4.8 Convex sets and functions

The purpose of this and the next section is to connect linear programming with what we know about constrained optimization (which because of its greater generality is sometimes called "nonlinear programming") and the Kuhn–Tucker conditions. Because the Kuhn–Tucker conditions concern *local* optima, we show in this section that for so-called *convex* optimization problems, which include linear programs, a local minimum is a global minimum.

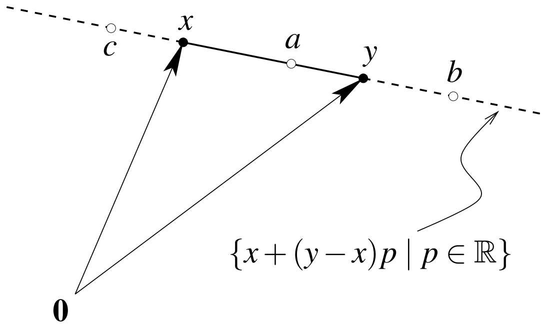

Figure 4.7 The line through the points x and y is given by the points $x+(y-x)p$ where $p \in \mathbb{R}$. Examples are point a for $p = 0.6$, point b for $p = 1.5$, and point c when $p = -0.4$. The line segment that connects x and y (drawn as a solid line) results when p is restricted to $0 \le p \le 1$.

Let x and y be two vectors in $\mathbb{R}^m$. Figure 4.7 shows two points x and y in the plane, but the picture may also be regarded as a suitable view of the situation in a higher-dimensional space. The line that goes through the points x and y is obtained by adding to the point x, regarded as a vector, any scalar multiple of the difference $y-x$. The resulting vector $x+(y-x)p$, for $p \in \mathbb{R}$, gives x when $p = 0$, and y when $p = 1$. Figure 4.7 gives some examples a, b, c of other points. When $0 \le p \le 1$, as for point a, the resulting points define the *line segment* that joins x and y. If $p > 1$, then one obtains points on the line through x and y on the other side of y relative to x, like the point b in Figure 4.7. For $p < 0$, the corresponding point, like c in Figure 4.7, is on that line but on the other side of x relative to y.

The expression $x+(y-x)p$ can be re-written as $x(1-p)+yp$, where the given points x and y appear only once. This special linear combination of x and y with nonnegative coefficients that sum to one is called a *convex combination* of x and y. It is useful to remember the expression $x(1-p)+yp$ in this order with $1-p$ as the coefficient of the first vector and p of the second vector, because then the line segment that joins x to y corresponds to the real interval $[0,1]$ for the possible values of p, with the endpoints 0 and 1 of the interval corresponding to the respective endpoints x and y of the line segment.

In general, a convex combination of points $z_1, z_2, \ldots, z_k$ in some space is given as any linear combination $z_1 p_1 + z_2 p_2 + \cdots + z_k p_k$ where the linear coefficients $p_1, \ldots, p_k$ are non-negative and sum to one. The previously discussed case corresponds to $z_1 = x$, $z_2 = y$, $p_1 = 1 - p$, and $p_2 = p \in [0,1]$.

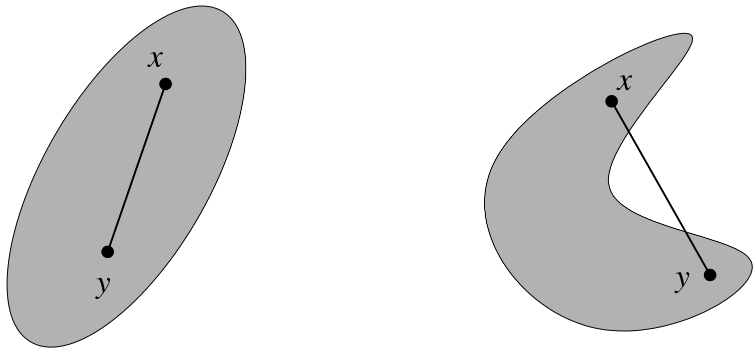

Figure 4.8 Examples of sets that are convex (left) and not convex (right).

A set of points is called *convex* if it contains with any points $z_1, z_2, \ldots, z_k$ also every convex combination of these points. Equivalently, one can show that a set is convex if it contains with any two points also the line segment joining these two points (see Figure 4.8). One can then obtain combinations of k points for $k > 2$ by iterating convex combinations of only two points.

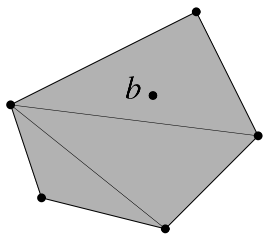

Figure 4.9 Illustration of Theorem 4.15 for $m = 2$. Any point in the pentagon belongs to one of the three shown triangles (which are not unique, there are others ways to "triangulate" the pentagon). A triangle is the set of convex combinations of its corners.

If a point is the convex combination of some points in the plane, then it is easy to see that it is already the convex combination of suitably chosen three of those points (see Figure 4.9). This is the case $m = 2$ of the following theorem, which we mention as an easy consequence of Lemma 4.4.

Theorem 4.15 (Carathéodory) *If a point b is the convex combination of some vectors in $\mathbb{R}^m$, then it is the convex combination of suitable $m+1$ of these vectors.*

Proof. If b is the convex combination of n vectors $A_1, \ldots, A_n$ in $\mathbb{R}^m$, then $b = A_1x_1 + \cdots + A_nx_n$ with $x_1, \ldots, x_n \geq 0$, $x_1 + \cdots + x_n = 1$. This is equivalent to saying that there are