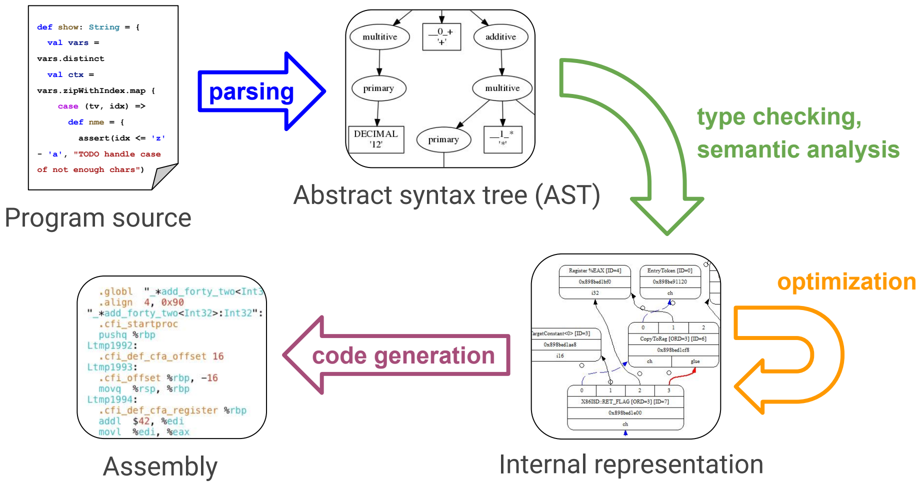
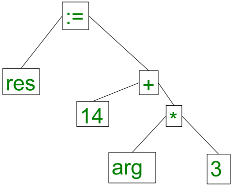
Recapitulation from Previous Lectures

Recap: Compilers



Recap: Compiler Phases

characters	`res = 14 + arg * 3`
↓ lexical analyzer	
words	`res` `=` `14` `+` `arg` `*` `3`
↓ parser	
trees	Assign(res, Plus(C(14), Times(V(arg),C(3))))
↓ name analyzer	
graphs	(variables mapped to declarations)
↓ type checker	
graphs	Assign(res:Int, Plus(C(14), Times(V(arg):Int,C(3)))):Unit
↓ intermediate code generator	
intermediate code	e.g. LLVM bitcode, JVM bytecode, Web Assembly
↓ JIT compiler or platform-specific back end	
machine code	e.g. x86, ARM, RISC-V



Recap: Transforming Text Into a Tree

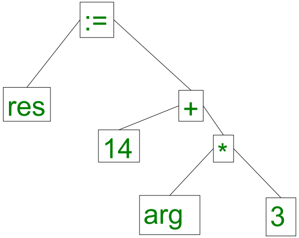
characters `res = 14 + arg * 3`

↓ lexical analyzer

words $\boxed{\texttt{res}}\ \boxed{\texttt{=}}\ \boxed{\texttt{14}}\ \boxed{\texttt{+}}\ \boxed{\texttt{arg}}\ \boxed{\texttt{*}}\ \boxed{\texttt{3}}$

↓ parser

trees Assign(res, Plus(C(14), Times(V(arg),C(3))))



First two phases:

1. lexical analyzer (lexer): sequence of characters → sequence of words
2. syntax analyzer (parser): sequence of words → tree

We will study *linear-time algorithms* for these problems.

We start with the underlying *theory of formal languages*.

Recap: Words

Let A be an alphabet $\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots\}$

We define words of length n, denoted A^n, as follows:

$A^0 = \{\varepsilon\}$ (only one word of length zero, always denoted ε)

For $n > 0$, $A^n = \{\, \mathrm{a}w \mid w \in A^{n-1} \,\}$

Notation '$\mathrm{a}\varepsilon$' abbreviated to 'a'.

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Concatenation: $u \cdot v$, also written uv, is associative.
We can decompose words arbitrarily, as in $w = u\mathrm{a}$ (i.e., $w = u \cdot \mathrm{a}\varepsilon$) if $|w| > 0$

Referring to letters by index: $w_{(0)} = \mathbf{1}$ $w_{(1)} = \mathbf{0}$ $w_{(2)} = \mathbf{1}$ $w_{(3)} = \mathbf{1}$

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Set of all words: $A^* = \bigcup_{n \geq 0} A^n$

which means: $w \in A^*$ if and only iff there exists n such that $w \in A^n$.

Recap: Languages

A *language* over alphabet A is a set $L \subseteq A^*$.

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Operations on languages:

- Set operations: union ($\cup$), intersection ($\cap$), difference ($\setminus$), etc.
- Language concatenation: $L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}$, also written $L_1 L_2$
- Language exponentiation: $L^0 = \{\varepsilon\}$ and $L^{n+1} = L \cdot L^n$

Recap: Regular Expressions

Syntax of regular expressions: $e ::= \varnothing \mid \varepsilon \mid c \mid (e_1 \,|\, e_2) \mid e_1 e_2 \mid e^*$

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The *semantics* of regular expression e is the language $L(e)$ – also written L_e – where:

$$
\begin{aligned}
L(\varnothing) &= \varnothing \\
L(\varepsilon) &= \{\varepsilon\} \\
L(c) &= \{c\} \qquad\qquad (c \in A) \\
L(e_1 \,|\, e_2) &= L(e_1) \cup L(e_2) \\
L(e_1 e_2) &= L(e_1) \cdot L(e_2) \\
L(e^*) &= L(e)^*
\end{aligned}
$$

Example: $\mathit{letter}(\mathit{letter} \,|\, \mathit{digit})^*$ where $\mathit{letter} = \mathrm{a} \,|\, \mathrm{b} \,|\, \mathrm{c} \,|\, \ldots$ and $\mathit{digit} = 0 \,|\, 1 \,|\, 2 \ldots |\, 9$

Recap: Closed Operations on Regular Expressions

Regular languages/expressions are closed under these operations:

- Shorthands for finite languages, such as $[\mathrm{a..z}] = \mathrm{a} \,|\, \mathrm{b} \,|\, ... \,|\, \mathrm{z}$
- Optionality $e^{?} = e \,|\, \varepsilon$
- Repeating at least once $e^{+} = ee^{*}$
- Other repetitions $e^{k..*} = e^{k}e^{*}$ and $e^{p..q} = e^{p}(e^{?})^{q-p}$
- Complementation: $!e$, denoting $A^{*} \setminus L(e)$ — Q: how to translate?
- Intersection $e_1 \,\&\, e_2 = !(!e \,|\, !e)$, denoting $L(e_1) \cap L(e_2)$

Recap: First and Nullable (Definitions)

Formal definitions of *first* and *nullable*:

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Examples:

$$\begin{aligned}
\mathit{first}(\mathrm{ab}^*) &= \{\mathrm{a}\} \\
\mathit{first}(\mathrm{ab}^* \,|\, \mathrm{c}) &= \{\mathrm{a}, \mathrm{c}\} \\
\mathit{first}(\mathrm{a}^*\mathrm{b}^*\mathrm{c}) &= \{\mathrm{a}, \mathrm{b}, \mathrm{c}\} \\
\mathit{first}((\mathrm{cb} \,|\, \mathrm{a}^*\mathrm{c}^*)\mathrm{d}^* e)) &= \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}
\end{aligned}$$

Recap: First and Nullable (Algorithms)

Algorithms for computing *first* and *nullable*:

$$
\begin{aligned}
\mathit{nullable}(\emptyset) &= \mathit{false} \\
\mathit{nullable}(\varepsilon) &= \mathit{true} \\
\mathit{nullable}(a) &= \mathit{false} \\
\mathit{nullable}(e_1|e_2) &= \mathit{nullable}(e_1) \vee \mathit{nullable}(e_2) \\
\mathit{nullable}(e^*) &= \mathit{true} \\
\mathit{nullable}(e_1 e_2) &= \mathit{nullable}(e_1) \wedge \mathit{nullable}(e_2)
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$$
\begin{aligned}
first(\emptyset) &= \emptyset \\
first(\varepsilon) &= \emptyset \\
first(a) &= \{a\}, \text{ for } a \in A \\
first(e_1|e_2) &= first(e_1) \cup first(e_2) \\
first(e*) &= first(e) \\
first(e_1 e_2) &= if\,(nullable(e_1))\ then\ first(e_1) \cup first(e_2) \\
&\ else\ first(e_1)
\end{aligned}
$$

Recap: Lexers

Definition of a *lexer* (aka *tokenizer*):
an ordered set of n labelled token definitions

$$\langle\ Token_1 := e_1\ ;\ Token_2 := e_2\ ;\ \ldots\ ;\ Token_n := e_n\ \rangle$$

Disambiguation rules:

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Disambiguation rules:

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- First-match

Trickiness of Lexing

Consider a lexer for these two classes of tokens:

$$\left\langle \mathsf{Token}_1 := \mathsf{aaa}\,;\ \mathsf{Token}_2 := \mathsf{a}^*\mathsf{b} \right\rangle$$

Example run on aaaab:

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How about a run on aaaaaa? And aaaaaaaaa? And aaaaaaaab?

General Approach to Automatic Lexing

Compile regexps into some sort of *automata* with *transitions* between *states*.

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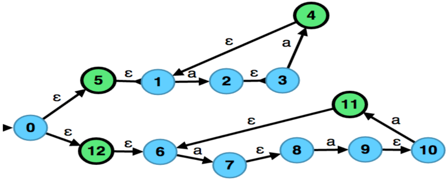

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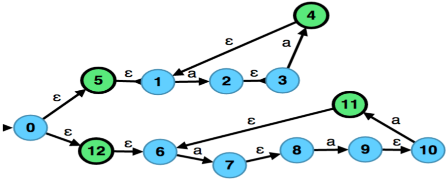

In this course: we'll look at a more straightforward approach based on *derivatives*

Automatically Recognizing Regular Languages

Regular Expression Matcher: First Trial

$$accepts : (e, w) \rightarrow \{\textbf{true}, \textbf{false}\}$$

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& \quad \textbf{if} \;\; e = e_1 \,|\, e_2, \;\; accepts(e_1, w) \lor accepts(e_2, w) \\
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- **if** $e = c$, **true if** $u = \varepsilon$, **false otherwise**
- **if** $e = d \ (d \neq c)$, **false**
- **if** $e = e_1 \mid e_2$, $accepts(e_1, w) \vee accepts(e_2, w)$
- **if** $e = e_1 e_2$, ...what to do? try all splits of w? (expensive)
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& \quad \textbf{if } \ e = \varepsilon, \ \textbf{false} \\
& \quad \textbf{if } \ e = c, \ \textbf{true if } \ u = \varepsilon, \ \textbf{false otherwise} \\
& \quad \textbf{if } \ e = d \ (d \neq c), \ \textbf{false} \\
& \quad \textbf{if } \ e = e_1 \mid e_2, \ accepts(e_1, w) \vee accepts(e_2, w) \\
& \quad \textbf{if } \ e = e_1 e_2, \ \ldots \text{what to do? try all splits of } w\text{? (expensive)} \\
& \quad \textbf{if } \ e = e_1^*, \ \ldots \text{similar problem here}
\end{aligned}$$

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Brzozowskis Derivatives, Definition

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The *derivative* of regexp e with respect to letter c, written $\delta^c(e)$, is defined as:

$$L(\delta^c(e)) = \{ w \mid cw \in L(e) \}$$

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Theorem

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Proof.

We'll give an algorithm for computing derivatives, which only yields regexps. $\square$

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Here are a few derivative examples:

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Derivative of a regexp e with respect to letter c, written $\delta^c(e)$, can be computed as:

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\end{aligned}
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Matching Regular Expressions by Derivation

Idea: given a **regexp** e and a **word** w, to know whether e **accepts** w,

simply ***derive*** e ***for all letters of*** w*, then check whether the* ***result is nullable****!*

$$accepts : (e, w) \rightarrow \{\textbf{true}, \textbf{false}\}$$

$$\begin{aligned} accepts(e, \varepsilon) &= nullable(e) \\ accepts(e, cw) &= accepts(\delta^c(e), w) \end{aligned}$$

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Important: need to *cache* each intermediate result to avoid duplicating work.

Regular Expressions in Scala

Regexp Data Type in Scala

```scala
enum RegExp:
  // empty language ∅
  case Failure
  // empty word ε
  case EmptyStr
  // character a such that predicate(a)
  case CharWhere(predicate: Character => Boolean)
  // union left | right
  case Union(left: RegExp, right: RegExp)
  // concatenation first | second
  case Concat(first: RegExp, second: RegExp)
  // Kleene star underlying*
  case Star(underlying: RegExp)
```

Regexp Methods

```
enum RegExp:
  case ...

  // is this regexp nullable?
  def acceptsEmpty: Boolean = ...

  // can this regexp possibly accept some words?
  def isProductive: Boolean = this match
    case Failure => false
    case EmptyStr | CharWhere(_) | Star(_) => true // approx.
    case Union(l, r) => l.isProductive || r.isProductive
    case Concat(l, r) => l.isProductive && r.isProductive
```

Regexp Combinators 1 (extension methods)

```
extension (expr: RegExpr):
  def ~ (that: RegExpr): RegExpr = Concat(expr, that)
  def | (that: RegExpr): RegExpr = Union(expr, that)
  def * : RegExpr = Star(expr)
  def ? : RegExpr = expr | EmptyStr
  def + : RegExpr = expr ~ expr.*
  def times(n: Int): RegExpr =
    if (n <= 0) EmptyStr else expr ~ expr.times(n - 1) }
```

Examples:

```
e1 ~ e2 ~ e3 | e4
e1.* | e2.+
```

Regexp Combinators 2 (top-level definitions)

```scala
def elem(pred: Char => Boolean): RegExpr = CharWhere(pred)

def elem(char: Char): RegExpr = CharWhere(_ == char)

def elem(chars: Iterable[Char]): RegExpr =
  chars.map(elem).foldLeft[RegExpr](Failure)(_ | _)

def word(chars: Iterable[Char]): RegExpr =
  chars.map(elem).foldLeft[RegExpr](EmptyStr)(_ ~ _)

def inRange(low: Char, high: Char): RegExpr =
  elem(c => c >= low && c <= high)

// Example:
elem(_.isLetter) ~ (elem(_.isLetter) | elem(_.isDigit)).*
```

Regexp Derivation in Scala

```scala
def derive(char: Character): RegExp =
   def work(expr: RegExp): RegExp = expr match
      case Failure | EmptyStr ⇒ Failure
      case CharWhere(pred) ⇒
         if pred(char) then EmptyStr else Failure
      case Union(left, right) ⇒ work(left) | work(right)
      case Concat(left, right) ⇒
         val w = work(left) ~ right
         if left.acceptsEmpty then w | work(right) else w
      case Star(inner) ⇒ work(inner) ~ expr
   work(this)
```

Efficiently Recognizing Regular Languages

Problem of Naive Derivation: Inefficiency

Pathological cases easily arise.

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$\boldsymbol{\delta}^{\mathsf{aaa}}(\mathsf{a}^*) = \varnothing\mathsf{a}^* \mid \varnothing\mathsf{a}^* \mid \boldsymbol{\varepsilon}\,\mathsf{a}^*$

...

Every step needs to go through the list of $\varnothing\mathsf{a}^*$, which grows without limit:

$\Rightarrow$ *unbounded number of "states" and wrong complexity*

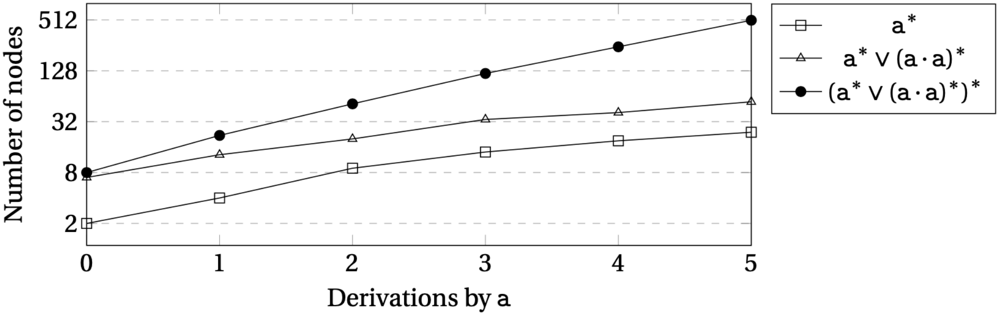
This is a simple case;

successive derivations are not always possible to "compact" on the fly...

– consider $(\mathsf{a}^* \mid (\mathsf{aa})^*)^*$

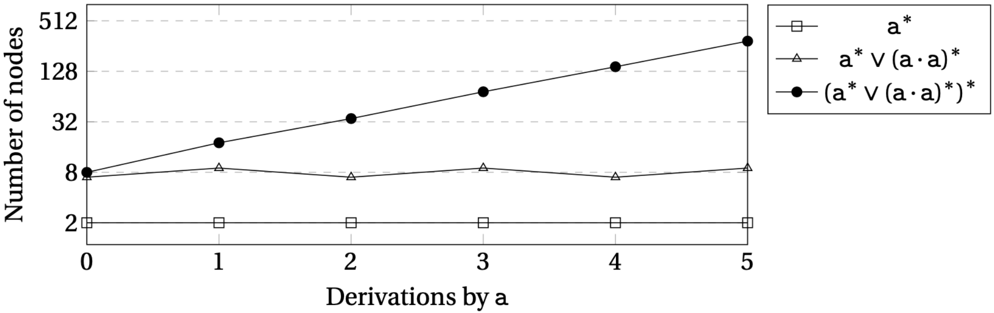
Problem of Naive Derivation: Inefficiency

Pathological cases easily arise.



Problem of Naive Derivation: Inefficiency

Not always possible to “compact” on the fly. Here it is with auto-compaction:



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To avoid getting into pathological cases,

it is sufficient to *normalize* the part of the expression we derive

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Normalization approach:

- associate all concatenations to the right: $(e_1 e_2) e_3 \Rightarrow e_1 (e_2 e_3)$
- avoid repetitions in unions: $(e_1 \mid e_2) \mid (e_3 \mid e_1) \Rightarrow e_1 \mid e_2 \mid e_3$

Normalizing Regexp Derivation in Scala

```
def deriveNorm(char: Character): RegExp =
  val disjuncted = collection.mutable.SortedSet[RegExp]()
  def work(expr: RegExp, rest: RegExp): Unit = expr match
    case CharWhere(pred) ⇒ if pred(char) then disjuncted += rest
    case Union(left, right) ⇒ work(left, rest); work(right, rest)
    case Concat(left, right) ⇒
      work(left, right ~ rest)
      if left.acceptsEmpty then work(right, rest)
    case Star(inner) ⇒ work(inner, expr ~ rest)
    case Failure | EmptyStr ⇒ ()
  work(this, EmptyStr) // register unions into `disjuncted`
  disjuncted.foldLeft[RegExp](Failure)(_ | _) // rebuild regexp
```

The Pumping Lemma

What is the Essence of Regular Expressions/Languages?

Although these languages are usually infinite,
they contain, at there core, *simple repeating patterns*.

Pumping Lemma

Recall: $w^n = \underbrace{w\,w \ldots w}_{n \ times}$

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Lemma (Pumping)

*For **all regexp** e, there **exists a constant** $p \geq 1$ such that*
for all $w \in L(e)$ of length at least $|w| \geq p$,
we have $w = w_0\,w_1\,w_2$ such that:

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Proof?

Proving the Pumping Lemma

Highly non-trivial.

By structural induction on e. Concatenation and Kleene star require careful case analysis on all possible length distributions.

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See §2.7 of Edelmann, Romain. Efficient Parsing with Derivatives and Zippers. PhD Thesis (EPFL). 2021.