COMP170 – Fall 2008
Midterm 1 Review

Question 1

8 men and 8 women are invited to a party at which they are seated at a long rectangular table

M_1	M_6	M_7	M_2	M_5	M_8	M_3	M_4
1	2	3	4	5	6	7	8
W_2	W_5	W_6	W_1	W_8	W_7	W_3	W_4

(i)

M_1	M_6	M_7	M_2	M_5	M_4	M_3	M_8
1	2	3	4	5	6	7	8
W_1	W_5	W_6	W_2	W_8	W_7	W_3	W_4

(ii)

a) How many different ways are there to seat the n guests at n seats?

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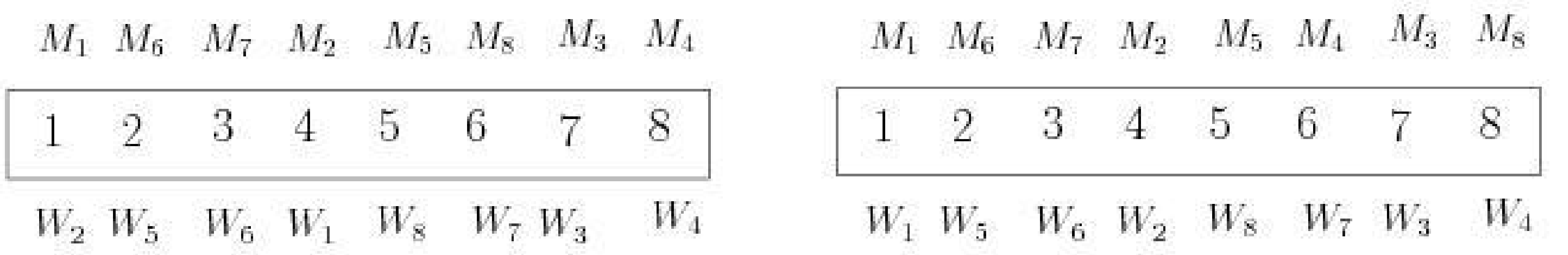

(i) (ii)

a) How many different ways are there to seat the n guests at n seats?

There are $8!$ ways to seat the men and $8!$ ways of seating the women. By the product principle,

$$(8!)^2 = 1,625,702,400$$

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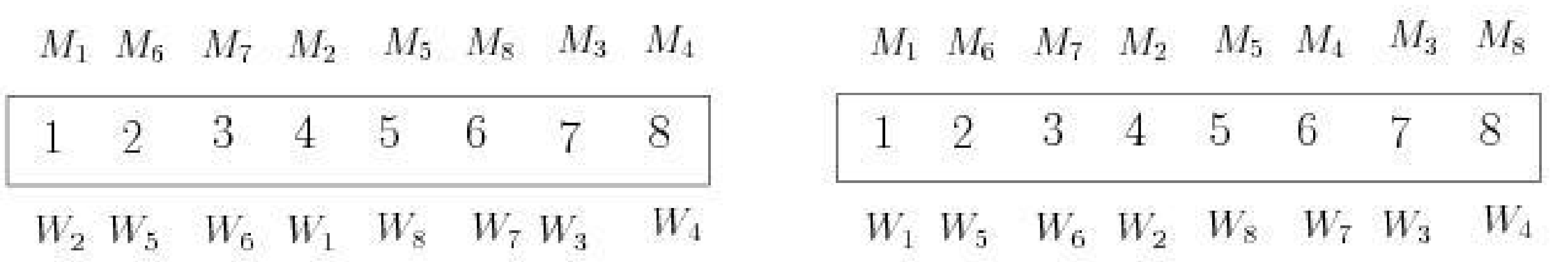

(i) (ii)

b) Suppose one man and one woman will not sit across from each other. How many ways are there to seat all of the men and women?

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b) Suppose one man and one woman will not sit across from each other. How many ways are there to seat all of the men and women?

Suppose W_1 and M_1 quarelled.
There are $8!$ ways of seating the men; then 7 possible locations to seat W_1, and after that, 7! ways of seating the remaining 7 women.
By the product principle,

$$8! \times 7 \times 7! = 1,422,489,600$$

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c) Suppose if a man and a woman are married, they sit exactly across from each other. If there are 3 married couple, how many ways are there to seat everyone?

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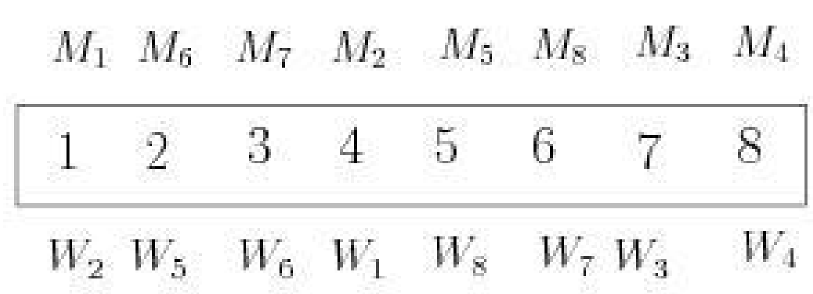

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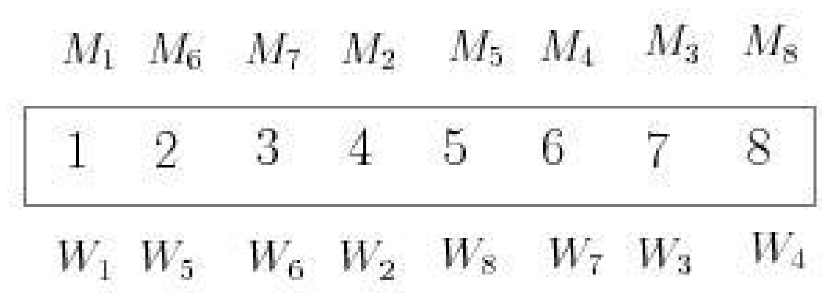

(ii)

c) Suppose if a man and a woman are married, they sit exactly across from each other. If there are 3 married couple, how many ways are there to seat everyone?

There are $8!$ ways of seating the men. After the men are seated, the 3 wives must sit across from their husbands. The remaining 5 women have $5!$ seating arrangements. So, by the product principle,

$$8! \times 5! = 4,838,400$$

Question 2

Give a combinatorial proof of the identity, for all $n \geq 11$.

$$\binom{n}{3}\binom{n-3}{5}\binom{n-8}{3} = \binom{n}{3}\binom{n-3}{3}\binom{n-6}{5}$$

Note: An algebraic proof of this identity will not be accepted as a solution.

Question 2

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Consider the problem of how to color n items so that 3 are red, 5 are green, 3 are blue and the remaining $n-11$ are yellow.
The left hand side obviously counts this

Consider the problem of how to color n items so that 3 are red, 3 are blue, 5 are green and the remaining $n-11$ are yellow.
The right hand side oObviously counts this

Notice that both problems are the same.
We only change the order in which we do the colorings.

Question 3

Starters (A List) 10 Types, $A_1, A_2, \ldots, A_{10}$

Main Courses (B List) 15 Types, $B_1, B_2, \ldots, B_{15}$

(a) How many different menus (3 from A, 2 from B) can you create?

(b) Suppose the restaurant lets you choose starters as main courses. Note, you can choose 2, 1 or 0 from A as your main courses. If you choose an item from the A list as a main course, then you cannot also choose it as a starter. Now how many different menus are there?

Examples:
$\{A_1, A_3, A_5, B_4, B_6\}$ is a legal menu for both questions (a) and (b)
$\{A_1, A_3, A_5, A_7, B_6\}$ is a legal menu for question (b) but not for (a).

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Main Courses (B List) 15 Types, $B_1, B_2, \ldots, B_{15}$

(a) How many different menus (3 from A, 2 from B) can you create?

$$\binom{10}{3}\binom{15}{2}$$

Question 3

Starters (A List)	10 Types, $A_1, A_2, \ldots, A_{10}$
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$$\binom{10}{3}\binom{15}{2} + \binom{10}{4}\binom{15}{1} + \binom{10}{5}$$

The solution is split into whether we choose 0, 1 or 2 A items as main courses.

Question 4

Let $n > 2$ be an integer and $a \in Z_n$. For each of the two following statements either
(i) prove that the statement is correct for all such a and n or
(ii) give a counterexample. A counterexample would be a pair a, n for which the statement is false.

(a) If the equation $a \cdot_n x = 1$ has a solution in Z_n, then the equation $a \cdot_n x = 2$ has a solution in Z_n,

(b) If the equation $a \cdot_n x = 2$ has a solution in Z_n, then the equation $a \cdot_n x = 1$ has a solution in Z_n.

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(a) If the equation $a \cdot_n x = 1$ has a solution in Z_n, then the equation $a \cdot_n x = 2$ has a solution in Z_n,

True. In class we proved that

if $a \cdot_n x = 1$ has a solution $x = a'$ in Z_n then $a \cdot_n x = b$ has a solution $x = a' \cdot_n b$.

Letting $b = 2$ proves the statement.

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(b) If the equation $a \cdot_n x = 2$ has a solution in Z_n, then the equation $a \cdot_n x = 1$ has a solution in Z_n.

False. For a counterexample, consider $n = 4$ and $a = 2$.

Then $a \cdot_n x = 2$ has solution $x = 1$ but $a \cdot_n x = 1$ has no solution.

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False. For a counterexample, consider $n = 4$ and $a = 2$.

Then $a \cdot_n x = 2$ has solution $x = 1$ but $a \cdot_n x = 1$ has no solution.

One way to see that $2 \cdot_4 x = 1$ has no solution is to note that $\gcd(4, 2) = 1 \neq 1$.

Question 5

(a) Does there exist an x in Z_{79} that solves

$$53 \cdot_{79} x = 1?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove the fact.

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Yes. $x = 3$ solves the equation.
One way to find this would be to use the Extended GCD algorithm to calculate

$$1 = 3 \cdot 53 - 2 \cdot 79.$$

Question 5

(b) Does there exist an x in Z_{147} that solves

$$12 \cdot_{147} x = 7?$$

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$$12 \cdot_{147} x = 7?$$

If yes, give the value of x (it is not necessary to show your work). If no, prove the fact.

No. If $12 \cdot_{147} x = 7$ then there is some integer q such that

$$12x = 147q + 7$$

or

$$3(4x - 49) = 7.$$

Since the left side of this equation is divisible by 3 and the right side isn't, this is impossible.

Question 6

Recall that $S_n = \{1, 2, 3, \ldots, n\}$.

(a) Let $n \geq 2$. How many onto functions are there from S_n to S_2?

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(a) Let $n \geq 2$. How many onto functions are there from S_n to S_2?

One way to define such an onto function is to let $X \subset S_n$ be all of the $x \in S_n$ such that $f(x) = 1$ (while if $x \notin X$ then $f(x) = 2$).

There is a bijection between onto-functions and the possible X.

The only constraint on X is that $X \neq \emptyset$ and $X \neq S_n$.

Since the total number of subsets of S_n is 2^n, and only two are not allowed, the answer is

$$2^n - 2.$$

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Recall that, for $i \in S_4$, $f^{-1}(i)$ is the set of all items $x \in S_6$ such that $f(x) = i$. This lets us note that f is uniquely determined by the ordered partition of S_6

$$\left(f^{-1}(1),\ f^{-1}(2), f^{-1}(3), f^{-1}(4)\right)$$

in which no set is empty So, we want to count the number of such ordered partitions.

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Recall that $S_n = \{1, 2, 3, \ldots, n\}$.

(b) How many onto functions are there from S_6 to S_4?

There are two possiblities.

- One of the $f^{-1}(i)$ contains 3 items and the other 3 contain one item. There are $\binom{6}{3}$ ways of choosing the 3 items and once they are chosen there are $4!$ ways of ordering the sets.
- Two of the $f^{-1}(i)$ contain 2 items and the other 2 contain one item. There are $\frac{1}{2}\binom{6}{2}\binom{4}{2}$ ways to choose these sets and then $4!$ ways of ordering them.

The total answer is therefore

$$\binom{6}{3}4! + \frac{1}{2}\binom{6}{2}\binom{4}{2}4! = 1560.$$

Question 7

Consider the following statement:

$$gcd(k, k - j) = gcd(k, k + j).$$

Is this statement always true for k, j with $k > j > 0$? Either prove that it is true for all k, j with $k > j > 0$, or give values for k, j with $k > j > 0$ such that $gcd(k, k-j) \neq gcd(k, k + j)$.

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(a) suppose that d is a common divisor of k and $k-j$. Then there are q_1 and q_2 such that $k = q_1 d$ and $k-j = q_2 d$. Then,

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(a) suppose that d is a common divisor of k and $k-j$. Then there are q_1 and q_2 such that $k = q_1 d$ and $k-j = q_2 d$. Then,

$$k + j = k - [(k - j) - k] = (2q_1 - q_2)d$$

so d is a divisor of $k + j$.

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The combination of (a) and (b) implies that the set of common divisors of k and $k+j$ is exactly the same as the set of common divisors of k and $k-j$.

Since the sets of common divisors are the same, the greatest common divisor of k and $k+j$ is the same as the greatest common divisor of k and $k-j$.

There are actually many different proofs of the statement. For one alternative proof recall that we proved in class (Lemma 2.13)

If j, k, q, and r are nonnegative integers such that $k = jq + r$, then $\gcd(j,k) = gcd(r,j)$.

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Note: This was not the "intended" solution but came, slightly modified, from one of the test books.