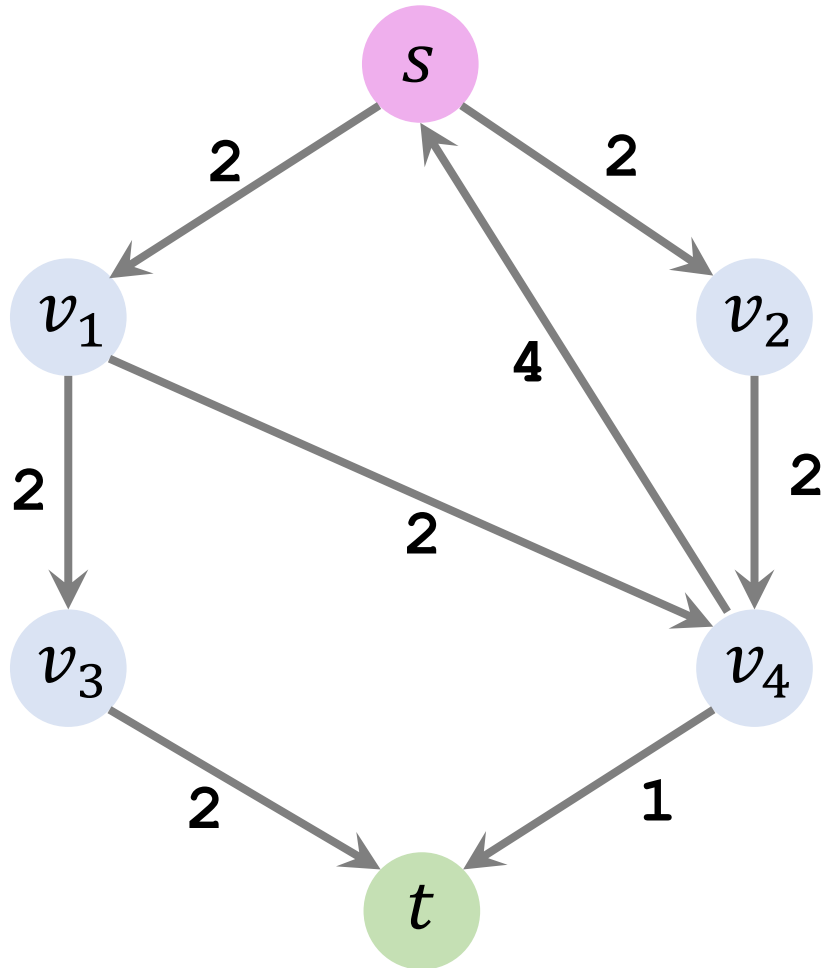


Minimum Cut Problem

Shusen Wang

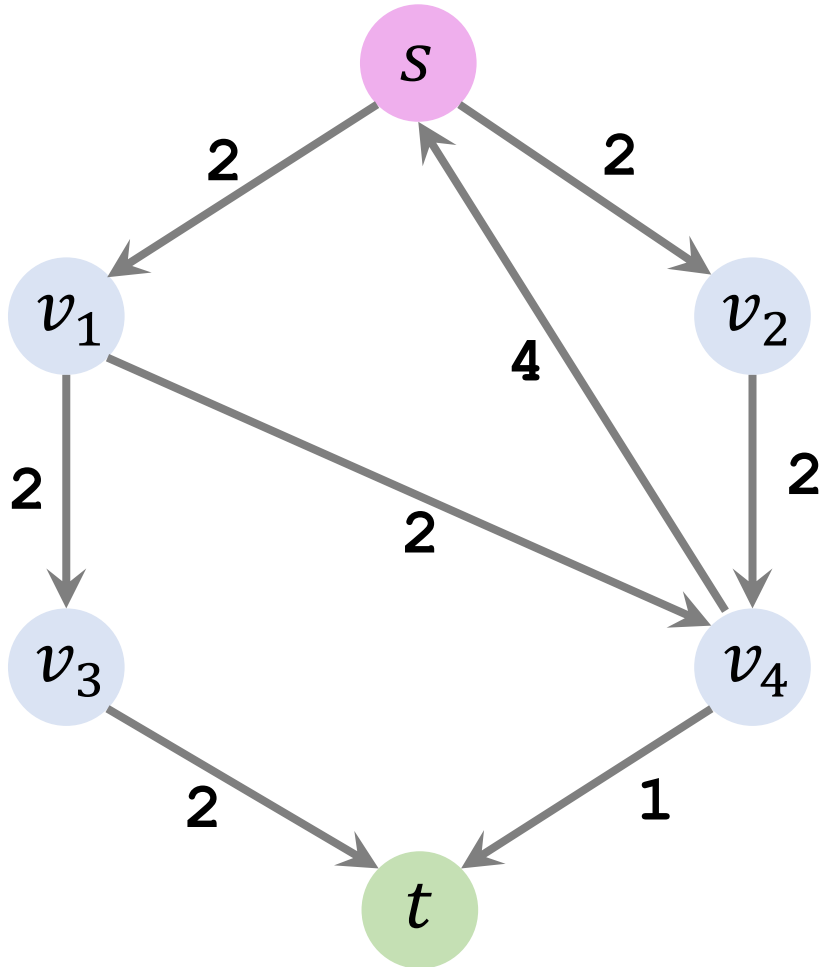
<http://wangshusen.github.io/>

Inputs



- Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.
- Source: $s \in \mathcal{V}$.
- Sink: $t \in \mathcal{V}$.

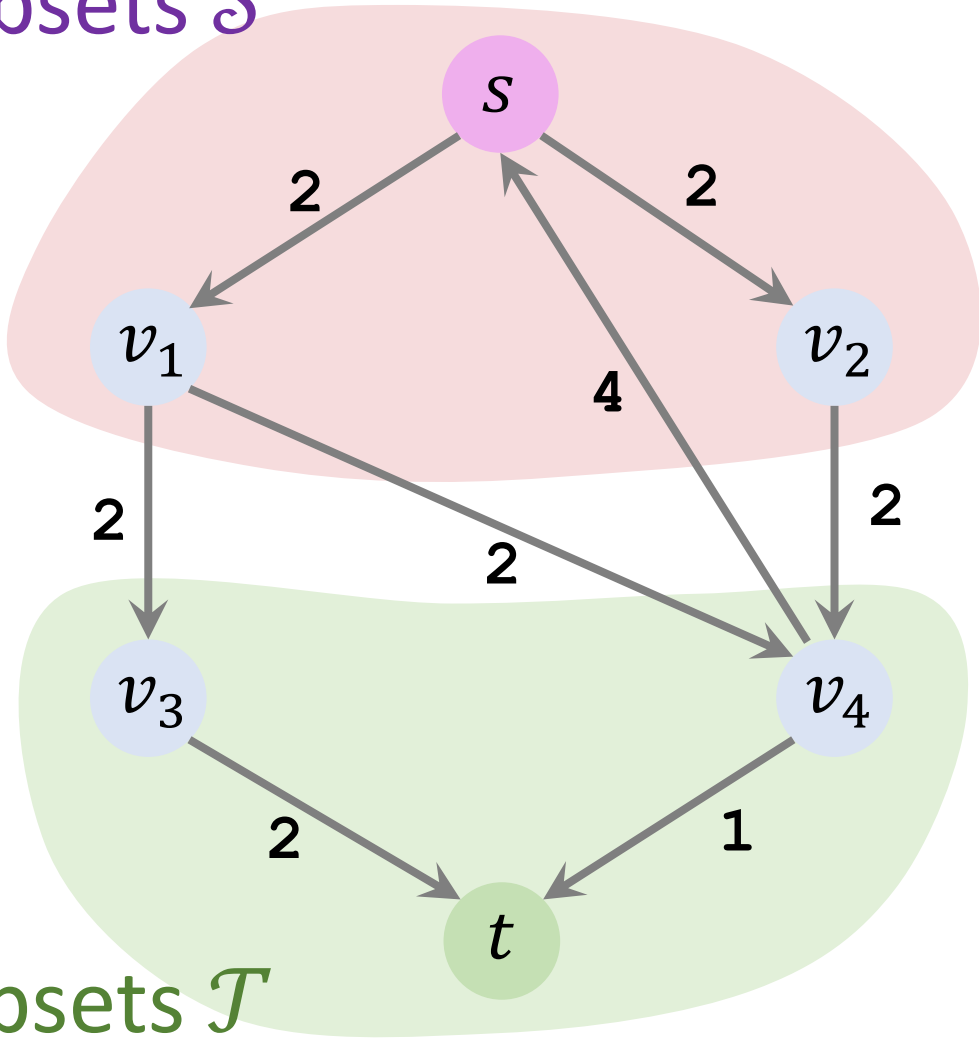
S-T Cut



- Split \mathcal{V} into two subsets: \mathcal{S} and \mathcal{T} .
 - $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.
 - $s \in \mathcal{S}$ and $t \in \mathcal{T}$.
- The pair $(\mathcal{S}, \mathcal{T})$ is called **s-t cut**.

S-T Cut

Subsets \mathcal{S}

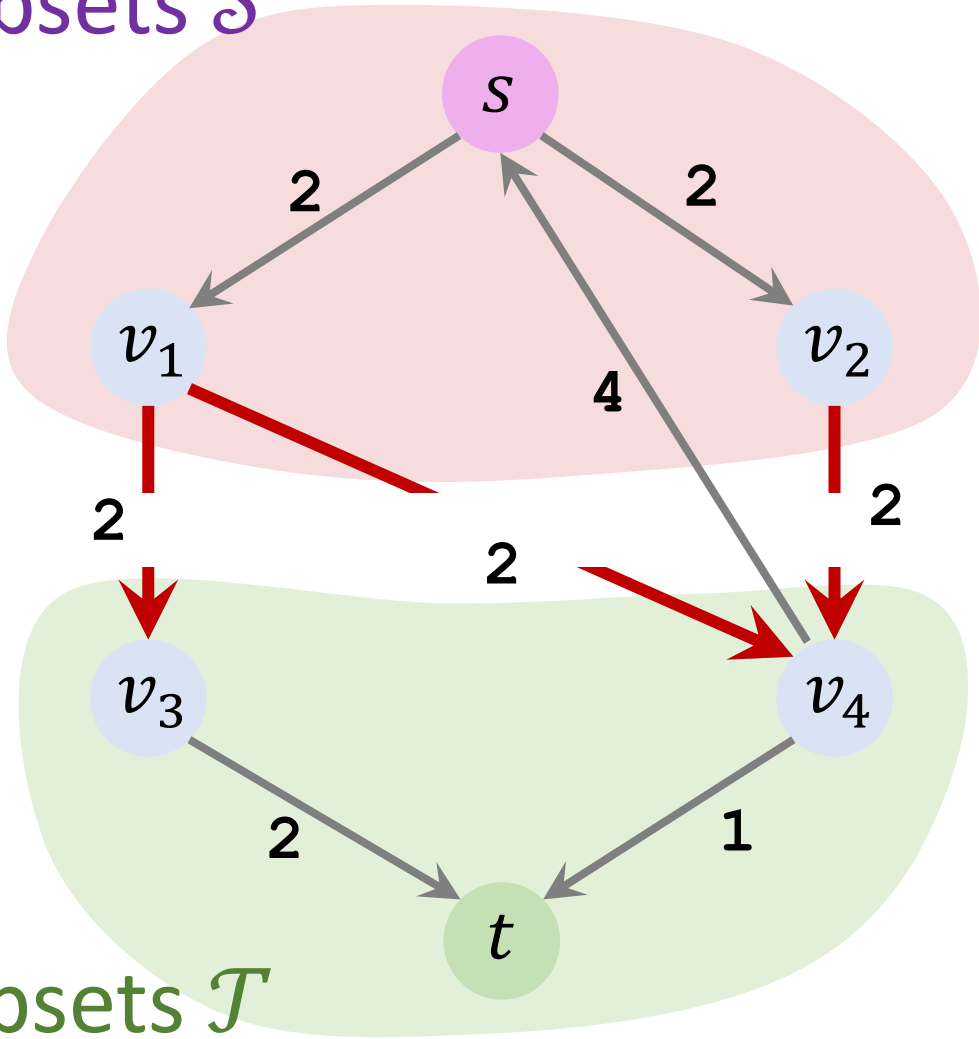


Subsets \mathcal{T}

- Split \mathcal{V} into two subsets: \mathcal{S} and \mathcal{T} .
 - $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.
 - $s \in \mathcal{S}$ and $t \in \mathcal{T}$.
- The pair $(\mathcal{S}, \mathcal{T})$ is called **s-t cut**.

S-T Cut

Subsets \mathcal{S}

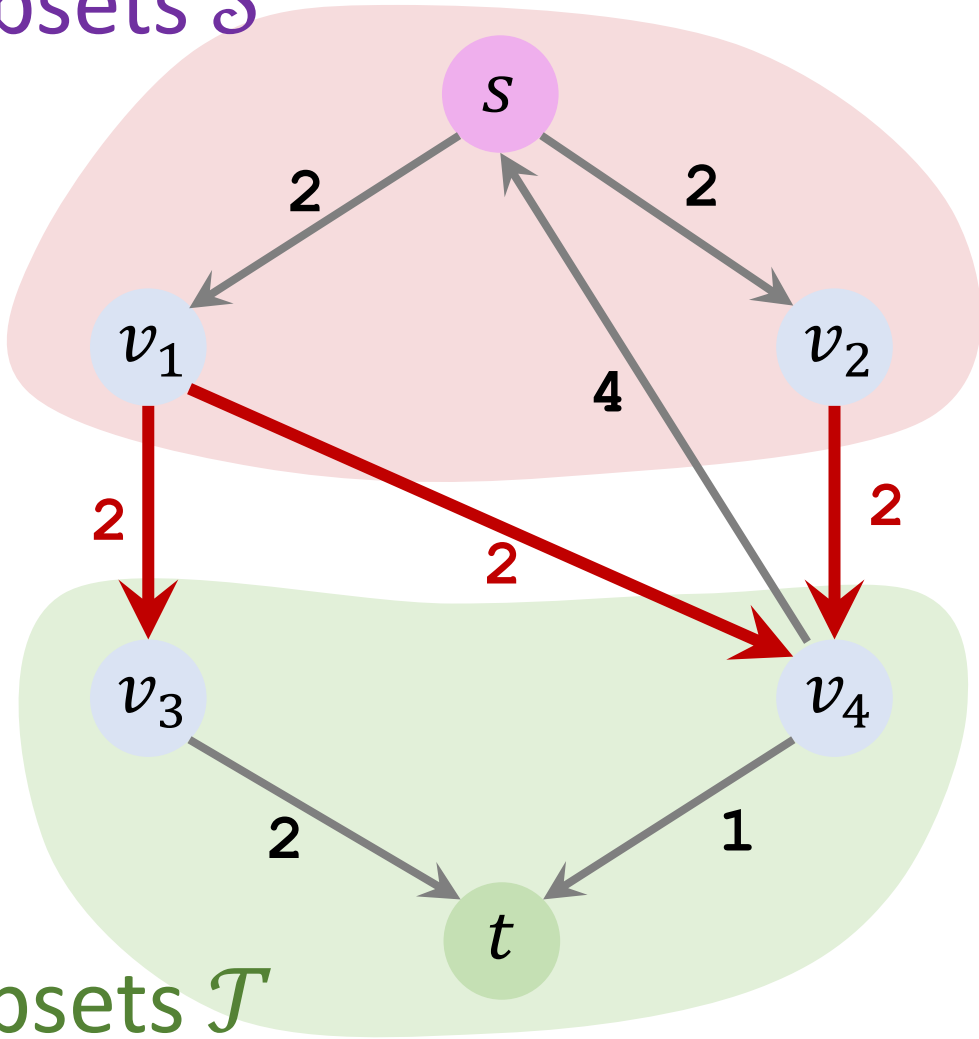


Subsets \mathcal{T}

- Split \mathcal{V} into two subsets: \mathcal{S} and \mathcal{T} .
 - $\mathcal{S} \cup \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \emptyset$.
 - $s \in \mathcal{S}$ and $t \in \mathcal{T}$.
- The pair $(\mathcal{S}, \mathcal{T})$ is called **s-t cut**.

S-T Cut

Subsets \mathcal{S}

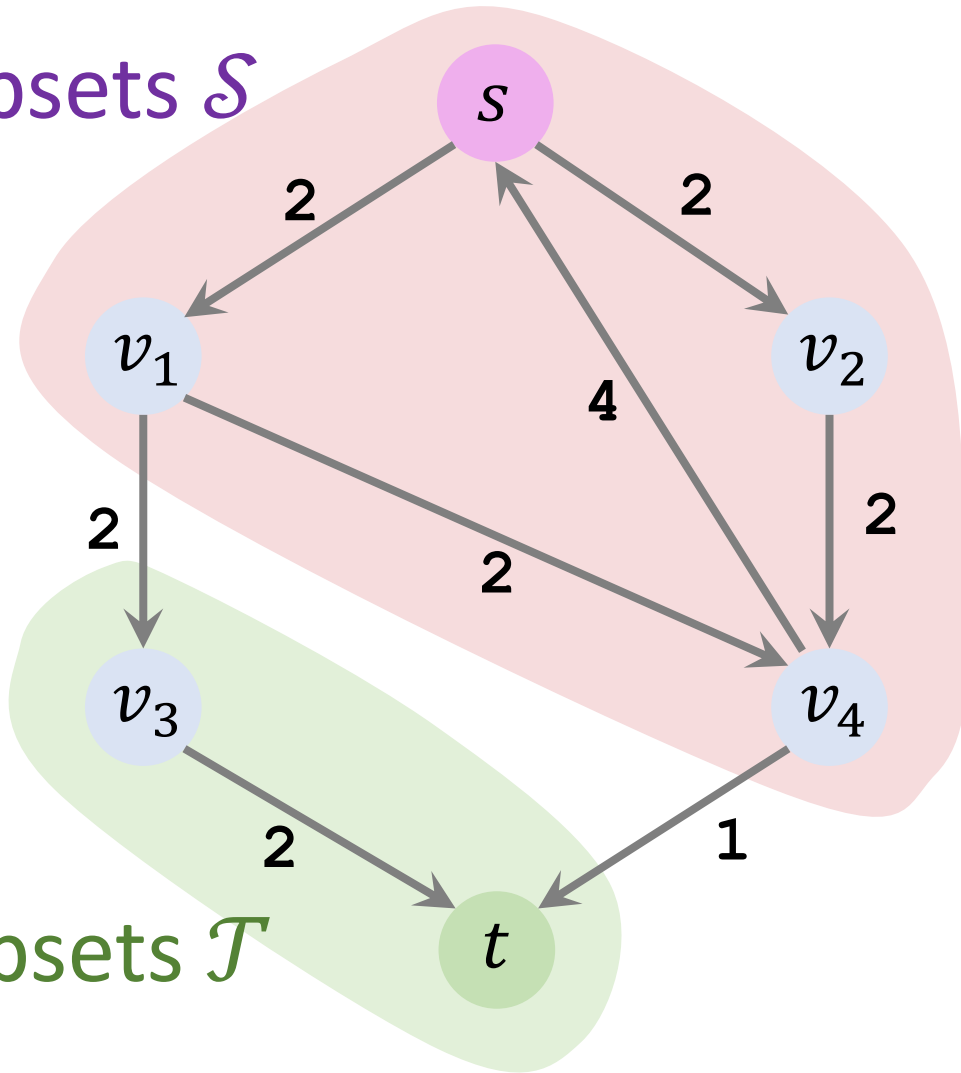


Subsets \mathcal{T}

- $\text{Capacity}(\mathcal{S}, \mathcal{T})$ = sum of weights of edges leaving \mathcal{S} .
- In the figure, three edges leave \mathcal{S} .
- $\text{Capacity}(\mathcal{S}, \mathcal{T}) = \underline{2 + 2 + 2 = 6}$.

S-T Cut: Another Example

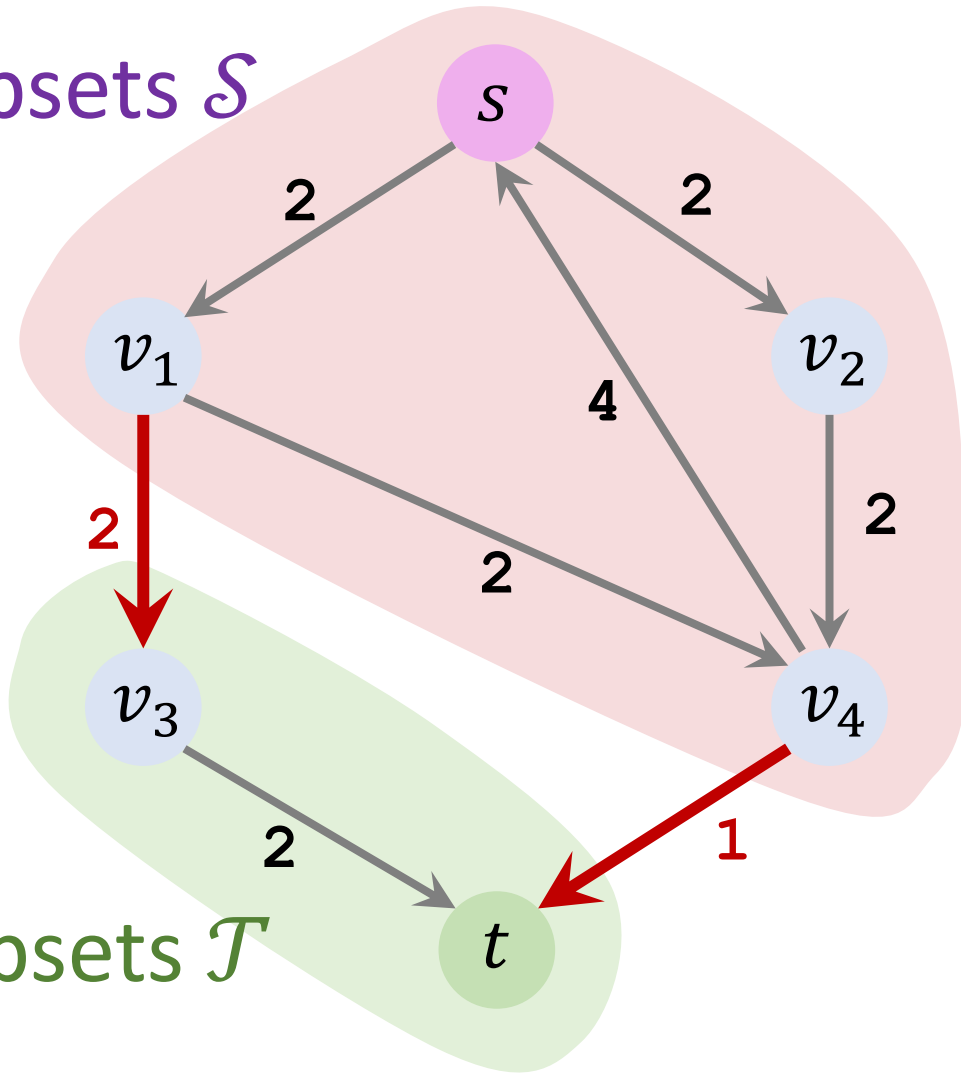
Subsets \mathcal{S}



- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.
- Subset $\mathcal{T} = \{t, v_3\}$.

S-T Cut: Another Example

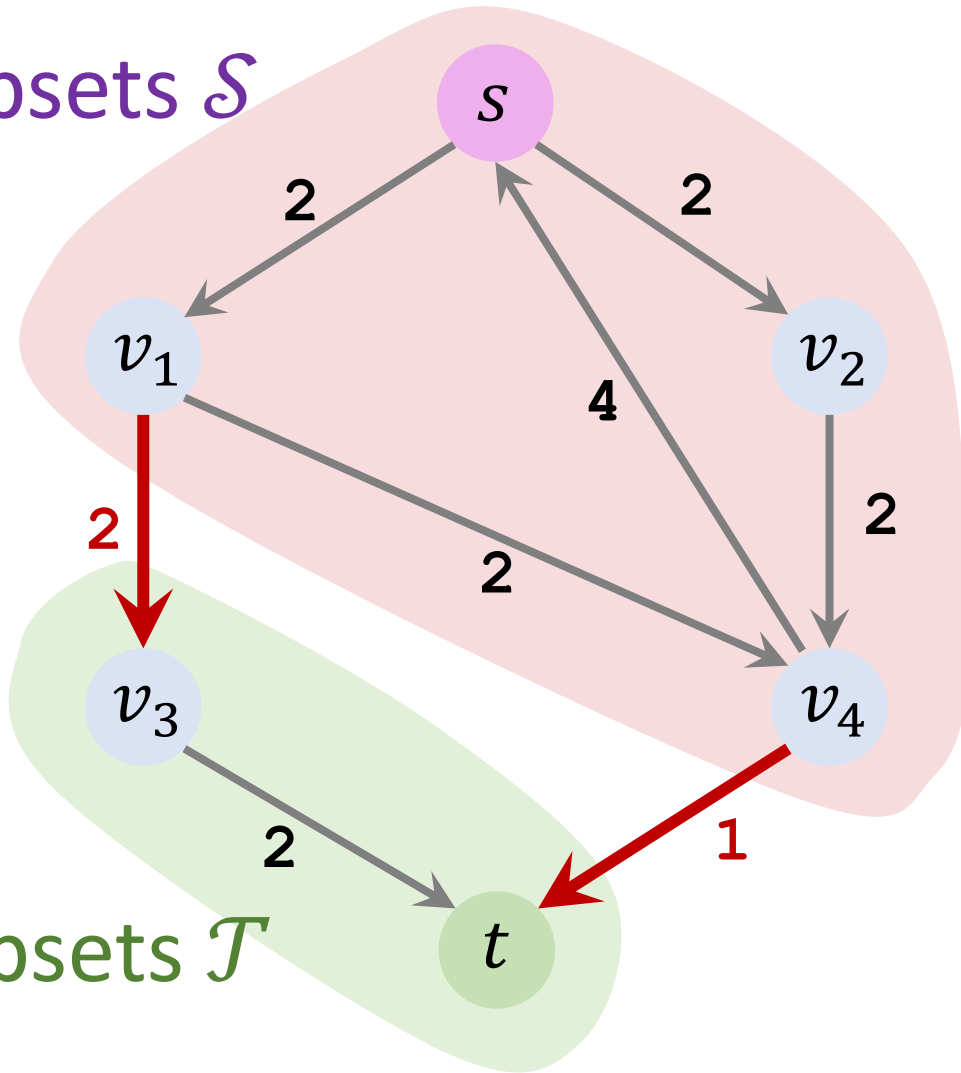
Subsets \mathcal{S}



- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.
- Subset $\mathcal{T} = \{t, v_3\}$.
- In the figure, two edges leave \mathcal{S} .
- $\text{Capacity}(\mathcal{S}, \mathcal{T}) = \underline{\underline{2 + 1 = 3}}$.

Minimum S-T Cut Problem

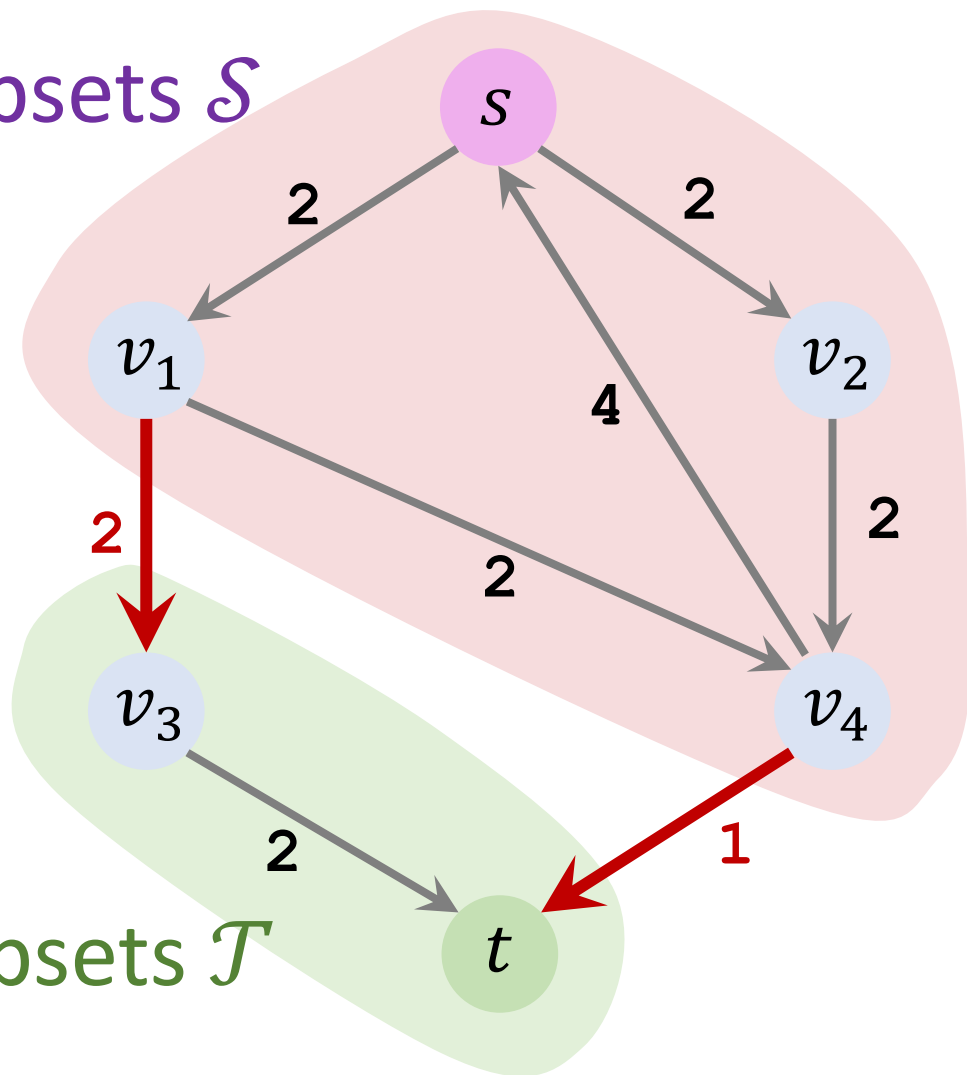
Subsets \mathcal{S}



Definition: Min-Cut

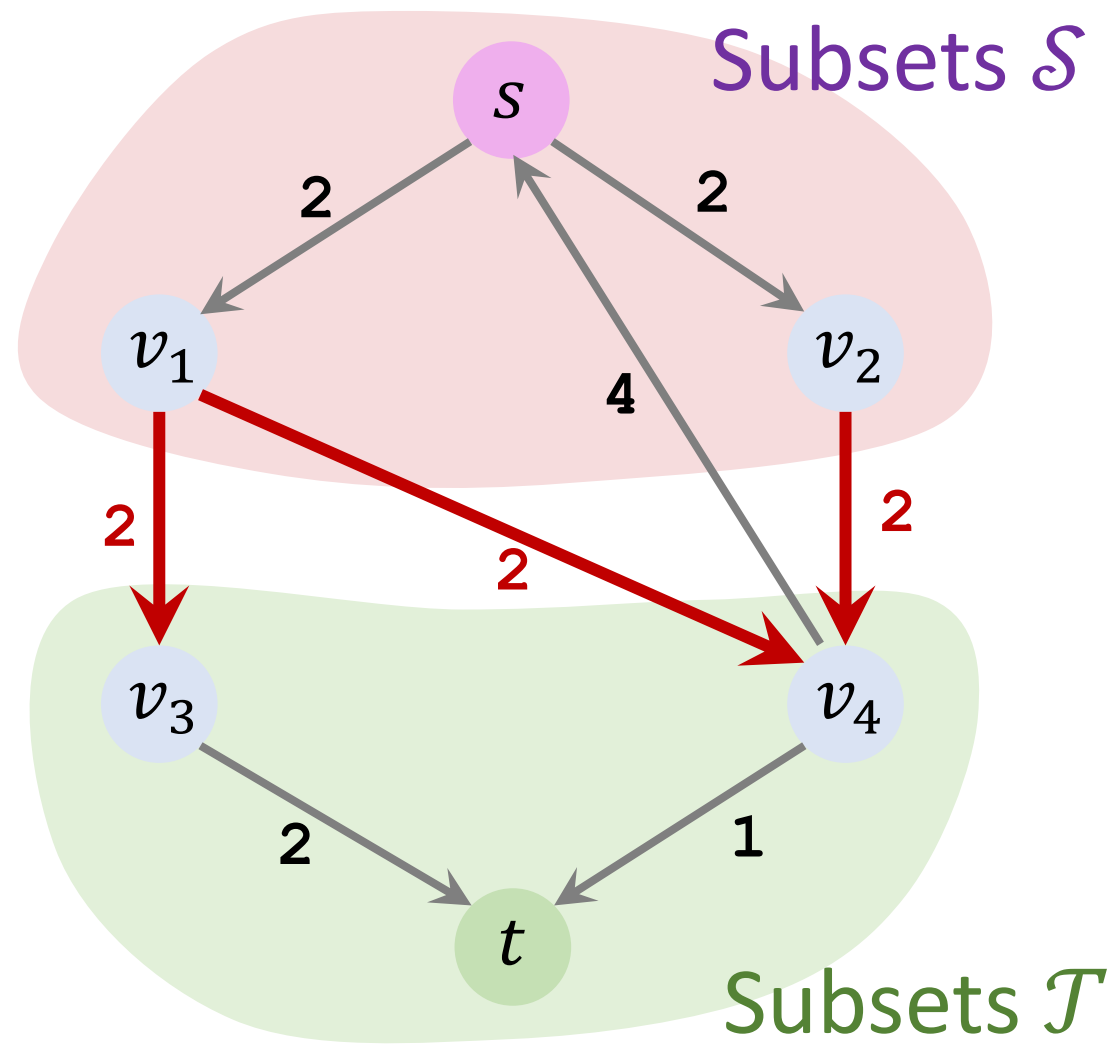
The s-t cut $(\mathcal{S}, \mathcal{T})$ that minimizes $\text{Capacity}(\mathcal{S}, \mathcal{T})$.

Subsets \mathcal{S}



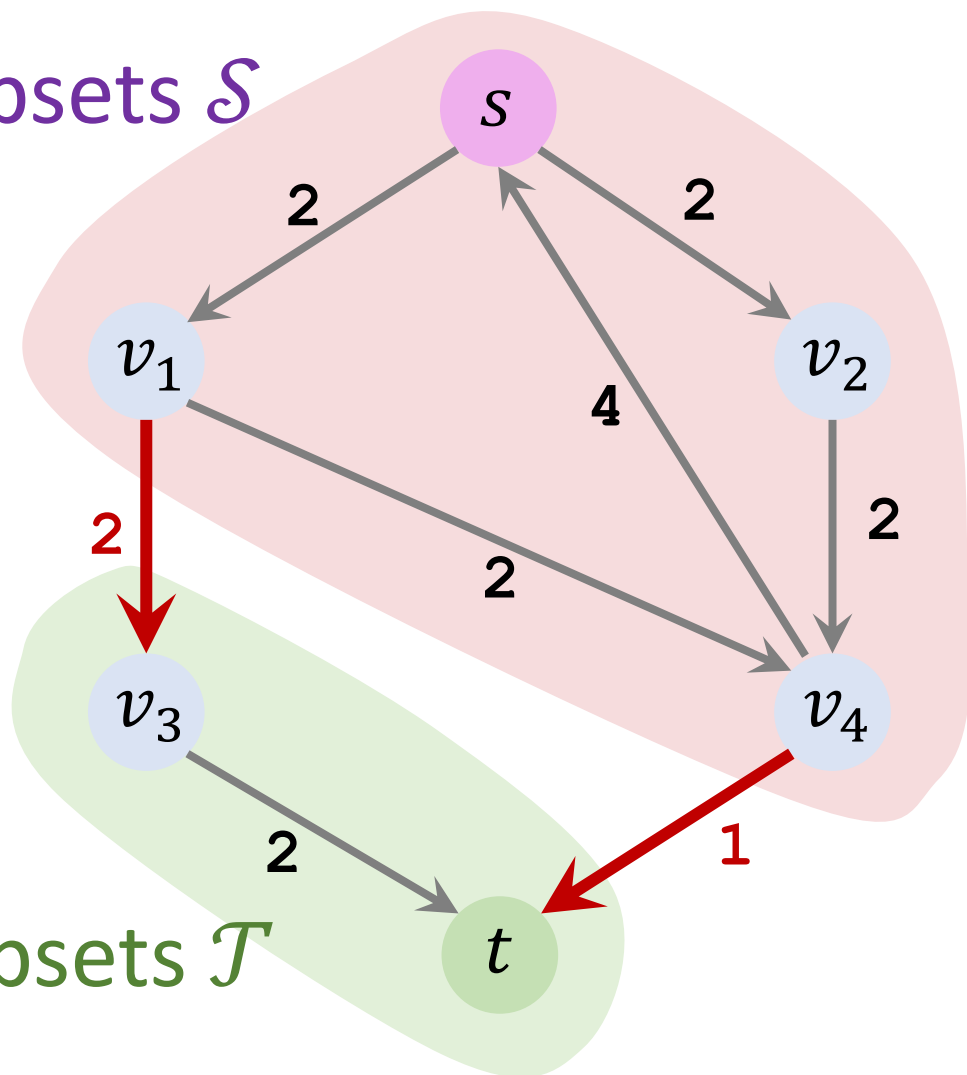
Min-cut

Subsets \mathcal{S}



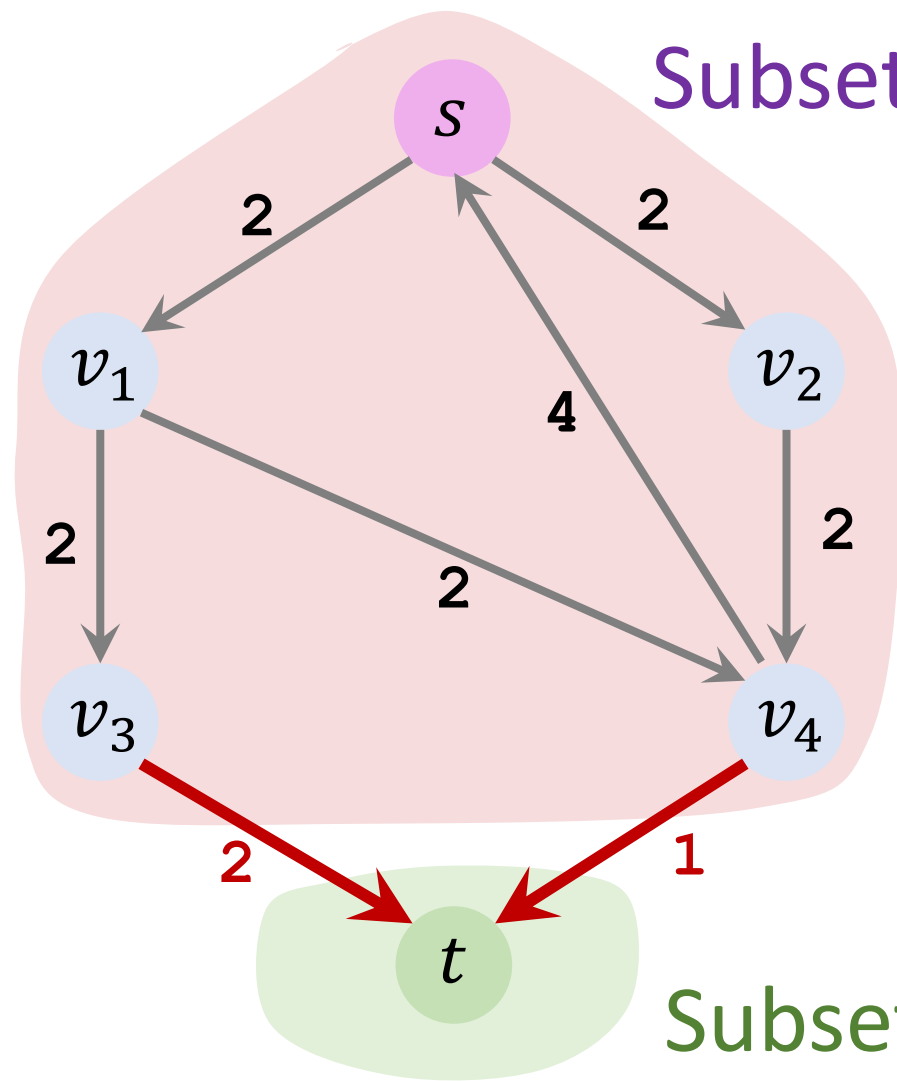
Not min-cut

Subsets \mathcal{S}



Min-cut

Subsets \mathcal{S}



Min-cut

Max-Flow Min-Cut Theorem

Max-Flow Min-Cut Theorem

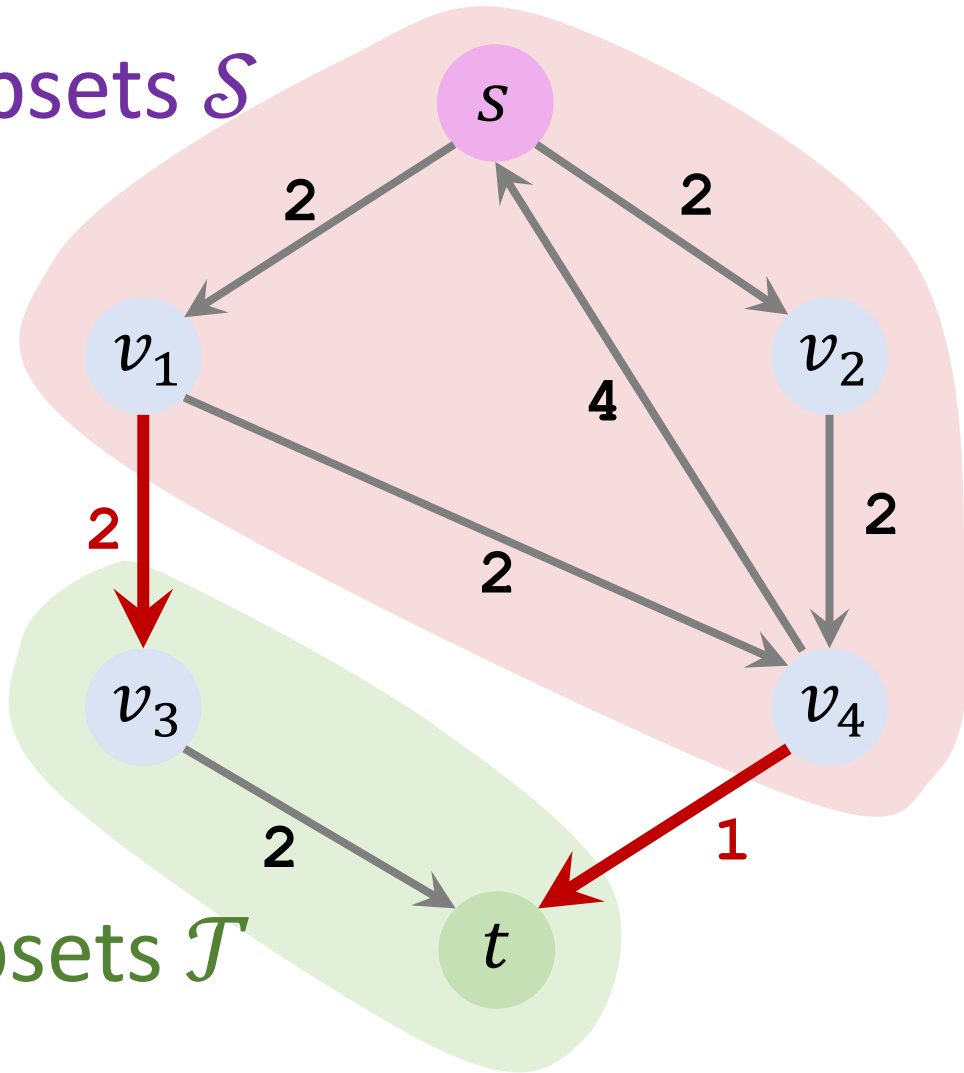
- In a flow network, the maximum amount of flow from s to t is equal to the capacity of the minimum s - t cut.
- In short, **amount of max-flow = capacity of min-cut.**

Reference

- L. R. Ford and D. R. Fulkerson. [Flows in Networks](#). *Princeton University Press*, 1962.

Max-Flow Min-Cut Theorem

Subsets \mathcal{S}

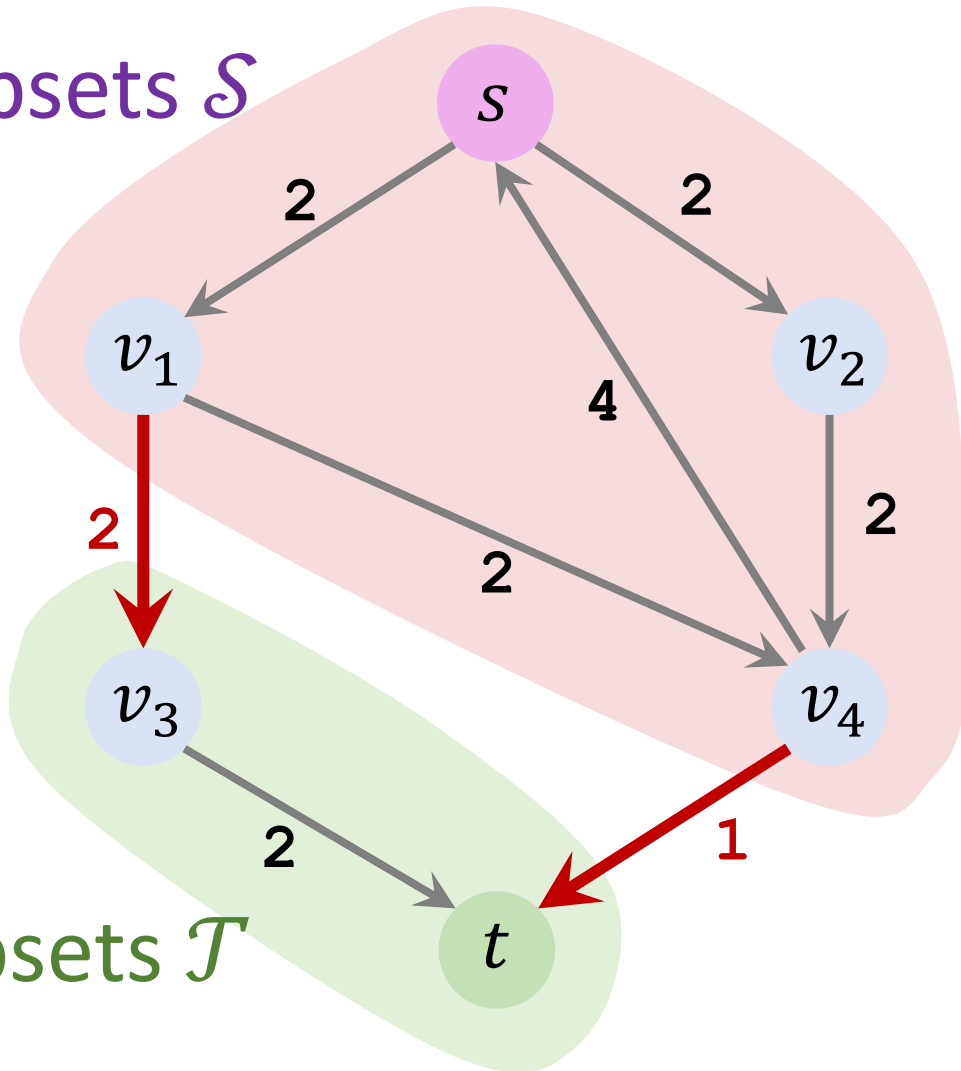


Subsets \mathcal{T}

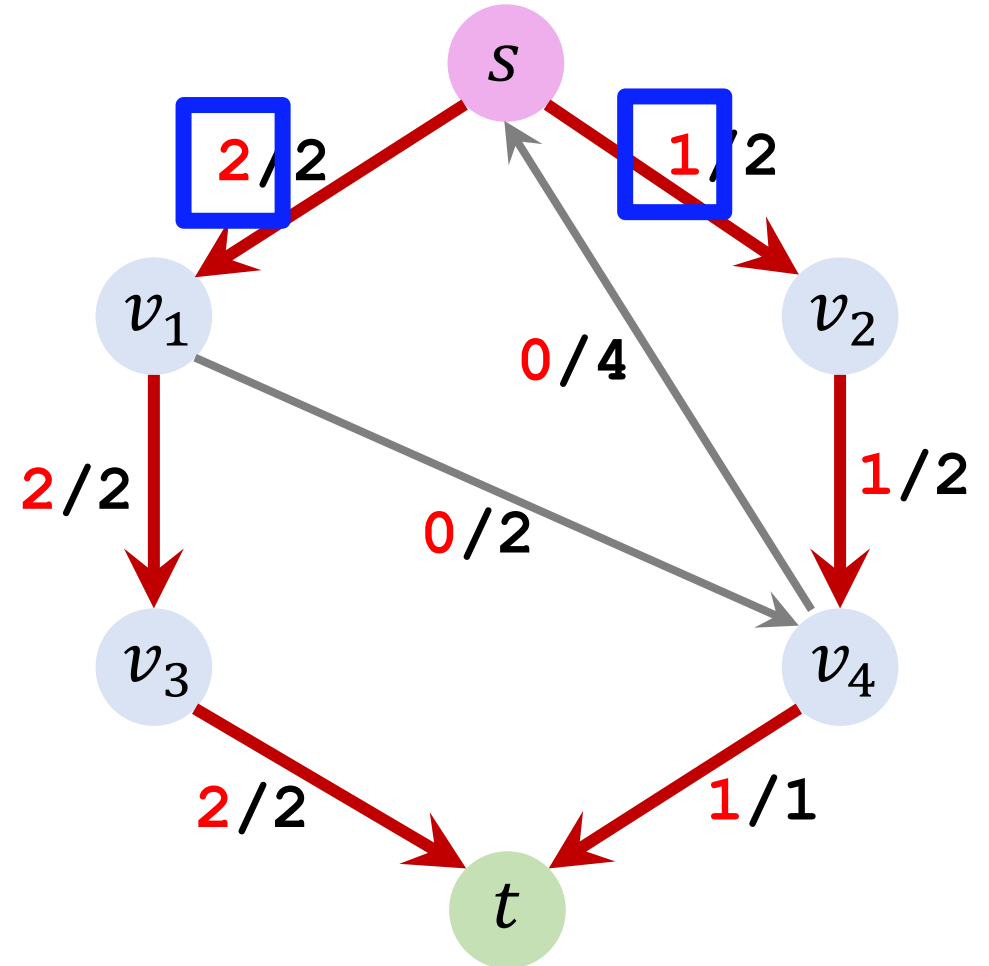
Capacity of min-cut = 3

Max-Flow Min-Cut Theorem

Subsets \mathcal{S}



Capacity of min-cut = 3



Amount of max-flow = 3

Find Min-Cut

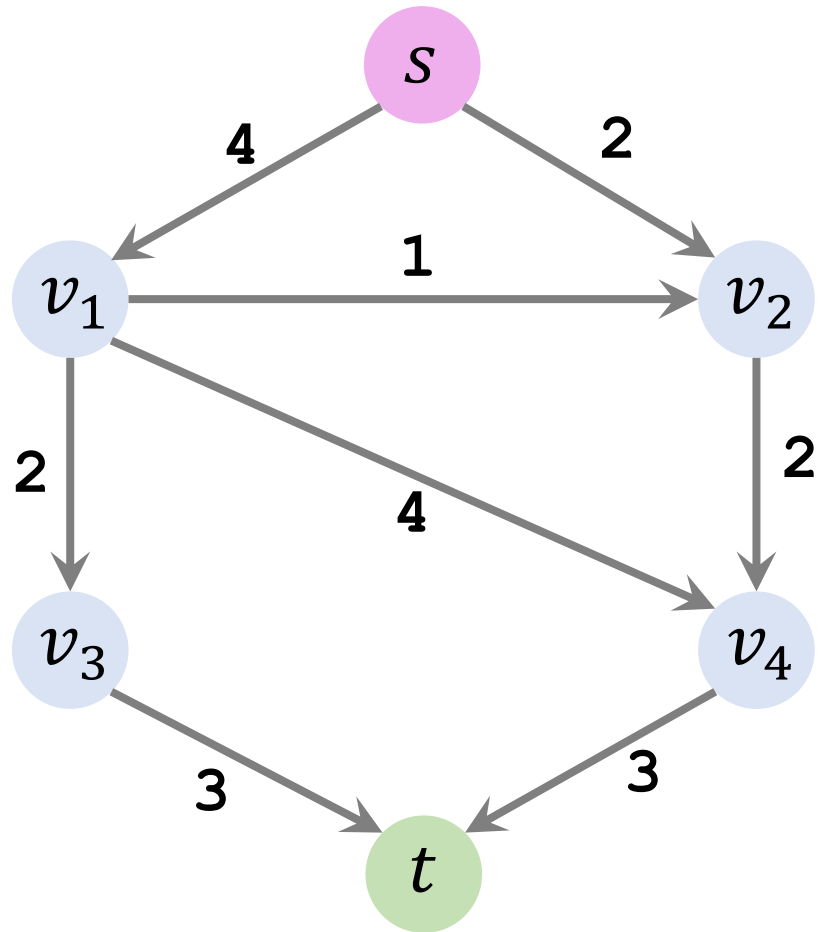
Algorithm

1. Run any **max-flow algorithm** to obtain the final residual graph.
 - E.g., Edmonds–Karp algorithm or Dinic's algorithm.
 - Ignore the backward edges on the final residual graph.

Algorithm

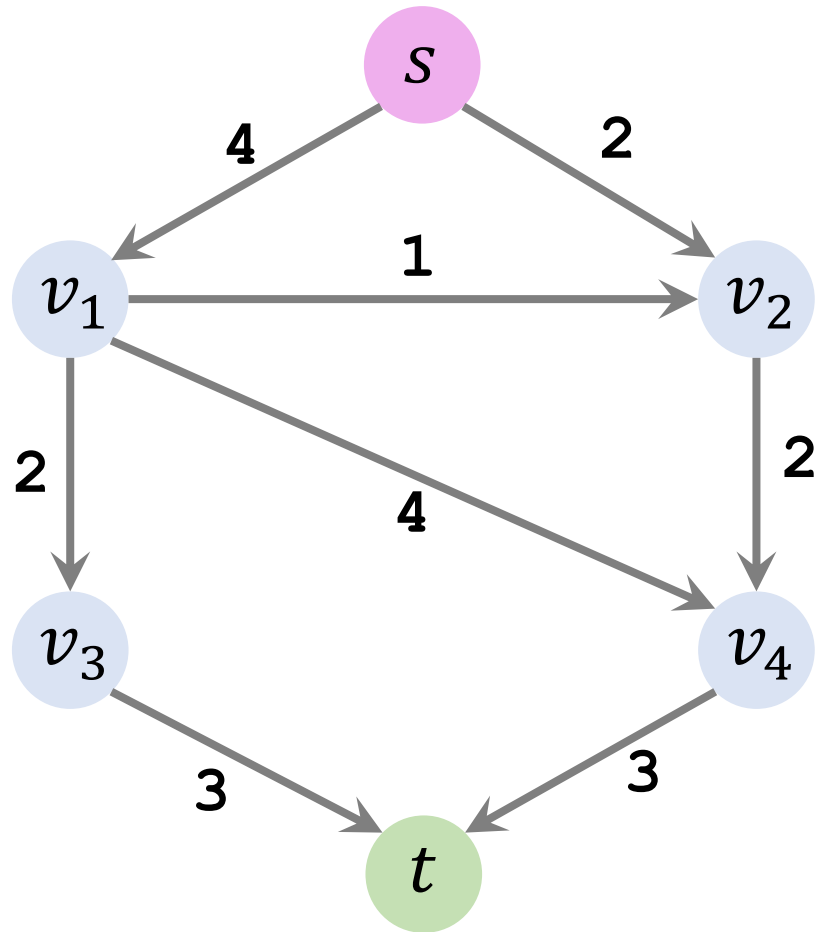
1. Run any max-flow algorithm to obtain the final residual graph.
 - E.g., Edmonds–Karp algorithm or Dinic’s algorithm.
 - Ignore the backward edges on the final residual graph.
2. Find the minimum s-t cut $(\mathcal{S}, \mathcal{T})$:
 - a. On the residual graph, find paths from source s to all the other vertices.
 - b. $\mathcal{S} \leftarrow$ all the vertices that has finite distance. (Reachable from s .)
 - c. $\mathcal{T} \leftarrow$ all the remaining vertices. (Not reachable from s .)

Example 1

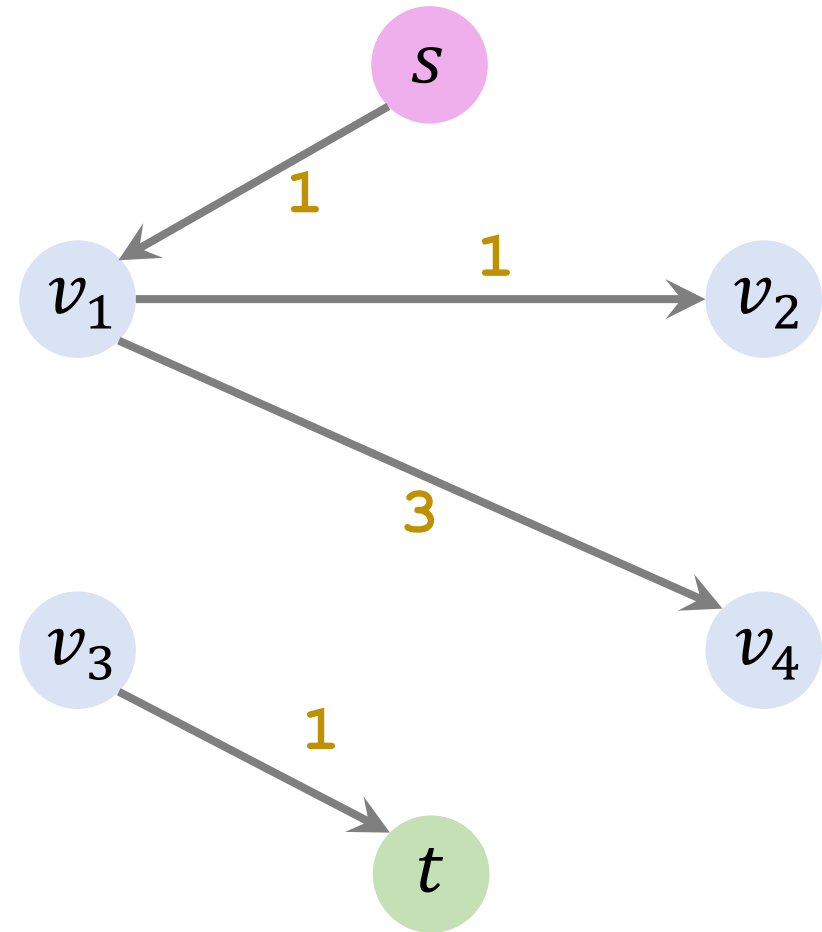


Original Graph

Example 1: Run a max-flow algorithm



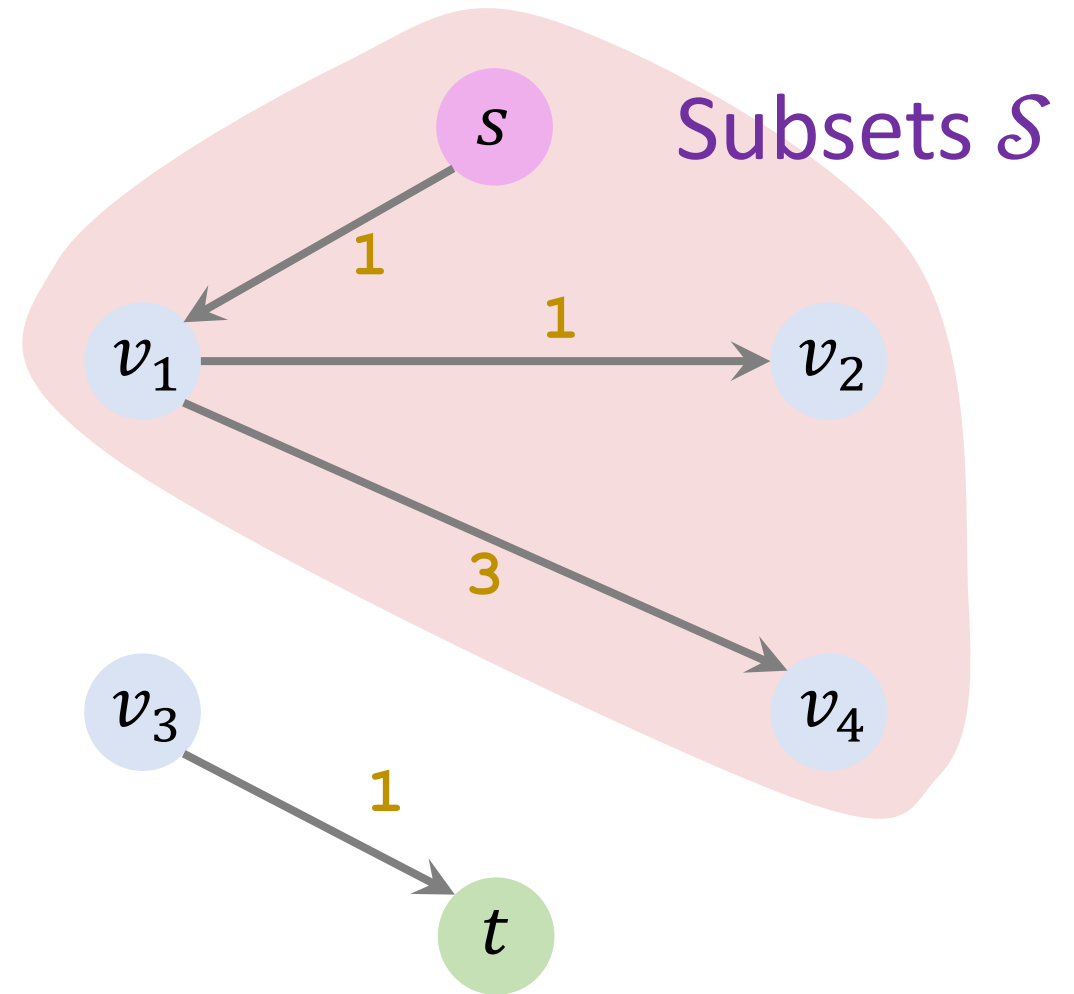
Original Graph



Residual Graph

Example 1: Find the min-cut

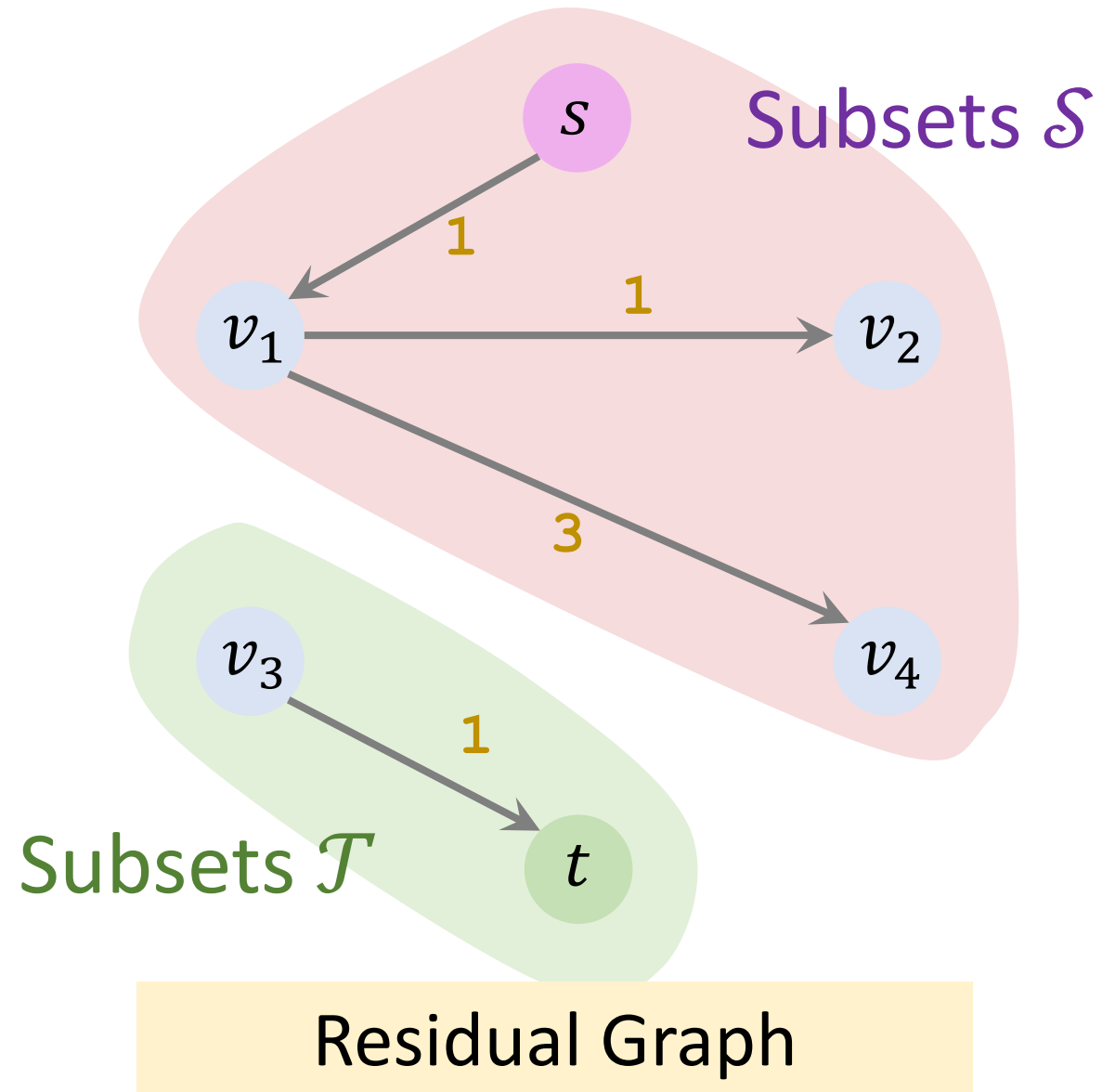
- Find the vertices reachable from s .
- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.



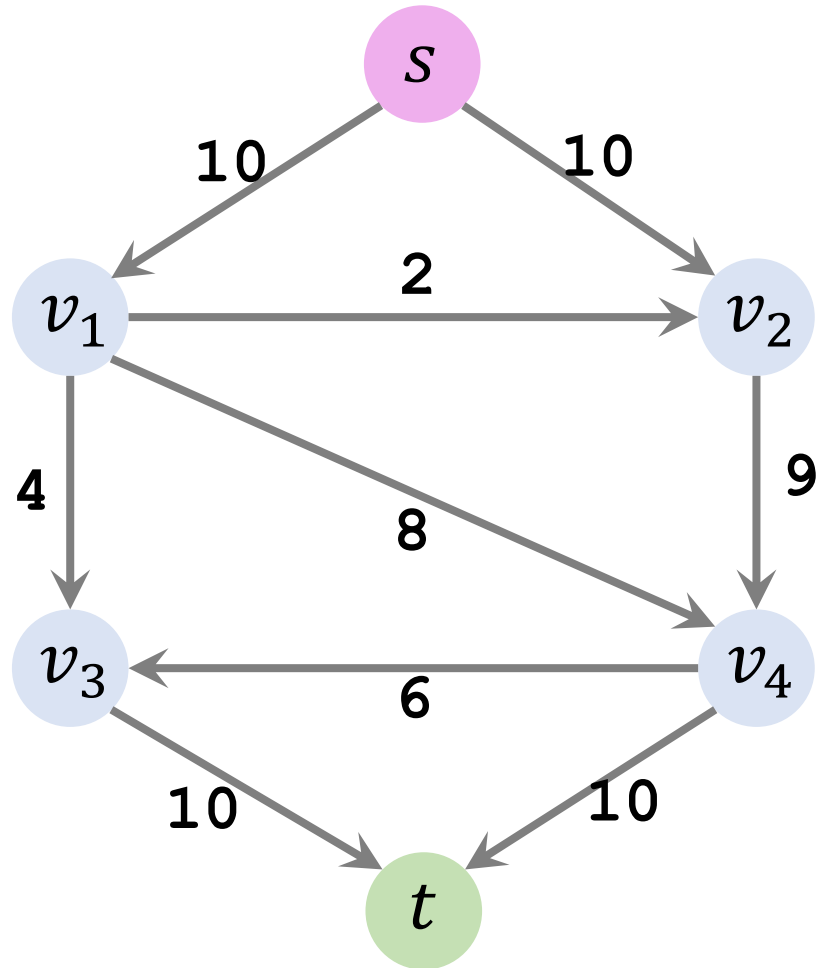
Residual Graph

Example 1: Find the min-cut

- Find the vertices reachable from s .
- Subset $\mathcal{S} = \{s, v_1, v_2, v_4\}$.
- The remaining vertices: t and v_3 .
- Subset $\mathcal{T} = \{t, v_3\}$.

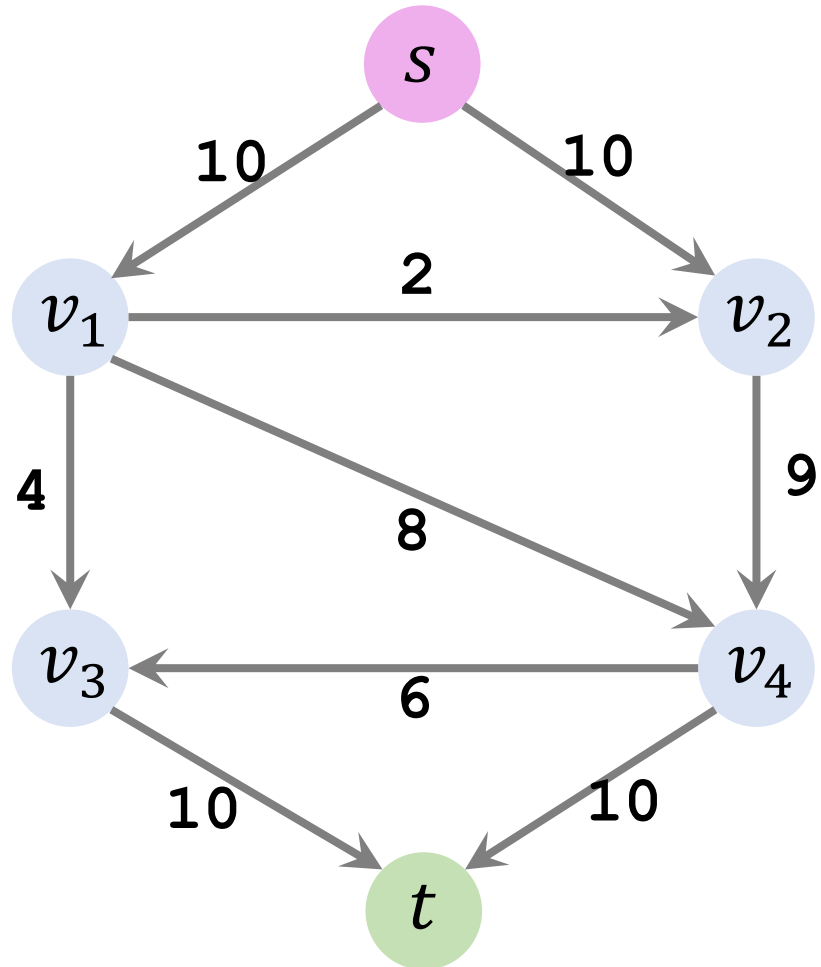


Example 2

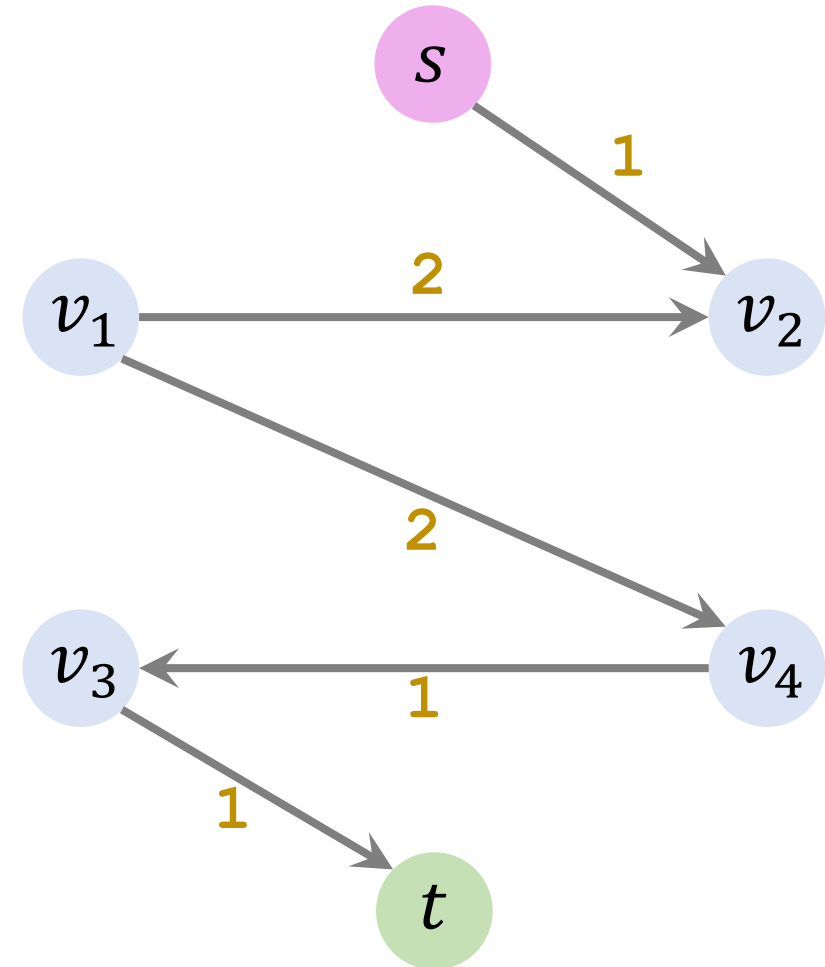


Original Graph

Example 2: Run a max-flow algorithm



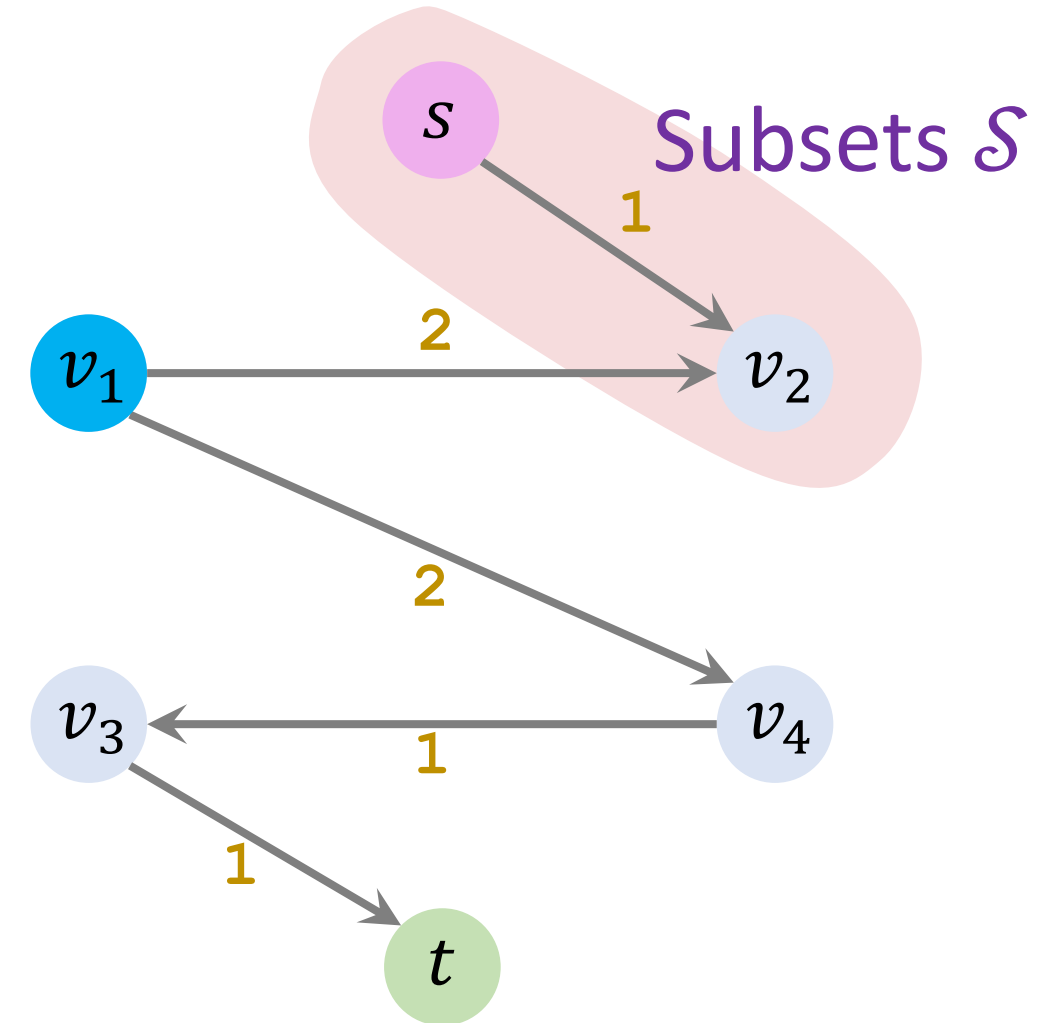
Original Graph



Residual Graph

Example 2: Find the min-cut

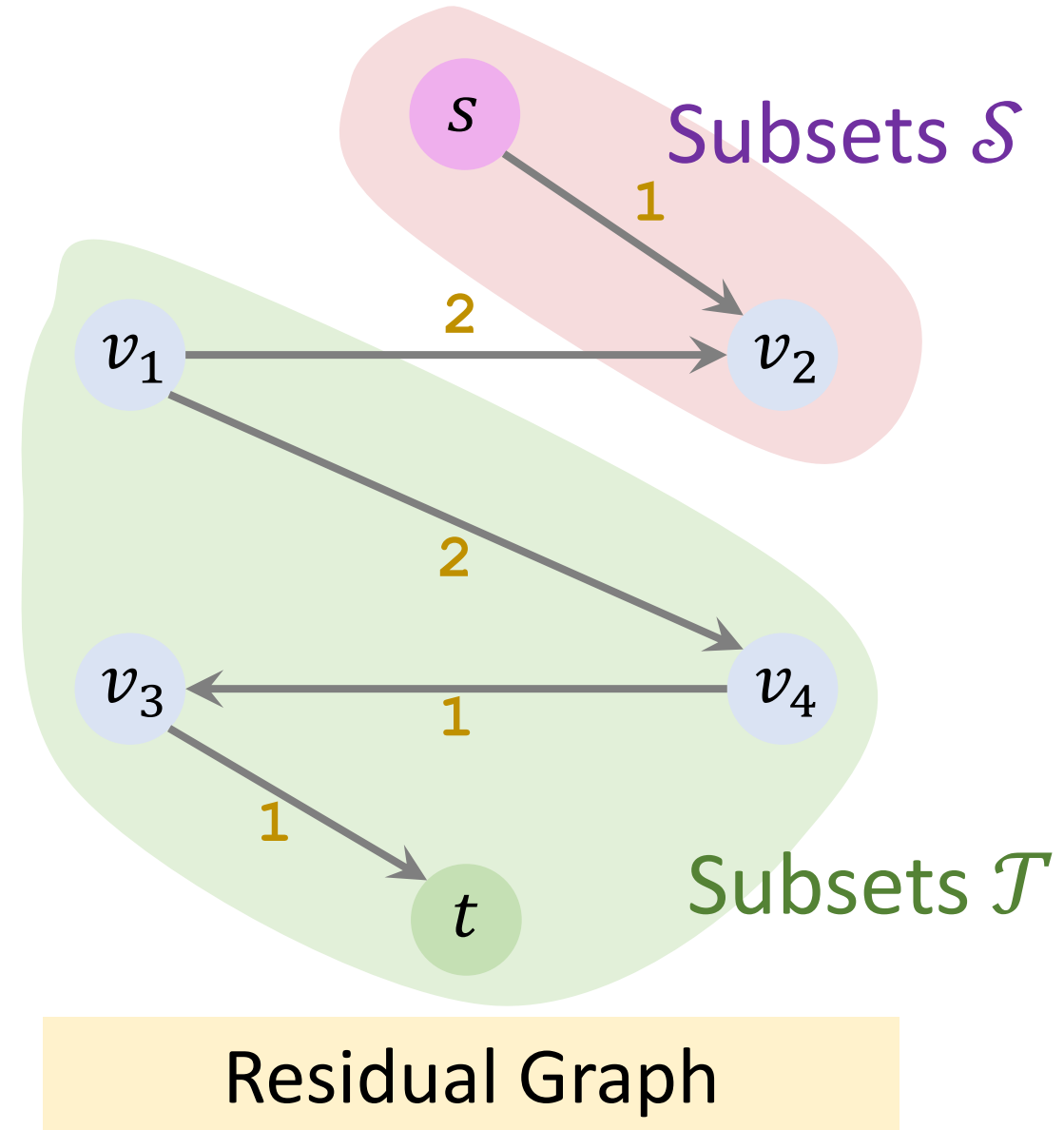
- Find the vertices reachable from s .
- Subset $\mathcal{S} = \{s, v_2\}$.



Residual Graph

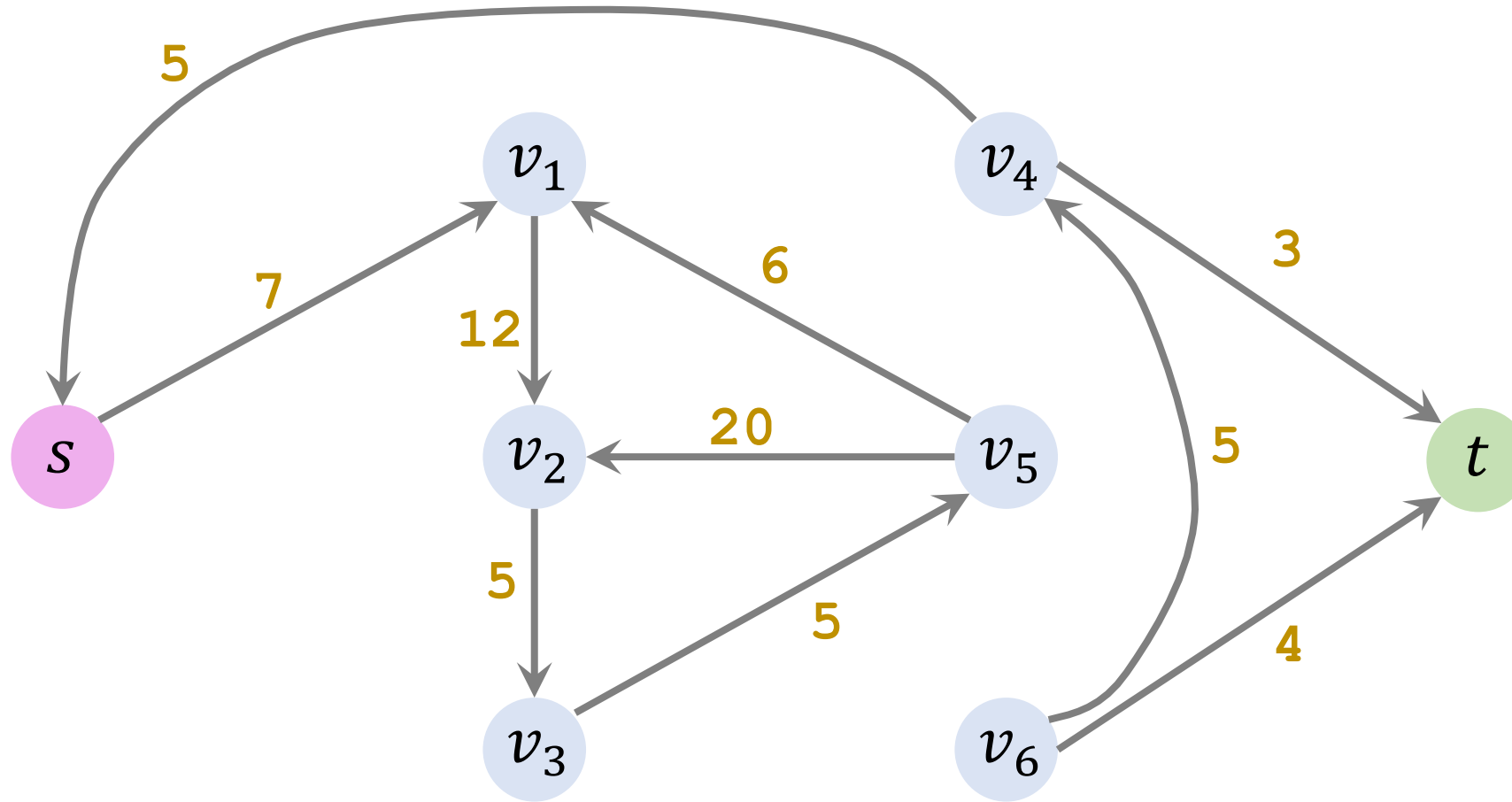
Example 2: Find the min-cut

- Find the vertices reachable from s .
- Subset $\mathcal{S} = \{s, v_2\}$.
- The remaining vertices: t, v_1, v_3, v_4 .
- Subset $\mathcal{T} = \{t, v_1, v_3, v_4\}$.



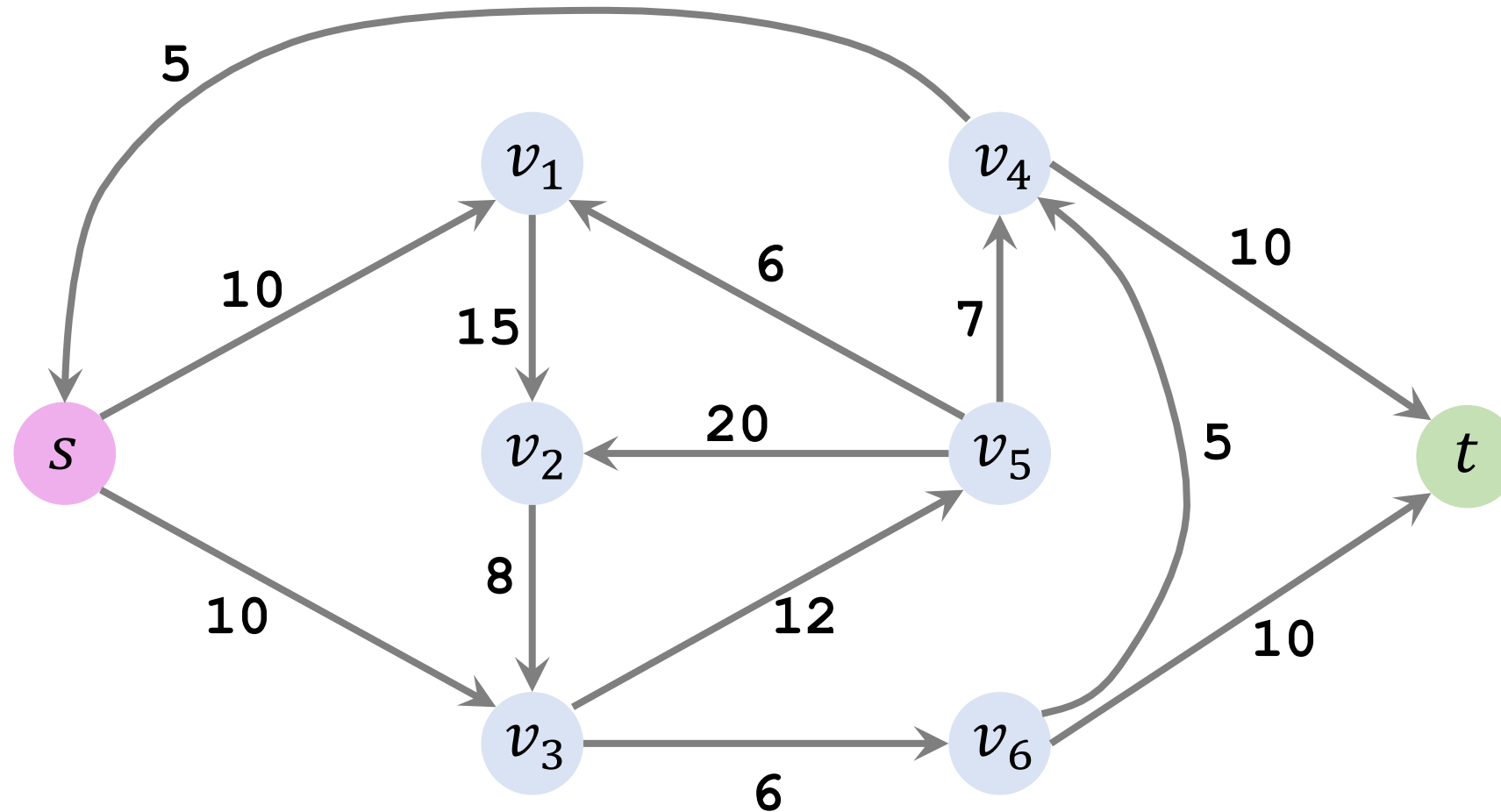
Questions

Q1: What is the min-cut?



This is the final **residual graph** found by Dinic's algorithm.

Q2: What is the capacity of the min-cut?



Hint: The min-cut is the same to the previous page.

Thank You!

<http://wangshusen.github.io/>