

Quick Review of Linearity of Expectation

COMP 3711H - HKUST
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Linearity of Expectation is one of the simplest and most useful tools used in the analysis of randomized algorithms.

In its easiest form it just says that,
if X, Y are *any* two random variables (not necessarily independent) then

$$E(X + Y) = E(X) + E(Y).$$

The iterated version is that
if X_1, X_2, \dots, X_n are *any* random variables, then

$$E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i).$$

Example: Let Z be the value seen when rolling two dice.
 $Z = X_1 + X_2$ where X_i is the value seen when rolling single die $i = 1, 2$. It's easy to calculate that

$$E(X_i) = \sum_{j=1}^6 j \Pr(X_i = j) = \sum_{j=1}^6 j \frac{1}{6} = \frac{7}{2}.$$

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Then

$$E(Z) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7.$$

When flipping n coins, what is the expected number of heads?

$Z = \sum_{i=1}^n X_i$, where

$X_i = 1$ if coin i is a head and 0 if it is a tail.

Set $\Pr(X_i = 1) = p_i$ and $\Pr(X_i = 0) = 1 - p_i$.

Then X_i is a *Bernoulli Random Variable* with probability p_i .

Note that $E(X_i) = 1 \cdot \Pr(X_i = 1) = p_i$ so

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$$p_i = p \text{ (all coins the same)} \Rightarrow E(Z) = pn$$

$$p_i = \frac{1}{i} \Rightarrow E(Z) = \sum_{i=1}^n p_i = \sum_{i=1}^n \frac{1}{i} = H_n \sim \ln n$$

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Let x_1, x_2, \dots, x_n be the list of coin tosses, i.e., $x_i \in \{\text{H(ead)}, \text{T(ail)}\}$.

$$Z = \sum_{i=3}^n X_i \text{ where } X_i = 1 \text{ iff } x_{i-2}x_{i-1}x_i = HHH$$

$$\Pr(X_i = 1) = \frac{1}{8}, \text{ so}$$

$$E(Z) = \frac{n-2}{8}.$$

Suppose an algorithm's input is a permutation of n numbers.

Let x_1, x_2, \dots, x_n be the input in its given order.

x_i is a *left to right maxima* if it's bigger than x_1, x_2, \dots, x_{i-1} .

For example, the red items in these two permutations are the l.t.r. maxima:

5 4 7 8 1 6 3 2 1 3 5 7 2 4 6 8

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One way of generating a random permutation is to first randomly choose the first i items equally likely among all possible $\binom{n}{i}$ subsets. Then choose a random permutation among the i possible permutations to order them as x_1, \dots, x_i . Then randomly order the remaining items as x_{i+1}, \dots, x_n .

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Probability x_i is l.t.r. maxima is prob it's largest in first i items which is now $\frac{1}{i}$. So X_i is Bernoulli Random Variable with $p_i = 1/i$ and $E(Z_i) = H_n$.

Suppose we flip n coins.

i th coin is Heads with probability $p_i = 1/i$.

If i th coin is Heads you get i dollars; Tails, you get nothing.
What is expected amount you receive?

Let Z be total amount. $Z = \sum_{i=1}^n X_i$ where $X_i = i$ if i th coin is heads and is otherwise 0. Then

$$E(X_i) = ip_i = 1$$

So,
$$E(Z) = E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n 1 = n$$

Make a minor change from the previous page.

Suppose we flip n coins. i th coin is Heads with probability $p_i = 1/i$.

If i th coin is Heads you run another random process Y_i to tell you how much money you receive. All you know is that $E(Y_i) = i$.

If i th coin is Tails, you get nothing. What is expected amount you receive?

Let Z be total amount. $Z = \sum_{i=1}^n X_i Y_i$ where

$X_i = 1$ if i th coin is Heads and is otherwise 0, so $E(X_i) = p_i = \frac{1}{i}$.

Then, *because X_i and Y_i are independent*

$$E(X_i Y_i) = E(X_i) E(Y_i) = \frac{1}{i} \cdot i = 1$$

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