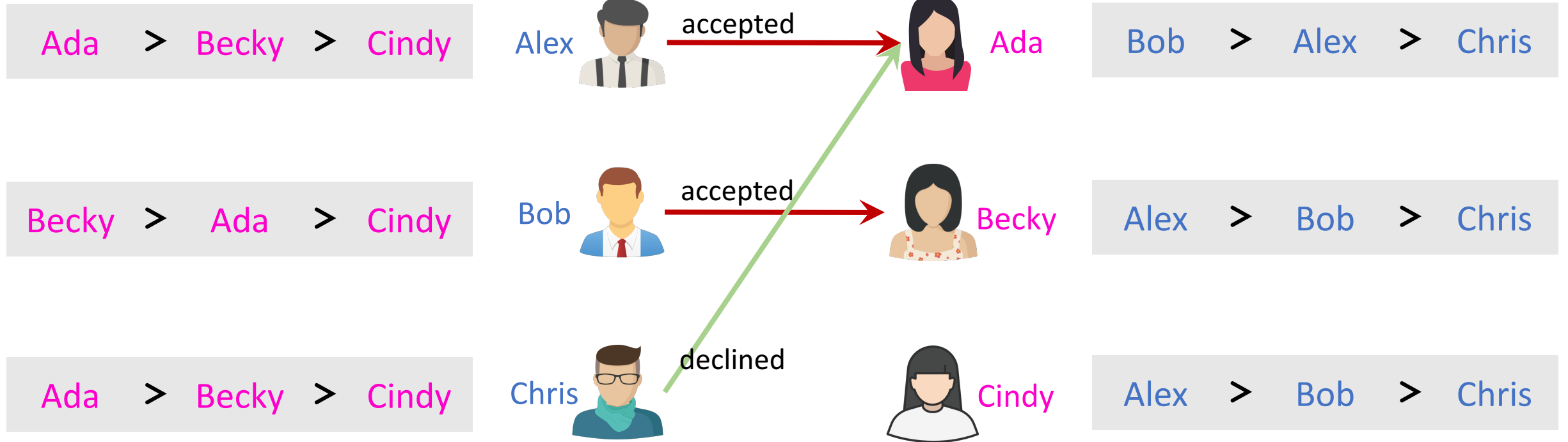Ada > Becky > Cindy   Chris [propose] ⟶ Cindy   Alex > Bob > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 1

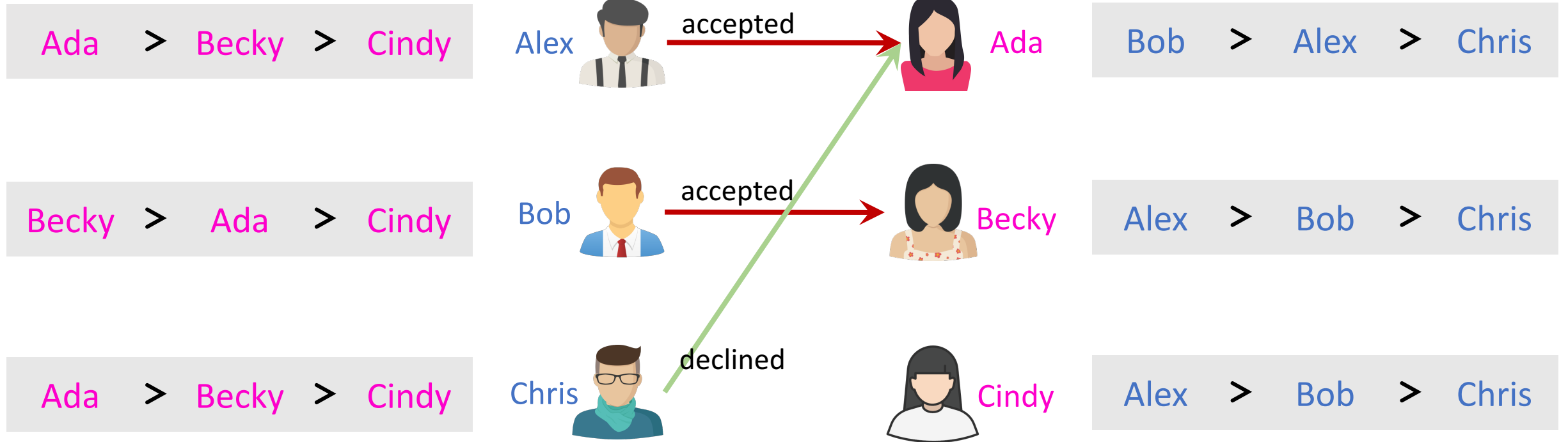|				
Ada > Becky > Cindy	Alex → **accepted** → Ada		Bob > Alex > Chris
Becky > Ada > Cindy	Bob → **accepted** → Becky		Alex > Bob > Chris
Ada > Becky > Cindy	Chris → **declined**	Cindy	Alex > Bob > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 1

Ada > Becky > Cindy

Alex — accepted → Ada

Bob > Alex > Chris

Becky > Ada > Cindy

Bob — accepted → Becky

Alex > Bob > Chris

Ada > Becky > Cindy

Chris — declined →

Cindy

Alex > Bob > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.
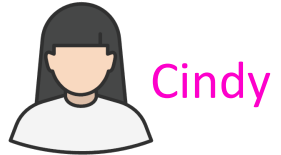
Iteration 1

~~Ada~~ > Becky > Cindy   Alex ❤ Ada   Bob > Alex > Chris
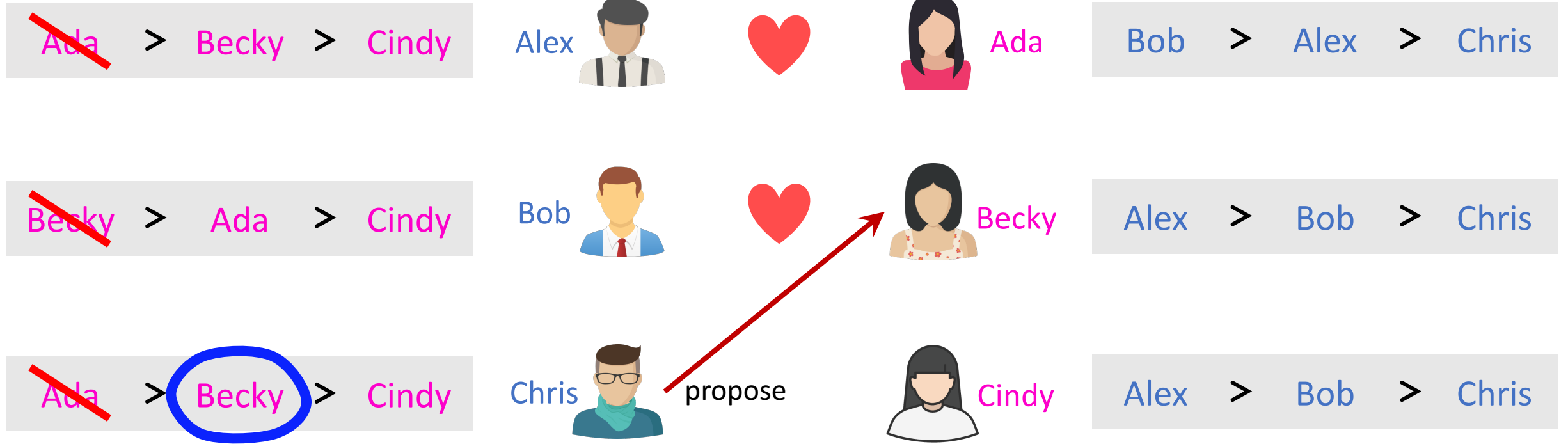
~~Becky~~ > Ada > Cindy   Bob ❤ Becky   Alex > Bob > Chris

~~Ada~~ > Becky > Cindy   Chris   Cindy   Alex > Bob > Chris

Iteration 2

Ada > Becky > Cindy   Alex ❤️ Ada   Bob > Alex > Chris

Becky > Ada > Cindy   Bob ❤️ Becky   Alex > Bob > Chris
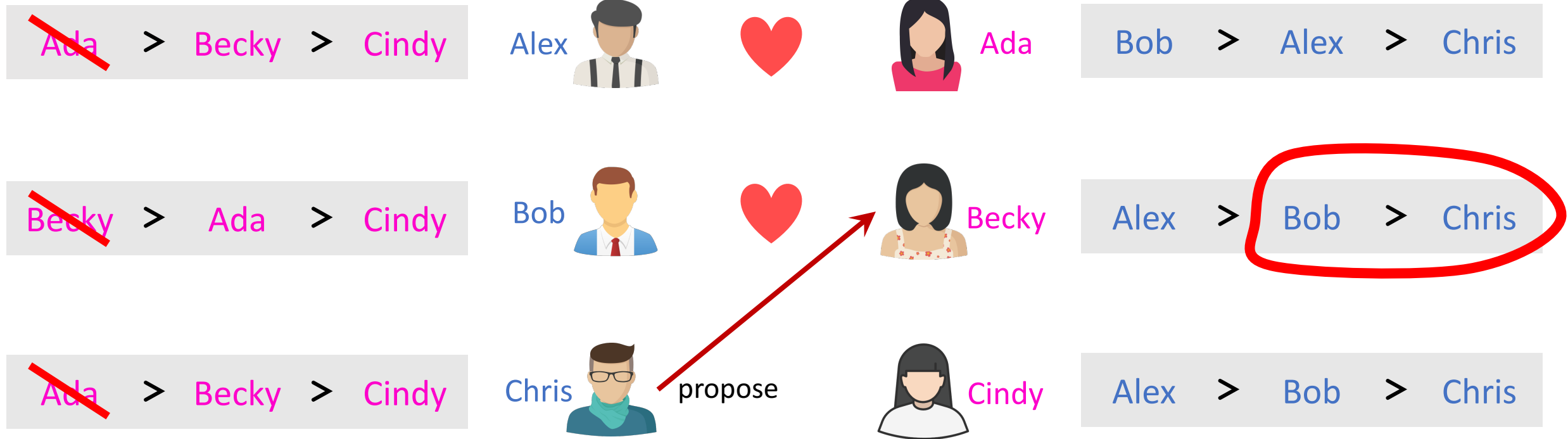
Ada > Becky > Cindy   Chris —propose→ Cindy   Alex > Bob > Chris

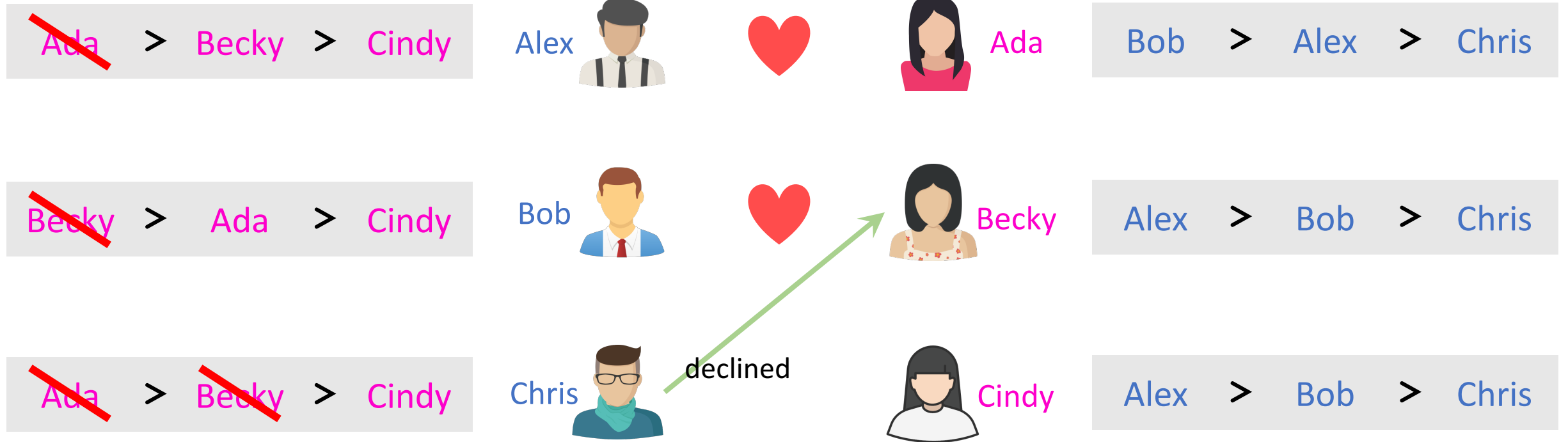**Step 1:** An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to.*

Iteration 2

~~Ada~~ > Becky > Cindy   Alex ❤️ Ada   Bob > Alex > Chris

~~Becky~~ > Ada > Cindy   Bob ❤️ Becky   Alex > Bob > Chris

~~Ada~~ > Becky > Cindy   Chris propose Cindy   Alex > Bob > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 2

~~Ada~~ > Becky > Cindy   Alex ❤️ Ada   Bob > Alex > Chris

~~Becky~~ > Ada > Cindy   Bob ❤️ Becky   Alex > Bob > Chris

~~Ada~~ > ~~Becky~~ > Cindy   Chris — declined → Becky   Cindy   Alex > Bob > Chris
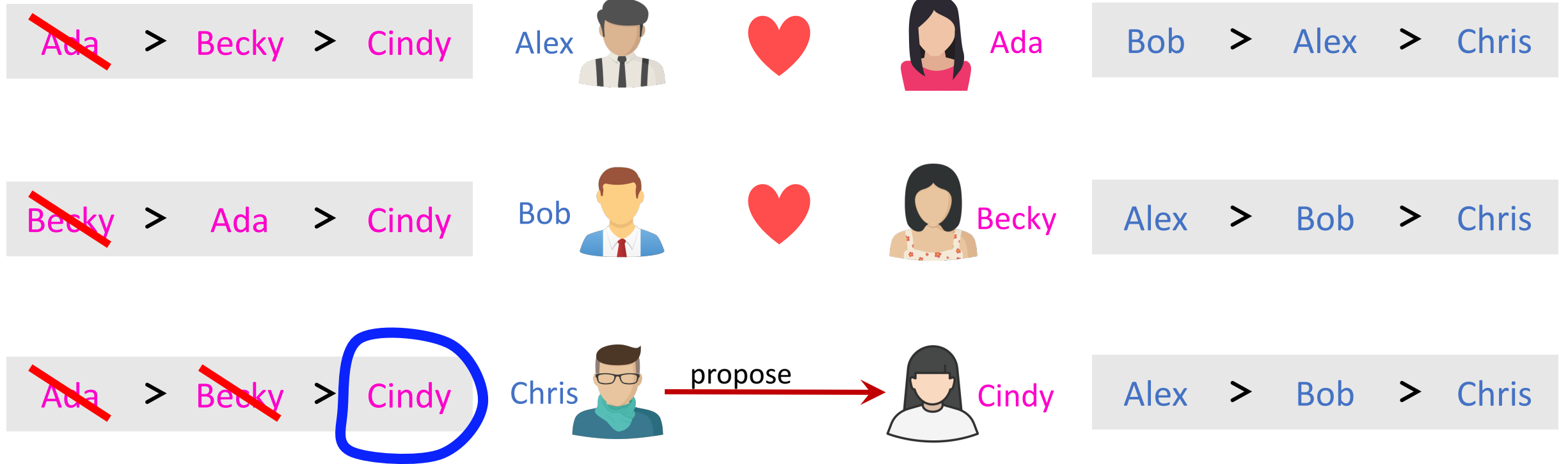
Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 3

Ada > Becky > Cindy

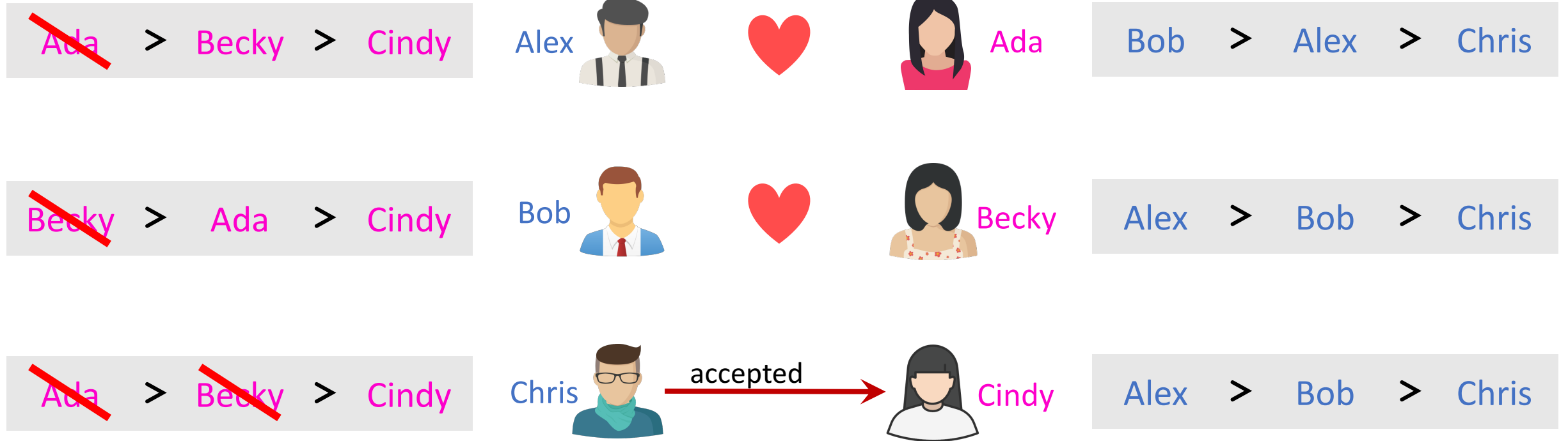Alex ♥ Ada   Bob > Alex > Chris

Becky > Ada > Cindy

Bob ♥ Becky   Alex > Bob > Chris

Ada > Becky > Cindy

Chris —propose→ Cindy   Alex > Bob > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred* among those he *has not proposed to*.

Iteration 3

~~Ada~~ > Becky > Cindy   Alex ❤️ Ada   Bob > Alex > Chris

~~Becky~~ > Ada > Cindy   Bob ❤️ Becky   Alex > Bob > Chris

~~Ada~~ > ~~Becky~~ > Cindy   Chris —accepted→ Cindy   Alex > Bob > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

End of Procedure

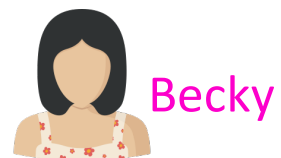~~Ada~~ > Becky > Cindy   Alex ❤ Ada   Bob > Alex > Chris

~~Becky~~ > Ada > Cindy   Bob ❤ Becky   Alex > Bob > Chris

~~Ada~~ > ~~Becky~~ > ~~Cindy~~   Chris ❤ Cindy   Alex > Bob > Chris

Stop if everyone is married.

Another Example

Inputs

Ada	>	Diana	>	Cindy	>	Becky	Alex
Ada	>	Becky	>	Cindy	>	Diana	Bob
Becky	>	Diana	>	Cindy	>	Ada	Chris
Cindy	>	Ada	>	Becky	>	Diana	David

Ada	Chris	>	Alex	>	Bob	>	David
Becky	Alex	>	David	>	Chris	>	Bob
Cindy	Bob	>	David	>	Chris	>	Alex
Diana	David	>	Bob	>	Alex	>	Chris

Iteration 1

Ada	>	Diana	>	Cindy	>	Becky		Alex →	Ada	Chris > Alex > Bob > David

Ada > Diana > Cindy > Becky   Alex   Ada   Chris > Alex > Bob > David

Ada > Becky > Cindy > Diana   Bob   Becky   Alex > David > Chris > Bob

Becky > Diana > Cindy > Ada   Chris   Cindy   Bob > David > Chris > Alex
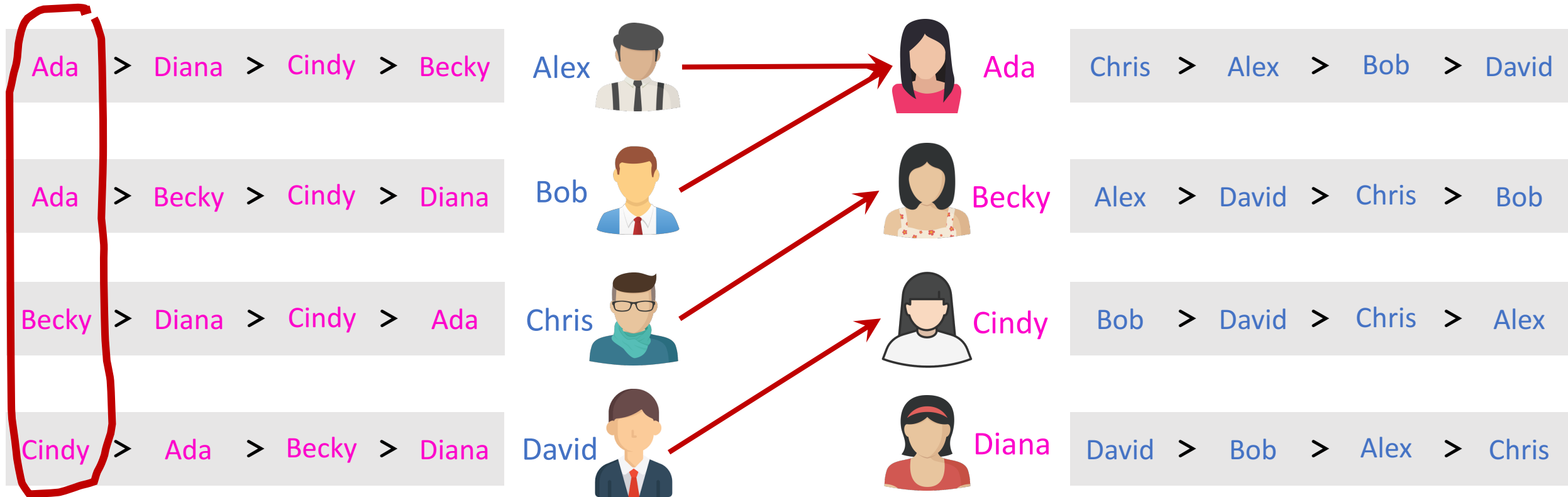
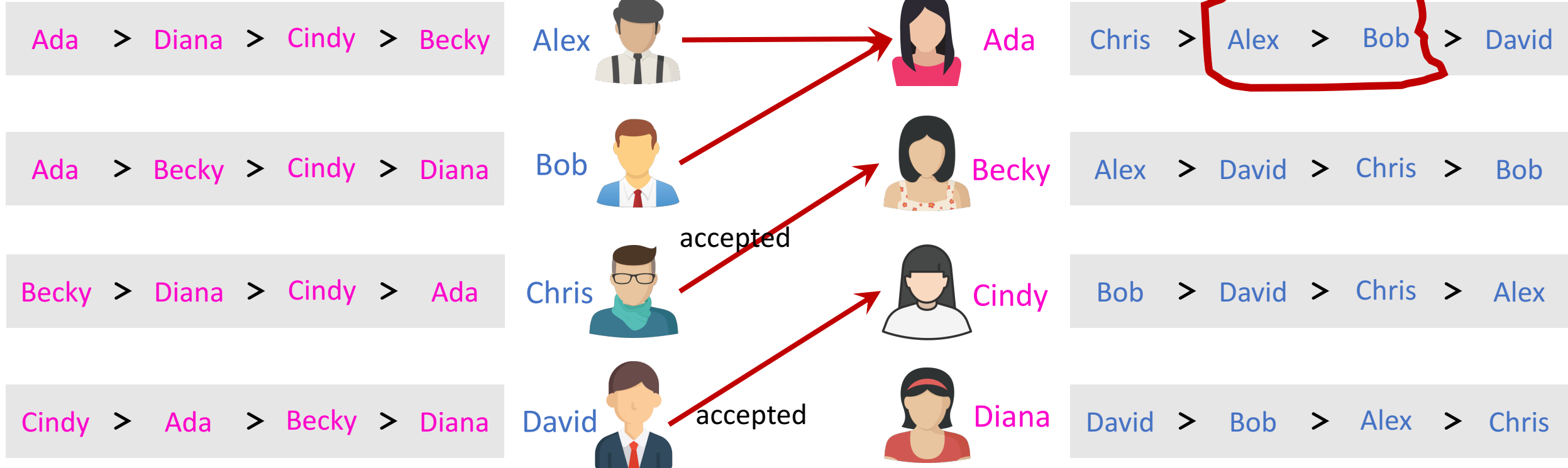Cindy > Ada > Becky > Diana   David   Diana   David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to.*

Iteration 1
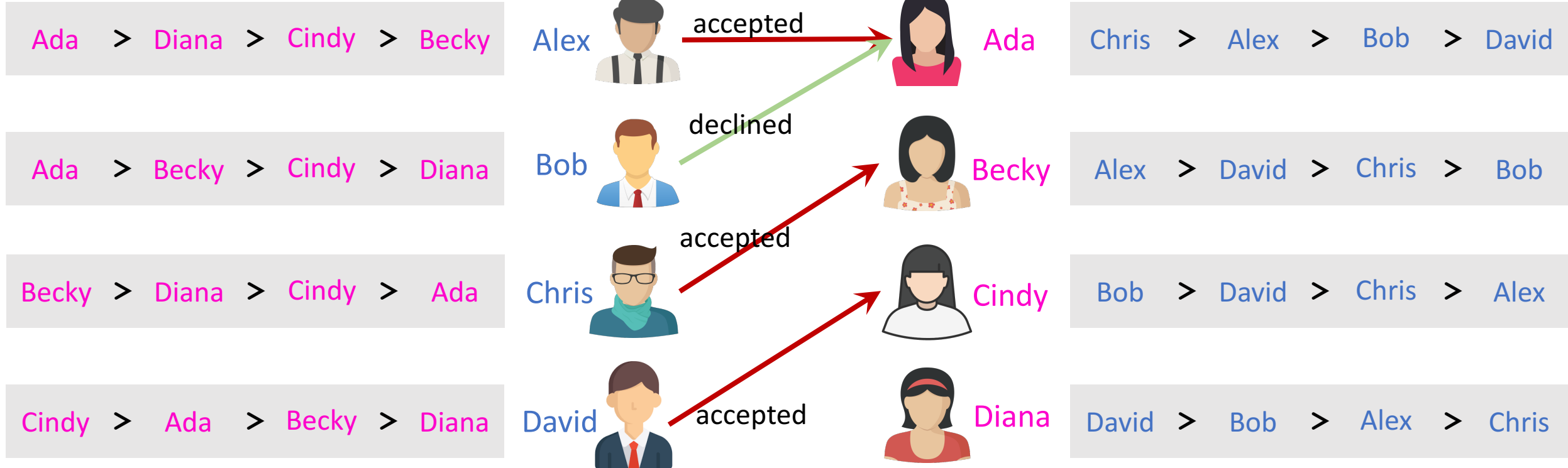
Ada	>	Diana	>	Cindy	>	Becky	Alex

Alex

Ada

Chris	>	Alex	>	Bob	>	David

Ada	>	Becky	>	Cindy	>	Diana	Bob

Bob

Becky

Alex	>	David	>	Chris	>	Bob

accepted

Becky	>	Diana	>	Cindy	>	Ada	Chris

Chris

Cindy

Bob	>	David	>	Chris	>	Alex

Cindy	>	Ada	>	Becky	>	Diana	David

David

accepted

Diana

David	>	Bob	>	Alex	>	Chris

Step 2: A woman accepts her best offer.
• If the woman finds better matching than her current spouse, then divorce.

Iteration 1

Ada > Diana > Cindy > Becky   Alex — accepted → Ada   Chris > Alex > Bob > David

Ada > Becky > Cindy > Diana   Bob — declined → Becky   Alex > David > Chris > Bob

Becky > Diana > Cindy > Ada   Chris — accepted → Cindy   Bob > David > Chris > Alex

Cindy > Ada > Becky > Diana   David — accepted → Diana   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 1

	Men's Preferences				Women's Preferences		
Alex	Ada > Diana > Cindy > Becky			Ada	Chris > Alex > Bob > David		
Chris	Becky > Diana > Cindy > Ada			Becky	Alex > David > Chris > Bob		
David	Cindy > Ada > Becky > Diana			Cindy	Bob > David > Chris > Alex		
Bob	Ada > Becky > Cindy > Diana			Diana	David > Bob > Alex > Chris		

Iteration 1

~~Ada~~ > Diana > Cindy > Becky   Alex   ❤️   Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris   ❤️   Becky   Alex > David > Chris > Bob

~~Cindy~~ > Ada > Becky > Diana   David   ❤️   Cindy   Bob > David > Chris > Alex

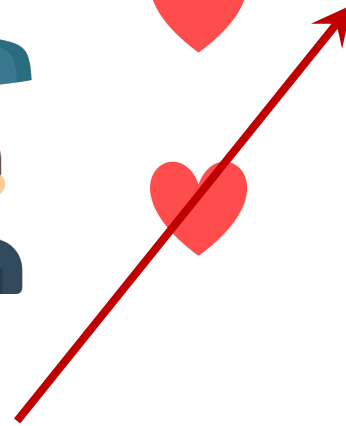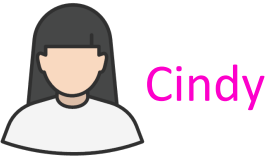~~Ada~~ > Becky > Cindy > Diana   Bob   Diana   David > Bob > Alex > Chris

Iteration 2

~~Ada~~ > Diana > Cindy > Becky   Alex ❤️ Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris ❤️ Becky   Alex > David > Chris > Bob

~~Cindy~~ > Ada > Becky > Diana   David ❤️ Cindy   Bob > David > Chris > Alex

~~Ada~~ > Becky > Cindy > Diana   Bob Diana   David > Bob > Alex > Chris

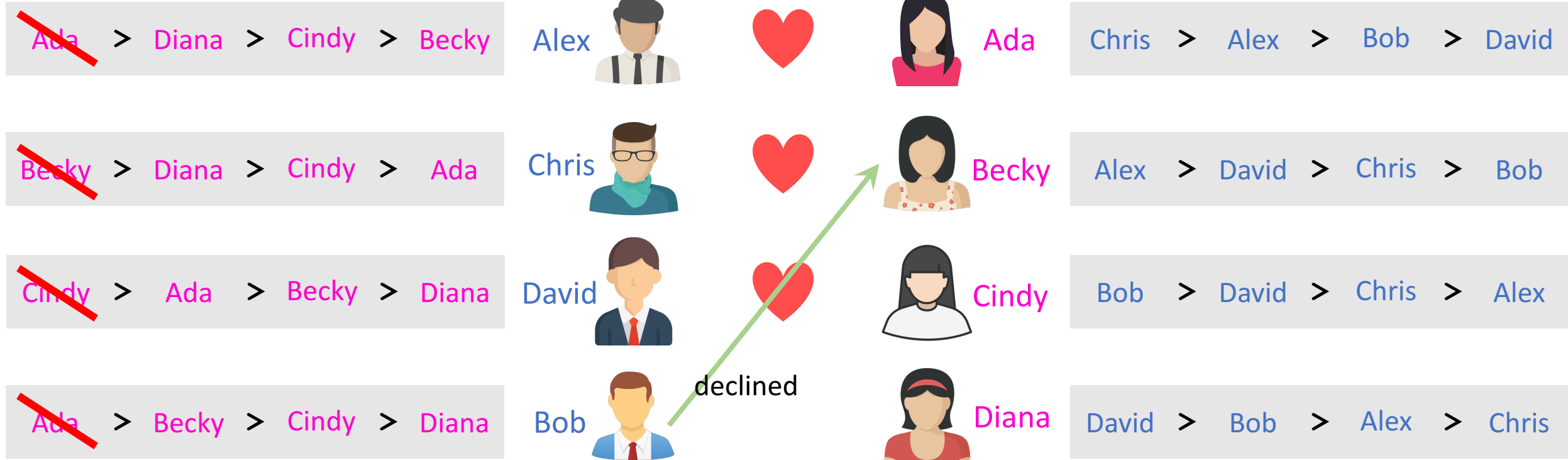**Step 1:** An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to.*

Iteration 2

~~Ada~~ > Diana > Cindy > Becky   Alex ❤️ Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris ❤️ Becky   Alex > David > **Chris > Bob**

~~Cindy~~ > Ada > Becky > Diana   David ❤️ Cindy   Bob > David > Chris > Alex

~~Ada~~ > Becky > Cindy > Diana   Bob   Diana   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 2

Alex: ~~Ada~~ > Diana > Cindy > Becky
Chris: ~~Becky~~ > Diana > Cindy > Ada
David: ~~Cindy~~ > Ada > Becky > Diana
Bob: ~~Ada~~ > Becky > Cindy > Diana

Ada: Chris > Alex > Bob > David
Becky: Alex > David > Chris > Bob
Cindy: Bob > David > Chris > Alex
Diana: David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
• If the woman finds better matching than her current spouse, then divorce.

Iteration 2

	Alex ❤ Ada	
~~Ada~~ > Diana > Cindy > Becky		Chris > Alex > Bob > David

	Chris ❤ Becky	
~~Becky~~ > Diana > Cindy > Ada		Alex > David > Chris > Bob

	David ❤ Cindy	
~~Cindy~~ > Ada > Becky > Diana		Bob > David > Chris > Alex

	Bob ❤ Diana	
~~Ada~~ > ~~Becky~~ > Cindy > Diana		David > Bob > Alex > Chris

Iteration 3

~~Ada~~ > Diana > Cindy > Becky   Alex ❤️ Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris ❤️ Becky   Alex > David > Chris > Bob

~~Cindy~~ > Ada > Becky > Diana   David ❤️ Cindy   Bob > David > Chris > Alex

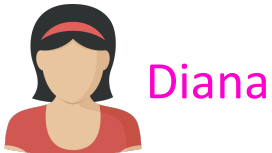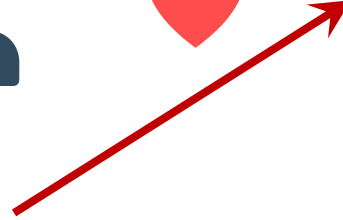~~Ada~~ > ~~Becky~~ > Cindy > Diana   Bob → Diana   David > Bob > Alex > Chris

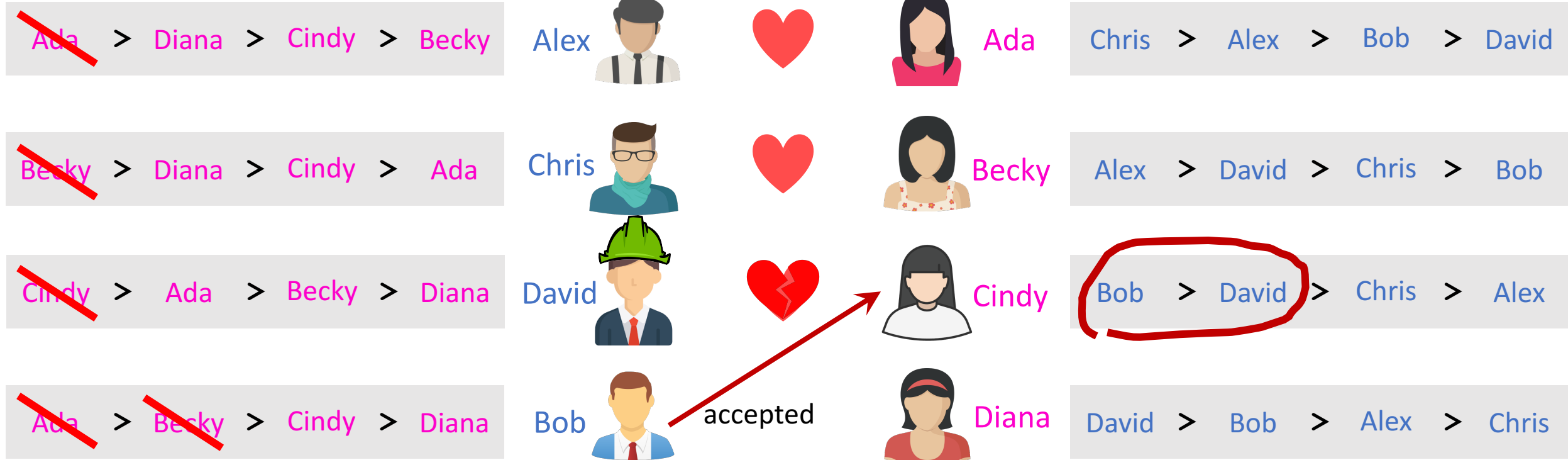**Step 1:** An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to.*

Iteration 3

~~Ada~~ > Diana > Cindy > Becky   **Alex** 🤎 **Ada**   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   **Chris** 🤎 **Becky**   Alex > David > Chris > Bob

~~Cindy~~ > Ada > Becky > Diana   **David** 🤎 **Cindy**   Bob > David > Chris > Alex

~~Ada~~ > ~~Becky~~ > Cindy > Diana   **Bob**   **Diana**   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 3

~~Ada~~ > Diana > Cindy > Becky   **Alex**   ❤️   **Ada**   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   **Chris**   ❤️   **Becky**   Alex > David > Chris > Bob

~~Cindy~~ > Ada > Becky > Diana   **David**   💔   **Cindy**   Bob > David > Chris > Alex

~~Ada~~ > ~~Becky~~ > Cindy > Diana   **Bob**   *accepted* →   **Diana**   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 3

| ~~Ada~~ | > | Diana | > | Cindy | > | Becky | Alex | ❤️ | Ada | Chris | > | Alex | > | Bob | > | David |

| ~~Becky~~ | > | Diana | > | Cindy | > | Ada | Chris | ❤️ | Becky | Alex | > | David | > | Chris | > | Bob |

| ~~Ada~~ | > | ~~Becky~~ | > | ~~Cindy~~ | > | Diana | Bob | ❤️ | Cindy | Bob | > | David | > | Chris | > | Alex |

| ~~Cindy~~ | > | Ada | > | Becky | > | Diana | David | | Diana | David | > | Bob | > | Alex | > | Chris |

Iteration 4

~~Ada~~ > Diana > Cindy > Becky   Alex   ❤️   Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris   ❤️   Becky   Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana   Bob   ❤️   Cindy   Bob > David > Chris > Alex

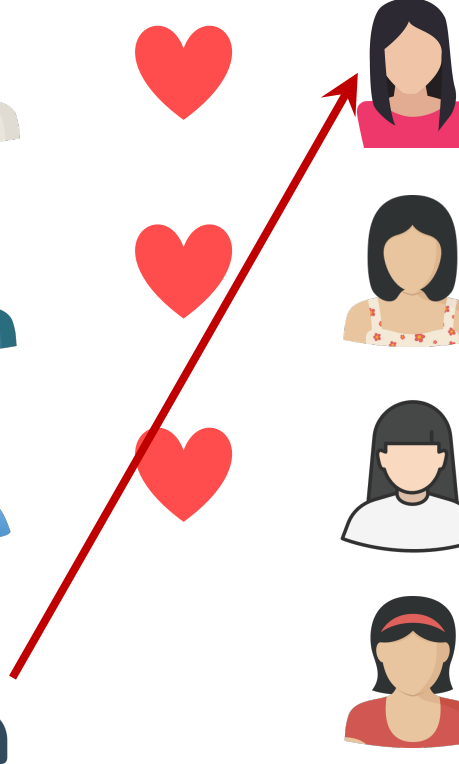~~Cindy~~ > (Ada) > Becky > Diana   David   Diana   David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to.*

Iteration 4

~~Ada~~ > Diana > Cindy > Becky	Alex	❤️	Ada	Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada	Chris	❤️	Becky	Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana	Bob	❤️	Cindy	Bob > David > Chris > Alex

~~Cindy~~ > Ada > Becky > Diana	David	declined	Diana	David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 4

	Alex	❤️	Ada	
~~Ada~~ > Diana > Cindy > Becky				Chris > Alex > Bob > David

	Chris	❤️	Becky	
~~Becky~~ > Diana > Cindy > Ada				Alex > David > Chris > Bob

	Bob	❤️	Cindy	
~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana				Bob > David > Chris > Alex

	David		Diana	
~~Cindy~~ > ~~Ada~~ > Becky > Diana				David > Bob > Alex > Chris

Iteration 5

~~Ada~~ > Diana > Cindy > Becky   **Alex** 💗 **Ada**   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   **Chris** 💗 **Becky**   Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana   **Bob** 💗 **Cindy**   Bob > David > Chris > Alex

~~Cindy~~ > ~~Ada~~ > Becky > Diana   **David** **Diana**   David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Iteration 5

~~Ada~~ > Diana > Cindy > Becky   Alex 💗 Ada   Chris > Alex > Bob > David

~~Becky~~ > Diana > Cindy > Ada   Chris 💗 Becky   Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana   Bob 💗 Cindy   Bob > David > Chris > Alex

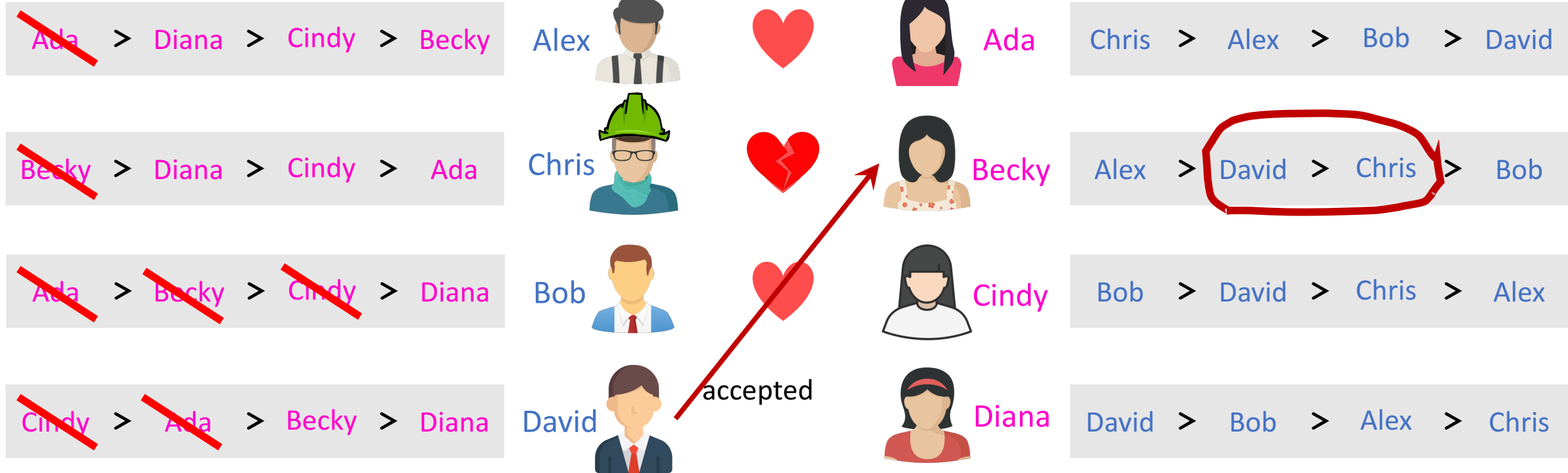~~Cindy~~ > ~~Ada~~ > Becky > Diana   David Diana   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

Iteration 5

~~Ada~~	>	Diana	>	Cindy	>	Becky	

Alex ❤️ **Ada**

Chris > Alex > Bob > David

~~Becky~~	>	Diana	>	Cindy	>	Ada	

Chris 💔 **Becky**

Alex > David > Chris > Bob

~~Ada~~	>	~~Becky~~	>	~~Cindy~~	>	Diana	

Bob ❤️ **Cindy**

Bob > David > Chris > Alex

~~Cindy~~	>	~~Ada~~	>	Becky	>	Diana	

David → accepted → **Diana**

David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
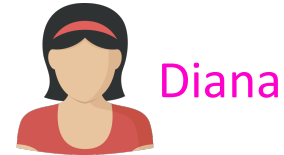- If the woman finds better matching than her current spouse, then divorce.

Iteration 5

~~Ada~~ > Diana > Cindy > Becky	Alex ❤️ Ada	Chris > Alex > Bob > David
~~Cindy~~ > ~~Ada~~ > ~~Becky~~ > Diana	David ❤️ Becky	Alex > David > Chris > Bob
~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana	Bob ❤️ Cindy	Bob > David > Chris > Alex
~~Becky~~ > Diana > Cindy > Ada	Chris Diana	David > Bob > Alex > Chris

Iteration 6

~~Ada~~ > Diana > Cindy > Becky **Alex** ❤️ **Ada** Chris > Alex > Bob > David

~~Cindy~~ > ~~Ada~~ > ~~Becky~~ > Diana **David** ❤️ **Becky** Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana **Bob** ❤️ **Cindy** Bob > David > Chris > Alex

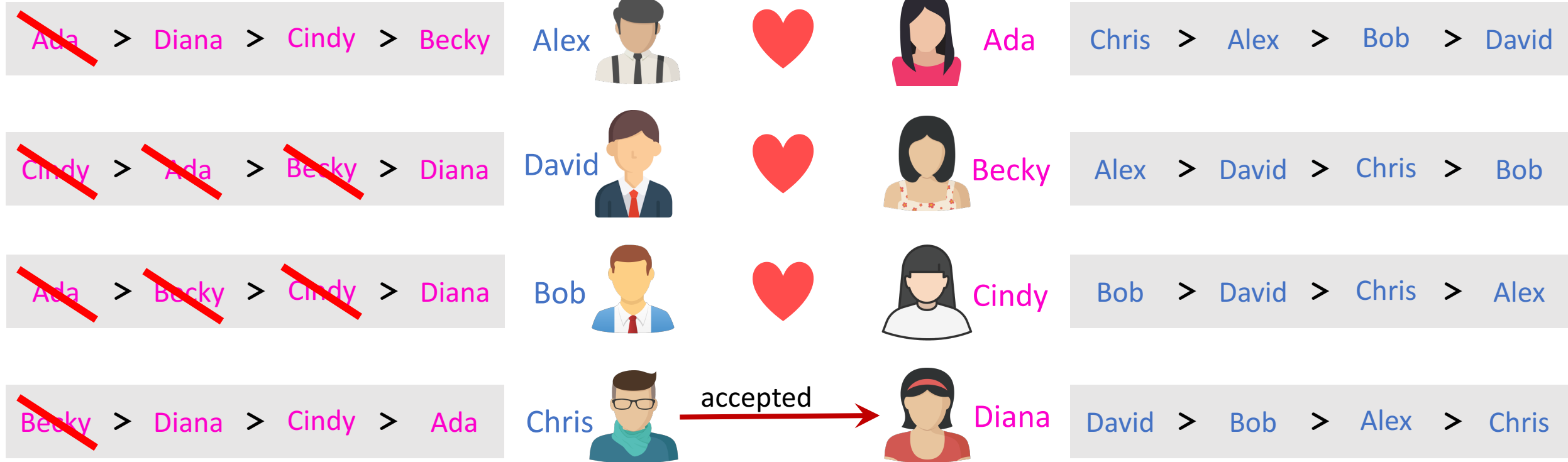~~Becky~~ > Diana > Cindy > Ada **Chris** ⟶ **Diana** David > Bob > Alex > Chris

Step 1: An unmarried man proposes to the woman who is *his most preferred among those he has not proposed to*.

Iteration 6

~~Ada~~ > Diana > Cindy > Becky   **Alex** ❤️ **Ada**   Chris > Alex > Bob > David

~~Cindy~~ > ~~Ada~~ > ~~Becky~~ > Diana   **David** ❤️ **Becky**   Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana   **Bob** ❤️ **Cindy**   Bob > David > Chris > Alex

~~Becky~~ > Diana > Cindy > Ada   **Chris** --accepted--> **Diana**   David > Bob > Alex > Chris

Step 2: A woman accepts her best offer.
- If the woman finds better matching than her current spouse, then divorce.

End of Procedure

~~Ada~~ > Diana > Cindy > Becky   Alex ❤️ Ada   Chris > Alex > Bob > David

~~Cindy~~ > ~~Ada~~ > ~~Becky~~ > Diana   David ❤️ Becky   Alex > David > Chris > Bob

~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana   Bob ❤️ Cindy   Bob > David > Chris > Alex

~~Becky~~ > ~~Diana~~ > Cindy > Ada   Chris ❤️ Diana   David > Bob > Alex > Chris

Stop if everyone is married.

Summary

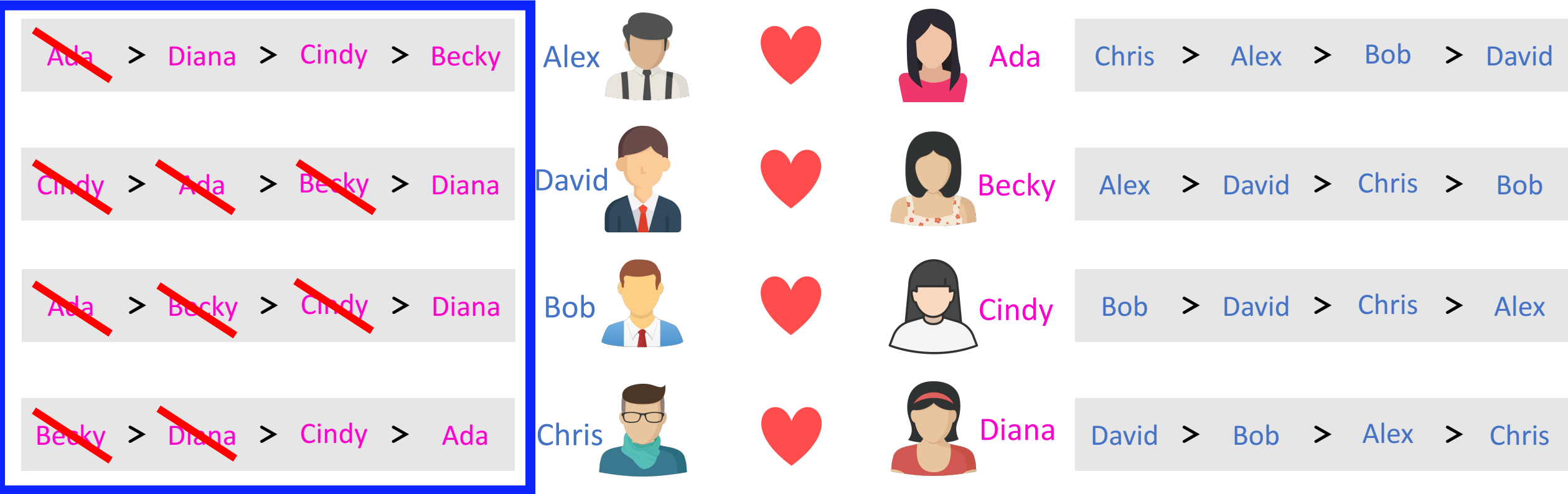# Gale-Shapley Algorithm

1. Every unmarried man proposes to a woman who is his most preferred among those he has not proposed to.

Gale-Shapley Algorithm

1. Every unmarried man proposes to a woman who is his most preferred among those he has not proposed to.

2. A woman marries her most preferred suitor.

   - A bad suitor is better than no suitor.

   - If the suitor is better than her current husband, then she will divorce.

3. Stop if everyone is married; otherwise, repeat Steps 1 and 2.

Time Complexity: $O(n^2)$

Alex ❤ Ada — Chris > Alex > Bob > David

David ❤ Becky — Alex > David > Chris > Bob

Bob ❤ Cindy — Bob > David > Chris > Alex

Chris ❤ Diana — David > Bob > Alex > Chris

Alex: ~~Ada~~ > Diana > Cindy > Becky

David: ~~Cindy~~ > ~~Ada~~ > ~~Becky~~ > Diana

Bob: ~~Ada~~ > ~~Becky~~ > ~~Cindy~~ > Diana

Chris: ~~Becky~~ > ~~Diana~~ > Cindy > Ada

- The total number of proposal is at most $O(n^2)$.
- A woman takes $O(1)$ time to decide whether to accept or decline.

Thank You!