

MAT1856/APM466 Assignment 1

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Fundamental Questions - 25 points

1.

- a) Governments issue bonds instead of printing money to prevent the risk of inflation, because increasing money supply can reduce the value of money. Issuing bonds allows governments to borrow money to finance government spending without immediately increasing money supply.
- b) A flattening of the long-term part of a yield curve is usually a sign of economic uncertainty. Investors may purchase more long-term bonds and drive down long-term yields because they anticipate slower economic growth or lower inflation in the future.
- c) Quantitative easing is a monetary policy in which a central bank purchases big amount of financial assets in order to push down long-term interest rates and boost economic activity. During the COVID-19 pandemic, the U.S. Federal Reserve implemented QE by acquiring Treasury bonds and mortgage-backed securities to calm financial markets and boost economic recovery.

2. These are 10 bonds that I chose:

Bond	Coupon	Maturity	ISIN	Issue Date
CAN 0.25 Mar 26	0.250%	03/01/2026	CA135087L518	10/09/2020
CAN 1.00 Sep 26	1.000%	09/01/2026	CA135087L930	04/16/2021
CAN 1.25 Mar 27	1.250%	03/01/2027	CA135087M847	10/15/2021
CAN 2.75 Sep 27	2.750%	09/01/2027	CA135087N837	05/13/2022
CAN 3.50 Mar 28	3.500%	03/01/2028	CA135087P576	10/21/2022
CAN 4.00 Mar 29	4.000%	03/01/2029	CA135087Q988	10/13/2023
CAN 3.50 Sep 29	3.500%	09/01/2029	CA135087R895	04/08/2024
CAN 2.75 Mar 30	2.750%	03/01/2030	CA135087S471	10/03/2024
CAN 2.75 Sep 30	2.750%	09/01/2030	CA135087T388	04/10/2025
CAN 2.75 Mar 31	2.750%	03/01/2031	CA135087T792	10/02/2025

They are all Government of Canada bonds with semi-annual coupon payments, which makes it easier to compare them and construct the yield, spot, and forward curves. Semi-annual coupons provide more cashflow points over time, which helps smooth the curve over the 0–5 year maturity range. They are also frequently traded, so prices are more reliable and consistent for plotting.

- 3. The eigenvectors of the covariance matrix describe the main directions the stochastic curve moves (e.g., shifting up/down, becoming steeper/flatter, or changing curvature). The eigenvalues tell us how important each pattern is, i.e., how much of the total variation is explained by each eigenvector. Therefore, the largest eigenvalue corresponds to the dominant movement pattern in the curve.

Empirical Questions - 75 points

4.

- a) On each day, we observe 10 yield-to-maturity (YTM) values, each corresponding to a different bond maturity. Because these maturities are unevenly spaced, we cannot directly draw a smooth yield curve over the 0–5 year range without estimating yields at intermediate terms. To do this, we apply piecewise linear interpolation in maturity. Let $\{(\tau_j, y_j)\}_{j=1}^{10}$ denote the observed maturities (in years, ACT/365) and their continuously-compounded YTMs. For any maturity τ between two adjacent observed maturities τ_k and τ_{k+1} , the yield is approximated by

$$y(\tau) = y(\tau_k) + \frac{y(\tau_{k+1}) - y(\tau_k)}{\tau_{k+1} - \tau_k}(\tau - \tau_k).$$

This approach is straightforward and stable, and it is reasonable given that government bond yields typically vary smoothly with maturity over short intervals.

- b) **Pseudo-code (bootstrap spot curve, ACT/365, semiannual coupons, continuous comp).** For each day d :

- i. For each bond j , convert clean to dirty: $AI_j \leftarrow (n_j/365)(Fc_j)$ and $P_j^{dirty} \leftarrow P_j^{clean} + AI_j$.
 - ii. For each bond j , generate coupon dates every 6 months to maturity; compute cashflows $CF_{j,i}$ (coupon, plus face at maturity) and times $t_{j,i} \leftarrow \text{days}(d, \text{pay}_{j,i})/365$.
 - iii. Sort bonds by increasing maturity time.
 - iv. Initialize known spot set $\mathcal{S} \leftarrow \emptyset$.
 - v. For each bond j in maturity order:
 - A. Let cashflows be $(t_1, CF_1), \dots, (t_n, CF_n)$ with maturity t_n .
 - B. Compute $PV_{known} \leftarrow \sum_{i=1}^{n-1} CF_i \exp(-r(t_i) t_i)$ using $r(t_i)$ from \mathcal{S} (linearly interpolate if needed).
 - C. Solve $D(t_n) \leftarrow (P_j^{dirty} - PV_{known})/CF_n$.
 - D. Set $r(t_n) \leftarrow -\ln(D(t_n))/t_n$ and add $(t_n, r(t_n))$ to \mathcal{S} .
 - vi. Interpolate \mathcal{S} to obtain the spot curve over $t \in [1, 5]$ (or spot points at $t = 1, 2, 3, 4, 5$).
 - vii. Plot the 1–5y spot curve for day d and superimpose curves for all days.
- c) **Pseudo-code (1-year forward curve from spot curve, continuous comp).** For each day d :
- i. From the spot curve in (b), obtain $r(1), r(2), r(3), r(4), r(5)$ (interpolate if needed).
 - ii. Compute 1-year forward rates using $f_{T,T+1} \leftarrow r(T+1)(T+1) - r(T)T$ for $T = 1, 2, 3, 4$: $f_{1,2}, f_{2,3}, f_{3,4}, f_{4,5}$.
 - iii. Store forward-curve points as $(2, f_{1,2}), (3, f_{2,3}), (4, f_{3,4}), (5, f_{4,5})$ (x-axis = end maturity).
 - iv. Plot the forward curve for day d and superimpose curves for all days.
5. The covariance matrix for the daily log-returns of yield: The covariance matrix for forward rates:

	X1	X2	X3	X4	X5
X1	5.462531e-05	4.231525e-06	1.534532e-06	1.335811e-05	1.079575e-05
X2	4.231525e-06	2.767099e-05	2.000900e-05	-1.485856e-06	-1.245030e-05
X3	1.534532e-06	2.000900e-05	1.576227e-05	3.572018e-06	-3.022616e-06
X4	1.335811e-05	-1.485856e-06	3.572018e-06	3.652394e-05	4.476728e-05
X5	1.079575e-05	-1.245030e-05	-3.022616e-06	4.476728e-05	5.962618e-05

Figure 1: log-returns of yield

	X1	X2	X3	X4
X1	1.252076e-04	1.475132e-05	-1.518390e-04	-9.725355e-05
X2	1.475132e-05	1.167729e-05	2.535662e-05	1.692211e-05
X3	-1.518390e-04	2.535662e-05	5.543492e-04	3.361380e-04
X4	-9.725355e-05	1.692211e-05	3.361380e-04	2.487606e-04

Figure 2: log-returns of forward rate

6. The eigenvalues and eigenvectors of both covariance matrices

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Eigenvalue      Value
lambda_1  1.01494e-04
lambda_2  5.41164e-05
lambda_3  3.81469e-05
lambda_4  3.28577e-07
lambda_5  1.22484e-07

Eigenvectors (each column v_k corresponds to lambda_k above):
Component      v1      v2      v3      v4      v5
1Y yield -0.322704  0.794782  0.508666  0.067904  0.028843
2Y yield  0.124665  0.468156 -0.608712 -0.492792  0.389763
3Y yield  0.025335  0.294674 -0.532152  0.757632 -0.235231
4Y yield -0.577196 -0.010873 -0.261197 -0.370751 -0.679004
5Y yield -0.739272 -0.249401 -0.138993  0.202691  0.575215

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Figure 3: the eigenvalues and eigenvectors of Yield covariance

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Eigenvalues (largest -> smallest):
Eigenvalue      Value
lambda_1  8.18563e-04
lambda_2  8.47789e-05
lambda_3  3.23184e-05
lambda_4  4.33434e-06

Eigenvectors (each column v_k corresponds to lambda_k above):
Component      v1      v2      v3      v4
1y-1y fwd -0.250923  0.930175 -0.013187  0.267653
1y-2y fwd  0.031960  0.284447  0.009470 -0.958112
1y-3y fwd  0.813499  0.189300 -0.544342  0.077956
1y-4y fwd  0.523676  0.134275  0.838706  0.065621

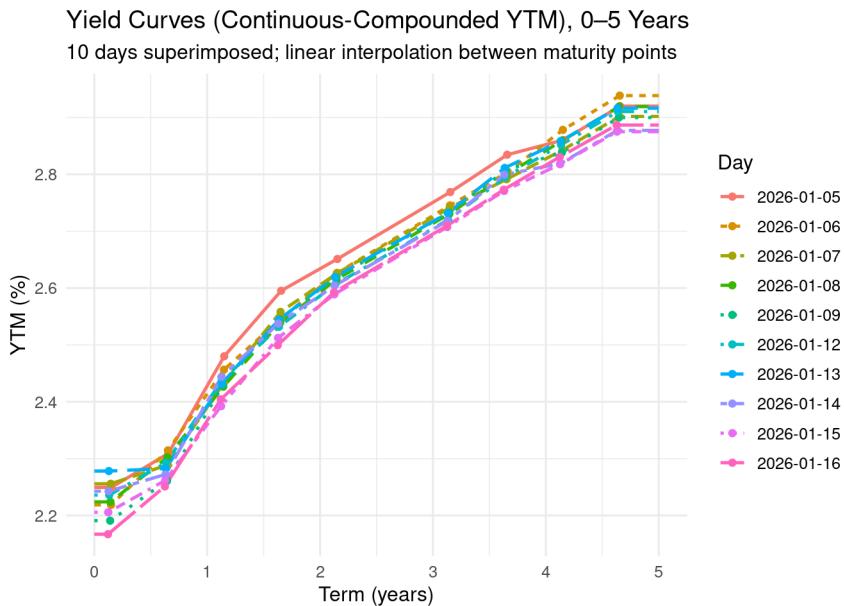
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Figure 4: the eigenvalues and eigenvectors of Forward covariance

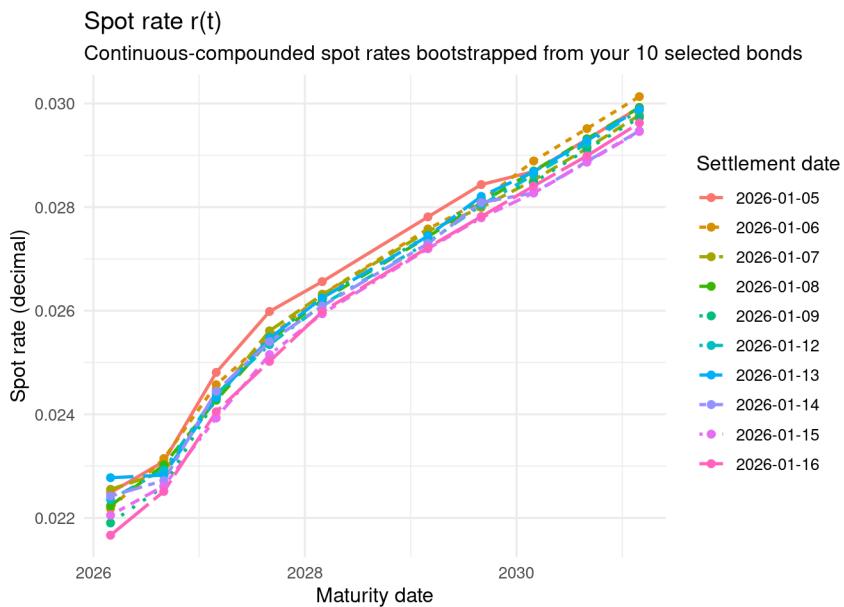
In both the yield and forward covariance matrices, the largest eigenvalue captures a dominant common movement in rates, while its associated eigenvector shows how that movement is distributed across maturities (1y–5y yields) and forward tenors (1y–1y to 1y–4y).

References and GitHub Link to Code

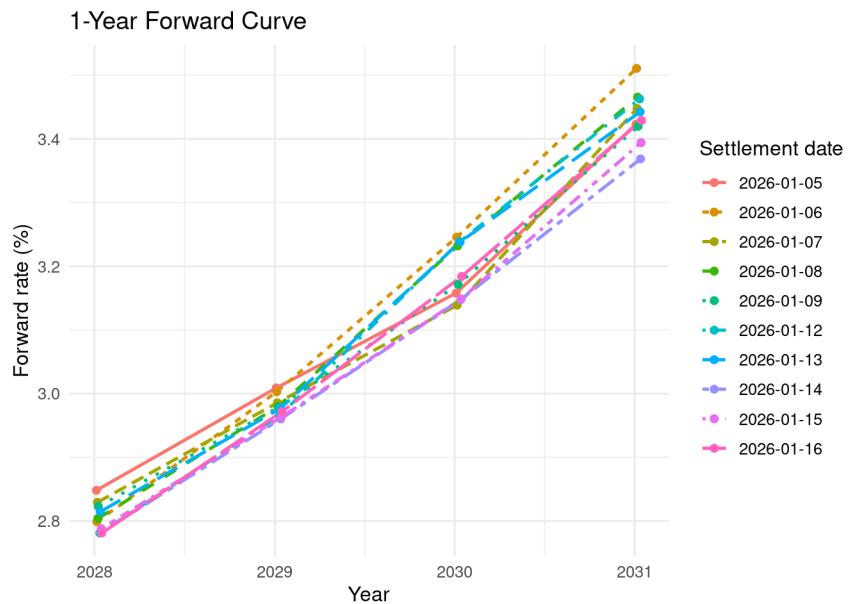
1. Plot for 4(a)



2. Plot for 4(b)



3. Plot for 4(c)



4. <https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=shortterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=¤cy=184&rating=&country=19>
5. <https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=midterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=¤cy=184&rating=&country=19>
6. Wessel, D., & Milstein, E. (2024, January 2). What did the Fed do in response to the COVID-19 crisis? Brookings. <https://www.brookings.edu/articles/fed-response-to-covid19/>
7. Chen, J. (2025, October 22). Understanding flat yield curves: Impacts on investors and strategies. Investopedia. <https://www.investopedia.com/terms/f/flatyieldcurve.asp>
8. <https://github.com/hklanh2111/APM466>