Probability: Part I

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Contents

Introduction

Definition 1

Probability is the measure of the likeliness that an event will occur.

Applications

- Risk analysis.
- Shannon information theory.
- Hashing in computer science and cryptography.
- The analysis of the average-case complexity of algorithms.

Experiments, Sample Spaces, and Events

Definition 2

- An experiment is a **procedure** that yields one of a given set of possible outcomes.
- The sample space of the experiment is the set of all possible outcomes.
- An event is a subset of the sample space.
- The event space is the power set of the sample space.

Example 3

Experiment: Toss a fair coin.

Sample space: $S = \{H, T\}$.

Event: e.g., $\{H\}$.

Event space: $P(S) = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$.

Experiments, Sample Spaces, and Events

Remark

Often, some of the more interesting events have special names.

Example 4

Experiment: Toss two fair coins.

Sample space: $S = \{HH, HT, TH, TT\}$.

Event: e.g., $\{HH\}$, $\{\text{TT}\}$. They are called matches.

Event space: $P(S)$.

Experiments, Sample Spaces, and Events

Definition 5

If S is a finite sample space in which all outcomes are equally likely, we say that S has uniform probability distribution.

In this case, we say that every event in S happens with **probability** $1/|S|$.

Example 6

The experiment: toss a fair coin. The probabilities of all outcomes are as follows:

Outcomes	H	T
Probability	$\frac{1}{2}$	$\frac{1}{2}$

The sample space $S = \{H, T\}$ has thus uniform probability distribution.

Experiments, Sample Spaces, and Events

Definition 7

The experiment of selecting an element from a sample space with uniform probability distribution is called **selecting an element of S at random** or **selecting an element of S randomly.**

Uniform Probability Measure

Definition 8

Let S be a finite sample space with uniform probability distribution, and E be an event in S. The probability of E is

$$p(E) = \frac{|E|}{|S|}.$$

Example 9

The experiment: toss a fair coin. The probability measure is below:

Event	$\emptyset$	$\{H\}$	$\{T\}$	$\{H, T\}$
Probability	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Uniform Probability Measure

Example 10

The experiment: toss two fair coins. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

Some events and there probabilities are:

Event	$\emptyset$	$\{HT, TH, HH\}$	$\{TH\}$	$\{HH, TT\}$
Probability	0	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

Probability of the Complement of an Event

Proposition 11

Let E be an event in a finite sample space S. Then the probability of the complement E^c of the event E is

$$p(E^c) = 1 - p(E).$$

Proof.

- $E \cap E^c = \emptyset$ and $S = E \cup E^c$.
- $|S| = |E| + |E^c|$.
- $\frac{|S|}{|S|} = \frac{|E|}{|S|} + \frac{|E^c|}{|S|}$.
- $1 = p(E) + p(E^c)$.

$\square$

The Probability of the Complement of an Event

Example 12

A fair coin is tossed 10 times. Find the probability of at least one tail.

Solution 13

Let E denote the event that the outcome is at least one tail. Then the complement E^c is 10 heads. It is clear that

$$p(E^c) = \frac{1}{2^{10}} = \frac{1}{1024}.$$

It then follows from Proposition 11 that

$$p(E)) = 1 - p(E^c) = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Probability of a Union of Events

Proposition 14

Let E_1 and E_2 be two events in a finite sample space S with uniform probability distribution. Then the probability of the union $E_1 \cup E_2$ of E_1 and E_2 is

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Proof.

- $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$ (by the Inclusion-exclusion Principle).
- $\frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$.
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

$\square$

Probability of a Union of Events

Example 15

An integer is chosen from the interval $[1, \ldots, 100]$ randomly. Find the probability that it is divisible either by 6 or by 15.

Solution 16

Let E_1 and E_2 be the event that the integer is divisible by 6 and 15, respectively. An integer is divisible by both 6 and 15 if and only if it is divisible by 30. Hence the event $E_1 \cap E_2$ is that the integer is divisible by 30.
Note that

$$p(E_1 \cap E_2) = \frac{\lfloor \frac{100}{30} \rfloor}{100} = \frac{3}{100}, \; p(E_1) = \frac{\lfloor \frac{100}{6} \rfloor}{100} = \frac{16}{100}, \; p(E_2) = \frac{\lfloor \frac{100}{15} \rfloor}{100} = \frac{6}{100}.$$

It then follows from Proposition 14 that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{16}{100} + \frac{6}{100} - \frac{3}{100} = \frac{19}{100}.$$

Probability of a Union of Events

Proposition 14 can be generalized as follows.

Proposition 17

Let $E_1, E_2, \ldots, E_n$ be n events in a finite sample space S with uniform probability distribution. Then

$$\begin{aligned} p(\cup_{i=1}^{n} E_i) &= \sum_i p(E_i) - \sum_{i<j} p(E_i \cap E_j) + \\ & \sum_{i<j<k} p(E_i \cap E_j \cap E_k) - \ldots + (-1)^{n+1} p(\cap_{i=1}^{n} E_i). \end{aligned}$$

Proof.

The desired conclusion follows from Definition 8 and the Inclusion-Exclusion Principle for sets, whose proof was left as an assignment problem. □

Rolling a Balanced (Fair) Dice

Problem 18

The experiment is to roll a balanced dice.

1. *The sample space* $S = \{1,2,3,4,5,6\}$.
2. *The probability of the event* $\{1\}$ *is* $\frac{1}{6}$.
3. *The probability of the event* $\{1,2,3\}$ *is*

$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}.$$

Rolling a Pair of Balanced Dice

Problem 19

The experiment is to roll a pair of balanced dice.

1. *The sample space* $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$*?*
2. *The probability of the event* $\{(1,1)\}$ *is* $\frac{1}{36}$.
3. *The probability of the event* $\{(1,1),(2,2)\}$ *is*

$$\frac{1}{36} + \frac{1}{36} = \frac{1}{18}.$$

Another Problem

Problem 20

*A base-10 numeral is **randomly** chosen from the range* $000\ldots999$.

1. *What is the probability that the numeral does not contain 3's and 5's?*
2. *What is the probability that the numeral contains one 3 and no 5's?*

Solution

1. There are 8^3 base-10 numerals containing no 3s and no 5s and 10^3 three-digit numerals altogether. Thus, the probability is

$$\frac{8^3}{10^3} = \left(\frac{4}{5}\right)^3.$$

2. The 3 could occur as each of the three digits. There would be 8^2 possibilities for the other two digits. Thus, the probability is $\frac{3\times 8^2}{10^3}$.

Limitations of Sample Spaces with Uniform Distribution

Question 1

Up to this point, we have calculated probabilities only for situations, such as tossing a fair coin or rolling a pair of balanced dice, where the outcomes in the sample space are all ***equally likely****. But coins are not always fair and dice are not always balanced.*
How is it possible to calculate probabilities for these more general situations?

Answer

We will deal with these more general situations with the general probability theory next time.

Online Problem

Definition 21

A **yes-no** question is any question whose answer is either Yes or No.

Problem 22

I have randomly chosen an integer in the set $\{1, 2, 3, \cdots, 62, 63, 64\}$*. Your task is to design a procedure that allows you to ask me a series of yes-no questions, so that you are able to determine the integer eventually. Your procedure should minimize the total cost, as I will charge you HK\$ 10000 for answering each yes-no question you ask.*