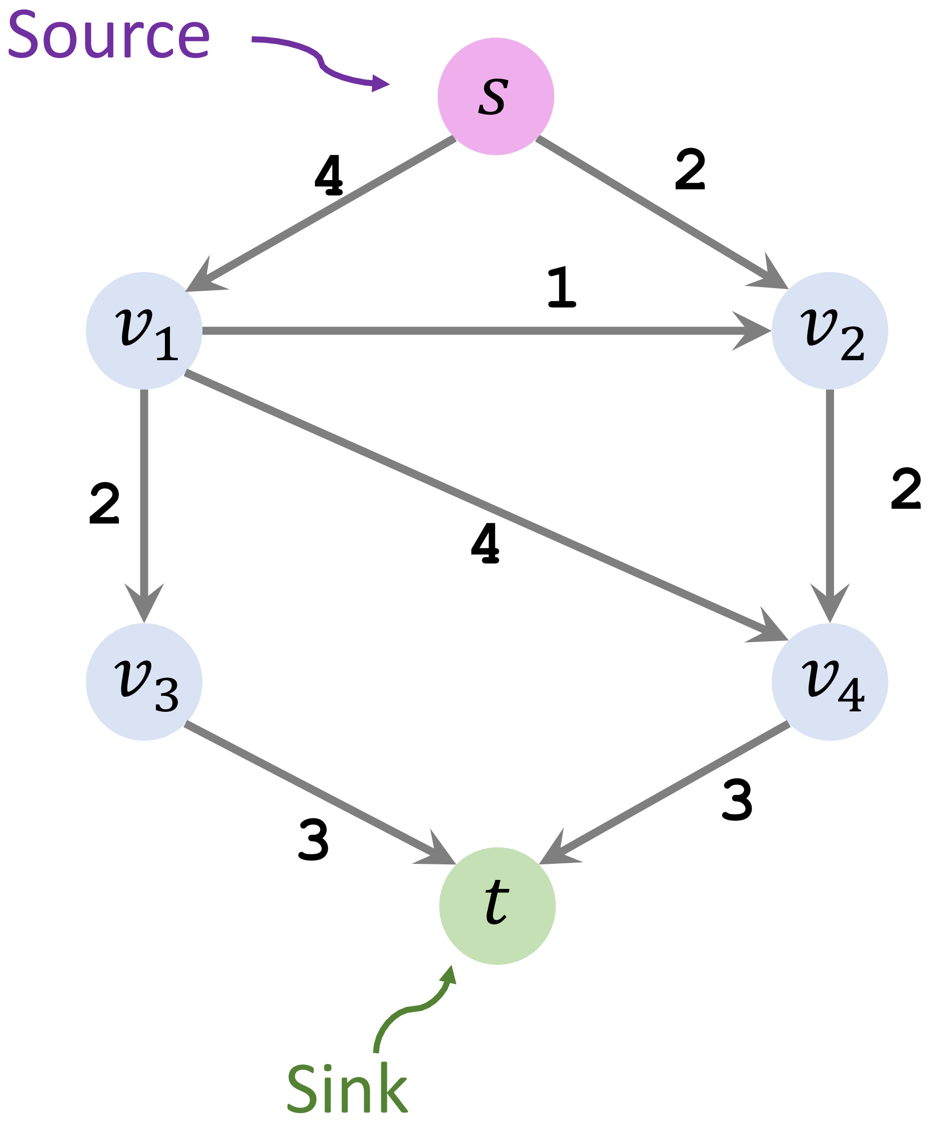
Network Flow Problems

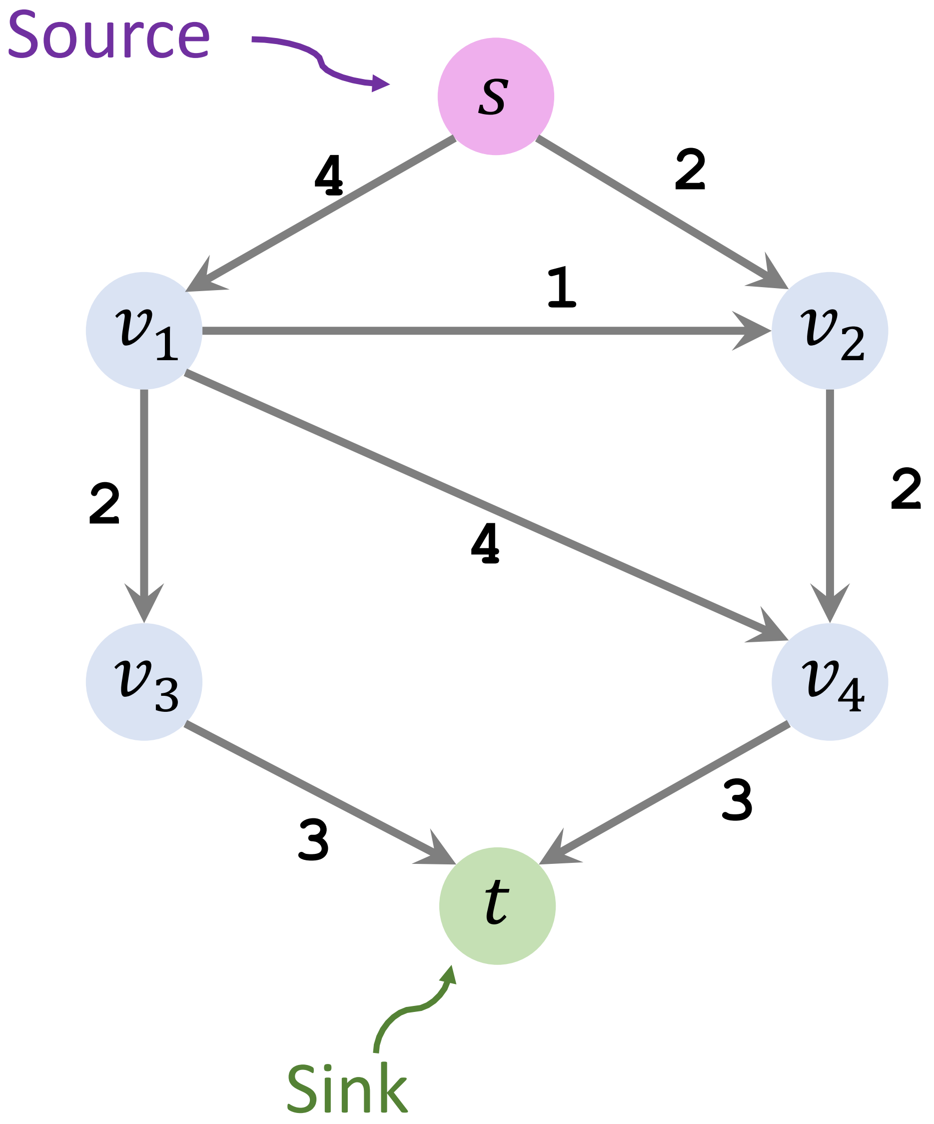
Shusen Wang

http://wangshusen.github.io/

Maximum Flow Problem

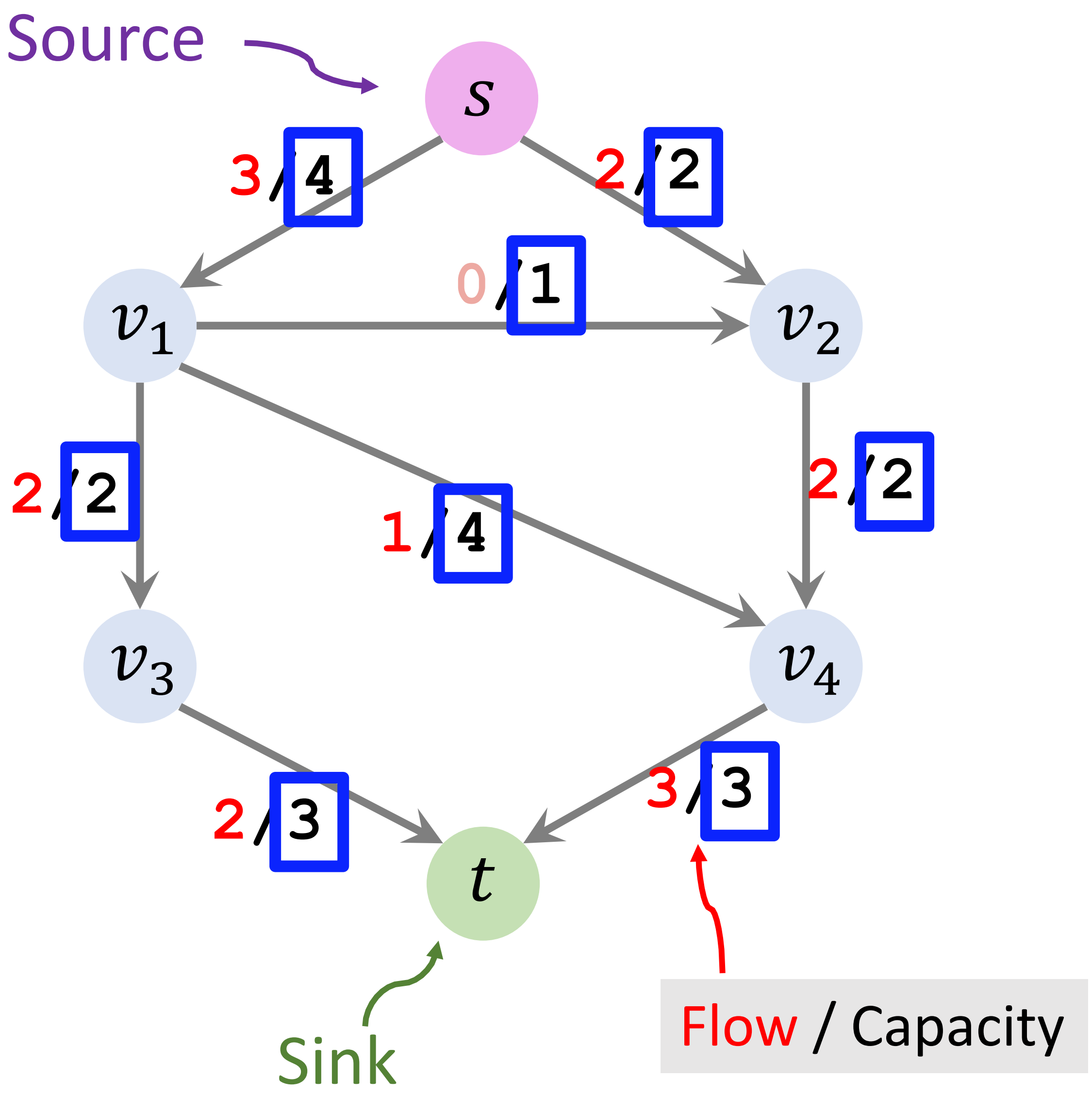


Maximum Flow Problem



- Send water from the source s to the sink t.
- The edges are pipes which have certain capacities, e.g., $4\ m^3/s$.
- How much water can flow from the source s to the sink t at most?

Maximum Flow Problem



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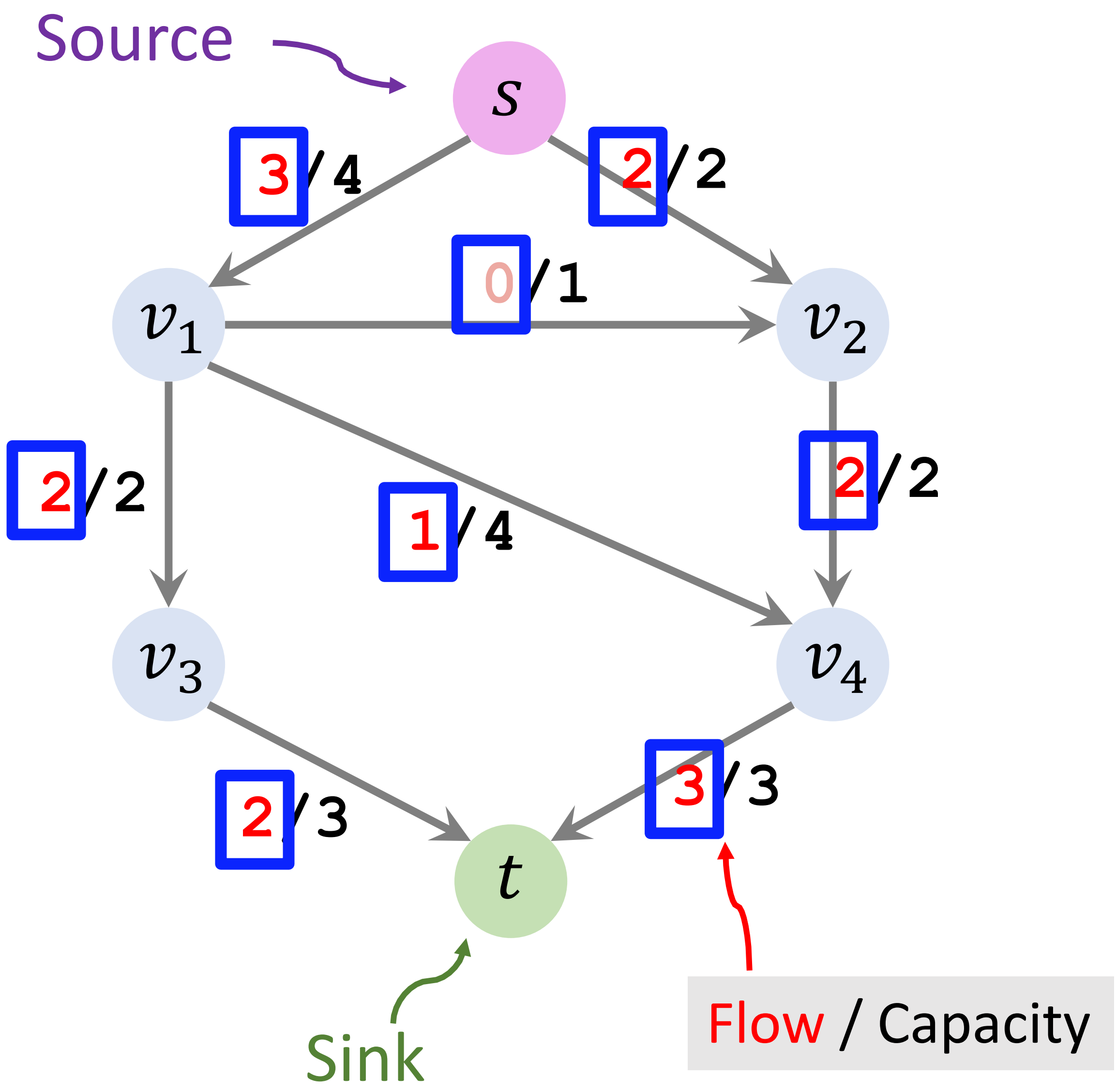
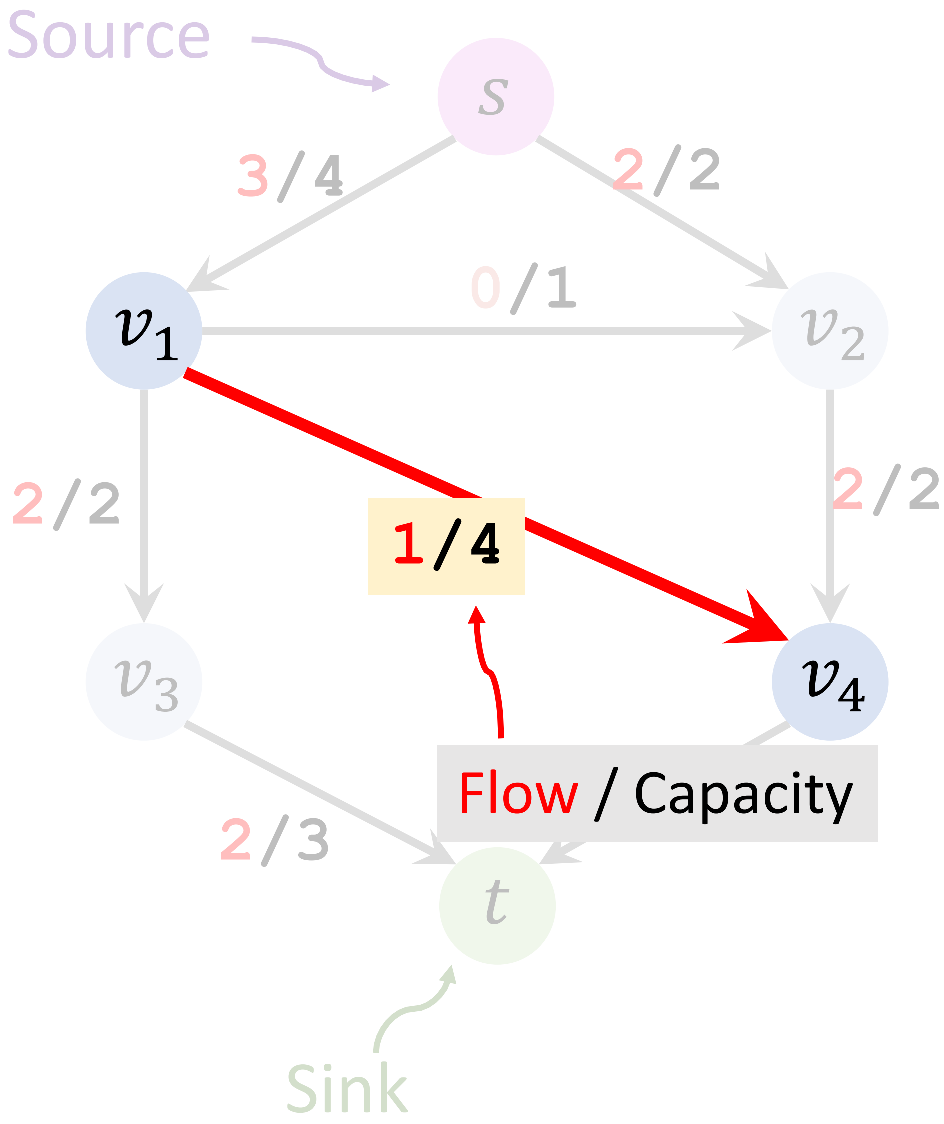
Maximum Flow Problem



- Send water from the source s to the sink t.
- The edges are pipes which have certain capacities, e.g., $4\ m^3/s$.
- How much water can flow from source s to the sink t at most?
- Flow $= 5$.

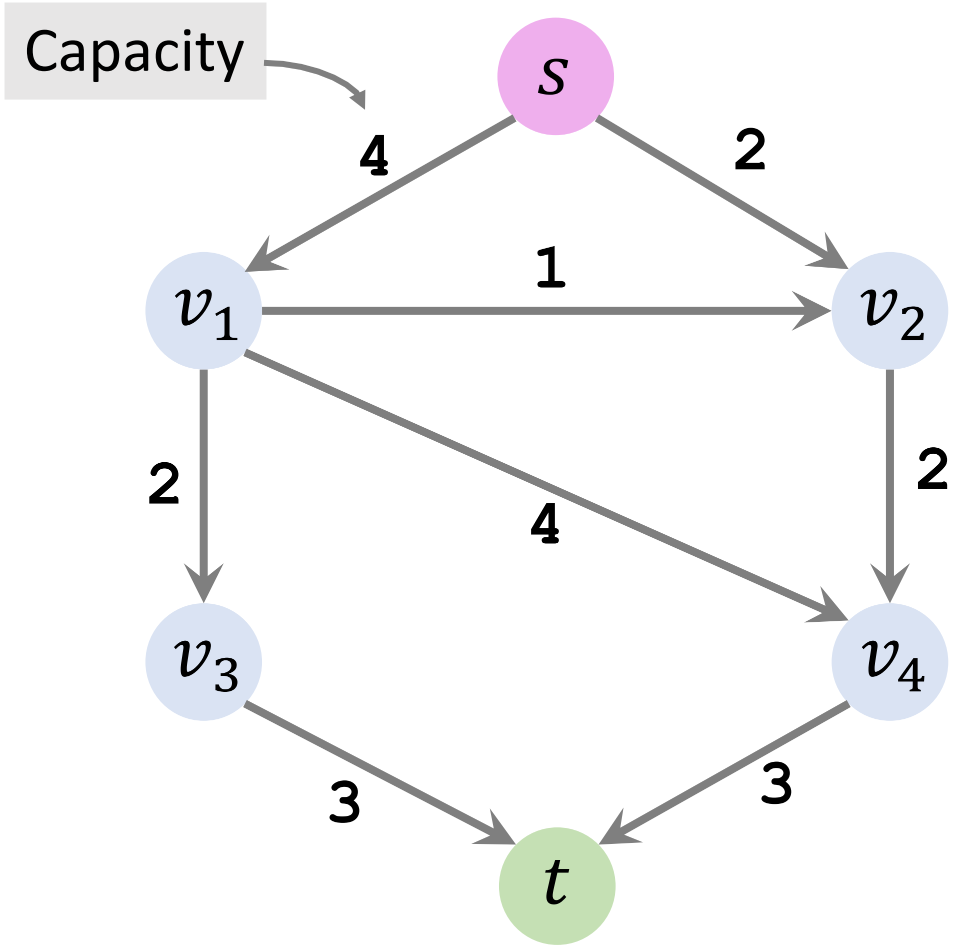
Maximum Flow Problem



- Capacity of the red pipe is $4\ m^3/s$.
- A flow of $1\ m^3/s$ goes through the red pipe.
- It has a residual of $3\ m^3/s$.

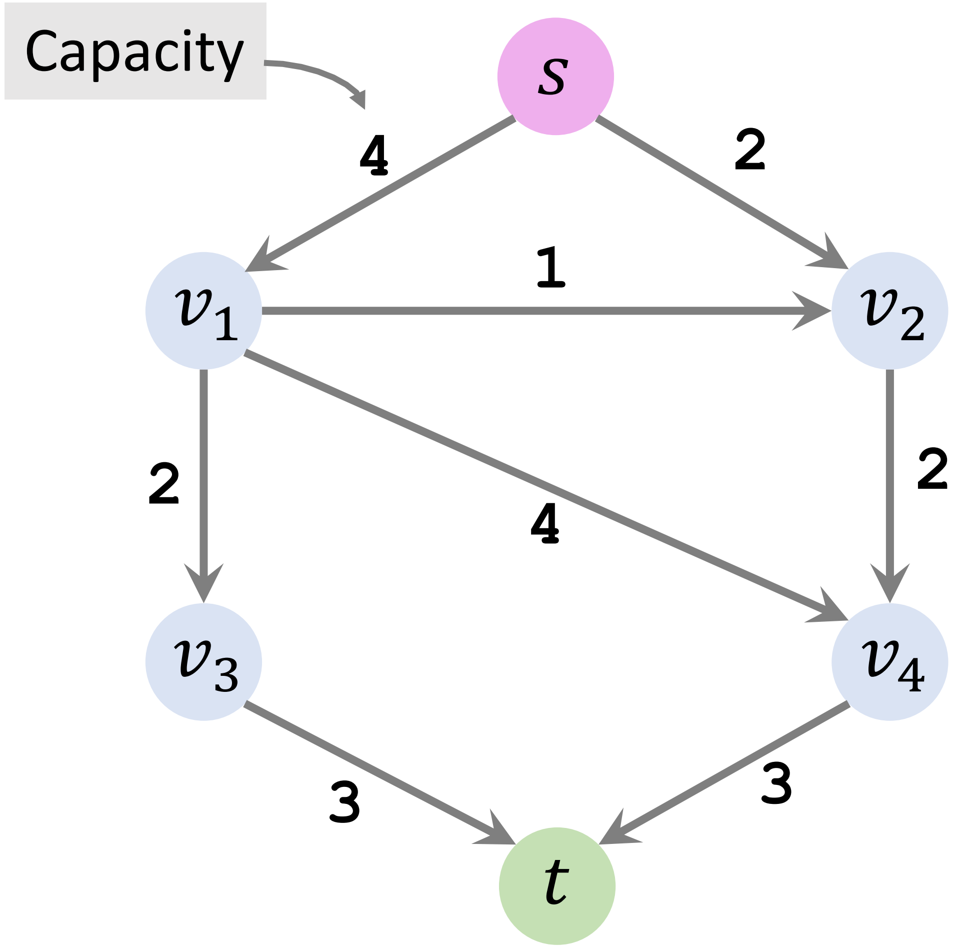
Naïve Algorithm

Initialization

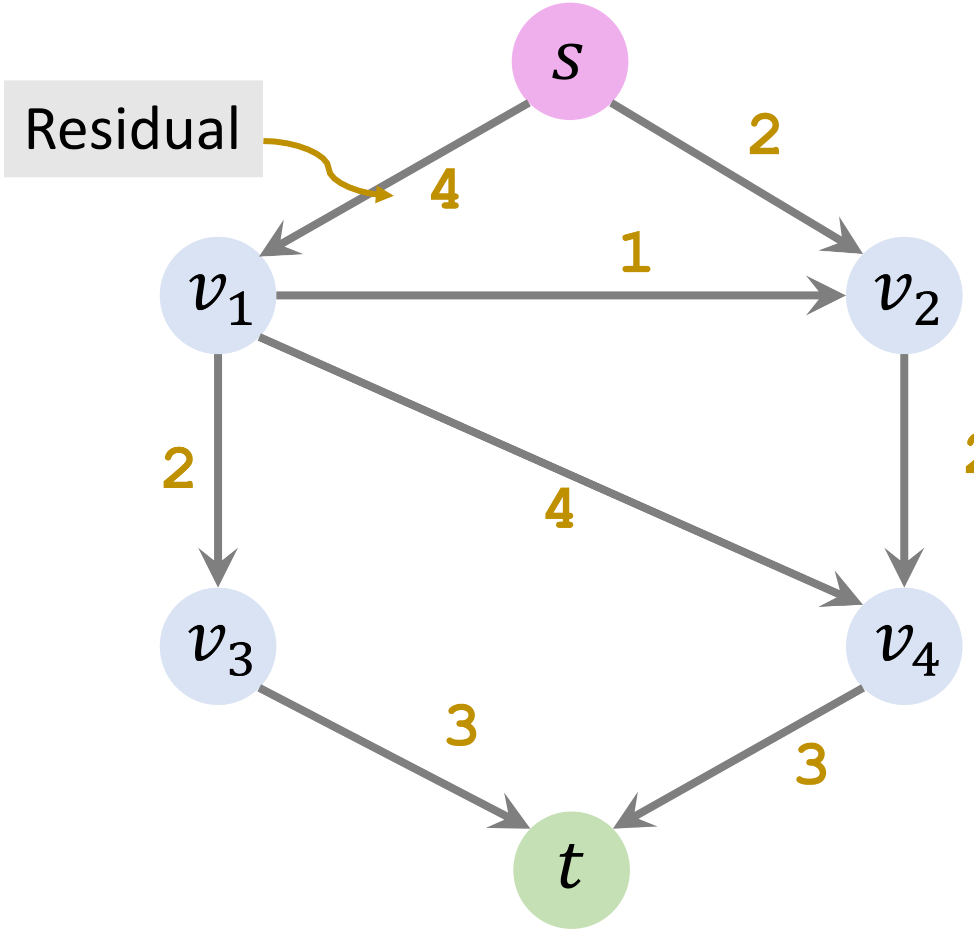


Original Graph

Initialization

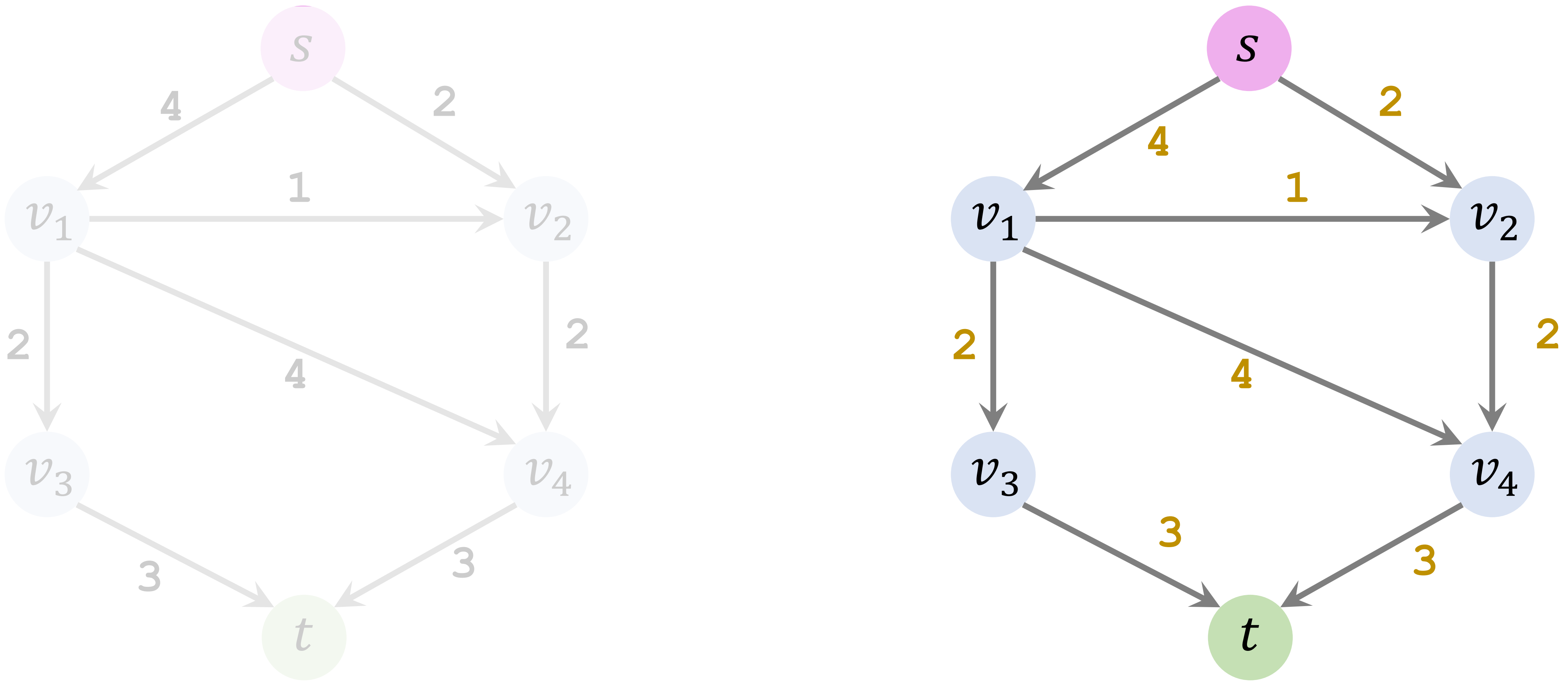


Original Graph



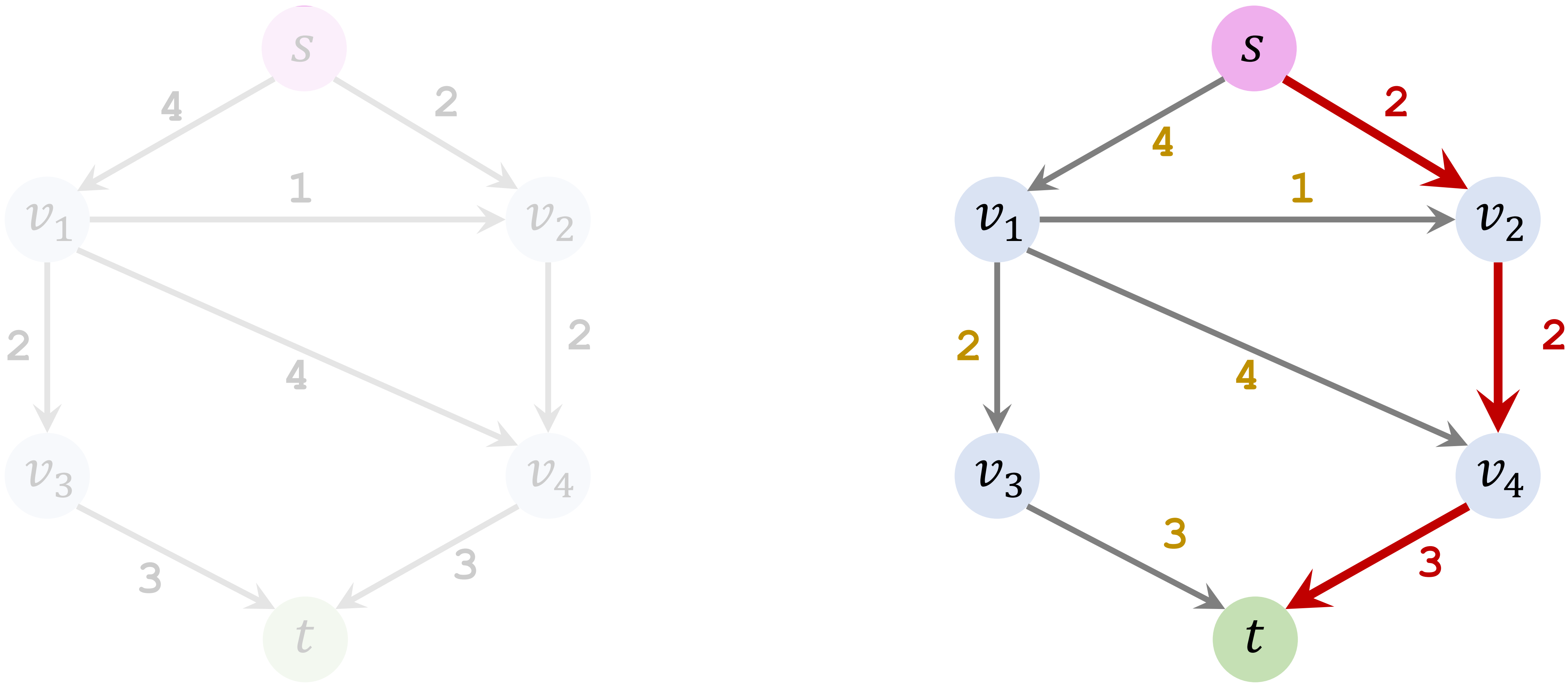
Residual Graph

Iteration 1: Find an augmenting path



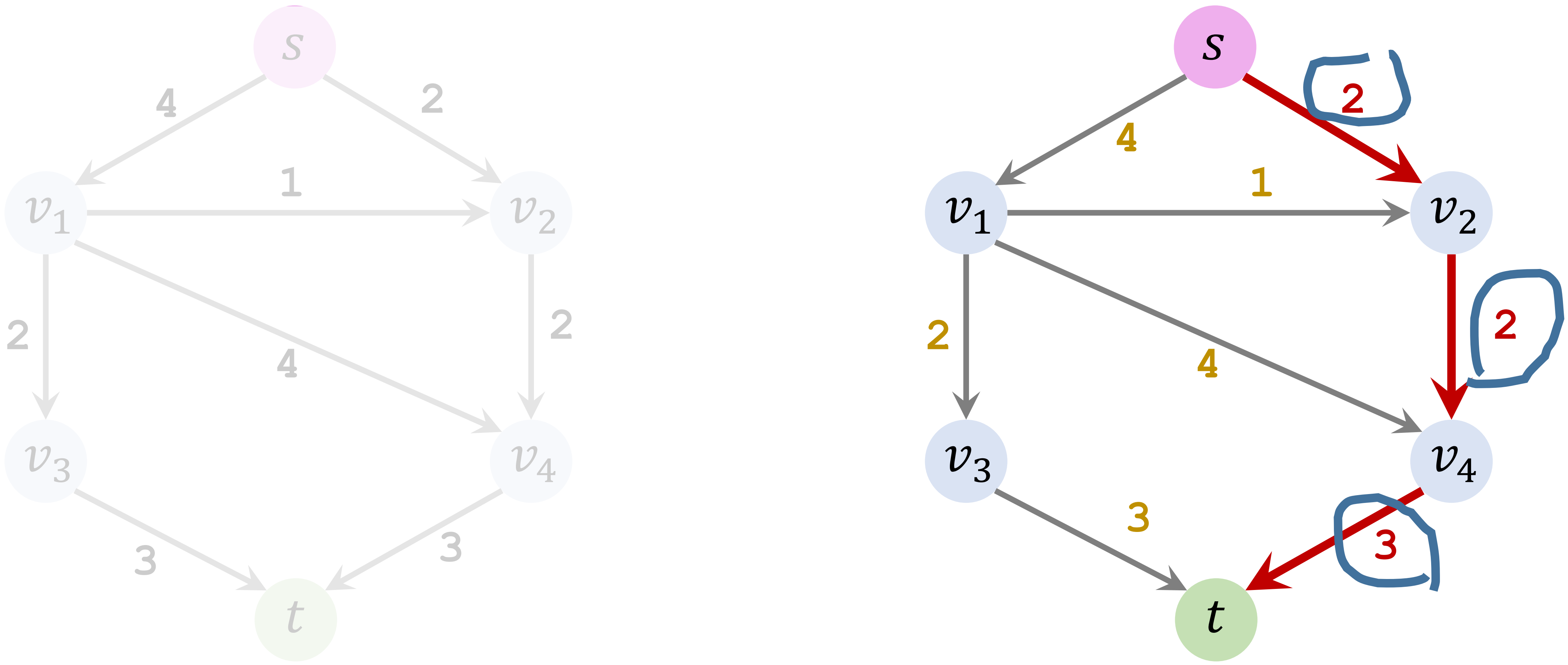
Augmenting path: a path from s to t that does not contain cycles.

Iteration 1: Find an augmenting path



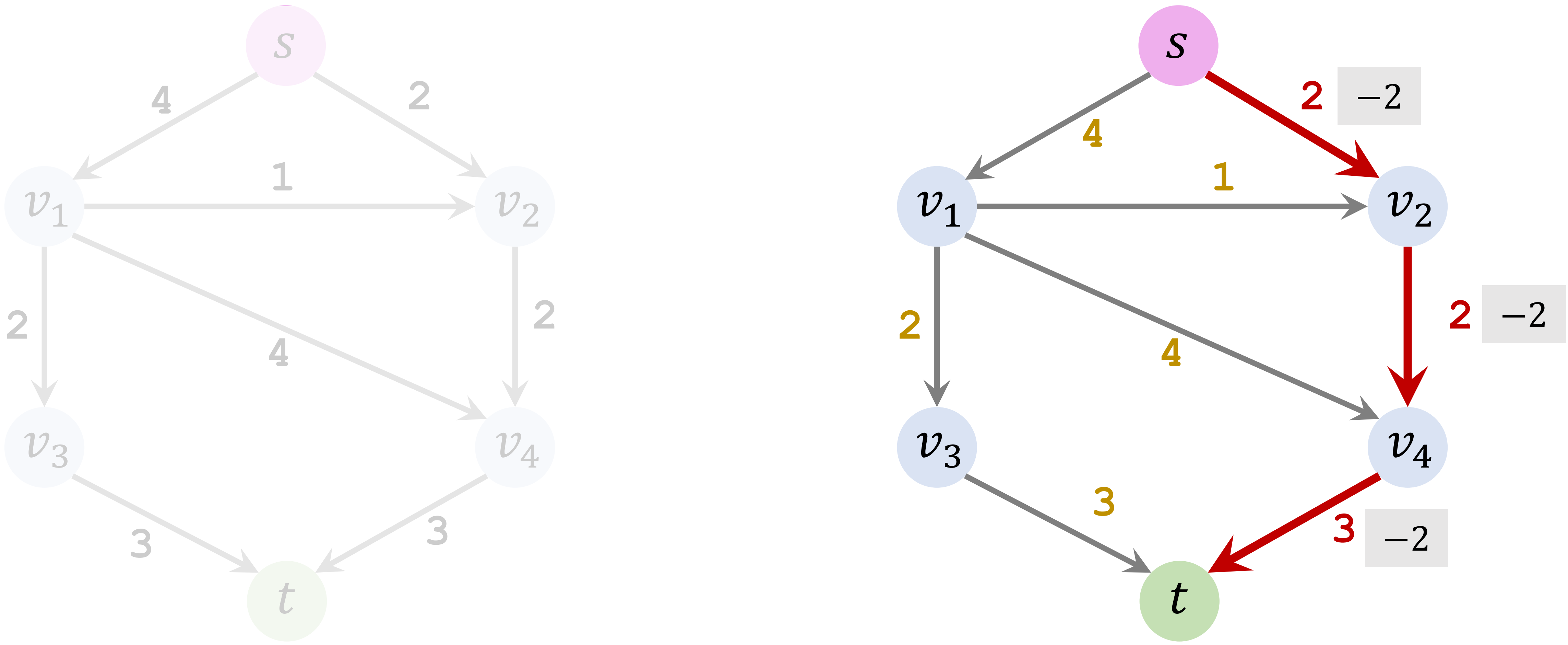
Found path $s \to v_2 \to v_4 \to t$.

Iteration 1: Find an augmenting path



Found path $s \to v_2 \to v_4 \to t$. (Bottleneck capacity $= 2$.)

Iteration 1: Update residuals



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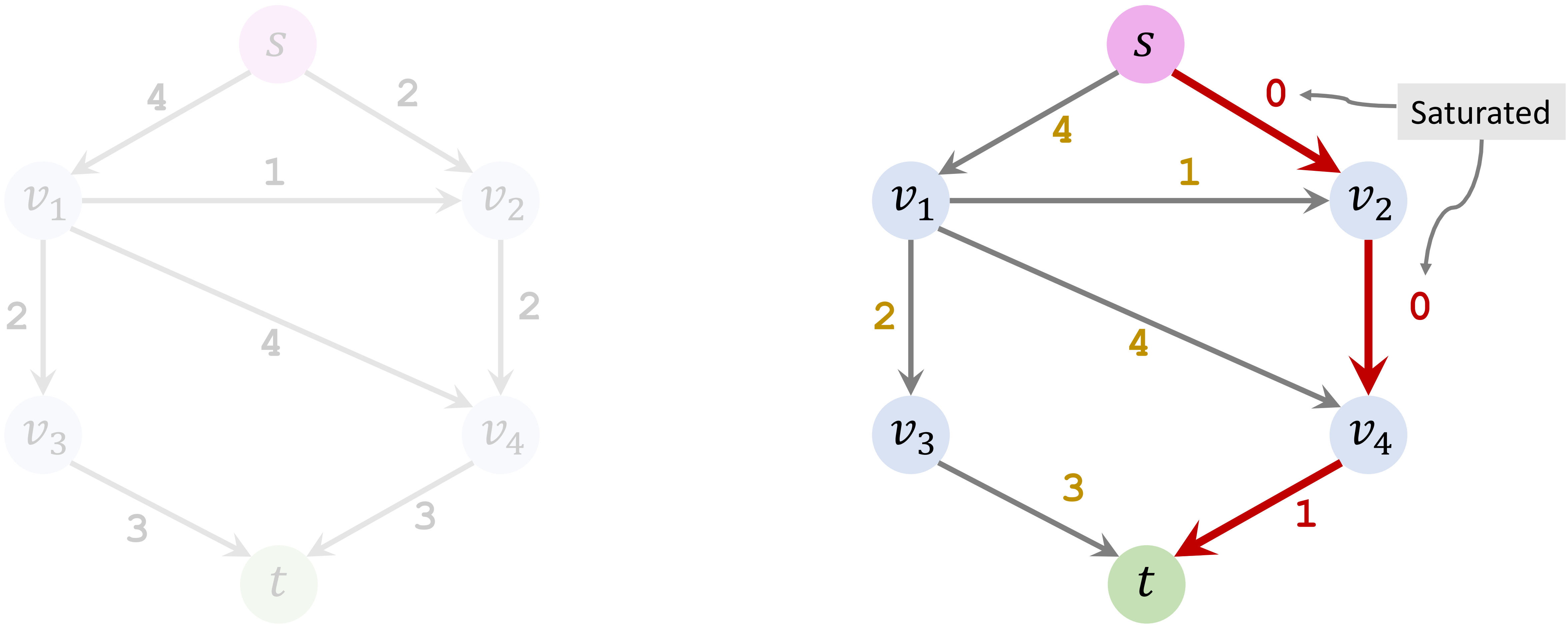
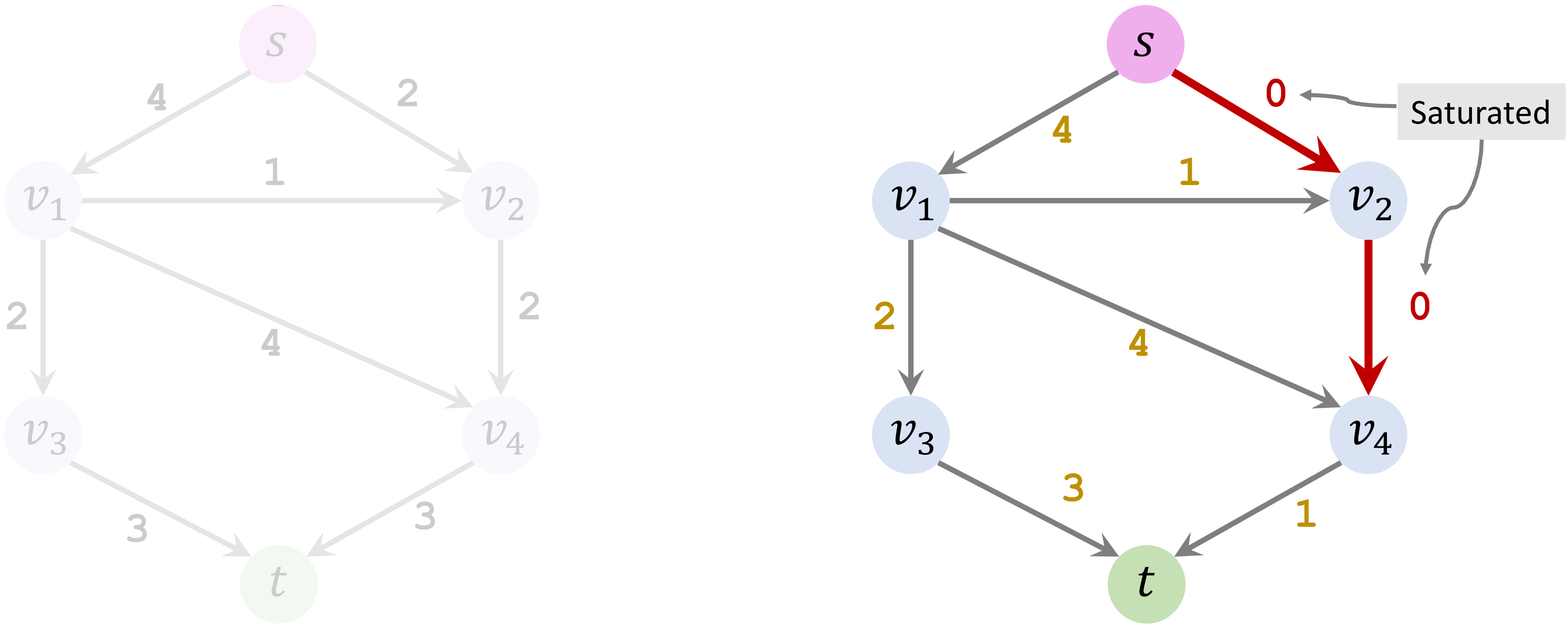
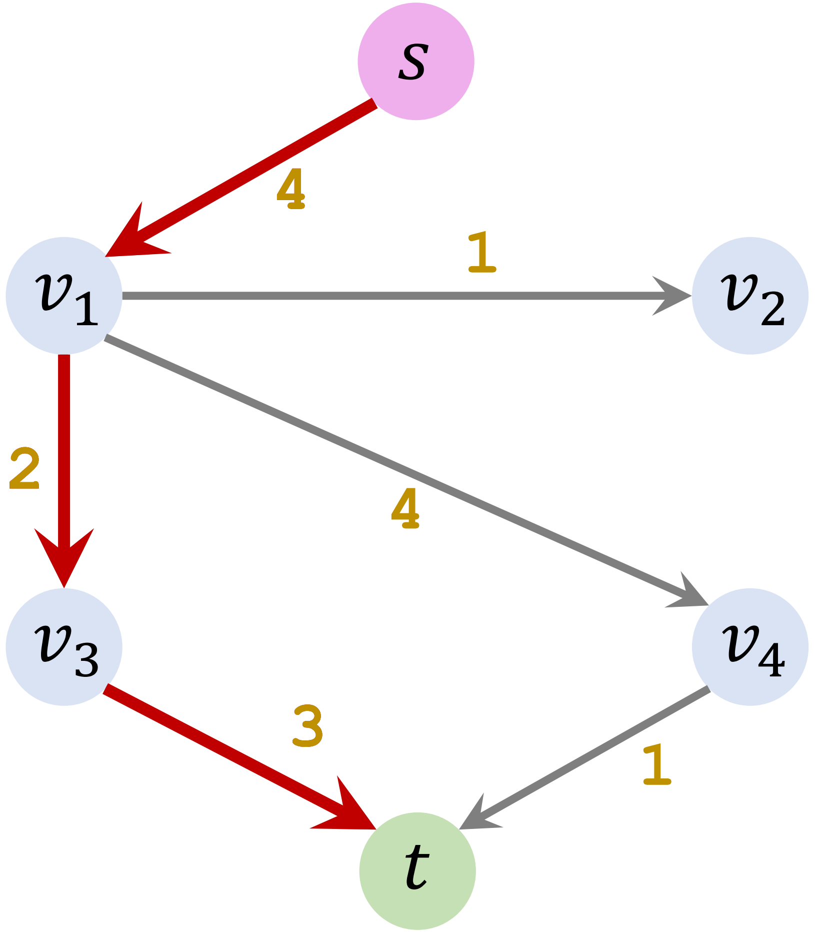


Iteration 1: Remove saturated edges

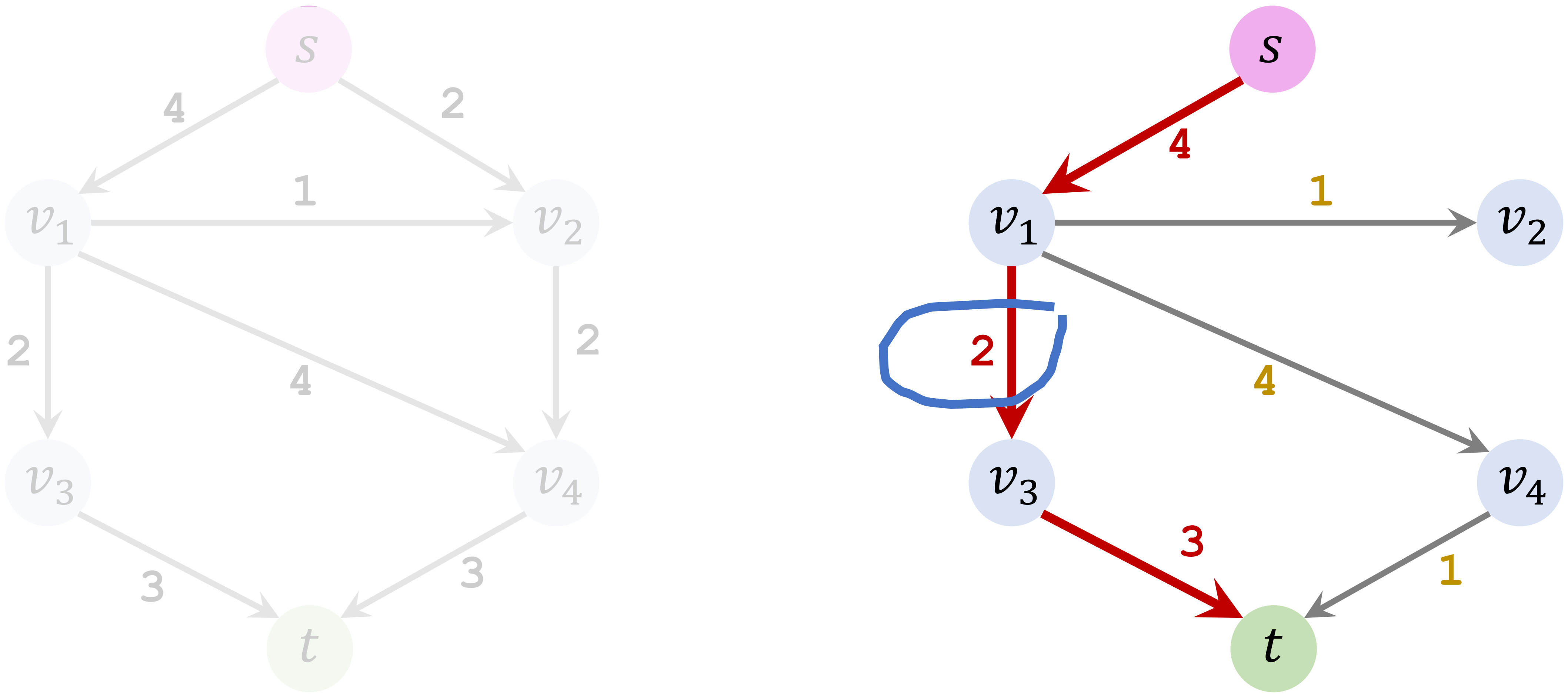


Iteration 2: Find an augmenting path



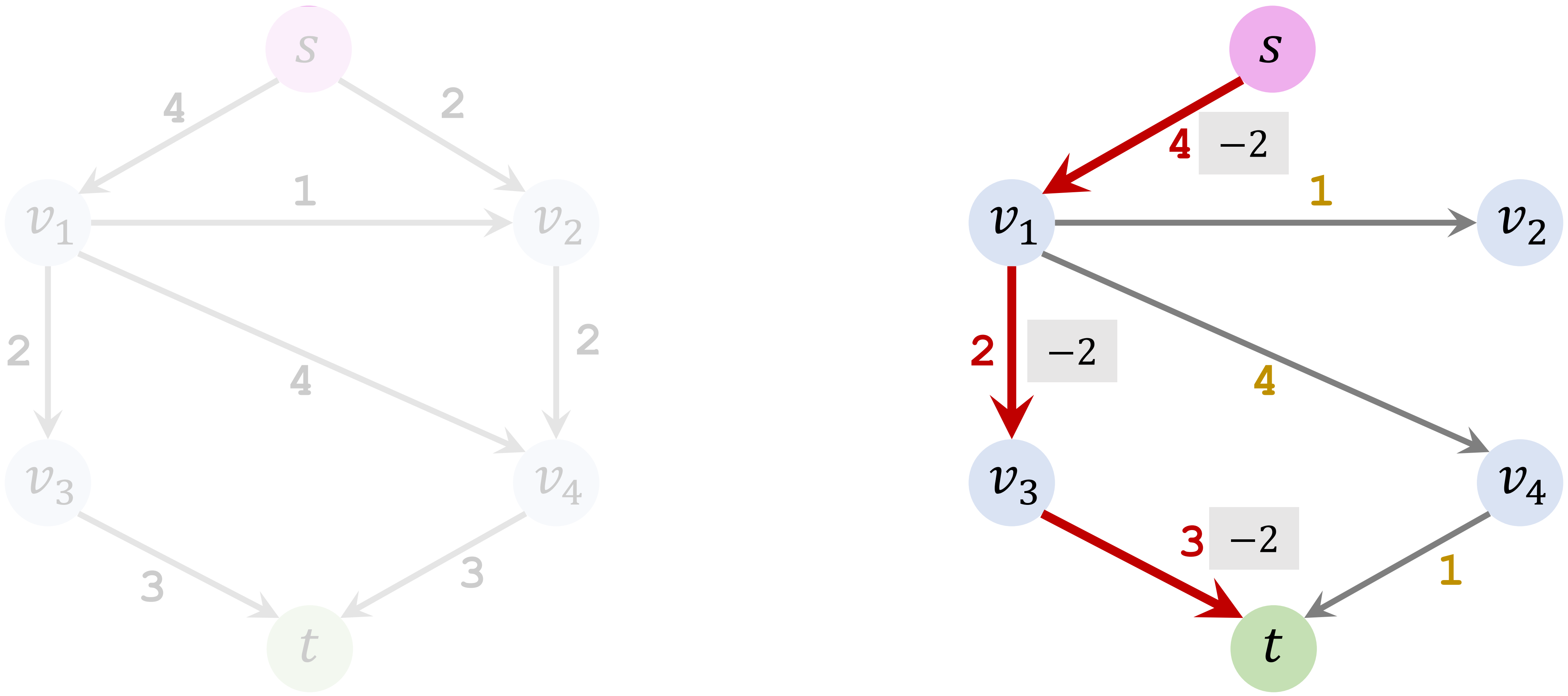
Found path $s \to v_1 \to v_3 \to t$.

Iteration 2: Find an augmenting path

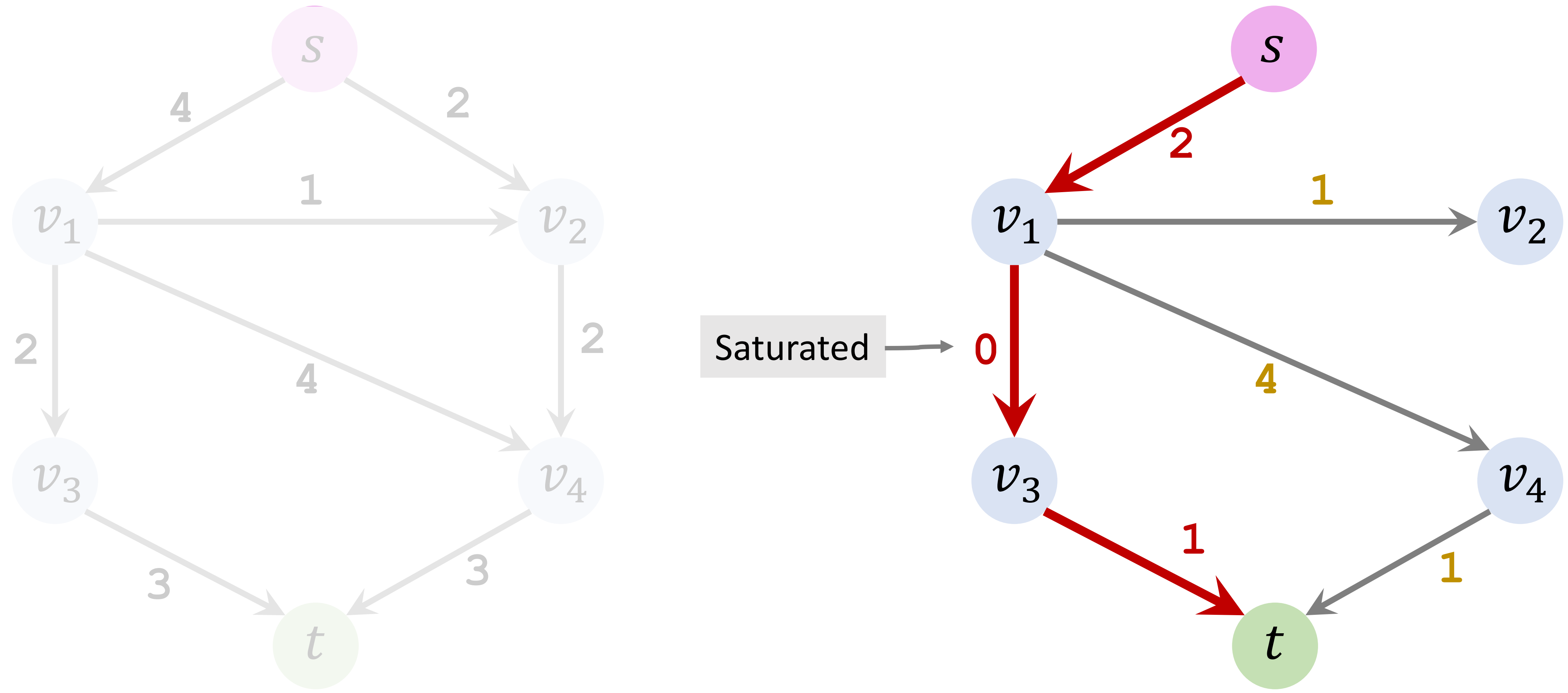


Found path $s \to v_1 \to v_3 \to t$. (Bottleneck capacity = 2.)

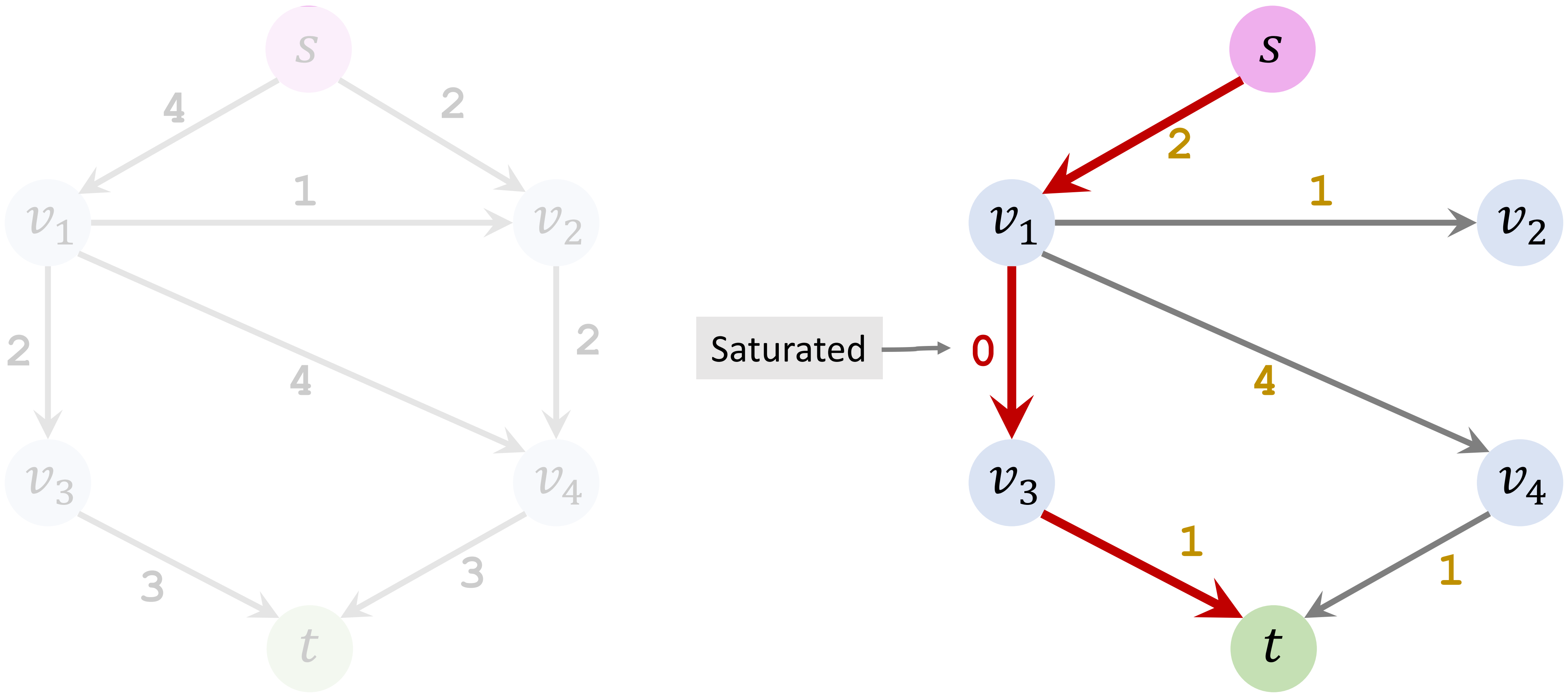
Iteration 2: Update residuals



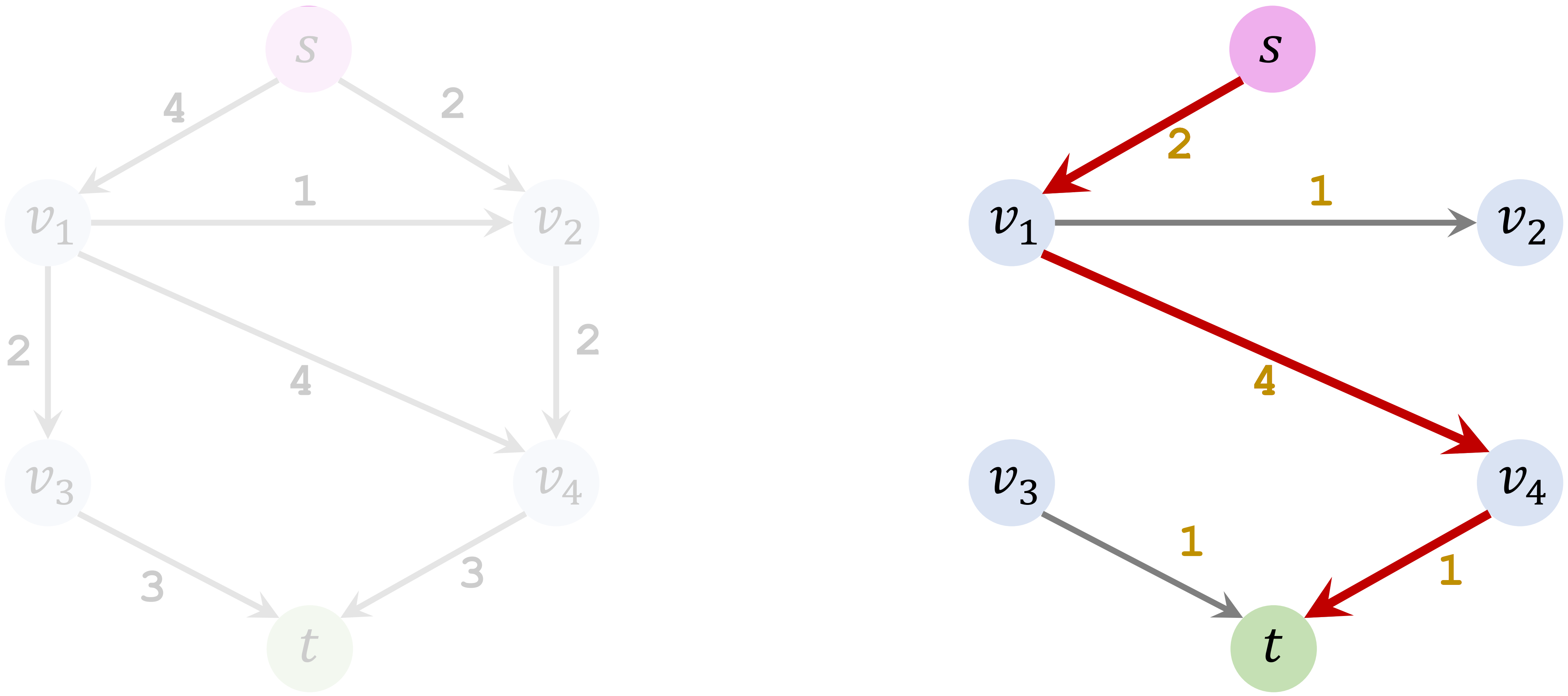
Iteration 2: Update residuals



Iteration 2: Remove saturated edges

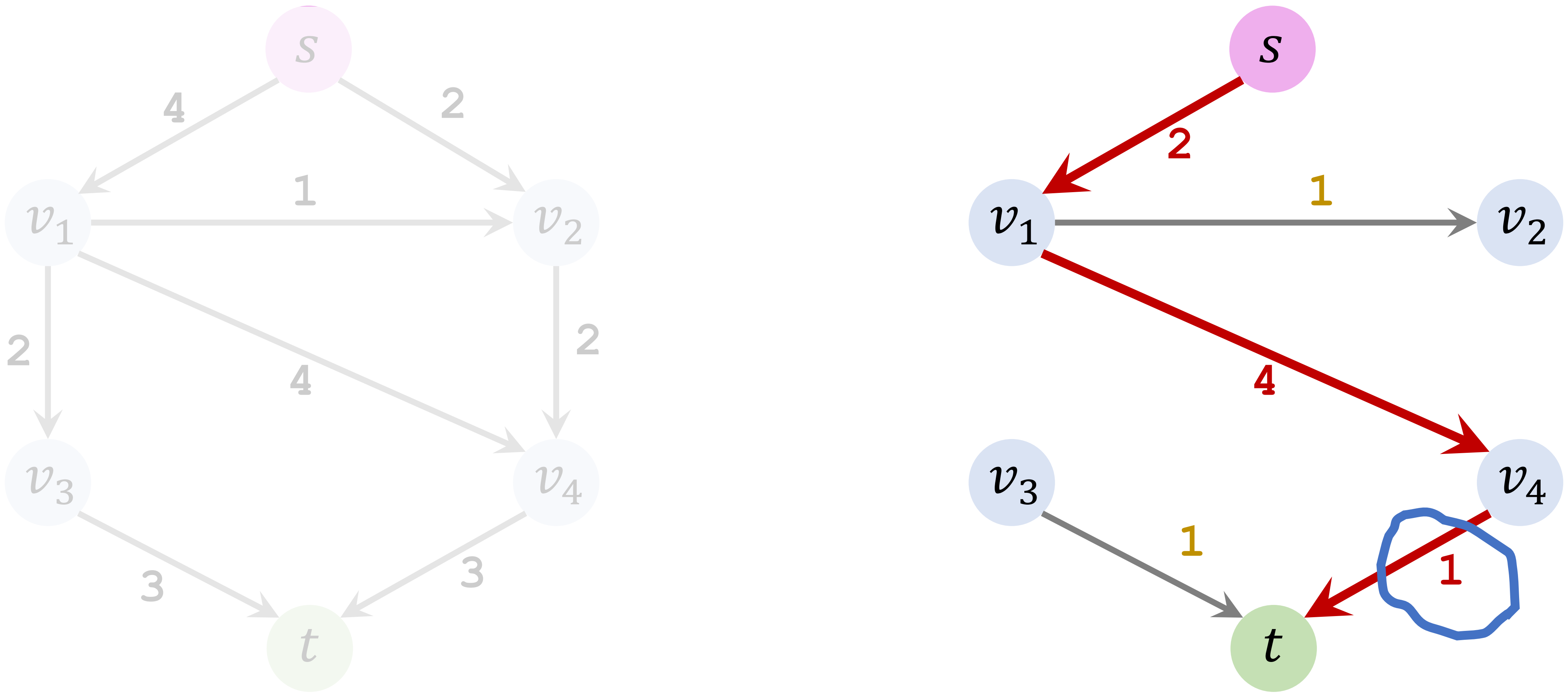


Iteration 3: Find an augmenting path



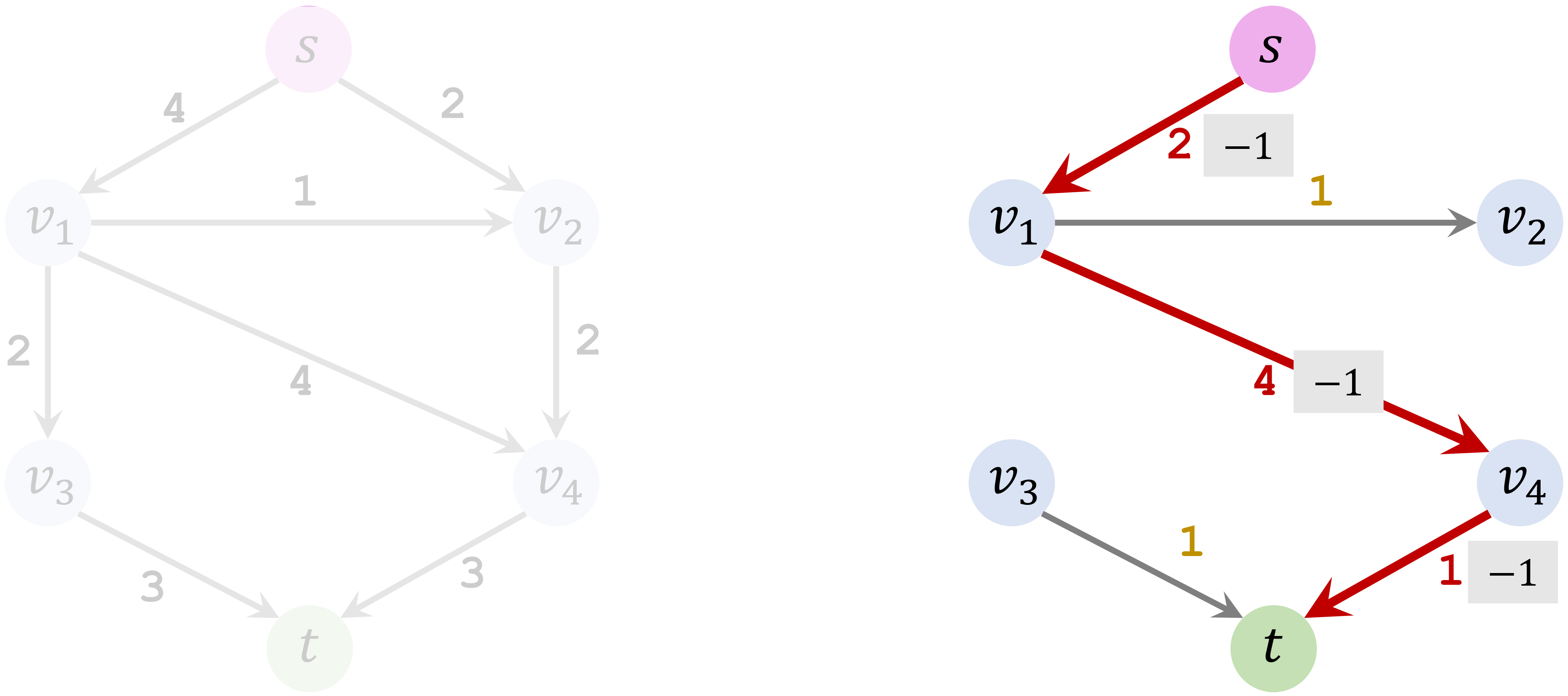
Found path $s \to v_1 \to v_4 \to t$.

Iteration 3: Find an augmenting path



Found path $s \to v_1 \to v_4 \to t$. (Bottleneck capacity $= 1$.)

Iteration 3: Update residuals



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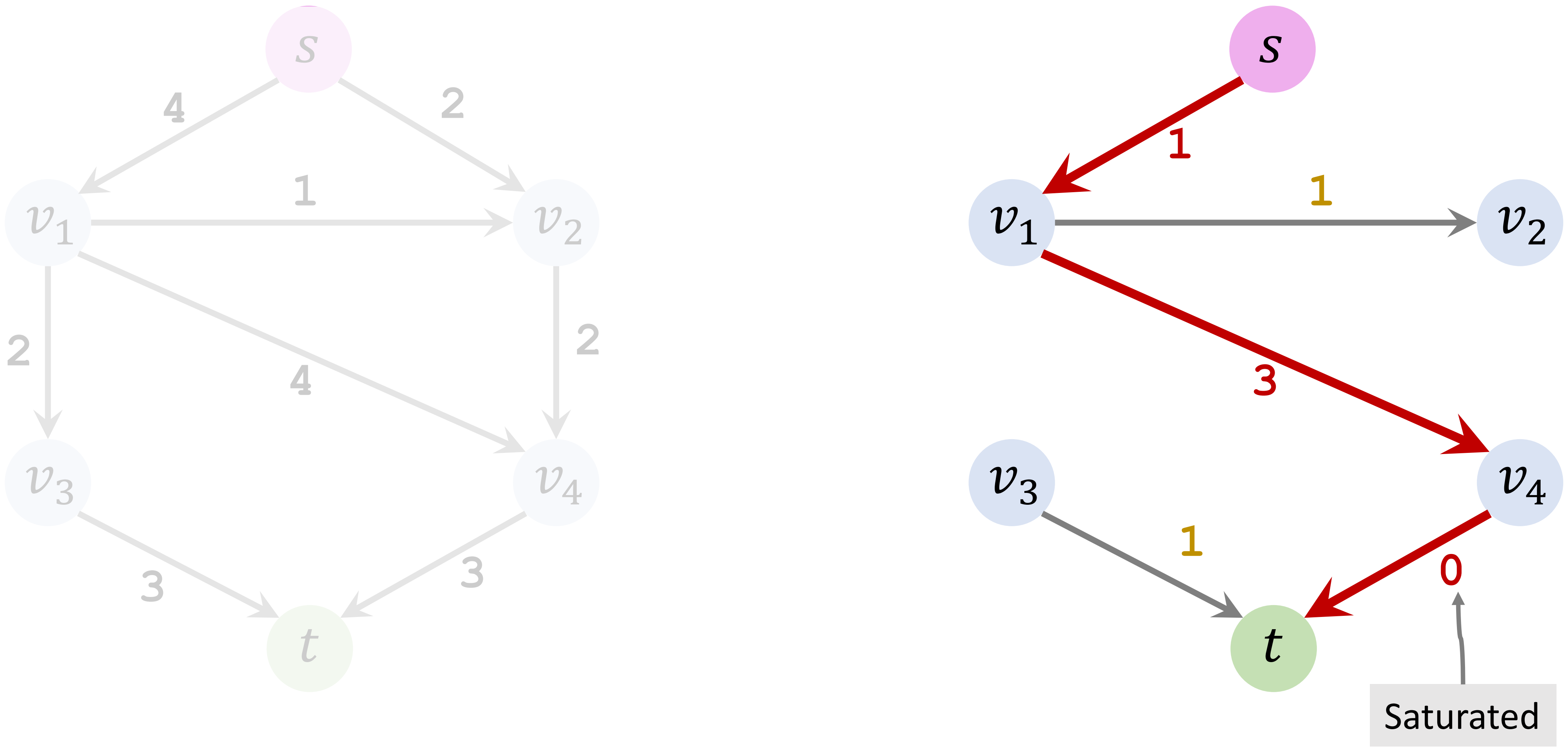
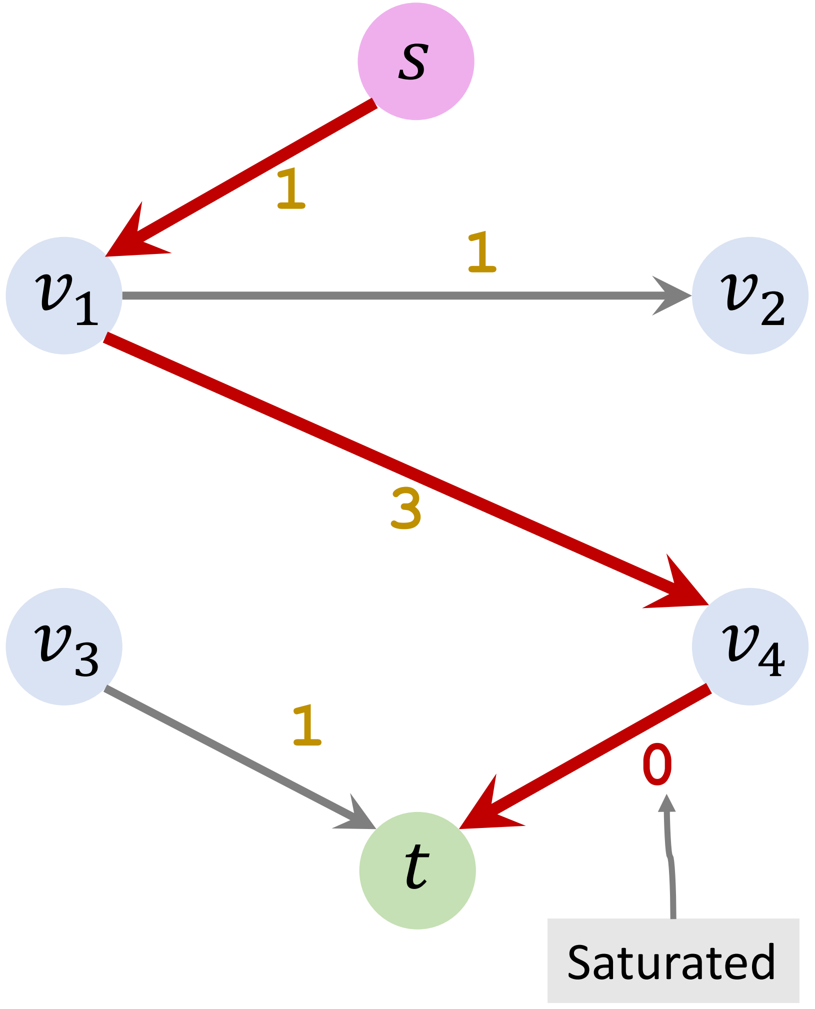
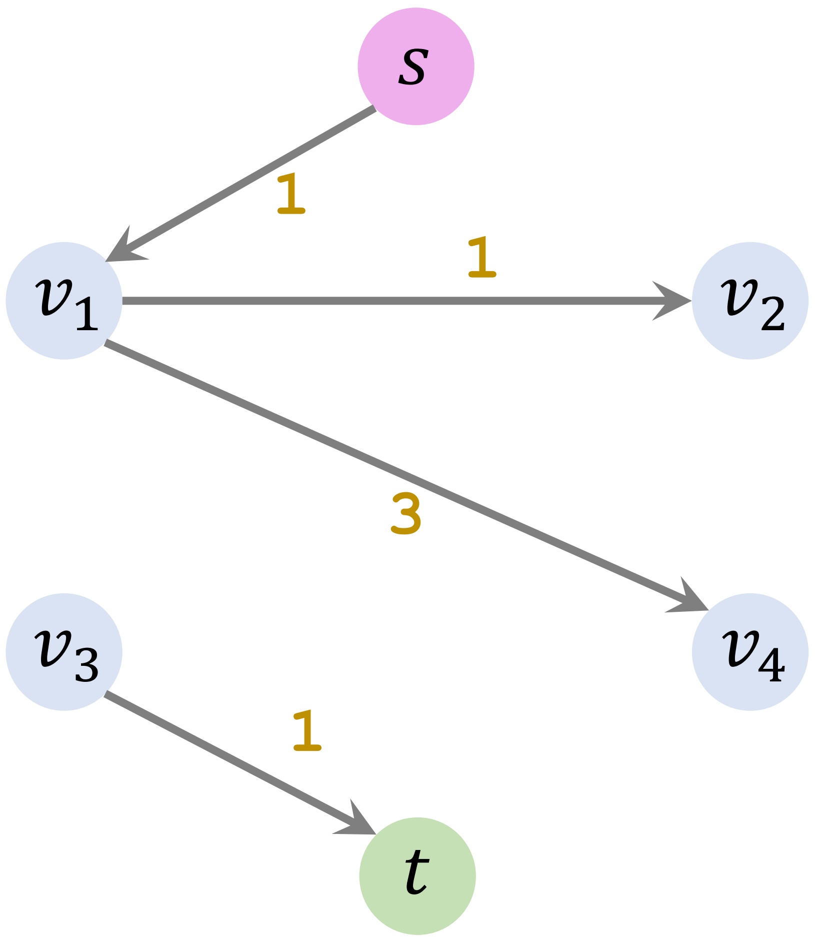


Iteration 3: Remove saturated edges

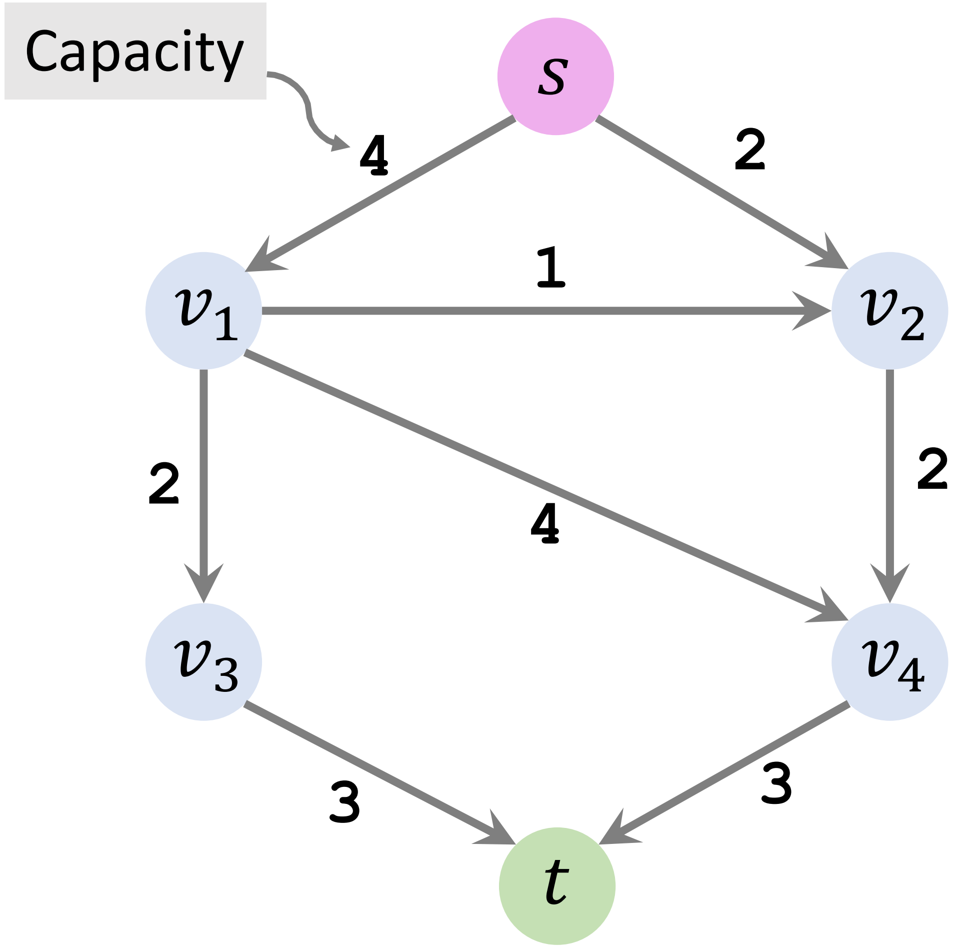


Iteration 4: Find an augmenting path

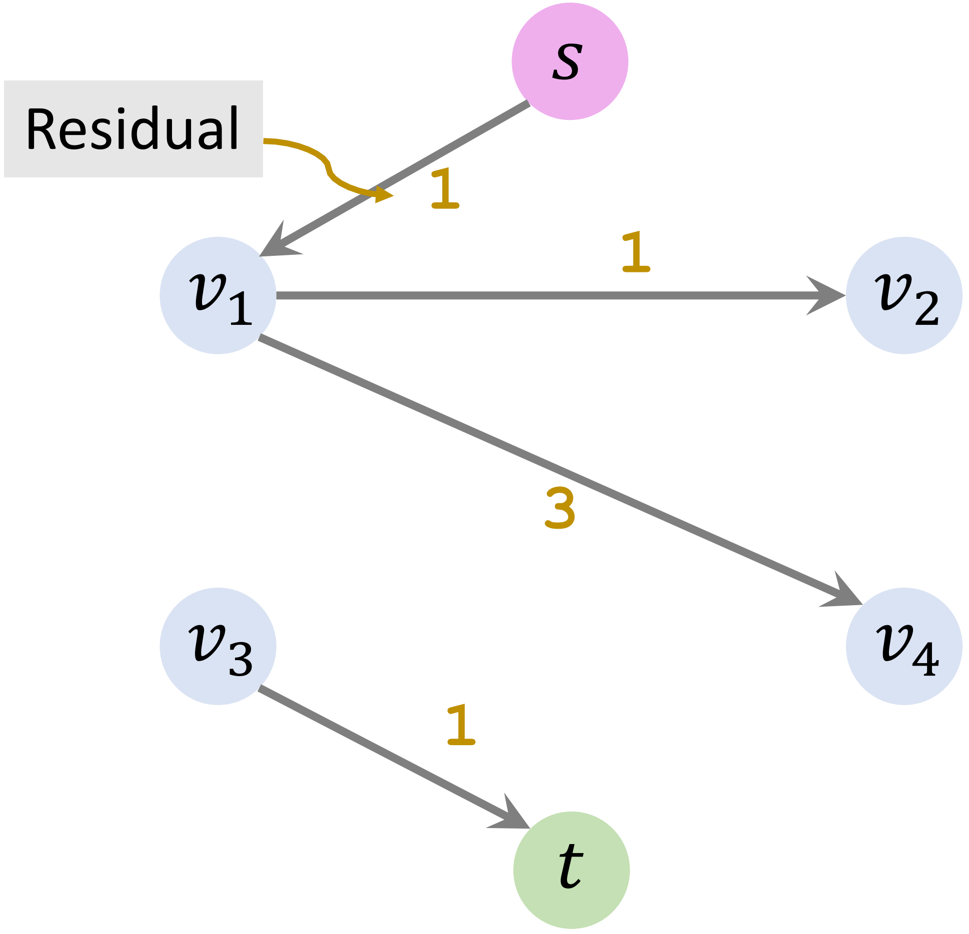


Cannot find any path from source to sink.

End of Procedure

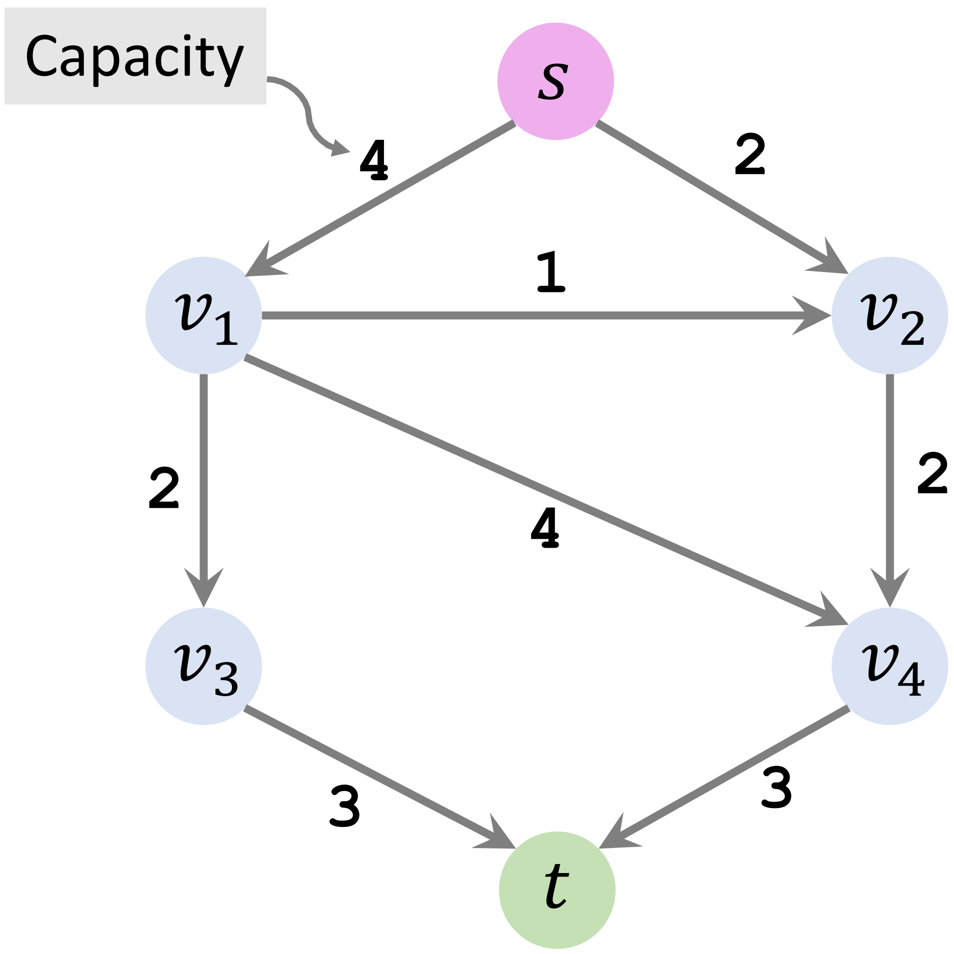


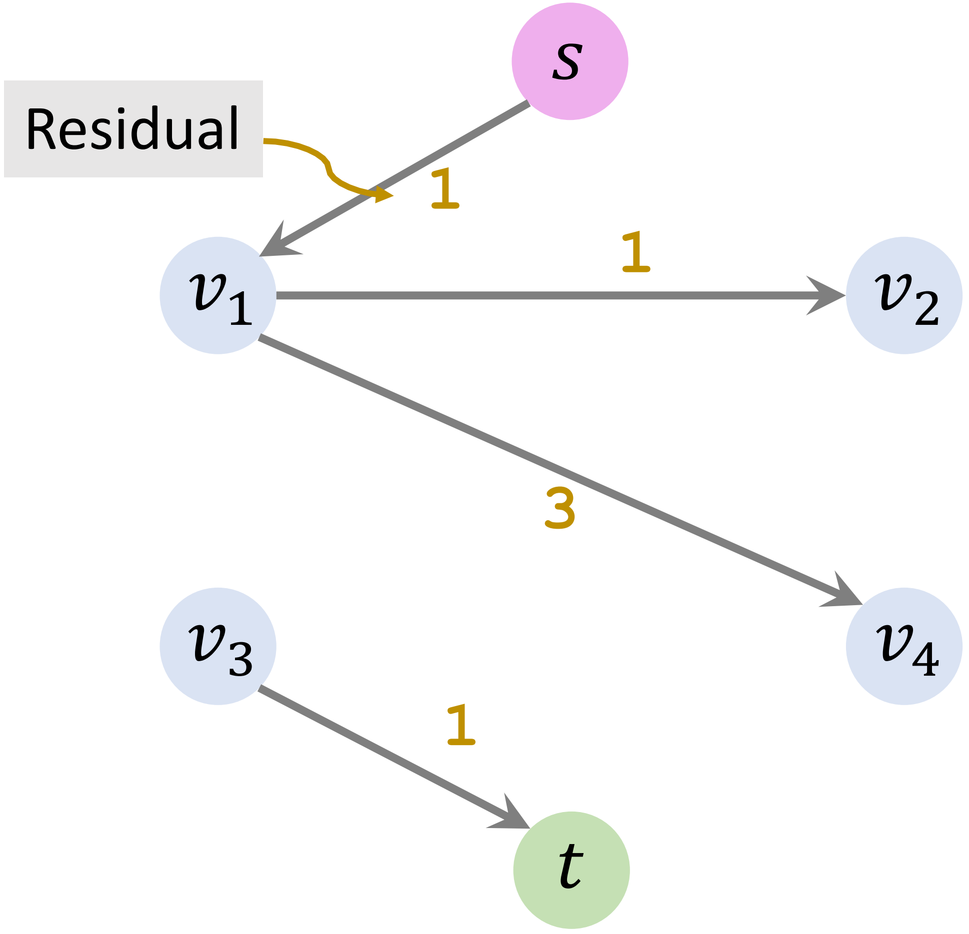
Original Graph



Residual Graph

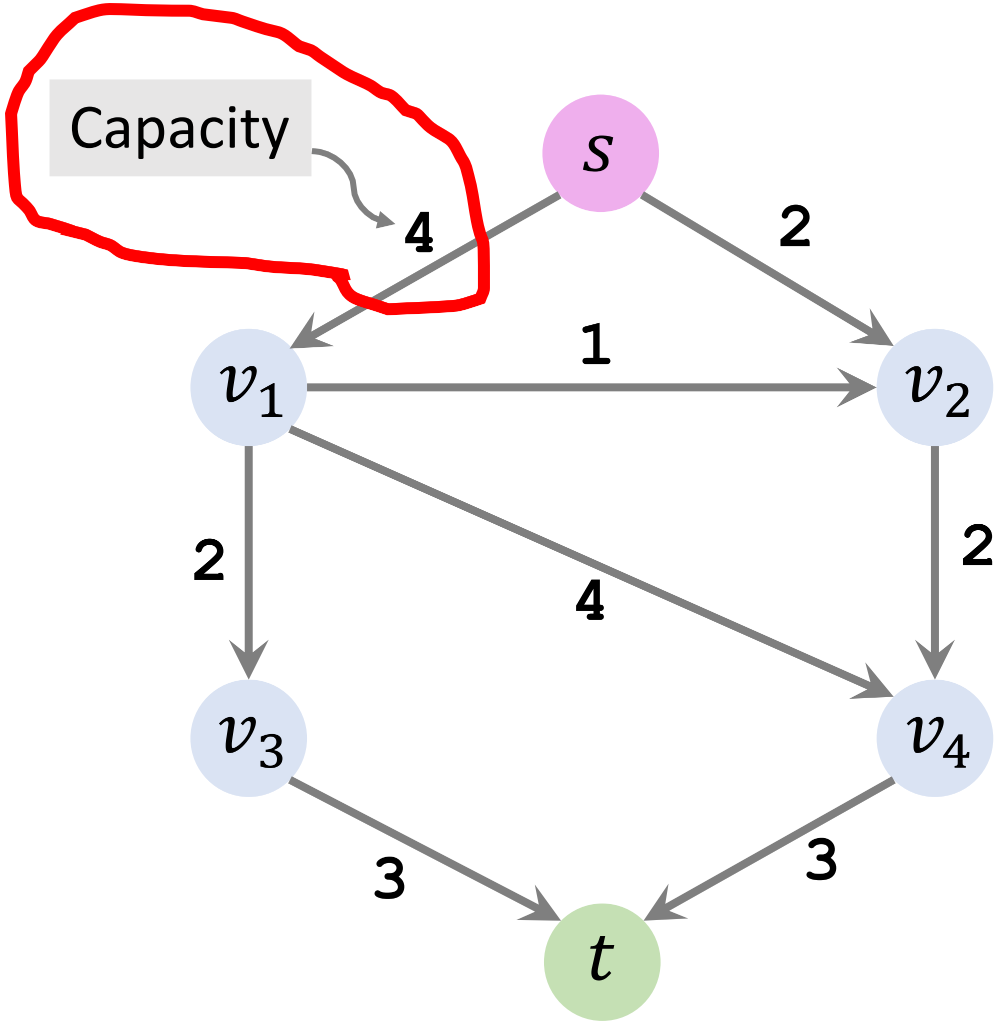
End of Procedure

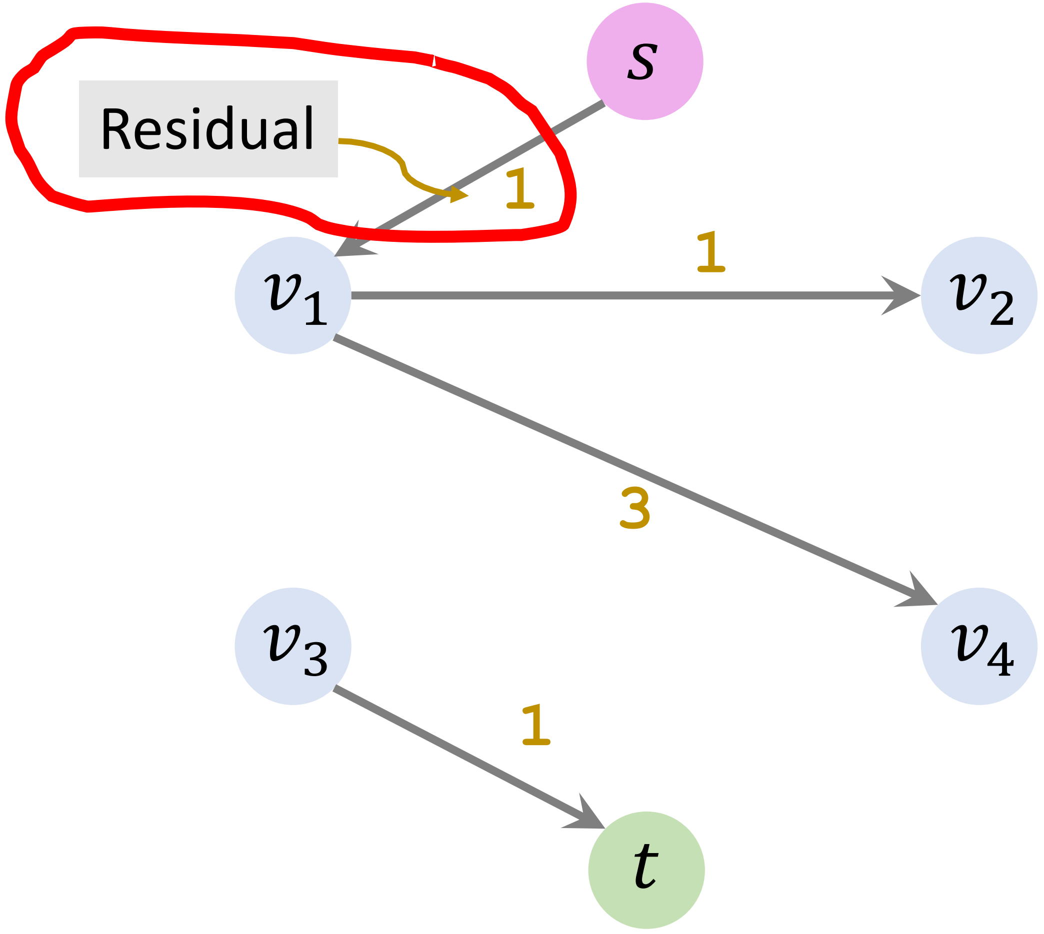




Flow $=$ Capacity $-$ Residual.

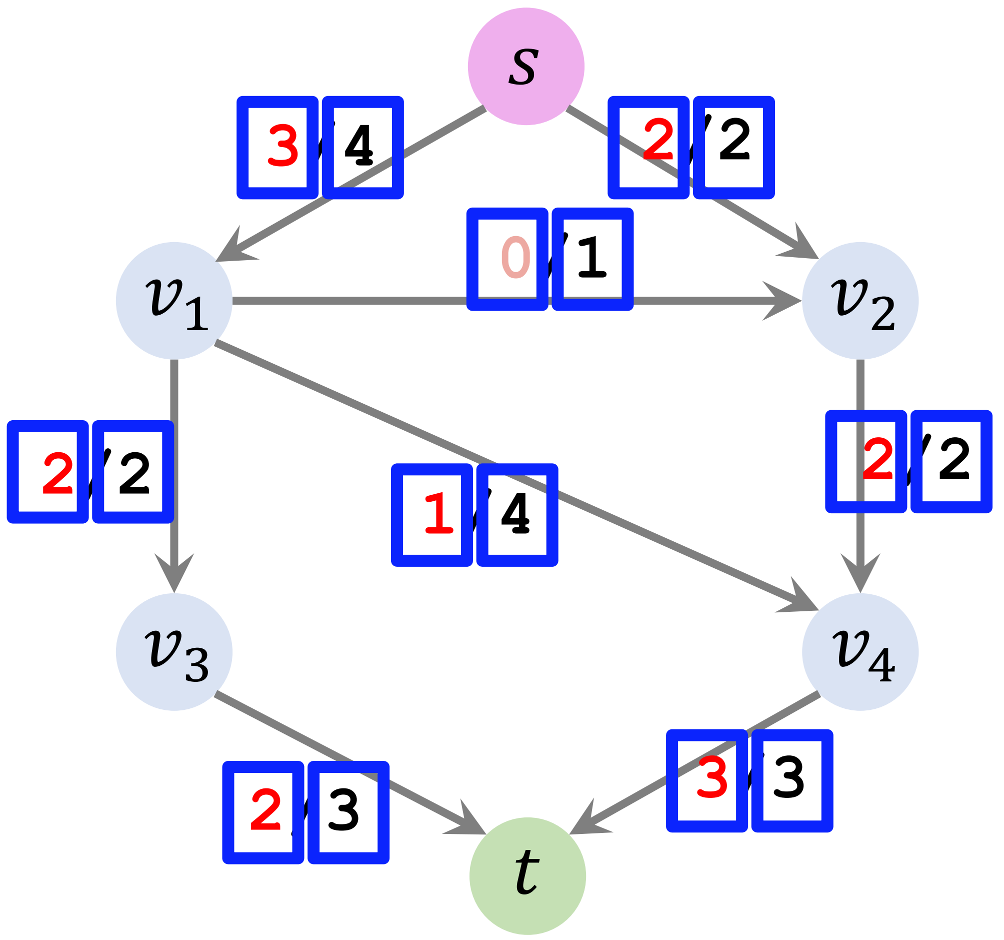
End of Procedure

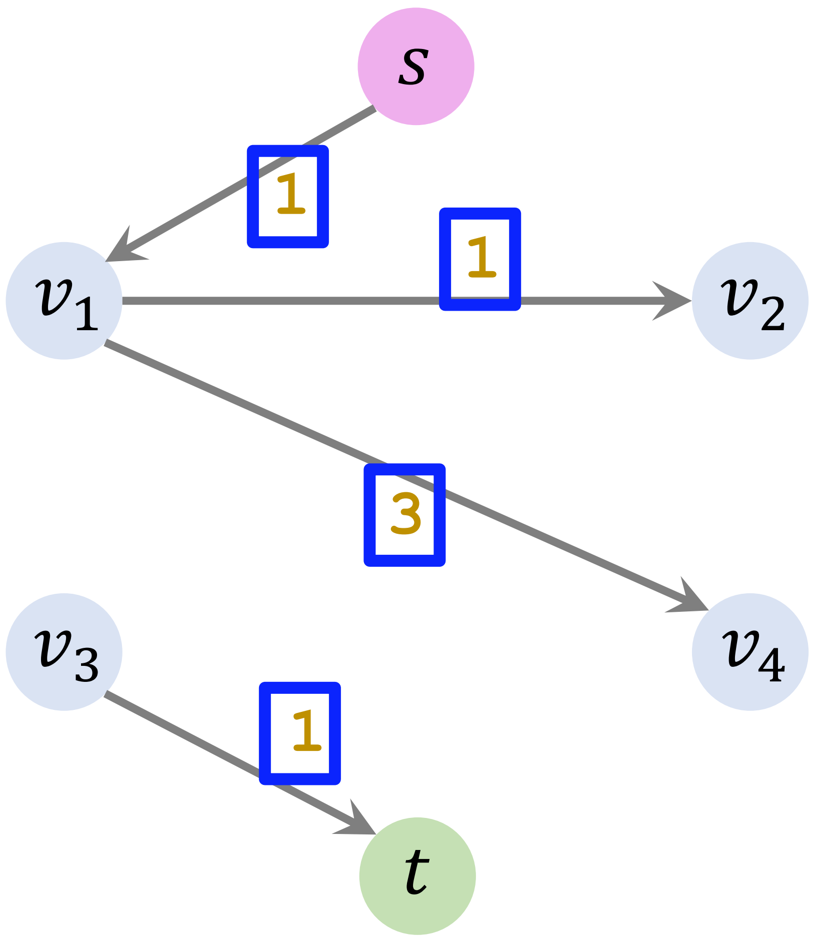




Flow = Capacity − Residual.

End of Procedure





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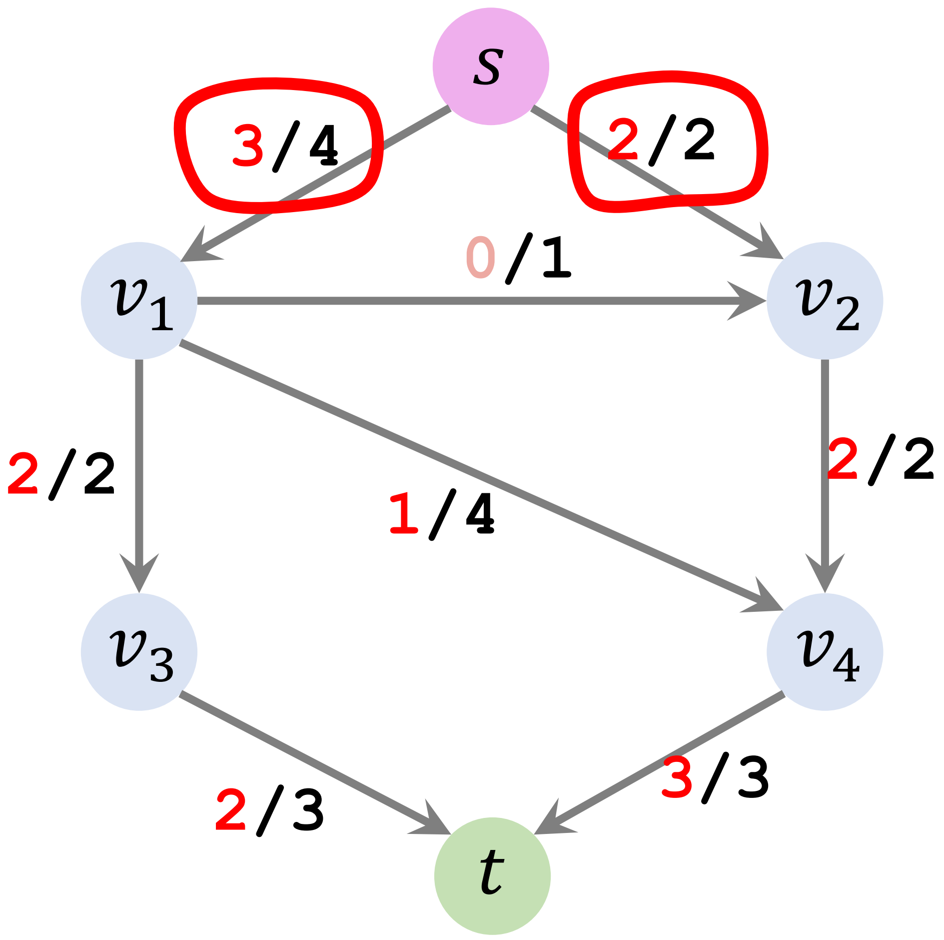
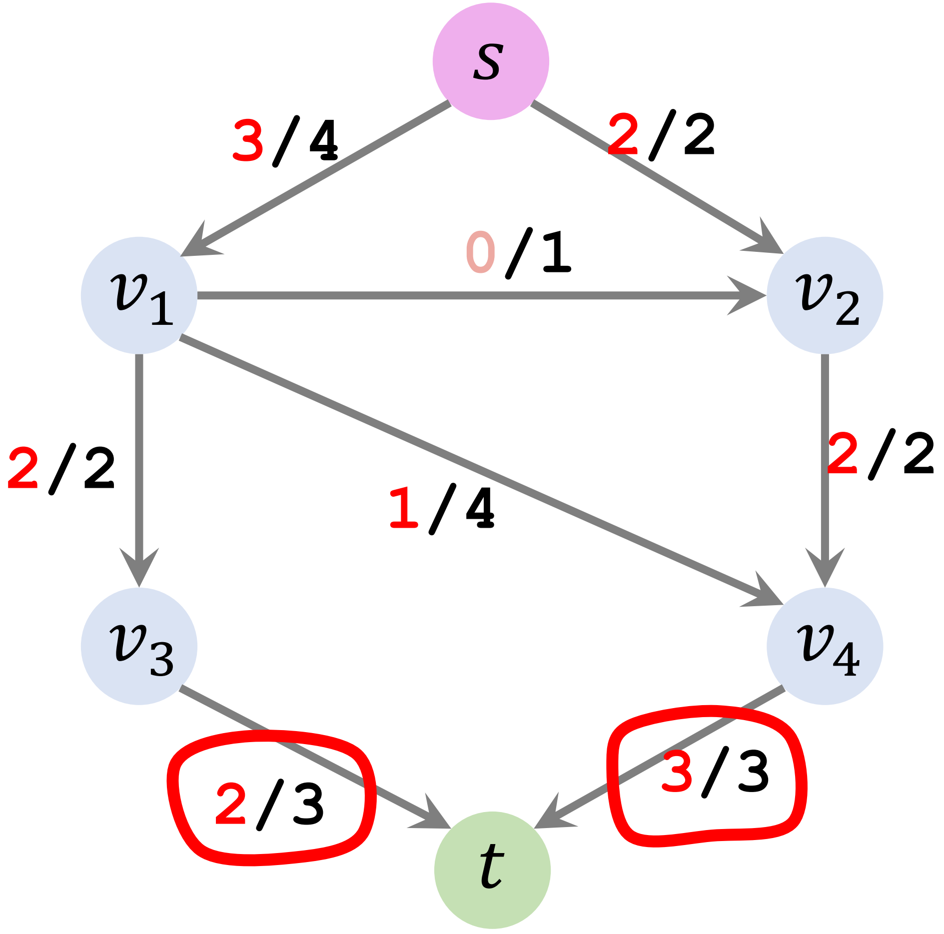


Amount of Flow = 5.

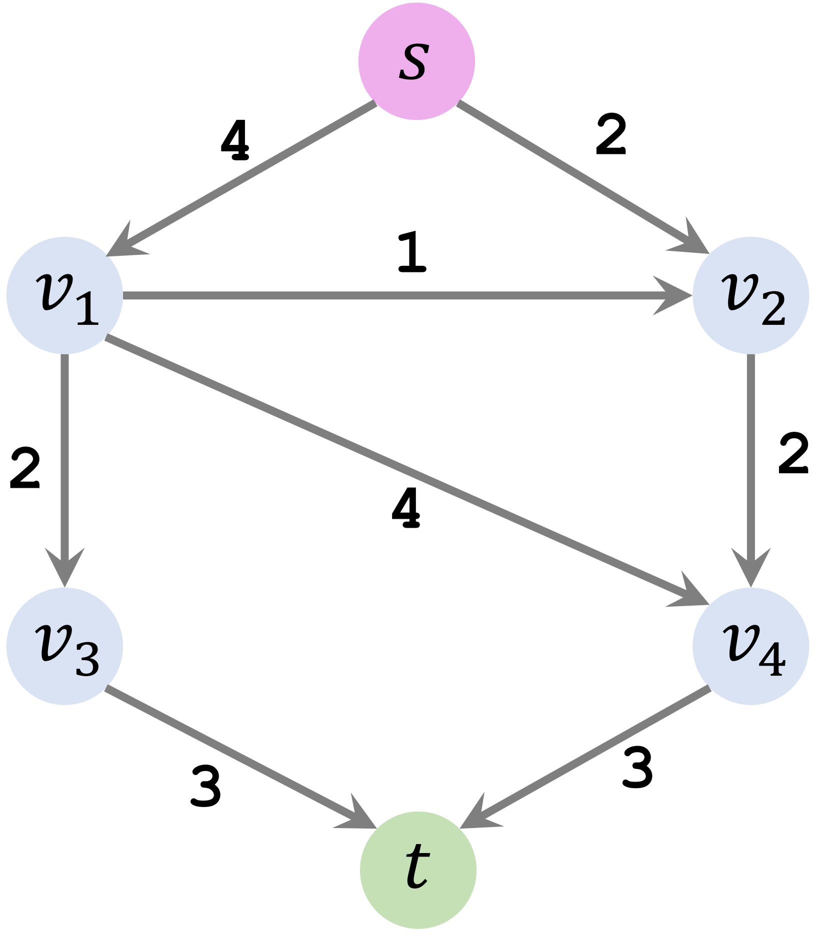
End of Procedure



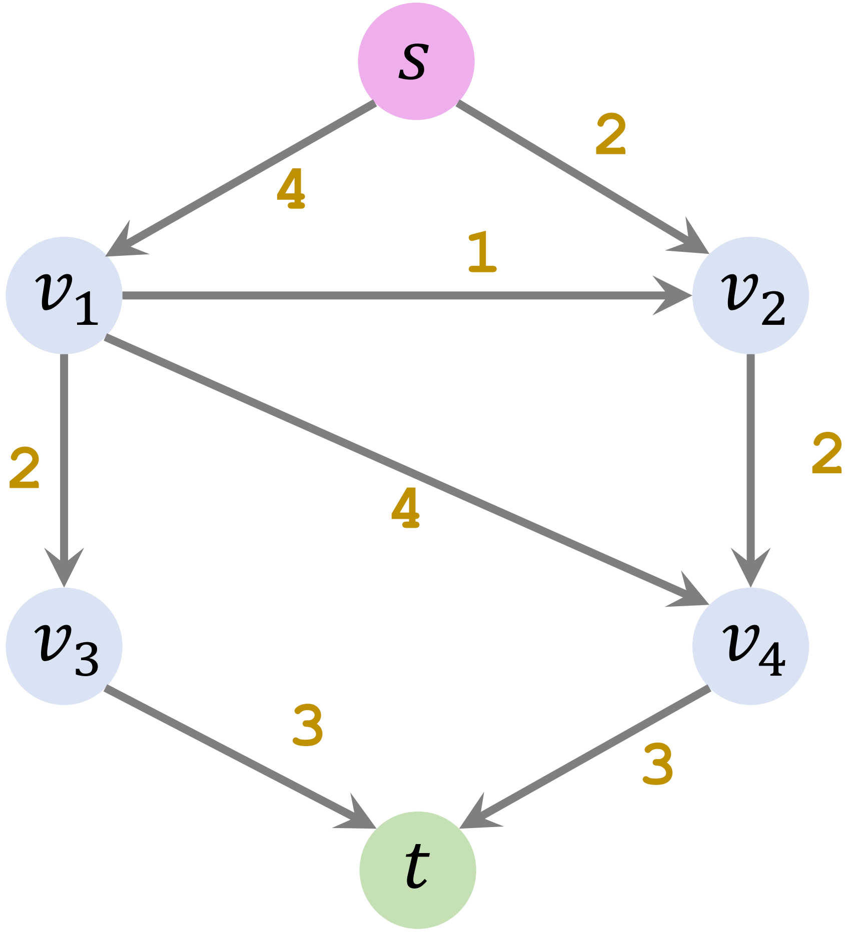
Amount of Flow = 5.

The naïve algorithm can fail!

Initial State

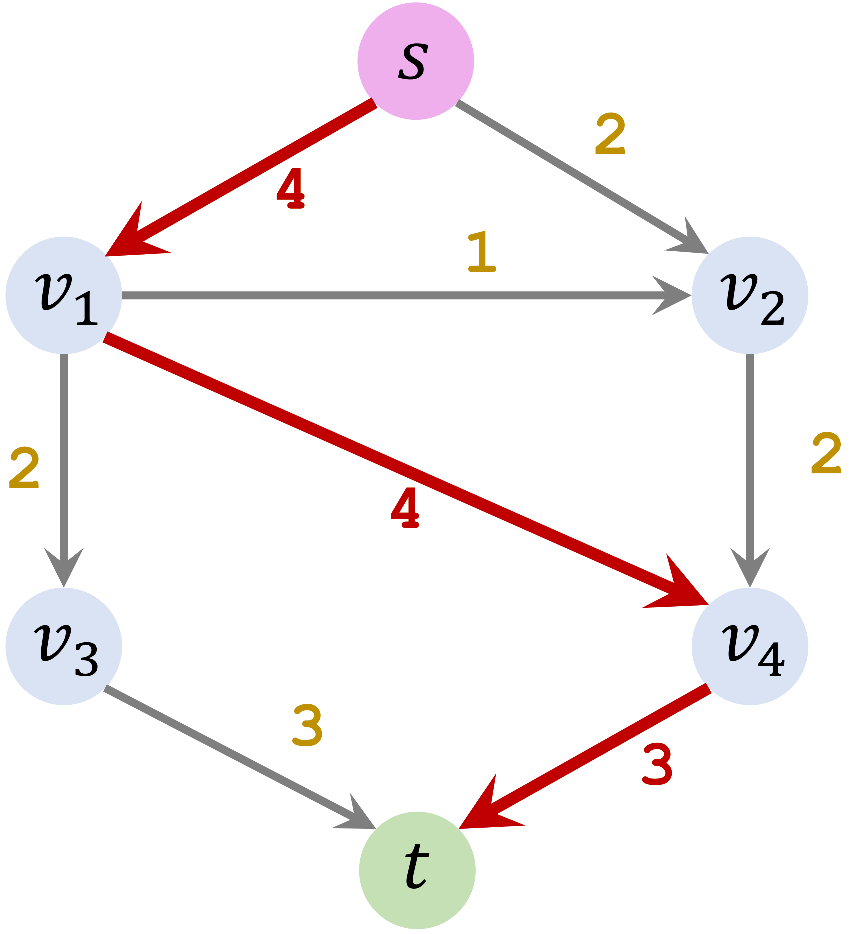


Original Graph



Residual Graph

Iteration 1: Find an augmenting path



Found path $s \to v_1 \to v_4 \to t$.

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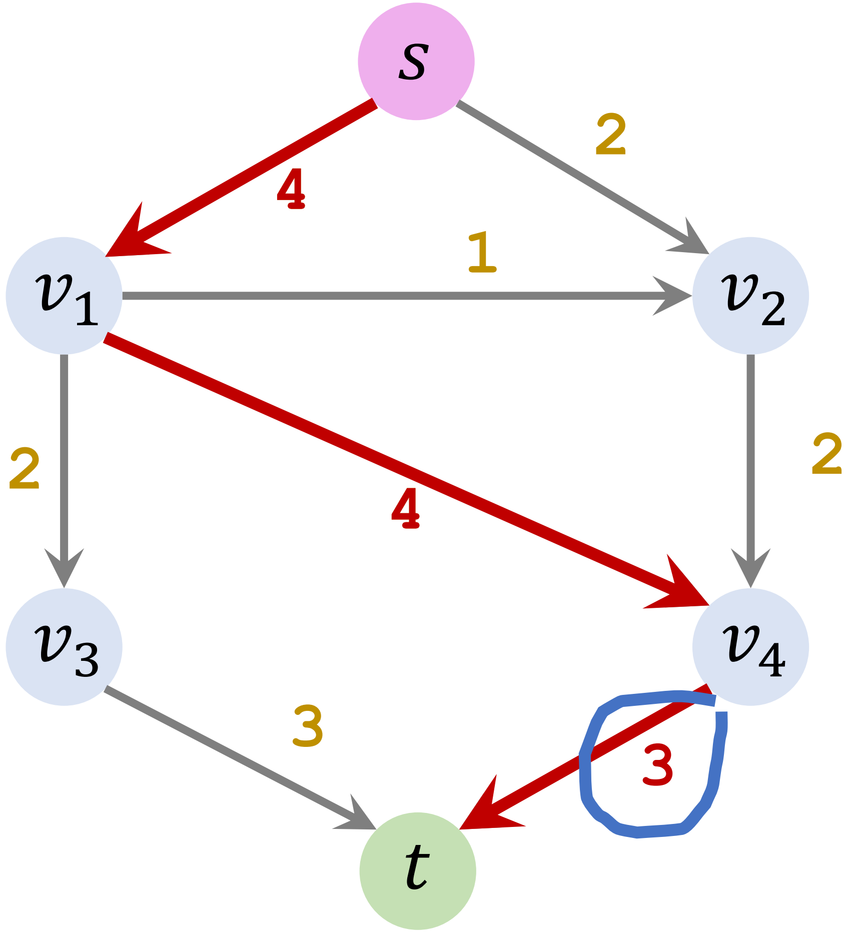
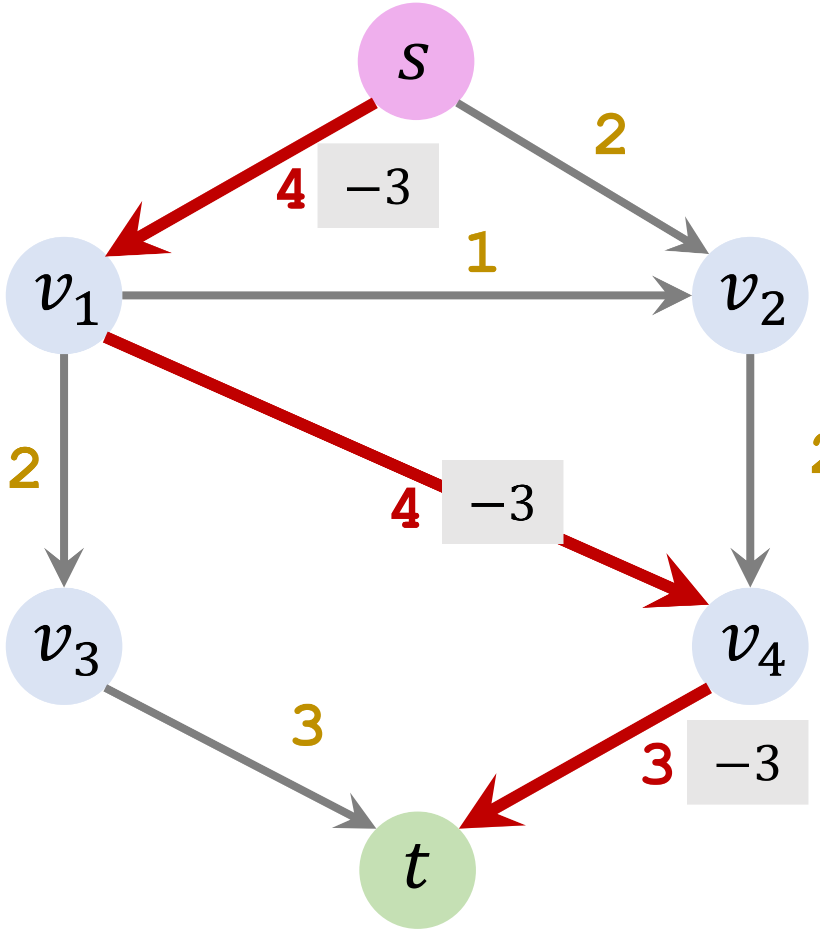


Found path $s \to v_1 \to v_4 \to t$. (Bottleneck capacity = 3.)

Iteration 1: Update residuals



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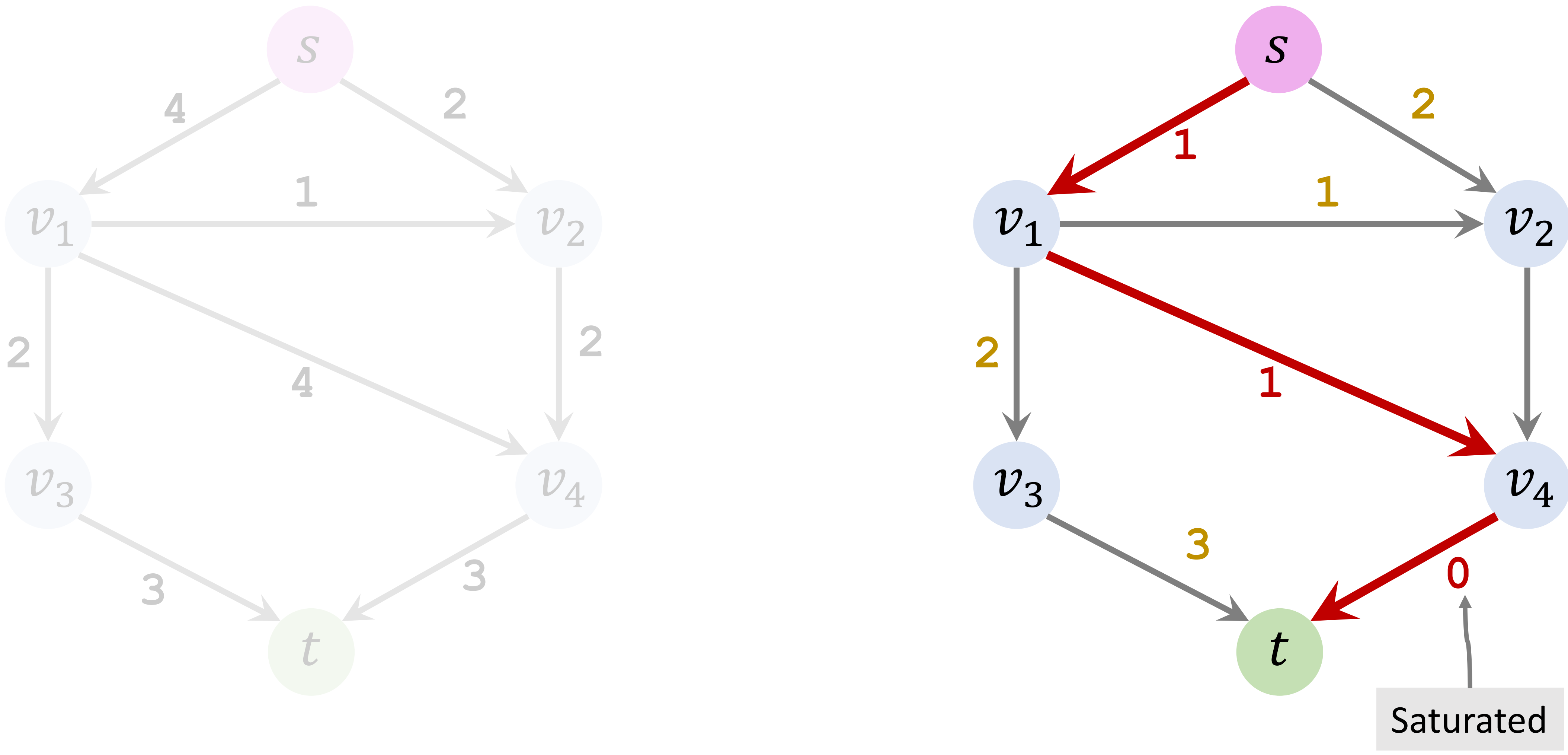
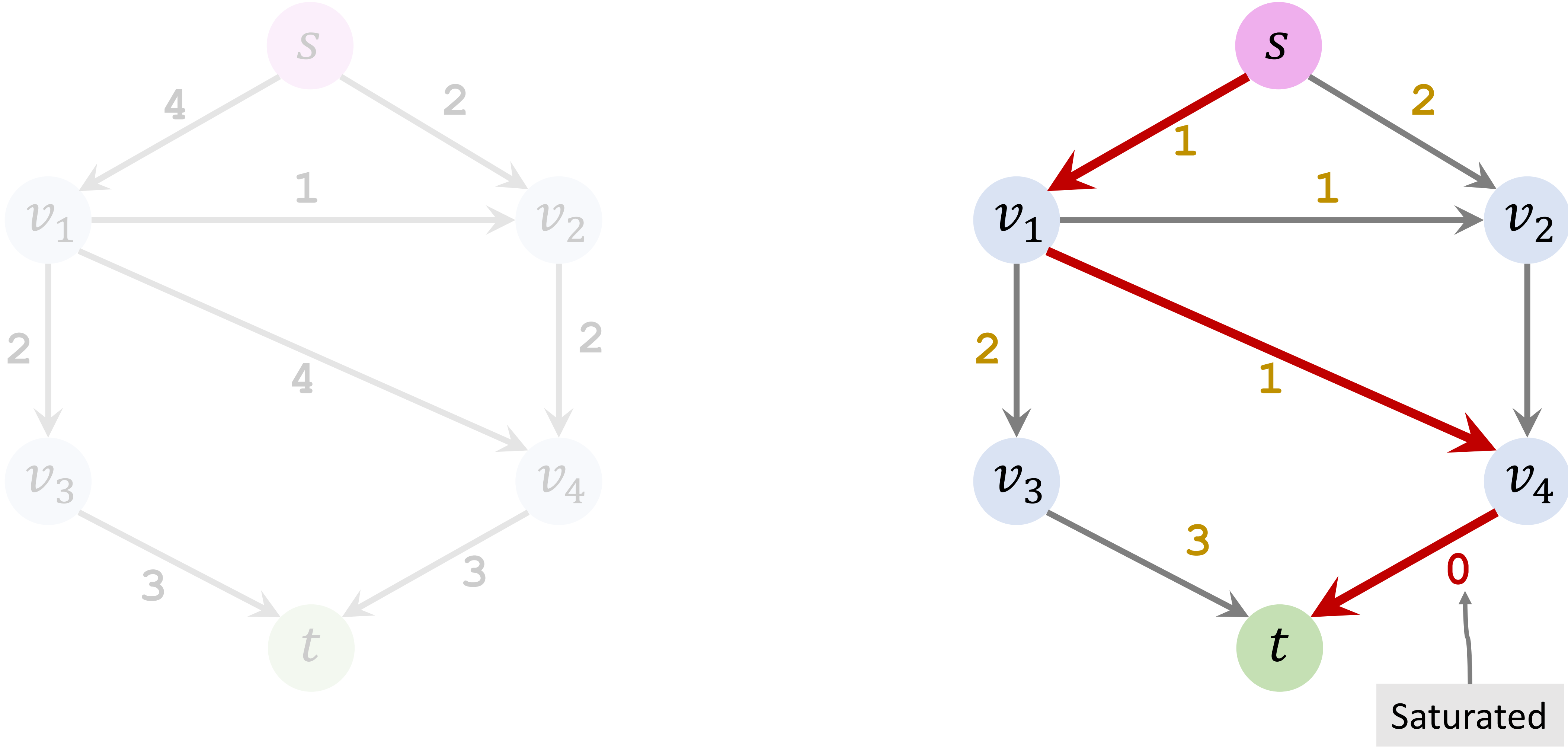
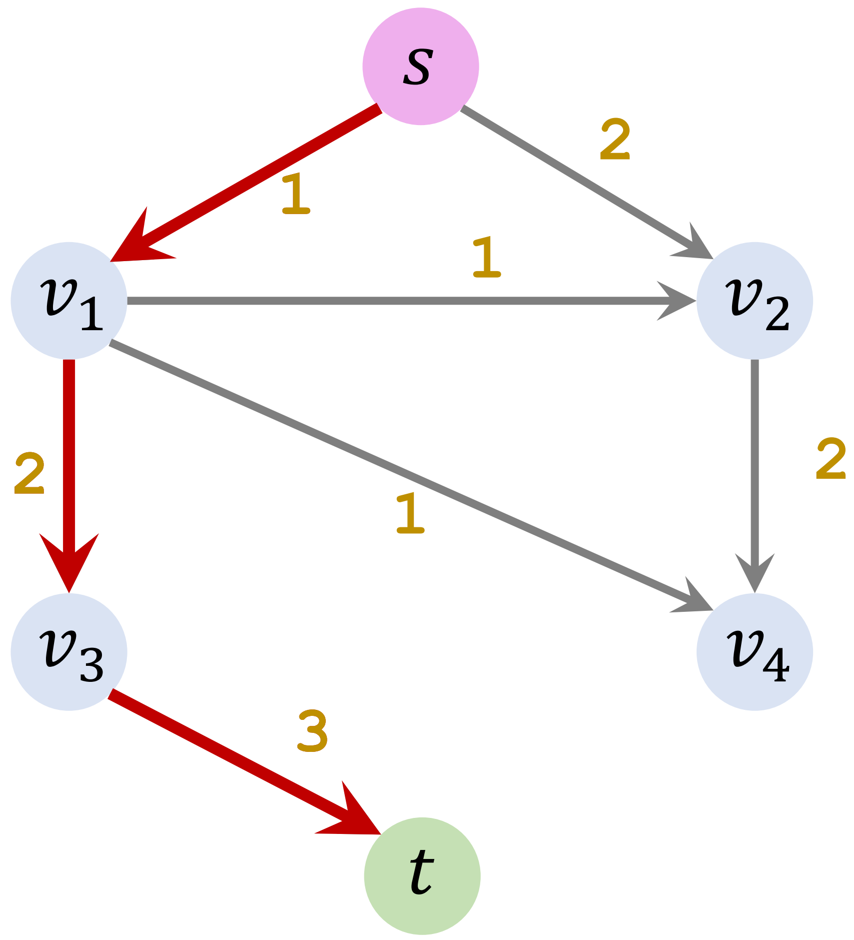


Iteration 1: Remove saturated edges

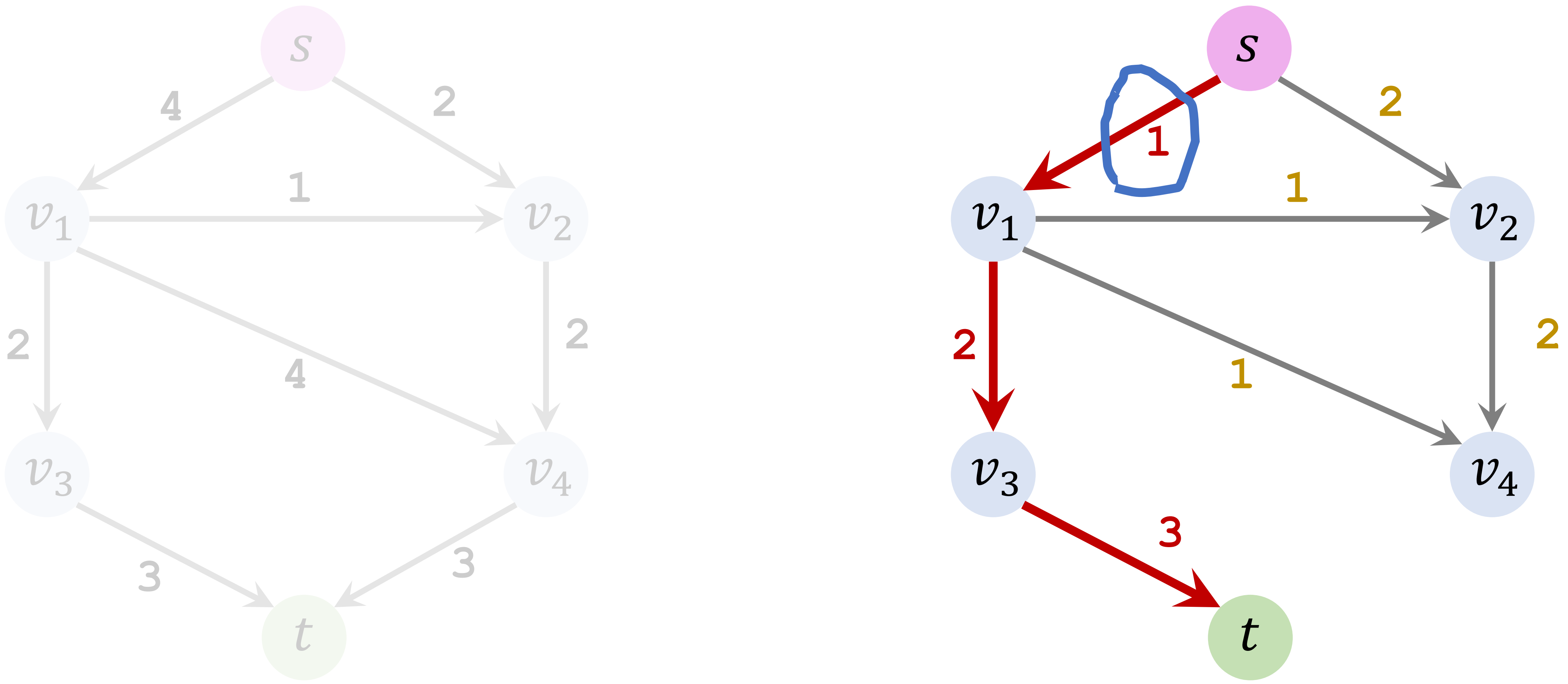


Iteration 2: Find an augmenting path



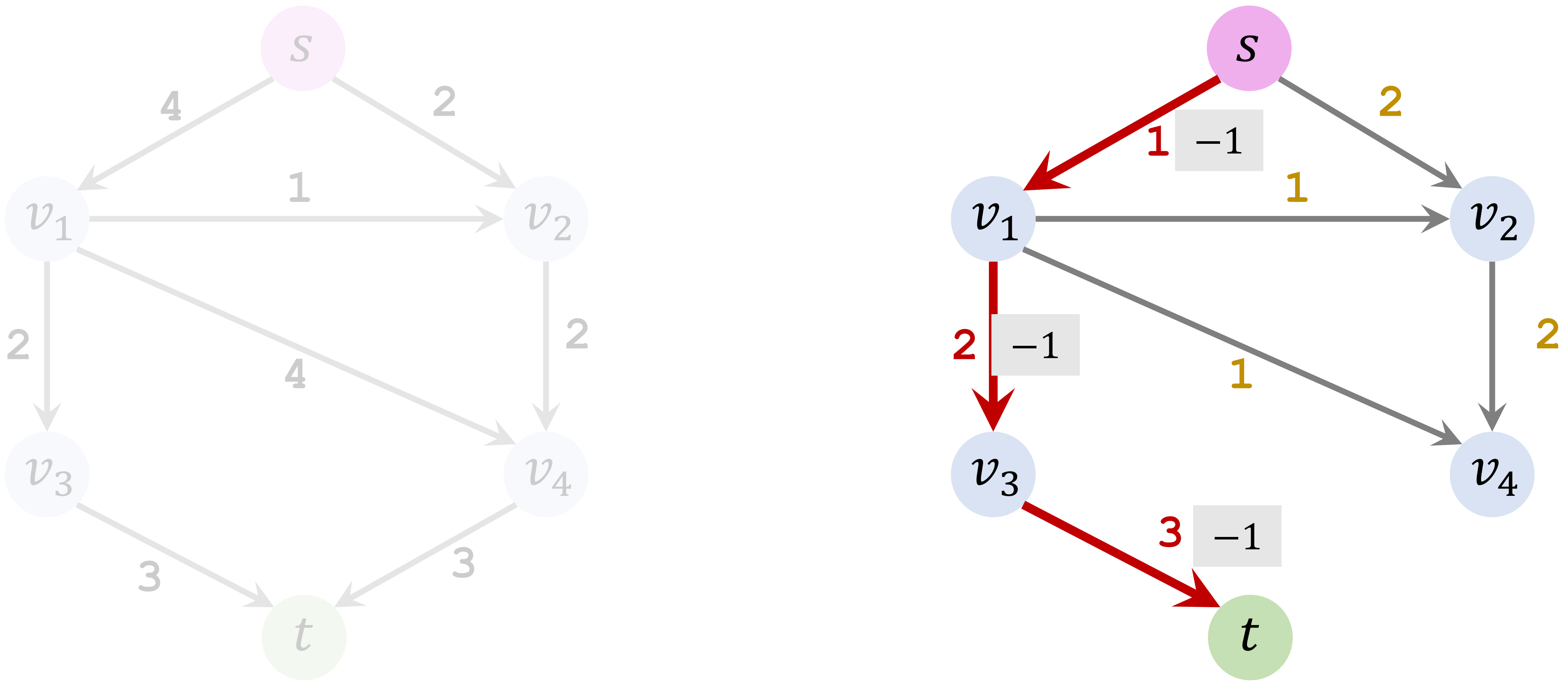
Found path $s \to v_1 \to v_3 \to t$.

Iteration 2: Find an augmenting path

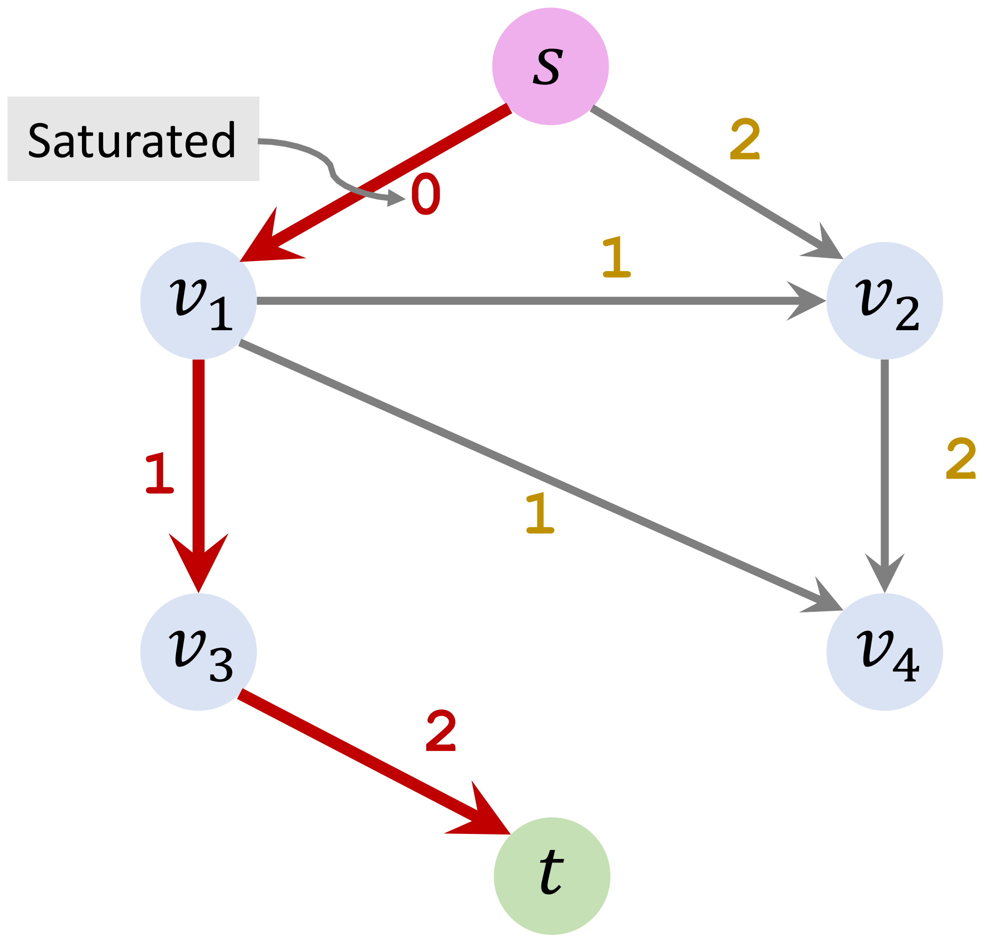


Found path $s \to v_1 \to v_3 \to t$. (Bottleneck capacity $= 1$.)

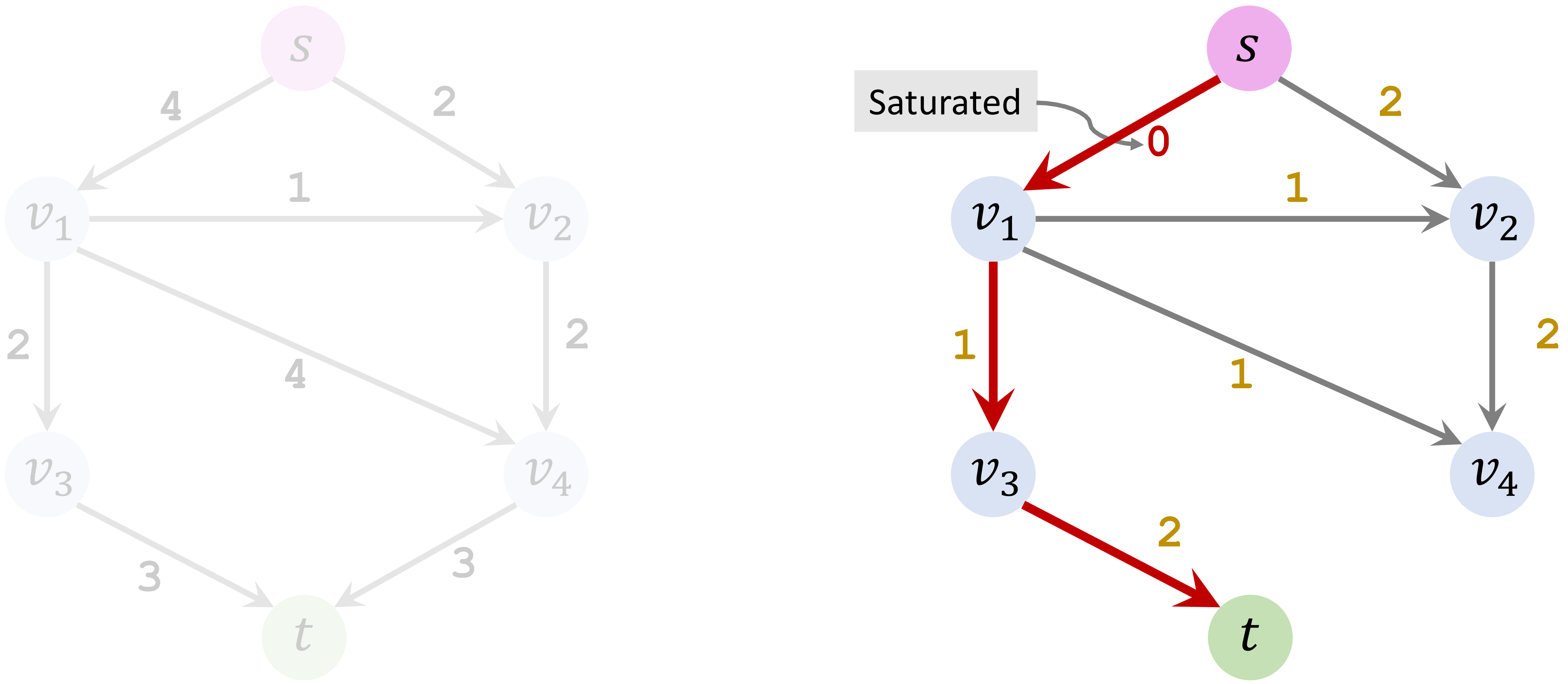
Iteration 2: Update residuals



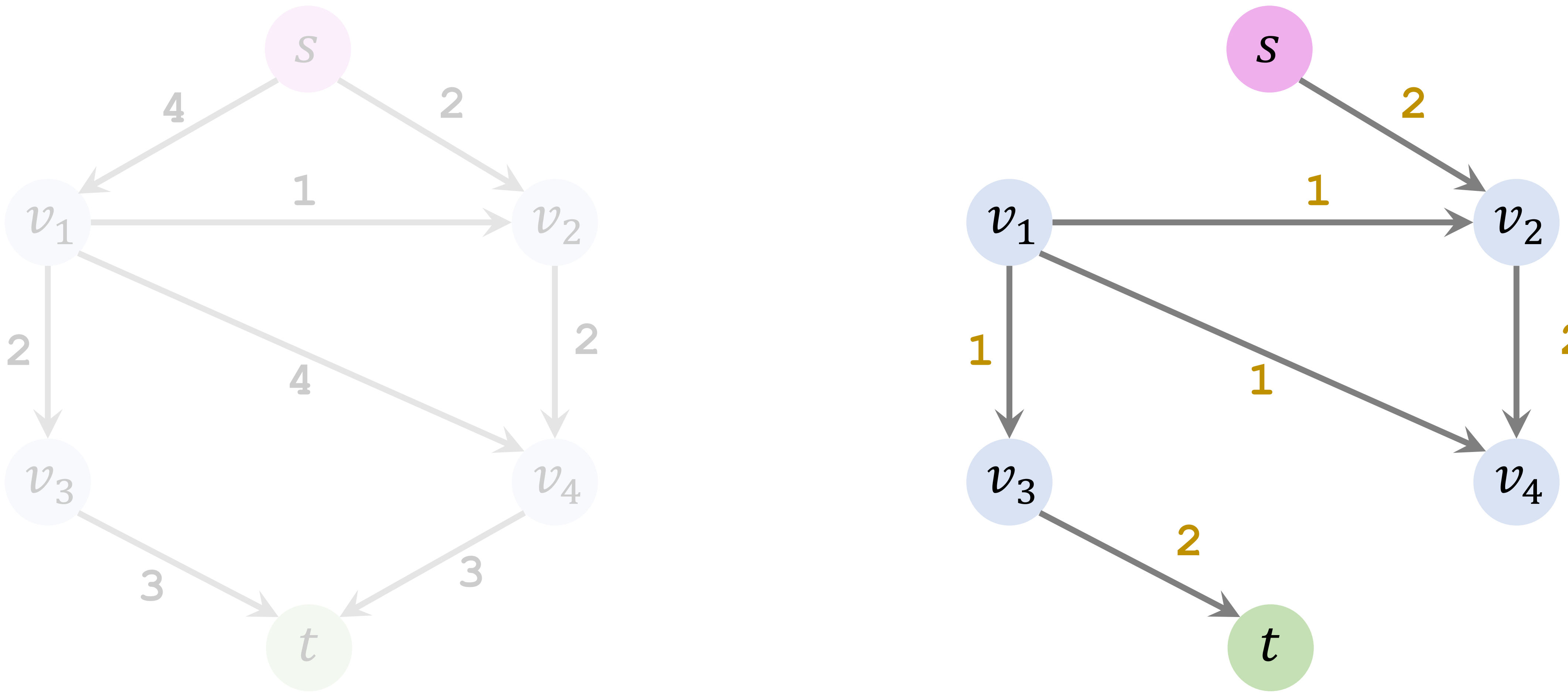
Iteration 2: Update residuals



Iteration 2: Remove saturated edges

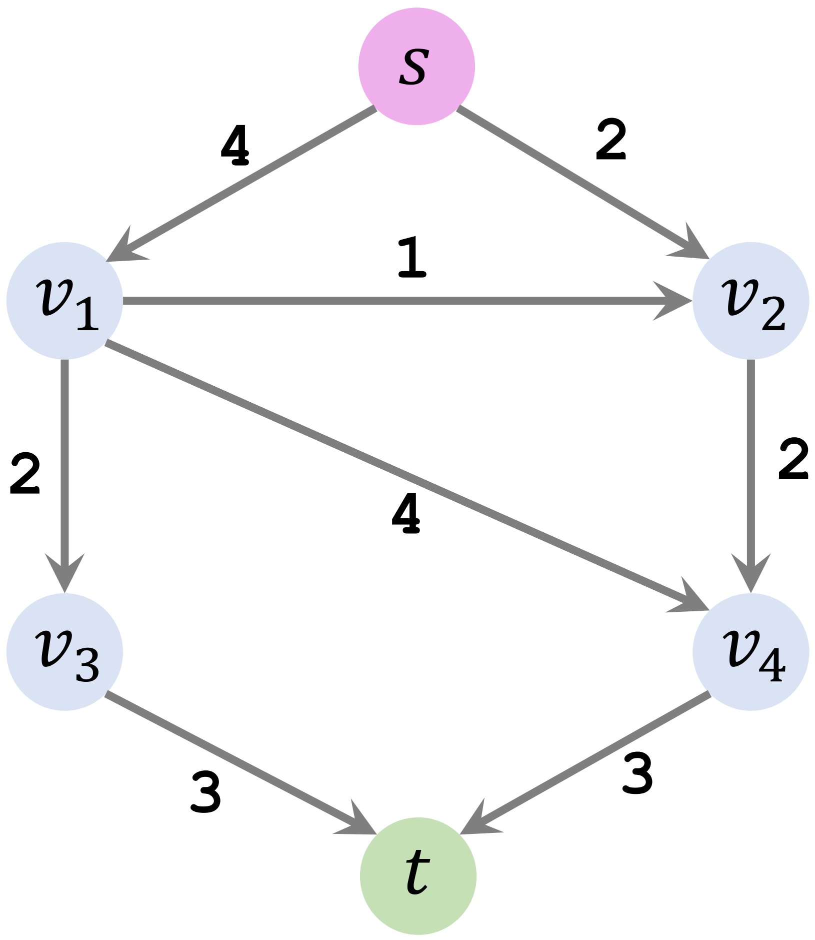


Iteration 3: Find an augmenting path

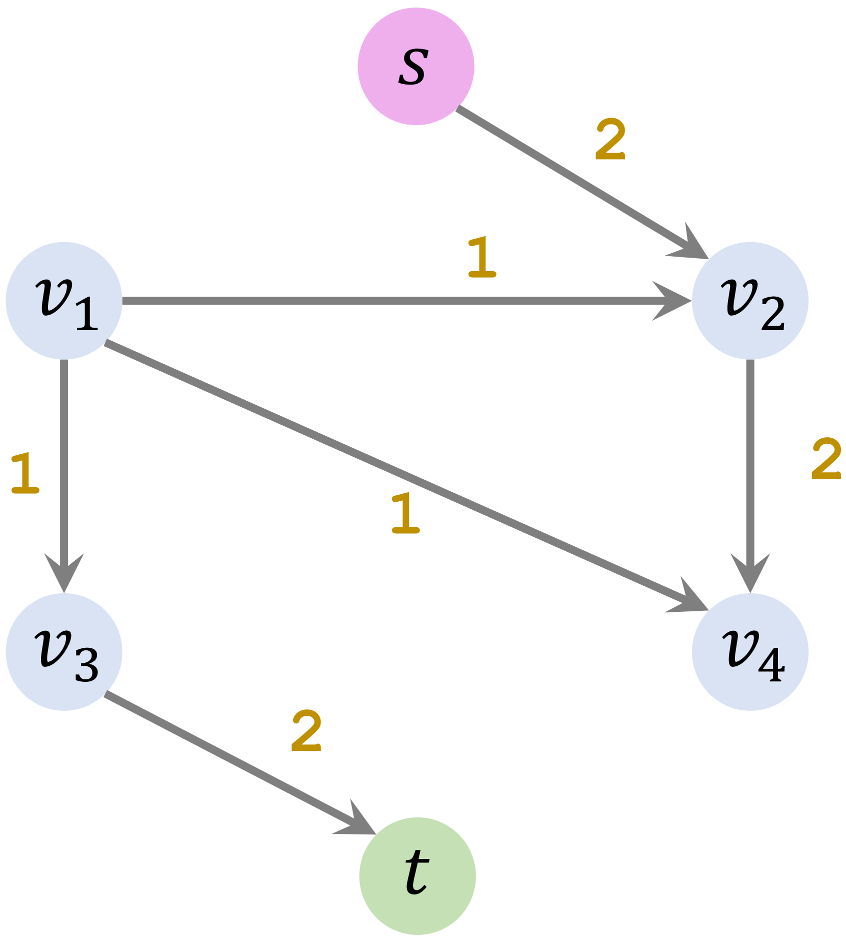


Cannot find any path from source to sink.

End of Procedure

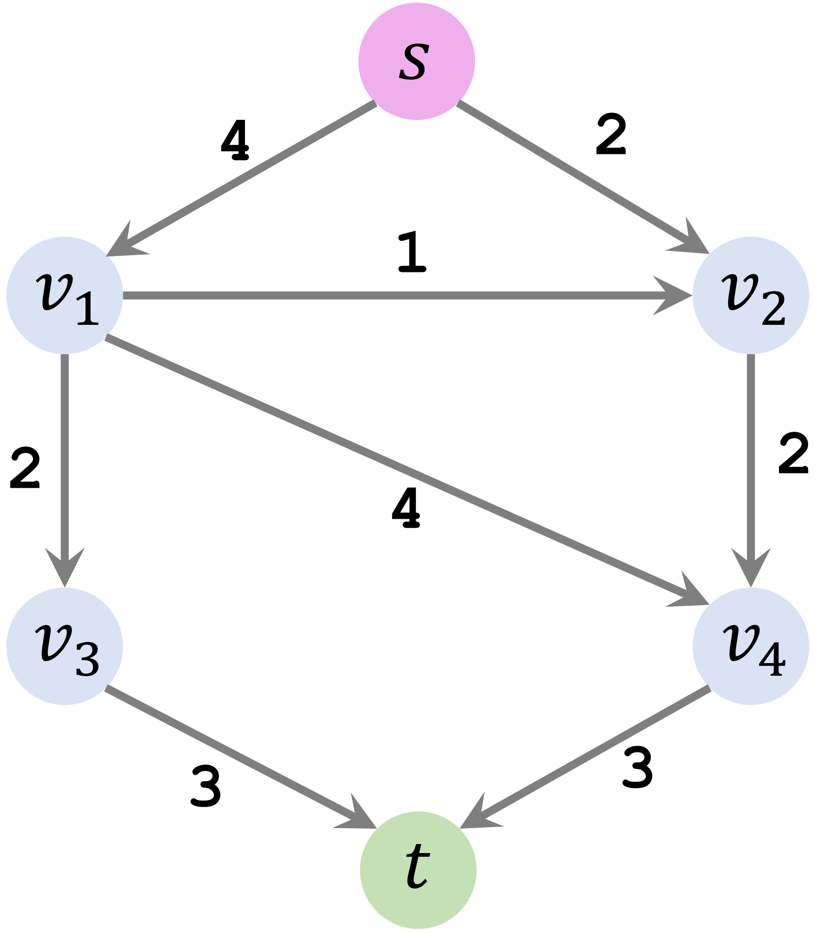


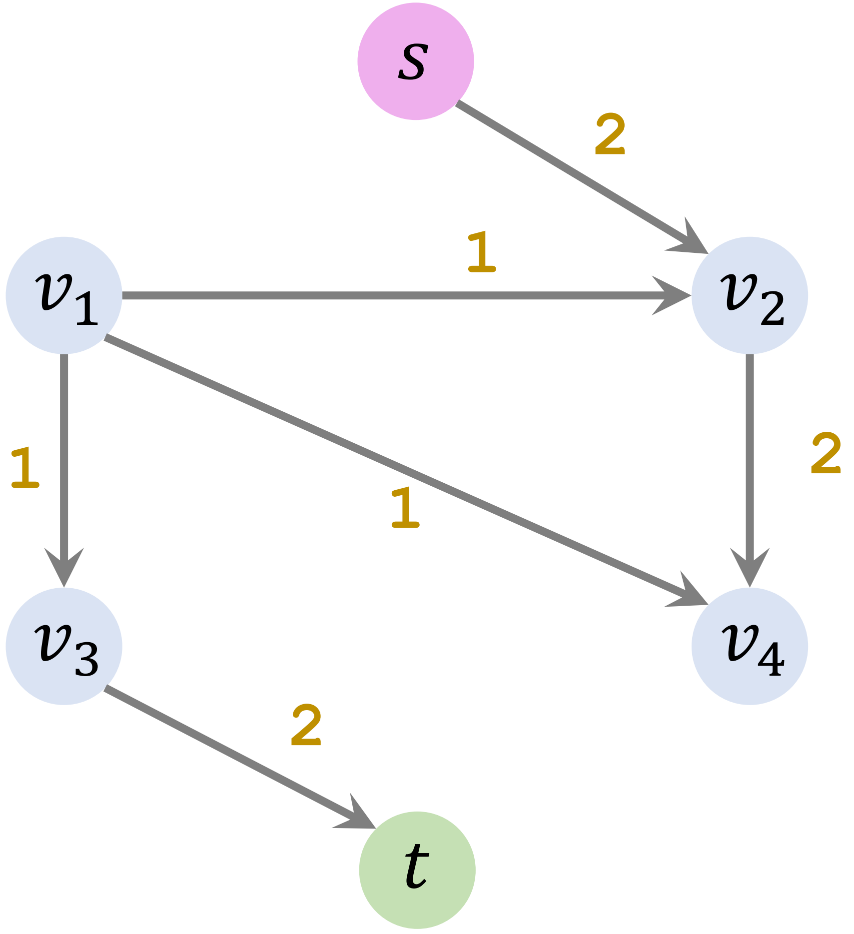
Original Graph



Residual Graph

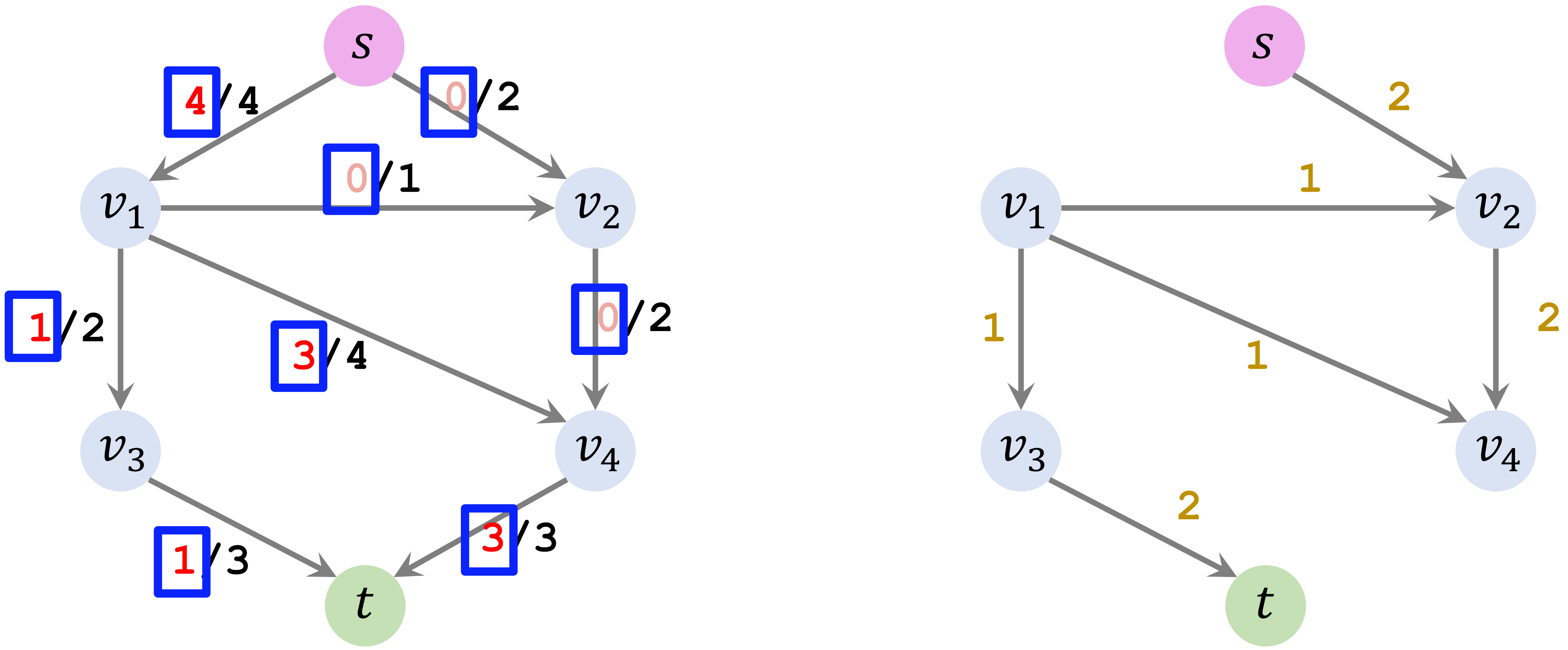
End of Procedure





Flow = Capacity − Residual.

End of Procedure



Flow = Capacity − Residual.

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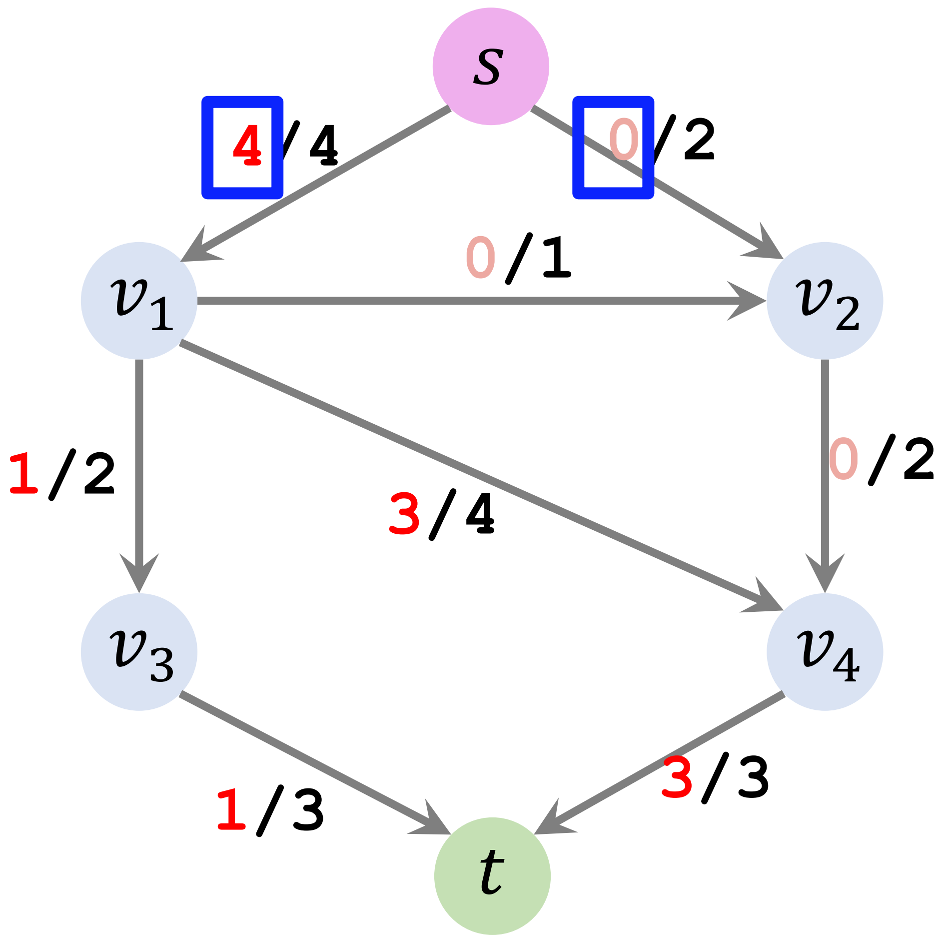
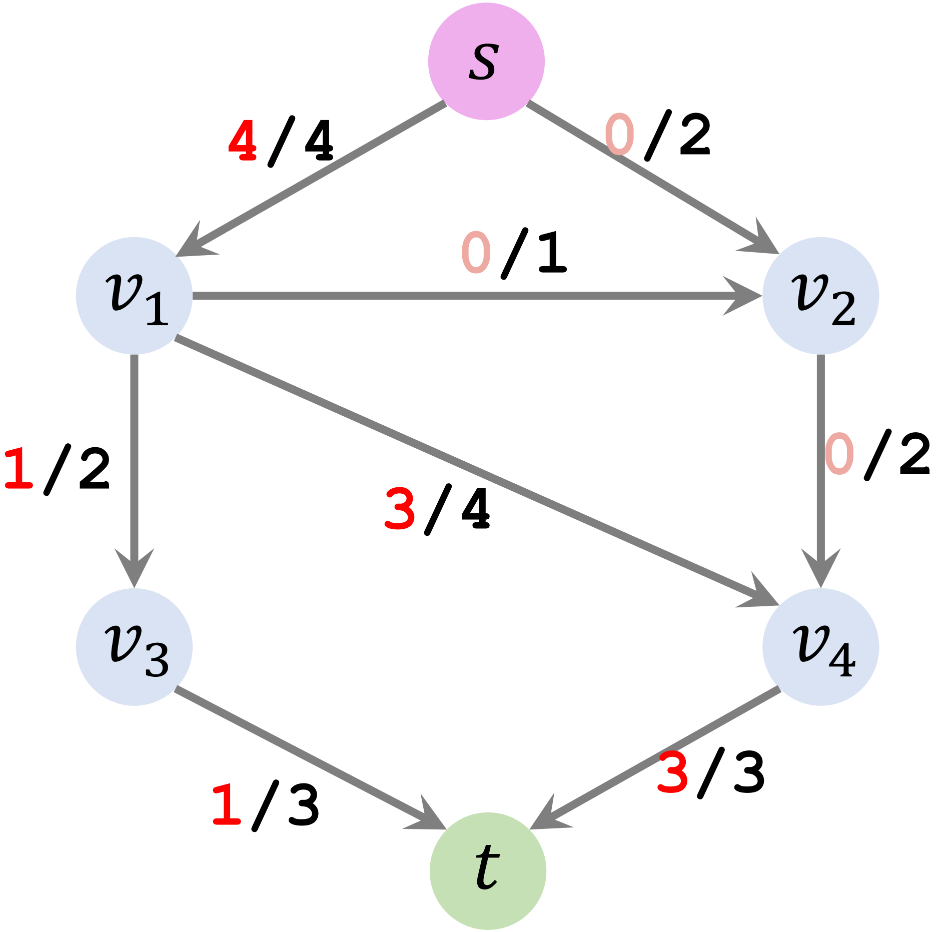
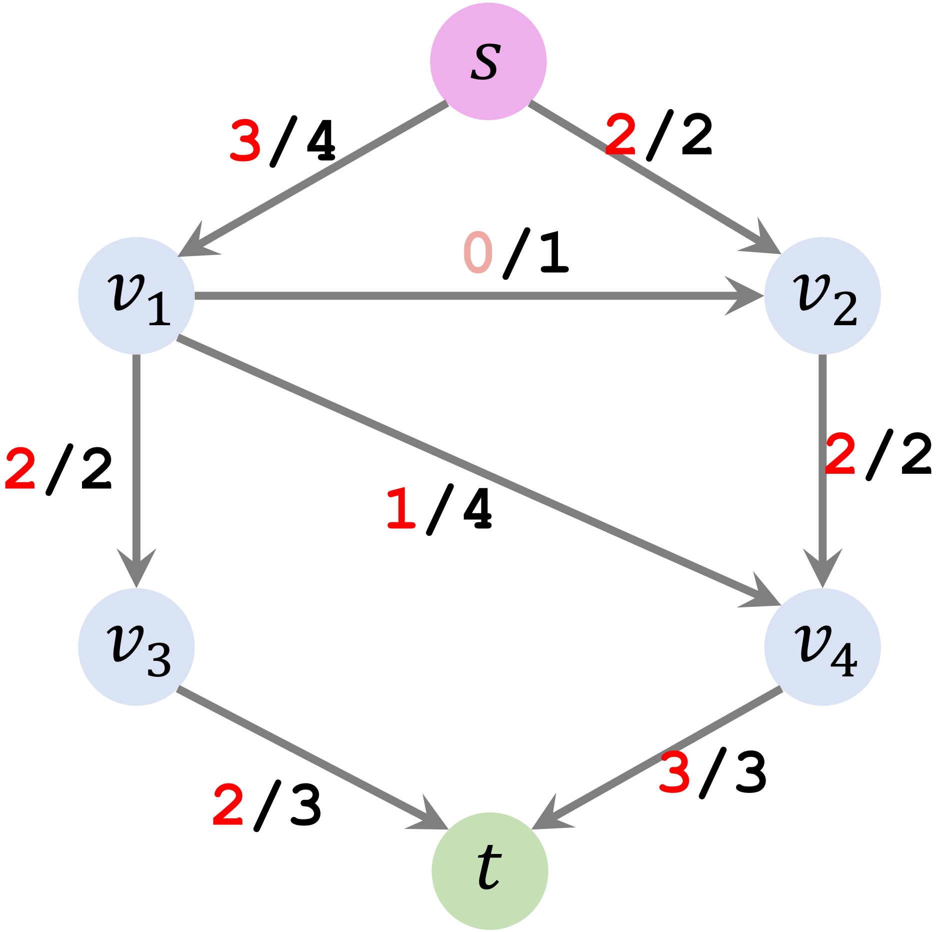


Amount of Flow = 4.

The result is not maximum flow!

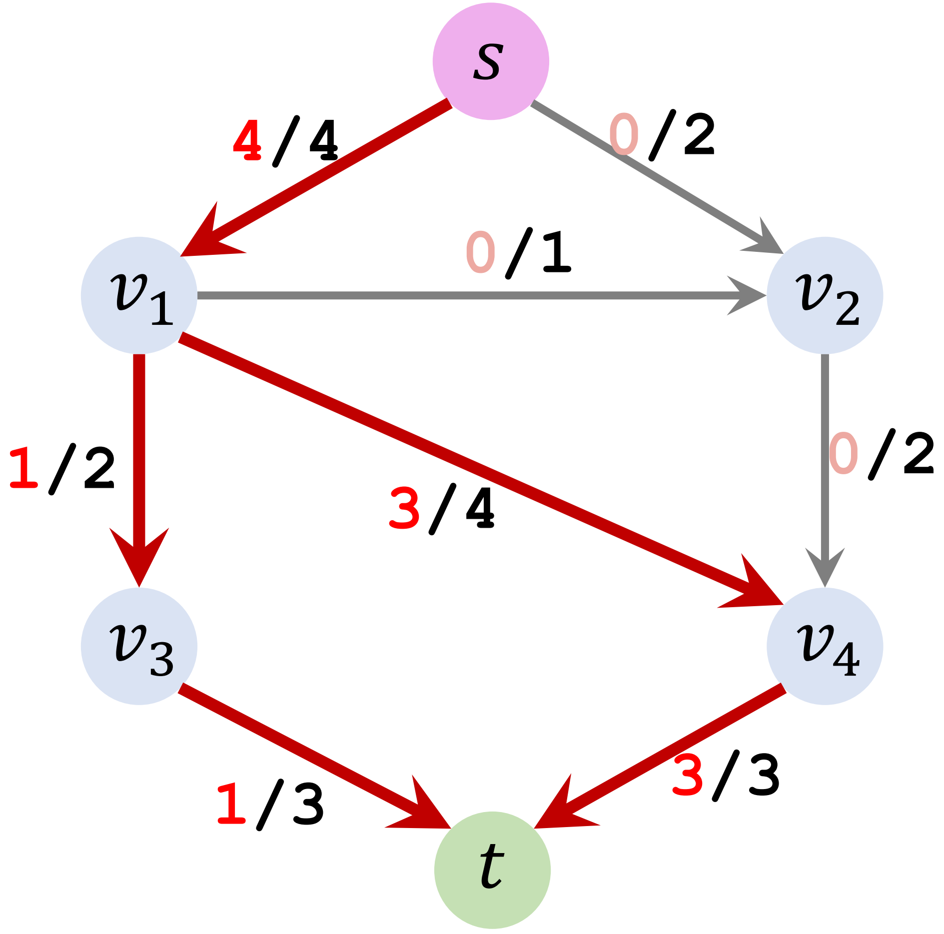




Flow = 4. (Not maximum!)

Flow = 5. (Maximum!)

Blocking Flow



- A flow is blocking flow if no more flow from source to sink can be found.
- The "pipes" are blocked.
- Maximum flow is also blocking flow.

Summary

Maximum Flow Problem

- **Inputs:** a weighted directed graph, the source s, and the sink t.
- **Goal:** Send as much water as possible from s to t.
- **Constraints:**
  - Each edge has a weight (i.e., the capacity of the pipe).
  - The flow must not exceed the capacity.

Naïve Algorithm

1. Build a residual graph; initialize the residuals to the capacity.

Naïve Algorithm

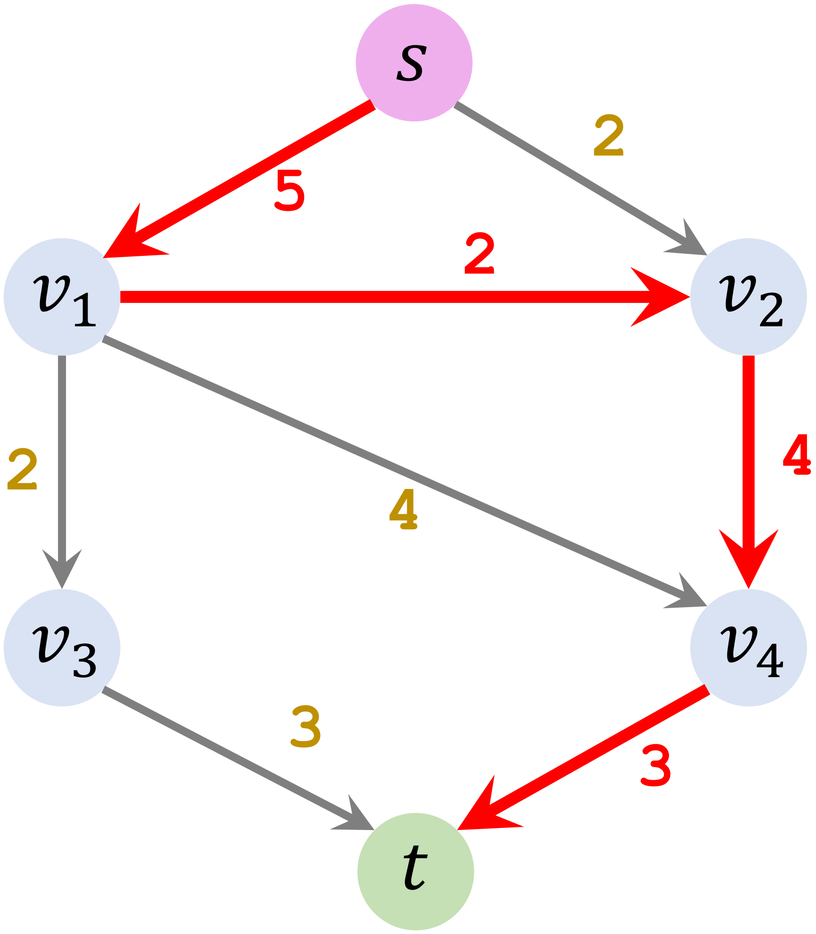
1. Build a residual graph; initialize the residuals to the capacity.
2. While augmenting path can be found:
   a. Find an augmenting path (on the residual graph.)
   b. Find the bottleneck capacity x in the augmenting path.
   c. Update the residuals. ($\text{residual} \leftarrow \text{residual} - x$.)

The naïve algorithm can fail

- The naïve algorithm always finds the blocking flow.
- However, the outcome may not be the maximum flow.

Questions

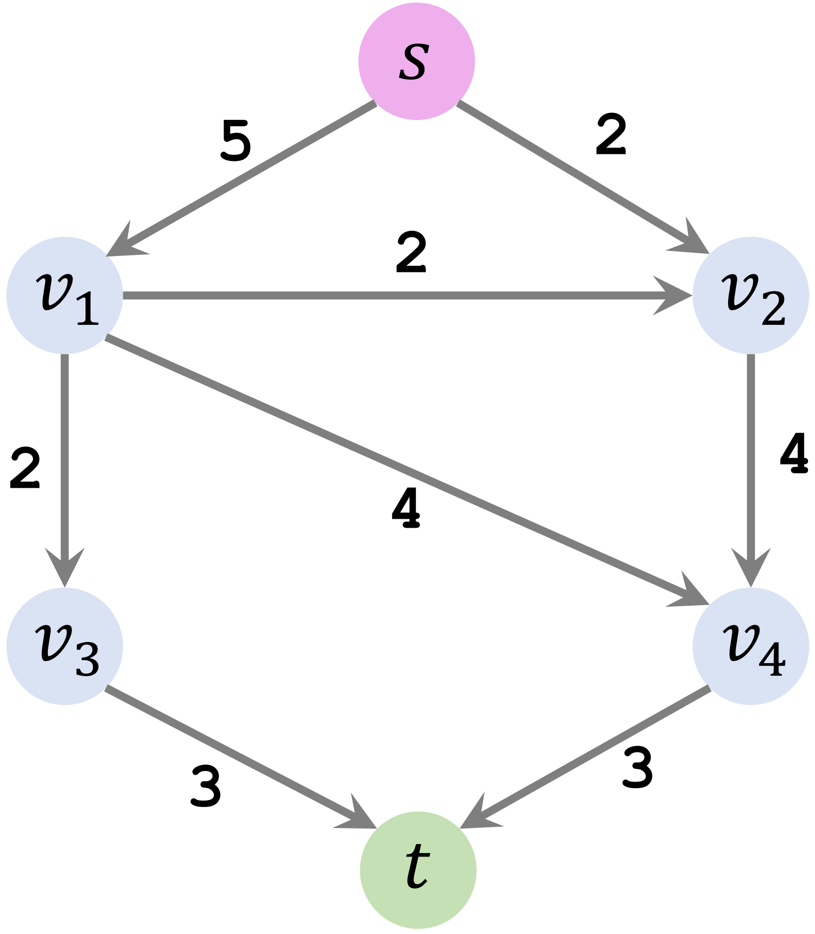
Q1: Bottleneck Capacity



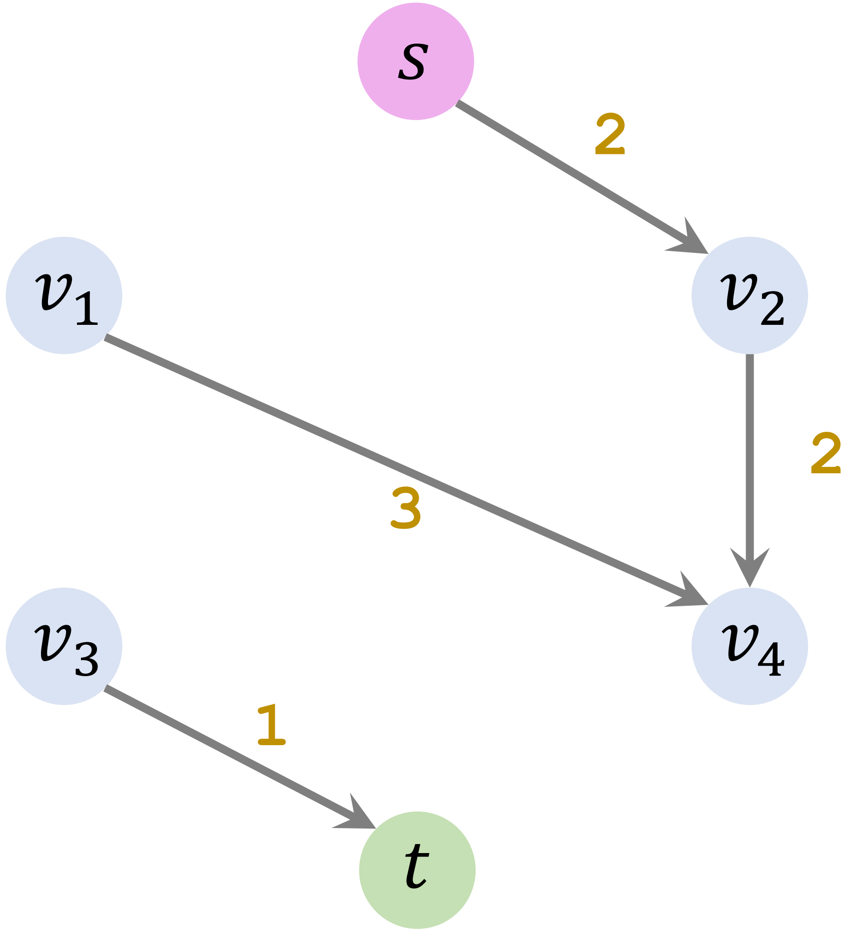
Residual Graph

- **Question:** What is the bottleneck of the red path?

Q2: What is the amount of flow from s to t?



Original Graph



Final Residual Graph

Thank You!