Permütasyonda sıralama veya diziliş söz konusudur. Seçilmiş nesnelerin sıralanışı veya dizilişi önemlidir.

Kombinasyonda ise, seçim veya seçme söz konusudur.

sıra önemli değilse
daha az elemanlı küme

Discrete Mathematics

Counting

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Permutations are for lists (order matters) and combinations are for groups (order doesn't matter).

Topics

sıra önemsiz

Combinatorics
   Introduction   çiğköfte yapma   5 kişilik takımdan 3 kişi seçme
   Basic Principles

Permutations
   Introduction   eldeki malzemelerle pasta yapma
   Circular Arrangements   5 kişilik takımda 3 kişi sıralama

Combinations
   Introduction
   With Repetition

Combinatorics

farklı nesne düzenleme (seçme) işidir ve bu süreçte sıranın bu düzenleme de önemi yoktur.

seçme, düzenleme

- combinatorics: study of arrangement of objects
- enumeration: counting of objects with certain properties

sayma

- to solve a complicated problem:
- break it down into smaller problems
- piece together solutions to these smaller problems

divide and conquer

Sum Rule

- $task_1$ can be performed in n_1 distinct ways
- $task_2$ can be performed in n_2 distinct ways
- $task_1$ and $task_2$ cannot be performed simultaneously

- performing either $task_1$ or $task_2$ can be accomplished in $n_1 + n_2$ ways

Sum Rule Example

- a college library has 40 textbooks on sociology, and 50 textbooks on anthropology

- to learn about sociology or anthropology a student can choose from $40 + 50 = 90$ textbooks

seçmeli ders: 10 bilg, 5 elektronikten var. 1 dersi 15 farklı yolla

elbise 3 üst, 5 alt var. 1 tane elbise seçeceksin üzerindeki takımdan farklı olabilmesi için yani ya alt ya üst seçeceksen eğer kaç farklı seçimin olur 3+5

Sum Rule Example

- a computer science instructor has two colleagues
- one colleague has 3 textbooks on "Introduction to Programming"
- the other colleague has 5 textbooks on the same subject

- n: maximum number of different books that can be borrowed
- $5 \leq n \leq 8$
- both colleagues may own copies of the same book

Product Rule

- a procedure can be broken down into $stage_1$ and $stage_2$
- n_1 possible outcomes for $stage_1$
- for each of these, n_2 possible outcomes for $stage_2$

- procedure can be carried out in $n_1 \cdot n_2$ ways

elbise 3 üst, 5 alt . kaç farklı üst-alt takımı oluşturabilirsin

3*5

Product Rule Example

- drama club is holding tryouts for a play
- 6 men and 8 women auditioning for the leading roles

- director can cast leading couple in $6 \cdot 8 = 48$ ways

  çift seçecek yani 1 kadın 1 erkek

Product Rule Example

- license plates with 2 letters, followed by 4 digits
- how many possible plates?

- no letter or digit can be repeated:
  $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3,276,000$

- repetitions allowed:
  $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$

- repetitions allowed, only vowels and even digits:
  $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$

Product Rule Example

- a byte consists of 8 bits
- a bit has two possible values: 0 or 1

- number of possible values for a byte:
  $2 \cdot 2 \cdots 2 = 2^8 = 256$

Counting Example

- pastry shop menu:
  6 kinds of muffins, 8 kinds of sandwiches
  hot coffee, hot tea, icea tea, cola, orange juice

- buy either a muffin and a hot beverage,
  or a sandwich and a cold beverage
- how many possible purchases?

- muffin and hot beverage: $6 \cdot 2 = 12$
- sandwich and cold beverage: $8 \cdot 3 = 24$
- total: $12 + 24 = 36$

Permutation

- permutation: a linear arrangement of distinct objects
- order important   sıralama, dizilenmesi

Permutation Example

- a class has 10 students: $A, B, C, \ldots, I, J$
- 4 students are to be seated in a row for a picture: $BCEF, CEFI, ABCF, \ldots$
- how many such arrangements?

- filling of a position: a stage of the counting procedure $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

Permutation Example

$$
\begin{aligned}
10 \cdot 9 \cdot 8 \cdot 7 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{10!}{6!}
\end{aligned}
$$

Permutations

n: total obje sayısı   r=10, r=4
r: sıralanacak obje

- n distinct objects
- number of permutations of size r (where $1 \leq r \leq n$):

$$
\begin{aligned}
P(n, r) &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

- if repetitions are allowed: n^r

Permutations Example

- if size equals number of objects: $r = n$

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

example

- number of permutations of the letters in "COMPUTER": 8!

Arrangements Example

- number of arrangements of the letters in "BALL"
- two L's are indistinguishable

A	B	L	L		L	A	B	L
A	L	B	L		L	A	L	B
A	L	L	B		L	B	A	L
B	A	L	L		L	B	L	A
B	L	A	L		L	L	A	B
B	L	L	A		L	L	B	A

- number of arrangements: $\frac{4!}{2} = 12$

Arrangements Example

- arrangements of all letters in "DATABASES"

  ayırt edilemeyen

- for each arrangement where A's are **indistinguishable**, $3! = 6$ arrangements where A's are **distinguishable**:

  $DA_1TA_2BA_3SES, DA_1TA_3BA_2SES, DA_2TA_1BA_3SES,$
  $DA_2TA_3BA_1SES, DA_3TA_1BA_2SES, DA_3TA_2BA_1SES$

- for each of these, 2 arrangements where S's are distinguishable:

  $DA_1TA_2BA_3S_1ES_2, DA_1TA_2BA_3S_2ES_1$

- number of arrangements: $\frac{9!}{2! \cdot 3!} = 30,240$

Generalized Rule
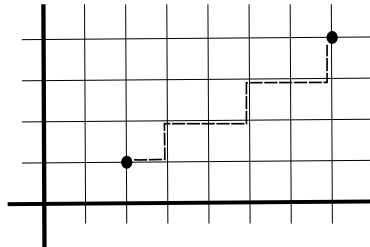
genel kural

- n objects   ayırt edilemeyen
- n_1 indistinguishable objects of $type_1$
  n_2 indistinguishable objects of $type_2$
  . . .
  n_r indistinguishable objects of $type_r$
- $n_1 + n_2 + ... + n_r = n$
- number of linear arrangements:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

Arrangements Example

- go from $(2, 1)$ to $(7, 4)$
- each step one unit to the right (R) or one unit upwards (U)
- *RURRURRU, URRRUURR*
- how many such paths?

asıl

- each path consists of 5 R's and 3 U's
- number of paths: $\frac{8!}{5! \cdot 3!} = 56$

Circular Arrangements Example

- 6 people seated around a round table: A, B, C, D, E, F
- arrangements considered to be the same when one can be obtained from the other by rotation: *ABEFCD, DABEFC, CDABEF, FCDABE, EFCDAB, BEFCDA*
- how many different circular arrangements?
- each circular arrangement corresponds to 6 linear arrangements
- number of circular arrangements: $\frac{6!}{6} = 120$

(n-1)!

Combination   Kombinasyon

sıra önemsiz seçme

- combination: choosing from distinct objects
- order not important   sıra önemsiz

elbise kombinasyonu

kot- tişört veya tişört-kot aynı kombinasyondur

Combination Example   permütasyon örneği

sinek   karo

- a deck of 52 playing cards   kupa   maça
- 4 suits: clubs, diamonds, hearts, spades
- 13 ranks in each suit: Ace, 2, 3, ..., 10, Jack, Queen, King
- draw 3 cards in succession, without replacement   vale, kız, papaz
- how many possible draws?

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = P(52, 3) = 132,600$$

P(52,3)

Combination Example

kupa, karo, sinek, maça

örnek

- one such draw: kupa as   sinek   karo papaz
  AH (ace of hearts), $9C$ (9 of clubs), KD (king of diamonds)

- if order doesn't matter

- 6 permutations of $(AH, 9C, KD)$ correspond to just one selection

C(52,3) $\dfrac{52!}{3! \cdot 49!} = 22,100$

sıralama önemli değil,

52 kartlık bir desteden 3 kart çekiliyor:
Örneğin AH (kupa ası), 9C (sinek dokuz), KD (karo papazı).

Eğer **kartların çekiliş sırası önemli değilse,**
bu 3 kartın 6 farklı dizilişi (permütasyonu) aslında **aynı seçime** karşılık gelir.

Buna göre, **52 kartlık bir desteden sırası önemli olmadan 3 kart seçmenin kaç farklı yolu vardır?**

Number of Combinations

- n distinct objects
- each combination of r objects: $r!$ permutations of size r
- number of combinations of size r (where $0 \le r \le n$):

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n - r)!}$$

Number of Combinations

- number of combinations:

$$C(n, r) = \frac{n!}{r! \cdot (n - r)!}$$

- note that:

$$C(n, 0) \ = 1 = \ C(n, n)$$
$$C(n, 1) \ = n = \ C(n, n - 1)$$

Number of Combinations Example

- Lynn and Patti buy a powerball ticket
- match five numbers selected from 1 to 49
- and then match powerball, 1 to 42
- how many possible tickets?

- Lynn selects five numbers from 1 to 49: $C(49, 5)$
- Patti selects the powerball from 1 to 42: $C(42, 1)$
- possible tickets: $\binom{49}{5}\binom{42}{1} = 80,089,128$

5 ana sayı: 1 ile 69 arasında seçilir. 1 Powerball (kırmızı top): 1 ile 26 arasında seçilir.

Number of Combinations Examples

C(53,9)= 53!/(44!*9!)

- for a volleyball team, gym teacher must select nine girls from junior and senior classes
- 28 junior and 25 senior candidates
- how many different ways?

(50*49*48*47*46*45) / 6!

C(50,6)

- if no restrictions: $\binom{53}{9} = 4,431,613,550$
- if two juniors and one senior are best spikers and must be on the team: $\binom{50}{6} = 15,890,700$
- if there has to be four juniors and five seniors: $\binom{28}{4}\binom{25}{5} = 1,087,836,750$

C(28,4)* C(25,5)

C(53,9)

Binomial Theorem

Theorem

if x and y are variables and n is a positive integer, then:

$$(x+y)^n = \binom{n}{0}x^0 y^n + \binom{n}{1}x^1 y^{n-1} + \binom{n}{2}x^2 y^{n-2} + \cdots$$

$$+ \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^n y^0$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3

- $\binom{n}{k}$: binomial coefficient

Binomial Theorem Examples

- in the expansion of $(x+y)^7$, coefficient of $x^5 y^2$: $\binom{7}{5} = \binom{7}{2} = 21$

C(7,2)

Multinomial Theorem

Theorem

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_t!}$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

Multinomial Theorem Examples

- in the expansion of $(x + y + z)^7$, coefficient of $x^2y^2z^3$:

$$\binom{7}{2, 2, 3} = \frac{7!}{2! \cdot 2! \cdot 3!} = 210$$

Combinations with Repetition Example

Tekrarlı Kombinasyon

- 7 students visit a restaurant
- each of them orders one of the following:
  cheeseburger (c), hot dog (h), taco (t), fish sandwich (f)
- how many different purchases are possible?

Kaç farklı satın alma işlemi yapılabilir?

Combinations with Repetition Example

n=4, r=7   r arrangement'ı yapılacak nesne

```
c c h h t t f | x x | x x | x x | x
c c c c h t f | x x x x | x | x | x
c c c c c c f | x x x x x x | | | x
h t t f f f f | | x | x x | x x x x
t t t t t t t | | | x x x x x x x |
f f f f f f f | | | | x x x x x x x
```

- enumerate all arrangements of 10 symbols
  consisting of seven x's and three |'s
- number of different purchases: $\frac{10!}{7! \cdot 3!} = \binom{10}{7} = 120$

C(7+4-1, 7)

ayıraç sayısı mı?

3 tanesini söyleyince son kalana gerek var mı ?

Number of Combinations with Repetition

4 yemek içinden 7 kişiye sipariş

- select, with repetition, r of n distinct objects
- considering all arrangements of r x's and $n - 1$ |'s

$$\frac{(n + r - 1)!}{r! \cdot (n - 1)!} = \binom{n + r - 1}{r}$$

7 tane yemek dağıtılıyor
r=7

7 kişiye farklı şekilde 4 yemek çeşidini dağıt
r=7
n= 4

Number of Combinations with Repetition Example

3 tane muz dağıtılıyor, r=3

- ▸ distribute 7 bananas and 6 oranges among 4 children
- ▸ each child receives at least one banana
- ▸ how many ways?

4 çocuk

- ▸ step 1: give each child a banana
- ▸ step 2: distribute 3 bananas to 4 children

```
1 1 1 0    b | b | b |
1 0 2 0    b | | b b |
0 0 1 2    | | b | b b
0 0 0 3    | | | b b b
```

- ▸ $C(6,3) = 20$ ways

n=4, r=3

3 muzu farklı şekilde
r=3, n=4
3 muz, 4 çocuğa. distinct object muz

Number of Combinations with Repetition Example

- ▸ step 3: distribute 6 oranges to 4 children

```
1 2 2 1    o | o o | o o | o
1 2 0 3    o | o o | | o o o
0 3 3 0    | o o o | o o o |
0 0 0 6    | | | o o o o o o
```

- ▸ $C(9,6) = 84$ ways

- ▸ step 4: by the rule of product: $20 \cdot 84 = 1,680$ ways

References

Required Reading: Grimaldi

- ▸ Chapter 1: Fundamental Principles of Counting
    - ▸ 1.1. The Rules of Sum and Product
    - ▸ 1.2. Permutations
    - ▸ 1.3. Combinations
    - ▸ 1.4. Combinations with Repetition