COMP 170 Discrete Mathematical Tools for CS
2007 Fall Semester – Written Assignment # 3
Distributed: Sept 20, 2007 – Due: Sept 27, 2007

At the top of your solution, please write your (i) name, (ii) student ID #, (iii) email address and (iv) tutorial section.

Some Notes:

- Please write clearly and briefly. For all questions you should also provide a short explanation as to *how* you derived the solution. That is, if the solution is 20, you shouldn't just write down 20. You need to explain *why* it's 20.
- Please follow the guidelines on doing your own work and avoiding plagiarism given on the class home page. Don't forget to *acknowledge* individuals who assisted you, or sources where you found solutions.
- These problems are taken (some modified) from section 1.3 of the textbook.
- Please make a *copy* of your assignment before submitting it. If we can't find your paper in the submission pile, we will ask you to resubmit the copy.
- Your solutions can either be submitted at the end of your Thursday lecture section or, before 5PM, in the collection bin in front of Room 4213A.

Problem 1: In class we stated that
each row of Pascal's triangle first increases and then decreases.
In this question you will prove this statement.

(a) Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ prove that

$$\text{if } 0 < k \le n/2 \text{ then } \binom{n}{k-1} < \binom{n}{k}.$$

(b) Using part (a) and the fact that $\binom{n}{k} = \binom{n}{n-k}$ prove that
each row of Pascal's triangle first increases and then decreases.

Problem 2: (a) If you have eleven distinct chairs to paint, in how many ways can you paint seven of them orange and four of them red?

(b) Now, how many ways can you paint three of them green, four of them blue, and four of them red?

Problem 3: In a Cartesian coordinate system, how many paths are there from the origin to the point with integer coordinates (m, n) if the paths are built up of exactly $m + n$ horizontal and vertical line segments, each of length 1?

You should assume that all of the horizontal edges go from left to right and all the vertical edges from bottom up. You should also assume that $m, n > 0$.

That is, you start at point $(0, 0)$ and each edge either goes
(i) vertically, from (i, j) to $(i, j + 1)$ or
(ii) horizontally, from (i, j) to $(i + 1, j)$.

Problem 4: In class we proved, for $0 < k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

using the sum principle. Now prove this equation algebraically (directly) by plugging in the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and showing that the left side equals the right side.

Problem 5: Give two proofs that

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

Your first proof should be purely algebraic, i.e., just plug in the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and show that the left side equals the right side. Your second proof should be combinatorial, i.e., it should show that the left and right sides are just two different ways to count the same thing.

Problem 6: Explain why

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

An algebraic proof would be sufficient (a combinatorial proof is not necessary).

Problem 7: Prove that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

for $2 \leq k \leq n - 2$. In your proof, you may make use of any result already proved in class. Your proof may be algebraic or combinatorial.

Challenge problem: (a) Suppose that $n = 2k$ is even. Evaluate $\sum_{i=0}^{k} \binom{n}{2i}$.

(b) Now suppose that $n = 3k$ for some integer k. Evaluate $\sum_{i=0}^{k} \binom{n}{3i}$.