Depth-First Search

Version of September 23, 2016

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4. $f[u]$: finishing time
   - a counter indicating when the processing of vertex u (and all its descendants) is finished

The DFS Algorithm

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- It starts from an initial vertex.
- After visiting a vertex, it recursively visits *all* of its neighbors.
- The strategy is to search "deeper" in the graph whenever possible

DFS Algorithm

DFS(G)

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// Initialize
foreach u in V do
    color[u] = WHITE; // undiscovered
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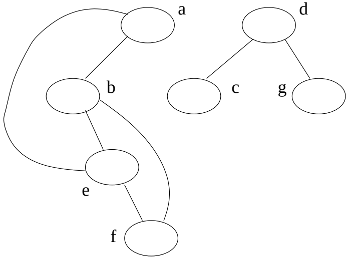
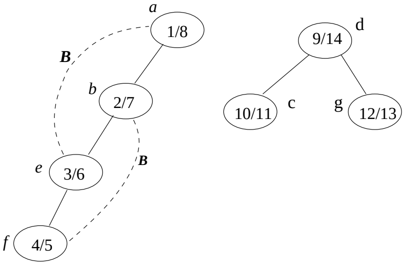
DFS Example





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- Use $pred[v]$ to define a graph $F = (V, E_f)$ as follows:

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$$E_f = \{(pred[v], v) | v \in V, pred[v] \neq \mathrm{NULL}\}$$

- This is a graph with no cycles, and hence a forest, i.e. a collection of trees.
- Called a DFS Forest.
- Vertices in the subtree rooted at u are those discovered while u is gray.

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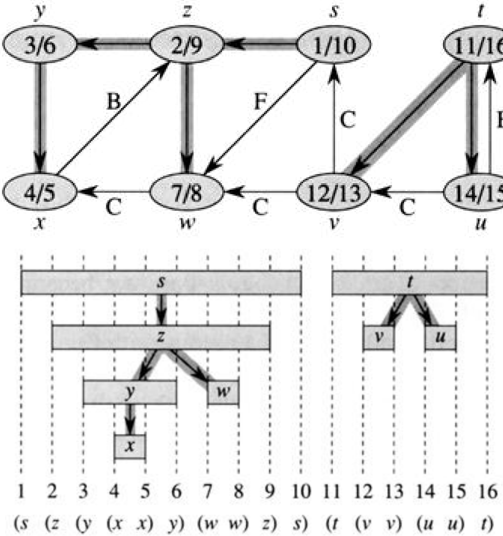
Hence, the running of DFS on a graph with V vertices and E edges is $O(V + E)$

Time-Stamp Structure



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- u is a descendant (in DFS trees) of v, if and only if $[d[u], f[u]]$ is a subinterval of $[d[v], f[v]]$ (Example)

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- u is an ancestor of v, if and only if $[d[u], f[u]]$ contains $[d[v], f[v]]$ (Example)
- u is unrelated to v, if and only if $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint intervals (Example)

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The argument for other case, where $d[v] > d[u]$, is similar.

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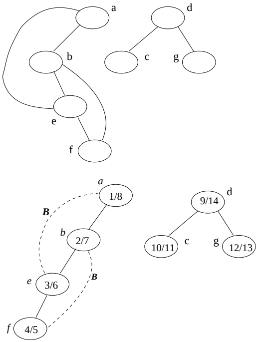

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- Then v is discovered while u is gray (why?).

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    - If $pred[v] = u$, then (u, v) is a tree edge.

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- Without loss of generality, assume $d(u) < d(v)$.
- Then v is discovered while u is gray (why?).
- Hence v is in the DFS subtree rooted at u.
    - If $pred[v] = u$, then (u, v) is a tree edge.
    - if $prev[v] \neq u$, then (u, v) is a back edge.

An Application of DFS: Cycle Finding

Question

Given an undirected graph G, how to determine whether or not G contains a cycle?

Lemma

G is acyclic if and only if a DFS of G yields no back edges.

Proof.

$\Rightarrow$: Suppose that there is a back edge (u, v). Then, vertex v is an ancestor (excluding predecessor) of u in the DFS trees. There is thus a path from v to u in G, and the back edge (u, v) completes a cycle.

$\Leftarrow$: If there is no back edge, then since an edge in an undirected graph is either a tree edge or a back edge, there are only tree edges, implying that the graph is a forest, and hence is acyclic. $\square$

Cycle Finding

Cycle(G)

```
foreach u in V do
    color[u] = WHITE;
    pred[u] = NULL;
end
foreach u in V do
    if color[u] = WHITE then
        Visit(u);
    end
end
output "No Cycle";
```

Visit(u)

```
color[u] = GRAY;
foreach v in Adj(u) do
    // consider (u, v)
    if color[v] = WHITE then
        // v unvisited
        pred[v] = u;
        Visit(v); // visit v
    else if v != pred[u] then
        // back edge detected
        output "Cycle found!";
        exit; // terminate
    end
end
color[u] = BLACK;
```

Running time: $O(V)$

- only traverse tree edges, until the first back edge is found
- at most $V - 1$ tree edges