ASSIGNMENT 4: COMP2711H

FALL 2015

Q1 Assume that the equation

$$x^\ell - c_1x^{\ell-1} - c_2x^{\ell-2} - \cdots - c_{\ell-1}x - c_\ell = 0$$

has distinct roots $r_1, r_2, \ldots, r_\ell$, where all c_i are real numbers and $c_\ell \neq 0$.

Define a sequence $(s_i)_{i=0}^{\infty}$ by

$$s_i = \alpha_1 r_1^i + \alpha_2 r_2^i + \ldots + \alpha_\ell r_\ell^i \text{ for integers } i \geq 0,$$

with initial conditions $s_0, s_1, \cdots, s_{\ell-1}$, where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are constants.

Show that the sequence $(s_i)_{i=0}^{\infty}$ satisfies the following linear recurrence relation:

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell.$$

(15 marks)

Q2 Solve the following linear recurrence relation

$$s_i = 6s_{i-1} - 11s_{i-2} + 6s_{i-3} \text{ for all } i \geq 3$$

with initial conditions $s_0 = 2$, $s_1 = 5$ and $s_2 = 15$. (15 marks)

Q3 Solve the following linear recurrence relation

$$s_i = 4s_{i-1} - 4s_{i-2} \text{ for all } i \geq 2$$

with initial conditions $s_0 = -2$ and $s_1 = 2$. (15 marks)

Q4 Let $(s_i)_{i=0}^{\infty}$ be a sequence with least period ℓ.

(a) Show that the sequence has a linear homogeneous recurrence relation. (5 marks)

(b) Find a linear homogeneous recurrence relation of degree as small as possible for the sequence. (You need to write down the linear recurrence relation of the sequence explicitly.) (10 marks)

Q5 Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and $a_n \neq 0$. Prove that

$$x^n \text{ is } O(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0).$$

(15 marks)

Q6 Define a sequence $(s_i)_{i=1}^{\infty}$ recursively as follows:

$$s_i = 2s_{\lfloor i/2 \rfloor} + i, \quad i \geq 2$$

with the initial condition $s_1 = 0$. Use strong mathematical induction to prove that s_n is $O(n \log_2 n)$ (recall that sequences are in fact functions defined on the set of nonnegative integers). (15 marks)

Q7 Show that

$$f(n) := \frac{2n}{3} + \frac{2n}{3^2} + \frac{2n}{3^3} + \cdots + \frac{2n}{3^n}$$

is $\Theta(n)$. (10 marks)

Date: Handed out on October 7, due on October 14 at the *beginning* of the lecture.