

Illustration of the Proof of Lemma 5.28

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In these slides, we illustrate the proof of Lemma 5.28 with an example.

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$$P(X = 1) = 1/3 \quad P(Y = 1) = 1/2$$

$$P(X = 2) = 1/3 \quad P(Y = 2) = 1/4$$

$$P(X = 4) = 1/3 \quad P(Y = 4) = 1/4$$

Suppose that we have two **independent** random variables, X, Y that each can take on the values $1, 2, 4$, but with different probability weights.

$$\begin{array}{lll} P(X = 1) = 1/3 & P(Y = 1) = 1/2 & \Rightarrow E(X) = 7/3 \\ P(X = 2) = 1/3 & P(Y = 2) = 1/4 & E(Y) = 2 \\ P(X = 4) = 1/3 & P(Y = 4) = 1/4 & E(X)(EY) = 14/3 \end{array}$$

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$Z = XY$ can **only** take on the values $1, 2, 4, 8, 16$.

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$Z = XY$ can **only** take on the values 1, 2, 4, 8, 16.

$$P(Z = 1) = P(X = 1 \wedge Y = 1) = \frac{1}{6}$$

$$P(Z = 2) = P(X = 1 \wedge Y = 2) + P(X = 2 \wedge Y = 1) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$$

$$\begin{aligned} P(Z = 4) &= P(X = 1 \wedge Y = 4) + P(X = 4 \wedge Y = 1) \\ &\quad + P(X = 2 \wedge Y = 2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} \end{aligned}$$

$$P(Z = 8) = P(X = 2 \wedge Y = 4) + P(X = 4 \wedge Y = 2) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$P(Z = 16) = P(X = 4 \wedge Y = 4) = \frac{1}{12}$$

$Z = XY$ can only take on the values 1, 2, 4, 8, 16. So

$$\begin{aligned} E(XY) &= E(Z) \\ &= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\ &\quad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\ &= \frac{14}{3} \\ &= E(X) \cdot E(Y) \end{aligned}$$

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On the next page, we mimic the proof of Lemma 5.28, using these X, Y . Reading the proof with this example in mind, might make the proof more understandable.

$$E(X)E(Y)$$

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E(X)E(Y) &= \sum_{x \in \{1,2,4\}} xP(X=x) \sum_{y \in \{1,2,4\}} yP(Y=y) \\
&= \sum_{x \in \{1,2,4\}} \sum_{y \in \{1,2,4\}} xyP(X=x)P(Y=y) \\
&= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x,y \in \{1,2,4\} \\ xy=z}} P(X=x)P(Y=y)
\end{aligned}$$

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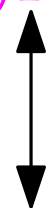
$$= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x,y \in \{1,2,4\} \\ xy=z}} P((X = x) \wedge (Y = y))$$

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Ind of X, Y

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&\quad \quad \quad \updownarrow \text{Ind of } X, Y \\
&= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x,y \in \{1,2,4\} \\ xy=z}} P((X=x) \wedge (Y=y)) \\
&= \sum_{z \in \{1,2,4,8,16\}} zP(Z=z) = E(Z) = E(XY)
\end{aligned}$$

$$\begin{aligned}
\boxed{E(X)E(Y)} &= \sum_{x \in \{1,2,4\}} xP(X=x) \sum_{y \in \{1,2,4\}} yP(Y=y) \\
&= \sum_{x \in \{1,2,4\}} \sum_{y \in \{1,2,4\}} xyP(X=x)P(Y=y) \\
&= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x,y \in \{1,2,4\} \\ xy=z}} P(X=x)P(Y=y) \\
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&= \sum_{z \in \{1,2,4,8,16\}} zP(Z=z) = E(Z) \boxed{= E(XY)}
\end{aligned}$$