RC Circuits' Dynamics in Filtering White and Line Noise

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***Abstract*—To model current flow through an electrical circuit with constant and sinusoidal inputs, an analytical and numerical analysis of differential equations relating to the subject was applied. We then generalized the circuit model to noise filtering, allowing for the exploration of the relationship between amplitude and frequency.**

I. Part 1

A. Simulation of RC Circuit Current with Fixed Voltage Source

For finding the current traveling through the circuit, we first solved for the current analytically, getting the following solution (Equation 1). We then compared with the simulated solution to verify our solved answer.

$$I = \frac{v_{in}(1 - e^{\frac{-t}{\tau}})}{R} \tag{1}$$

When we solved with time approaching infinity, $v_{in} = 5V$, and $R = 1000\Omega$, the result was a current of 0.005 A. Upon simulating in MatLab, we generated Fig. 1, with our analytical solution matching the simulated.

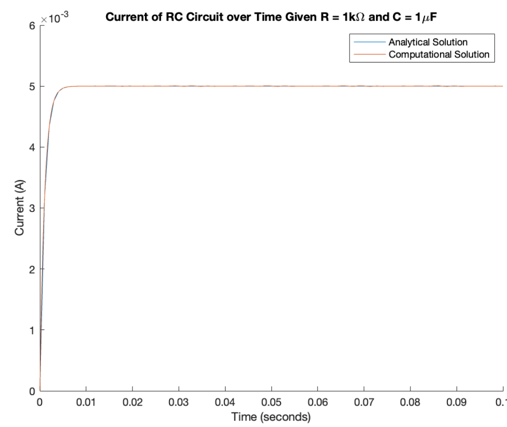

Fig. 1. Analytical and simulated solutions for the current in the circuit with a fixed $v_{in} = 5V$ and $R = 1000\Omega$

B. Simulation of RC Circuit's Voltage across Capacitor

For finding v_{in} for the alternate circuit, solving analytically resulted in the following solution.

$$v_{out} = v_{in}(1 - e^{\frac{-t}{\tau}}) \tag{2}$$

We verified this solution with the simulated solution, found with ode45, as seen in Figure 2.

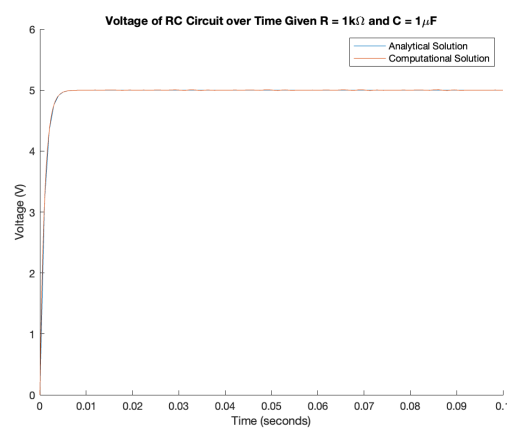

Fig. 2. Analytical and simulated solutions for the voltage across the resistor in the circuit with $v_{in}(t) = 5sin(2\pi ft)$ and $R = 1000\Omega$

C. Varying Resistance and Capacitance

After finding out how output voltage and current generally behaved in the given RC circuit, we investigated how current behaves for different values of resistance and capacitance. We used the previously found analytical solutions, changing τ as appropriate, results shown in Figure 3.

We found that when τ is close to 1 or greater, current linearly increases with time until reaching 0.005 A. The more τ is less than 1, the faster current increases until it reaches the equilibrium current. There was some variance in what the equilibrium current was as well.

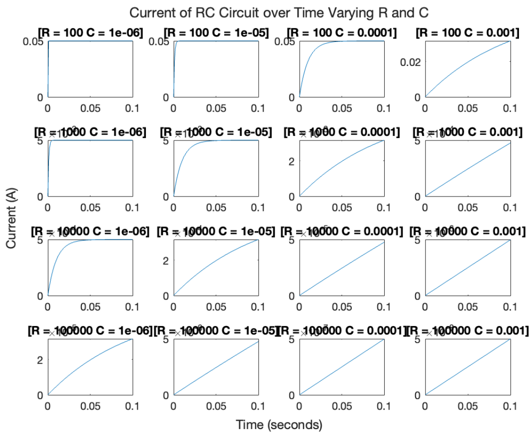

Fig. 3. Current through circuit for varying values of resistance and capacitance in the RC circuit, found with analytical solutions

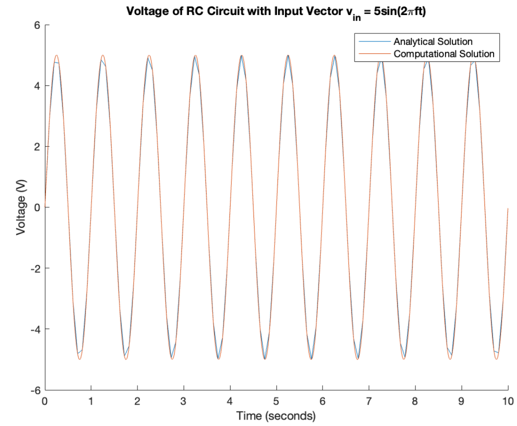

Fig. 4. Analytical and numerical solutions to the voltage across the resistor as a function of time with sinusoidal input given f = 1

II. Part II

A. Switching Capacitor and Resistor with Sinusoidal Voltage Source

Next, we consider the voltage across the resistor after switching the position of the capacitor and resistor in the RC circuit system. The defining differential equation becomes

$$C * \frac{dv_{out}}{dt} + \frac{v_{out}}{R} = \frac{v_{in}}{R} \tag{3}$$

where the voltage source is a sinusoidal function of time

$$v_{in} = 5sin(2\pi ft) \tag{4}$$

An analytical solution of Eq. 3 can then be found as a superposition of the particular solution and the homogeneous solution. The particular solution will be of the form

$$Asin(2\pi ft)) + Bcos(2\pi ft) \tag{5}$$

where f is the frequency in Hertz. Substituting Eq. 5 and its derivative into our equation, grouping coefficients, and solving the system of equations yields

$$A = \frac{5}{1 + 4C^2R^2\pi^2f^2} \tag{6}$$

$$B = \frac{-10CR\pi f}{1 + 4C^2R^2\pi^2f^2} \tag{7}$$

Combined with the homogeneous solution, our complete analytical solution to Eq. 3 becomes

$$v_{out} = e^{-\frac{1}{RC}t}(-B - \frac{B}{RC}t - 2A\pi ft) + Asin(2\pi ft) + Bcos(2\pi ft) \tag{8}$$

The analytical solution can then be verified through a comparison with the numerical solution which can be simulated in MATLAB using the ode45 function. The plots of both solutions are pictured in Figure 4.

B. Amplitude and Phase Lag as a Function of Frequency

From here it was possible to analyze the ensuing amplitude and phase lag as a function of the frequency. Note that the phase lag is relative to the voltage input function. The particular portion of the analytical solution obtained in part a can be combined to form a singular sinusoid of the form

$$\sqrt{A^2 + B^2}sin(2\pi ft - tan^{-1}(-\frac{1}{2CR\pi f}) + \frac{\pi}{2}) \tag{9}$$

where the amplitude is $\sqrt{A^2 + B^2}$ and the phase lag is $tan^{-1}(-\frac{1}{2CR\pi f}) + \frac{\pi}{2}$. Both the amplitude and the phase lag are thus expressed as functions of frequency. Analytically, we see that as frequency increases, amplitude and the relative amount of phase lag decrease, phenomena we may verify numerically (Figure 5).

III. Part III

A. A Solution with Two Sinusoidal Inputs

We then investigated the RC circuit with the capacitors and second resistor in parallel, and the input being the sum of the two sinusoids (Equation 10). In solving analytically via the method of undetermined coefficients, we found the particular solution (Equation 11) and the characteristic solution (Equation 13).

The input is as seen below, where $f_1 = 50$Hz and $f_2 = 100$kHz

$$a_1 sin(2\pi f_1 t) + a_2 sin(2\pi f_2 t) \tag{10}$$

The circuit's behavior is characterized as

$$\ddot{v} + \frac{R_2(C_1 + C_3) - R_4C_3}{R_2R_4C_1C_3}\dot{v} - \frac{1}{R_2R_4C_1C_3}v = \ddot{v_{in}} \tag{11}$$

where the coefficient of $\ddot{v}$ = a, $\dot{v}$ = b, and v = c

To solve, because $R_2 = R_4 = 330\Omega$ and $C_1 = C_3 = 0.68\mu F$, the discriminant,

$$\Delta = i^{-1}\sqrt{\frac{R_2(C_1 + C_3) - R_4C_3}{R_2R_4C_1C_3}} = \sqrt{\frac{1}{R^2C^2}(-3)} \tag{12}$$

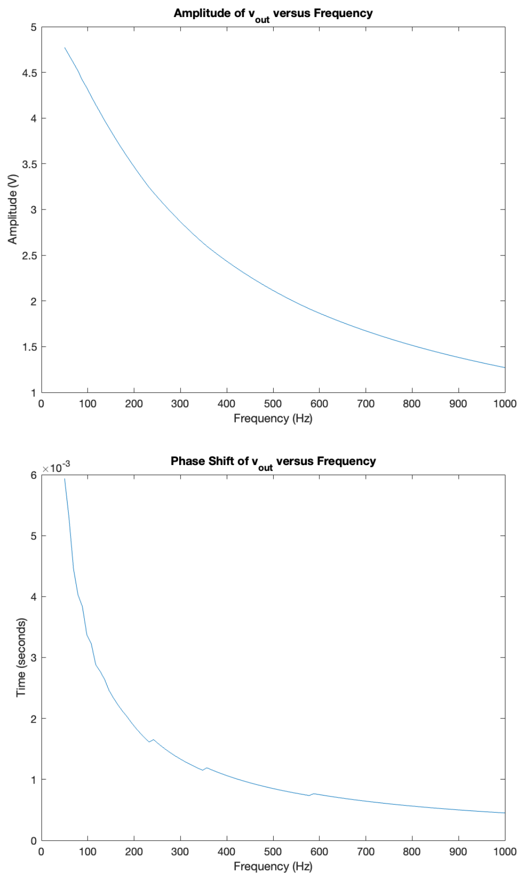

Fig. 5. Amplitude and relative phase lag of v_{out} decrease as frequency increases

with $\beta = -(\frac{R_2(C_1+C_3)-R_4C_3}{R_2R_4C_1C_3})$

$$y_c(t) = C_3e^{\beta}cos(\Delta t) + C_4e^{\beta}sin(\Delta t) \tag{13}$$

$$A_n = a_n\frac{c-f_n^2}{(c-f_n^2)^2+(bf_n)^2}B_n = a_n\frac{cf_n}{(c-f_n^2)^2+(bf_n)^2} \tag{14}$$

$$y_p(t) = A_1sin(2\pi f_1t) + B_1cos(2\pi f_1t) + A_2sin(2_2t) + B_2cos(2\pi f_2t) \tag{15}$$

So then the general solution,

$$y(t) = y_p(t) + y_c(t) \tag{16}$$

is the sum of Equation 15 and Equation 13.

B. Signal Component Attenuation

To counteract the second piece of the input, $a_2sin(2\pi f_2t)$, because A_2 and B_2 are dependent on the characteristics of the circuit, by plotting A_2 and B_2 as Bode Amplitude plots with varying values of C where $C_1 = C_3$, it is possible to determine a value for C with frequency as a constant that will reduce A_n and B_n to some arbitrarily small number, thereby reducing the amplitude of any higher frequency value sinusoid while allowing lower frequency sinusoids to retain their potency.

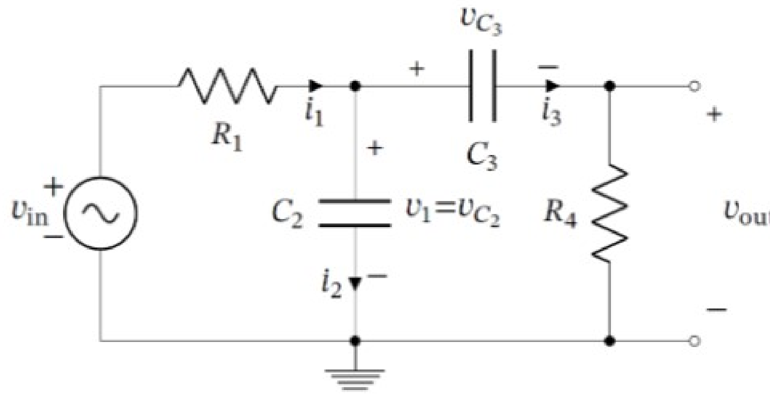

Fig. 6. Chosen circuit layout for Part 3B. It is identical to the layout for Part 3.A, so the solution for the general solution is also identical.

We found that the response resembled that of an RLC circuit with the damped oscillatory response. Given the difference in frequency between the two sinusoids, we aimed to design a circuit that would filter out anything above a frequency higher than f_1 so it would attenuate the component at f_2, like a low pass filter.

To reduce simplify the circuitry, the circuit layout in Part 3.A was used to make the capacitance the only free parameter.

IV. Part IV

A. Circuit Behavior

The chosen circuit diagram is the same as Figure 6's, where $R_1 = R_4 = 16\Omega$ and $C_2 = C_3$ The accompanying equations are then, for the circuit, assuming some sinusoidal input:

$$\ddot{v} + \frac{R_2(C_1+C_3)-R_4C_3}{R_2R_4C_1C_3}\dot{v} - \frac{1}{R_2R_4C_1C_3}v = \ddot{v_{in}} \tag{17}$$

The solution to this differential equation type is seen in Equations 11-15.

B. Noise Filtering

Using the MATLAB function lsim, the differential equations for the circuit were used alongside the input, the sound file, to filter out noise in the background of the audio.

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ c & -b\end{bmatrix} * \begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ -Fs^2\end{bmatrix} * f(t) \tag{18}$$

$$v(t) = \begin{bmatrix}1 & 0\end{bmatrix} * \begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{19}$$

It should be stated $v = x_1$, $\dot{v} = x_2$, f(t) represents the input audio and Fs represents the sampling frequency, which was a result of the input being equal to its double derivative in the circuit differential equation. Using the Bode amplitude plots from Equation 14, a general idea for the capacitance of $C_1 = C_3$ were found, and then refined experimentally.