Recapitulation from Previous Lectures

Recap: Naive Derivation of Regular Expressions

$$
\begin{aligned}
\delta^c(\varnothing) &= \varnothing \\
\delta^c(\varepsilon) &= \varnothing \\
\delta^c(d) &= \begin{cases} \varepsilon & \textbf{if } d = c \\ \varnothing & \textbf{if } d \neq c \end{cases} \\
\delta^c(e_1 \mid e_2) &= \delta^c(e_1) \mid \delta^c(e_2) \\
\delta^c(e_1\, e_2) &= \begin{cases} \delta^c(e_1)\, e_2 \mid \delta^c(e_2) & \textbf{if } \mathit{nullable}(e_1) \\ \delta^c(e_1)\, e_2 & \textbf{otherwise} \end{cases} \\
\delta^c(e_1^*) &= \delta^c(e_1)\, e_1^*
\end{aligned}
$$

Problem: produces “dumb” repeated patterns, resulting in poor performance:

$$\delta^{\mathrm{aaaa}}(\mathrm{a}^*) = \varnothing\mathrm{a}^* \mid \varnothing\mathrm{a}^* \mid \varnothing\mathrm{a}^* \mid \varepsilon\,\mathrm{a}^*$$

Recap: Naive Derivation in Scala

```
def derive(expr: RegExp, c: Character): RegExp =
  expr match
    case Failure | EmptyStr ⇒ Failure
    case CharWhere(pred) ⇒
      if pred(c) then EmptyStr else Failure
    case Union(left, right) ⇒ derive(left, c) | derive(right, c)
    case Concat(left, right) ⇒
      val w = derive(left, c) ~ right
      if left.acceptsEmpty then w | derive(right, c) else w
    case Star(inner) ⇒ derive(inner, c) ~ expr
```

Recap: Normalizing Derivation

Idea: while differentiating,

- "flatten" disjunction structures (take them out of sequences)
- remove duplicates from generated disjunctions

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Example: consider a^*

$\delta^{\mathsf{a}}(\mathsf{a}^*) = \varepsilon \mathsf{a}^*$

$\delta^{\mathsf{aa}}(\mathsf{a}^*) = \varnothing \mathsf{a}^* \mid \varepsilon \mathsf{a}^*$

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$\delta^{a}(a^*) = \varepsilon a^*$
$\delta^{aa}(a^*) = \emptyset a^* \mid \varepsilon a^*$
$\delta^{aaa}(a^*) = \emptyset a^* \cancel{\mid \emptyset a^*} \mid \varepsilon a^*$

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...

Recap: Normalizing Derivation in Scala

```
def deriveNorm(char: Character): RegExp =
  val disjuncted = collection.mutable.Set[RegExp]()
  def work(expr: RegExp, rest: RegExp): Unit = expr match
    case CharWhere(pred) ⇒ if pred(char) then disjuncted += rest
    case Union(left, right) ⇒ work(left, rest); work(right, rest)
    case Concat(left, right) ⇒
      work(left, right ~ rest)
      if left.acceptsEmpty then work(right, rest)
    case Star(inner) ⇒ work(inner, expr ~ rest)
    case Failure | EmptyStr ⇒ ()
  work(this, EmptyStr) // register unions into `disjuncted`
  disjuncted.foldLeft[RegExp](Failure)(_ | _) // rebuild regexp
```

Theory of Normalizing Derivation

Sets of Regular Expressions

We will now be deadline with sets E of regular expressions.

The ascribed semantics will be that of a disjunction of all e in E.

Since $\cdot | \cdot$ is commutative, associative, and idempotent (just like sets),

this is a good representation.

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Lexicographic Sorting

We can totally order any inductive data type *lexicographically*.
For example, $\varnothing < \varepsilon < \varnothing\varnothing < \varnothing\varepsilon < \varepsilon\varepsilon < \varnothing|\varnothing < \ldots$ etc.

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Definition (disjunction of set of regexp)

Define $\mathit{disj}(E)$ as the (left-associated) disjunction of the elements in E,
taken in lexicographic order.

Example: $\mathit{disj}(\{\varepsilon, \mathrm{ab}, \mathrm{a}\}) \;=\; \varepsilon\,|\,\mathrm{a}\,|\,\mathrm{ab}$

Normalizing Derivation, Formally

Let the *normalizing derivation* of e w.r.t. c be defined as: $\underline{\delta}^c(e) = \mathit{disj}(\mathit{derive}_c(e, \varepsilon))$

$$
\begin{aligned}
\mathit{derive}_c(\varnothing, e) &= \emptyset \\
\mathit{derive}_c(\varepsilon, e) &= \emptyset \\
\mathit{derive}_c(c, e) &= \{e\} \\
\mathit{derive}_c(d, e) &= \emptyset \qquad (d \neq c) \\
\mathit{derive}_c(e_1 \,|\, e_2, e) &= \mathit{derive}_c(e_1, e) \cup \mathit{derive}_c(e_2, e) \\
\mathit{derive}_c(e_1\, e_2, e) &= \mathit{derive}_c(e_1, e_2\, e) \cup \begin{cases} \mathit{derive}_c(e_2, e) & \text{if } \mathit{nullable}(e_1) \\ \emptyset & \text{otherwise} \end{cases} \\
\mathit{derive}_c({e_1}^*, e) &= \mathit{derive}_c(e_1, {e_1}^* e)
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Easy to verify that $L(\underline{\delta}^c(e)) = L(\delta^c(e))$.

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Notations: $\mathit{derive}_\varepsilon(e) = \{e\}$ and $\mathit{derive}_{cw}(e) = \bigcup\{\mathit{derive}_w(e_0) \mid e_0 \in \mathit{derive}_c(e)\}$

We can show that $\mathit{disj}(\mathit{derive}_w(e)) = \underline{\delta}^w(e)$

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Intuition: *over-approximate* the set of *any* successive derivations of a regular expression

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Theorem

For any e, $max(e, \varepsilon)$ over-approximates all successive derivations of e:

$$\forall c, w.\ derive_{cw}(e) \subseteq max(e, \varepsilon)$$

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Not tight: $max(\varnothing \mathrm{a}, \varepsilon) = \{\mathrm{a}\}$ but for any c and w we have $derive_{cw}(\varnothing \mathrm{a}, \varepsilon) = \varnothing$

Max of Regular Expressions, Proof

Theorem

$$\forall e, c, w.\ derive_{cw}(e) \subseteq max(e, \varepsilon)$$

Proof.

By induction on w, showing that: $\forall e.\ derive_{w}(e) \subseteq max(e, \varepsilon) \cup \{e\}$

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- When $w = \varepsilon$, we have

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- When $w = cw'$, assuming $\forall e'.\ derive_{w'}(e') \subseteq max(e', \varepsilon) \cup \{e'\}$, we have

$$\begin{aligned} derive_w(e) &= \bigcup \{ derive_{w'}(e') \mid e' \in derive_c(e) \} \\ &\subseteq \bigcup \{ max(e', \varepsilon) \mid e' \in derive_c(e) \} \cup \{e\} \end{aligned}$$

and we can show by induction on e that the latter is $\subseteq max(e, \varepsilon) \cup \{e\}$

$\square$

Max of Regular Expressions, Consequence

Theorem

$$\forall e, c, w.\ derive_{cw}(e) \subseteq max(e, \varepsilon)$$

$\Rightarrow$ Number of distinct regexps generated by normalizing derivation of any w **is *bounded*!**

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$\Rightarrow$ Number of distinct regexps generated by normalizing derivation of any w **is *bounded*!**

More specifically, for any e, we have: (remember $\underline{\delta}^{w}(e) \;=\; disj(derive_{w}(e))$)

$$|\{\underline{\delta}^{w}(e) \mid w \in A^{*}\}| \;\leq\; 1 + 2^{|max(e,\varepsilon)|}$$

(Better bounds can be derived.)

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$\Rightarrow$ Number of distinct regexps generated by normalizing derivation of any w **is *bounded*!**

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(Better bounds can be derived.)

Opens the door to efficient memoization (caching)
allowing regexp matching in ***constant space*** and ***linear time*** w.r.t. size of words

Memoizing successive derivations

Algorithm regexp matching:

- Start with empty mapping $M := \emptyset$ and with regexp e
- For each i^{th} character c_i in w,
  - If $(e, c_i) \notin domain(M)$, set $M(e, c_i) := \underline{\delta}^{c_i}(e)$
  - Set $e := M(e, c_i)$
- Test whether $nullable(e)$

In Scala types $M :$ mutable.Map[(RegExp, Char), RegExp]

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Conceptually builds a deterministic finite-state automaton on the fly

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More or less how **Silex** is implemented! (library used in the Amy project)

Expressiveness Limitation of Regular Expressions

Consider the language $L_{\mathsf{ab}} = \{\, \mathsf{a}^n\mathsf{b}^n \mid n \geq 0 \,\}$

Is L_{ab} regular?

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Let $K = 1 + 2^{|max(e_{\mathrm{ab}}, \varepsilon)|}$

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By definition, $\underline{\delta}^{\mathrm{a}^i}(e_{\mathrm{ab}})$ must accept b^i because $\mathrm{a}^i\mathrm{b}^i \in L_{\mathrm{ab}}$
which also means $\underline{\delta}^{\mathrm{a}^{K+1}}(e_{\mathrm{ab}})$ must accept b^i

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which also means $\underline{\delta}^{\mathrm{a}^{K+1}}(e_{\mathrm{ab}})$ must accept b^i ... ***but*** $\mathrm{a}^{K+1} \mathrm{b}^i \notin L_{\mathrm{ab}}$

$\Rightarrow$ ***Contradiction.***

$\square$

Introduction to Grammars

Regular *Grammars*

An equivalent way of defining regular languages – name intermediate productions:

$$\begin{aligned}
\text{start} &\rightarrow \text{letter}(\text{letter} \,|\, \text{digit})^* \\
\text{letter} &\rightarrow \text{a} \,|\, \text{b} \,|\, \text{c} \,|\, \ldots \,|\, \text{z} \\
\text{digit} &\rightarrow 0 \,|\, 1 \,|\, 2 \,|\, 3 \,|\, 4 \,|\, 5 \,|\, 6 \,|\, 7 \,|\, 8 \,|\, 9
\end{aligned}$$

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"start", "letter", "digit" are part of a distinct set of identifiers called *non-terminals*,

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where S is often used as the *S*tart symbol

Regularity requirement: *no recursion!*

Definitions should form a *directed acyclic graph* (DAG)

Context-Free Grammars (CFG)

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What if we allowed *recursion* in grammar definitions?

$$S \;\rightarrow\; \varepsilon \mid \mathrm{a}\,S\,\mathrm{b}$$

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Semantics given by *rewriting derivations* (note: not to be confused with *derivative*!)

Example derivation:

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Semantics given by *rewriting derivations* (note: not to be confused with *derivative*!)

Example derivation:

$$S \Rightarrow \mathrm{a}S\mathrm{b} \Rightarrow \mathrm{aa}S\mathrm{bb} \Rightarrow \mathrm{aaa}S\mathrm{bbb} \Rightarrow \mathrm{aaa}\varepsilon\mathrm{bbb} = \mathrm{aaabbb}$$

Definition of Context-Free Grammars (CFG)

Formally: a tuple $G = (A, N, S, R)$

- A – terminals (alphabet for generated words $w \in A^*$), usually *tokens*
- N – non-terminals – symbols with (recursive) definitions
- R – grammar rules as pairs (n, v), written $n \rightarrow v$ where $n \in N$ is a non-terminal
  $v \in (A \cup N)^*$ – sequence of terminals and non-terminals
- A derivation in G starts from the starting symbol S
- Each step replaces a non-terminal with one of its right hand sides

Example from before: $S \;\rightarrow\; \varepsilon \mid \mathrm{a}\, S\, \mathrm{b}$

$$G \;=\; (A = \{a, b\},\; N = \{S\},\; S,\; R = \{(S, \varepsilon),\; (S, aSb)\})$$

$$G \;=\; (\{a, b\},\; \{S\},\; S,\; \{(S, \varepsilon),\; (S, aSb)\})$$

Parse Trees

Definition (parse tree)

A tree t is a *parse tree* of $G = (A, N, S, R)$ iff t is a node-labelled tree with ordered children that satisfies:

- root is labeled by S
- leaves are labelled by elements of A
- each non-leaf node is labelled by an element of N
- for each non-leaf node labelled by n
  whose children left to right are labelled by $p_1 \dots p_k$,
  there is a rule $(n \rightarrow p_1 \dots p_k) \in R$

Parse Trees

Definition (parse tree)

A tree t is a *parse tree* of $G = (A, N, S, R)$ iff t is a node-labelled tree with ordered children that satisfies:

- root is labeled by S
- leaves are labelled by elements of A
- each non-leaf node is labelled by an element of N
- for each non-leaf node labelled by n
  whose children left to right are labelled by $p_1 \dots p_k$,
  there is a rule $(n \;\rightarrow\; p_1 \dots p_k) \in R$

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Easy to check: "is t parse tree of G?"; **harder**: "are there parse trees for word w?"

Parse Tree Example

Given grammar:

$$\begin{aligned} S &\rightarrow PQ \\ P &\rightarrow \mathrm{a} \mid \mathrm{a}P \\ Q &\rightarrow \varepsilon \mid \mathrm{a}Q\mathrm{b} \end{aligned}$$

Show a parse tree for aaaabb

Balanced Brackets Grammar

Consider language L of words made up of square brackets “[” and “]” that are balanced (each opening bracket has a matching closing bracket)

Example sequences of brackets:

- [[[ ]] [ ]] – balanced, belongs to the language
- [ ] [ ]] – not balanced, does not belong

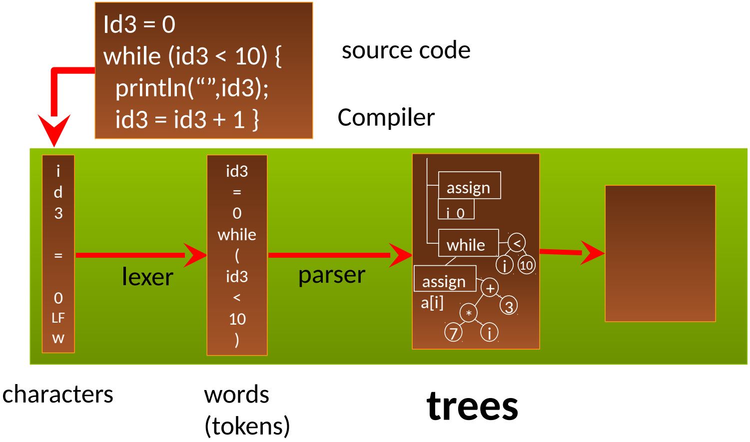
Exercise: give the grammar

Balanced Brackets Grammar

All these grammars define the same language of balanced brackets:

$$
\begin{array}{ll}
G_1 & S \rightarrow \varepsilon \mid S[S]S \\
G_2 & S \rightarrow \varepsilon \mid [S]S \\
G_3 & S \rightarrow \varepsilon \mid S[S] \\
G_4 & S \rightarrow \varepsilon \mid SS \mid [S]
\end{array}
$$

Syntax Trees



Parse Trees and Abstract Syntax Trees

Difference between parse trees and abstract syntax trees

- Node children in ***parse trees*** correspond precisely to RHS of grammar rules

  Definition of parse trees is fixed given the grammar

  Often, compilers never actually build parse trees in memory

- Nodes in ***abstract syntax tree*** (AST) contain only useful information

  We can choose our own syntax trees,
  to facilitate both construction and processing in later stages of compiler

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Pretty printer: takes AST and outputs the leaves of one possible parse tree

Abstract Syntax Trees for Expressions

An expression grammar:

$$
\begin{aligned}
\text{expr} &\rightarrow \text{intLiteral} \mid \text{ident} \mid \text{expr op expr} \mid \text{`('expr`)'} \\
\text{op} &\rightarrow + \mid * \\
&\dots
\end{aligned}
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A possible AST for it:

```
enum Expr:
  case IntLit(n: Int)
  case Var(name: String)
  case Plus(e1: Expr, e2: Expr)
  case Times(e1: Expr, e2: Expr)
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Notice: no parenthesis case; no "op"

Ambiguous Grammars

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Multiple possible ASTs:

- `Times(Var("x"), Plus(IntLit(42), Var("x")))`
- `Plus(Times(Var("x"), IntLit(42)), Var("x"))`

Which is "correct"?

Ambiguous Grammars

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Can we change the grammar to reject the latter?

Layering the Grammar by Priorities

Idea: go from

$$\text{expr} \;\rightarrow\; \text{intLiteral} \mid \text{ident} \mid \text{expr op expr} \mid \text{`('expr`)'}$$

to the *layered* form

$$\begin{aligned} \text{expr} \;&\rightarrow\; \text{expr} + \text{expr} \mid \text{multi} \\ \text{multi} \;&\rightarrow\; \text{intLiteral} \mid \text{ident} \mid \text{multi} * \text{multi} \mid \text{`('expr`)'} \end{aligned}$$

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Associativity

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Multiple interpretations:

- "$(x + 42) + y$"
- "$x + (42 + y)$"

Left Associativity – Left Recursion

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Idea: keep layering the syntax...

$$
\begin{aligned}
\text{expr} &\rightarrow \text{expr} + \text{multi} \mid \text{multi} \\
\text{multi} &\rightarrow \text{multi} * \text{factor} \mid \text{factor} \\
\text{factor} &\rightarrow \text{intLiteral} \mid \text{ident} \mid \text{`('expr`)'}
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Note: we say such grammars are ***left-recursive***

because they "recurse" immediately on the left of a rule: $\textbf{expr} \rightarrow \textbf{expr} + \text{multi}$

Parsing: From Grammars to Abstract Syntax Trees

Parsing – An Old Problem

Generally performed from context-free grammars (CFGs) or *simpler subclasses* of CFGs

- LL(k) grammars such as LL(1), LL($*$)
- LR(k) grammars, SLR(1), LALR...
- ...

More recently, parsing-expression grammars (PEGs)

Many algorithms have been devised!

- Automaton-based LL(1) parsing (Lewis and Stearns, 1968)
- CYK algorithm for general CFG (Cocke, Younger and Kasami, ca 1967)
- Earley parsing for general CFG (Earley, 1970)
- Generalized LR parsing (Tomita, 1987)
- Parsing by derivatives for general CFG (Might et al., 2011)
- Packrat for PEG (Ford, 2002)
- etc. etc.

In this course: focus on LL(1), Pratt, Packrat, mention LR(1) in passing

Recursive Descent LL(1) Parsing

Recursive *descent* is a *decent* parsing technique

- Can be **easily implemented** manually based on the grammar
- **Efficient** – linear in the size of the token sequence
- Direct correspondence between grammar and code

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In next lecture: how to parse by recursive descent *manually* and then *automatically*