Combinatorics I

Cunsheng Ding

HKUST, Hong Kong

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Contents

The Addition Rule (1)

Proposition 1

(The Addition Rule) Given n pairwise disjoint sets, $A_1, A_2, \ldots A_n$, we have then

$$|\cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

Proof by induction on n.

When $n = 1$, it is obviously true. When $n = 2$, the conclusion follows from Proposition 29 in the lecture on sets. Suppose it is true for any $k \geq 1$. Note that $\cup_{i=1}^{k} A_i$ and A_{k+1} are disjoint. We have

$$\begin{aligned}
\left|\cup_{i=1}^{k+1} A_i\right| &= \left|\left(\cup_{i=1}^{k} A_i\right) \cup A_{k+1}\right| \text{ (by associativity of } \cup) \\
&= \left|\cup_{i=1}^{k} A_i\right| + |A_{k+1}| \text{ (by the proved conclusion for } n = 2) \\
&= \left(\sum_{i=1}^{k} |A_i|\right) + |A_{k+1}| \text{ (by the induction basis)} = \sum_{i=1}^{n} |A_i|.
\end{aligned}$$

□

The Addition Rule (2)

Example 2

A student must take another course, in order to complete her degree. She can take one of the computer science courses in the set

$$A_1 = \{\text{CS2601, CS2605, CS2606}\},$$

or one of the mathematics courses in

$$A_2 = \{\text{MA2101, MA2333, MA2888, MA2909}\}.$$

So there are $3 + 4 = 7$ ways in which this student can register for the last course because A_1 and A_2 are disjoint.

The Addition Rule (3)

Definition 3

1. The outcome of any process or experiment is called an event. For example, your score of Test 1 is 17 is an event.
2. Events are mutually exclusive if no two of them can occur together. For example, your score of Test 1 is 17 and your score of Test 1 is 18 are mutually exclusive.

In terms of events, the Addition Rule can also be stated in the following form.

Proposition 4

The number of ways in which precisely one of a collection of mutually exclusive events can occur is the sum of the number of ways in which each event can occur.

The Addition Rule (4)

Example 5

In how many ways can one get a total of six when rolling two dice?

Solution 6

The event "get a six" is the union of the following mutually exclusive ***subevents****:*

1. E_1 : *"two 3s".*
2. E_2 : *"a 2 and a 4".*
3. E_3 : *"a 1 and a 5".*

Event E_1 *can occur in one way,* E_2 *can occur in two ways, and* E_3 *can occur in two ways. So the number of ways to get a six is* $1 + 2 + 2 = 5$*.*

The Multiplication Rule (1)

Proposition 7

(The Multiplication Rule) Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|$$

Proof: We first prove that $|A_1 \times A_2| = |A_1| \cdot |A_2|$. Let

$$A_1 = \{a_{11}, a_{12}, \cdots, a_{1m}\}, \;\; A_2 = \{a_{21}, a_{22}, \cdots, a_{2t}\}.$$

Then $A_1 \times A_2 = \cup_{i=1}^{m} E_i$, where

$$E_i = \{(a_{1i}, a_{21}), (a_{1i}, a_{22}), \ldots, (a_{1i}, a_{2t})\}.$$

Then $E_1, E_2, \ldots, E_m$ are pairwise disjoint. Hence by addition rule,

$$|A_1 \times A_2| = \sum_{i=1}^{m} |E_i| = \sum_{i=1}^{m} t = m \cdot t = |A_1| \cdot |A_2|.$$

We now prove the conclusion by induction on n (continued on the next page).

The Multiplication Rule (2)

Proof of Proposition 6 (continued): By induction on n.
Basis case: When $n = 1$, the conclusion is obvious.
Inductive case: Assume that the conclusion is true for $n = k$. We prove that it is also true for n. Note that

$$A_1 \times A_2 \times \cdots \times A_k \times A_{k+1} = (A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}.$$

Then by the induction hypothesis and the above conclusion for $n = 2$, we obtain

$$\begin{aligned} |A_1 \times A_2 \times \cdots \times A_k \times A_{k+1}| &= |(A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}| \\ &= |(A_1 \times A_2 \times \cdots \times A_k)| \times |A_{k+1}| \\ &= |A_1 \times A_2 \times \cdots \times A_k| \times |A_{k+1}| \\ &= (\prod_{i=1}^{k} |A_i|)|A_{k+1}| = \prod_{i=1}^{k+1} |A_i|. \end{aligned}$$

Thus the general conclusion is also true for n.

The Multiplication Rule (3)

Thinking of A_i as the set of ways a certain event can occur, we obtain a variant of the multiplication rule.

Proposition 8

The number of ways in which a sequence of events can occur is the product of the number of ways in which each individual event can occur.

The Multiplication Rule (4)

Example 9

How many numbers in the range 1000 - 9999 do not have any repeated digits?

Solution 10

Each number in the range has four digits: The first digit has 9 choices (any of 1 - 9). Once the first is selected, the second has 9 choices (any of 0 - 9 other than the first). Similarly, the third has 8 choices and the last has 7 choices. Since the choice of digits for the four positions are sequential, by multiplication rule, there $9 \times 9 \times 8 \times 7 = 4536$ *possible numbers.*

The Pigeon-hole Principle (1)

Proposition 11

(The pigeon-hole principle) If n pigeons fly into m pigeon-holes and $n > m$, then at least one hole must contain two or more pigeons.

Proof.

By contradiction. Suppose that no hole contains more than one pigeon. Then each hole has no pigeon or one pigeon. Thus the total number of pigeons in all the m holes is at most $m < n$. This is a contradiction, since all n pigeons fly into the holes. $\square$

The Pigeon-hole Principle (2)

The pigeon-hole principle (PHP) can also be stated in the following forms:

1. If n objects are put into m boxes and $n > m$, then at least one box contains two or more of the objects.
2. A function from one finite set to a smaller finite set cannot be one-to-one.

The Pigeon-hole Principle (3)

Example 12

Show that among $(n+1)$ arbitrary chosen integers, there must exist two whose difference is divisible by n.

Solution 13

Define the following n sets.

$$[i] = \{i + nx \mid x \in \mathbb{Z}\}, \ i = 0, 1, \ldots, (n-1)$$

where $\mathbb{Z}$ is the set of all integers. Clearly, $[0], [1], \ldots, [n-1]$ are pairwise disjoint and

$$[0] \cup [1] \cup \cdots \cup [n-1] = \mathbb{Z}$$

So by the pigeon-hole principle, at least two integers x and y among the $(n+1)$ chosen integers must be in the same set $[k]$. Hence n divides $x-y$.

The Pigeon-hole Principle (3)

Definition 14

For any real number x, the ceiling function $\lceil x \rceil$ means the least integer which is greater than or equal to x. For example, $\lceil 3.5 \rceil = 4$, $\lceil -2.9 \rceil = -2$.

Proposition 15

(Strong Form of PHP) If n objects are put into m boxes and $n > m$, then some box must contain at least $\lceil \frac{n}{m} \rceil$ objects.

Proof.

Clearly, we have

$$\left\lceil \frac{n}{m} \right\rceil < \frac{n}{m} + 1.$$

If a box contains fewer than $\lceil \frac{n}{m} \rceil$ objects, then it contains at most $\lceil \frac{n}{m} \rceil - 1$ and so fewer than $\frac{n}{m}$ objects. If all m boxes are like this, we account for fewer than $m \times \frac{n}{m} = n$ objects. This is a contradiction. □

The Pigeon-hole Principle (4)

Example 16 (Strong PHP)

In a group of 100 people, several will have their birthdays in the same month. At least how many of them must have birthdays in the same month?

Solution

Note that there are 12 months. By the strong form of PHP, at least $\left\lceil \frac{100}{12} \right\rceil = 9$ people have their birthdays in the same month.

Permutations (1)

Definition 17

A permutation of a set of distinct symbols is an arrangement of them in a line in some order.

Example 18

The set of elements a, b, and c has six permutations:

$$abc, \quad acb, \quad cba, \quad bac, \quad bca, \quad cab$$

Permutations (2)

Proposition 19

For any integer $n \geq 1$, the number of permutations of a set of n distinct elements is $n!$, where $n! = 1 \times 2 \times \cdots \times n$.

We need to put the n distinct elements in the following n positions in a row:

$$\square \quad \square \quad \square \quad \cdots \quad \square \quad \square$$

For the first position, we have n choices. Once the first position is filled, the second position has $(n-1)$ choices. Generally, once the first, $\cdots$, and $(i-1)$th positions are fixed, we have $(n+1-i)$ choices for the ith position. Since the filling of the positions is sequential, by the Multiplication Rule, the number of permutations of a set of n distinct elements is

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Permutations (3)

Example 20

1. How many ways can the letters in the word "COMPUTER" be arranged in a row?
2. How many ways can the letters in the word "COMPUTER" be arranged if the letter "CO" must remain next to each other (in order) as a unit?

Solution 21

1. *All the letters in the word COMPUTER are distinct, so the number of ways to arrange the letters equals the number of permutations of a set of 8 elements. This equals* $8!$.
2. *Since CO must remain together in order, this is to arrange the 7 distinct objects CO, M, P, U, T, E, R in a row. So the number of ways of arrangements is* $7! = 5040$.

Permutations (4)

Definition 22

An r-permutation of a set of n distinct elements is an **ordered selection** of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.

Example 23

Let $S = \{a, b, c\}$. All the 2-permutations of S are the following:

$$ab, \; ba, \; ac, \; ca, \; bc, \; cb.$$

Example 24

An n-permutation of a set S with size n is a permutation of S defined earlier.

Permutations (5)

Proposition 25

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n distinct elements is given by the formula

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

or equivalently,

$$P(n,r) = \frac{n!}{(n-r)!}$$

A proof of this proposition is given in the next slide.

Permutations (6): Proof of Proposition 25

Proof.

We prove this result by the Multiplication Rule. We have r positions in a row to be filled with r elements in the set of n elements:

$$\square \quad \square \quad \square \quad \cdots \quad \square \quad \square$$

The first position has n choices, i.e. it can be filled with any of the n elements. The second one has $(n-1)$ choices once the first position is filled. Generally, the ith position has $(n+1-i)$ choices once all the preceding positions are filled. Since this is done sequentially, by the multiplication rule, there are

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

number of r-permutations of a set of n distinct elements. Clearly,

$$P(n,r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

$\square$

Permutations (7)

Example 26

How many different ways can two of the letters of the word "WORK" be chosen and written in a row?

Solution 27

The answer equals the number of 2-permutations of a set of four elements. This equals

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{1 \times 2 \times 3 \times 4}{1 \times 2} = 12.$$