Catalan numbers

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The Catalan numbers (1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, ...), named after Eugène Charles Catalan (1814–1894), arise in a number of problems in combinatorics. They can be computed using this formula:

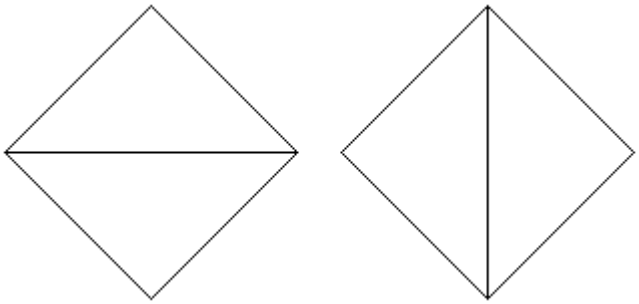
$$\frac{\binom{2n}{n}}{n+1}$$

Among other things, the Catalan numbers describe:

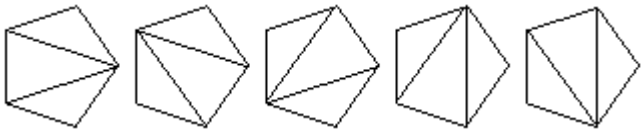
- the number of ways a polygon with n+2 sides can be cut into n triangles
- the number of ways to use n rectangles to tile a stairstep shape (1, 2, ..., n−1, n).
- the number of ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time
- the number of planar binary trees with n+1 leaves
- the number of paths of length 2n through an n-by-n grid that do not rise above the main diagonal

Polygon diagrams:

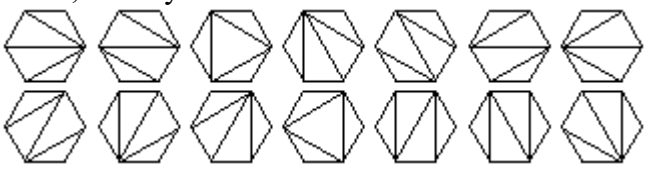
4 sides, 2 ways:



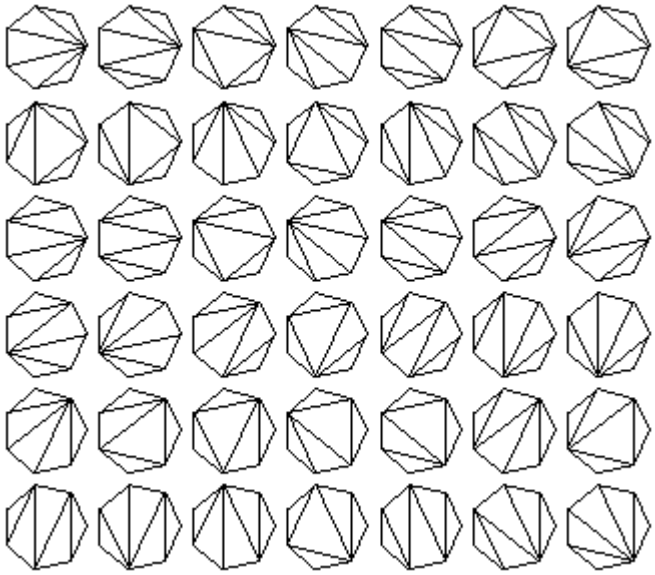
5 sides, 5 ways:



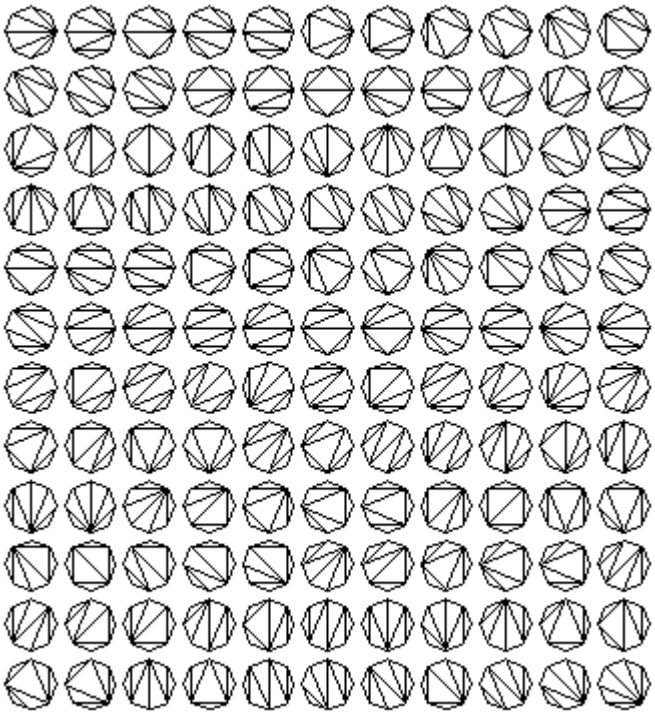
6 sides, 14 ways:



7 sides, 42 ways:



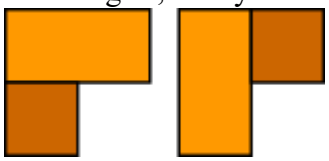
8 sides, 132 ways:



9 sides, 429 ways:
(Hidden in file catalan9.png.)

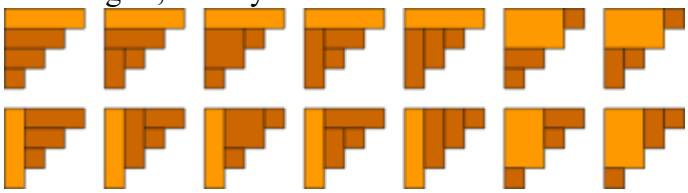
Step diagrams:

2 rectangles, 2 ways:

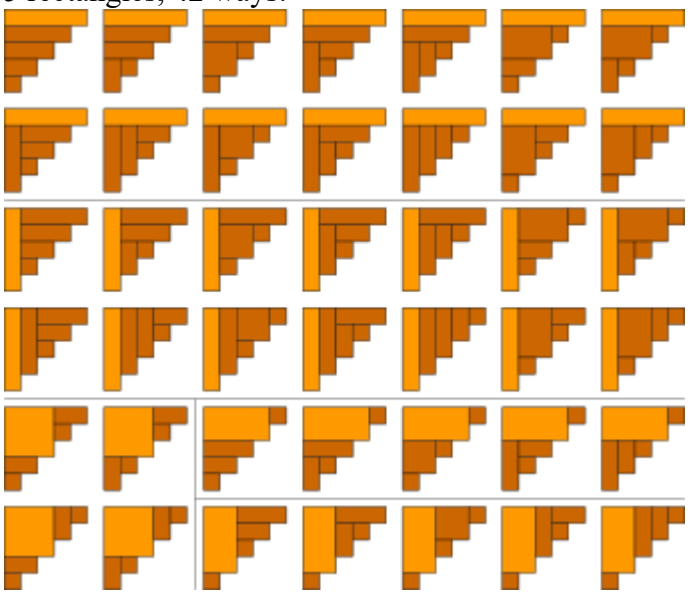


3 rectangles, 5 ways:

4 rectangles, 14 ways:



5 rectangles, 42 ways:



6 rectangles, 132 ways:

Multiplication diagrams:

3 numbers:

```
(1 (2 3))   ((1 2) 3)
```

4 numbers:

```
(1 (2 (3 4)))   (1 ((2 3) 4))
((1 2) (3 4))   ((1 (2 3)) 4)
(((1 2) 3) 4)
```

5 numbers:

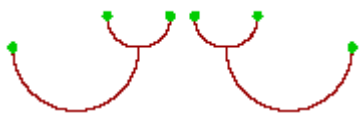
```
(1 (2 (3 (4 5))))   (1 (2 ((3 4) 5)))
(1 ((2 3) (4 5)))   (1 ((2 (3 4)) 5))
(1 (((2 3) 4) 5))   ((1 2) (3 (4 5)))
((1 2) ((3 4) 5))   ((1 (2 3)) (4 5))
((1 (2 (3 4))) 5)   ((1 ((2 3) 4)) 5)
(((1 2) 3) (4 5))   (((1 2) (3 4)) 5)
(((1 (2 3)) 4) 5)   ((((1 2) 3) 4) 5)
```

6 numbers:

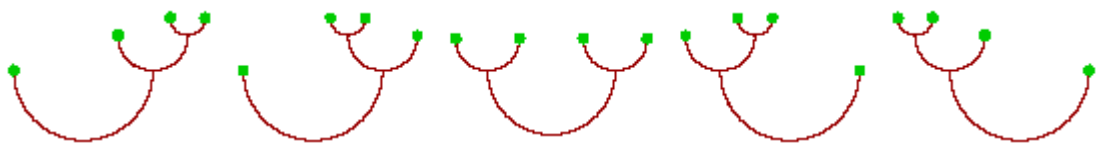
```
(1 (2 (3 (4 (5 6)))))   (1 (2 (3 ((4 5) 6))))
(1 (2 ((3 4) (5 6))))   (1 (2 ((3 (4 5)) 6)))
(1 (2 (((3 4) 5) 6)))   (1 ((2 3) (4 (5 6))))
(1 ((2 3) ((4 5) 6)))   (1 ((2 (3 4)) (5 6)))
(1 ((2 (3 (4 5))) 6))   (1 ((2 ((3 4) 5)) 6))
(1 (((2 3) 4) (5 6)))   (1 (((2 3) (4 5)) 6))
(1 (((2 (3 4)) 5) 6))   (1 ((((2 3) 4) 5) 6))
((1 2) (3 (4 (5 6))))   ((1 2) (3 ((4 5) 6)))
((1 2) ((3 4) (5 6)))   ((1 2) ((3 (4 5)) 6))
((1 2) (((3 4) 5) 6))   ((1 (2 3)) (4 (5 6)))
((1 (2 3)) ((4 5) 6))   ((1 (2 (3 4))) (5 6))
((1 (2 (3 (4 5)))) 6)   ((1 (2 ((3 4) 5))) 6)
((1 ((2 3) 4)) (5 6))   ((1 ((2 3) (4 5))) 6)
((1 ((2 (3 4)) 5)) 6)   ((1 (((2 3) 4) 5)) 6)
(((1 2) 3) (4 (5 6)))   (((1 2) 3) ((4 5) 6))
(((1 2) (3 4)) (5 6))   (((1 2) (3 (4 5))) 6)
(((1 2) ((3 4) 5)) 6)   (((1 (2 3)) 4) (5 6))
(((1 (2 3)) (4 5)) 6)   (((1 (2 (3 4))) 5) 6)
(((1 ((2 3) 4)) 5) 6)   ((((1 2) 3) 4) (5 6))
((((1 2) 3) (4 5)) 6)   ((((1 2) (3 4)) 5) 6)
((((1 (2 3)) 4) 5) 6)   (((((1 2) 3) 4) 5) 6)
```

Tree diagrams:

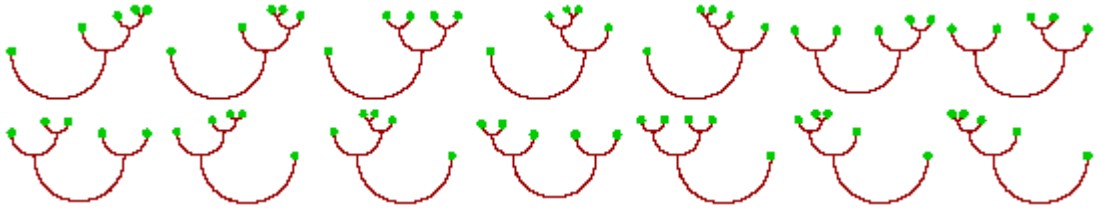
3 nodes:



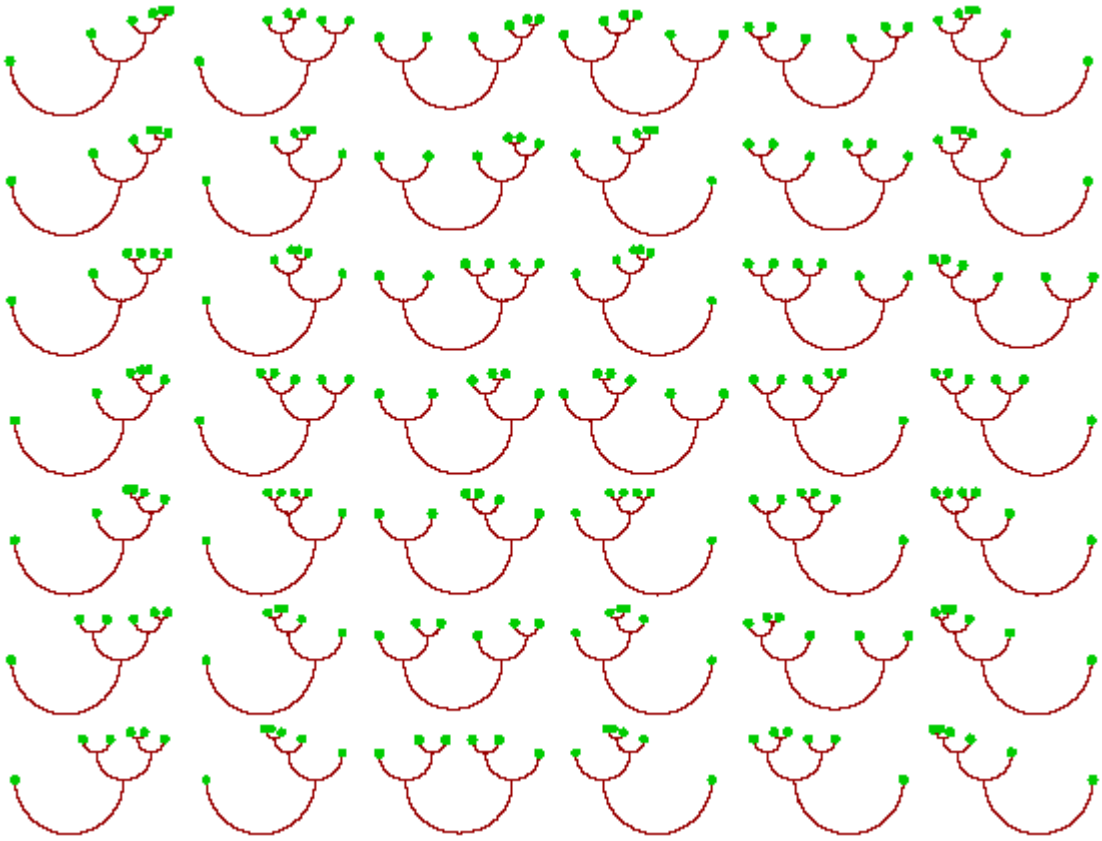
4 nodes:



5 nodes:

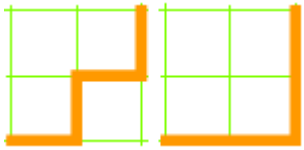


6 nodes:

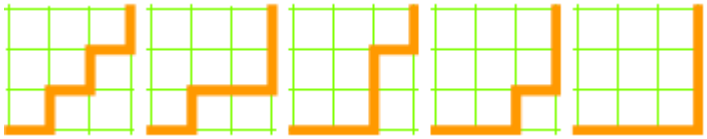


Path diagrams:

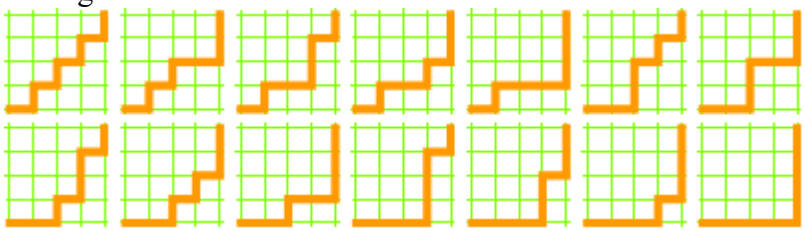
2 × 2 grid:



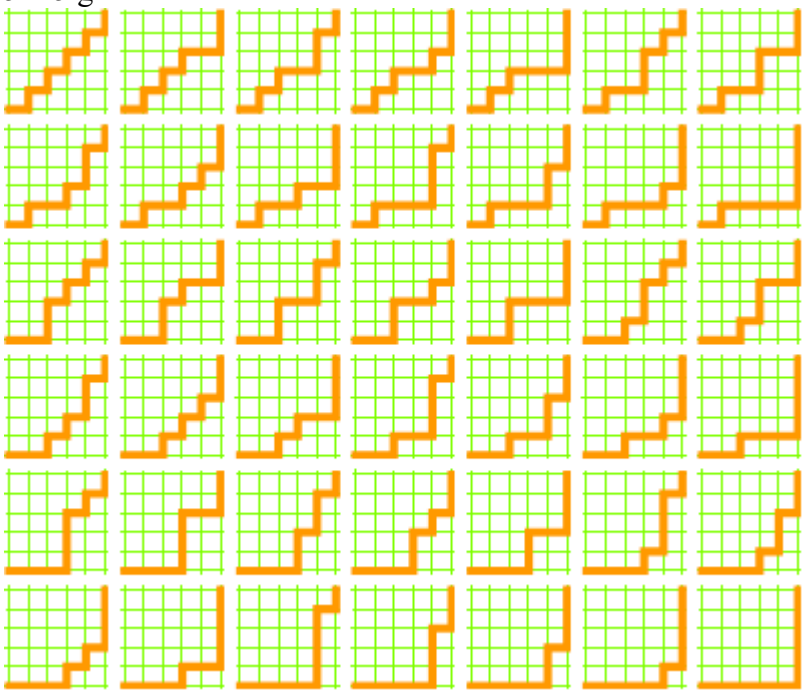
3 × 3 grid:



4 × 4 grid:



5 × 5 grid:



6 × 6 grid:
(Out of the way in file catpath6.png.)

Originally designed and rendered using *Mathematica* 3.0 for the Apple Macintosh. PNG conversions performed with an old version of ImageMagick.

Inspiration and facts (though not figures) by Brian Hayes, "A Question of Numbers", *American Scientist*, January–February 1996; Steven S. Skiena, *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, Addison-Wesley, 1990; Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984; and D. E. Knuth, *Sorting and Searching* (vol. 3 of *The Art of Computer Programming*), Addison-Wesley, 1973. Catalan dates from Florian Cajori, *A History of Mathematics*, The Macmillan Company, 1922.

See also Martin Gardner, *Time Travel and Other Mathematical Bewilderments*, Chapter 20, W. H. Freeman, 1988; and Ilan Vardi, *Computational Recreations in Mathematica*, Chapter 9, Addison-Wesley, 1991.

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