BELİRSİZ İNTEGRAL

1. Değişken Değiştirme
2. Kısmi integrasyon
3. Rasyonel Fonk integrali → $\frac{K}{(x-a)^n}$, $\frac{Mx+N}{(x^2+ax+b)^n}$

Basit Kesirlere Ayırma Metodu

$$\frac{mx+N}{x^2+ax+b} = \frac{Mx+N}{(x+k)^2+e^2}$$

i. $x^2+ax+b=u$

$(2x+a)\,dx = (Mx+N)\,dx$

ÖRNEKLER:

①

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \cos x = t \qquad -\sin x\,dx = dt$$

$$\int \tan x\,dx = \int \frac{-dt}{t} = -\int \frac{dt}{t}$$

②

$$\int \arctan x\,dx$$

$d(u.v) = v.du + u.dv$

$\arctan x = u \qquad \int dx = \int dv$

$d(\arctan x) = du \qquad x = v$

$\frac{1}{1+x^2} = du$

$u.v = \int v\,du + \int u\,dv$

$u.v - \int v\,du = \int u\,dv$

$u.v - \int u\,dv = \int v\,du$

$\int f(x)\,g(x)\,dx$, $f(x)=u$, $g(x)=u$; $g(x)\,dx = dv$, $f(x)\,dx = dv$

polinom fonk u denek kolaylık seçler derecesi düşer

ters trig. fonk u denek → türevi alındığında ters trig. kaybolur

$(\arctan x)' = \frac{1}{1+x^2}$

$u.v - \int v\,du$

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

Örnek: $\int \sec^3 x\,dx$ kısmi int.

③ $\int f(x) = \int \frac{P(x)}{Q(x)}$ paydanın türevi payı veriyorsa değişken değiştirme kullanılır.

Der $P(x) \geq$ Der $Q(x)$

$$\begin{array}{r|l} P(x) & Q(x) \\ \hline & B(x) \\ K(x) & \end{array} \quad \Big\} \quad \frac{P(x)}{Q(x)} = B(x) + \frac{K(x)}{Q(x)}$$

3 Örnek: $\int \frac{x}{(x-2)(x+1)}\,dx$ $\qquad \frac{x}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$, A, B sabit

$x \equiv A(x+1) + B(x-2)$

$$\int \frac{x}{(x-2)(x+1)}\,dx = A\int \frac{dx}{x-2} + B\int \frac{dx}{x+1} = A\ln|x-2| + B\ln|x+1| + C$$

$$\int \frac{x^2}{(x-2)^3(x+1)^2(x+3)} = \frac{A_1}{(x-2)} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1}{(x+1)} + \frac{B_2}{(x+1)^2} + \frac{C}{(x+3)}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$$

Ör:

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+2^2} = \int \frac{du}{2^2+u^2} = \frac{1}{2}\arctan\frac{x}{2}$$

Ör:

$$\int \frac{dx}{x(x^2+2x+5)}$$

$$\frac{1}{x(x^2+2x+5)} = \frac{A}{x} + \frac{Bx+C}{(x^2+2x+5)}$$

$$1 \equiv (x^2+2x+5)A + (Bx+C)x$$

$$1 \equiv (Ax^2+2Ax+5A) + (Bx^2+Cx)$$

$$1 \equiv (A+B)x^2 + (2A+C)x + 5A$$

$$1 \equiv 5A$$

$$A = \frac{1}{5}$$

$$0 \equiv 2(A+B)x + (2A+C)$$

$$0 \equiv 2(A+B)$$

$$0 \equiv 2A+2B \qquad -A=B$$

$$2A+C=0$$

$$C=-2A$$

$$C=-\frac{2}{5}$$

$$B=-\frac{1}{5}$$

$$\frac{1}{x(x^2+2x+5)} = \frac{1}{5x} + \frac{-\frac{x}{5}-\frac{2}{5}}{(x^2+2x+5)}$$

$$\frac{1}{x(x^2+2x+5)} = \frac{1}{5x} - \frac{x+2}{5(x^2+2x+5)}$$

$$\frac{\ln|x|}{5} - \int \frac{x+2}{5(x^2+2x+5)} = \frac{\ln|x|}{5} - \frac{\sum}{2}\arctan\left(\frac{x}{2}\right)$$

Basit kesir 2 türlü dür.

$$\frac{A}{(x-a)^n}, \quad \frac{Bx+C}{x^2+2x+5}$$

3. der polinom varsa bir kökü mutlaka reel köktür.

ÖR: $\displaystyle\int \frac{dx}{x(x^2+2x+5)^2}$

$$\frac{1}{x(x^2+2x+5)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2}$$

ÖR. $\displaystyle\int \frac{4x-11}{x^2-6x+10}\,dx$

$x^2-6x+10 = u$

$(2x-6)\,dx = du$

$4x-11 = 2\left(2x-\frac{11}{2}\right) = 2\left(2x-6+6-\frac{11}{2}\right) = 2(2x-6)+1$

ağırlıklı ortalama

$$2\int \frac{2x-6}{x^2-6x+10}\,dx + \int \frac{dx}{x^2-6x+10}$$

Rasyonel Hale Getirilebilen Fonksiyonların İntegrali

$$\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8} \qquad x=t^3$$

ÖR:

$$\int \frac{\sqrt{x^3}-\sqrt[3]{x}}{6\sqrt[4]{x}}\,dx \qquad x=t^{12} \qquad dx = 12t^{11}\,dt \longrightarrow \text{Rasyonel Fonk. integrali} \quad 2\int \frac{t^{18}-t^4}{t^3}\,t^{11}\,dt = \frac{2}{27}x^{27/12} - \frac{2}{13}x^{13/12} + c$$

ÖR.

$$\int \frac{dx}{(1+x)^{\frac{2}{3}} + (1+x)^{\frac{1}{2}}} \qquad u^6 = 1+x \qquad 6u^5 = x$$

$$6\int \frac{u^2}{u+1}\,du$$

$$= 6\int \left(u-1+\frac{1}{u+1}\right) du$$

$$= 3u^2 - 6u + 6\ln|u+1| + c$$

Trigonometrik fonksiyonların integrali

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$

$\int \cot x \, dx = \ln|\sin x| + C$

$\int \sec x \, dx = ?$

$\int \operatorname{cosec} x \, dx = ?$

* negatif sayıların logaritması tanımlı değil o sebeple mutlak değere alınır.

$\int \tan(ax+b)\, dx = -\dfrac{\ln|\cos(ax+b)|}{a}$

1. $\int \sin^m x \, dx$, $\int \cos^n x \, dx$

i m (veya n) tek terim ise

$m = 2k+1$

$$\int \sin^{2k+1} x \, dx = \int \sin^{2k} x \, \sin x \, dx$$
$$= \int (\sin^2 x)^k \cdot \sin x \, dx$$
$$= \int (1-\cos^2 x)^k \cdot \sin x \, dx$$
$$= -\int (1-u^2)^k \, du$$

$\cos x = u$

$-\sin x \, dx = du$

Ör: $$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$$
$$= \int (1-\cos^2) x \cdot \sin x \, dx$$
$$= \int (u^2-1)\, du = \frac{u^3}{3} - u + C$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

ÖR: $\int \cos^5 x \, dx$

$= \int \cos^2 x \; \cos^2 x \cdot \cos x \cdot dx$

$= \int (1-\sin^2 x)(1-\sin^2 x) \cos x \, dx$ $\qquad \sin x = u \qquad \cos x \, dx = du$

$= \int (1-u^2)^2 \, du$

$= \int (1 - 2u^2 + u^4) \, du$

ii m (n) çift terim

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

ÖR:

$\int \cos^2 x \, dx = \int \dfrac{1+\cos 2x}{2} \, dx$

$= \dfrac{1}{2} \int (1+\cos 2x) \, dx = \dfrac{1}{2}\left(x + \dfrac{\sin 2x}{2}\right) + C$

ÖR:

$\int \sin^2 x \, dx = \int \dfrac{1-\cos 2x}{2} \, dx = \dfrac{1}{2} \int (1-\cos 2x) \, dx = \dfrac{1}{2}\left(x - \dfrac{\sin 2x}{2}\right) + C$

$= \dfrac{x}{2} - \dfrac{\sin x + \cos x}{2} + C$

Soru:

$\int \sin^2 \dfrac{x}{4} \, dx = ?$ $\qquad \sin^2 x = \dfrac{1-\cos 2x}{2} \qquad \sin^2 \dfrac{x}{4} = \dfrac{1-\cos\left(2\frac{x}{4}\right)}{2}$

$\int \dfrac{1-\cos\left(\frac{x}{2}\right)}{2} \, dx = \dfrac{1}{2} \int dx - \int \dfrac{\cos\left(\frac{x}{2}\right)}{2} \, dx$

$= \dfrac{x}{2} - \dfrac{\sin\left(\frac{x}{2}\right)}{2 \cdot \frac{1}{2}} + C = \dfrac{x}{2} - \sin\left(\dfrac{x}{2}\right) + C$

ÖR:

$$\int \cos^4 x \, dx = ?$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\left(x - \frac{2\sin 2x}{2}\right) + \frac{1}{4}\int \cos^2 2x \, dx$$

2. $\int \cos^n x \cdot \sin^m x \, dx$ biçimindeki int n veya m tek terim ise

m tek terim olsun

$m = 2k+1$

$$\int \cos^n x \cdot \sin^{2k+1} x \, dx = ?$$

$$= \int \cos^n x \, (\sin^2 x)^k \cdot \sin x \, dx$$

$$= \int \cos^n x \, (1-\cos^2 x)^k \cdot \sin x \, dx \quad , \quad \boxed{\begin{array}{l} \cos x = u \\ -\sin x \, dx = du \end{array}}$$

$$= \int u^n (1-u^2)^k \cdot (-du)$$

n ve m çift terim ise

ÖR:

$$\int \cos^2 x \cdot \sin x \, dx = \int -u^2 \, du = -\frac{u^3}{3} + c = -\frac{\cos^3 x}{3} + c$$

Ör: $\int \cos^3 x \cdot \sin^5 x \, dx$

$$= \int \cos^2 x \cdot \cos x \cdot \sin^5 x \, dx$$

$$= \int (1-\sin^2 x) \cdot \sin^5 x \cdot \cos x \, dx$$

$$= \int (1-t^2) \cdot t^5 \, dt$$

$$= \int \cos^3 x \cdot \sin^4 x \cdot \sin x \, dx$$

$$= \int \cos^3 x \, (1-\cos^2 x)^2 \sin x \, dx$$

$$= \int u^3 (1-u^2)^2 (-du)$$

Çıkan iki sonuç arasındaki fark integral sabitidir.

Ör: $\int \cos^2 x \cdot \sin^2 x \, dx = ?$

1. yol

$$\int \left(\frac{1+\cos 2x}{2}\right) \cdot \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{4} \int 1-\cos^2 2x \, dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

2. yol

$$\int \cos^2 x \, (1-\cos^2 x) \, dx = \int (\cos^2 x - \cos^4 x) \, dx$$

Soru: $\int \sin^3 x \cdot \cos^{-5} x \, dx = ?$

$= \int \tan^3 x \cdot \sec^2 x \, dx$

$\sin x \cdot \cos x$ ile çözülen integraller

$$\left.\begin{aligned} \cos(A+B) &= \cos A \cdot \cos B - \sin A \cdot \sin B \\ \cos(A-B) &= \cos A \cdot \cos B + \sin A \cdot \sin B \end{aligned}\right\} \quad \cos A \cdot \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$A = mx \quad , \quad B = nx$

$$\left.\begin{aligned} \sin(A+B) &= \sin A \cos B + \sin B \cos A \\ \sin(A-B) &= \sin A \cos B - \sin B \cos A \end{aligned}\right\} \quad \sin A \cdot \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

ÖR: $\int \cos 4x \cdot \cos 2x \, dx = ?$

$= \frac{1}{2}\int (\cos 6x + \cos 2x)\,dx = \frac{1}{2}\int \cos 6x \, dx + \frac{1}{2}\int \cos 2x \, dx$

Öe: $\int \sin 3x \cdot \sin 2x \, dx$

Ör: $\int \sin 2x \cdot \sin 3x \cdot dx = ?$

clue: $\cos(-\theta) = \cos\theta$

çift fonksiyon

Ör: $\int \sin 3x \cos 5x \, dx = ?$

clue: $\sin(-\theta) = -\sin\theta$

tek fonk

Ör: $\int \sin x \cdot \cos x \, dx = ?$

1. yol

$\frac{1}{2}\int \sin 2x \, dx$

2. yol

replacer

3. yol

Tanjant ve Kotanjant integrali

3. $\int \tan^n x \, dx$, $\int \cot^n x \, dx$

Ör:

$$\int \tan^2 x \, dx = \int (\tan^2 x + 1 - 1)\, dx = \int (\tan^2 x + 1)\, dx - \int dx$$

$$= \tan x - x + C$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)'$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x}$$

ör: $\int \tan^3 x \, dx$

$$= \int \tan^2 x \cdot \tan x \cdot dx$$

$u = \tan x \qquad du = \sec^2 x \, dx$

$$\int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

evap $\frac{1}{2}\tan^2 x + \ln|\cos x| + c$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$$= -\ln|\cos x| + c$$

$$= \ln \frac{1}{|\cos x|} + c$$

$$= \ln|\sec x| + c$$

ör: $\int \tan^4 x \, dx = ?$

$$\tan^2 x = \sec^2 x - 1$$

$$= \int (\sec^2 x - 1)^2 \tan^2 x \, dx$$

$$= \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx$$

$u = \tan x \qquad du = \sec^2 x \, dx$

ikisi $\tan^2$ de $\sec^2$ olmak zorunda bu durum bol bol pratik ile anlaşılır.

$$= \int u^2 du - \int \tan^2 x \, dx$$

$$= \frac{1}{3}u^3 - \tan x + x + C \qquad = \frac{1}{3}\tan^3 x - \tan x + x + C$$

ör: $\int \cot^2 x \, dx = ?$

$$\int (\cot^2 x + 1 - 1) dx = \int (\cot^2 x + 1) dx - \int dx$$

$$= -\cot x + x + C$$

NOT : $\dfrac{-\sin^2 - \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$

Ör:

$\int \cot^5 x = ?$

$= \int \cot^2 x \cdot \cot^3 x \, dx$

$= \int (\operatorname{cosec}^2 - 1) \cot^3 x \, dx$

$= \int \csc^2 x \cot^3 x \, dx - \int \cot^3 x \, dx$

$\int \cot^m x \cdot \operatorname{cosec}^n x \, dx$

$\int \tan^m x \cdot \sec^n x \, dx$

$\int \csc^2 x \cot^3 x \, dx - \int \cot^2 x \cot x \, dx$

$= \int \csc^2 x \cot^3 x \, dx - \int (\csc^2 x - 1) \cdot \cot x \, dx$

$- \int (\csc^2 x \cdot \cot x - \cot x) \, dx$

$\csc x = u$

$-\csc x \cdot \cot x \, dx = du$

$\int \csc^2 x \cdot \cot^2 x \cdot \cot x \, dx - \int (\csc^2 x \cdot \cot x) \, dx + \int \cot x \, dx$

$\int \csc^2 x \, (\csc^2 x - 1) \cdot \cot x \, dx$

$+ \int u \, du + \int \cot x \, dx$

$\cot x = u$

$\csc^2 x \cdot dx = -du$

$\dfrac{u^2}{2}$

$\dfrac{-\cot^4}{4} + \dfrac{\csc^2 x}{2} + \ln|\sin x| + c$

$-\int u^3 \, du = -\dfrac{u^4}{4}$

$+ \dfrac{\csc^2 x}{2} + \ln|\sin x| + c$

1. Hafta Örnekler

Rasyonel Trigonometrik Fonk. integrali

$f(x) = \dfrac{P(x)}{Q(x)}$, $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$, $f(x) = \dfrac{\sin x + \cos x}{1 - \cos 2x}$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

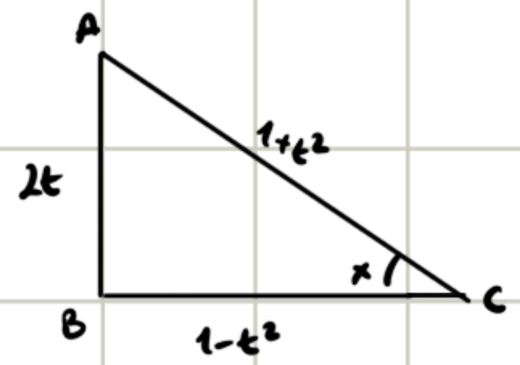
$\tan\left(\dfrac{x}{2} + \dfrac{x}{2}\right) = \tan x = \dfrac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$

$t = \tan\frac{x}{2}$ diyelim

$t = \tan\frac{x}{2} \Rightarrow dx = ?$

$\tan x = \frac{2t}{1-t^2}$

$\frac{x}{2} = \arctan t$



$\frac{dx}{2} = \frac{1}{1+t^2}dt$

$dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

Rasyonel Trigonometrik Fonk. integralini çözmek için

$$\int f(\sin x, \cos x, \tan x)\,dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}$$

$\boxed{\tan\frac{x}{2} = t}$, $\tan x = \tan\left(\frac{x}{2}+\frac{x}{2}\right) = \frac{2t}{1-t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$\int \sin x\,dx$

$\int \cos x\,dx$

$\int \tan x\,dx$

$\int \cot x\,dx$

$\int \sec x\,dx = ?$

$\int \operatorname{cosec} x\,dx = ?$

ör: $\int \sec x\,dx$

$$\int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{1-t^2}$$

çözüm

$$\int \frac{dx}{\cos x} = 2\int \frac{dt}{1-t^2}$$

$t = \tan\frac{x}{2}$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right|$$

çözüm

$\int \frac{dt}{a^2+t^2}$, $\int \frac{dt}{a^2-t^2}$, $\int \frac{dt}{t^2-a^2}$

$\int \frac{dt}{\sqrt{a^2-t^2}}$

$$\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{1+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} \qquad \frac{\frac{\cos x+\sin x}{\cancel{\cos x}}}{\frac{\cos x-\sin x}{\cancel{\cos x}}} \Rightarrow \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}$$

$$= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \frac{\cos x}{\cos^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2}+\sin^2\frac{x}{2}}$$

$$= \frac{\cos x}{1-\sin x} = \frac{\cancel{\cos x}\,(1+\sin x)}{1-\cancel{\sin^2 x}}$$

$$= \frac{1}{\cos x}+\frac{\sin x}{\cos x}$$

$$= \sec x + \tan x$$

Soru: $\int \operatorname{cosec} x \, dx = ?$

$$\int \frac{1}{\cos x}\,dx = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \ln\left|\tan\frac{x}{2}\right| + c = \ln|\csc x - \cot x| + c$$

$t = \tan\frac{x}{2}$

$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

ör: $\int \sec^2 x \, dx = ?$

$$\int (1+\tan^2 x)\,dx = \int dx + \int \tan^2 x\,dx = \tan x + c$$

Kesmiş int kullanılır

ör: $\int \sec^3 x \, dx = ?$

$$\int \sec x + \tan^2 x \cdot \sec x \, dx$$

$\sec x = u$

$\tan x \cdot \sec x \, dx = du$

$$\int \sec^2 x \cdot \sec x \, dx$$

$$\int \sec x \, dx + \int \tan x \cdot \tan x \cdot \sec x \, dx$$

$$\int (1+\tan^2 x) \cdot \sec x \, dx$$

Soru:

$$\int \frac{\cos x}{1+\cos x}\,dx$$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{\frac{1-t^2}{\cancel{1+t^2}}}{\frac{1+t^2+1-t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1-t^2}{\cancel{2}} \cdot \frac{\cancel{2}dt}{1+t^2} = \int \frac{1-t^2}{1+t^2}\,dt$$

$$\begin{array}{r|l} 1-t^2 & 1+t^2 \\ \hline -1-t^2 & -1 \\ \hline 2 & \end{array}$$

$$= -1 + \frac{2}{1+t^2}$$

$$= \int -1\,dt + \int \frac{2}{1+t^2}\,dt$$

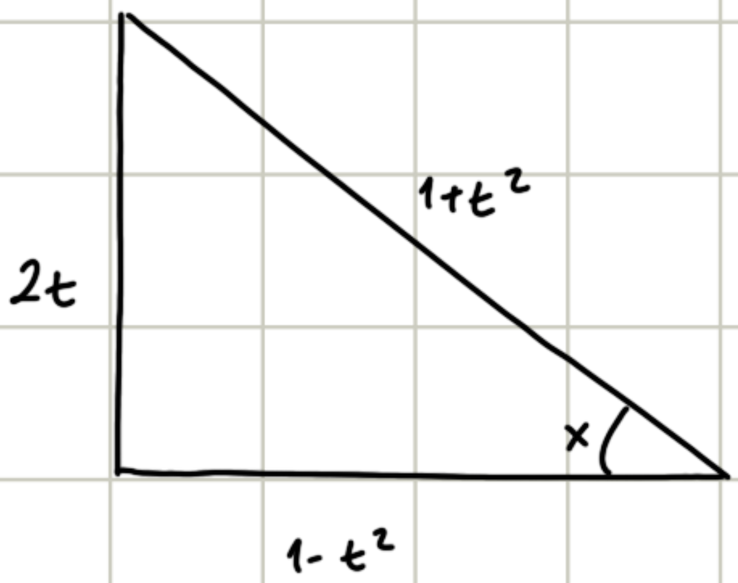
$$= -t + 2\arctan t + c$$

$$= -\tan\frac{x}{2} + 2\arctan\left(\tan\frac{x}{2}\right) + c$$

$$= -\tan\frac{x}{2} + 2\left(\frac{x}{2}\right) + c$$

$$= -\tan\frac{x}{2} + x + c$$

Ödev: $\tan\frac{x}{2}$ yerine sin ve cos yazarak sonucu bul

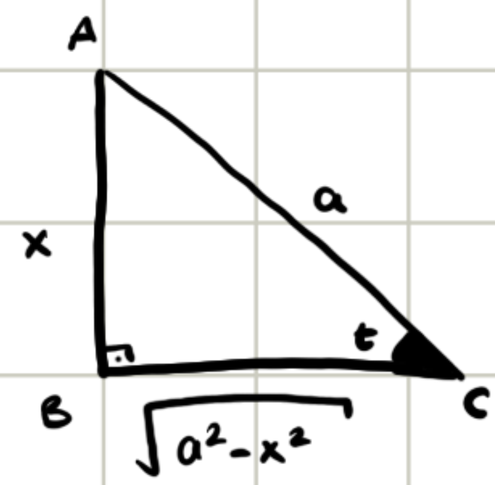


Çeşitli Değişken Değiştirmeler

$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ ifadelerini ihtiva eden ifadeler

i. $\sqrt{a^2-x^2}$, $x = a.\sin t$

$$a^2-x^2 = a^2 - a^2\sin^2 t = a^2\cos^2 t$$



$$\sqrt{a^2-x^2} = a.\cos t \quad , \quad dx = a\cos t\, dt$$

Ör:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$x = \sin t$, $dx = \cos t . dt$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 x} = \cos x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos t\, dt}{\cos t} = \int dt = t + C = \arcsin x + C$$

$x = \sin t$

$\arcsin x = t$

ÖR:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = ?$$

$x = a.\sin t$

$dx = a.\cos t\,.dt$

$$= \int \frac{a.\cos t\,.dt}{\sqrt{a^2-(a.\sin t)^2}}$$

$$= \int \frac{a.\cos t\,.dt}{\sqrt{a^2-(a^2.\sin^2 t)}} = \int \frac{a\cos t\,dt}{a\cos t} = t + c$$

$$= \arcsin\frac{x}{a} + c$$

$\sqrt{a^2-u^2}$, $u = a.\sin t$

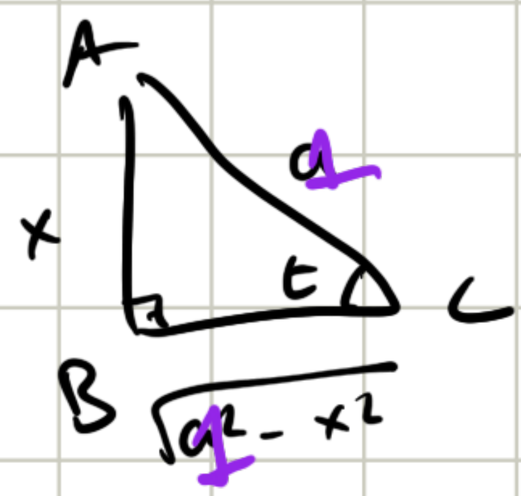
ÖR: $\int \sqrt{1-x^2}\,dx = ?$

$$\int \cos t\,.\cos t\,dt = \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt$$

$$\frac{1}{2}\int (1+\cos 2t)\,dt = \frac{t}{2} + \frac{1}{2}\int \cos 2t$$

$$\frac{t}{2} + \frac{1}{2}\cdot\frac{\sin 2t}{2} = \frac{t}{2} + \frac{\sin 2t}{4} + c$$

$$\frac{t}{2} + \frac{\cancel{2}.\sin t.\cos t}{\cancel{4}\,2} + c$$



$\sin t = \frac{x}{a}$

$$+ \frac{1}{2}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}\,x\sqrt{1-x^2} + c$$

Ör: $\displaystyle\int \frac{dx}{x\sqrt{25-16x^2}} = ?$

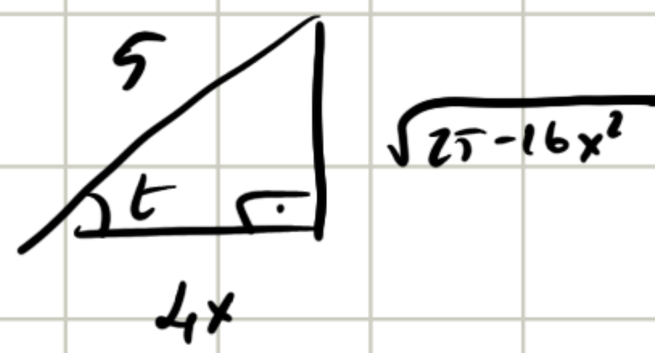
$$\int \frac{dx}{x\sqrt{25-(4x)^2}} = \frac{1}{4}\int \frac{dx}{x\sqrt{\left(\frac{5}{4}\right)^2 - x^2}}$$

$x = \frac{5}{4}\sin t$

dönüşümü

$$\frac{1}{4}\int \frac{\frac{5}{4}\cos dt}{\frac{5}{4}\sin t \cdot \frac{5}{4}\cos t}$$

$\frac{4x}{5}$



$dx = \frac{5}{4}\cos t\, dt$

$$\int \frac{\cancel{\cos t}\, dt}{\sin t \cdot \cancel{\cos t}}$$

$$= \frac{1}{5}\int \csc t\, dt$$

$$= \frac{1}{5}\ln|\csc t - \cot t| + C$$

$$= \frac{1}{5}\ln\left|\frac{5-\sqrt{25-16x^2}}{4x}\right| + C$$

ii. $\sqrt{a^2+x^2}$, intiva eden integraller

$\sqrt{a^2+x^2}$, $x = a \tan t$

$$a^2+x^2 = a^2 + a^2 \tan^2 t$$
$$= a^2 (1+\tan^2 t)$$
$$= a^2 \sec^2 t$$

$$\sqrt{a^2+x^2} = a \sec t$$

$x = a \tan t$, $dx = a \sec^2 t \, dt$

$\sqrt{a^2-u^2}$, $u = a \cdot \sin t$

$\sqrt{a^2+u^2}$, $u = a \cdot \tan t$

Ör:

$$\int \frac{dx}{\sqrt{1+x^2}} = ?$$

$x = a \cdot \tan t$

$x = \tan t$

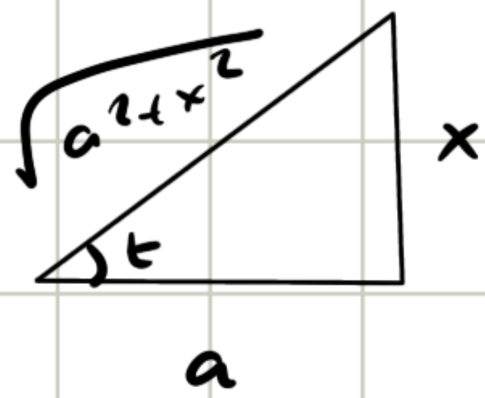
$dx = \sec^2 t \, dt$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec t \, dt}{\sec t}$$

$a = 1$

a sonda 1 dir

$\tan t = \frac{x}{a}$



$$= \int \sec t \, dt$$

$$= \ln |\sec t + \tan t| + c$$

$$= \ln \left| \sqrt{1+x^2} + x \right| + c$$

Ör:

$$\int \frac{dx}{x^2\sqrt{x^2+25}} = ?$$

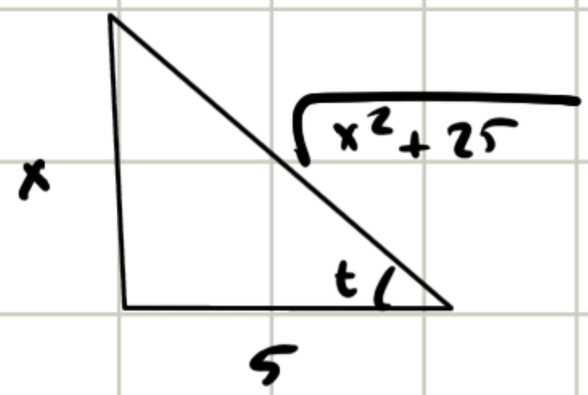
Dönüşüm

$x = 5\tan t$

$\frac{x}{5} = \tan t$

$x = 5\tan t$

$dx = 5\sec^2 t \, . dt$



$$\int \frac{5\sec^2 t . dt}{(5\tan t)^2 \sqrt{25\tan^2 t + 25}}$$

$$\int \frac{\cancel{5}\sec^{\cancel{2}} t \, dt}{25\tan^2 t \, . \cancel{5} \cancel{\sec t}} = \frac{1}{25}\int \frac{\sec t \, dt}{\tan^2 t} = \frac{1}{25}\int \frac{\frac{1}{\cancel{\cos t}}}{\frac{\sin^2 t}{\cos^{\cancel{2}} t}}$$

$$\frac{1}{25}\int \frac{\cos t}{\sin^2 t} dt$$

$\sin t = u$

$\cos t \, dt = du$

$$\frac{1}{25}\int \frac{du}{u^2} = \frac{1}{25}\int u^{-2} du$$

$$\frac{1}{25}\frac{u^{-1}}{-1} = -\frac{1}{25\sin t} = \frac{\sqrt{x^2}}{\cancel{x}}$$

Hafta 2 - Ders 2 ~ Sah

NOT

i $\sqrt{a^2-u^2}$, $u = a\sin t$

ii $\sqrt{a^2+u^2}$, $u = a\tan t$

iii $\sqrt{u^2-a^2}$, $u = a\sec t$

Örnek :

$$\int \frac{dx}{x^2\sqrt{x^2+25}} = ?$$

$x = 5\tan t$

$dx = 5\sec^2 t\, dt$

$$\int \frac{5\sec^2 t\, dt}{25\tan^2 t\sqrt{25\tan^2 t+25}} = \frac{1}{5}\int \frac{\sec^{\cancel{2}} t\, dt}{\tan^2 t \cdot \cancel{\sec t} \cdot 5} = \frac{1}{25}\int \frac{\sec t\, dt}{\tan^2 t}$$

$$= \frac{1}{25}\int \frac{\frac{1}{\cancel{\cos t}}dt}{\frac{\sin t}{\cancel{\cos t}}\cdot\frac{\sin t}{\cos t}} = \frac{1}{25}\int \frac{\cos t\, dt}{\sin^2 t}$$

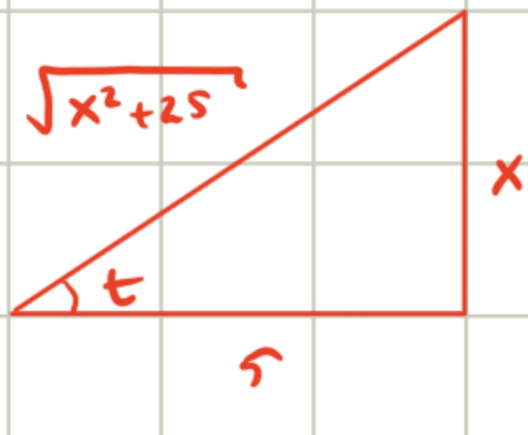
$\sin t = u$

$\cos t\, dt = du$

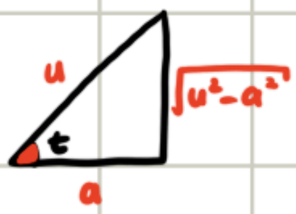
$$= \frac{1}{25}\int \frac{du}{u^2}$$

$$= \frac{1}{25}\cdot\frac{u^{-1}}{-1} + C = -\frac{1}{25u} + C = -\frac{1}{25\sin t} = -\frac{1}{25\frac{x}{\sqrt{x^2+25}}} =$$

$$= \frac{-\sqrt{x^2+25}}{25x} + C$$



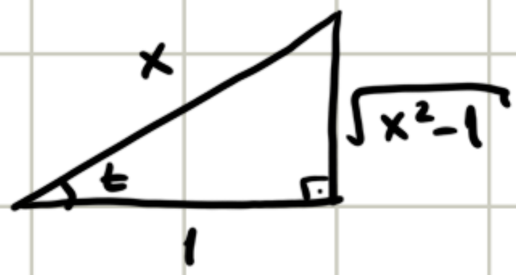
iii $\sqrt{u^2-a^2}$, $u = a\sec t$ dönüşümü yapılır.



Örnek : $\int \frac{dx}{\sqrt{x^2-1}} = ?$

$x = \sec t$

$dx = \sec t \cdot \tan t\, dt$



$$\int \frac{\sec t \cdot \tan t\, dt}{\sqrt{\sec^2 t - 1}} = \int \frac{\sec t \cdot \cancel{\tan t}\, dt}{\cancel{\tan t}} = \int \sec t\, dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left|x + \sqrt{x^2-1}\right| + C$$

Örnek: $\int \frac{dx}{x\sqrt{x^2-4}} = ?$

$x = 2\sec t$

$dx = 2\sec t \cdot \tan t\, dt$

$\frac{x}{2} = \sec t$

$\frac{2}{x} = \cos t$

$\arccos\left(\frac{2}{x}\right) = t$

$$\int \frac{\cancel{2\sec t} \cdot \tan t\, dt}{\cancel{2\sec t}\sqrt{4\sec^2 t - 4}} = \int \frac{\tan t\, dt}{2 \cdot \tan t} = \int \frac{dt}{2} = \frac{1}{2}\int dt = \frac{t^2}{4} + C = \frac{\arccos^2\left(\frac{2}{x}\right)}{4} + C$$

Örnek: $\int \sqrt{4+x^2}\,dx = ?$

$x = 2\tan t$

$dx = 2\sec^2 t\,dt$

$\sqrt{a^2+u^2}$, $u = a\tan t$

$$= \int \sqrt{4+4\tan^2 t}\,.2\sec^2 t\,dt$$

$$= \int 2\sec t\,.\,2\sec^2 t\,dt$$

$$= 4\int \sec^3 t\,dt$$

$$\boxed{\int \sec^3 t\,dt = ?}$$

Kısmi integrasyon

$\sec t = u$, $\sec^2 t\,dt = du$

$\int \sec^2 t\,.\,\sec t\,.\,dt$

$\sec t\,.\,\tan t\,.\,dt = du$, $\tan t = v$

$$\int \sec^3 t\,dt = \sec t\,.\,\tan t - \int \sec t\,.\,\tan^2 t\,.\,dt$$

$$= \sec t\,.\,\tan t - \int \sec t\,.\,(\sec^2 t - 1)\,dt$$

$$- \int (\sec^3 t - \sec t)\,dt$$

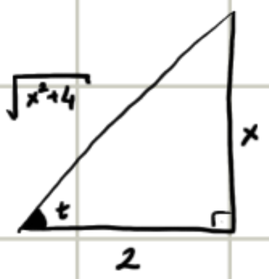
$$- \int \sec^3 t\,dt + \int \sec t\,dt$$

$$\int \sec^3 t\,dt = \sec t\,.\,\tan t - \int \sec^3 t\,dt + \int \sec t\,dt$$

$$\int \sec^3 t\,dt = \frac{\sec t\,.\,\tan t + \ln|\tan t + \sec t|}{2} + c$$

$$= \int \sqrt{4+x^2}\,dx = 4\int \sec^3 t\,dt = \frac{\frac{\sqrt{x^2+4}}{2}\cdot\frac{x}{2} + \ln\left|\frac{x}{2} + \frac{\sqrt{x^2+4}}{2}\right|}{2} + c$$

$\tan t = \frac{x}{2}$



$\cos t = \frac{2}{\sqrt{x^2+4}}$

$\sin t = \frac{x}{\sqrt{x^2+4}}$

$$= \frac{x\sqrt{x^2+4}}{8} + \frac{\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right|}{2} + c$$

Örnek: $\int \sqrt{x^2+6x+5}\,dx = ?$

$x+3 = 2\sec t$

$dx = 2\sec t \cdot \tan t \cdot dt$

$= \int \sqrt{x^2+6x+9-4}\,dx$

$= \int \sqrt{(x+3)^2-4}\,dx$

$= \int \sqrt{4\sec^2 t - 4}\;\; 2\sec t \cdot \tan t\,dt \;=\; \int 2\tan t \cdot 2\sec t \cdot \tan t\,dt$

$= 4\int \tan^2 t \cdot \sec t\,dt$

$= 4\int (\sec^2 t - 1)\cdot \sec t\,dt$

$= 4\int (\sec^3 t - \sec t)\,dt$

$= \underbrace{4\int \sec^3 t}\; - 4\int \sec t\,dt$

$= \qquad - 4\ln|\tan t + \sec t| + c$

Örnek:

$\int \frac{3x+2}{\sqrt{4x^2-4x+5}}\,dx = ?$

★ Payı paydanın türevini verecek şekilde düzenle

$4x^2-4x+5 = u$

$(8x-4)\,dx = du$

$\frac{3}{8}(8x-4) = 3x+2$

$8x-4 = \frac{8}{3}(3x+2) - \frac{28}{3}$

$4x^2-4x+5$

$4\left[x^2 - x + \frac{5}{4}\right] = 4\left[x^2 - x + \frac{5}{4}\right] = 4\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + \frac{5}{4}\right]$

$= 4\left[\left(x-\frac{1}{2}\right)^2 + 1\right] = \frac{8}{3}\int du$

$2\sqrt{\underbrace{\left(x-\frac{1}{2}\right)^2}_{u^2}+1}$

Örnek:

$$\int \frac{2x\,dx}{(4x^2-24x+27)^3} = ?$$

$u = 4x^2 - 24x + 27$ → paydanın türevi payı veriyor mu?

$du = (8x-24)\,dx$

$$\frac{1}{4}\int \frac{8x-24x+24}{(4x^2-24x+27)^3} = \frac{1}{4}\int \frac{8x-24}{(4x^3-24x+27)^3}\,dx + 6\int \frac{dx}{\sqrt{(4x^3-24x+27)^3}}$$

$$\frac{1}{4}\int \frac{du}{\sqrt{u^3}} + 6\int \frac{dx}{\sqrt{(4x^2-24x+27)^3}}$$

$$= \frac{3}{4}\int \frac{dx}{\left(x^2-6x+\frac{27}{4}\right)^3}$$

$$\boxed{(x-3)^2 - 9 - \frac{27}{3}}$$

$$= \frac{3}{4}\int \frac{dx}{\sqrt{\left(x^2-6x+3^2-3^2-\frac{27}{4}\right)^3}}$$

$$= \frac{3}{4}\int \frac{dx}{\sqrt{\left((x-3)^2-\left(\frac{3}{2}\right)^2\right)^3}}$$

$$\boxed{x-3 = \frac{3}{2}\sec t}$$

Soru: Köşeler olmasaydı nasıl çözülürdü?

Örnek:

$$\int \sqrt{4x^2-24x+27}\,dx = ?$$

$$4\left(x^2-6x+\frac{27}{4}\right)$$

$$4\left((x-3)^2-9+\frac{27}{4}\right)$$

$$\sqrt{4\left((x-3)^2-\frac{9}{4}\right)}$$

$$2\int \sqrt{\left((x-3)^2-\frac{9}{4}\right)} \rightarrow x-3 = \frac{3}{2}\sec t$$

$$dx = \frac{3}{2}\sec t\tan t\,dt$$

$$2\int \sqrt{\frac{9}{4}\sec^2 t - \frac{9}{4}}\cdot\left(\frac{3}{2}\sec t\cdot\tan t\,dt\right)$$

$$= \frac{3}{2}\cdot 3\int \tan t\cdot\sec t\cdot\tan t\,dt$$

$$= \frac{9}{2}\int (\sec^2 t - 1)\sec t\,dt$$

$$= \frac{9}{2}\int (\sec^3 t - \sec t)\,dt$$

HAFTA - 3 DERS 1

Uğur Er, Görünmü

integral teknikleri

$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$

$a = \sin t$, $a = \tan t$, $a = \sec t$

Örnek: $\displaystyle\int \frac{dx}{(1+x^2)^2} = ?$

$x = \tan t$

$dx = \sec^2 t \, dt$

$$\int \frac{\sec^2 t \, dt}{(\tan^2 t + 1)^2} = \int \frac{\sec^2 t \, dt}{\sec^4 t}$$

$$= \int \cos^2 t \, dt$$

$$= \int \frac{1 + \cos 2t}{2} \, dt$$

$$= \frac{1}{2} t + \frac{1}{2} \sin t \cdot \cos t + c$$

$$\frac{1 + \cos 2t}{2} = \cos^2 t$$

Örnek: $\displaystyle\int \frac{\sqrt{x^2 + 4x}}{x^2} \, dx$

$(x+2)^2 - 4 = x^2 + 4x$

$x + 2 = u$, $dx = du$

$$\int \frac{\sqrt{(x+2)^2 - 4}}{x^2} \, dx = \int \frac{\sqrt{u^2 - 4}}{(u-2)^2} \, du$$

$u = 2 \sec t$

$du = 2 \sec t \cdot \tan t \, dt$

$$\int \frac{2 \tan t}{(2 \sec t - 2)^2} \, 2 \sec t \cdot \tan t \, dt = \int \frac{\sec t \cdot \tan^2 t \, dt}{(\sec t - 1)^2} = \int \frac{\sec t \cdot \tan^2 t \, dt}{\sec^2 t - 2 \sec t + 1}$$

$$= \int \frac{\sec t \, (\sec^2 t - 1) \, dt}{(\sec t - 1)^2} = \int \frac{\sec^3 t - \sec t}{(\sec t - 1)^2}$$

2.40L

$$\int \frac{t^2 + 4t}{2t + 4} \, \frac{(2t+4)^2}{t^4} \cdot \frac{2(t^2 + 4t)}{(2t+4)^2} \, dt = \int \frac{(t+4)^2}{t^2 (t+2)} \, dt$$

$\sqrt{x^2 + 4x} = t - x$

$$= \int \frac{t^2 + 8t + 16}{t^2 (t+2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+2}$$

$\cancel{x^2} + 4x = t^2 - 2tx + \cancel{x^2}$

$t^2 + 8t + 16 = At(t+2) + B(t+2) + Ct^2$

$x = \dfrac{t^2}{2t + 4}$

$dx = \dfrac{2t(2t+4) - 2t^2}{(2t+4)^2}$

$\equiv At^2 + 2A + Bt + 2B + Ct^2$

$y = \sqrt{ax^2+bx+c}$, $\int R(x,y)\,dx$ biçimindeki terimleri ihtiva eden integraller için aşağıdaki dön. kolaylık sağlar.

Soru:

$\int \sqrt{1+x^2}\,dx$

1. YOL

$x = \tan t$ dönüşümü yapılarak çözülür

2. YOL

$y = \sqrt{ax^2+bx+c}$ $\int R(x,y)$

a nın pozitif veya negatif olusuna göre ayrı ayrı hesaplanır.

$a>0$ $t - \sqrt{a}\,x = \sqrt{ax^2+bx+c}$

$a<0$ ise $b^2-4ac>0$ ax^2+bx+c iki köke sahiptir (α, β)

$\sqrt{ax^2+bx+c} = t(x-\alpha)$ veya $t(x-\beta)$

Örnek:

$$\int \frac{x\,dx}{(5-4x-x^2)^{3/2}} = ?$$

$\sqrt{ax^2+bx+c}$ $a=-1<0$

$\sqrt{5-4x-x^2} = t(1-x)$ veya $t(x+5)$

$5-4x-x^2 = (1-x)(x+5) = t^2(1-x)^2$

Eğer bu dönüşümler doğru ise başlangıçta irrasyonel olan fonksiyonlar rasyonel fonk. dönüşür

$x = \frac{t^2-5}{t^2+1}$ $dx = \frac{2t(t^2+1) - (t^2-5)(2t)}{(t^2+1)^2}$

$= \frac{\cancel{2t^3}+2t - (\cancel{2t^3}+10t)}{(t^2+1)^2} = \frac{12t}{(t^2+1)^2}\,dt$

$$= \int \frac{\frac{t^2-5}{t^2+1}\cdot\frac{12t}{(t^2+1)^2}}{t^3\left(1-\frac{t^2-5}{t^2+1}\right)^3}$$

$$= \int \frac{\frac{t^2-5}{\cancel{t^2+1}}\cdot\frac{12t}{\cancel{(t^2+1)^2}}\,dt}{\frac{6^3t^3}{\cancel{(t^2+1)^3}}} = \int \frac{(t^2-5)\cancel{12}}{\underset{18}{\cancel{6^3}}\,t^2}\,dt$$

$$= \frac{1}{18}\int \frac{(t^2-5)\,dt}{t^2} = \frac{1}{18}\int dt - \frac{5}{18}\int \frac{dt}{t^2}$$

$$= \frac{t}{18} - \left(\frac{5}{18}\,\frac{t^{-1}}{-1}\right) + c$$

$$= \frac{t}{18} + \frac{5}{18t} + c = \frac{1}{18}\left(\frac{t^2+5}{t}\right) + c$$

NOT: integrali alınacak ifade $\dfrac{1}{(Ax+B)\sqrt{ax^2+bx+c}}$ ise uygun dönüşümü yapmak gereklir.

NOT: Binom diferansiyelleri α, β, γ rasyonel sabitler, a ve b reel sabitler olduğuna göre

$\int \ln x$ kısmi int ile hesaplar.

$\int \sqrt{\sin x}\, dx$, $\int \frac{e^x}{x} dx$, $\int \frac{dx}{\log x}$, $\int \sqrt{x} \cdot \sin x\, dx$ her zaman elementer integ. yöntemleri ile hesaplanamaz

$\int R\left(x, \sqrt{P(x)}\right) dx$

$\int \cosh x\, dx = \operatorname{sh} x + C$

$\int \sinh x\, dx = \operatorname{ch} x + C$

$\int \tanh x\, dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x} dx =$

$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$

$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$

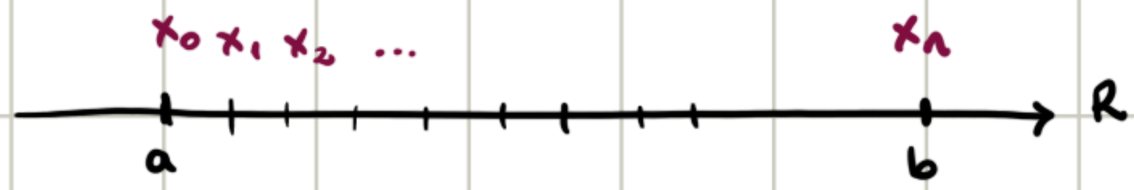
tanımlarını yazarak integral almak daha uygun olur.

Örnek: $\int \frac{dx}{\operatorname{ch} x - \operatorname{ch} a} = ?$

$$\int \frac{\frac{2du}{u}}{u + u^{-1} - e^a - e^{-a}} = \int \frac{2du}{(u-e^a)(u-e^{-a})} \quad \dots$$

BELİRLİ İNTEGRAL (DEFİNİTE İNTEGRAL)

$[a, b]$



$[a,b]$ aralığını

$x_1 < x_2 < x_3 \dots < x_n = b$ olacak şekilde

$[x_0, x_1]$, $[x_1, x_2]$, ... alt aralıklara bölelim

$P = \{ a = x_0, x_1, \dots, x_n = b \}$ kümesine $[a,b]$ aralığının bir bölüntüsü (parçası) denir.

NOT: Bir aralığın bölüntüsünü çok sayıda oluşturabiliriz.

Bir $[a,b]$ aralığını n tane eşit uzunlukta alt aralığa bölersem her bir alt aralığın uzunluğu $\frac{b-a}{n}$ olur.

Her bir alt aralık uzunluğunu Δx_i ile ayırt edelim.

$$\Delta x_i = x_i - x_{i-1}$$

ÖR: $[1,3]$, $P = \{ 1, 1.5, 1.7, 2, 2.5, 3 \}$

$[1, 1.5]$	$\Delta x_1 = 0.5$
$[1.5, 1.7]$	$\Delta x_2 = 0.3$
$[1.7, 2]$	$\Delta x_3 = 0.3$
$[2, 2.5]$	$\Delta x_4 = 0.5$
$[2.5, 3]$	$\Delta x_5 = 0.5$

Düzgün bölüntü: P, tüm alt aralık uzunlukları eşit ise bölünlüye düzgün bölüntü denir.

Bölüntünün normu: Δx_i uzunluklarının en büyüğüne bölüntünün normu denilir.

★ Düzgün bölüntünün normu alt aralık uzunluğuna eşit olur.

Riemann Toplamı ve Belirli İntegral

$y = f(x)$, $[a,b]$ aralığında sınırlı bir fonksiyon olsun.

($f(x) \to \infty$ olmayacak)

$a = x_0 < x_1 < \dots < x_n = b$ ile $p = \{x_0, \dots, x_n\}$ bölüntüsünü oluşturalım.

$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$

$x_0 \le z_0 \le x_1$, $x_1 \le z_1 \le x_2$, ...

$z_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} f(z_i)\,\Delta x_i = f(z_1)\Delta x_1 + f(z_2)\Delta x_2 + \dots + f(z_n)\Delta x_n$$

(Δx_i , i inci aralık uzunluğu)

Bu toplama , $f(x)$, $[a,b]$ aralığındaki Riemann toplamı denir.

ÖR: $[0,3]$ aralığını , n-tane eşit uzunlukta alt aralığa bölerek R_n toplamını bulun.

$$\Delta x = \Delta x_1 = \dots = \Delta x_n = \frac{3-0}{n} = \frac{3}{n}$$

$$\underbrace{\left[0, \frac{3}{n}\right]}_{z_1 = 0}, \underbrace{\left[\frac{3}{n}, 2\cdot\frac{3}{n}\right]}_{z_2 = \frac{3}{n}}, \underbrace{\left[2\cdot\frac{3}{n}, 3\cdot\frac{3}{n}\right]}_{z_3 = 2\cdot\frac{3}{n}}, \dots, \left[(n-1)\frac{3}{n}, n\cdot\frac{3}{n}\right]$$

z_i'ler her bir aralıktaki başlangıç noktası seçelim.

$z_1 = 0 \quad , \quad z_2 = \frac{3}{n} \quad , \quad z_3 = 2\cdot\frac{3}{n} \quad , \quad \dots \quad , \quad z_n = (n-1)\frac{3}{n}$

$f(z_1) = 0 \quad , \quad f(z_2) = \frac{3}{n} \quad , \quad f(z_3) = 2\cdot\frac{3}{n} \quad , \quad \dots \quad ,$

$f(z_n) = (n-1)\cdot\frac{3}{n}$

$f(z_1)\cdot\Delta x_1 + f(z_2)\Delta x_2 + f(z_3)\Delta x_3 + \dots + f(z_n)\Delta x_n$

$= 0\cdot\frac{3}{n} + \frac{3}{n}\left(\frac{3}{n}\right) + 2\cdot\frac{3}{n}\cdot\frac{3}{n} + \dots + (n-1)\frac{3}{n}\frac{3}{n}$

$= \frac{9}{n^2}\left[1+2+\dots+(n-1)\right]$

$= \frac{9}{n^2}\,\frac{(n-1)\,n}{2}$

NOT: z_i'ler alt aralıkların bitiş noktası olsun.

$z_1 = \frac{3}{n} \quad , \quad z_2 = 2\cdot\frac{3}{n} \quad , \quad \dots \quad , \quad z_n = n\cdot\frac{3}{n}$

$f(z_1) = \frac{3}{n} \quad , \quad f(z_2) = 2\cdot\frac{3}{n} \quad , \quad \dots \quad , \quad f(z_n) = n\cdot\frac{3}{n}$

$f(z_1)\Delta x_1 + \dots + f(z_n)\Delta x_n =$

$$= \frac{3}{n}\cdot\frac{3}{n} + 2\cdot\frac{3}{n}\frac{3}{n} + \dots + n\cdot\frac{3}{n}\cdot\frac{3}{n}$$

$$= \frac{9}{n^2}(1+2+\dots+n)$$

$$= \frac{9}{n^2}\cdot\frac{n(n+1)}{2}$$

Belirli İntegral

$y = f(x)$, $[a,b]$ aralığındaki sınırlı $R_n(f,P)$

$\|P\| = \max\{\Delta x_i\}$ Düzgün bölüntü için $\|P\| = R$

$\|P\|$ nin normunun sıfır olması halinde veya

n 'in sonsuza gitmesi halinde

$$\lim_{\|P\|\to 0} R_n(f,P) = \lim_{n\to\infty} R_n(f,P)$$

$$\Delta x = \frac{b-a}{n} \qquad = \lim_{n\to\infty}\sum_{i=1}^{n} f(z_i)\,\Delta x_i$$

Limiti varsa $y = f(x)$, $[a,b]$ aralığında integrallenebilir denir.

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n(f,P)$$

a alt sınır, b üst sınır adını

Ör: $\int_0^3 x\,dx = \lim_{n\to\infty} \frac{9}{n^2}\frac{(n-1)n}{2} = \frac{9}{2}$

$$\lim_{x\to\infty} \frac{9}{x^2}\frac{(x-1)x}{2} = \int_0^3 x\cdot dx = \frac{9}{2}$$

Belirli İntegral Özellikleri

integrallenebilir $\equiv$ limiti var

f ve g fonksiyonları [a,b] aralığında integrallenebilen iki fonksiyon olsun.

i. $\int_a^a f(x)\,dx = 0$

ii. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

iii. $c \in [a,b]$ $\quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

iv. $k \in \mathbb{R}$

$$\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$$

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

v. f fonksiyonu $[a,b]$ aralığında hem integrallenebilir hem $f(x) \geq 0$ ise

$$\int_a^b f(x)\,dx \geq 0$$

* $f(x) \leq g(x)$ ise $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

Ortalama Değer Teoremi

İntegral Hesabının Birinci Temel Teoremi

$f: [a,b] \to \mathbb{R}$ sürekli bir fonksiyon ve $F: [a,b] \to \mathbb{R}$ fonksiyonu

$$F(x) = \int_a^x f(t)\,dt$$ olarak tanımlasın

$f(x)$, $F(x)$ $\qquad F'(x) = f(x) \qquad \int f(x)\,dx = F(x) + C$

İntegral Hesabının İkinci Temel Teoremi

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Örnek:</u>

$$\int_0^1 (x+2)\,dx = \int_0^1 x\,dx + \int_0^1 2\,dx = \frac{x^2}{2}\Big|_0^1 + 2x\Big|_0^1$$

<u>Örnek:</u>

$-\cos x$

$$\int_{-\pi}^{\pi} \sin x\,dx = -\cos\pi + \cos(-\pi) = 0$$

Örnek: $\int_{-2}^{-1} \frac{dx}{x^3} = ?$ $\qquad \int \frac{dx}{x^3} = \int x^{-3} dx = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$$-\frac{1}{2.1} - \left(-\frac{1}{2.4}\right) = \frac{-1}{2} + \frac{1}{8}$$

$\int_{-2}^{1}$ olsa sınırlılık özelliği kaybolur integral hesaplanamaz

Örnek: $\int_{-2}^{1} \frac{dx}{x^3}$, $x=0$ noktasında, $f(x) = \frac{1}{x^3}$ sınırlı değildir.

$$\int_a^b f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f(z_i) \Delta x_i$$

Soru: $\int_0^1 (x^2+2x+3)$ $\qquad \frac{1}{3} + 1 + 3 = \frac{13}{3}$

4. Hafta - Pazartesi -

11.03.2025

Parçalı Fonk. integreli

$\int_{-1}^{2} f(x)dx$, $\int_a^b f(x)dx = \int_e^c f(x)dx + \int_c^b f(x)dx$

Örnek:

$$f(x) = \begin{cases} 2x+1 & 0 < x < -1 \\ x^2+1 & 0 < x < 2 \end{cases}$$

$$\int_{-1}^{2} f(x)dx = \int_{-1}^{0} f(x)dx + \int_0^2 f(x)dx = \int_{-1}^{0} (2x+1)\,dx + \int_0^2 (x^2+1)dx$$

Eğer bu parçaların birinde sınırsızlık varsa, bu int. çözülmez diyeceğiz veya başka bir yöntemle çözmeye çalışacağız.

Örnek:

$$|x| = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x \le 2 \end{cases}$$

$$\int_{-2}^{2} |x|\,dx = \int_{-2}^{0} (-x) + \int_{0}^{2} x\,dx$$

Örnek: $\displaystyle\int_0^1 x\sqrt{x+1}\,dx$

$$\boxed{x+1=u}$$

$$dx = du$$

$$= \int (u-1)\,u^{1/2}\,du$$

$$= \int \left(u^{\frac{3}{2}} - u^{1/2}\right) du$$

$$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c$$

$$\frac{2(x+1)^{\frac{5}{2}}}{5} - \frac{2(x+1)^{\frac{3}{2}}}{3} + c$$

Örnek:

$$\int_{x=0}^{x=1} \left(u^{3/2} - u^{1/2}\right) du = \int_1^2 \left(u^{3/2} - u^{1/2}\right) du$$

$x+1=u$, $x=0$ için $u=1$

$x=1$ " $u=2$

! Sınırları belirlerken değişkene dikkat et

Örnek: $\displaystyle\int_0^{\pi/2} x\cos x^2\,dx$

$x=\pi/2$ $\quad u=\left(\frac{\pi}{2}\right)^2$

$x=0$ $\quad u=0$

1- En basit yol → u-sub , kısmi int

$x^2=u \qquad 2x\,dx = du$

$$\int \cos u\,\frac{du}{2} = \frac{1}{2}\sin u$$

$$= \frac{\sin x^2}{2} + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \operatorname{sh} u\,du = \operatorname{ch} u + C$$

$$\int \operatorname{ch} u\,du = \operatorname{sh} u + C$$

u-sub

Örnek: $\int_1^4 x^2 e^x \, dx = ?$

Örnek: $\int_1^4 x e^x \, dx = ?$

x ve τ		e^x ve int
x	+	e^x
1	−	e^x
0		e^x

$$x e^x \Big|_1^4 - e^x \Big|_1^4$$

$$= 4e^4 - e - (e^4 - e)$$

* Eğer fonk. birinin türevi sıfıra gidiyorsa bu tablo kolaylık sağlar.

Ödev: $x \cdot e^x$ fonk. grafiği çizinize.

Tk = Reel sayılar

Asimptotta 3 özellik

$x \to 0$, $y \to \infty$ D.A.

$x \to \infty$, $y \to b$ Y.A

$x \to \infty$, $y \to \infty$ Eğik veya Eğik " Rasyonel Fonk. veya irrasyonel Fonk"

$y = mx + n$

Örnek: $\int_0^{\pi/2} x^2 \sin x \, dx = ?$

x^2, τ		$\sin x$, i
x^2	+	$\sin x$
$2x$	−	$-\cos x$
2	+	$-\sin x$
0		$\cos x$

$$\left(-x^2\cos x + 2x\sin x + 2\cos x\right)\Big|_0^{\pi/2}$$

$$= 0 + 2\left(\frac{\pi}{2} - 0 + 0 - 1\right)$$

$$= \pi - 2$$

Örnek: $\displaystyle\int \frac{2x+1}{x^2+10x+21}\,dx$

$$\frac{2x+1}{(x+7)(x+3)} = \frac{A}{(x+7)} + \frac{B}{(x+3)}$$

$$A(x+3) + B(x+7) = 2x+1$$

$$Ax + 3A + Bx + 7B = 2x+1$$

$$A+B = 2 \qquad 3A+7B = 1$$

$$B = 2-A$$

$$3A + 7(2-A) = 1$$

$$3A + 14 - 7A = 1$$

$$-4A = -13$$

$$A = \frac{13}{4}$$

$$B = 2 - \frac{13}{4} = \frac{-5}{4}$$

$$\int \left(\frac{\frac{13}{4}}{x+7} - \frac{\frac{5}{4}}{x+3} \right) dx$$

$$\frac{13}{4} \ln|x+7| - \frac{5}{4} \ln|x+3| + C$$

$$\int_4^6 \frac{2x+1}{x^2+10x+21}$$

$$\left(\frac{13}{4} \ln 13 - \frac{5}{4} \ln 9 \right)$$

$$- \left(\frac{13}{4} \ln 11 - \frac{5}{4} \ln 7 \right)$$

$$\frac{13}{4} \ln\left(\frac{91}{11}\right) - \frac{5}{4} \ln\left(\frac{7}{4}\right)$$

sabit yok belirli

Örnek: $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{1+\sin x - \cos x}$ integralini hesaplayınız

Rasy. trig. bu dönüşümden sonra rasyonele dönüşecek.

$$dx = \frac{2dt}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad t = \tan\left(\frac{x}{2}\right)$$

$$\int \frac{\frac{2dt}{1+t^2}}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{1+t^2+2t-(1-t^2)}{\cancel{1+t^2}}}$$

$$= \int \frac{dt}{t(t+1)}$$

$$= \int \frac{dt}{t(t+1)} = \ln\left|\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right| \Bigg|_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

Örnek: $\int_0^1 \sqrt{1-x^2}\, dx$

$x = \sin t$

$dx = -\cos t\, dt$

$$\int \sqrt{1-\sin^2 t}\; -\cos t\, dt = -\int \cos^2 t\, dt$$

$$-\int \frac{1+\cos 2t}{2}\, dt$$

$$= -\frac{1}{2}\left(\arcsin x + \frac{\sin 2(\arcsin x)}{2}\right)$$

$$= -\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)$$

$0 = \sin \pi, \sin 0$

$1 = \sin\frac{\pi}{2}$

$$-\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Bigg|_{0}^{\frac{\pi}{2}}$$

$$-\frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(\frac{-1}{2}\left(\pi + \frac{\sin 2\pi}{2}\right)\right)$$

$$-\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{4}$$

Sorular - ipuçları -

1- $\frac{1+\cos 2x}{2}$

2- $\frac{1-\cos 2x}{2}$

3- $\cos x = u$

$-\sin x\,dx = du$

İnce alıyorsa - yi unutma

4- $2+4x = t$

$x = \sqrt{\frac{t-2}{4}}$

5- Tablo

6- Gibi

7- Rasyonel böme işlemi

8- Kısmi integrasyon

9- Rasyonel trigonometrik

$\tan\left(\frac{x}{2}\right) = t$

10 - $\frac{2x}{(1+x^2)(3+x^2)} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{3+x^2}$

11- Basit kesirler

$\frac{x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{1+x^2}$

12 $\tan\frac{x}{2} = t$

13

Dönüşüm dere

12

15- $\ln x = u$

$\frac{du}{u}$

Belirli integral Uygulamaları

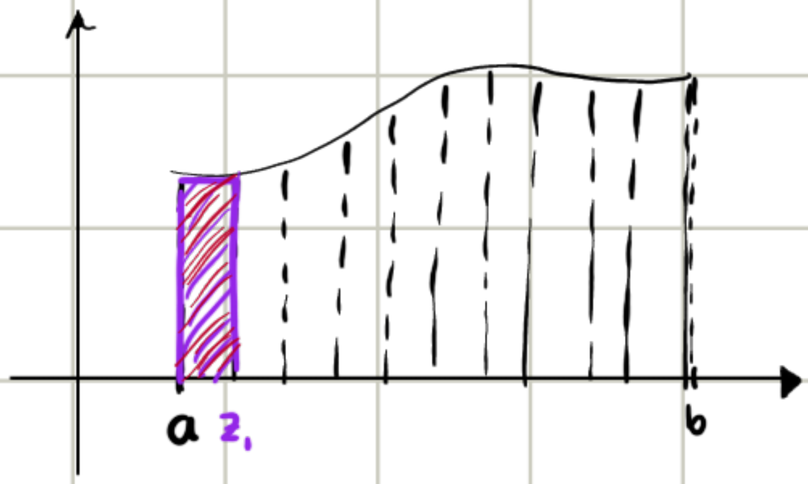
4. Hafta Salı

1. Alan Hesabı

1. dik. Alan

$$f(z_1)\Delta x_1 + \dots + f(z_n)\Delta x_n = R_n(f,P)$$

$$\lim_{n\to\infty} R_n(f,P) = \int_a^b f(x)\,dx$$



$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$

$z_1 \in [x_0, x_2]$, z

$$\Delta x_i = \frac{b-a}{n} = \Delta x$$

$$\frac{dx}{\sin x + \cos x + 2} \qquad \frac{\frac{2t}{1-t^2}}{\frac{t^2+2t+1}{1+t^2}}$$

n değeri sonsuza yaklaştıkça

Δx ler sıfıra yaklaşır.

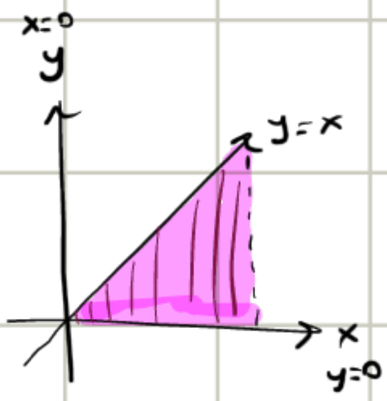
Kenarları düzlemsel olmayan bölgelerin alanını bulamazı sağlar.

Nedir Belirli integral

Alanlar toplamıdır.

Örn:

$$\int_0^1 x\,dx = \frac{x^2}{2} = \frac{1}{2}$$



Temel üçgen Alan

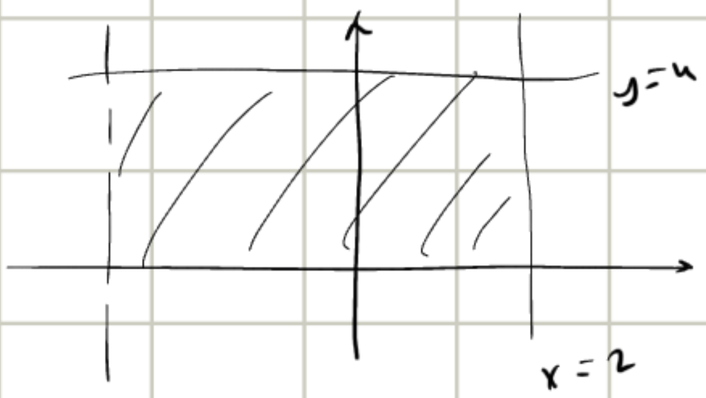
Örnek: $x^2+y^2=r^2$ $\qquad y=\pm\sqrt{r^2-x^2}$

① Bağıl gore süz taşı

$y=\sqrt{r^2-x^2}$ $\qquad$ max, min geçtiği noktalar

② Önce artılı olan alınır.

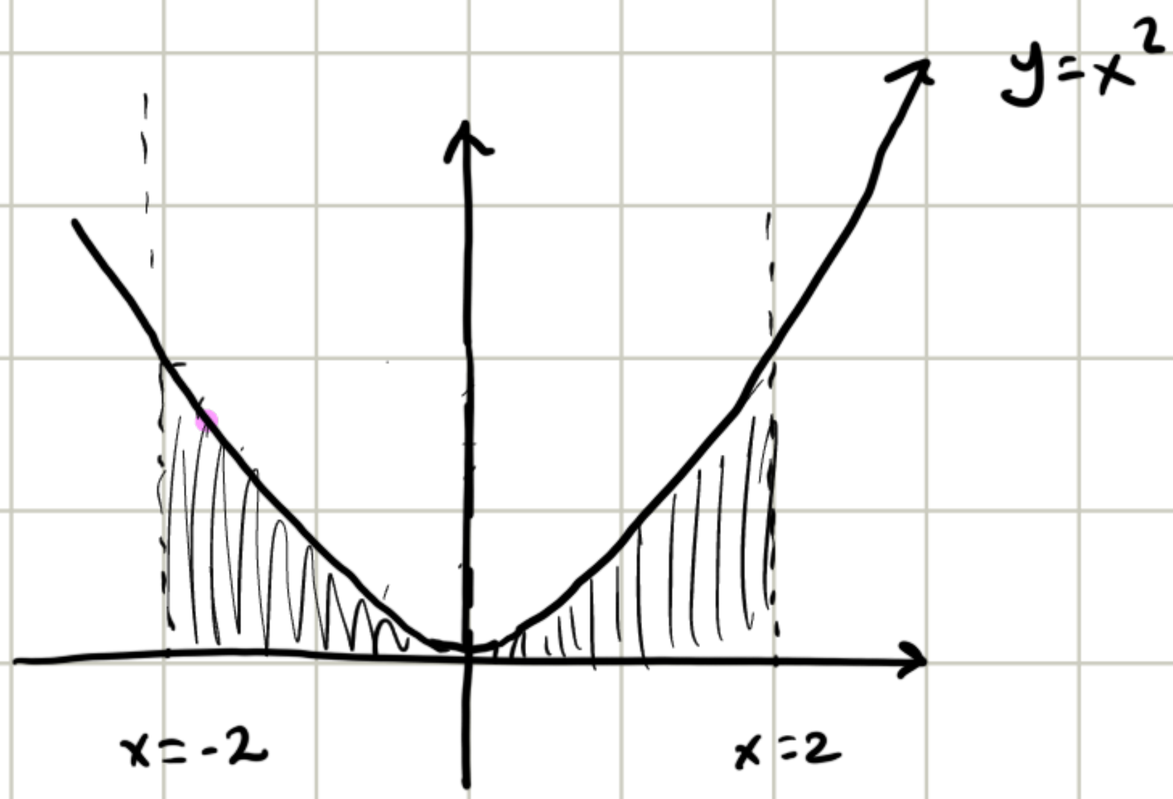
<u>Örnek</u> $\quad y=4\ ,\ x=-3\ ,\ x=2$



Alan=20

$$A=\int_{-3}^{2}4\,dx=4x\Big|_{-3}^{2}$$

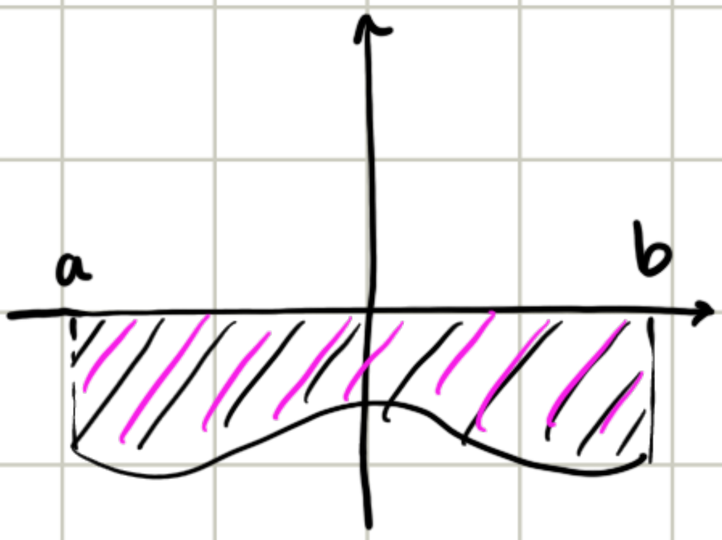
Örnek:



$$\int_{-2}^{2}x^2dx=2\int_{0}^{2}x^2dx$$

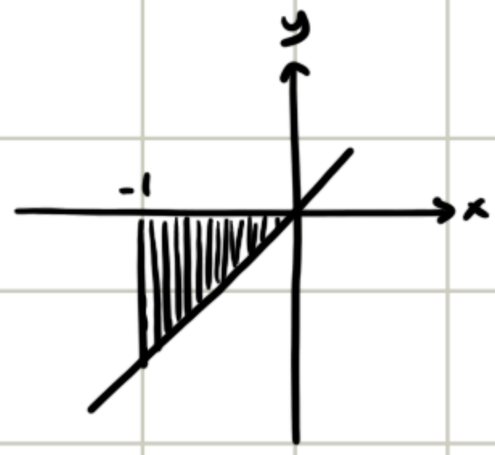
$$=\frac{2\cdot x^3}{3}=\frac{16}{3}$$

$f(x)<0$ $\qquad [a,b]$ ise,

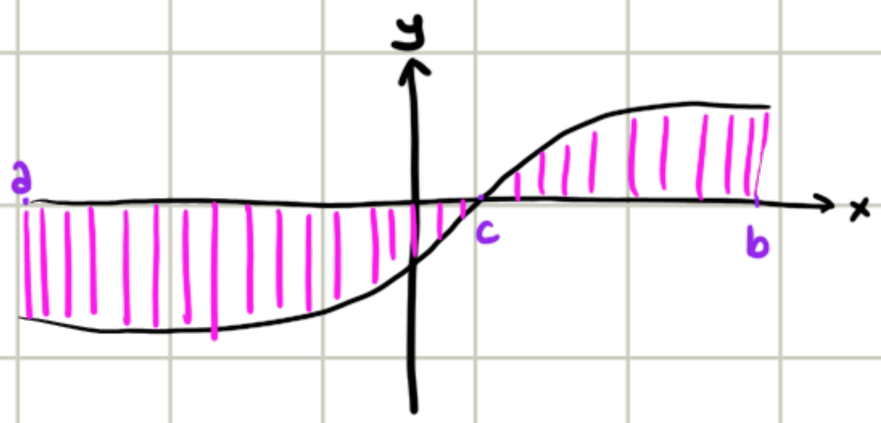


$$T.A.=\left|\int_a^b f(x)\right|=-\int_a^b f(x)\,dx$$

Örnek: $y=x$, $x=-1$, $x=0$ ve x-ek ile sınırlı bölgenin alanını bulunuz.

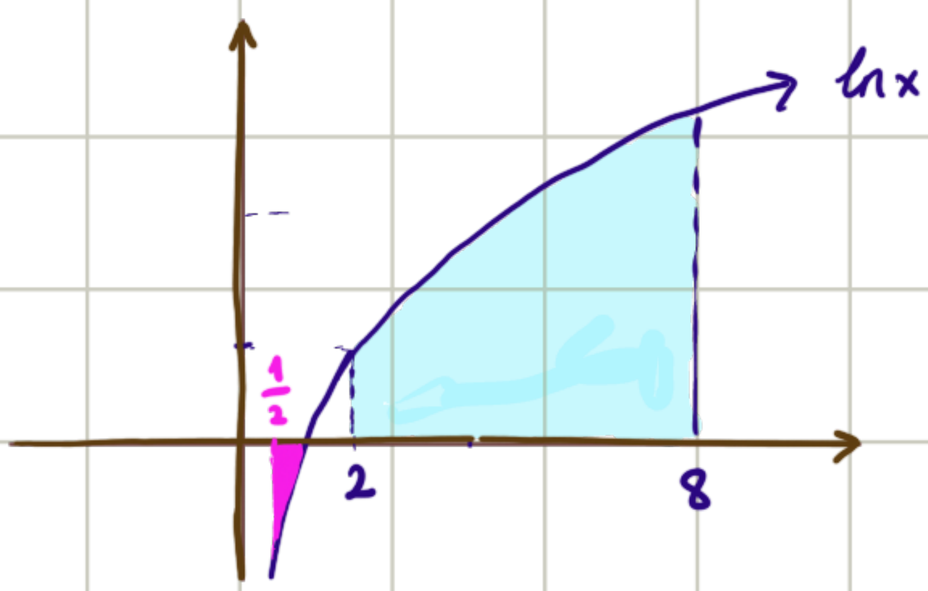


$$\int_{-1}^{0} x\,dx = \frac{x^2}{2}\Big|_{-1}^{0} = -\frac{1}{2} \text{ br}^2$$



T.A : $-\int_a^c f(x)dx + \int_c^b f(x)dx$

Örnek: $y=\ln x$, x-ekseni, $x=2$ ve $x=e^2$ doğruları ile sınırlanan alanı bulunuz.



$$A = \int_2^{e^2} \ln x\,dx = x\ln x\Big|_2^{e^2} - \int_2^{e^2} dx$$

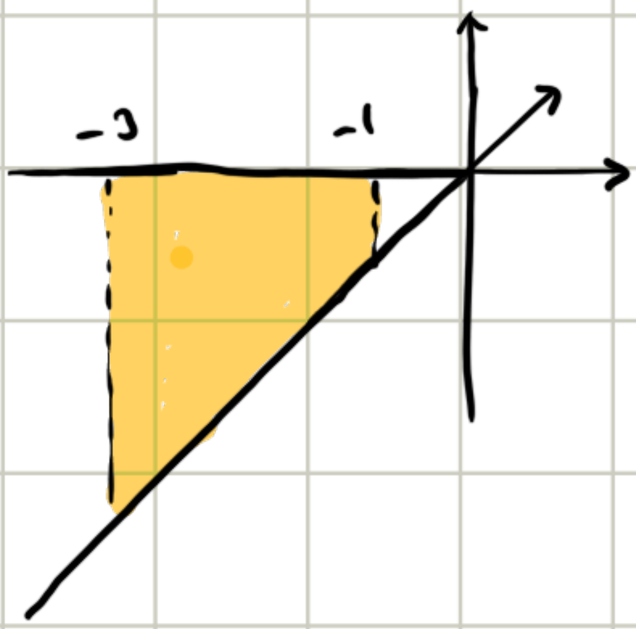
$$= x\ln x\Big|_2^{e^2} - x\Big|_2^{e^2}$$

Örnek: $y = \ln x$, $x = 1/2$, $x = e^2$, x-eks ile sınırlı bileşenin alanını bulunuz.

$$\int_{1/2}^{e^2} \ln x \, dx \quad , \quad \text{ilgili alanı vermez.}$$

$$TA = \left| \int_{1/2}^{1} \ln x \, dx \right| + \int_{1}^{e^2} \ln x \, dx$$

Örnek: $y = x$, $x = -3$, $x = -1$, x-eks



$$A = -\int_{-3}^{-1} x \, dx = \frac{x^2}{2} \Big|_{-3}^{-1} = 4$$

Örnek: $y = \sin x$ eğrisinin $[\pi, 2\pi]$ arasındaki kısmı ile x eks arasında kalan kısmı hesaplayınız.

$$\int_{\pi}^{2\pi} \sin x$$

$$-\cos x \Big|_{\pi}^{2\pi} = -(-\cos 2\pi + \cos \pi)$$

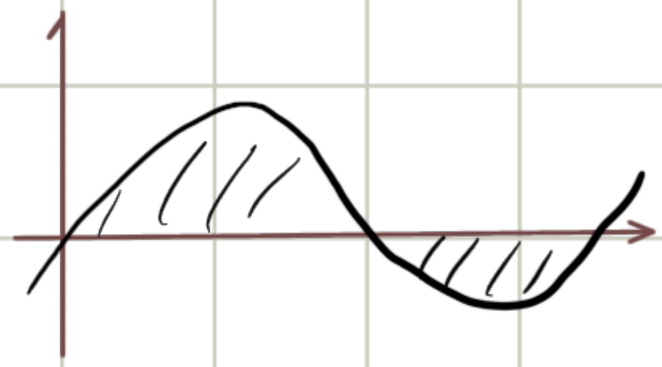
$$= -(-1 - 1) = 2$$

Örnek:

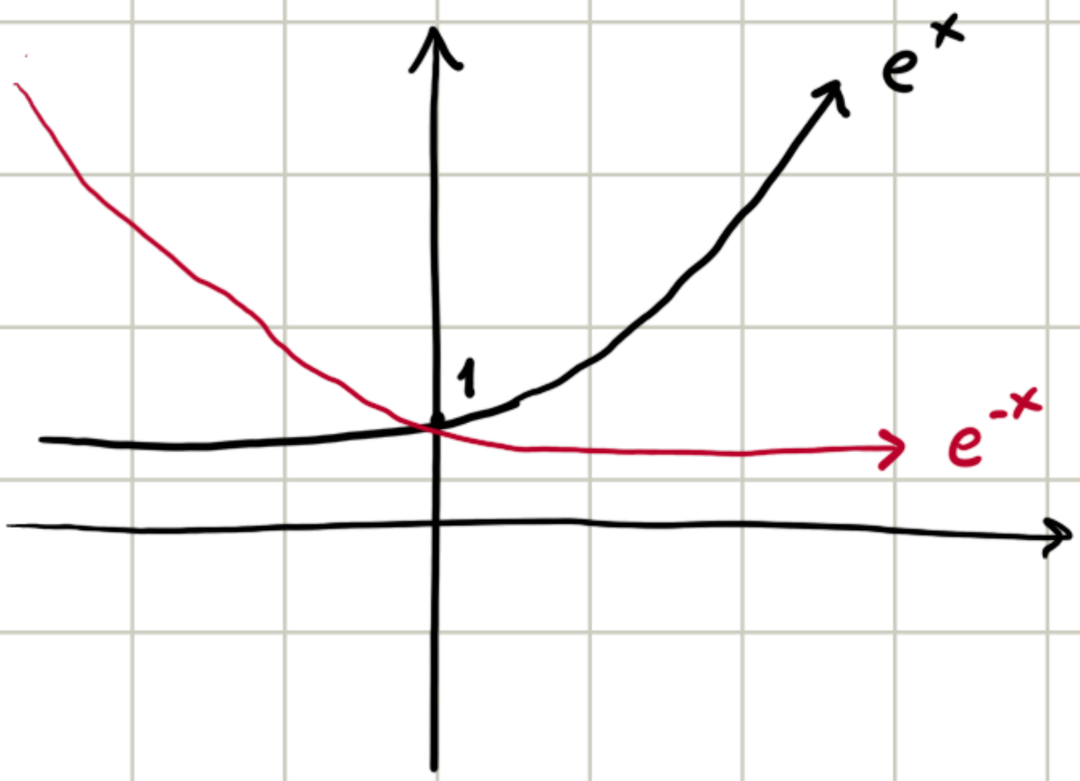
i. $\int_0^{2\pi} \sin x \, dx = ?$ $\qquad -\cos x \Big|_0^{2\pi} = -\cos 2\pi + \cos 0 = 0$

ii. $y = \sin x$, $[0, 2\pi]$ x-eks arasında kalan alanı bulunuz.

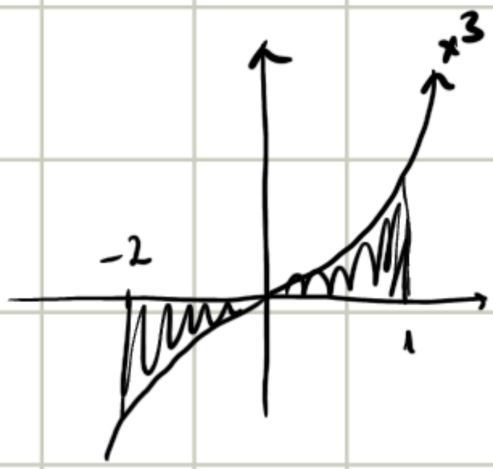
$$\int_0^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx = \text{Alan}$$



Örnek: $y = e^x$, $y = e^{-x}$, $y = -e^x$ grafikleri ile x-ekseni arasında



Örnek: $y = x^3$ eğrisi $[-2, 1]$ aralığındaki kısmı ile x-eks arasındaki alanı bulunuz.

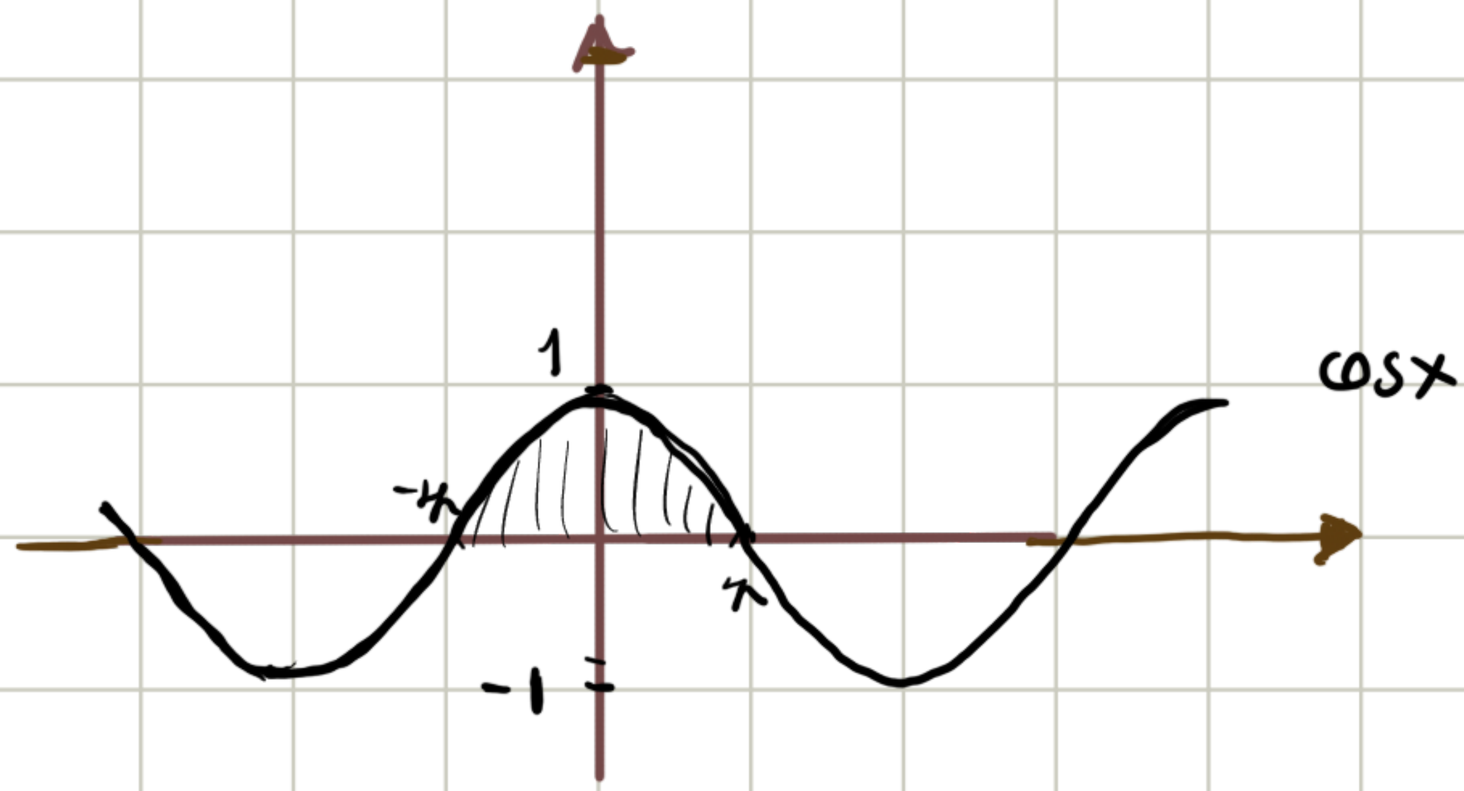


$$\int_{-2}^{0} x^3 dx + \int_0^1 x^3 dx$$

$$\frac{x^4}{4}\Big|_{-2}^{0} + -\frac{1}{4}$$

$$\frac{16}{4} - 0 + -4 + \frac{1}{4} \rightarrow \frac{17}{4}$$

Örnek: $y = \cos x$ $[-\pi, \pi]$ aralığında kalan kısmı ile x eksi arasındaki alanı bulunuz.



$$\int_{-\pi}^{\pi} \cos x \, dx = \sin x \Big|_{-\pi}^{\pi} = 0$$

Alan hesabı yapılmamış olan $-\pi$ den π ye cos integrali "0" dır.

$$\int_{-\pi}^{-\pi/2} \cos x \, dx + \int_{-\pi/2}^{\pi/2} \cos x \, dx = 4 \text{ br}^2$$

~ Bayram Öncesi Son Hafta

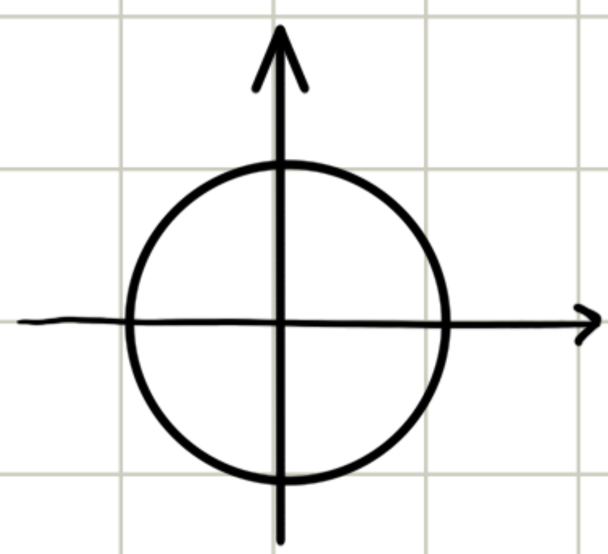
① Kapalı Fonk. / Daire

Eğer fonksiyon $y = f(x)$ şeklinde değilse

$x^2 + y^2 = r^2$

$x = \pm\sqrt{r^2 - y^2}$

$y = \pm\sqrt{r^2 - x^2}$



$$\int_c^d g(y) \, dy$$

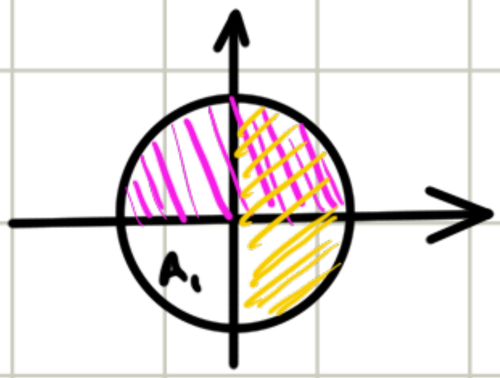
$x = g(y)$, $y = c$, $y = d$

ve y ekseni ile sınırlanan bölgenin alanı.

$$TA = \int_c^d g(y) \, dy$$

Örnek: $x^2+y^2=r^2$ ile sınırlanan bölgenin (dairenin) alanını bulalım.

Çözüm :



$y=\sqrt{r^2-x^2}$

$$\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

$x=\sqrt{r^2-y^2}$

$$\int_{-r}^{r}\sqrt{r^2-y^2}\,dy$$

$$A_1=\int_0^r\sqrt{r^2-x^2}\,dx \qquad DA=4A_1$$

A_1 'i bulalım.

$$\int_0^r\sqrt{r^2-x^2}\,dx$$

$x=r.\sin t$

$dx=r.\cos t.dt$

$$=\int_0^r\sqrt{r^2-r^2\sin^2 t}\;r.\cos t.dt$$

$$=\int_0^r r^2(\cos^2 t)\,dt=r^2\int_0^r\left(\frac{1+\cos 2t}{2}\right)dt$$

$$=r^2.\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Bigg|_0^r$$

sınırları düzenle

$x=r.\sin t$

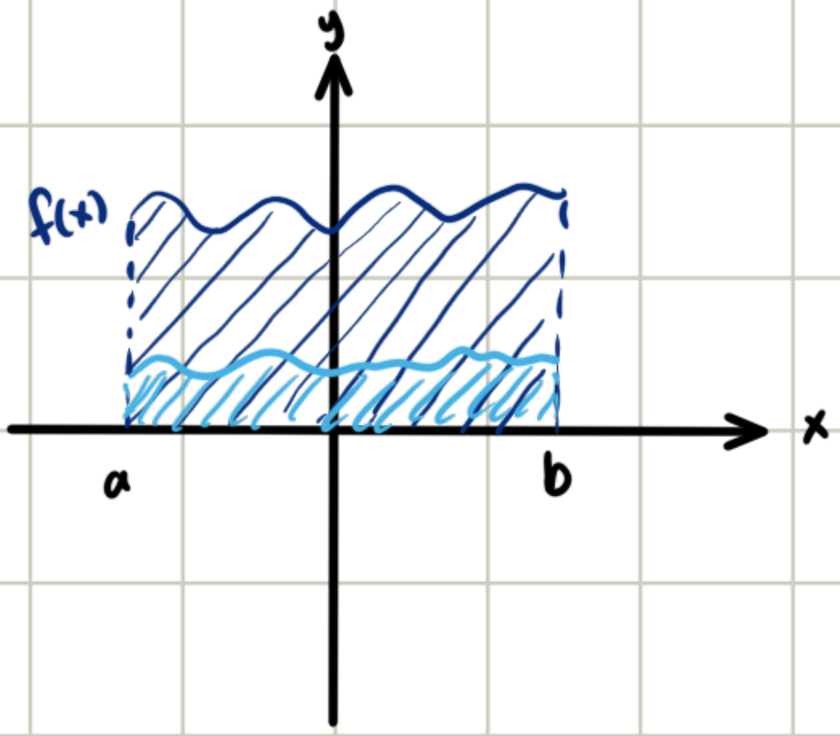
$x=0 \quad t=0$

$x=r \quad t=\pi/2$

$$\frac{r^2}{2}\left(t+\frac{\sin 2t}{2}\right)\Bigg|_{\frac{\pi}{2}}^{0}=\frac{\pi r^2}{4}=A_1$$

$4A_1=\pi r^2$ (Dairenin Alanı)

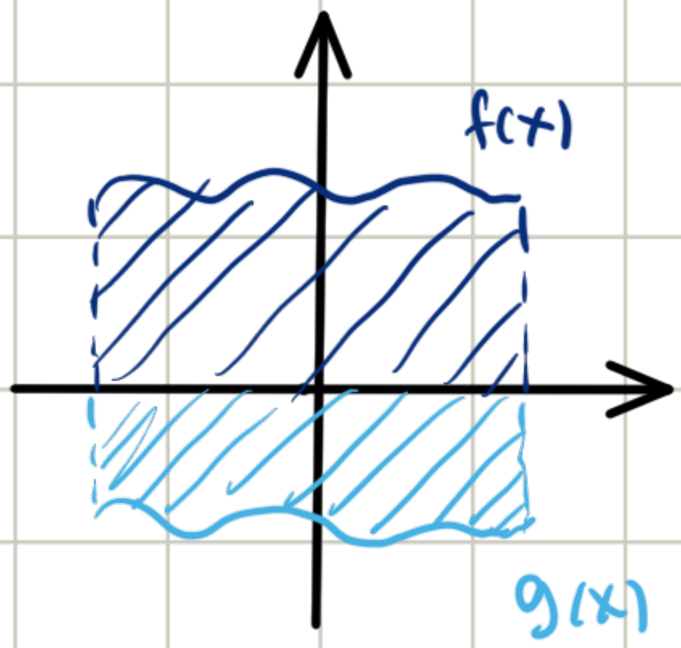
② İki Eğri Arasında Kalan Alan



$y=f(x)$, $y=g(x)$, $x=a$ ve $x=b$ ile sınırlanan alan

$$\int_a^b f(x)\,dx$$

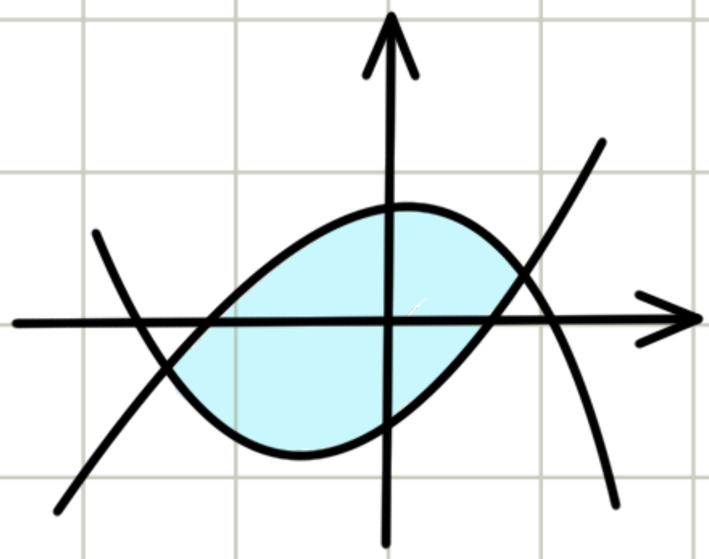
$$=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx=\text{Alan}$$



$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \text{Alan}$$

NOT: $[a,b]$, $f(x) > g(x)$ kabul edilmiştir.

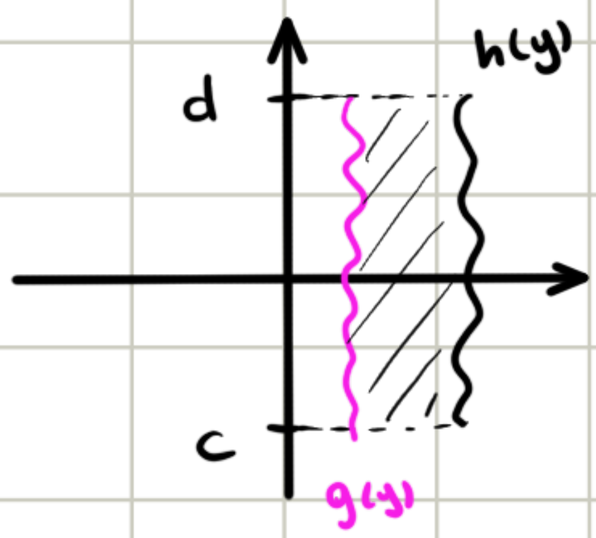
NOT:



$$\int_a^b \left[f(x) - g(x) \right] dx \qquad f(x) = g(x)$$

kesişme noktaları a ve b olarak alınır.

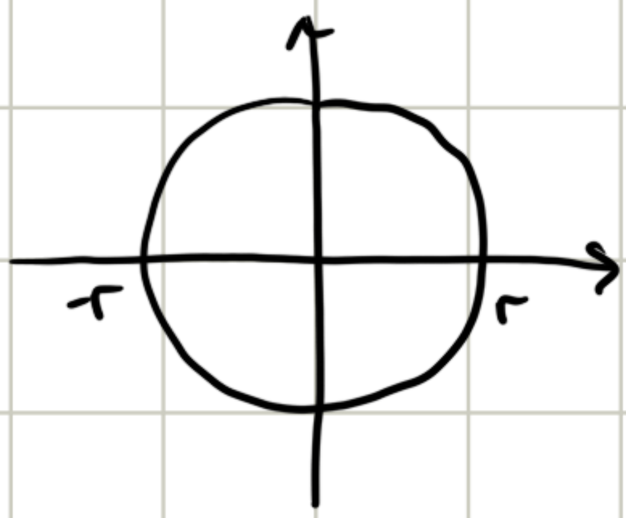
NOT: $x = h(y)$, $x = g(y)$, $y = c$, $y = d$ ile sınırlı bölgenin alanı



$$\text{T.A} = \left[\int_c^d h(y) - g(y) \right] dy$$

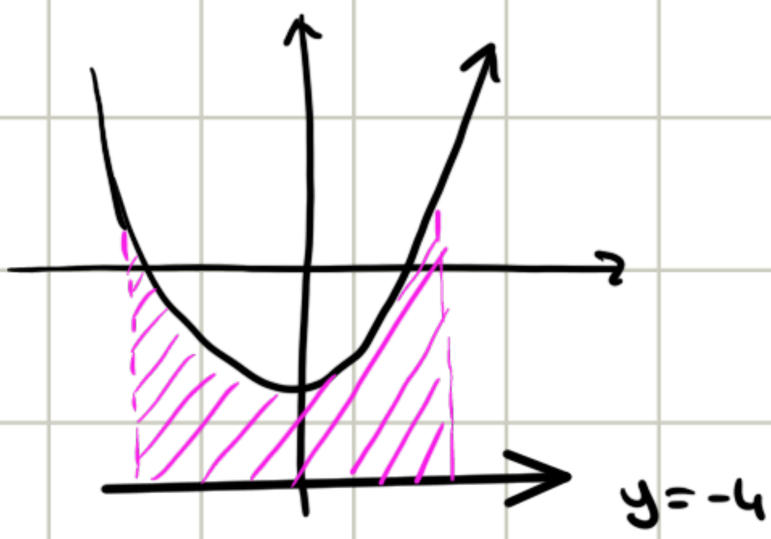
Örn: $x^2 + y^2 = r^2$ çemberinin sınırladığı alanı bulunuz.

(iki eğrinin sınırladığı alan ile)



$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx - \int_{-r}^{r} -\sqrt{r^2 - x^2}\,dx$$

ÖR: $y = x^2-1$, $y=-4$, $x=-1$, $x=1$ doğruları arasında kalan bölgenin alanı



$$\int_{-1}^{1} \left[x^2-1 - (-4) \right] dx = \frac{20}{3} \text{ br}^2$$

1. tür

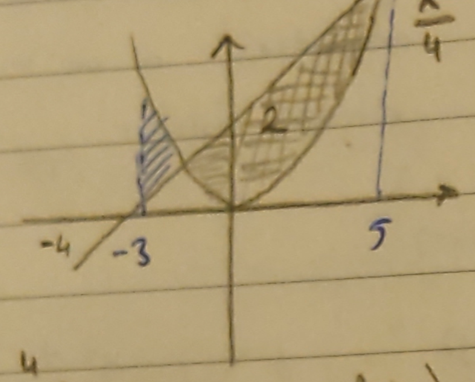
yukarıda sınırlayan

2. tür

bir eğri sağdan

bir eğri soldan

ÖR: $f(x) = \frac{x^2}{4}$, $y = \frac{x}{2}+2$ eğrileri tarafında sınırlanan alanı bulunuz



$$\frac{x^2}{4} = \frac{x}{2}+2$$

$$x^2 = 2x+8$$

$$0 = -x^2+2x+8$$

$$\begin{matrix} -x & 4 \\ x & 2 \end{matrix}$$

$$x=4 \quad x=-2$$

$$\int_{-2}^{4} \left(\left(\frac{x}{2}+2\right) - \left(\frac{x^2}{4}\right) \right) dx$$

$$= \int_{-2}^{4} \left(\left(\frac{x+4}{2}\right) - \frac{x^2}{4} \right) dx$$

$$= \int_{-2}^{4} \left(\frac{-x^2+2x+8}{4} \right) dx = \frac{1}{4} \int_{-2}^{4} (-x^2+2x+8)$$

$$= \frac{1}{4} \left(\frac{-x^3}{3} + \frac{2x^2}{2} + 8x \right) \Big|_{-2}^{4}$$

$$\frac{1}{4} \left(-\frac{64}{3} + 16 + 32 \right) - \left(\frac{1}{4} \left(\frac{8}{3} + \frac{8}{2} - 16 \right) \right)$$

$$-\frac{16}{3} + 4 + 8 - \left(\frac{2}{3} + \frac{2}{2} - 4 \right)$$

$$-\frac{16}{3} + 12 - \frac{2}{3} - 1 + 4$$

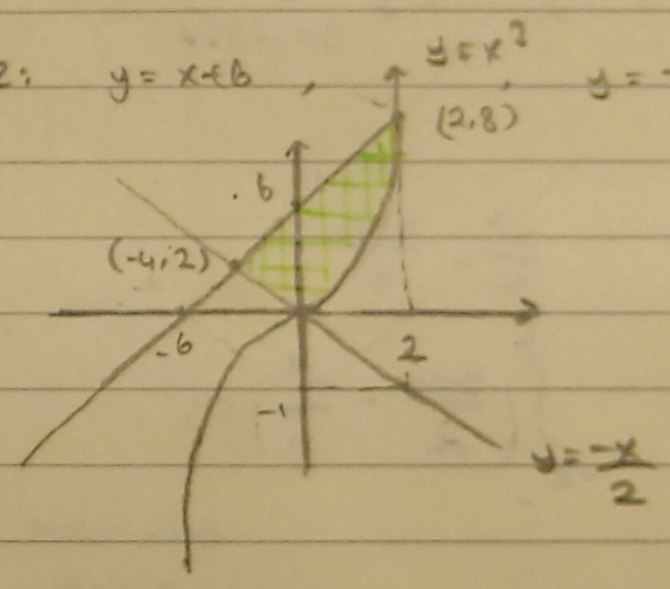
$$-6 + 15 = 9 \text{ br}^2$$

ii) yukarıda verilen eğrilerin $[-3,5]$ aralığında kalan kısımları arasındaki alanı bulunuz.

$$\int_{-3}^{-2} \left| \frac{x^2}{4} - \left(\frac{x}{2}+2\right) \right| dx , \quad \int_{4}^{5} \left| \frac{x^2}{4} - \left(\frac{x}{2}+2\right) \right| dx$$

③ Bölge birden fazla eğri ile sınırlanıyorsa

ÖR: $y = x+6$, $y = x^3$, $y = \frac{-x}{2}$



İki eğri yukarıdan

İki eğri aşağıdan sınırlıyor

Alanı parçalara bölmek gerekir

$$\frac{-x}{2} = x+6$$

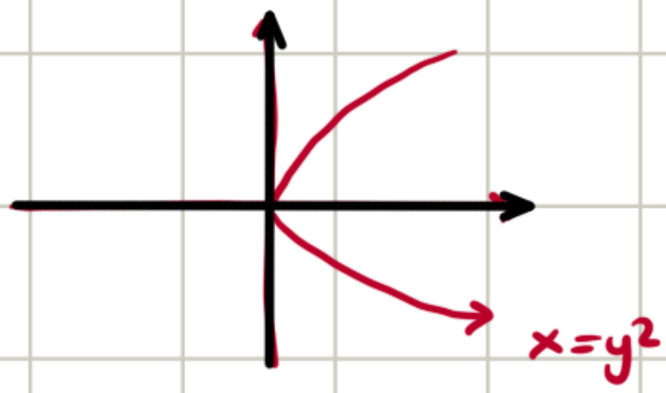
$$-6 = \frac{3x}{2}$$

$$-\frac{12}{3} = x$$

$$\int_{-4}^{0} \left| x+6 - \left(-\frac{x}{2}\right) \right| dx + \int_{0}^{2} \left| x+6 - x^3 \right| dx = 22$$

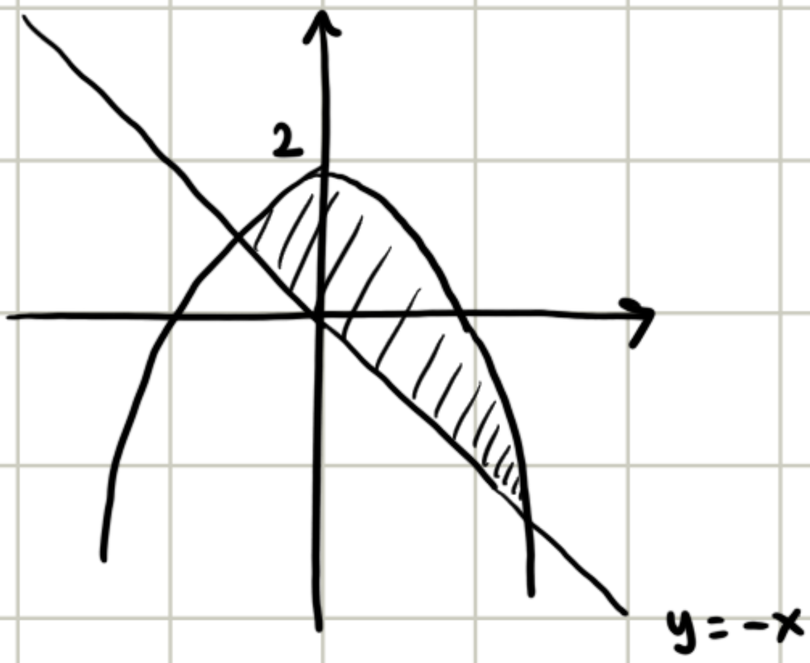
Ör: $y=-2\sqrt{x}$, $y=2\sqrt{x}$, $y=2x-4$ tarafından sınırlanan bölgenin alanını bulunuz.

NOT:



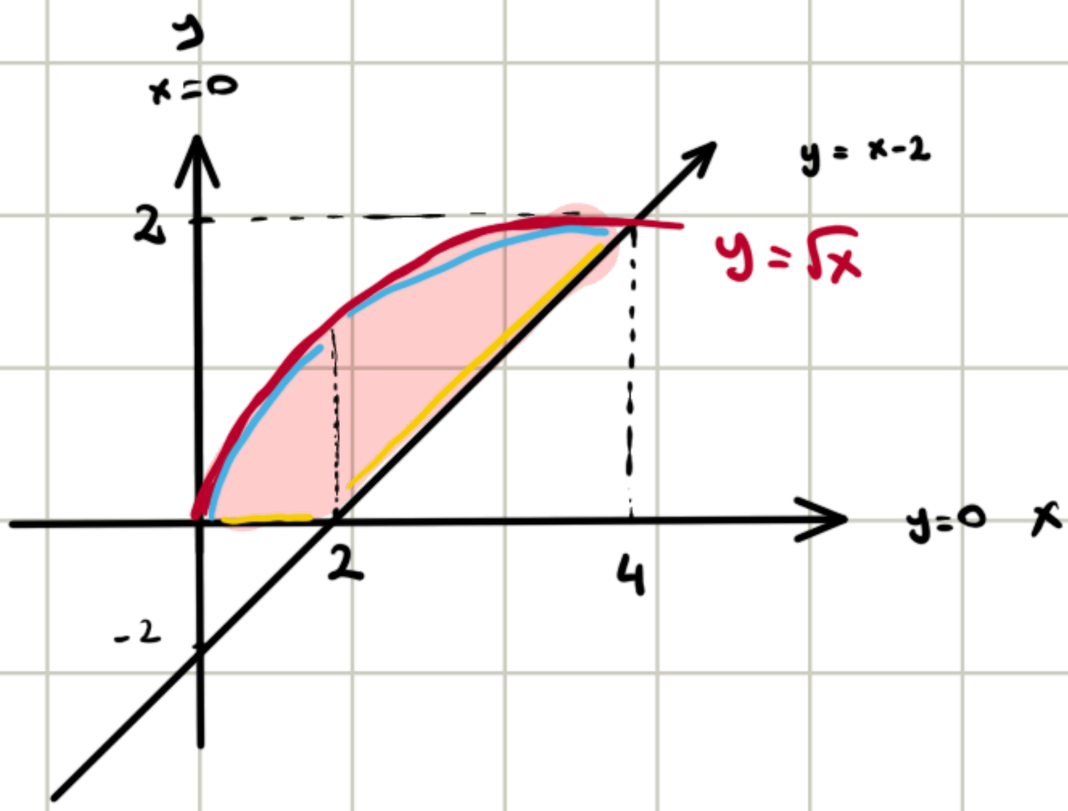
Ör

Ör: $y = 2-x^2$, $y=-x$ doğrusuyla sınırlayan bölgenin alanını bulunuz.



$$\int_{-1}^{2} \left[(2-x^2) - (-x)\right] dx = \frac{9}{2}$$

Ör: $y=\sqrt{x}$, $y = x-2$, x ekseni



$x-2 = \sqrt{x}$

$x-\sqrt{x} = 2$

$\sqrt{x}\,(\sqrt{x}-1) = 2$

2

$\sqrt{x} = 2 \qquad x=4$

Aşağıdan ve yukarıdan biner eğri ile sınırlamıyor.

$$\int_0^2 \sqrt{x}\,dx + \int_2^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$\frac{2x^{\frac{3}{2}}}{3} + \int_2^4 \left[\sqrt{x} - x + 2\right] dx$$

$$\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2} + 2x\right]\Bigg|_2^4 =$$

sağdan $x-2$ ile soldan $\sqrt{x}$ ile sınırlıyor

$$\int_0^2 \left[y+2 - (y^2)\right] dy = \int_0^2 \left[-y^2+y+2\right] dy = \frac{10}{3}\ br^2$$

Ör: $x = 12y^2 - 12y^3$ eğrisini çiziniz.

$y = 12x^2 - 12x^3$.

TK: Reel Sayılar

$x=0 \quad y=0 \quad , \quad x=1 \quad y=0$

noktanın bir çift katlı kökü iki kök vardır

Asimptot kesebi → Polinom Fonk
Asimptot yok!

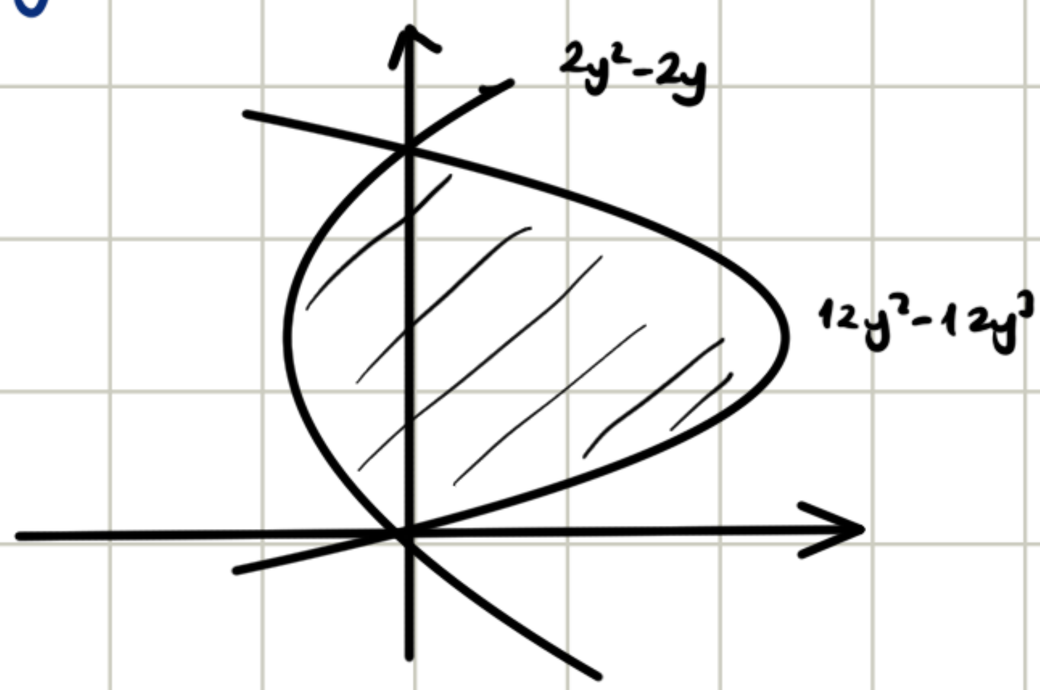
$y' = 24x - 36x^2 \longrightarrow 12x(2-3x) = 0 \qquad x=0 \quad , \quad x = \frac{2}{3}$ max, min

$y'' = 24 - 72x = 0 \qquad x = 3$ bükülme noktası

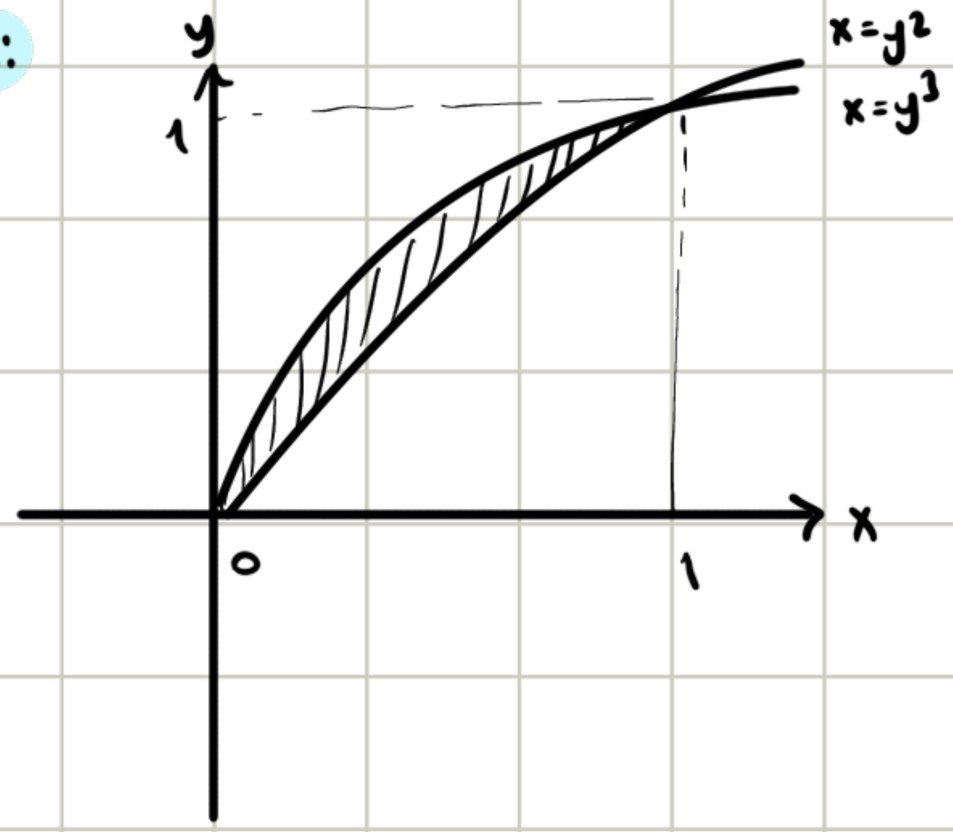
x	$-\infty$		$+\infty$
y'			
y''			

NOT: $y=x^2$, $y=ax^2+bx+c$ parabol $x=y^2$ $x=a_1y^2+b_1y+c_1$ parabol

$$\int_0^1 \left[(12y^2 - 12y^3) - (2y^2 - 2y)\right] dy$$



ÖR:



$x=y^2$ sağdan $\quad$ $x=y^3$ soldan

$y=\sqrt[3]{x}$ yukarıdan $\quad$ $y=\sqrt{x}$ aşağıdan

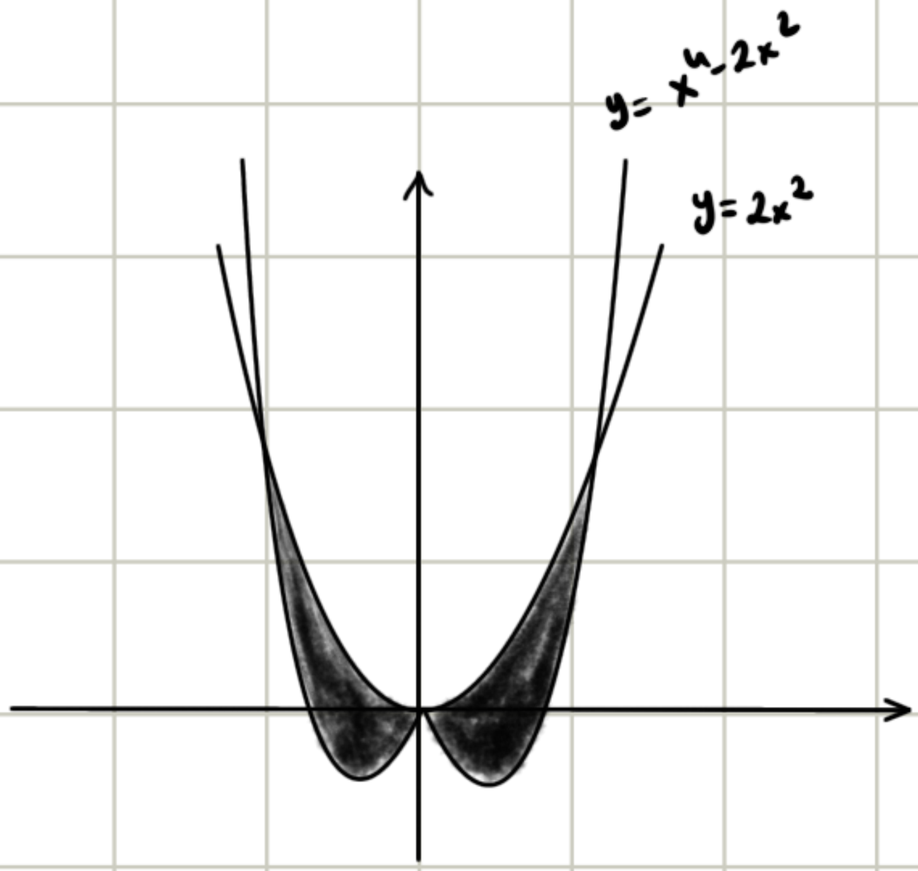
$$\int_0^1 (y^2-y^3)\,dy = \frac{y^3}{3}-\frac{y^4}{4}$$

$$= \frac{1}{3}-\frac{1}{4} = \frac{1}{12}$$

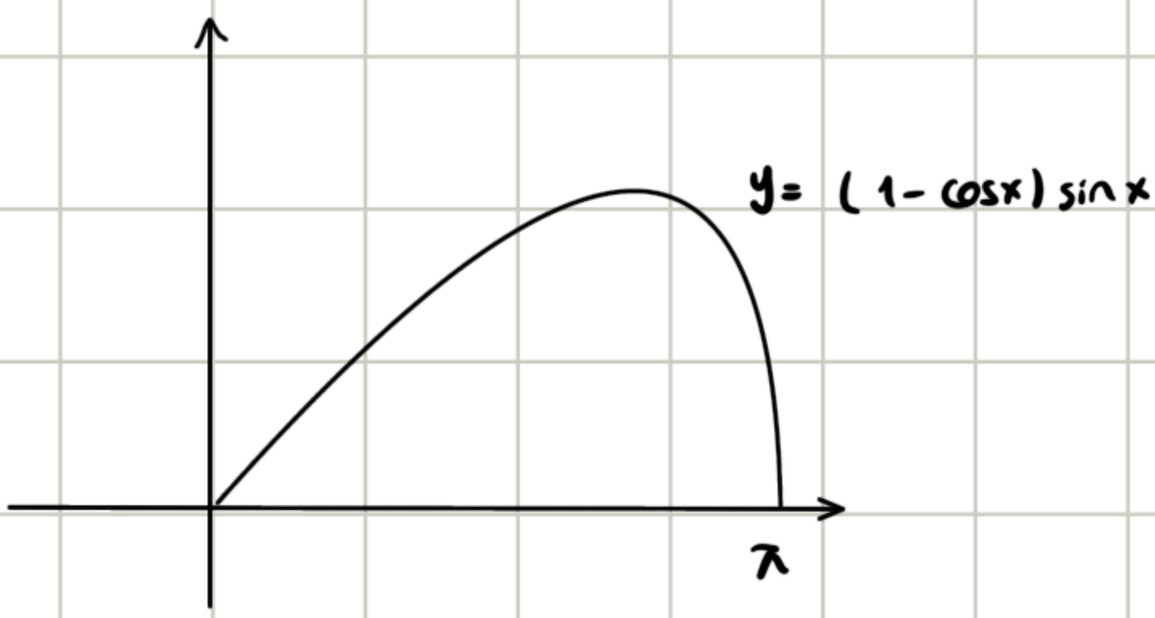
$$\int_0^1 (\sqrt[3]{x}-\sqrt[2]{x})\,dx$$

$$\left.\frac{x^{\frac{1}{3}+1}}{\frac{4}{3}}-\frac{x^{\frac{1}{2}+1}}{\frac{3}{2}}\right|_0^1 = \frac{3x^{\frac{4}{3}}}{4}-\frac{2x^{\frac{3}{2}}}{3} = \underset{(3)}{\frac{3}{4}}-\underset{(4)}{\frac{2}{3}} = \frac{9-8}{12} = \frac{1}{12}$$

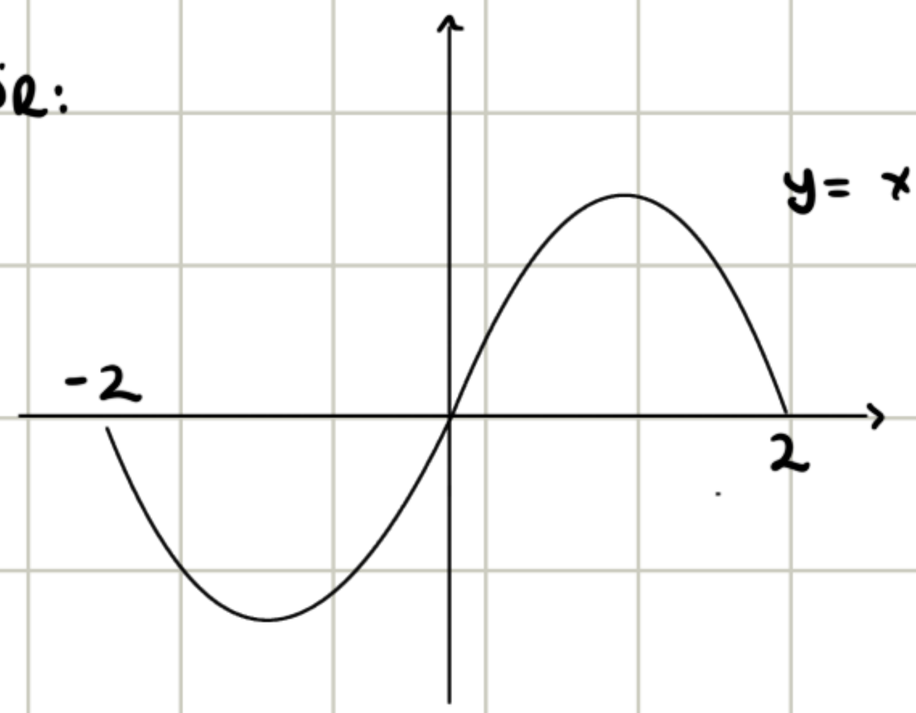
ÖR:



ÖR:



ÖR:



$y = x\sqrt{4-x^2}$

$$-\int_{-2}^{0} x\sqrt{4-x^2}$$

$$\int x\sqrt{4-x^2}\,dx$$

$x = 2\sin t$

$dx = 2\cos t\,dt$

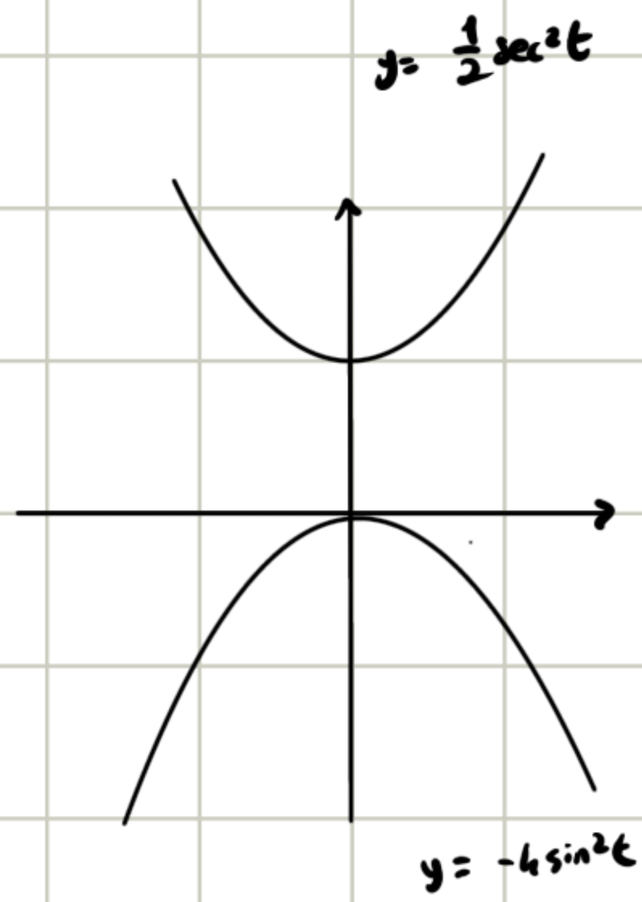
$$\int 2\sin t \cdot 2\cos t \; 2\cos t\,dt$$

$$8\int \sin t \cos^2 t\,dt$$

Ödev: $y = 3(\sin x)\sqrt{1+\cos x}$ grafiğinin x ekseni ile sınırladığı alan

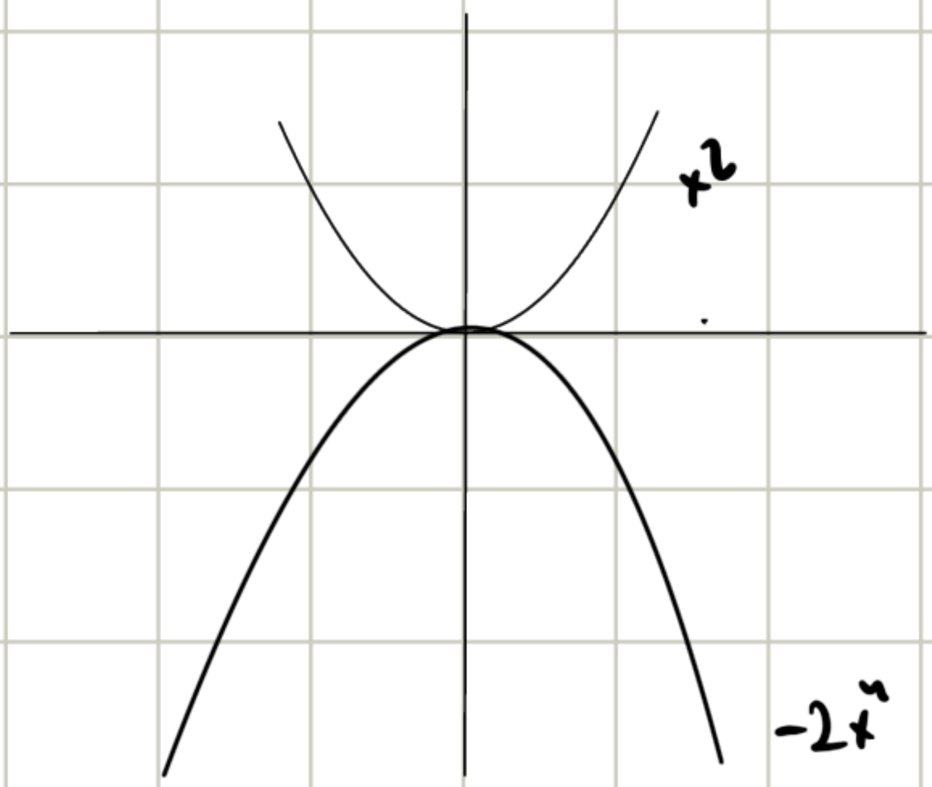
Ödev: $y = \frac{x}{2}\cos x\,(\sin(x + x\sin x))$ $\quad y' = ?$ $\quad (y')' = ?$

$y = \frac{x}{2}\cos x\,(\sin(x + x\sin x))$ $\quad y' =$

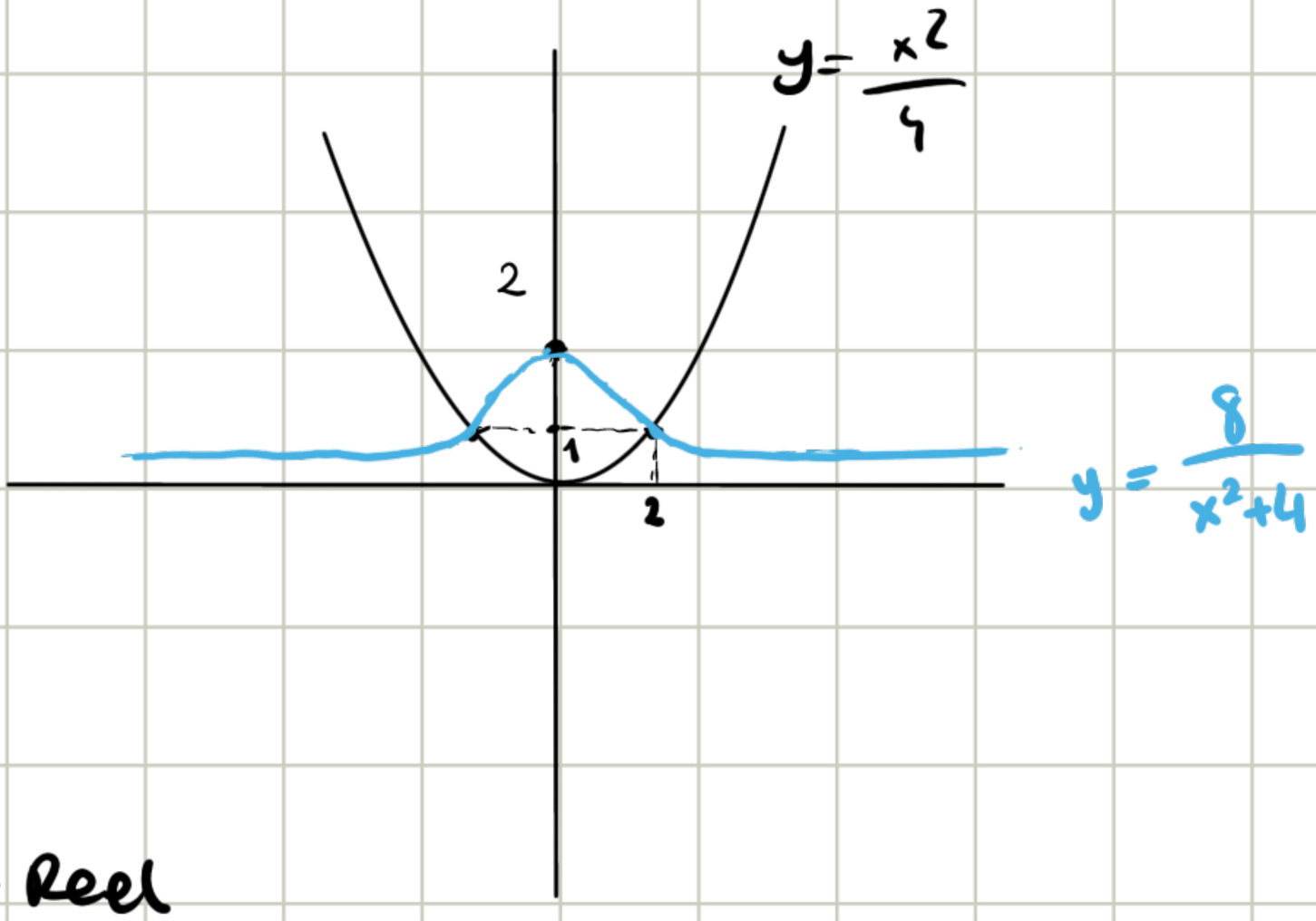


Ödev: $y = -4\sin^2 t$ $\quad y = \frac{1}{2}\sec^2 t$

Ödev:



örn: $x^2 = 4y$ parabolü ile $y = \dfrac{8}{x^2+4}$ eğrisinin sınırladığı alanı bulunuz



$y = \dfrac{8}{x^2+4}$

$x = 0 \quad y = 2$

$\dfrac{8}{x^2+4} = \dfrac{x^2}{4}$

TK : Reel

x ekseni yatay asimptot

$32 = x^4 + 4x^2$

$32 = x^2(x^2+4)$

$4 \quad 4$

$y' = \left(\dfrac{8}{x^2+4}\right)' = \dfrac{-16x}{(x^2+4)^2}$

$x^2 = 4 \quad x = 2 \quad x = -2$

$x < 0 \quad , y' > 0$

$x > 0 \quad , y' < 0$

Bükülme Noktası

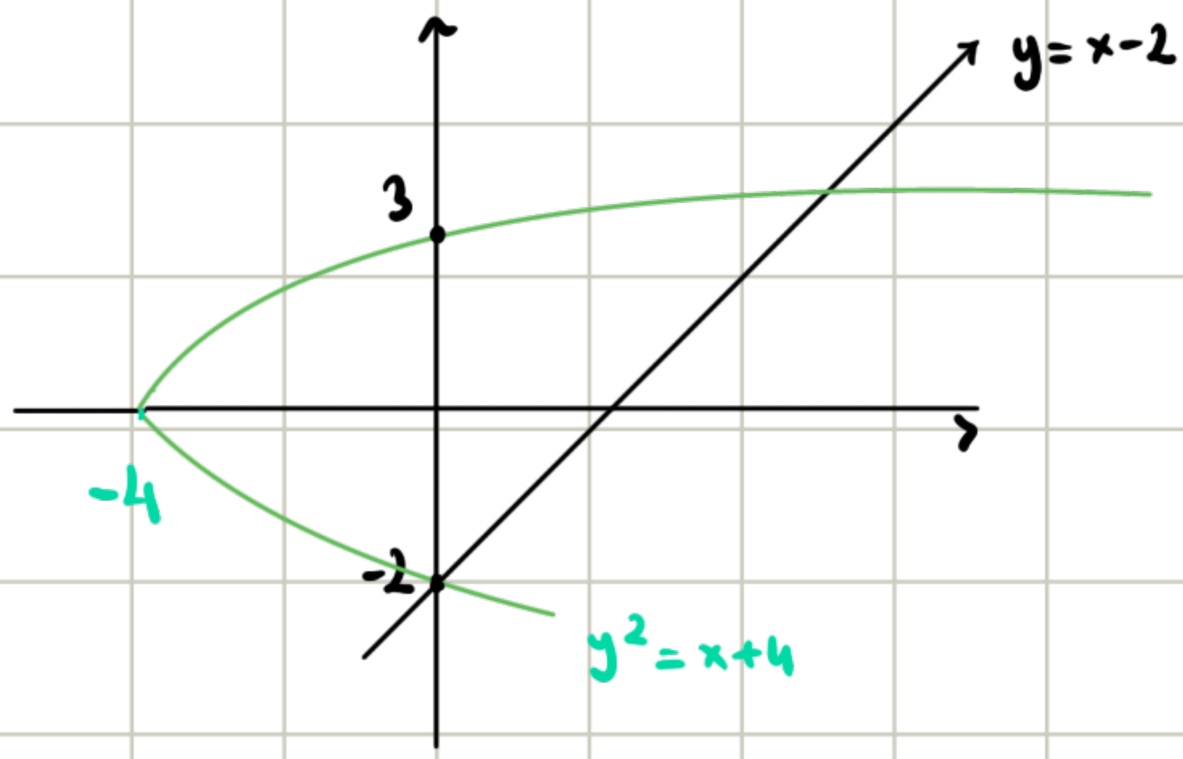
x	$-\infty$		0		$+\infty$
y'		$+$	$\|\|$	$-$	
y	0	$\nearrow$	2 (max)	$\searrow$	0

$y'' =$

$$\text{Alan} = 2\int_0^2 \frac{8}{x^2+4} - \frac{x^2}{4}\,dx = 2\cdot 8\int \frac{1}{x^2+4}\,dx$$

Eğer bora çalışıyorsa ter sil

ÖR: $y^2 = x+4$ paraboli ile $y = x-2$ doğrusu arasında kalan alanı hesaplayınız.



$y+2 = x$

$y^2-4 = x$

$y^2-4 = y+2$

$y^2-y-6=0$

$y \quad -3$

$y \quad +2$

$$\int_{-2}^{3} \left[y+2 - (y^2-4) \right] dy$$

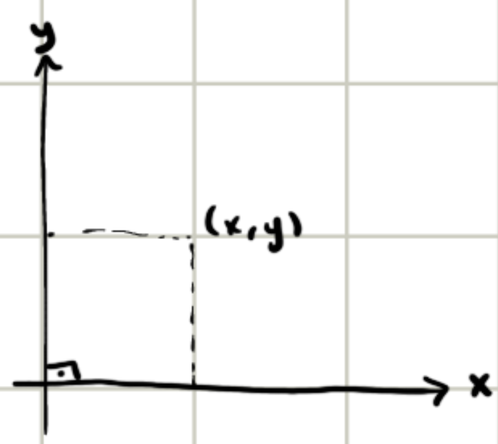
Dönen Cisimlerin Hacimleri

Burada oluşan cismin denklemini bulmuyoruz. $y = f(x)$

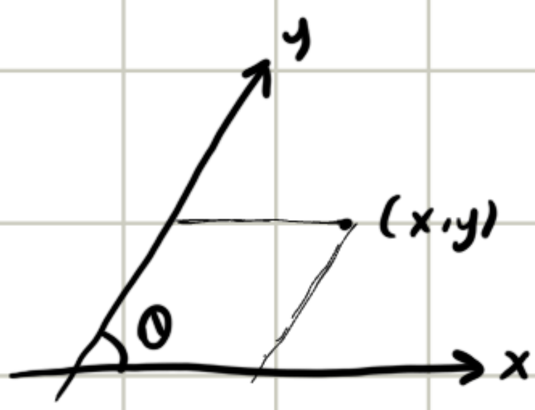
$x=a$, $x=b$ ve x - ekseni ile sınırlı bölgenin x - ekseni et. döndürülmesiyle oluşan cismin hacmini bulacağız

$R \times R = R^2 = \{ (x,y) : x, y \in R \}$ (Düzlem)

Dik Koordinat Sistemi



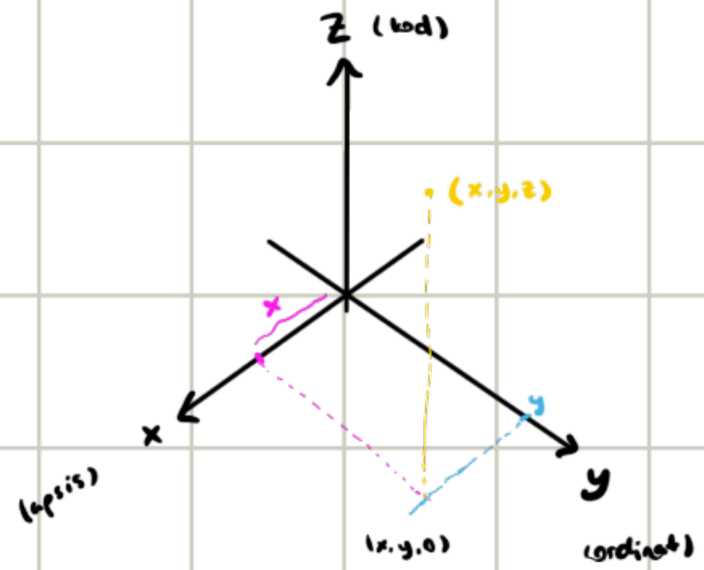
Eğik Koordinat Sistemi



Uzay

$R^3 = R \times R \times R = \{(x, y, z) : x, y, z \in R\}$

saat yönünün tersinde (genellikle)

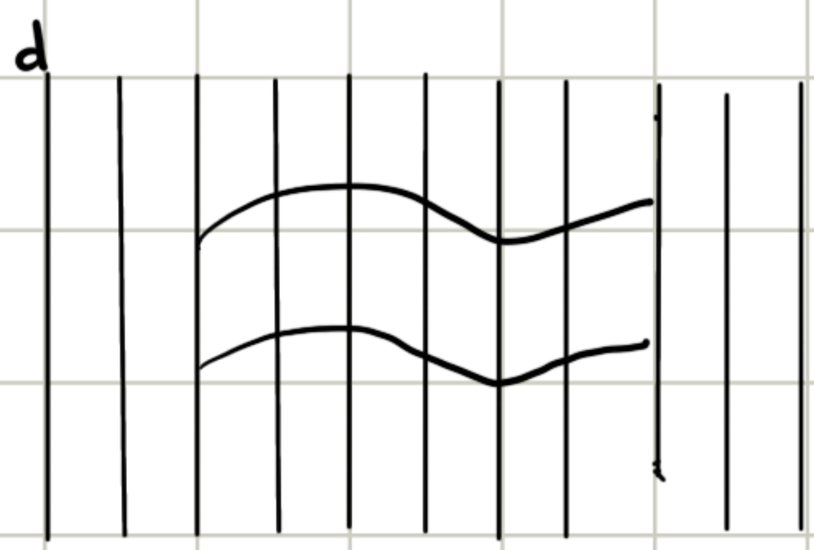


1. Disk Metodu

2. Silindir Kalınlığı (Tabaka Metodu)

Silindirin hacmi; V = Taban alanı × yükseklik

Silindir: Bir doğrunun bir eğri boyunca bir doğruya veya doğultuya paralel kalarak hareket etmesiyle oluşturduğu yüzeye silindir yüzeyi denir.

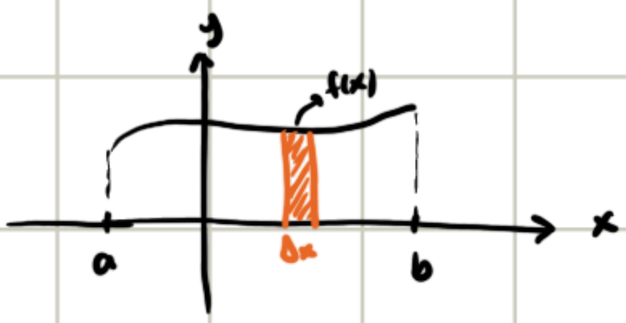


$V = \pi r^2 h$

Dairenin Alanı: Eğrinin sınırladığı alandır. İntegralle bulunur.

1. Disk metodu ile Dönel cisimlerin Hacmi

$y = f(x)$, $x = a$, $x = b$ ve x-ekseni ile sınırlı bölgenin x-ekseni etrafında (360° dönd.) meydana gelen cismin hacmi.



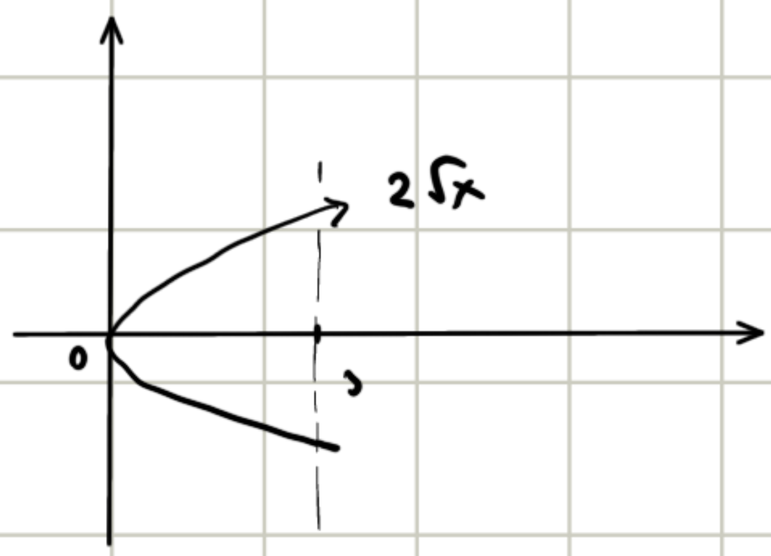
temsili dikdörtgenin döndürülmesiyle bir silindir oluşur

i. dikdörtgen için

$$V_i = \pi\,[f(x)]^2 \cdot \Delta x \qquad V = \int_a^b \pi\,[f(x)]^2 \cdot dx = \pi \int_a^b y^2\,dx$$

Örnek: $y = 2\sqrt{x}$ eğrisinin $[0,3]$ aralığında kalan kısmı ile x-eks arasındaki bölgenin x ekseni etrafında dönmesiyle oluşan cismin hacmini bulunuz.

$x = \frac{1}{4}y^2$



$$\int_0^3 \pi\,(2\sqrt{x})^2\,dx$$

$$= \int_0^3 \pi\,4x\,dx = \pi \int_0^3 4x\,dx$$

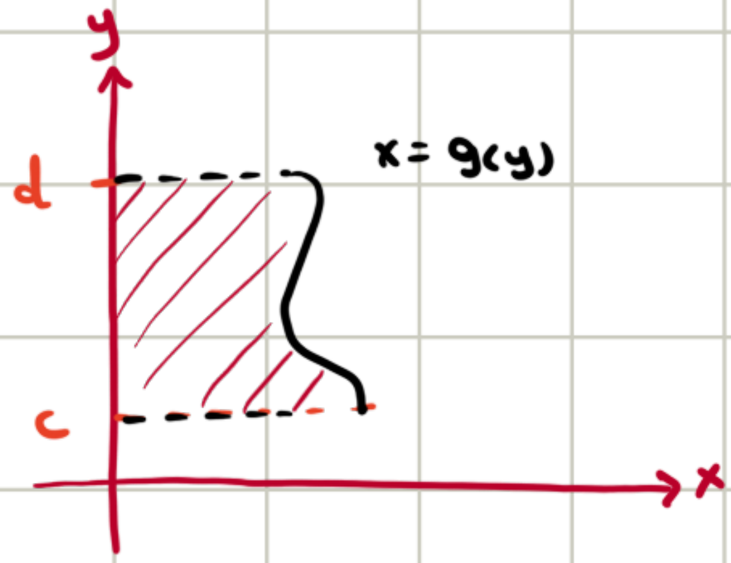
$$4\pi \int_0^3 x\,dx$$

$$4\pi \frac{x^2}{2}\Big|_0^3 = \frac{4\pi\,9}{2}$$

$$= 18\pi$$

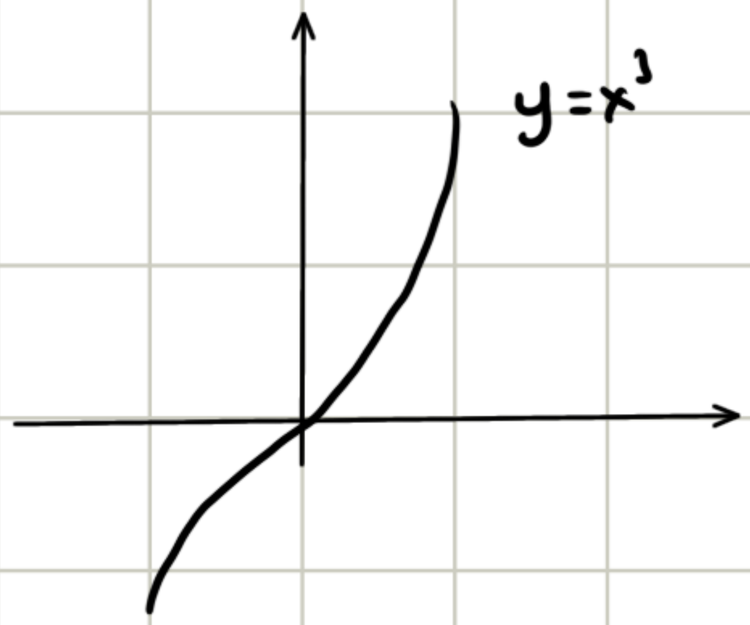
Örnek: $y = x^3$ eğrisi, y ekseni $y=1$ ve $y=3$ doğruları arasında kalan bölgenin y ekseni et. dön. ile oluşan hacmi bulunuz.

NOT:



$x = g(y)$, $y=c$, $y=d$ doğruları ve y eks. arasında kalan bölge y-eks etr. dönd. ile oluşan cismin hacmi

$$V = \pi \int_c^d x^2\,dy = \pi \int_c^d [g(y)]^2\,dy$$



$x = y^{\frac{1}{3}}$

$$V = \pi \int_1^3 x^2\, dy = \pi \int_1^3 y^{2/3}\, dy$$

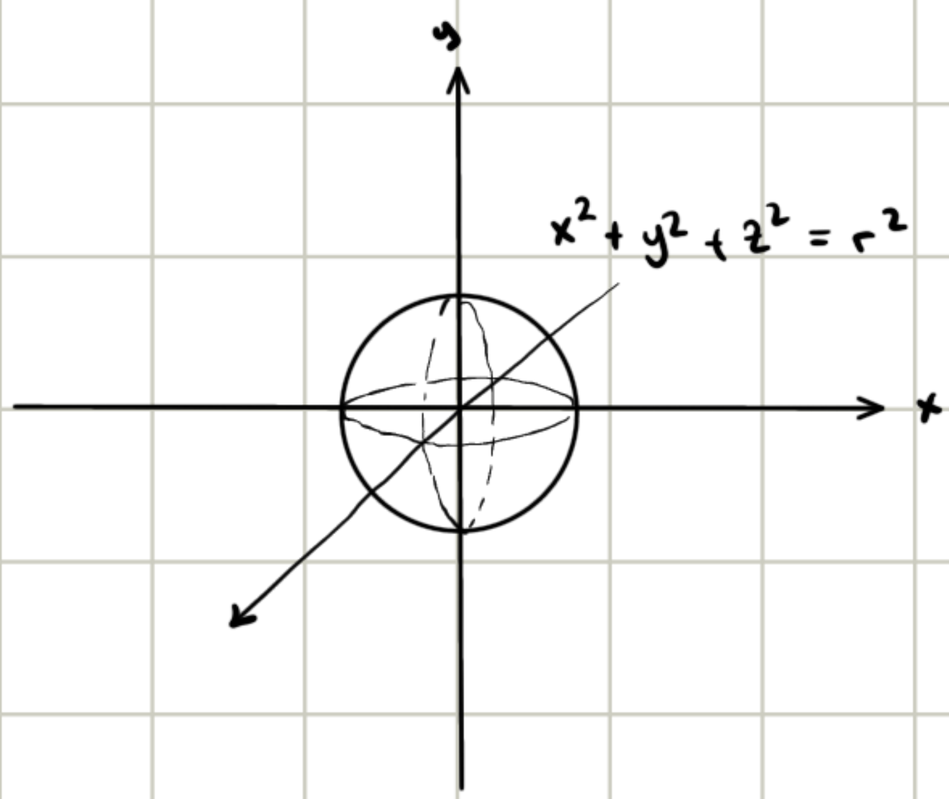
$$= \pi \left(\frac{9}{5} \cdot 3^{\frac{2}{3}} - \frac{3}{5}\right) br^3$$

Kürenin Hacmi

Örnek: Merkezi orjinde yarıçapı r olan çemberin üst yarı düzlemde kalan kısmının x eksi etrafında dönmesi ile oluşan cismin (küre) hacmini bulunuz.

Çember : düzlemde verilen bir noktaya sabit bir uzaklıktaki noktalar kümesi

Küre : uzayda verilen bir noktaya sabit uzaklıktaki noktalar kümesi



üst yarı kısım $\quad y = \sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} y^2\, dx = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \frac{4}{3} \pi r^3 \; br^3$$

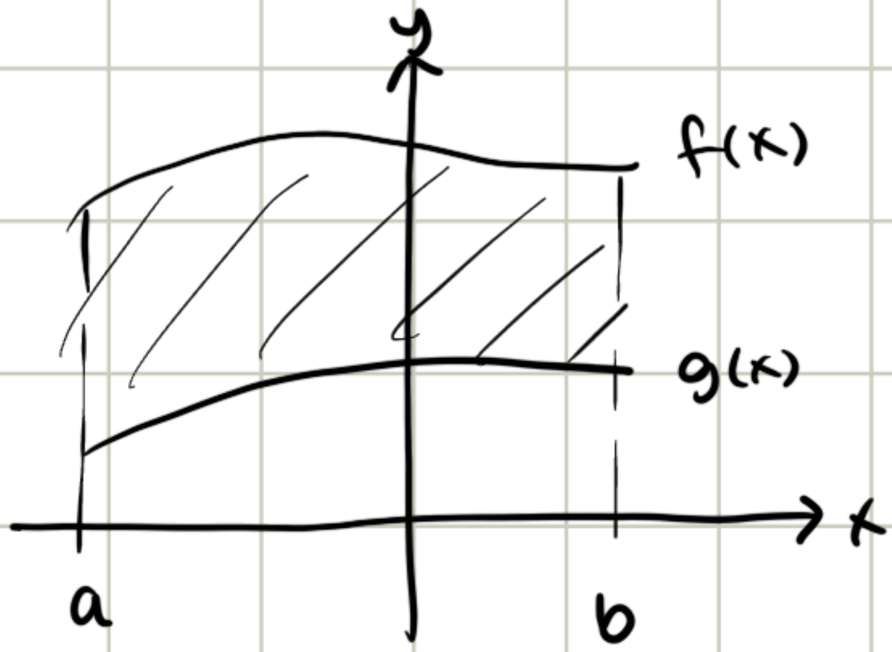
2. yol : y eksi etrafında döndürülmesi ile

$$V = \pi \int_{-r}^{r} x^2\, dy$$

$x^2 + y^2 = r^2$

$x^2 = r^2 - y^2$

İki Eğri Arasında Kalan Bölgenin x - eks etr. dön ile Oluşan Cismin Hacmi

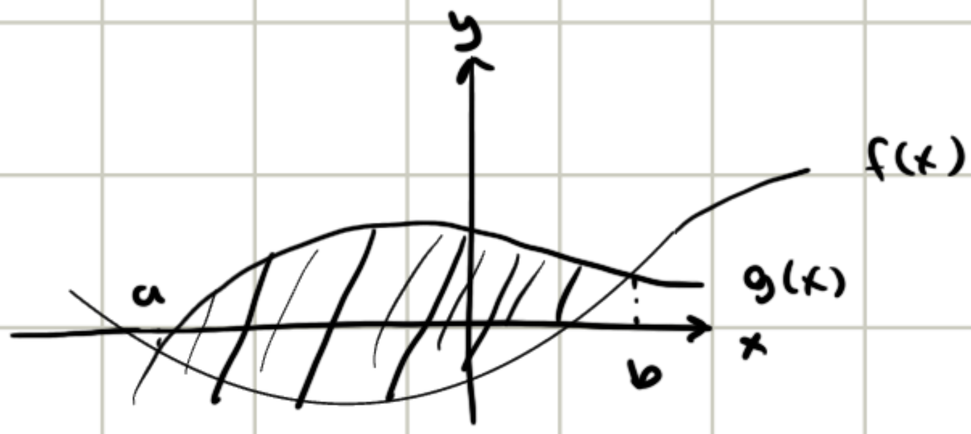


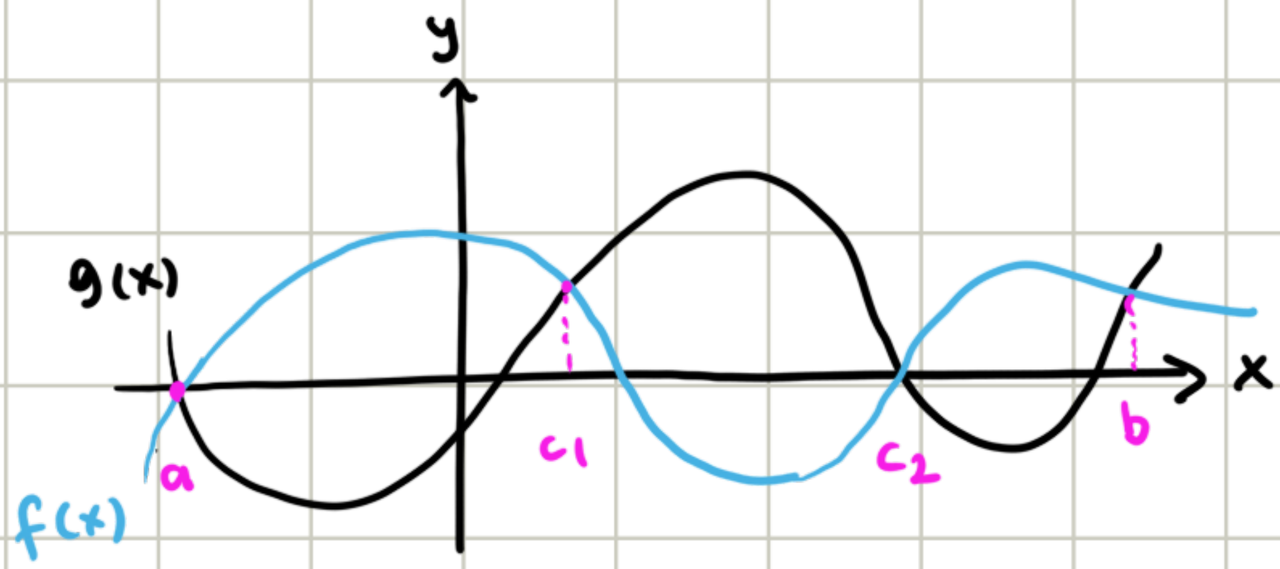
$$\pi \int_a^b [f(x)]^2 dx - \pi \int_a^b [g(x)]^2 dx$$

$$V = \left[\text{dıştaki eğrinin hacmi}\right] - \left[\text{içteki eğrinin hacmi}\right]$$

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

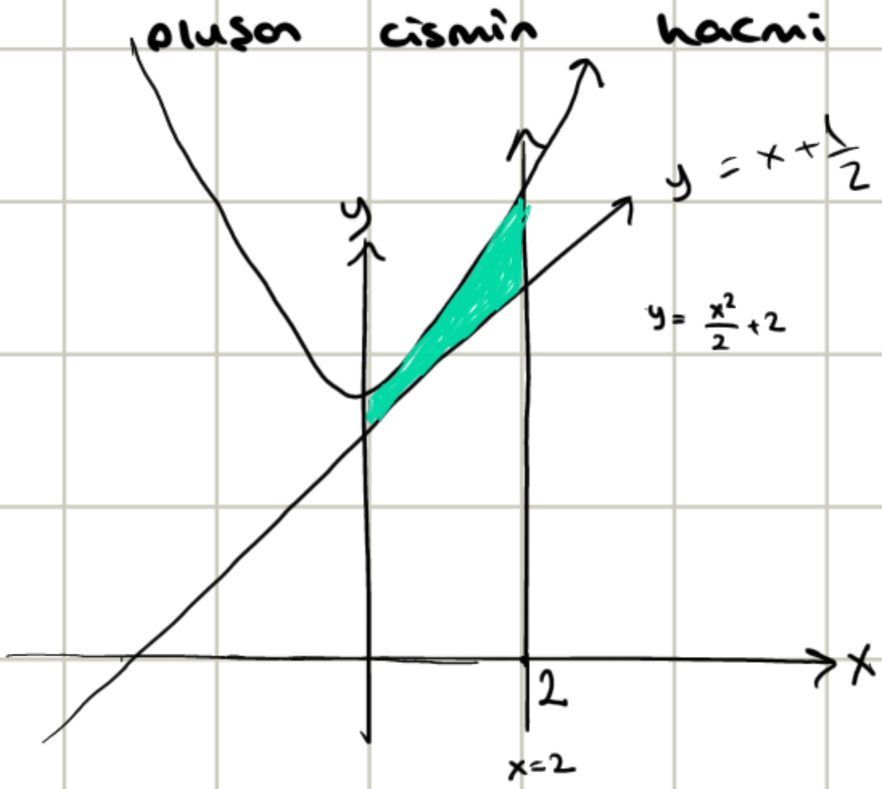
Örnekler :





$$V = \pi . \left[\int_a^{c_1} [f(x)]^2 - [g(x)]^2 + \int_{c_1}^{c_2} \quad + \int_{c_2}^{b} \right.$$

Soru : $y = \frac{x^2}{2} + 2$ eğrisi , $y = x + \frac{1}{2}$, $x = 0$ ve $x = 2$ doğruları ile sınırlanan bölgenin alanı x- eks. etrafında döndürülüyor oluşan cismin hacmi nedir ?



$$V = \pi \int_0^2 \left[\left(\frac{x^2}{2} + 2 \right)^2 - \left(x + \frac{1}{2} \right)^2 \right] dx = \frac{293}{30} \pi \, br^3$$

$$\int \left(\frac{x^2}{2} + 2 \right)^2 dx - \int \left(x + \frac{1}{2} \right)^2 dx$$

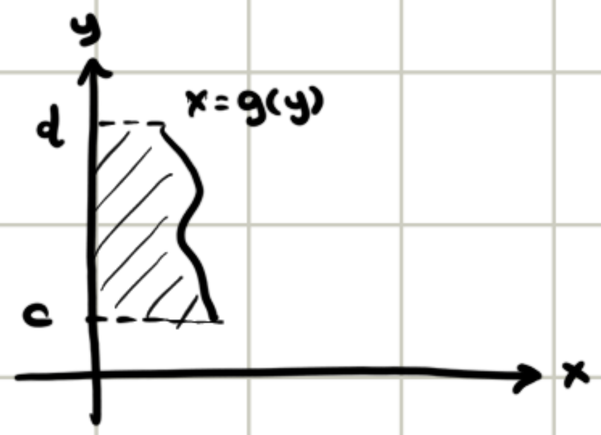
$$\int \left(\frac{x^4}{4} + 2x^2 + 4 \right) dx - \int \left(x^2 + x + \frac{1}{4} \right) dx$$

Örnek: $y = \frac{1}{4}x^3$ eğrisi ve $y=x$ doğrusu ile sınırlanan bölgenin koordinat düz. 1. bölg. kalan kısmının

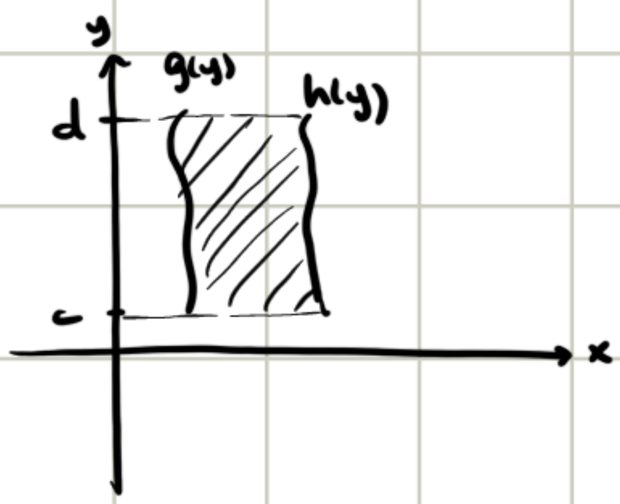
a. y ekseni etrafında

b. x ekseni etrafında

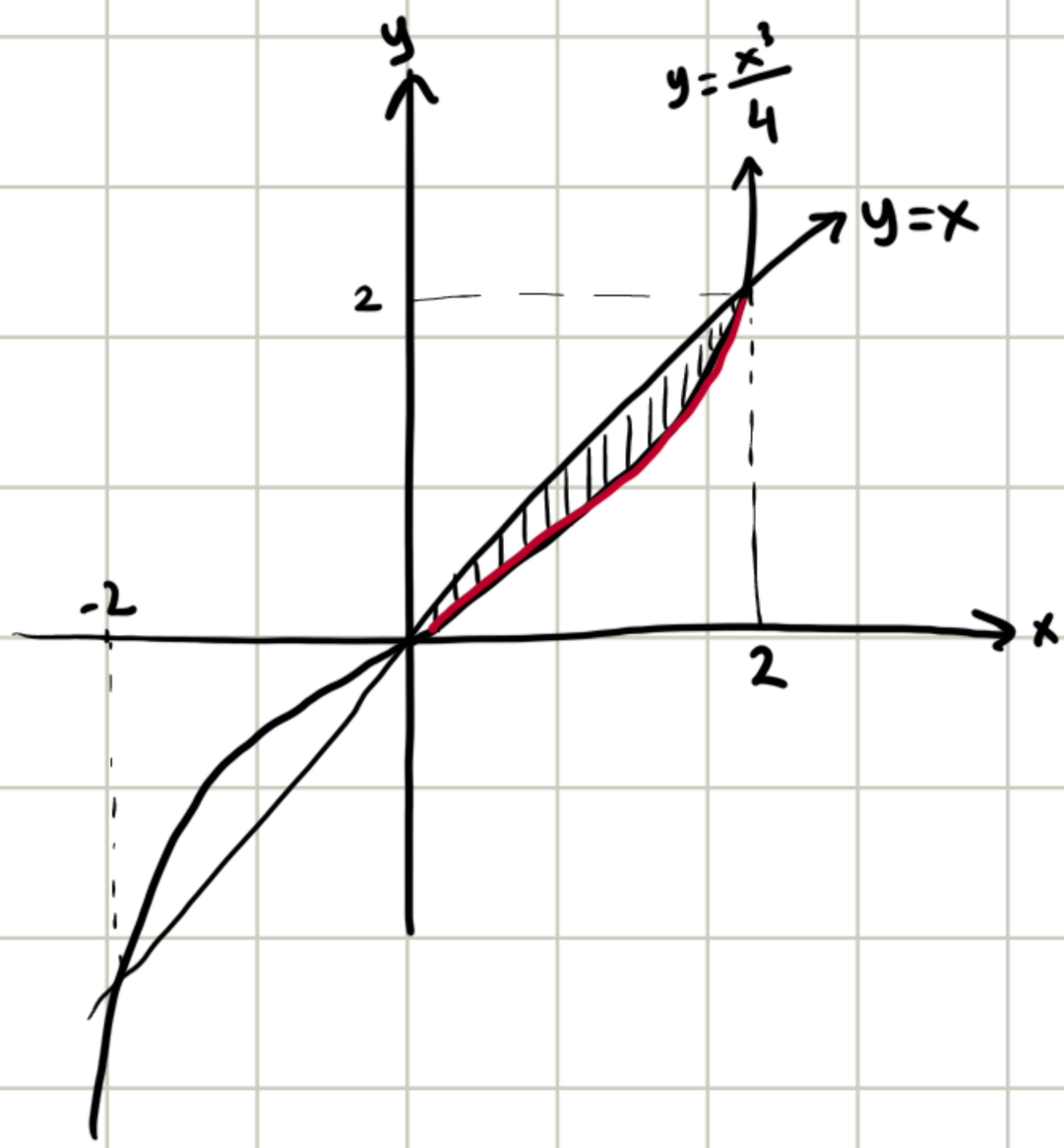
NOT:



$$V = \pi \int_c^d [g(y)]^2 \, dy$$



$$V = \pi \int_c^d \left([h(y)]^2 - [g(y)]^2 \right) dy$$



a)

$$V_{ox} = \pi \int_0^2 \left[x^2 - \left(\frac{1}{4}x^3\right)^2 \right] dx$$

b)

$$V_{oy} = \pi \int_0^2 \left[\left(\sqrt[3]{4y}\right)^2 - y^2 \right] dy$$

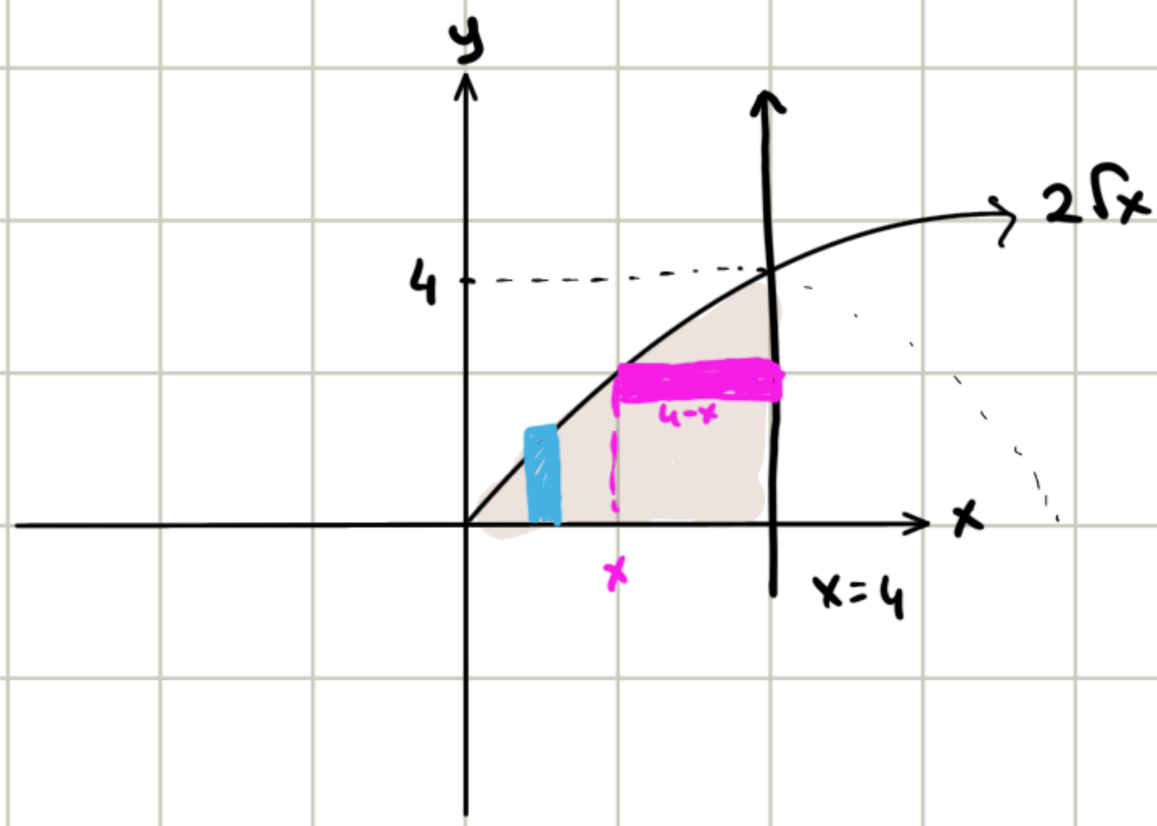
sağdan sınırlayan

Soru: Grafikte bulunan tüm bölge x ekseni etr. dönd. oluşan cismin hacmi

$$2\pi \int_0^2 \left[x^2 - \left(\frac{x^3}{4}\right)^2 \right] dx$$

$$\int_0^2 x^2 \quad - \quad \int_0^2 \frac{x^6}{8}$$

Örnek: $y = 2\sqrt{x}$ eğrisi, x eks. ve $x=4$ doğrusu ile sınırlanan bölge $x=4$ doğrusu etrafında dönmesi ile oluşan silindir hacmi?



$x=4$ etr. döndürülürse

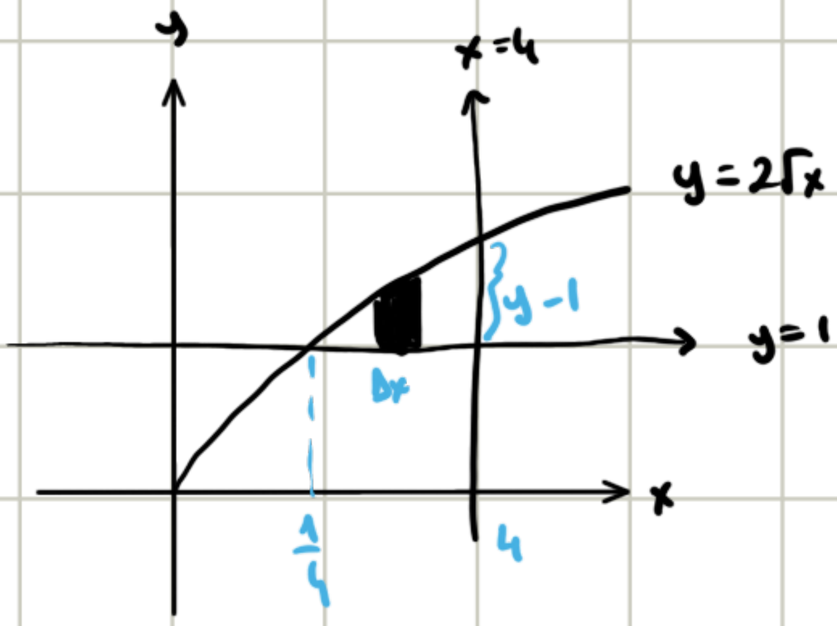
$$V = \pi \int_0^4 (4-x)^2 \, dy = \pi \int_0^4 \left(4 - \frac{y^2}{4}\right)^2 dy$$

$$= \frac{512}{15} \pi \, br^3$$

$y=4$ etr. döndürme

$$V_{ox} = \pi \int_0^4 y^2 \, dy = \pi \int_0^4$$

Örnek: $y = 2\sqrt{x}$ eğrisi $y=1$, $x=4$ $y=1$ etrf. döndürülürse.

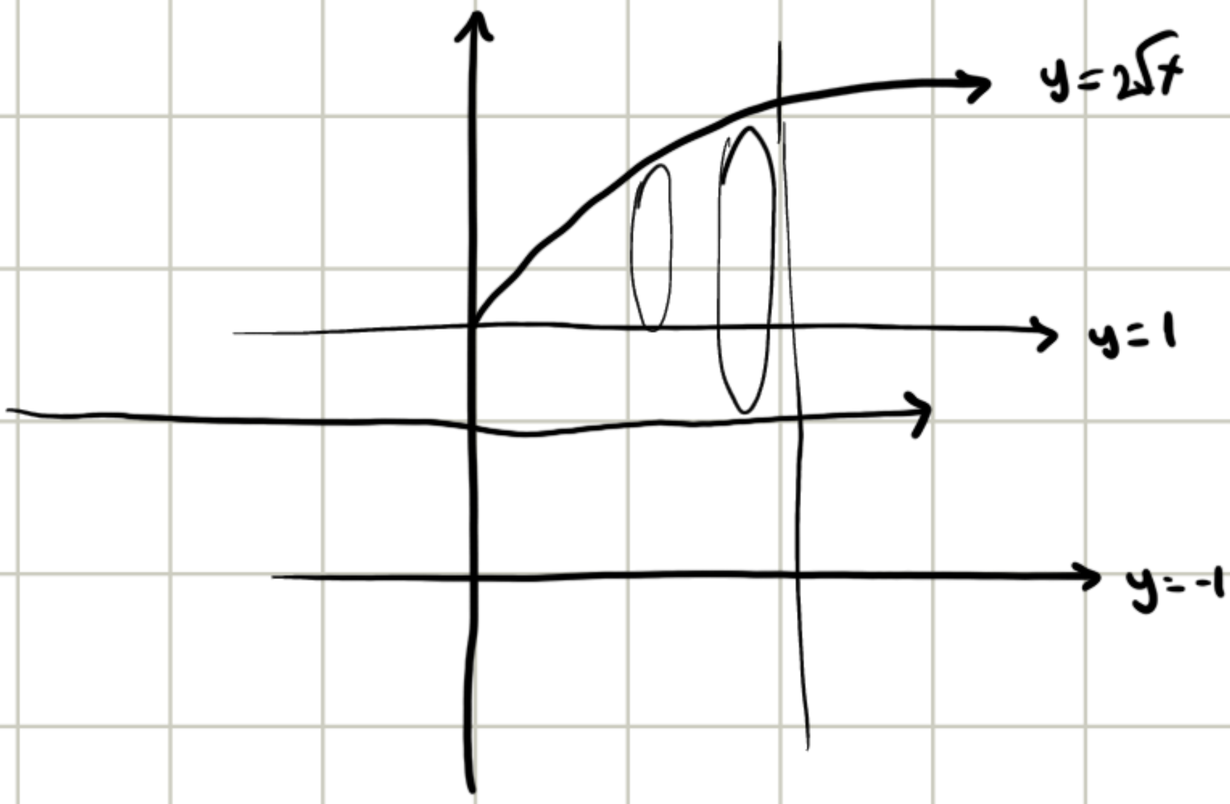


$$V = \pi \int_{1/4}^4 (y-1)^2 \, dx = \pi \int_{1/4}^4 (2\sqrt{x}-1)^2 \, dx = \frac{117}{8}\pi \; br^3$$

$$\pi \int_{1/4}^4 (2\sqrt{x}-1)^2 = \int_{1/4}^4 4x - 4\sqrt{x} + 1$$

(*) Temsili dik. oluşturduğu silindirin yan alanı önemlidir.

Örnek: $y=-1$ doğrusu etr. döndürülecek



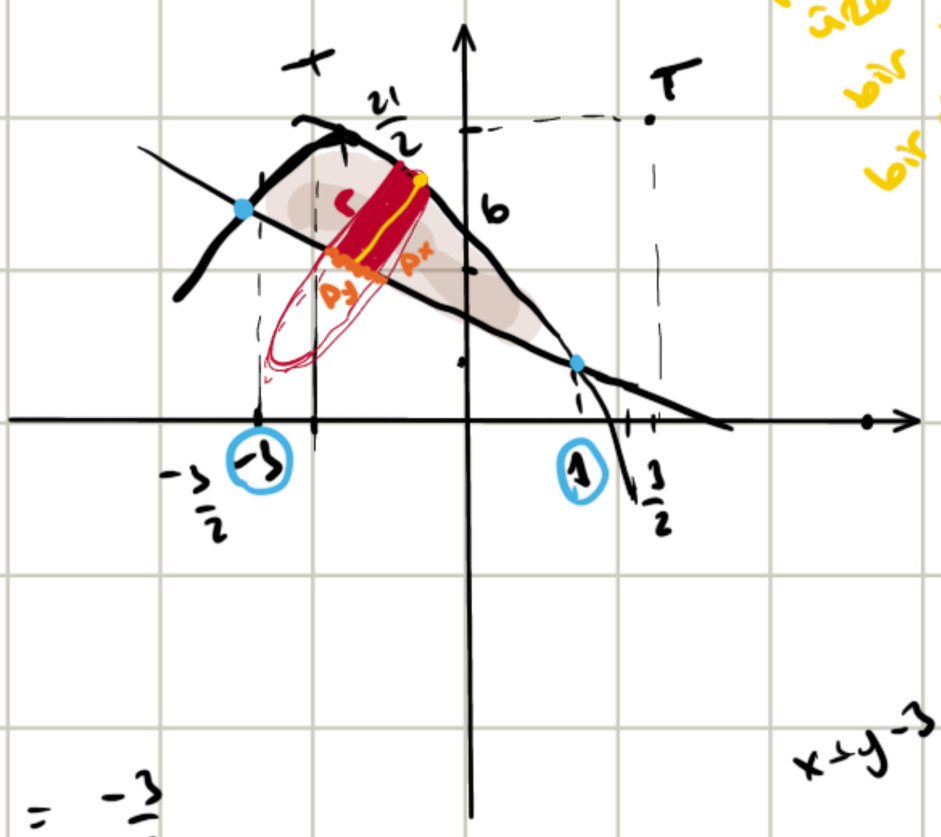
Pistakinden izleki çıkarılarak alan bulunur.

Örnek: $y=-x^2-3x+6$ parabolü ile $x+y-3=0$ doğrusu ile sınırlanan bölge $x+y-3=0$ doğrusu etr. döndürülüyor.

$V_i = \pi \cdot r^2 \cdot h$

(x_i, y_i)

yancap parabol üzerindeki bir noktanın bir doğruya uzaklığıdır.



$-x^2-3x+6$

0, 6

$\frac{-3}{2(-1)} = \frac{3}{2}$

$T\left(\frac{3}{2}, \frac{-3}{4}\right)$

$-\frac{9}{4} - \frac{9}{2} + 6$

$\frac{-9-18}{4}$

$\frac{-27}{4} + 6 = \frac{-3}{4}$

$\frac{-(-3)}{2(-1)} = \frac{-3}{2}$

$x+y-3=0$

$y=3-x$

0, 3

3, 0

$3-x = -x^2-3x+6$

$0 = -x^2-2x+3$

$x \quad 3$

$-x \quad 3$

$-\frac{3}{2}, \quad -\frac{9}{4} \quad \frac{9}{2}+6 \quad \frac{12+9}{2} = \frac{21}{2}$

r bulunması

Bir nok. bir doğruya uzaklığı

(x_i, y_i) nok. $Ax+By+C=0$

$$\frac{|Ax_i + By_i + C|}{\sqrt{A^2+B^2}} = r$$

$h = \sqrt{(\Delta y_i)^2 + (\Delta x_i)^2}$

$V_i = \pi \cdot r^2 \cdot \sqrt{2}\,(\Delta x)$

$x + y - 3 = 0 \Rightarrow$ dif alınır.

$\Delta y + \Delta x = 0 \Rightarrow \Delta y = -\Delta x$

$h = \sqrt{(\Delta x)^2 + (\Delta x)^2} = \sqrt{2} \cdot (\Delta x)$

tanev alınıp yerine Δx

eğer sel işlem yapılırsa dx yazılır.

NOT: (x_i, y_i) parabol üzerinde seçildi ikisi arasında böyle bir bağıntı var

r neye eşit?

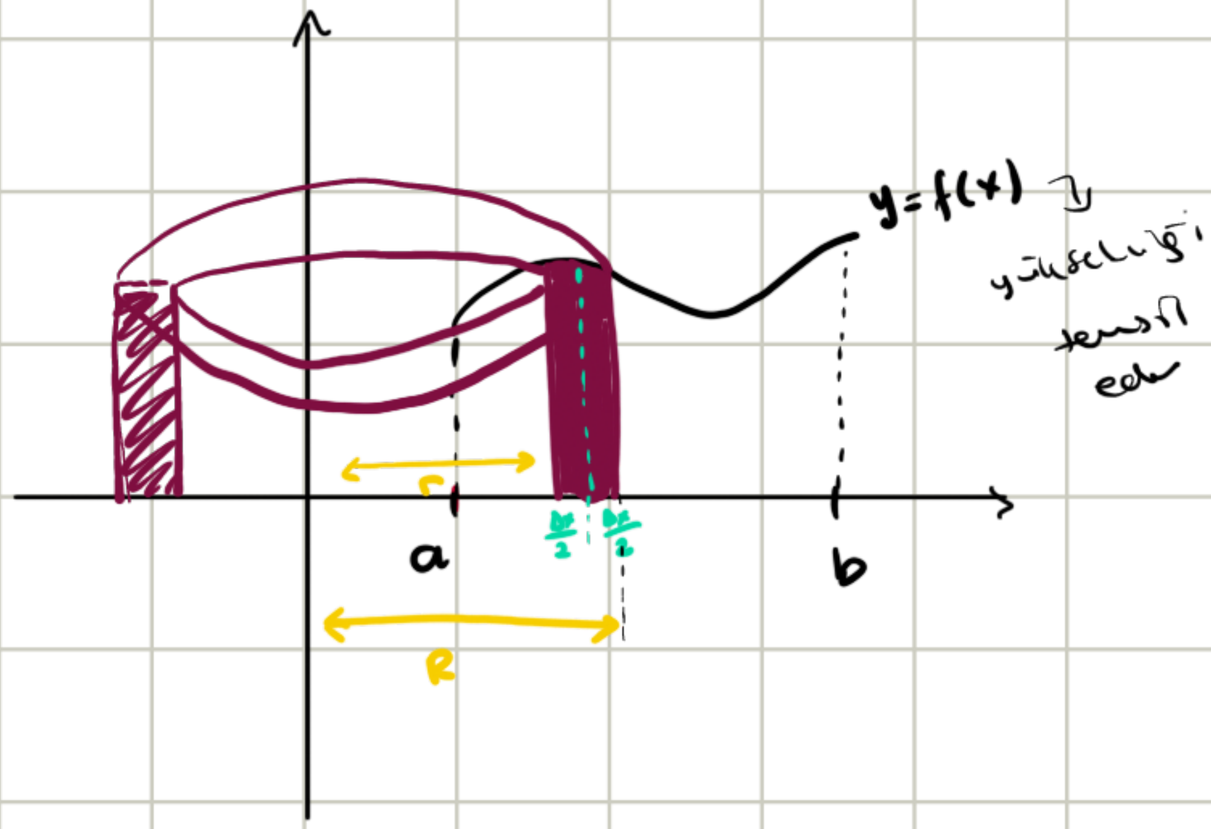
$x + y - 3 = 0$

$$r^2 = \frac{|Ax_i + By_i + C|^2}{A^2 + B^2} \Rightarrow \frac{|x_i + y_i - 3|^2}{1 + 1} = \frac{|x_i + (-x_i^2 - 3x_i + 6) - 3|^2}{2}$$

$$= \frac{(-x_i^2 - 2x_i + 3)^2}{2} = \frac{(-x^2 - 2x + 3)^2}{2} = r$$

$$\boxed{V = \pi \int_{-3}^{1} \frac{(-x^2 - 2x + 3)^2}{2} \cdot \sqrt{2}\, dx}$$

2. metot Silindir Kabuğu (tabaka) metodu ile hacim hesabı



$y = f(x)$, $x=a$, $x=b$ ve x-eks. ile sınırlı bölge y eks etr. döndürülüyor.

$$R = x + \frac{\Delta x}{2}$$

$$r = x - \frac{\Delta x}{2}$$

yüksekliği temsil eder

dış silindir

$$V = \pi . R^2 . f(x) - \pi . r^2 . f(x)$$

$$V = \pi f(x) \left[R^2 - r^2\right]$$

$$= \pi f(x) (R-r)(R+r)$$

$$= \pi f(x) \left(\cancel{x} + \frac{\Delta x}{2} - \cancel{x} + \frac{\Delta x}{2}\right) . 2x = \pi f(x) \Delta x . 2x = 2\pi . x . f(x) . \Delta x$$

Bir dik dikdörtgen in ile geçen hacim

$$V = 2\pi \int_a^b x . f(x) . dx$$

$$= 2\pi \int_a^b x . y . dx$$

Hangi metodu kullanacağınıza karar verirken

Diskte → bir silindir

Kabukta → iç içe geçmiş iki silindir göreceksiniz.