Illustration of the Proof of Lemma 5.28

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If X and Y are independent random variables on sample space S with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

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In these sides, we illustrate the proof of Lemma 5.28 with an example.

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$$P(X = 4) = 1/3 \qquad P(Y = 4) = 1/4$$

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$$P(X = 4) = 1/3 \quad P(Y = 4) = 1/4$$

$$\Rightarrow \begin{array}{l} E(X) = 7/3 \\ E(Y) = 2 \\ E(X)(EY) = 14/3 \end{array}$$

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$Z = XY$ can only take on the values $1, 2, 4, 8, 16$.

$$P(Z=1) = P(X=1 \ \wedge Y=1) = \tfrac{1}{6}$$

$$P(Z=2) = P(X=1 \ \wedge \ Y=2) + P(X=2 \ \wedge \ Y=1) = \tfrac{1}{12} + \tfrac{1}{6} = \tfrac{1}{4}$$

$$\begin{aligned} P(Z=4) = {} & P(X=1 \ \wedge \ Y=4) + P(X=4 \ \wedge \ Y=1) \\ & + P(X=2 \ \wedge \ Y=2) = \tfrac{1}{12} + \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{1}{3} \end{aligned}$$

$$P(Z=8) = P(X=2 \ \wedge \ Y=4) + P(X=4 \ \wedge \ Y=2) = \tfrac{1}{12} + \tfrac{1}{12} = \tfrac{1}{6}$$

$$P(Z=16) = P(X=4 \ \wedge Y=4) = \tfrac{1}{12}$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$. So

$$\begin{aligned} E(XY) &= E(Z) \\ &= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\ &\qquad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\ &= \tfrac{14}{3} \\ \\ &= E(X) \cdot E(Y) \end{aligned}$$

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On the next page, we mimic the proof of Lemma 5.28, using these X, Y. Reading the proof with this example in mind, might make the proof more understandable.

$E(X)E(Y)$

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$$\Big\updownarrow \ \text{Ind of } X, Y$$

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