An Introduction to Hashing

(Following CLRS)

COMP 3711H - HKUST

Version of 25/11/2014

M. J. Golin

Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends

Introduction

Known: A set $U = \{1, 2, \ldots, u-1\}$ of the universe of possible keys that could exist.

Goal: To maintain a dictionary that permits the following operation on keys

- **Search(x):** Find the record with key x or report that it does not exist
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Would like $O(1)$ (average) time per operation.

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For now, assume uniform hashing, that, every key is equally likely to hash to any of the m slots,

$$\forall x, i, \quad \Pr\left(h(x) = i\right) = \frac{1}{m}.$$

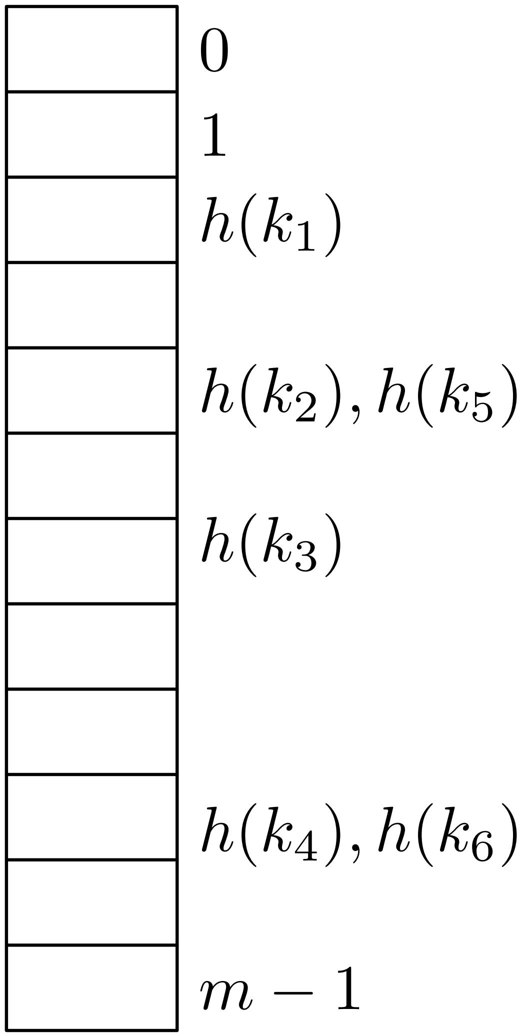
Introduction (iii)

$h : U \to \{0, 1 \ldots, m-1\}$

h maps the set of keys into a "small" table. Key k is stored in table slot $h(k)$.

Fnding key k is then just a matter of going to table location $h(k)$.

Problem is that, since m is small, many keys might be mapped to same slot, creating collision.



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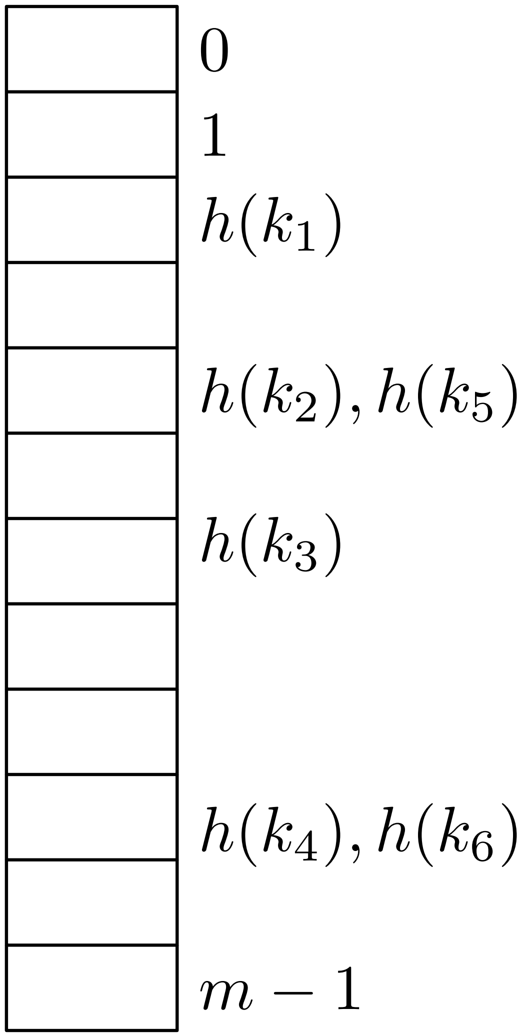

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Two major approaches to addresing collisions:
(1) Chaining
(2) Open Addressing

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Chaining

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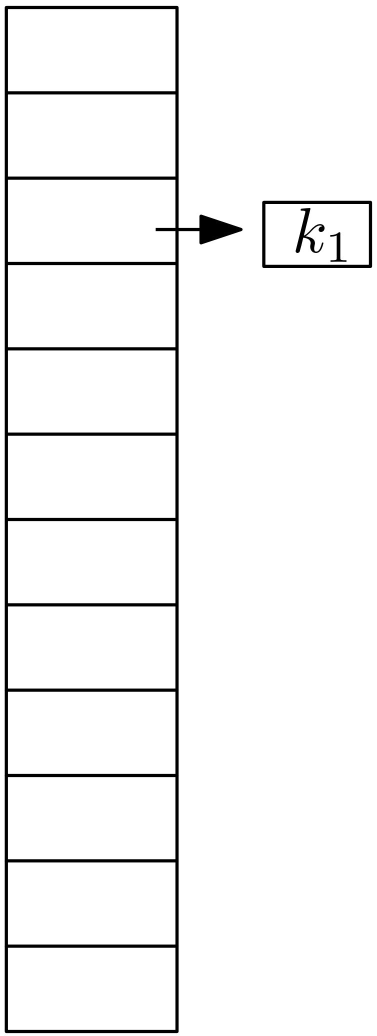

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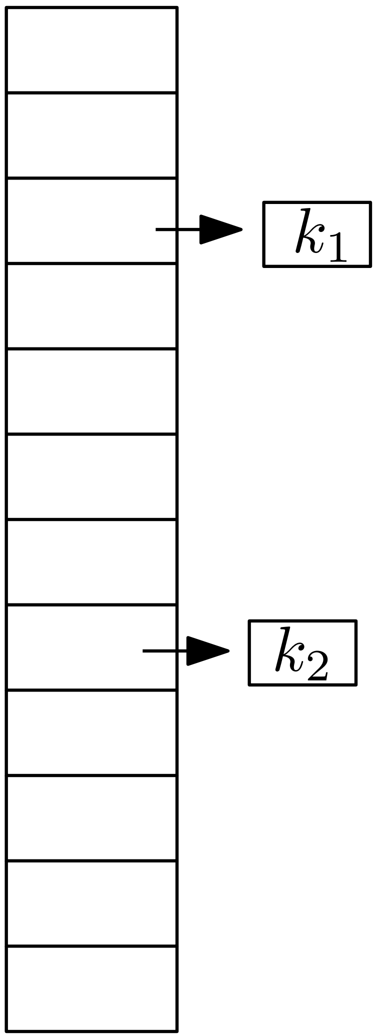

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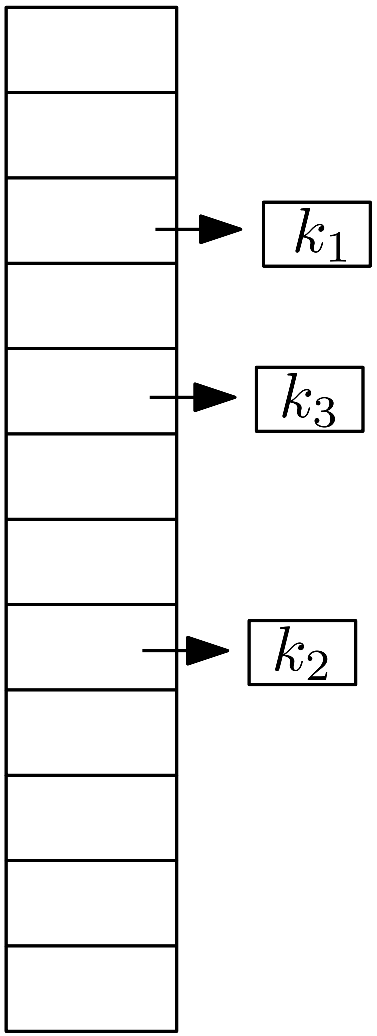

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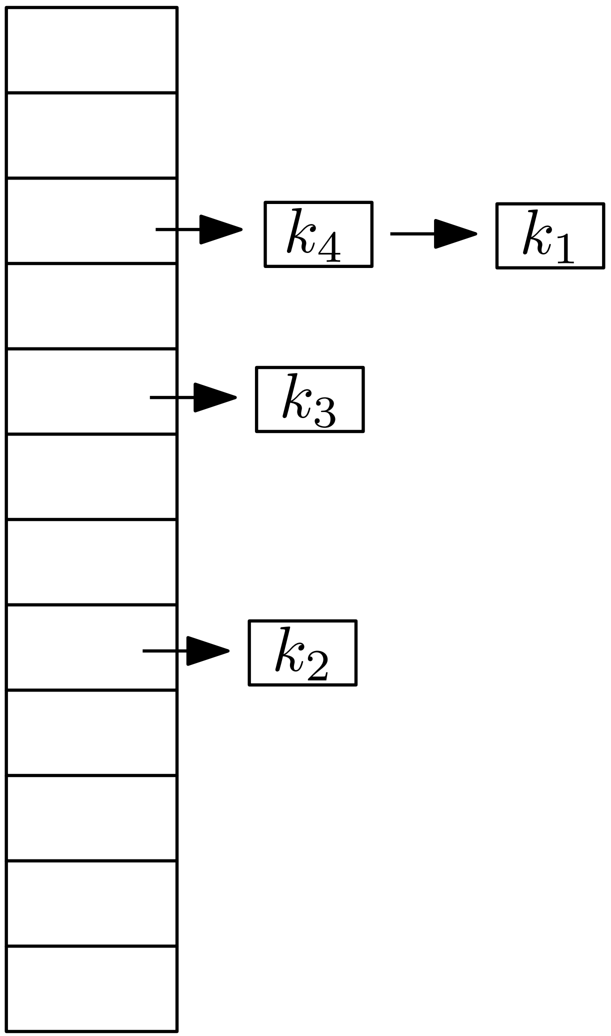

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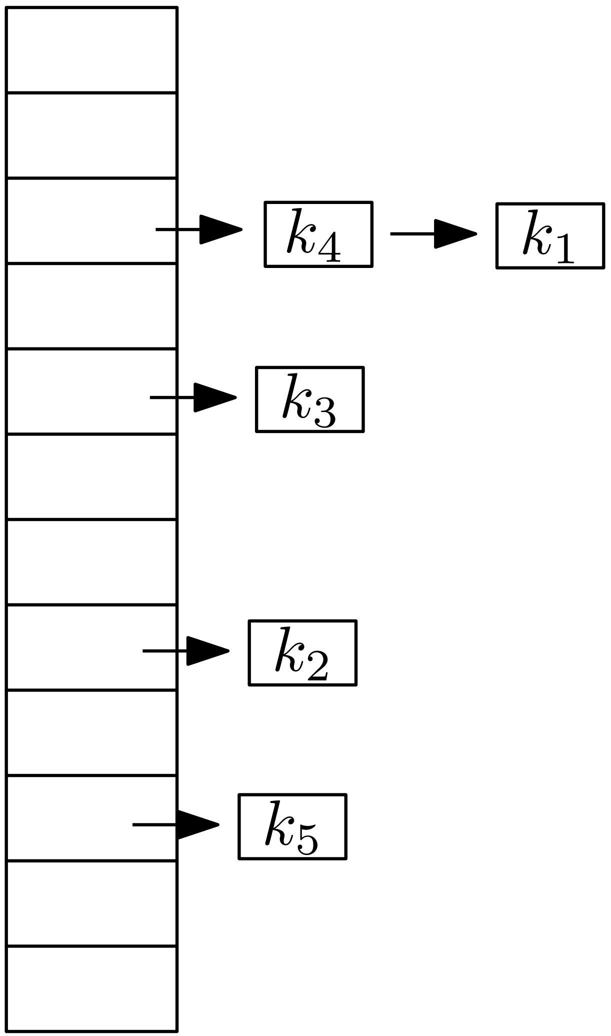

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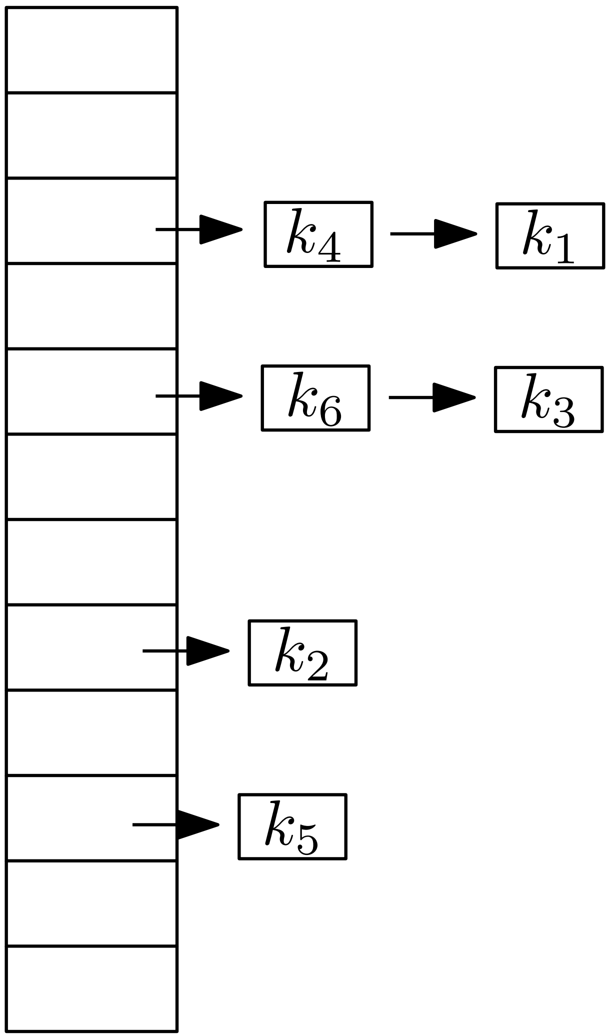

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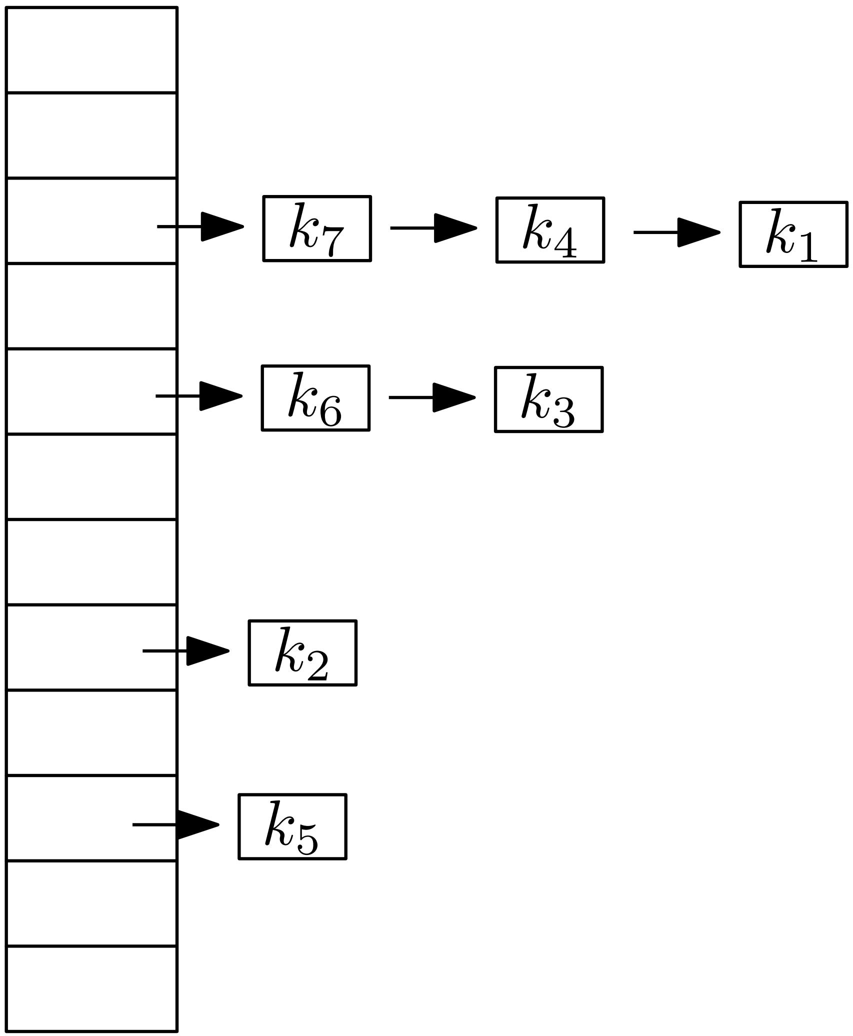

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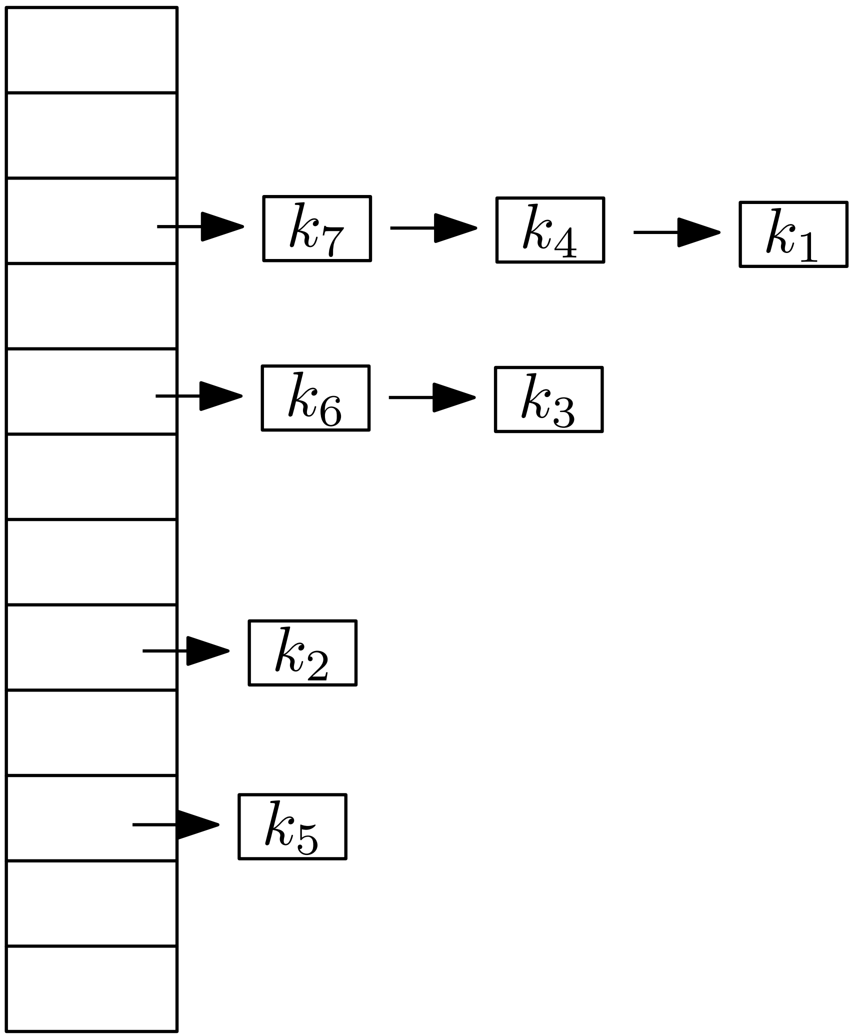

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Insert(x):Insert x into front of list for slot $h(x)$

Delete(x): Delete x from list for slot $h(x)$, if it's there.
Use doubly linked lists

Search(x): Search for x in list for $h(x)$

Chaining

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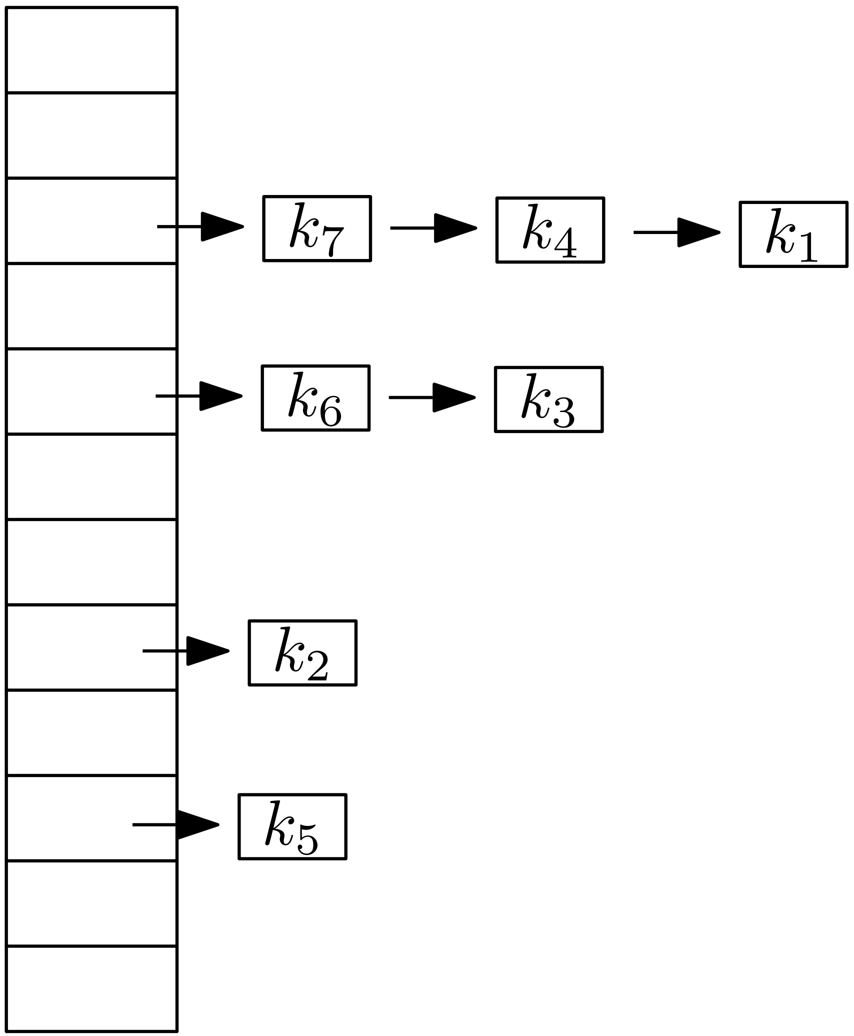

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Insert(x):Insert x into front of list for slot $h(x)$ $O(1)$

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Use doubly linked lists $O(1)$

Search(x): Search for x in list for $h(x)$ $O(\textit{length of list})$

Chaining: Unsuccessful Search

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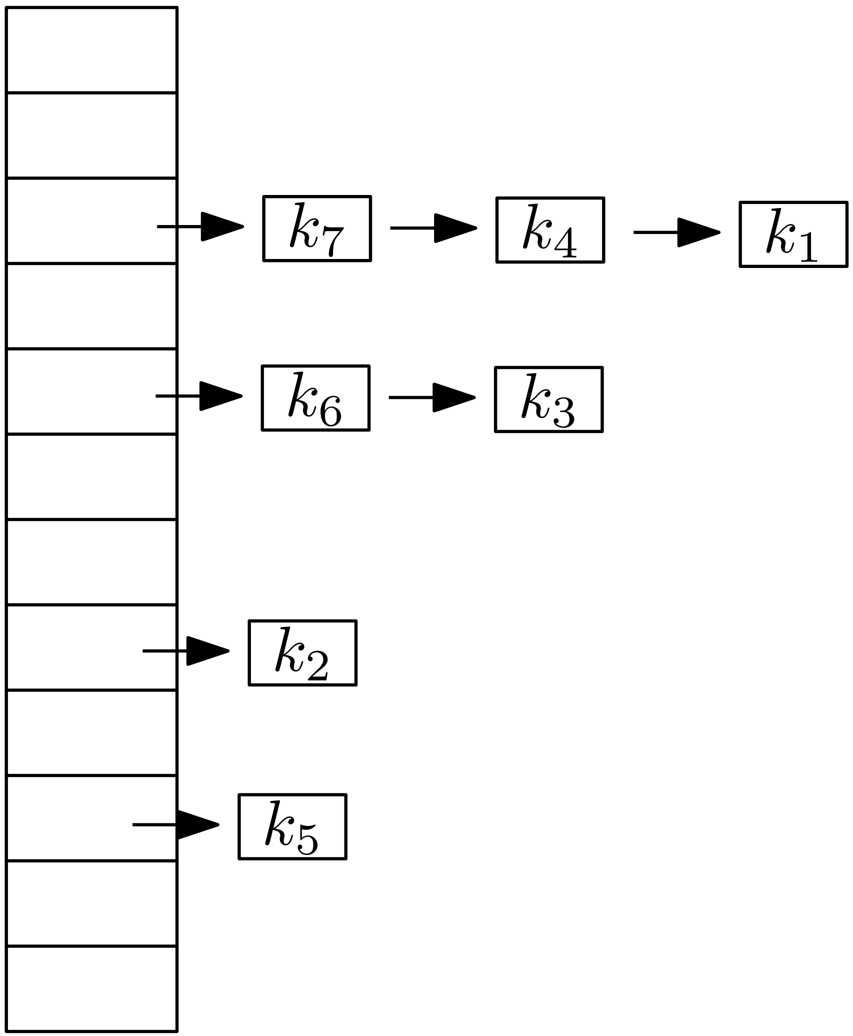

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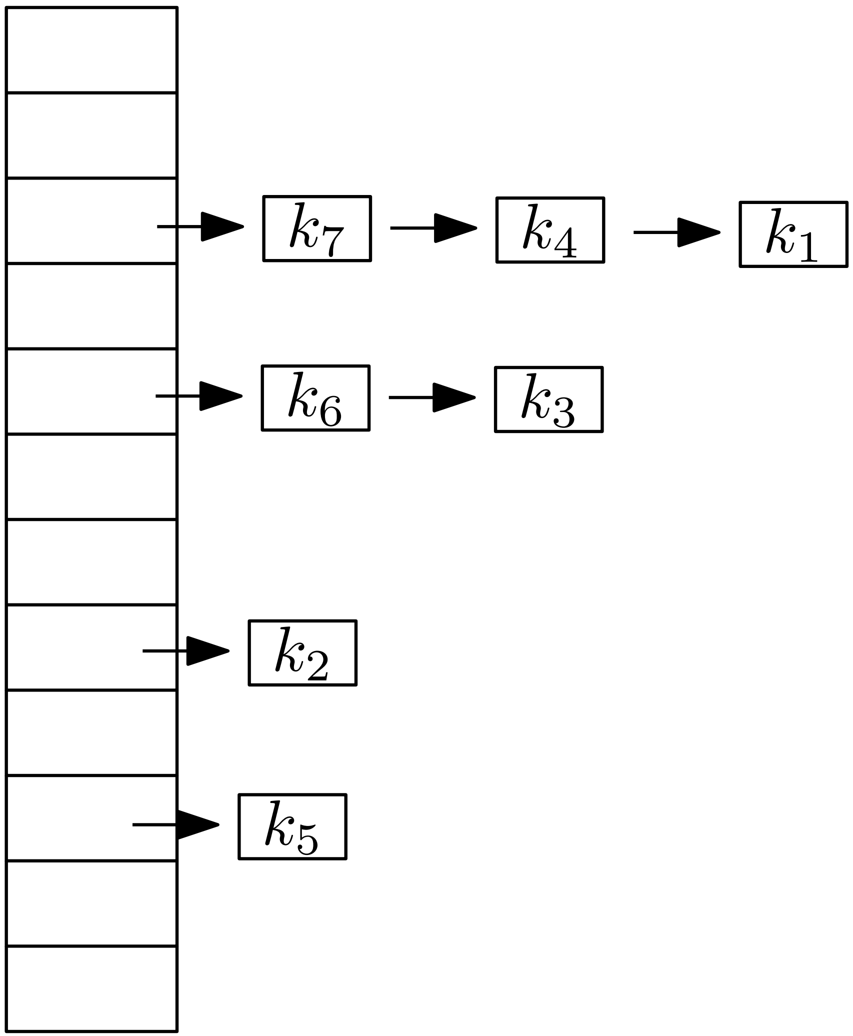

Search(x): Search for x in list for $h(x)$ $\quad$ *O(length of list)*

Recall load factor $\alpha = \frac{n}{m}$.

This is *average # items per list*.

Unsucessful search for x not in table will require searching entire list for $h(x)$.

Worst case length is $O(1)$.
Average case length is $O(\alpha)$.

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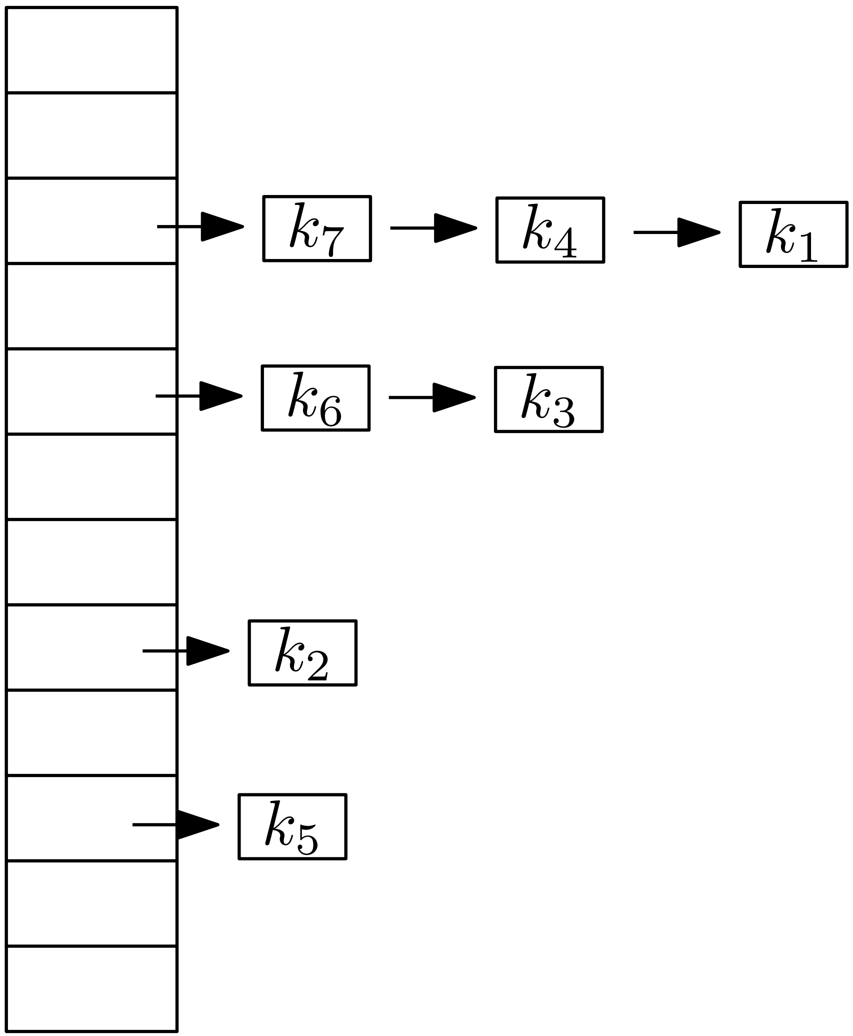

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Worst case length is $O(1)$.
Average case length is $O(\alpha)$.

Average Unsucessful Search time is

$O(1 + \alpha)$

where 1 is amount of time to calculate $h(x)$.

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If x is i'th item inserted

$\Rightarrow n - i$ items inserted after x

$\Rightarrow \alpha - \frac{i}{m} = \frac{n-i}{m}$ items inserted on average into $h(x)$ after x

x is equally likely (with prob $1/n$) to be i'th inserted item.

Average $\#$ of items ahead of x in list $h(x)$ is

$$\frac{1}{n}\sum_{i=1}^{n}\left(\alpha - \frac{i}{m}\right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}$$

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Adding 1 unit of time to calculate $h(x)$

Average cost of successful search is $\Theta(1+\alpha)$.

Chaining

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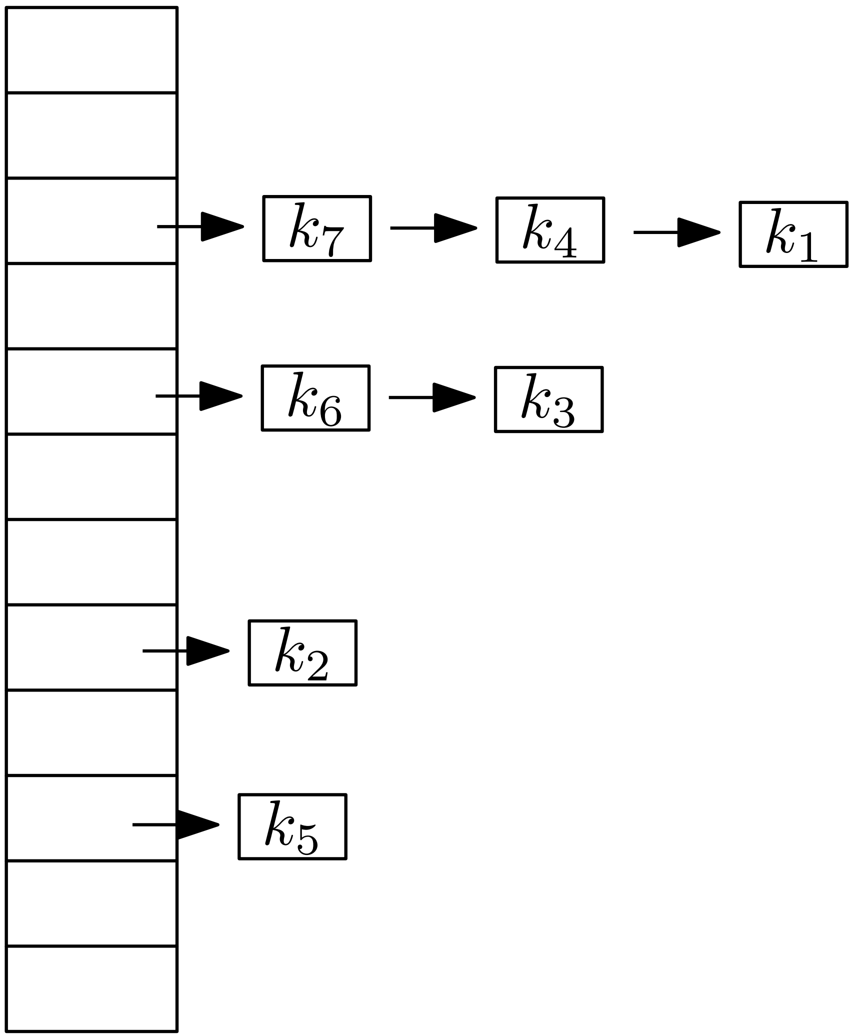

Search(x): Search for x in list for $h(x)$ $O(\textit{length of list})$

Both Successful and Unsuccessful Search require $O(1+\alpha)$ time on average

where $\alpha = \frac{n}{m}$ is the load factor

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Open Addressing

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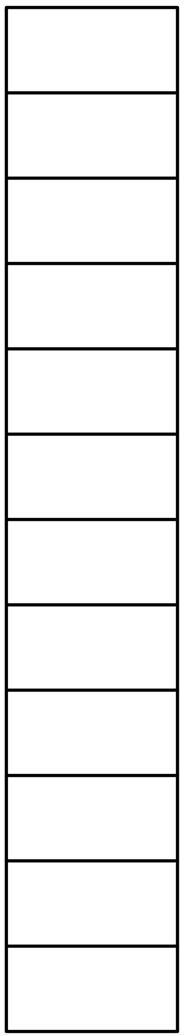


- No lists. All keys stored in hash table itself.
- For insertion, *probe* hash table until empty slot for insertion is found.
- *Probe Sequence* is part of hash function.
- Hash function is now

$$h : U \times \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$$

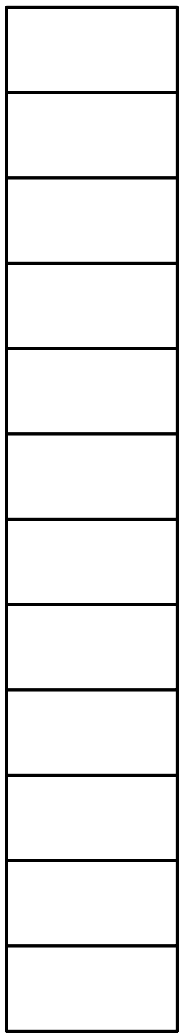
- Probe sequence for x is,

  $$h(x, 0),\ h(x, 1), \ldots,\ h(x, m-1)$$

  which is a permutation of $\{0, 1, \ldots, m\}$
- For search(x), *probe* hash table using probe sequence for $h(x)$ until either x or empty slot for insertion is found.

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x,i) = \big(h'(x) + i\big) \bmod m$ where $h'(x)$ is original hash function.
- Insert: Attempts insertion at $h'(x)$, then $h'(x)+1$, $h'(x)+2$, etc., (wrapping around to 0 after reaching end of table) until empty slot is found and x inserted there.
- Search(x): Examines probe sequence until it finds x or an empty slot.
  If empty slot is found then x wasn't previously inserted and search unsuccessful
- Deletion: More complicated.

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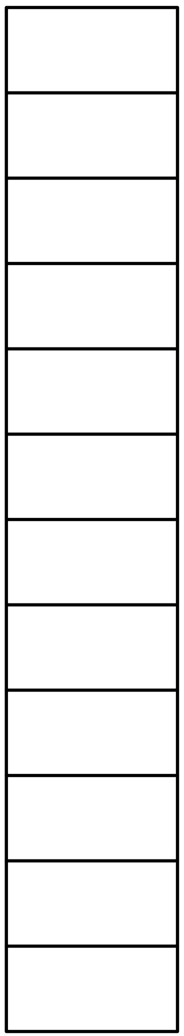

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- Search(x): Examines probe sequence until it finds x or an empty slot.
  If empty slot is found then x wasn't previously inserted and search unsuccessful
- Deletion: More complicated.
  Can't actually delete item and reset slot as 'empty"
  That would mess up Search(x).
  Can mark *slot as (used but) deleted.*
  Deletion in open addressing does cause difficulties.
  Better to use chaining.

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

As example, let $h(x) = x \bmod m$ with $m = 12$.

Open Addressing: Linear Probing

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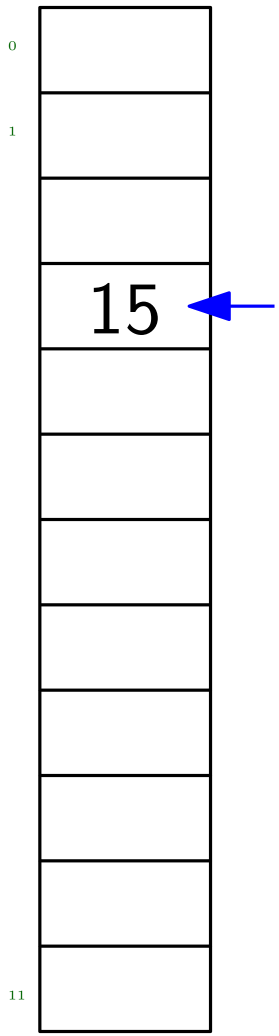

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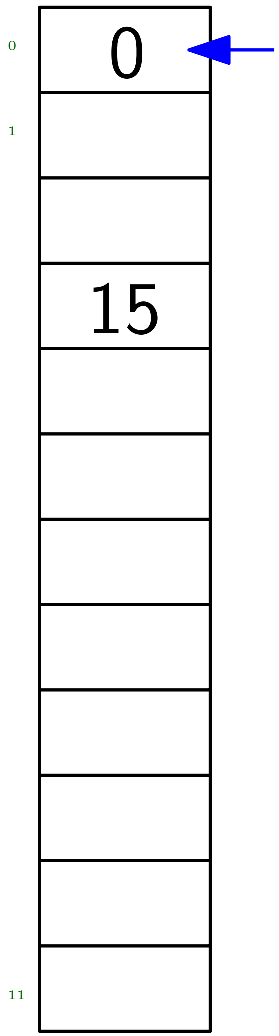

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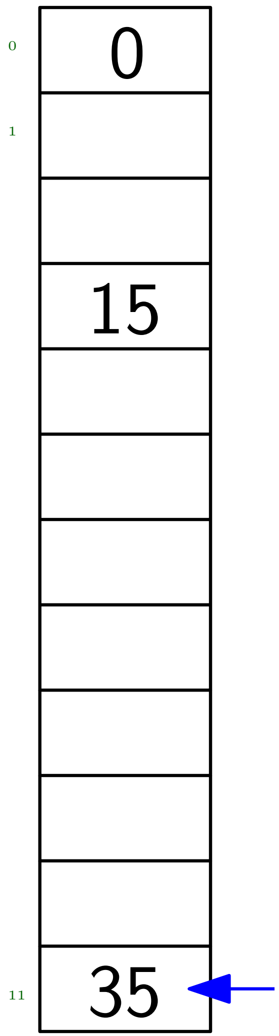

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1	
	15
11	35

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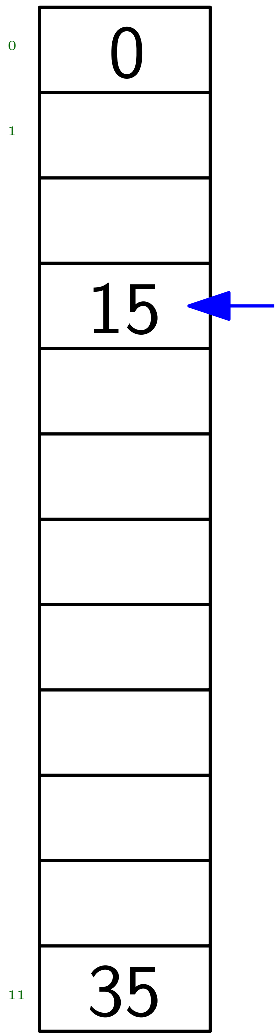

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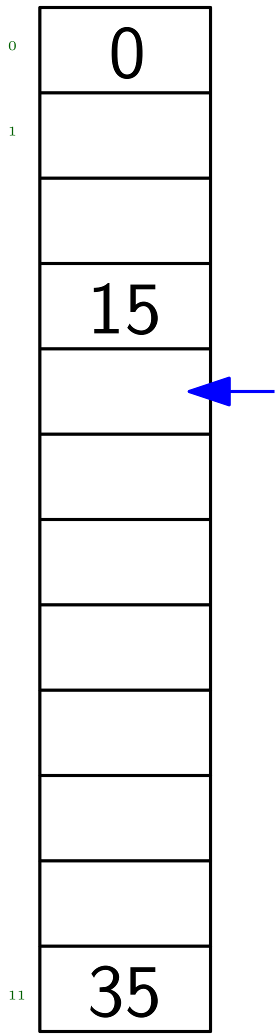

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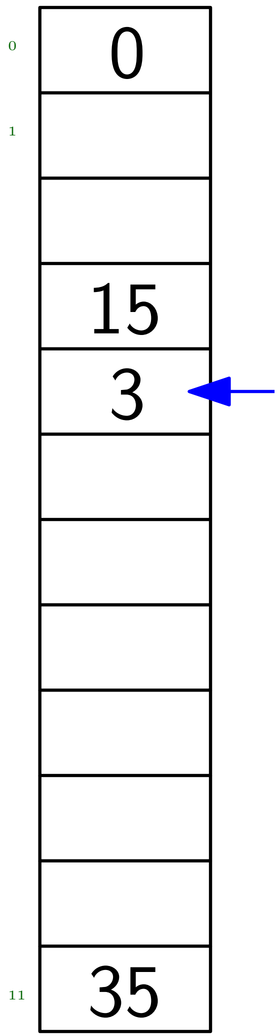

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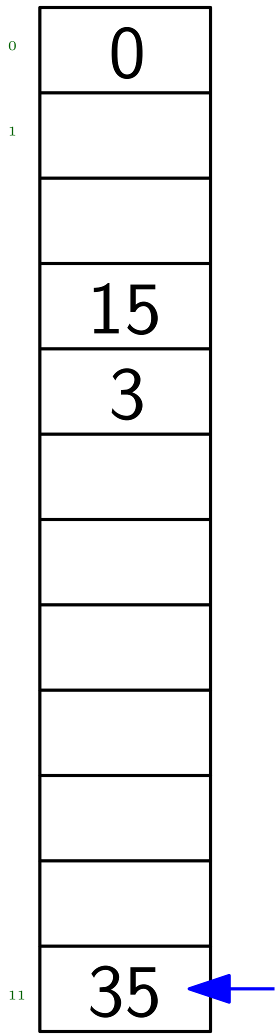

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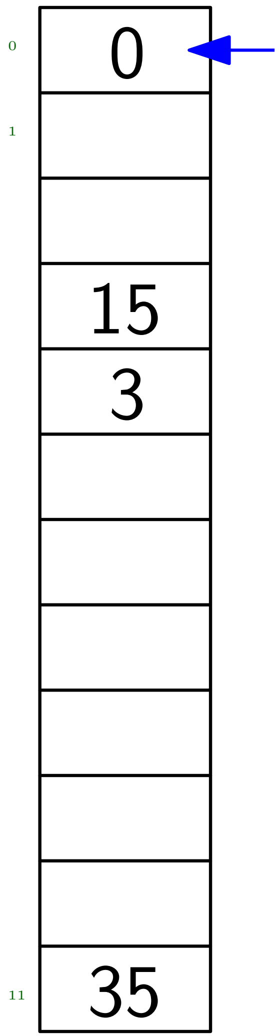

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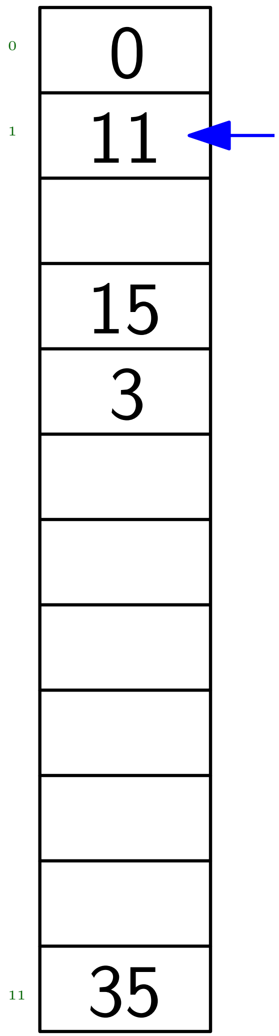

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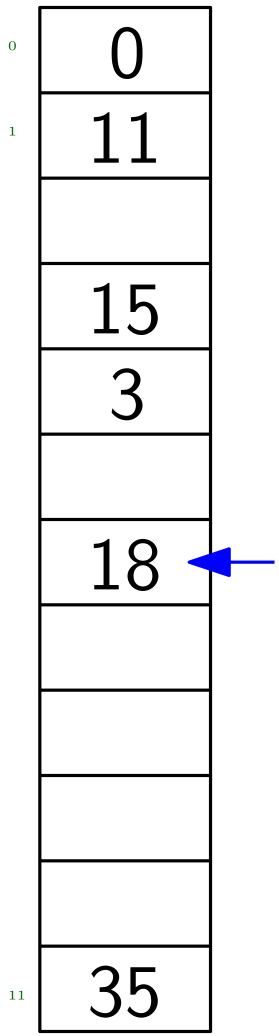

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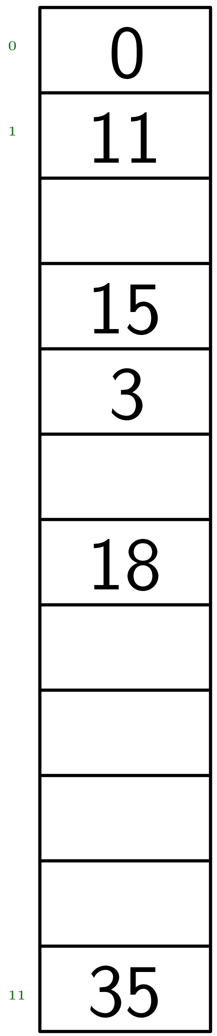

As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash function

Insert(15)

Insert(0)

Insert(35)

Insert(3)

Insert(11)

Insert(18)

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

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Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

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Search(11)

Open Addressing: Linear Probing

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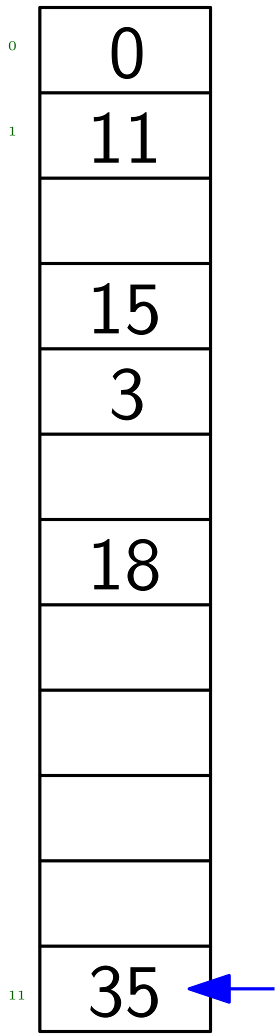

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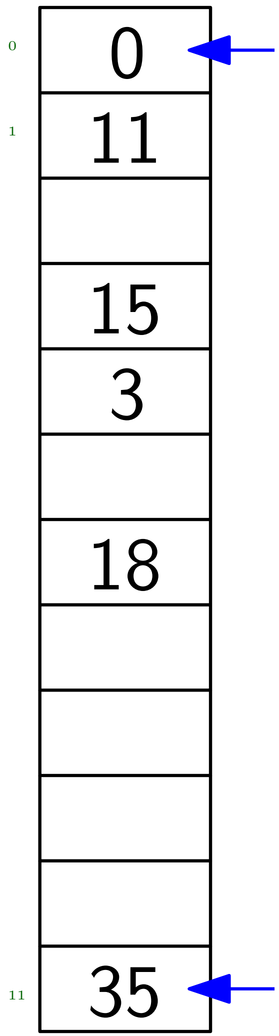

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$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

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Search(11) Exists

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash functionn

Search(11) Exists

Search(3)

Open Addressing: Linear Probing

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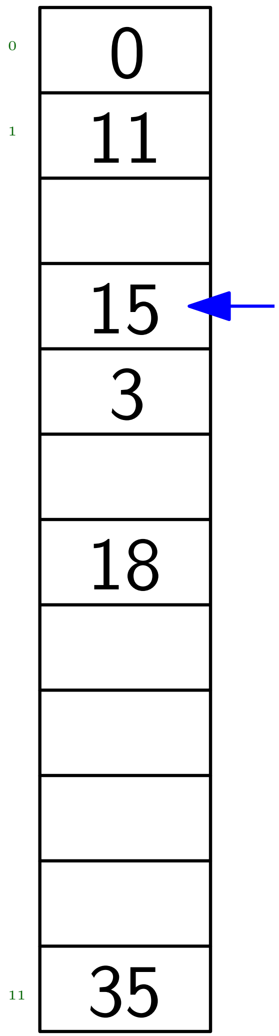

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Search(11) $\qquad$ Exists

Search(3)

Open Addressing: Linear Probing

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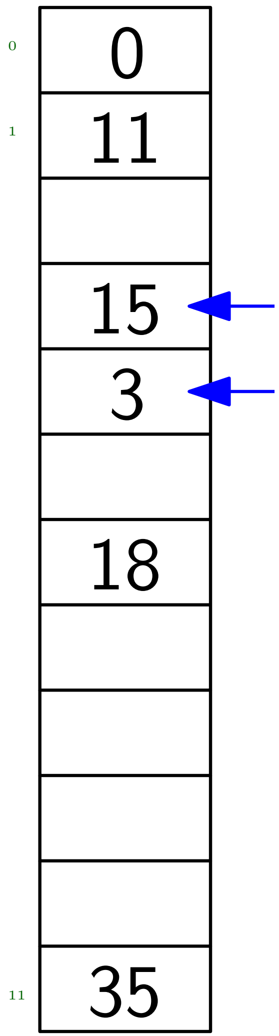

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Search(11) Exists

Search(3) Exists

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash functionn

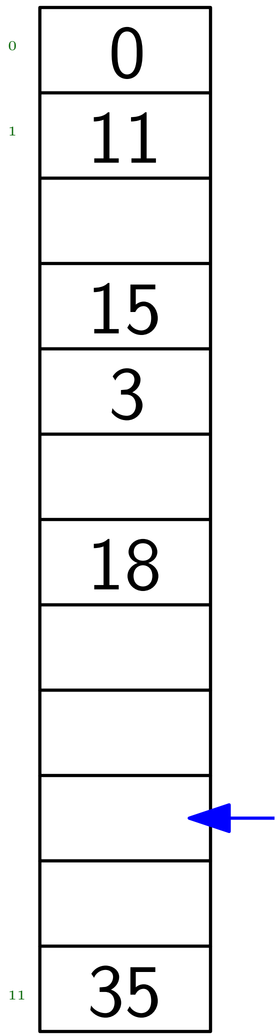
Search(11) Exists

Search(3) Exists

Search(9)

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



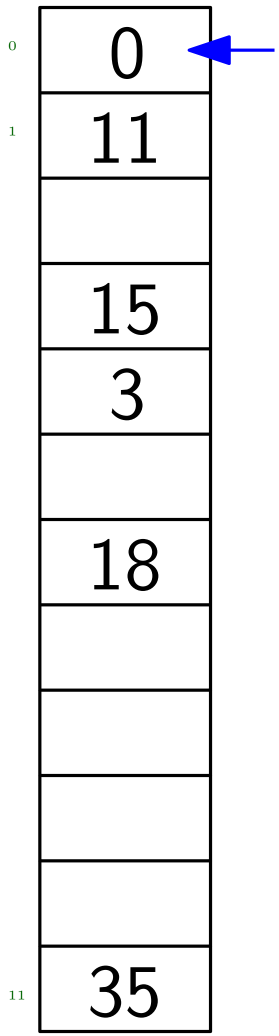
As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash functionn

Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



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Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	

Open Addressing: Linear Probing

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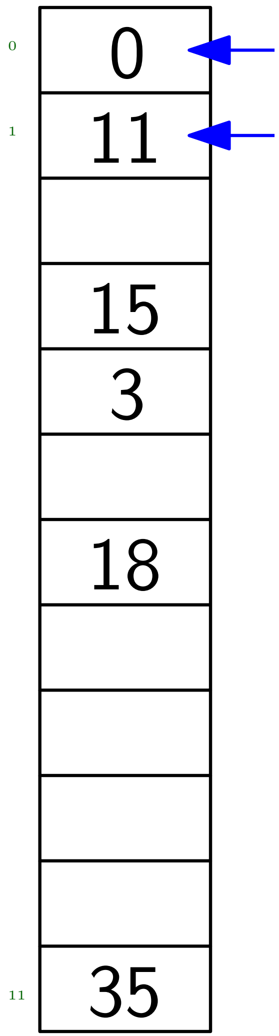

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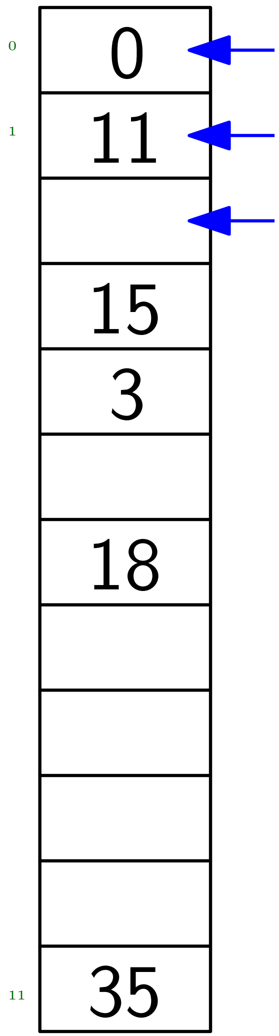
Only for illustration. This is a BAD hash functionn

Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



As example, let $h(x) = x \bmod m$ with $m = 12$.

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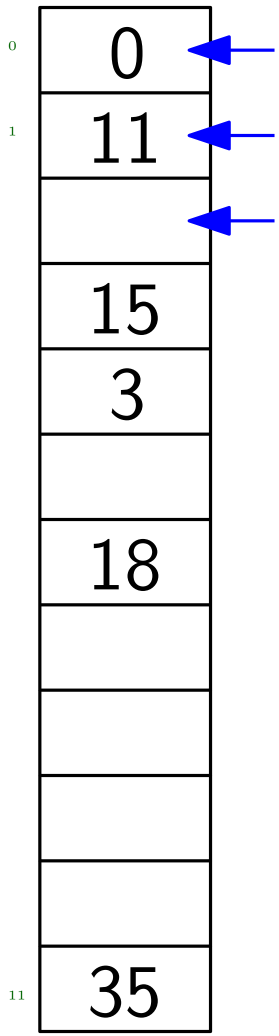
Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



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Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	Does not exist

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Easy to code but suffers from primary clustering.
Long runs build up, increasing average search time

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As example, let $h(x) = x \bmod m$ with $m = 12$.

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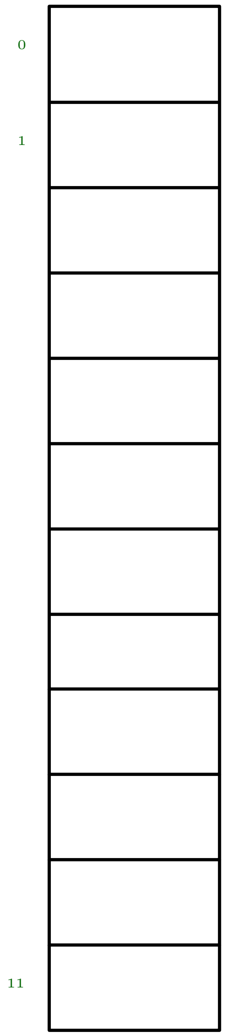
Easy to code but suffers from primary clustering.
Long runs build up, increasing average search time

One fix is to change probe sequence to no longer be linear.

Open Addressing: Quadratic Probing

$h' : U \to \{0, 1 \ldots, m-1\}$

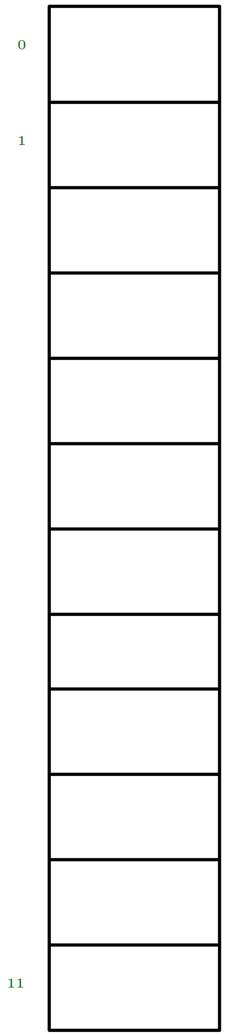


- Hash Function is $h(x,i) = \left(h'(x) + c_1 i + c_2 i^2\right) \bmod m$ where $h'(x)$ is original hash function and c_1, c_2 fixed constants.

- As example we will set $h'(x) = x \bmod 12$, $c_1 = 0$ and $c_2 = 1$ so $h(x,i) = \left(x + i^2\right) \bmod 12$

Open Addressing: Quadratic Probing

$h' : U \to \{0, 1 \ldots, m-1\}$



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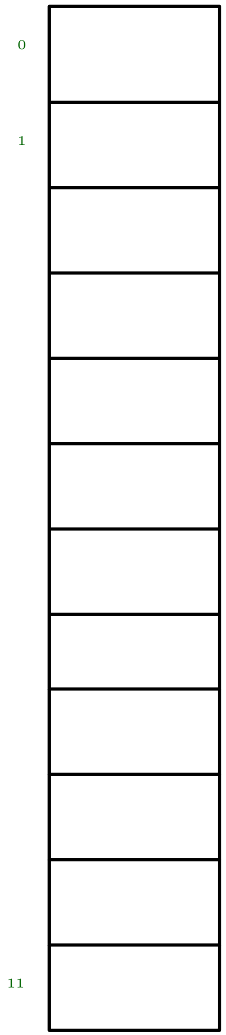

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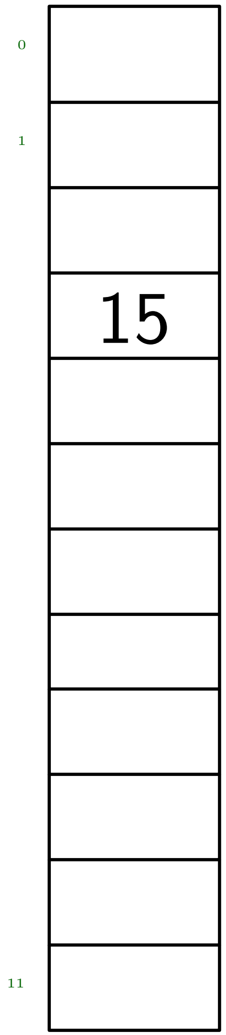

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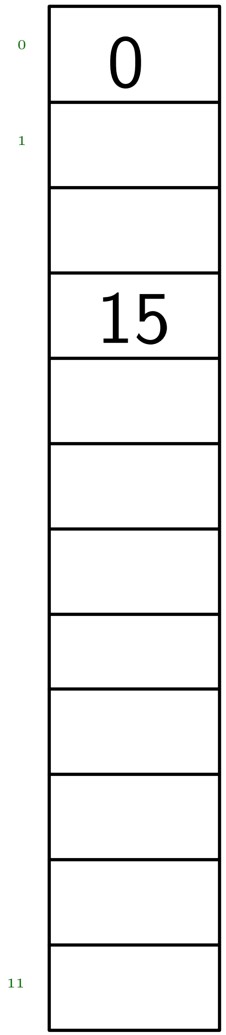

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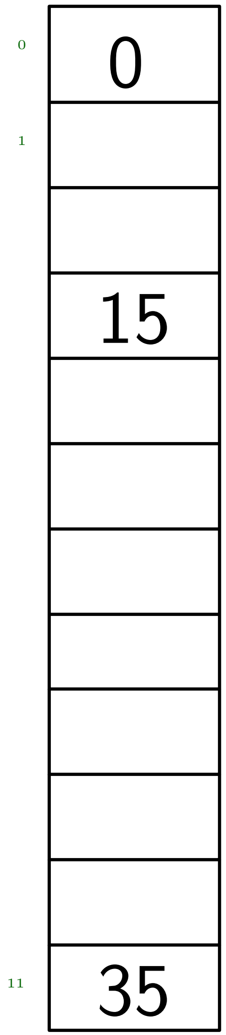

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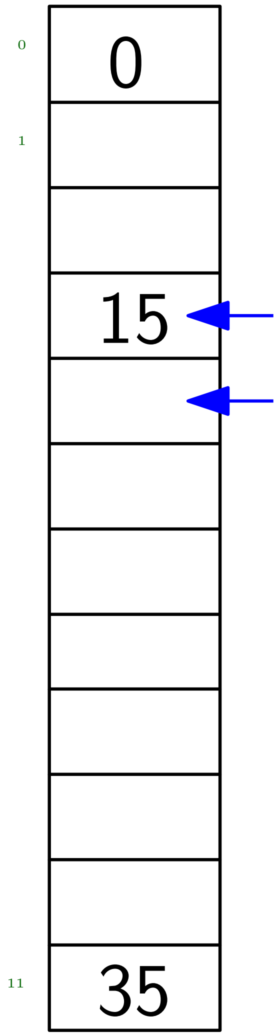

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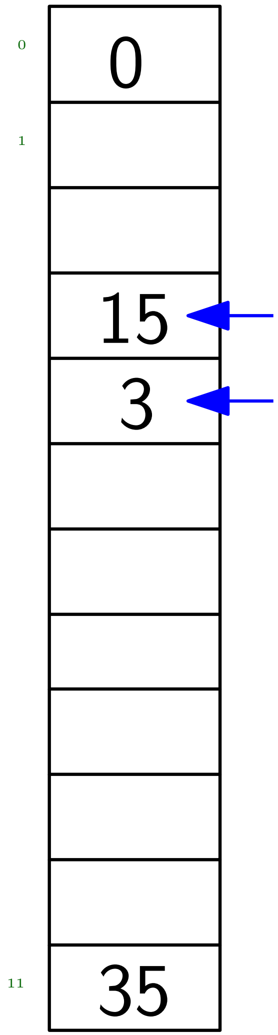

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Open Addressing: Quadratic Probing

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0	0
1	
	15
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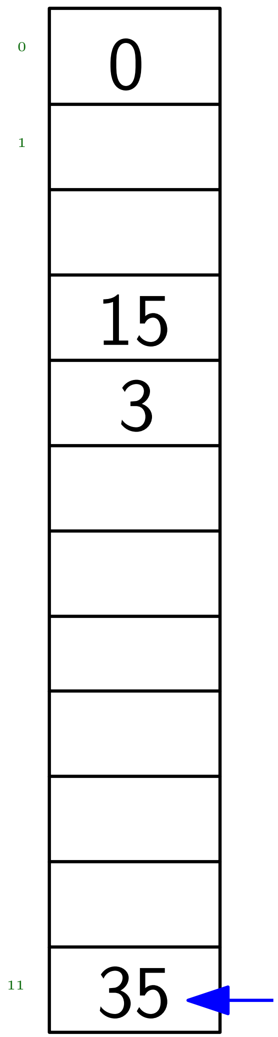

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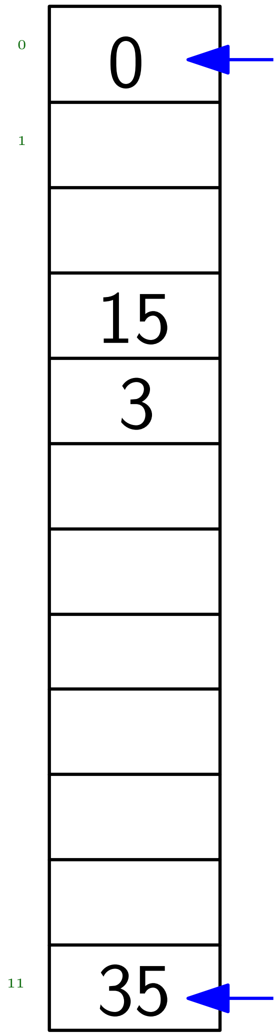

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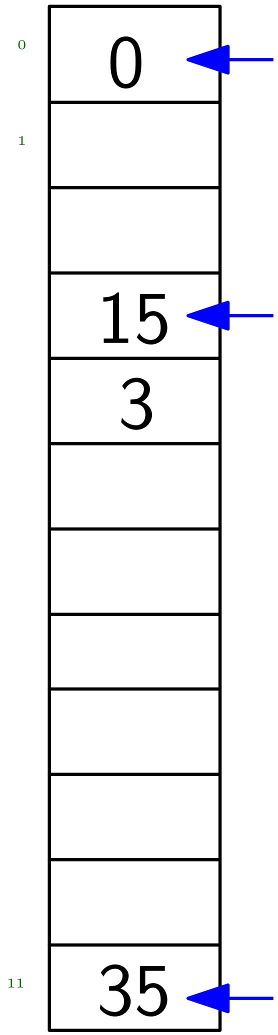

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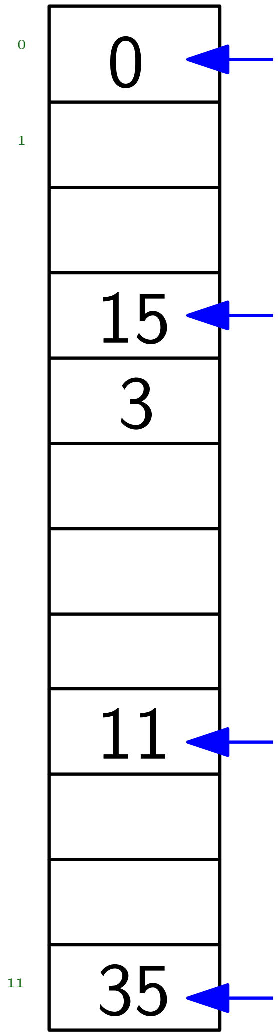

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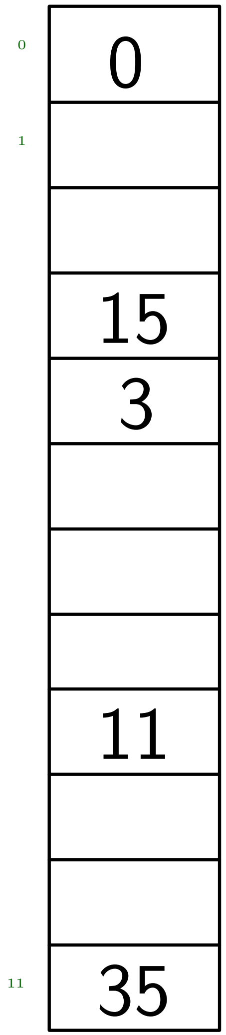

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Open Addressing: Quadratic Probing

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0	0
1	
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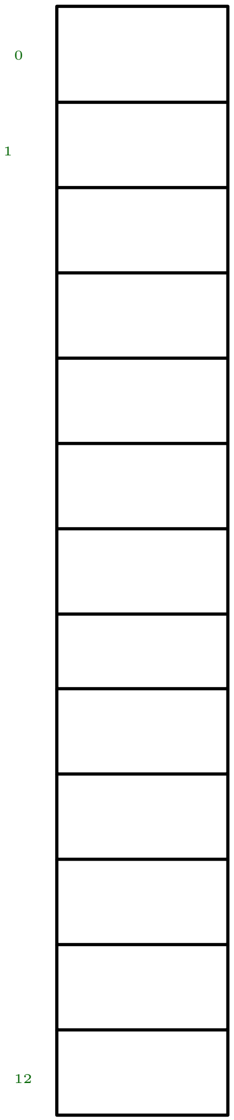
Open Addressing: Double Hashing

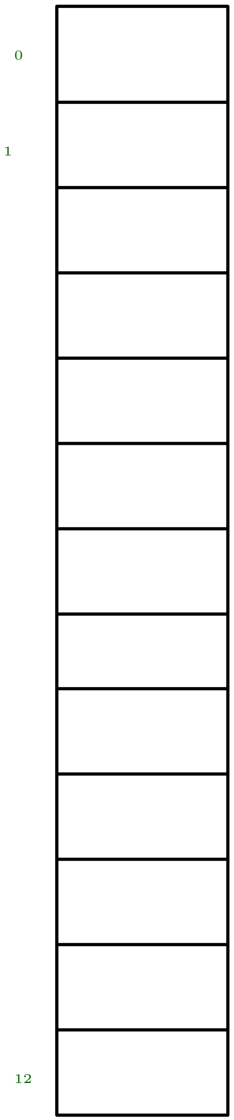
$h' : U \to \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x, i) = \big(h_1(x) + ih_2(x)\big) \bmod m$
- $h_1(x)$ and $h_2(x)$ are auxillary hash functions
- Note that (unlike before) probe sequence depends upon x
- In order for probe sequence to check entire table, must have $h_2(x)$ be relatively prime to m, e.g.,
  - m a power of 2; $h_2(x)$ always odd
  - m prime; $h_2(x)$ always less than m.

Open Addressing: Double Hashing

$h' : U \rightarrow \{0, 1 \ldots, m-1\}$



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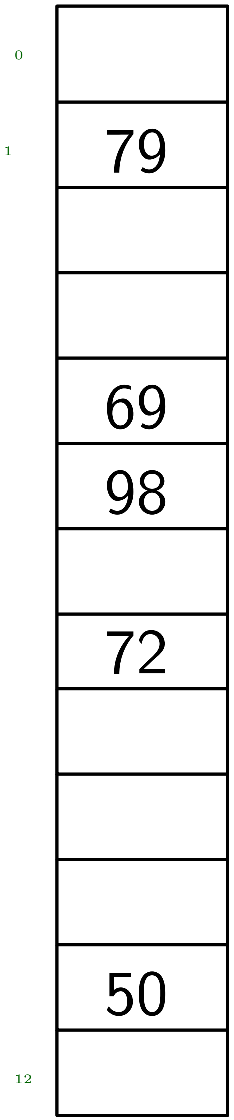
Example: $m = 13$
$h_1(x) = x \bmod m$
$h_2(x) = 1 + (x \bmod 11)$

Open Addressing: Double Hashing

$h' : U \rightarrow \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x, i) = \big(h_1(x) + i h_2(x)\big) \bmod m$
- $h_1(x)$ and $h_2(x)$ are auxillary hash functions
- Note that (unlike before) probe sequence depends upon x
- In order for probe sequence to check entire table, must have $h_2(x)$ be relatively prime to m, e.g.,
  - m a power of 2; $h_2(x)$ always odd
  - m prime; $h_2(x)$ always less than m.

Example: $m = 13$
$h_1(x) = x \bmod m$
$h_2(x) = 1 + (x \bmod 11)$

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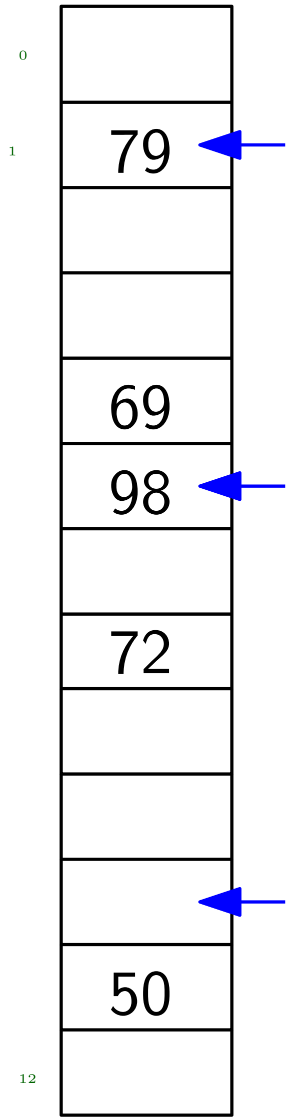

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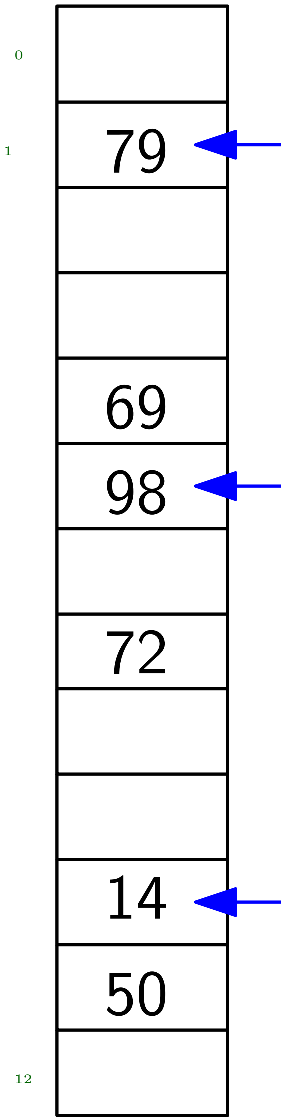

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Open Addressing: Analysis Results

We have seen 3 different open addressing collision resolution methods:

- Linear Probing: $h(x,i) = (h'(x) + 1) \bmod m$
- Quadratic Probing $h(x,i) = (h'(x) + c_1 i + c_2 x^2) \bmod m$
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For analysis, we often assume uniform hashing.
This states that the probe sequence

$$h(x,0),\ h(x,1), h(x,2),\ \ldots,\ h(x,m)$$

is equally likely to be any of the $m!$ permutations of $1, 2, \ldots, m$.

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Uniform Hashing is not actually realizable.
The more random our probe sequence, though, the closer actual behavior is to theory.

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Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends

Hash Functions & Universal Hashing

Returning to chained hashing, notice that our analysis assumed that the hashed keys were equally distributed among the slots.

- If all keys hashed to same slot, performance would be very bad.
- If the hash function $h(x)$ is given in advance and $n << U$, very easy to construct bad case in which all keys map to the same slot.

- We sidestep this issue by choosing a *random hash function*
- More specifically, we will have a *collection* of hash functions $\mathcal{H}$

- Given any set of keys, we will choose a random hash function $h \in \mathcal{H}$ and then hash using $h(x)$.

- *On average*, the set of n keys will be hashed so that each slot will get $O(n/m) = O(\alpha)$ keys.

- Our $O(1+\alpha)$ sucessful/unsuccessful search times for chained hashing will then hold on average.

- One class $\mathcal{H}$ of hash functions having this property are the *Universal* ones; they permit *Universal Hashing*

Universal Hashing

- Let $\mathcal{H}$ be a set of hash functions, such that each $h \in \mathcal{H}$ maps
  $h : U \to \{0, 1, \ldots, m-1\}$
- $\mathcal{H}$ is Universal if, for every two different keys k, ℓ, the number of hash functions in $\mathcal{H}$ that map k, ℓ to the same slot is at most $|\mathcal{H}|/m$, i.e.,

$$\forall k \neq \ell \in U, \quad |\{h \in \mathcal{H} \,:\, h(k) = h(\ell)\}| \leq \frac{|\mathcal{H}|}{m}.$$

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Let $k_1, k_2, \ldots, k_n$ be the n keys. Let i be any fixed index.
Then, for $j \neq i$, if $h \in \mathcal{H}$ is chosen uniformly at random,

$$\Pr(h(k_i) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}.$$

From linearity of expectation, if $h \in \mathcal{H}$ is chosen uniformly at random, average # of other keys mapping to the same slot as k_i is then

$$\sum_{j \neq i; 1 \leq j \leq n} \Pr(h(k_i) = h(k_j)) \leq \frac{n-1}{m} < \alpha$$

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Similarly, if k is not one of the n keys then, for all j,
$\Pr(h(k) = h(k_j)) \leq \frac{1}{|\mathcal{H}|}\frac{|\mathcal{H}|}{m} = \frac{1}{m}$ and average $\#$ of keys mapping to same slot as k is $\sum_{j=1}^{n} \Pr(h(k) = h(k_j)) \leq n/m = \alpha$

Construction of Universal Hash Functions

- Choose prime $p > U$
- Set $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ and $Z_p = \{0, 1, 2, 3, \ldots, p-1\}$
- Define

$$\forall a \in Z_p^*,\, b \in Z_p, \quad h_{a,b}(x) = \Big((ax + b) \bmod p \Big) \bmod m$$

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$$h_{3,4}(8) = ((3 \cdot 8 + 4) \bmod 17) \bmod 6 = 5$$

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Proof: Need to show that for all $k \neq \ell$, number of pairs (a, b) with $h_{a,b}(k) = h_{a,b}(\ell)$ is $\leq p(p-1)/m$

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$\forall a \in Z_p^*,\, b \in Z_p, \quad h_{a,b}(x) = \Big((ax+b) \bmod p \Big) \bmod m$

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(1) Let $k \neq \ell \in U$. For given $(a, b) \in Z_p^* \times Z_p$ set

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(2) Every different (a, b) pair generates a unique (r, s) pair

This is because for a given (r, s) pair we can (uniquely) solve

$$a = (r - s)(k - \ell)^{-1} \bmod p, \quad b = (r - ak) \bmod p.$$

where $(k-\ell)^{-1}$ is the multiplicative inverse base p. Since, for fixed p, k, ℓ, we must have $r \neq s$, there are are $p(p-1)$ (r, s) pairs. Since there are also $p(p-1)$ (a, b) pairs, there is a one-one correspondence between them, with every (a, b) pair generating a diffferent (r, s).

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$\boxed{\Rightarrow \mathcal{H} \text{ is Universal}}$

Universal Hashing: Wrap Up

- Just saw that the set of Hash functions

$$\mathcal{H} = \left\{h_{a,b} \, : \, a \in Z_p^*, \, b \in Z_p\right\}$$

  is *Universal*

- This implies that for *any* set of n keys $K = \{k_1, k_2, \ldots, k_n\}$, an effective way of storing the keys is to
  - Choose a random pair (a, b) uniformly at random from the $p(p-1)$ pairs in $Z_p^* \times Z_p$
  - Hash the items in K using hash function $h_{a,b}$

- Because $\mathcal{H}$ is Universal, average time for storing the data will be $O(n\alpha)$ where $\alpha = n/m$ is the load factor

- Average time for doing a search will be $(1 + \alpha)$

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Odds & Ends

- Hashing first recognized as a technique in the 1950's
- Comes from English word implying *chop and mix*
- Many different types of hashing for dictionary storage out there. This introduction only scratched the surface
- A *Cryptographic Hash Function* is a hash function that is *almost* impossible to invert efficiently, i.e, given $h(x)$ very dfifficult to find x.
  - Almost by necessity requires that function h distributes keys pretty "randomly" over $0, 1, 2, \ldots, m$. If not true, then would have first step towards guessing value of x that produces $h(x)$.
  - Example: Password protection. System password file only stores $h(\text{password})$ and not the password itself.
    * When user logs in and types password p, system checks $h(p)$ against file.
    * If an attacker steals the file it wouldn't be helpful, since attacker can't invert hashed password to get original one