Predicate Logic

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Contents

Predicates

Definition 1

- A predicate is a **statement** $P(x_1, x_2, \ldots, x_n)$ that contains n variables $x_1, x_2, \ldots, x_n$ and becomes a proposition when specific values are substituted for the variables x_i, where $n \geq 1$ is a positive integer.
- P is called an n-ary predicate.
- The domain D of the **predicate variables** $(x_1, x_2, \ldots, x_n)$ is the set of all values that may be substituted in place of the variables.
- The truth set of $P(x_1, x_2, \ldots, x_n)$ is defined to be

$$\{(x_1, x_2, \ldots, x_n) \in D \mid P(x_1, x_2, \ldots, x_n) \text{ is true}\}.$$

Warning

By definition, a predicate is a family of related propositions. Understanding the difference between predicates and propositions is a must.

Examples of Predicates

Example 2 (Predicate with One Variable)

Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbb{R}$ of all real numbers.

1. What are the truth values of the propositions $P(2)$ and $P(1)$?
2. What is the truth set of $P(x)$?

Answers

1. $P(2) = T$ and $P(1) = F$.
2. The truth set of $P(x)$ is $\{a > 1 : a \in \mathbb{R}\} \cup \{b < 0 : b \in \mathbb{R}\}$.

Examples of Predicates

Example 3 (Predicate with Two Variables)

Let $Q(x,y)$ be the predicate "$x = y+3$" with the domain $\mathbb{R} \times \mathbb{R}$.

1. What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?
2. What is the truth set of $Q(x,y)$?

Answers

1. $Q(1,2) = F$ and $Q(3,0) = T$.
2. The truth set of $Q(x,y)$ is $\{(a, a-3) : a \in \mathbb{R}\}$.

The Universal Quantifier

Definition 4

The symbol $\forall$ denotes "for all" and is called the universal quantifier.

Example 5

Let H be the set of all human beings. Let $P(x)$ be the predicate "x is mortal" with domain H. We have the following statement:

$$\forall\ x \in H, x \textit{ is mortal.}$$

Universal Statements

Definition 6

- Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form "$\forall x \in D, Q(x)$.
- It is defined to be true if, and only if, $Q(x)$ is true for every $x \in D$. It is defined to be false if, and only if, $Q(x)$ is false for at least one $x \in D$.
- A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example 7

Let $D = \{1, 2, 3, 4, 5\}$. Consider the following statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

Example 8

Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

The Existential Quantifier

Definition 9

The symbol $\exists$ denotes "there exists" and is called the existential quantifier.

Example 10

Let D be the set of all people. Let $P(x)$ be the predicate "x is a student in COMP2711H" with domain D. We have the following statement:

$$\exists\, x \in D \textit{ such that } x \textit{ is a student in COMP2711H.}$$

Existential Statements

Definition 11

- Let $Q(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form "$\exists x \in D$ such that $Q(x)$."
- It is defined to be true if, and only if, $Q(x)$ is true for at least one $x \in D$. It is false if, and only if, $Q(x)$ is false for all $x \in D$.

Example 12

Let $E = \{5, 6, 7, 8\}$. Consider the following statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Example 13

Let $\mathbb{N}$ be the same as before. Consider the statement

$$\exists m \in \mathbb{N} \text{ such that } m^2 = m.$$

Show that the statement is true.

Universal Conditional Statements

Definition 14

A <u>universal conditional statement</u> is of the form

$$\forall x, \textit{ if } P(x) \textit{ then } Q(x).$$

Example 15

1. $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$.

The Implicit Quantification

Definition 16

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$.
- The notation $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

Example 17

Let

- $P(n)$ be "n is a multiple of 8,"
- $Q(n)$ be "n is a multiple of 4,"

with the common domain $\mathbb{Z}$. Then $P(x) \Rightarrow Q(x)$.

The Implicit Quantification

Problem 18

Let

- $Q(n)$ *be "n is a factor of 4,"*
- $R(n)$ *be "n is a factor of 2,"*
- $S(n)$ *be "$n < 5$ and $n \neq 3$,"*

with the common domain $\mathbb{N}$*, the set of positive integers. Use the* $\Rightarrow$ *and* $\Leftrightarrow$ *symbols to indicate true relationships among* $Q(n)$*,* $R(n)$*, and* $S(n)$*.*

Solution 19

- *The truth set of* $Q(n)$ *is* $\{1,2,4\}$.
- *The truth set of* $R(n)$ *is* $\{1,2\}$.
- *The truth set of* $S(n)$ *is* $\{1,2,4\}$.

Hence,

$$R(n) \Rightarrow Q(n),$$
$$R(n) \Rightarrow S(n),$$
$$Q(n) \Leftrightarrow S(n).$$

Negation of a Universal Statement

Definition 20

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is a statement of the form

$$\exists x \in D \text{ such that } \sim Q(x).$$

Example 21

The negation of the following statement

$$\forall n \in \mathbb{N},\ P(n) > 0$$

is the statement that

$$\exists n \in \mathbb{N} \text{ such that } P(n) \leq 0.$$

Negation of an Existential Statement

Definition 22

The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is a statement of the form

$$\forall x \in D, \sim Q(x).$$

Example 23

The negation of the following statement

$$\exists n \in \mathbb{N} \text{ such that } P(n) \leq 0$$

is the statement that

$$\forall n \in \mathbb{N},\ P(n) > 0.$$

Variants of Universal Conditional Statements

Definition 24

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

- Its <u>contrapositive</u> is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
- Its <u>converse</u> is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
- Its <u>inverse</u> is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example 25

Consider a statement of the form: $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$.

Contrapositive: $\forall x \in \mathbb{R}$, if $x^4 \leq 4$ then $x \leq 2$.

Converse: $\forall x \in \mathbb{R}$, if $x^2 > 4$ then $x > 2$.

Inverse: $\forall x \in \mathbb{R}$, if $x \leq 2$ then $x^2 \leq 4$.

Statements with Multiple Quantifiers

A statement may involve multiple quantifiers.

Example 26

The following is an statement involving two quantifiers:

$$\forall x \text{ in set } D,\ \exists y \text{ in set } E \text{ such that } x \text{ and } y \text{ satisfy property } P(x, y).$$

An instance of the example above is the following.

Example 27

$$\forall x \text{ in set } \mathbb{Z},\ \exists y \text{ in set } \mathbb{Z} \text{ such that } x \text{ and } y \text{ satisfy property } x + y = 1.$$

Question 1

What is the negation of the statement with two quantifiers in Example 26?

Axioms

Definition 28

An axiom or postulate is a statement or proposition which is regarded as being established, accepted, or self-evidently true.

Example 29

- It is possible to draw a straight line from any point to any other point.
- It is possible to describe a circle with any center and any radius.

Theorems

Definition 30

A theorem is a statement that can be proved to be true.

Theorem 31

There are infinitely many primes.

Remark

A **theorem** contains usually a more important result, compared with a **proposition**.

Lemmas

Definition 32

A lemma is a statement that can be proved to be true, and is used in proving a theorem or proposition.

Lemma 33

The only even prime is 2.