Closest Pairs

COMP 3711H - HKUST
Version of 27/11/2014
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Outline

- Introduction
- The Gridding Lemma
- An $O(n \log n)$ sweep line algorithm
- An $O(n)$ randomized algorithm
  - Hashing a point set
  - The algorithm

Introduction

- Most algorithms seen so far have assumed that items have 1-dimensional weights or keys, i.e., they can be ordered.

- There is often a need, e.g., graphics, data bases, to manipulate multi-dimensional data. The area of algorithms that treats geometric (multi-dimensional) data is called computational geometry.

- In this set of slides, we will see two algorithms for solving the 2-dimensional closest-pair problem.

- The first algorithm will take $O(n \log n)$ worst case time to find the closest pair among n points. The second algorithm will be randomized and only require $O(n)$ average time.

Problem Definitions

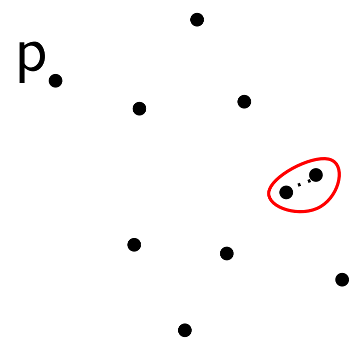
- Problem input is $P = \{p_1, p_2, \ldots, p_n\}$, a set of n 2-dimensional points, where $p = (p.x, p.y)$

- distance between p and p' is
$$d(p,p') = \sqrt{(p.x - p'.x)^2 + (p.y - p'.y)^2},$$

- Set $d(p,P) = \min_{p' \in P - \{p\}} d(p,p')$ to be distance of nearest point to p.

- Set $\delta(P) = \min_{p \in P} d(p,P)$ to be the *closest pair distance* in P.

- The Closest-Pair Problem is to find $p, p' \in P$ such that $d(p,p') = \delta(P)$



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The Gridding Lemma

The Gridding Lemma:

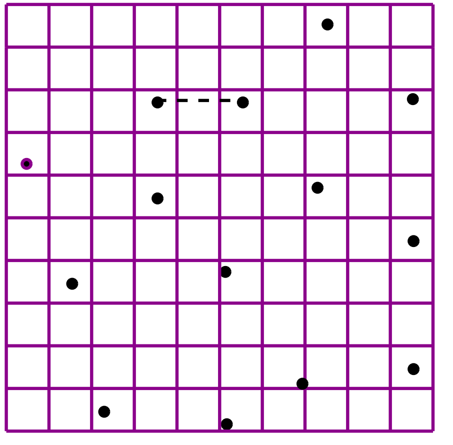
Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of points and
$\delta = \delta(P)$ their closest-pair distance.
Grid the plane into $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes using horizontal and vertical lines. Then

(a) No box holds more than 1 point

(b) If $d(p, p') = \delta$ then the boxes containing p and p'

are within two horizontal and two vertical boxes of each other.

Proof:



(a) if two points are in same box they can be at most $\delta/\sqrt{2} < \delta$ away from each other, contradicting def of δ.

(b) if two points p, p' are more than two boxes horizontally (or vertically) away from each other then

$|p.x - p'.x| > \delta$ $(|p.y - p'.y| > \delta)$, so $d(p, p') > \delta$

contradicting definition of δ.

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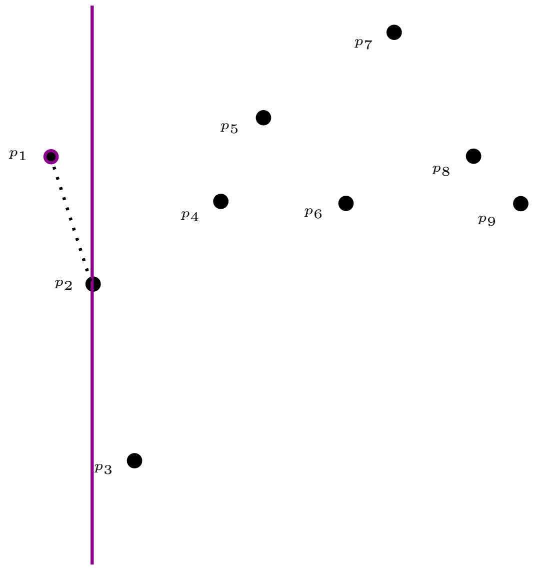
A Sweep Algorithm



$\delta(P_2) = d(p_2, p_1)$

Sort the points by x coordinate.

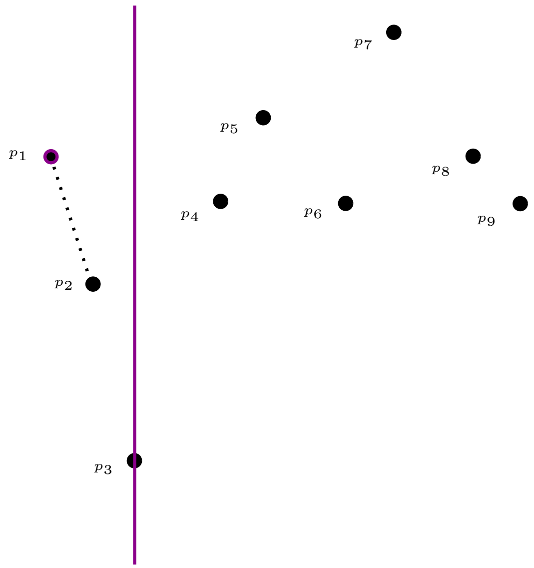
Set $P_i = \{p_1, \ldots, p_i\}$.

Note $\delta(P_i) = \min(\delta(P_{i-1}), d(p_i, P_{i-1}))$

Algorithm sweeps a vertical line through points, i.e., $p_2, p_3, \ldots, p_n$, at each step using equation to calculate $\delta(P_i)$.

At end, $\delta = \delta(P_n)$

A Sweep Algorithm



$\delta(P_2) = d(p_2, p_1)$
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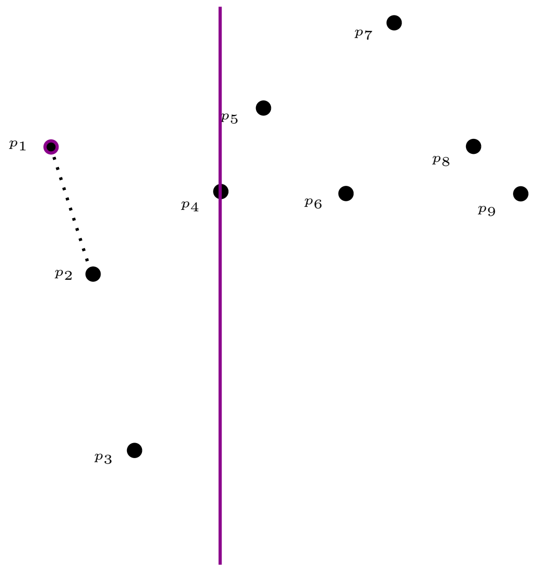

$\delta(P_2) = d(p_2, p_1)$
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Sort the points by x coordinate.

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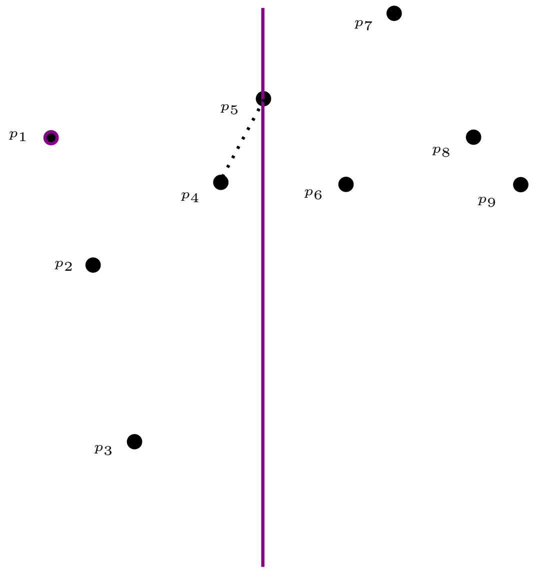
A Sweep Algorithm



$\delta(P_2) = d(p_2, p_1)$
$\delta(P_3) = d(p_2, p_1)$
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$\delta(P_5) = d(p_5, p_4)$

Sort the points by x coordinate.

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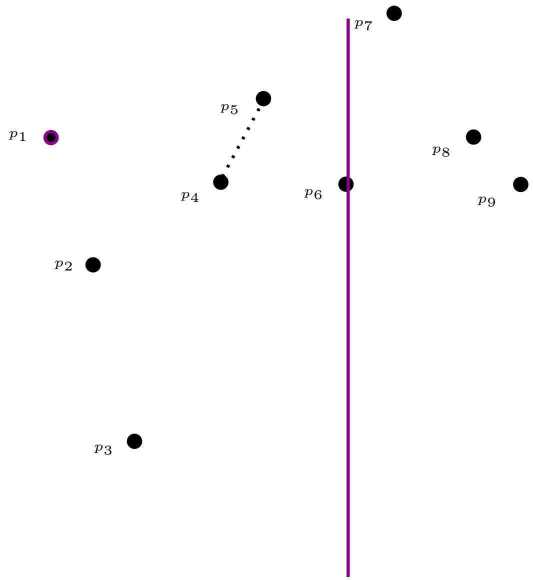
A Sweep Algorithm



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Sort the points by x coordinate.

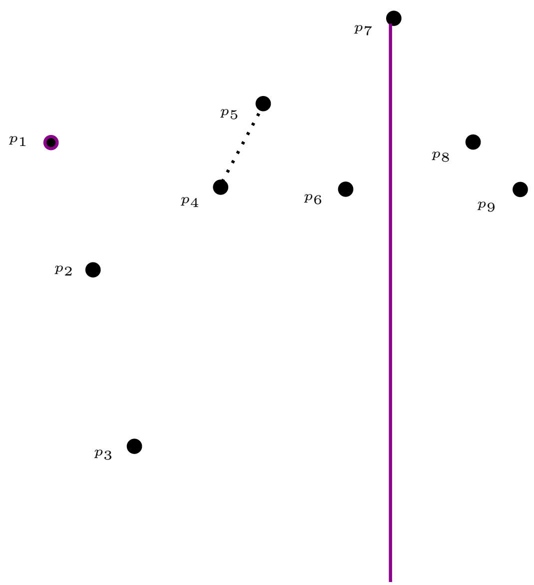
Set $P_i = \{p_1, \ldots, p_i\}$.

Note $\delta(P_i) = \min(\delta(P_{i-1}), d(p_i, P_{i-1}))$

Algorithm sweeps a vertical line through points, i.e., $p_2, p_3, \ldots, p_n$, at each step using equation to calculate $\delta(P_i)$.

At end, $\delta = \delta(P_n)$

A Sweep Algorithm



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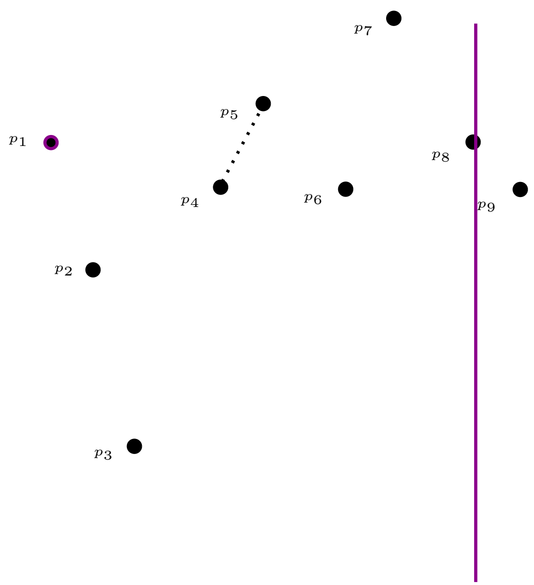
At end, $\delta = \delta(P_n)$

A Sweep Algorithm



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Sort the points by x coordinate.

Set $P_i = \{p_1, \ldots, p_i\}$.

Note $\delta(P_i) = \min(\delta(P_{i-1}), d(p_i, P_{i-1}))$

Algorithm sweeps a vertical line through points, i.e., $p_2, p_3, \ldots, p_n$, at each step using equation to calculate $\delta(P_i)$.

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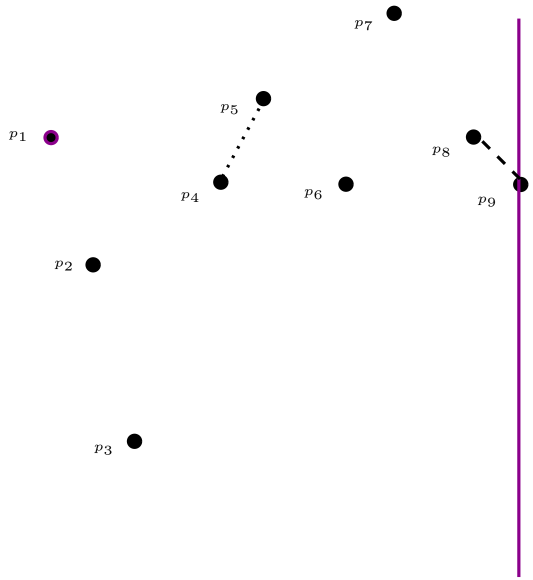
A Sweep Algorithm



$\delta(P_2) = d(p_2, p_1)$
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$\delta(P_8) = d(p_5, p_4)$
$\delta(P_9) = d(p_9, p_8)$

$\delta = \delta(P_9) = d(p_9, p_8)$

Sort the points by x coordinate.

Set $P_i = \{p_1, \ldots, p_i\}$.

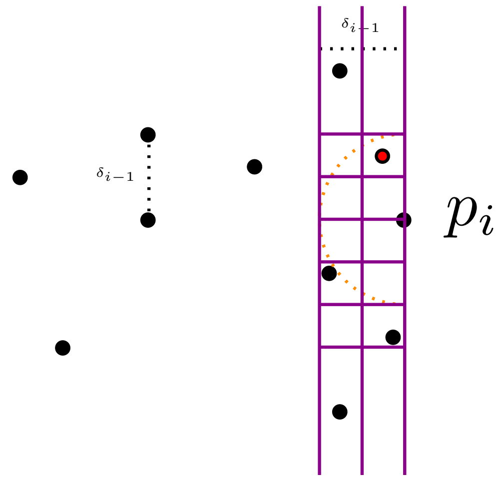
Note $\delta(P_i) = \min(\delta(P_{i-1}), d(p_i, P_{i-1}))$

Algorithm sweeps a vertical line through points, i.e., $p_2, p_3, \ldots, p_n$, at each step using equation to calculate $\delta(P_i)$.

At end, $\delta = \delta(P_n)$

Updating One Sweep Step

$\delta(P_i) = \min(\delta(P_{i-1}), d(p_i, P_{i-1}))$
Set $\delta_i = \delta(P_i)$ and $d_i = d(p_i, P_{i-1})$.



If $d_i < \delta_{i-1}$ then
$d_i = d(p_i, p_j)$ for some $j < i$ with

$$|p_i.x - p_j.x| \le d_i = \delta_i < \delta_{i-1}$$

If we grid the plane with a $\frac{\delta_{i-1}}{2} \times \frac{\delta_{i-1}}{2}$ grid with p_i as the origin, then p_j must be within 2 horizontal grid lines and 2 vertical grid lines of p_j.

Gridding Lemma tells us there is at most one point per box.

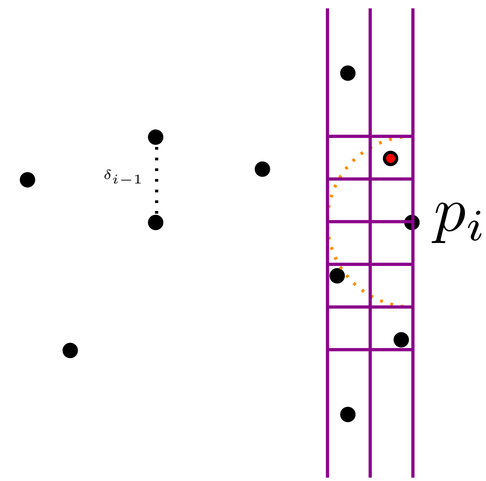
Let $S_i = \{p \in P_{i-1} \, : \, p_i.x - p.x < \delta_{i-1}\}$

This implies
if $d_i < \delta_{i-1}$ then
$d_i = (p_i, p)$ where
(a) $p \in S_i$ and
(b) p is one of the 4 points above p_i in S_i
or
p is one of the 4 points below p_i in S_i.

Updating One Sweep Step (ii)

$\delta_i = \min(\delta_{i-1}, d(p_i, P_{i-1}))$

Let $S_i = \{p \in P_{i-1} \,:\, p_i.x - p.x < \delta_{i-1}\}$



if $d_i < \delta_{i-1}$ then $d_i = (p_i, p)$ where
(a) $p \in S_i$ and
(b) p is one of the 4 points above p_i in S_i or
p is one of the 4 points below p_i in S_i.

Suppose we knew δ_{i-1} and had the points in S_i kept in a balanced binary search tree ordered by increasing y coordinate.

We could then calculate δ_i in $O(\log n)$ time as follows

- In S_i, find the 4 points above p_i and four points below it
  This takes $O(\log n)$ time
- Find the distances between these (at most) 8 points and p_i.
  This takes $O(1)$ time. Call this minimum distance d'
  (note that there might be no points in S_i. In this case $d' = \infty$)
- Set $\delta_i = \min(\delta_{i-1}, d')$

The Full Sweep Algorithm

Let $S_i = \{p \in P_{i-1} \,:\, p_i.x - p.x < \delta_{i-1}\}$

We just developed the following algorithm to calculate $\delta_2, \delta_3, \ldots, \delta_n$

A Set $\delta_2 = d(p_1, p_2)$;
  Create a y-sorted balanced BST on $S_2 = \{p_1\}$

B For $i = 3$ to n
  1. Update the BST from storing S_{i-1} to storing S_i
  2. In $O(\log n)$ time use the BST storing S_i to calculate δ_i

Step (A) uses $O(1)$ time. Since step (B)(2) uses $O(\log n)$ time per step and it's called $O(n)$ times, all the calls to it use only $O(n \log n)$ time.

If we can implement step (B)(1) in $O(n \log n)$ total time the entire algorithm then is $O(n \log n)$.

The Final Piece of Analysis

The Problem: Given $p_1, p_2, \ldots, p_n$ sorted by nondecreasing x-coordinate and a sequence $\delta_2 \geq \delta_3 \geq \cdots \geq \delta_{n-1}$, iteratively construct balanced BSTs storing the sets $S_2, S_3, \ldots, S_n$ sorted by y-coordinate. Use only $O(n \log n)$ time in total.

For $k < i$, if $p_k \notin S_i$ then $p_{k-1} \notin S_i$ because

$$p_i.x - p_{k-1}.x \geq p_i.x - p_k.x \geq \delta_{i-1}$$

so S_i is a continuous sequence of points $p_k, p_{k+1}, \ldots, p_{i-1}$.

Furthermore, if $p_k \notin S_{i-1}$ then $p_k \notin S_i$, because

$$p_i.x - p_k.x \geq p_{i-1}.x - p_k.x > \delta_{i-2} \geq \delta_{i-1}$$

$\Rightarrow$ S_{i-1} can be updated to S_i by (a) adding p_{i-1} to the BST and then (b) walking rightwards from the leftmost point in S_{i-1}, removing points from the BST, until finding the new leftmost point in S_i.

Every point is added once and removed at most once, so we perform $O(n)$ inserts and $O(n)$ deletes from the BST, requiring only $O(n \log n)$ time

Odds & Ends

- This approach of *sorting the points by x then updating solution by moving from left to right* is a common one in computational geometry. It is called the sweep-line technique.

- It is possible to prove that $O(n \log n)$ is a lower bound for solving this problem in the algebraic decision tree model of computation (a generalization of the comparison tree model we used for sorting). So, our algorithm is optimal in this model.

- With randomization and the use of the floor function ($\lfloor x \rfloor$) this can be impoved to an $O(n)$ average case time randomized algorithm.

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A Randomized Algorithm

- In this section we will see a randomized algorithm for finding the closest pair in $O(n)$ average case time.
- Input is n two-dimensional points.
- The randomization will come from two places
  - The first step of the algorithm will be to randomly order the input. That is, it will choose one of the $n!$ possible orderings of the n points uniformly at random and label the points as $p_1, p_2, \ldots, p_n$ in that order
  - The only other place randomization is used will be in the use of (universal) hashing
- Before starting the main part of the algorithm we perform an $O(n)$ scan finding x_{min}, x_{max}, y_{min} y_{max}, the smallest and largest x and y coordinates among all of the points.

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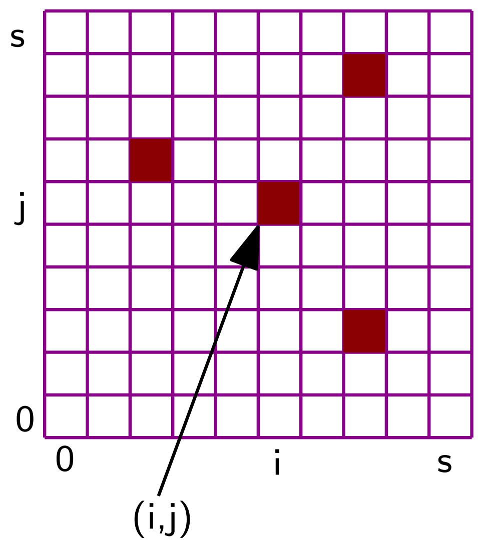
Hashing a Sparse Array

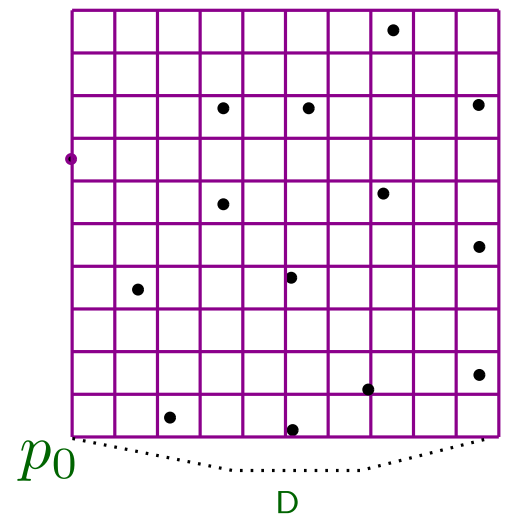
- Given an $s \times s$ array in which at most n items exist (are non-zero).



- We would like to be able to perform the following two operations in $O(1)$ time each.
  - Insert(x): Insert new item x into the array
  - Search(x): If x exists, return it (else return *nil*)
- In practice, array item is a pointer to a record. Sucessful search returns pointer to record's data.
- This can be solved using hashing
- Array index is (i, j) with $0 \leq i, j \leq s$. Can map array indices uniquely to integer $i + js < s^2 + 1$
- Applying **universal hashing** with $U = s^2 + 1$ and $m = n$ gives $O(1)$ average case insert and search

Bucketing: Hashing a Grid

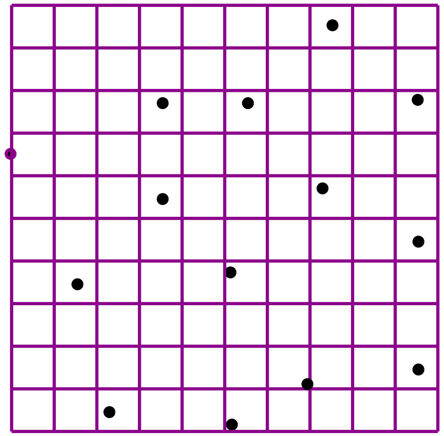
- Let $p_0 = (x_{\text{min}}, y_{\text{min}})$;
  $D = \max(x_{\text{max}} - x_{\text{min}}, y_{\text{max}} - y_{\text{min}})$.
- The entire point set is contained in the square of side D with lower-left corner p_0.
- A *grid of size* d will denote a horizontal/vertical grid with grid separation d between adjacent parallel lines and p_0 as the origin.



- The grid box labelled (X, Y) will contain all points (x, y) with $X \leq x \leq X + d$ and $Y \leq y \leq Y + d$
  (and appropriate tie breaking rule for points on the boundaries).
- (x, y) will be in the box $\left(\lfloor \frac{x - x_{\text{min}}}{d} \rfloor, \lfloor \frac{y - y_{\text{min}}}{d} \rfloor\right)$
- Maximum box coordinate is $\lceil D/d \rceil$

Hashing the Nearest Neighbor Grid

- (x, y) will be in the box $\left(\lfloor \frac{x - x_{\text{min}}}{d} \rfloor, \lfloor \frac{y - y_{\text{min}}}{d} \rfloor\right)$
- Maximum box coordinate is $\lceil D/d \rceil$
- Using the Array Hashing technique discussed before, we can hash the points by storing them in their corresponding array element (bucket). Non-empty buckets will return a pointer to list of points in that grid box.



- Inserting point into grid takes average $O(1)$ time. Searching for points in grid box takes average $O(1)$ time plus $\#$ of points in box.
- Let $P_i = \{p_1, p_2, \ldots, p_n\}$, $\delta_i = \delta(P_i)$ and grid size is $d = \delta_i/2$
- Gridding Lemma says that each grid box contains at most 1 point
- $\Rightarrow$ Finding points in a grid box takes average $O(1)$ time

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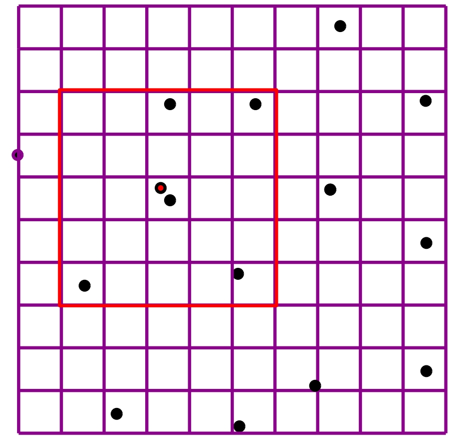
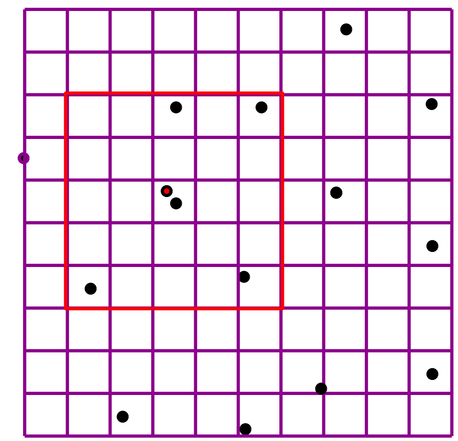
One Algorithmic Step



- Suppose we know δ_i and have hashed and stored the pointset P_i with grid size $d = \delta_i/2$.
- How can we calculate δ_{i+1} and, after completion, have P_{i+1} hashed and stored with grid size $\delta_{i+1}/2$?
- Note that $\delta_{i+1} = \min(\delta_i, d(p_{i+1}, P_i))$
- $\delta_{i+1} < \delta_i$ iff $d(p_{i+1}, P_i) = d(p_{i+1}, p) < \delta_i$
  for $p \in P_i$ with p in one of the 25 boxes
  within distance 2 of the box containing p_{i+1}
- Using the hash search, these 25 boxes and their contents (at most one point each) can be found in $O(1)$ time. In $O(1)$ further time, find d', the min distance between p_{i+1} and these ≤ 25 points.
- If $d' \geq \delta_i$ then $\delta_{i+1} = \delta_i$
  $\Rightarrow$ insert p_{i+1} into the grid in $O(1)$ average time
  If $d' < \delta_i$ then $\delta_{i+1} = d'$
  $\Rightarrow$ Throw away the entire old grid and
  in $O(i)$ average time, regrid P_{i+1} with the new $d = d'/2$

The Complete Algorithm



- Randomly label the points as $p_1, p_2, \ldots, p_n$
  Perform an $O(n)$ scan finding x_{min}, x_{max}, y_{min} y_{max}, the smallest and largest x and y coordinates among all of the points
- Set $\delta_2 = d(p_1, p_2)$.
  Hash P_2 into the grid with $d = \delta_2/2$.
- For $i = 3$ to n do

(A) Find the min distance d' between p_{i+1} and the ≤ 25 points in the 25 grid boxes surrounding p_{i+1}

(B) If $d' > \delta_i$ then

(i) Set $\delta_{i+1} = \delta_i$ and insert p_{i+1} into old grid

else $(d' < \delta_i)$

(ii) Set $\delta_{i+1} = d'$ and
Regrid all of P_{i+1} with grid size $d = \delta_{i+1}/2$.

Average Run Time

1st Step: $O(n)$

2nd Step: $O(1)$

All 3(A): $O(n)$

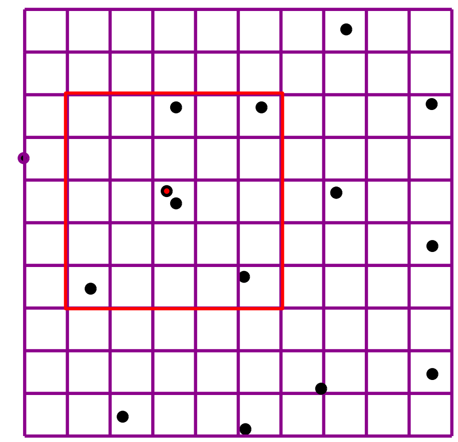
All 3(B)(i): $O(n)$

$= O(n)+$ time for all 3(B)(ii)

Final Piece

- We just saw the algorithm is $O(n)$ average running time with exception of 3(B)(ii)
- 3(B)(ii)can be rephased as, in step i
  if $\delta_{i+1} > \delta_i$ do nothing
  else do $O(i+1)$ average work
- Note that trivially $\delta_{i+1} = d(p_j, p_k)$ for some $j, k \le i+1$.
  $\delta_{i+1} < \delta_i$ only if $j = i+1$ or $k = i+1$.
- Points are in random order, so prob of this happening is $\le 2/(i+1)$
  i.e., $\Pr(\delta_{i+1} < \delta_i) \le 2/(i+1)$
- Let X_i be amount of work done by 3(B)(ii) at step $i+1$
  $X_i = 0$ with Prob $1 - \frac{2}{i+1}$; $X_i = O(i+1)$ with Prob $\frac{2}{i+1}$.
  $\Rightarrow$ expected value $E(X_i) = \frac{2}{i+1} O(i+1) = O(1)$
- Total work by 3(B)(ii) is $Y = \sum_{i=2}^{n-1} X_i$. Then total average work is
  $E(Y) = E\left(\sum_{i=2}^{n-1} X_i\right) = \sum_{i=2}^{n-1} E(X_i) = O(n)$

The Complete Algorithm



- Randomly label the points as $p_1, p_2, \ldots, p_n$
  Perform an $O(n)$ scan finding x_{min}, x_{max}, y_{min} y_{max}, the smallest and largest x and y coordinates among all of the points
- Set $\delta_2 = p_1, p_2$.
  Hash P_2 into the grid with $d = \delta_2/2$.
- For $i = 3$ to n do

(A) Find the min distance d' between p_{i+1} and the ≤ 25 points in the 25 grid boxes surrounding p_{i+1}

(B) If $d' > \delta_i$ then

(i) Set $\delta_{i+1} = \delta_i$ and insert p_{i+1} into old grid

else $(d' < \delta_i)$

(ii) Set $\delta_{i+1} = d'$ and
Regrid all of P_{i+1} with grid size $d = \delta_{i+1}/2$.

Average Run Time

1st Step: $O(n)$

2nd Step: $O(1)$

All 3(A): $O(n)$

All 3(B)(i): $O(n)$

All 3(B)(ii): $O(n)$

$\boxed{= \mathbf{O(n)}}$