COMP170
Discrete Mathematical Tools for Computer Science

Quantifiers

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Discrete Math for Computer Science
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3.2 Variables and Quantifiers

- Variables and Universes
- Quantifiers
- Standard Notation for Quantification
- Statements about Variables
- Proving Quantified Statements True or False
- Negation of Quantified Statements
- Implicit Quantification

Variables and Universes

Consider the statement:

$$(*) \qquad m^2 > m$$

Is (*) True or False?

This is an ill-posed question!

For some values of m, e.g., $m = 2$, (*) is True

For other values of m, e.g., $m = 1/2$, (*) is False

In statements such as $m^2 > m$, variable m is *not constrained*.
Unconstrained variables are called *free variables*.

Each possible value of a free variable gives a new statement.
The Truth or Falsehood of this new statement, is determined by substituting in the new value for the variable.

Again consider the statement: (*) $m^2 > m$

- For which values of m is (*) True
  and for which values is it False?

- This statement is also ill-defined!
  The answer depends upon which universe we assume

  - For the universe of non-negative integers, the statement is True for every value of m except $m = 0, 1$.

  - For the universe of real numbers, the statement is True for every value of m except for $0 \leq m \leq 1$

Two main points:

- Clearly state the universe

- A statement about a variable can be True for some values of a variable and False for others.

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Quantifiers

The statement

$$(*) \qquad \text{For every integer } m, \;\; m^2 > m$$

is False.

- While $m^2 > m$ is True for values such as $m = -3$ or $m = 9$ it is False for $m = 0$ or $m = 1$.

- Thus, it is not True that $m^2 > m$ for every integer m, so (*) is False

Quantifiers

The statement

$$(*) \quad \text{For every integer } m, \;\; m^2 > m$$

is False.

- A phrase like for every integer m that converts a symbolic statement about potentially any member of our universe into a statement about the universe is called a **quantifier**

- A quantifier that asserts a statement about a variable is true for every value of the variable in its universe, is called a **universal quantifier**.

Examples of universal quantifiers

The statement

(**) For every integer m, $2m$ is even

is True.

The statement

(**) For every real number m, $2m$ is even

is False.

The statement

(***) There exists an integer m, such that $m^2 > m$

is True.

- An **existential quantifier** asserts that at least one element of the universe exists that makes the individual statement True.

- To show that a statement with an existential quantifier is True, we need only exhibit *one* value of the variable being quantified that makes the statement True.
  - Example for (***): set $m = 2$

- What would you have to do to show that a statement about one variable with an existential quantifier is False?
  - You would have to show that every element of the universe makes the statement being quantified False

- What would you have to do to show that a statement about one variable with a universal quantifier is True?
  - You would have to show that every element of the universe makes the satement being quantified True

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Standard Notation for Quantification

A quantified statement about x asserts either that

- the statement is True for all x in the universe, or
- there exists an x in the universe that makes the statement True

Notation: $\forall$ for for all and $\exists$ for there exists.

Examples: Use Z for universe of all integers

- For all integers n, $n^2 \geq n$ becomes $\forall n \in Z \ (n^2 \geq n)$
- There exists an integer n such that $n^2 \not\geq n$ becomes $\exists n \in Z \ (n^2 \not\geq n)$

Using the given notation let's rewrite the part of Euclid's division theorem (Theorem 2.12) that states

> For every positive integer n and every nonnegative integer m, there are integers q and r, with $0 \leq r < n$, such that $m = qn + r$

Let Z^+ be the positive integers and N the nonnegative integers.

$$\forall n \in Z^+ \ \ (\forall m \in N \ \ (\exists q \in N \ \ (\exists r \in N$$

$$((r \ \ < \ \ n) \wedge (m \ \ = \ \ qn + r)) \ \ )))$$

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Statements about Variables

Use $p(n)$ to stand for the statement $n^2 > n$.

$p(4)$ and $p(-3)$ are True; $p(1)$ and $p(0.5)$ are False

We now rewrite Euclid's division theorem.
Let $p(m, n, q, r)$ denote $m = nq + r$ with $0 \le r < n$.
Leave out references to universes to clearly see the order in which the quantifiers occur.

$$\forall n \ (\forall m \ (\exists q \ (\exists r \ p(m, n, q, r) \ )))$$

Rewriting Statements to Encompass Larger Universes

It is sometimes useful to rewrite a quantified statement so that the universe is larger while the statement itself focuses on a subset of the new universe.

Let R be the real numbers & R^+ the positive reals.
Consider the following two statements.

a) $\forall x \in R^+ \ (x > 1)$

b) $\exists x \in R^+ \ (x > 1)$

Now rewrite (a) and (b) so that the universe is R but the statements say the same thing

a') $\forall x \in R \ (\ (x > 0) \ \Rightarrow \ (x > 1)\ )$

b') $\exists x \in R \ (\ (x > 0) \ \wedge \ (x > 1)\ )$

Theorem 3.2:
Let U_1, U_2 be two universes with $U_1 \subseteq U_2$.
Suppose that $q(x)$ is a statement such that

$$(*) \quad U_1 = \{x \in U_2 \,|\, q(x) \text{ is true}\}.$$

Then, if $p(x)$ is a statement about U_2, it may also be interpreted as a statement about U_1, and

a. $\forall x \in U_1\ (p(x))$ is equiv. to $\forall x \in U_2\ (q(x) \Rightarrow p(x))$, and

b. $\exists x \in U_1\ (p(x))$ is equiv. to $\exists x \in U_2\ (q(x) \wedge p(x))$.

Proof:

By (*), $q(x)$ must be True for all $x \in U_1$ and False for all $x \in U_2$ but $x \notin U_1$.

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Proving Quantified Statements True or False

Let R be the real numbers & R^+ the positive real numbers.

For each of the following, state T or F and explain why.

a) $\forall x \in R^+ \ (x > 1)$ — **F**, because $1/2 \leq 1$.

b) $\exists x \in R^+ \ (x > 1)$ — **T**, because $2 > 1$.

c) $\forall x \in R \ (\exists y \in R \ (y > x))$ — **T**. Let $y = x + 1$.

d) $\forall x \in R \ (\forall y \in R \ (y > x))$ — **F**. Let $x = 1, y = 0$

e) $\exists x \in R \ (\ (x \geq 0) \ \wedge \ \forall y \in R^+ \ (y > x))$

T. Let $x = 0$.

Principle 3.2 (The Meaning of Quantified Statements)

- The statement $\exists x \in U \ (p(x))$ is True if there exists at least one value of $x \in U$ for which the statement $p(x)$ is True.

- The statement $\exists x \in U \ (p(x))$ is False if there is no $x \in U$ for which $p(x)$ is True.

- The statement $\forall x \in U \ (p(x))$ is True if $p(x)$ is True for every value of $x \in U$.

- The statement $\forall x \in U \ (p(x))$ is False if $p(x)$ is False for at least one value of $x \in U$.

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Negation of Quantified Statements

What is the meaning of the statement

It is not the case that $n^2 > 0$ for all integers n

$\neg\forall n \in Z\ (n^2 > 0)$ asserts that

it is not the case that $n^2 > 0$ for all integers n.

Then, there must be some integer n such that $n^2 \not> 0$.

i.e., there exists some integer n s.t. $n^2 \leq 0$,

i.e., $\exists n \in Z\ (n^2 \leq 0)$

Thus, the negation of our for all ($\forall$) statement is a there exists ($\exists$) statement.

The following theorem formalizes the example.

Theorem 3.3: The statements

$$\neg\forall x \in U(p(x)) \quad \text{and} \quad \exists x \in U(\neg p(x))$$

are equivalent.

Proof:

$p(x)$	$\neg p(x)$	$\forall x \in U(p(x))$	$\neg\forall x \in U(p(x))$	$\exists x \in U(\neg p(x))$
always true	always false	true	false	false
not always true	not always false	false	true	true

Example: Let $p(x)$ be the statement $x^2 > 0$. Then

$$\neg\forall n \in Z\ (n^2 > 0) \qquad \text{is equivalent to} \qquad \exists n \in Z\ (n^2 \leq 0)$$

Corollary 3.4: The statements

$\neg\exists x \in U(p(x))$ and $\forall x \in U(\neg p(x))$ are equivalent.

Proof:

From Theorem 3.3

$\neg\forall x \in U(q(x))$ and $\exists x \in U(\neg q(x))$ are equivalent.

Negating both statements gives

$\forall x \in U(q(x))$ and $\neg\exists x \in U(\neg q(x))$ are equivalent.

Now, setting $q(x) = \neg p(x)$ gives

$\forall x \in U(\neg p(x))$ and $\neg\exists x \in U(p(x))$ are equivalent,

and proves the corollary.

Corollary 3.4: The statements
$\neg\exists x \in U(p(x))$ and $\forall x \in U(\neg p(x))$ are equivalent.

Example:

Let $p(x)$ be $2x$ is odd.

Then $\neg p(x)$ is $2x$ is even.

The corollary then says that

$$\neg\exists x \in Z\ (2x \text{ is odd})$$

is equivalent to

$$\forall x \in Z\ (2x \text{ is even})$$

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Implicit Quantification

Are there any quantifiers in the statement

The sum of even integers is even?

Yes! When we write this out mathematically we see that there are.

Let $p(n)$ be the statement n is even.

Our original statement really means that

For every two even integers, m, n, $m + n$ is even

In symbols

$$\forall m \in Z\ (\forall n \in Z\ (\ (p(m) \wedge p(n)) \Rightarrow p(m+n)\ )\ )$$