COMP170
Discrete Mathematical Tools for Computer Science

Lecture 14 *Version 5 Last updated, Nov 17 2005*

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 5.1, pp. 213-221

Introduction to Probability

- Why Study Probability?
- Probability Spaces and Distributions
- Complementary Probabilities
- Probability and Hashing
- The Uniform Probability Distribution

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In Computer Science we often deal with random events.
Some involve randomness imposed from the outside, e.g., networking, when requests from computers on the network enter the network at “random” time.

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Studying the performance of computer systems in the presence of these types of randomness, requires understanding randomness, which is the study of probability.

Hashing

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The records are stored in a table. each table location, called a bucket or slot, holds a list of records. We are also given a hash function $h(x)$. A record with key key is stored in the bucket with index $h(key)$.

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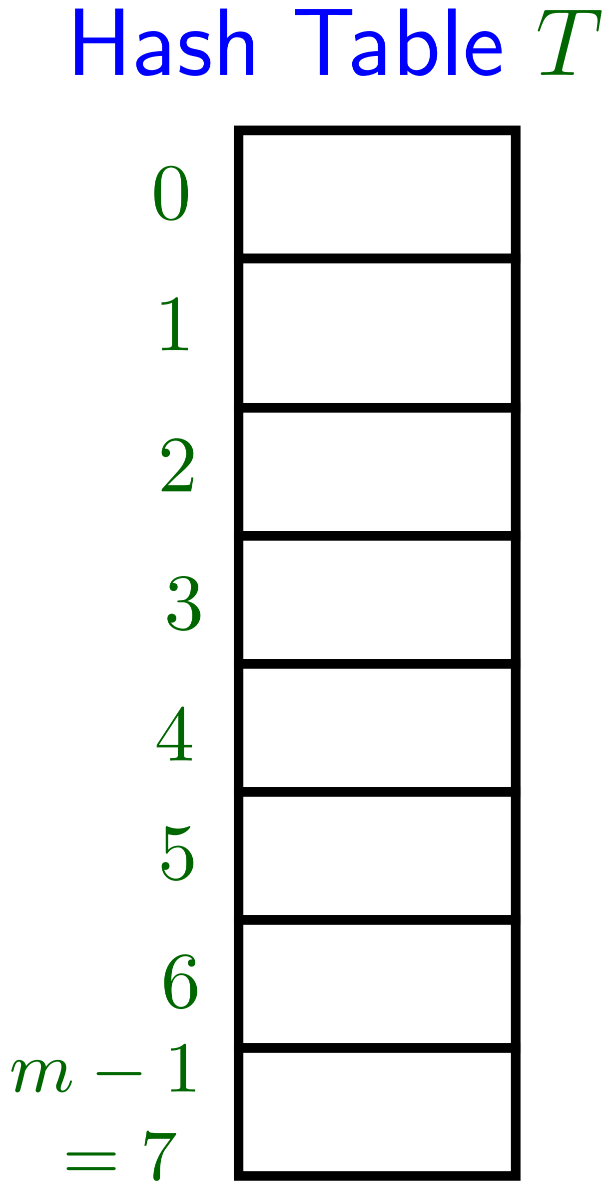

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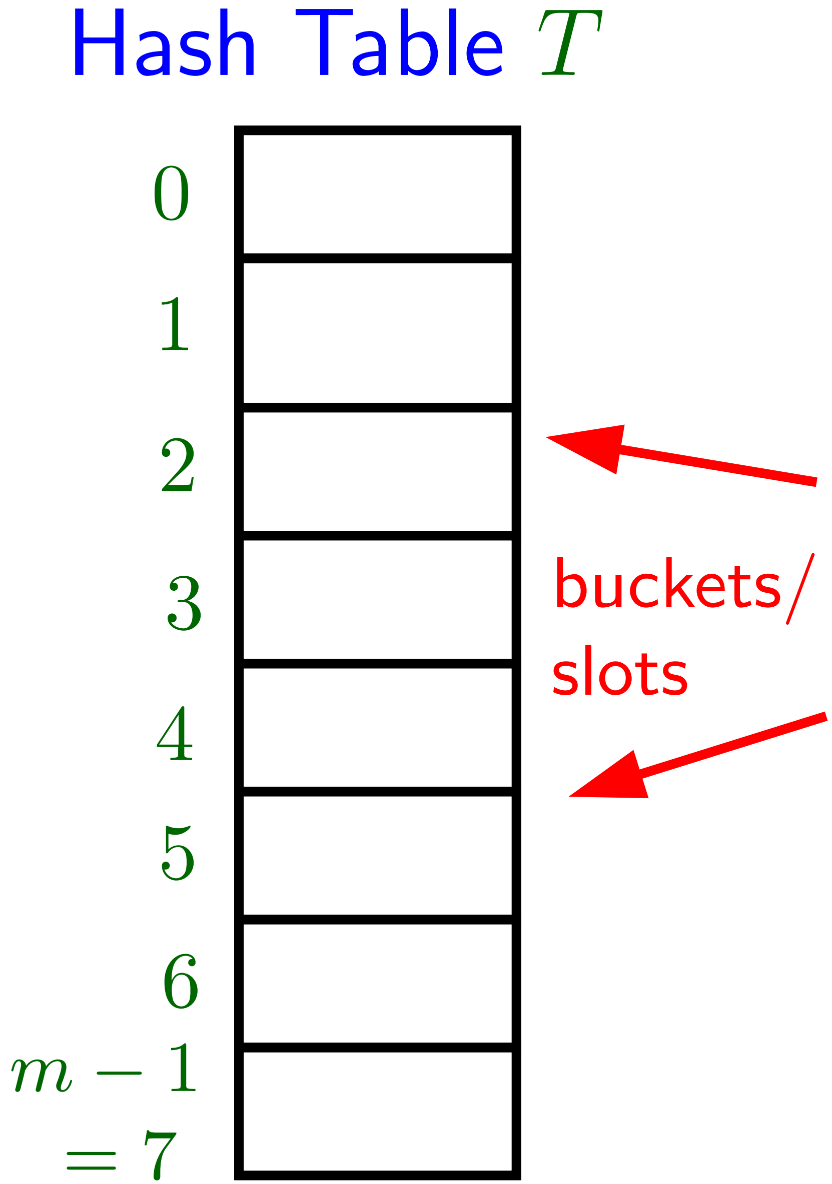

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Our Hash Function:

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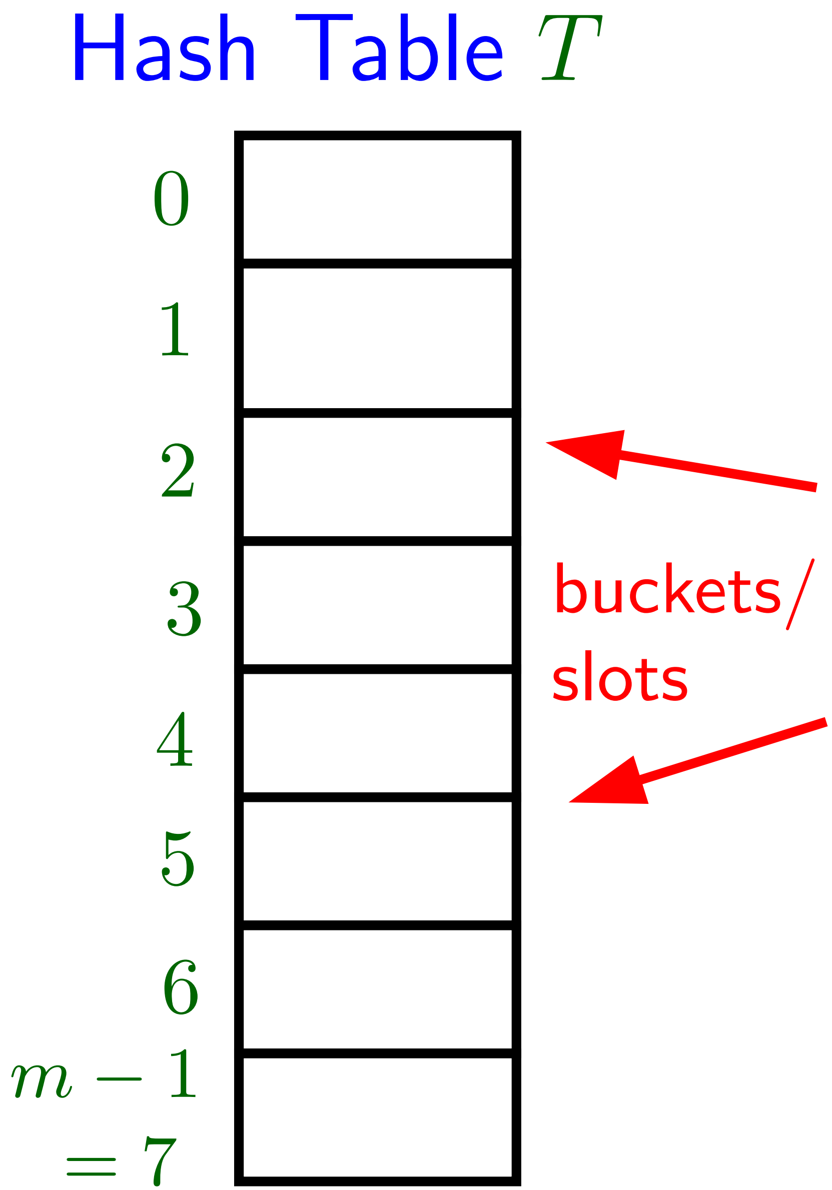

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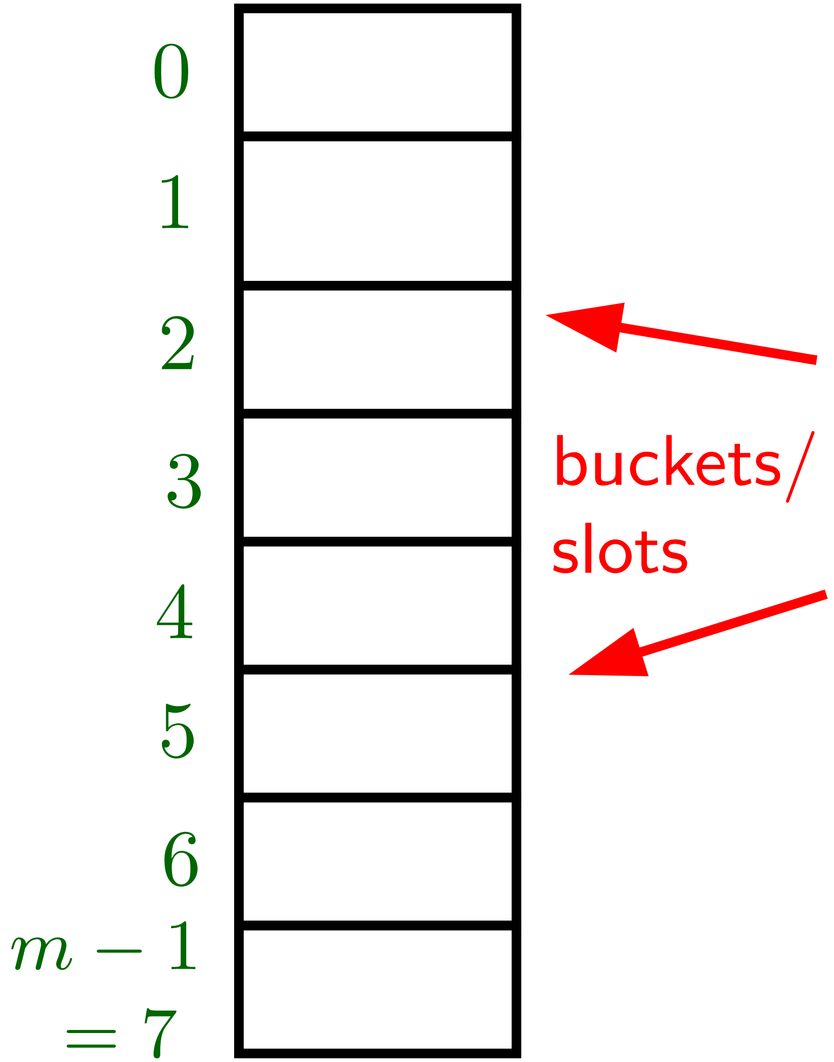
Data (with Keys)

Hash Table T



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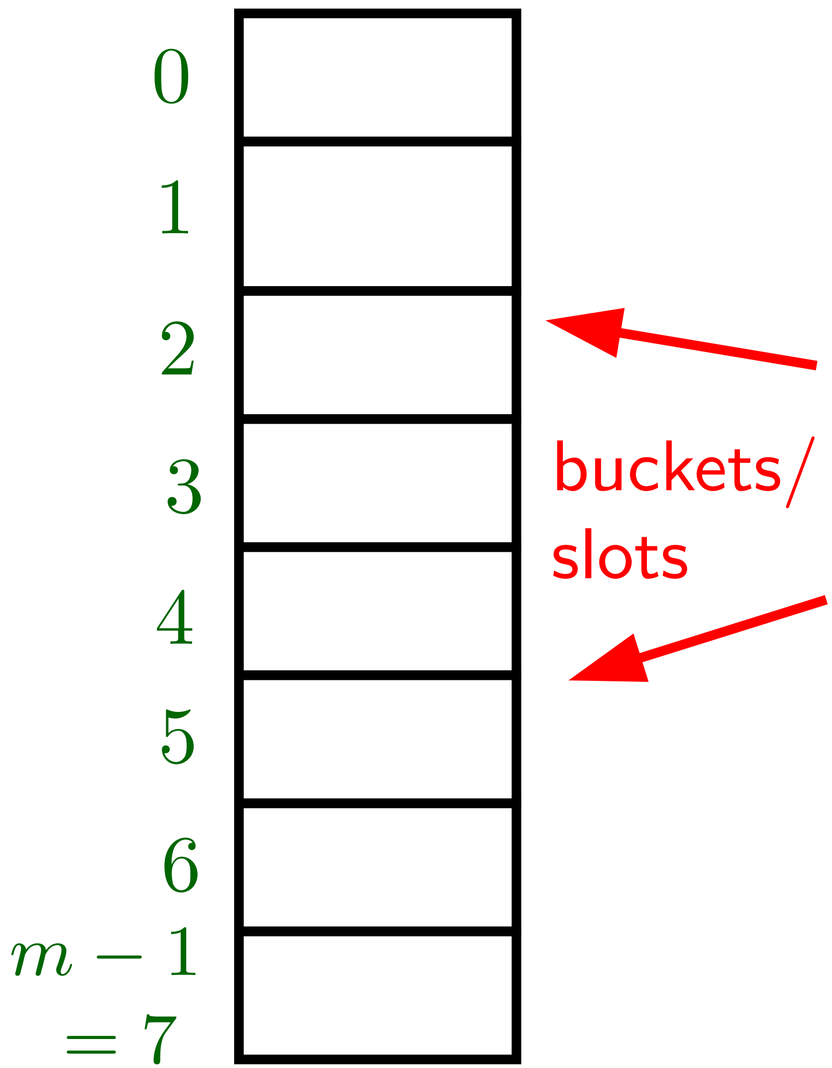

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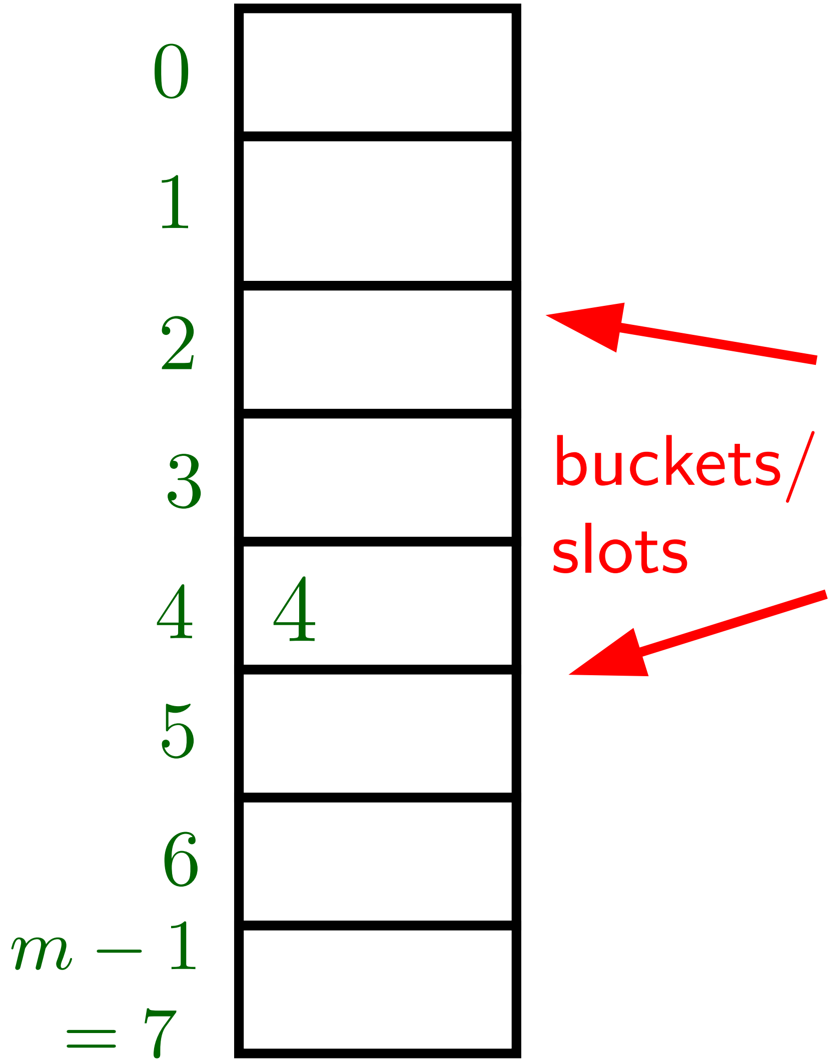

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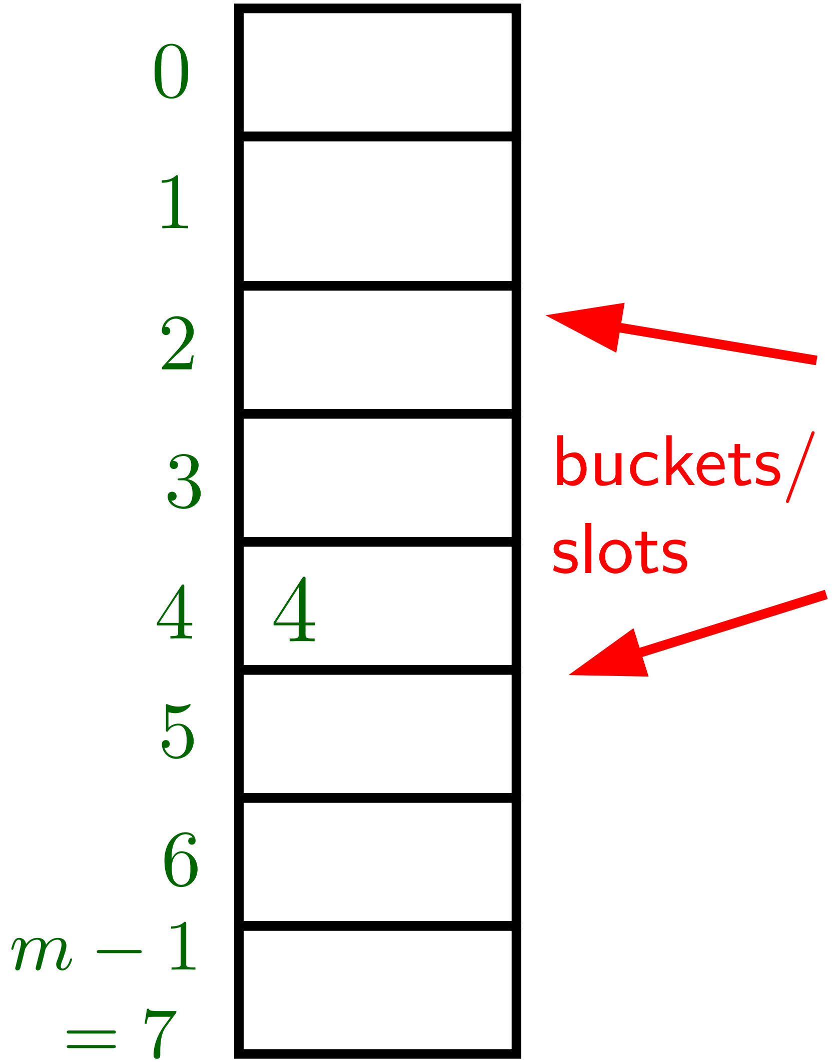

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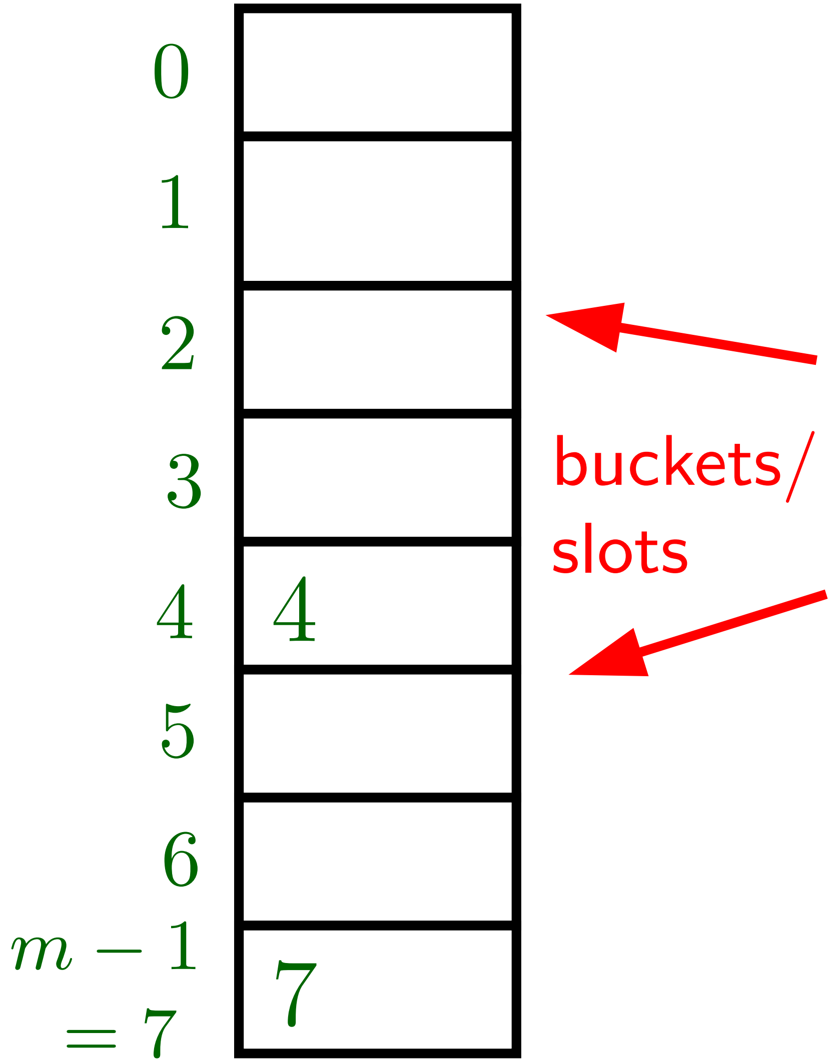

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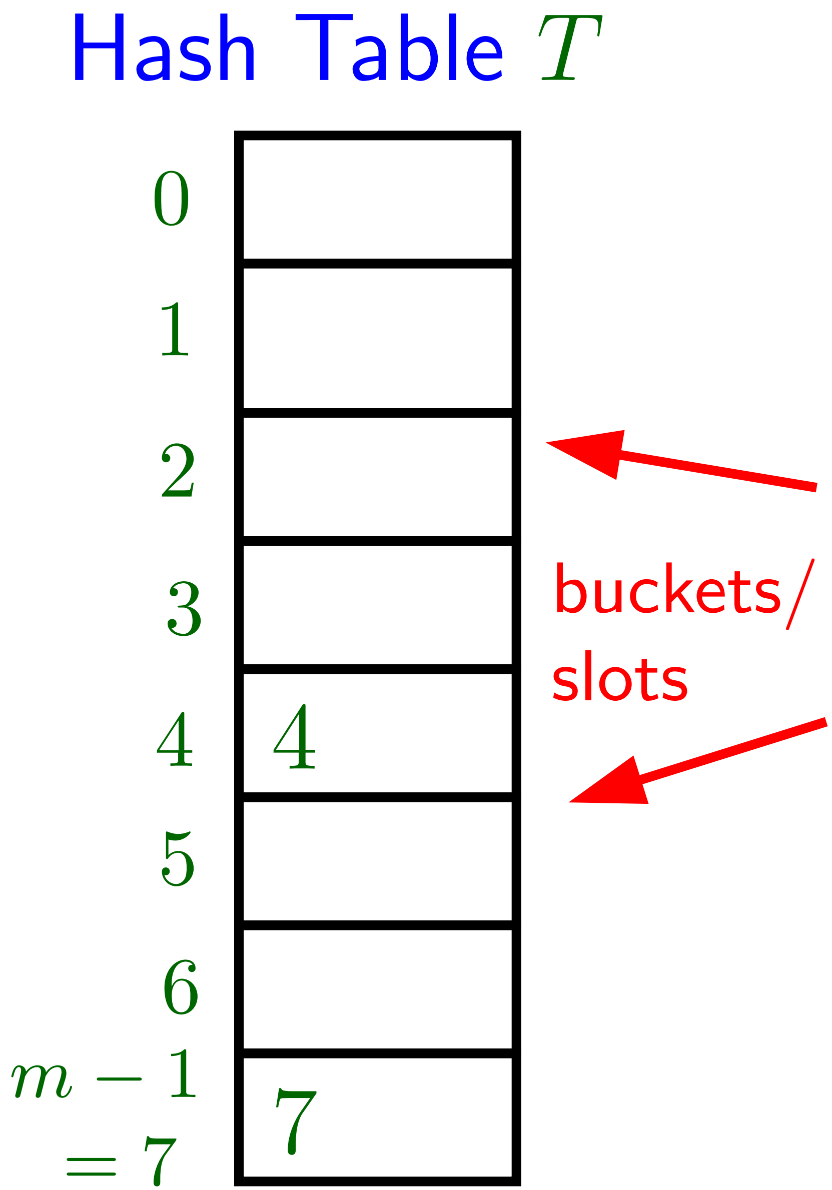

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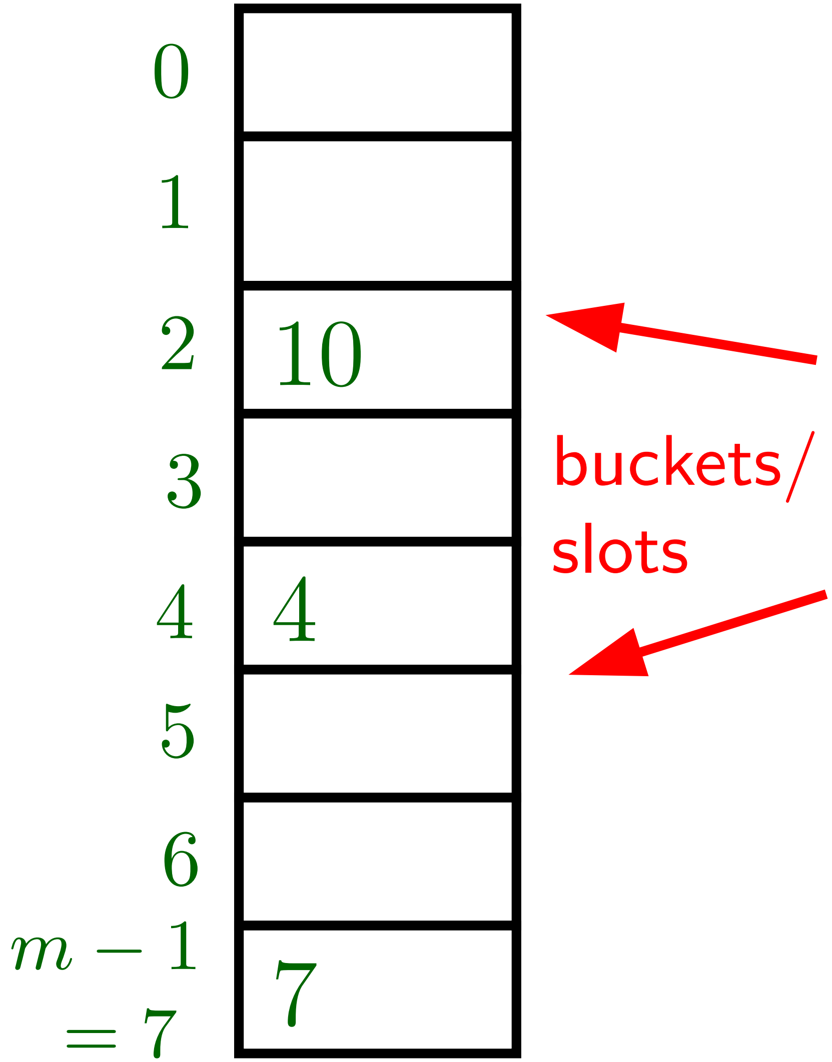

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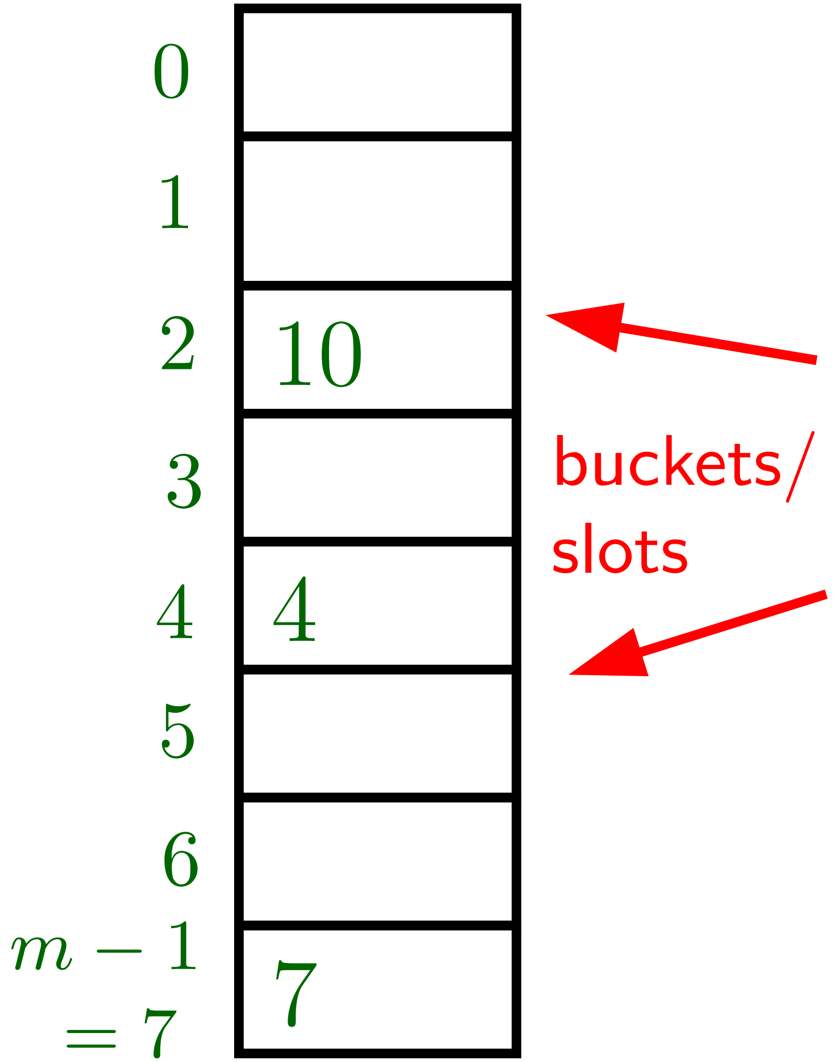

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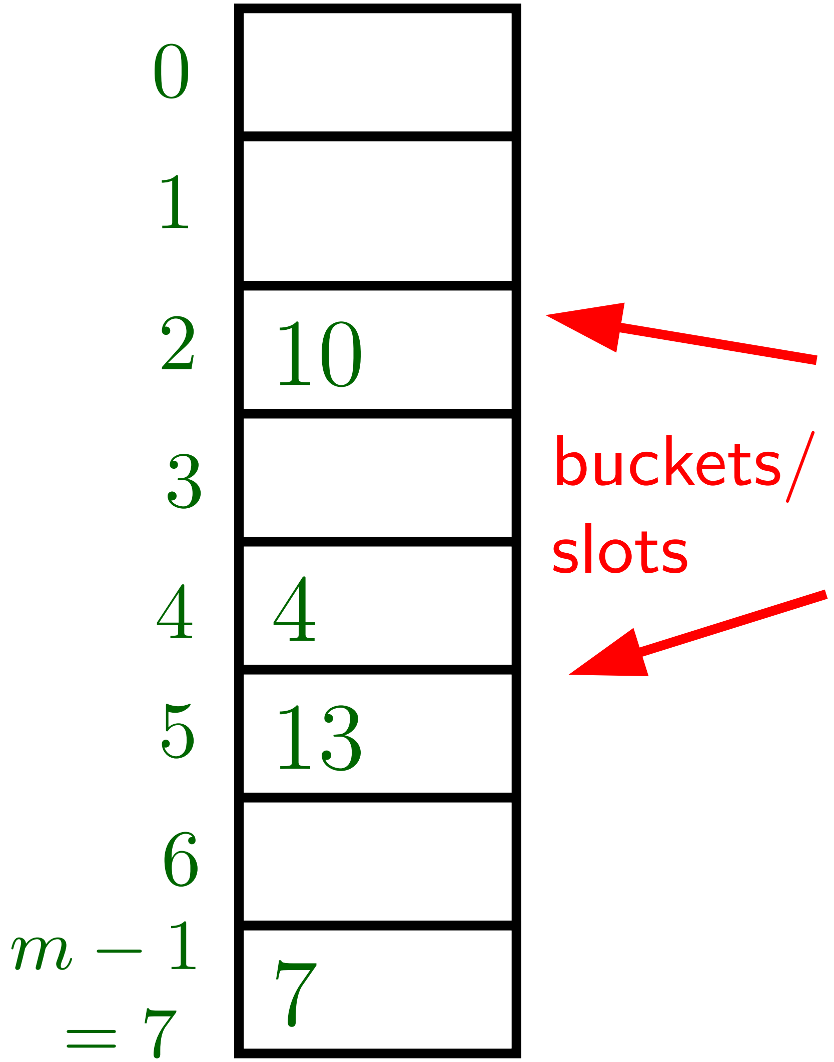

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$4 \quad 7 \quad 10 \quad 13 \quad 15$

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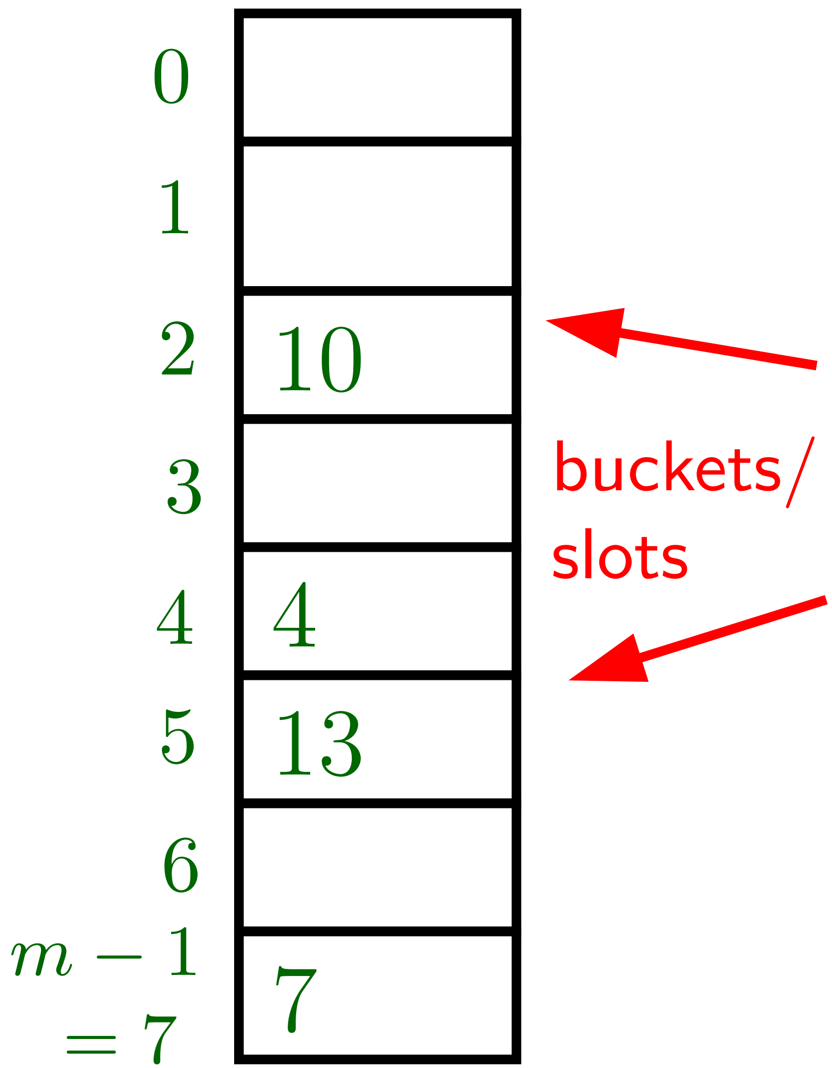

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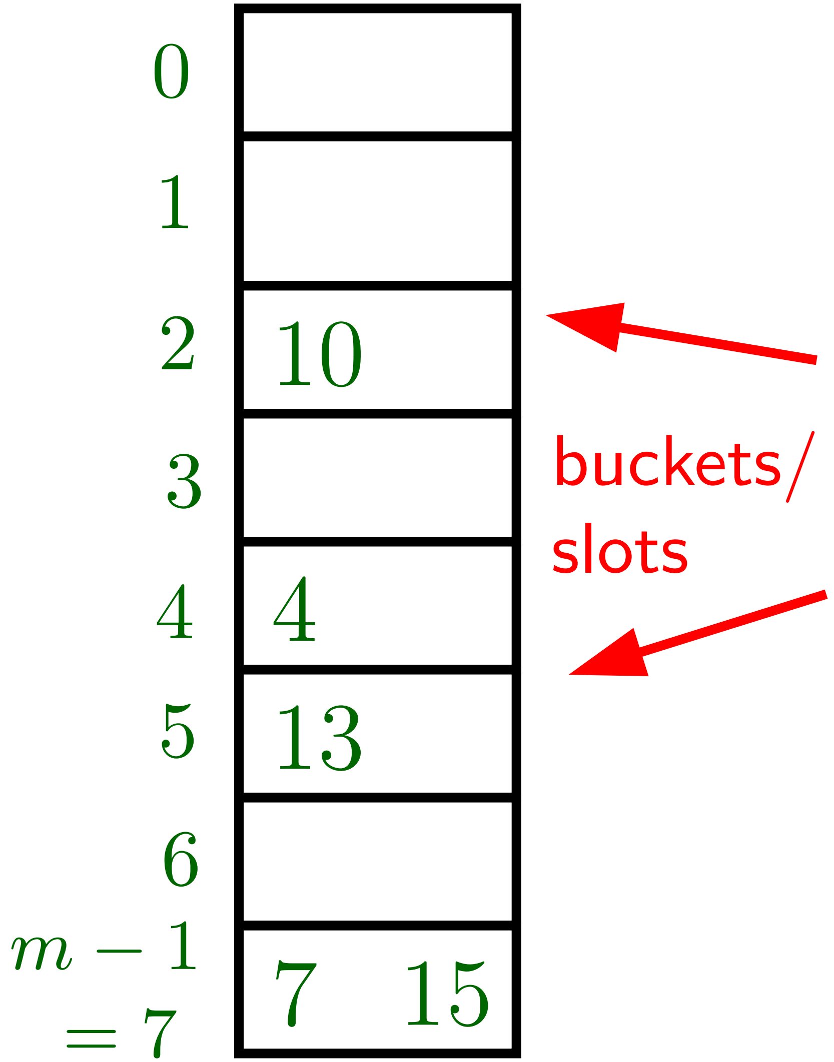

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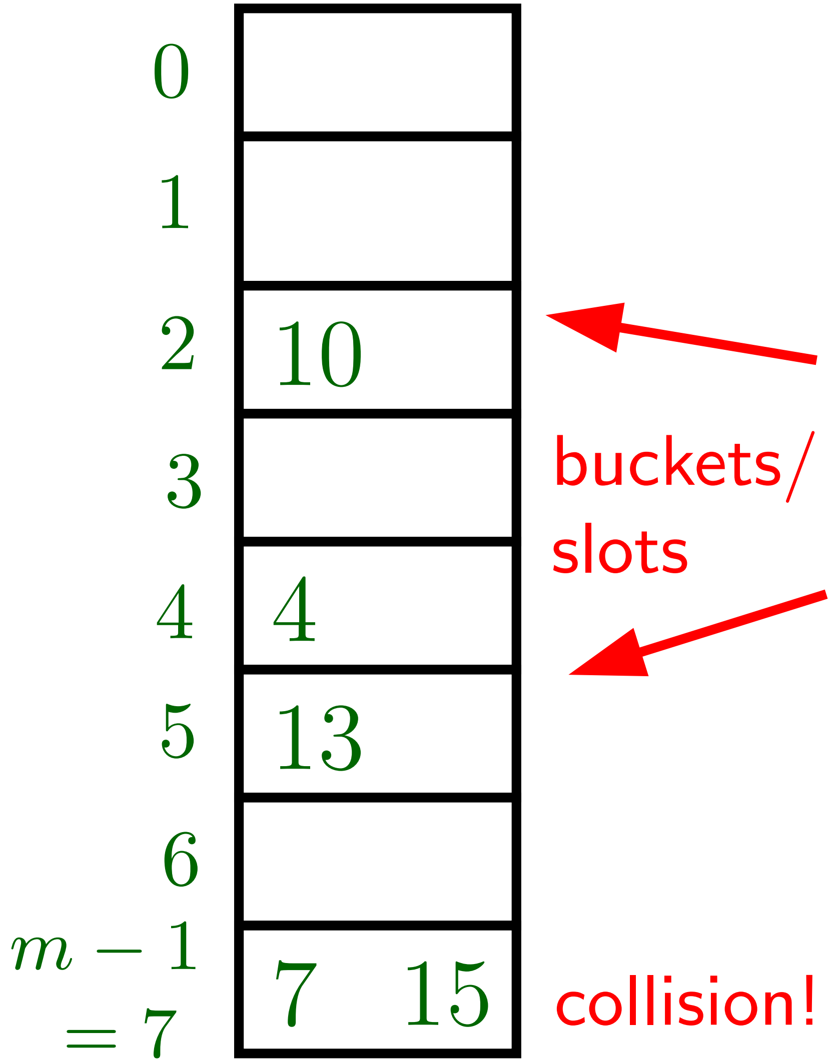

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When searching for a record you might have to look at *every* record in the appropriate bucket, so

Good hash function spreads keys evenly among buckets.

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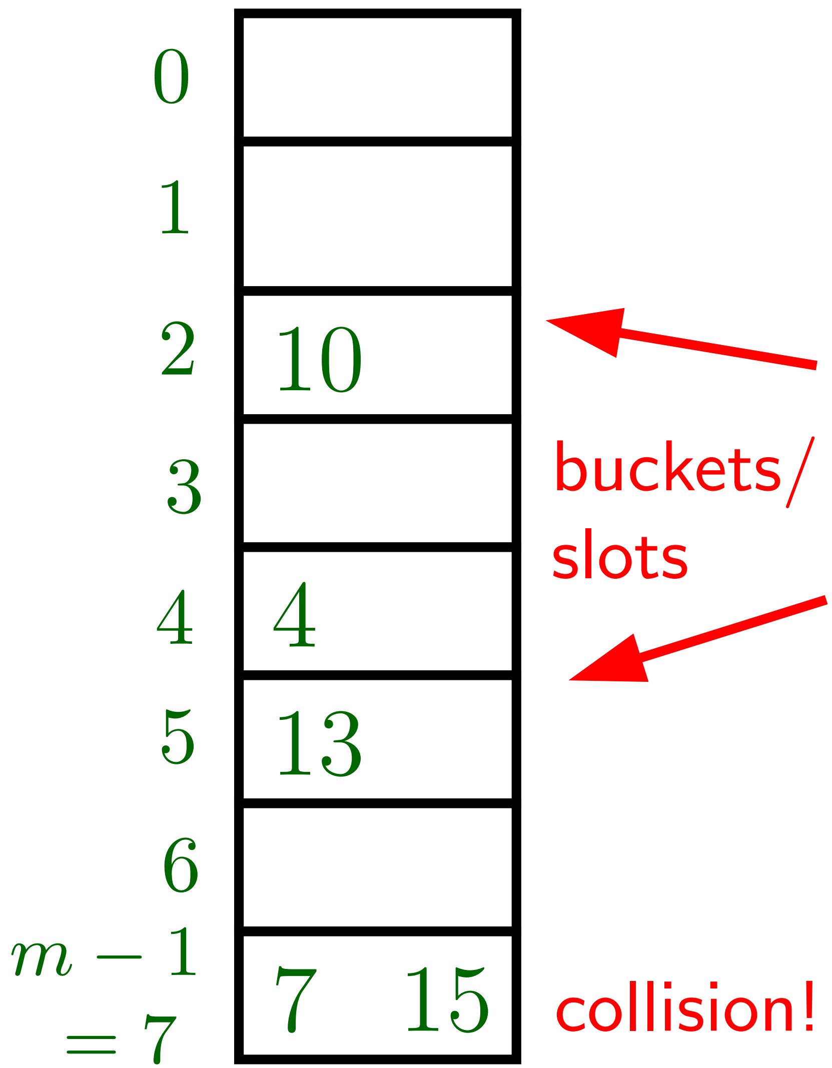

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How can we calculate likelihood of such events?

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$\longrightarrow$ Study of Probability

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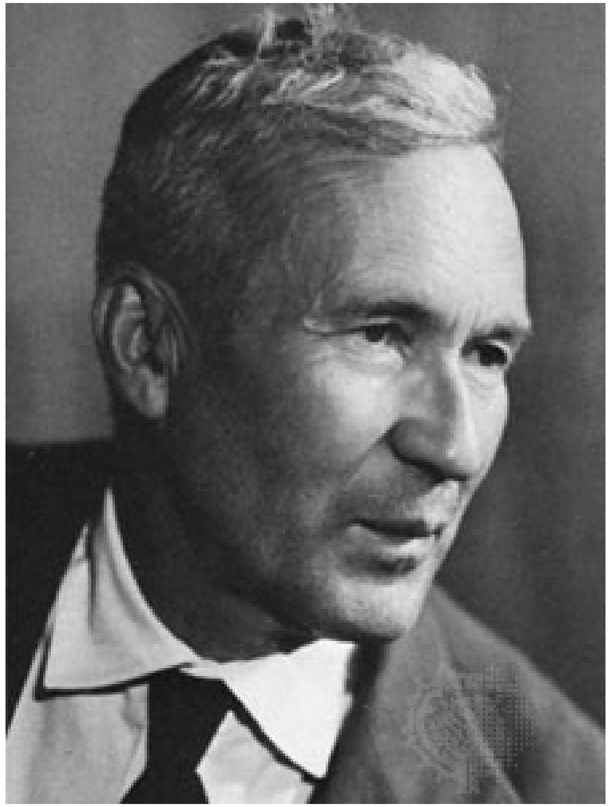
- The Weight of an element in the sample space
  Gives a Probability Distribution (Measure)

Andrei Nikolaevich Kolmogorov

Russian Mathematician

b. 1903. d. 1987



The birth of probabiity theory is often dated to 1654, when Pascal and Fermat, trying to solve a gambling problem, developed the fundamentals.

It wasn't until the work of Kolmogorav in 1933, though, that we had a "definition of probability that is precise enough for use in mathematics, yet comprehensive enough to be applicable to a wide range of phenomena".

Ref. A Short History of Probability by Tom Apostol
http://www.cc.gatech.edu/classes/cs6751_97_winter/Topics/stat-meas/probHist.html

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$=$ set of possible outcomes of a process.

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Professor starts each class
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The sample space of all possible patterns
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$$\texttt{S = \{TTT, TTF, TFT, FTT, TFF, FTF, FFT, FFF\}}.$$

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Note: TTT corresponds to all answers being true, etc..

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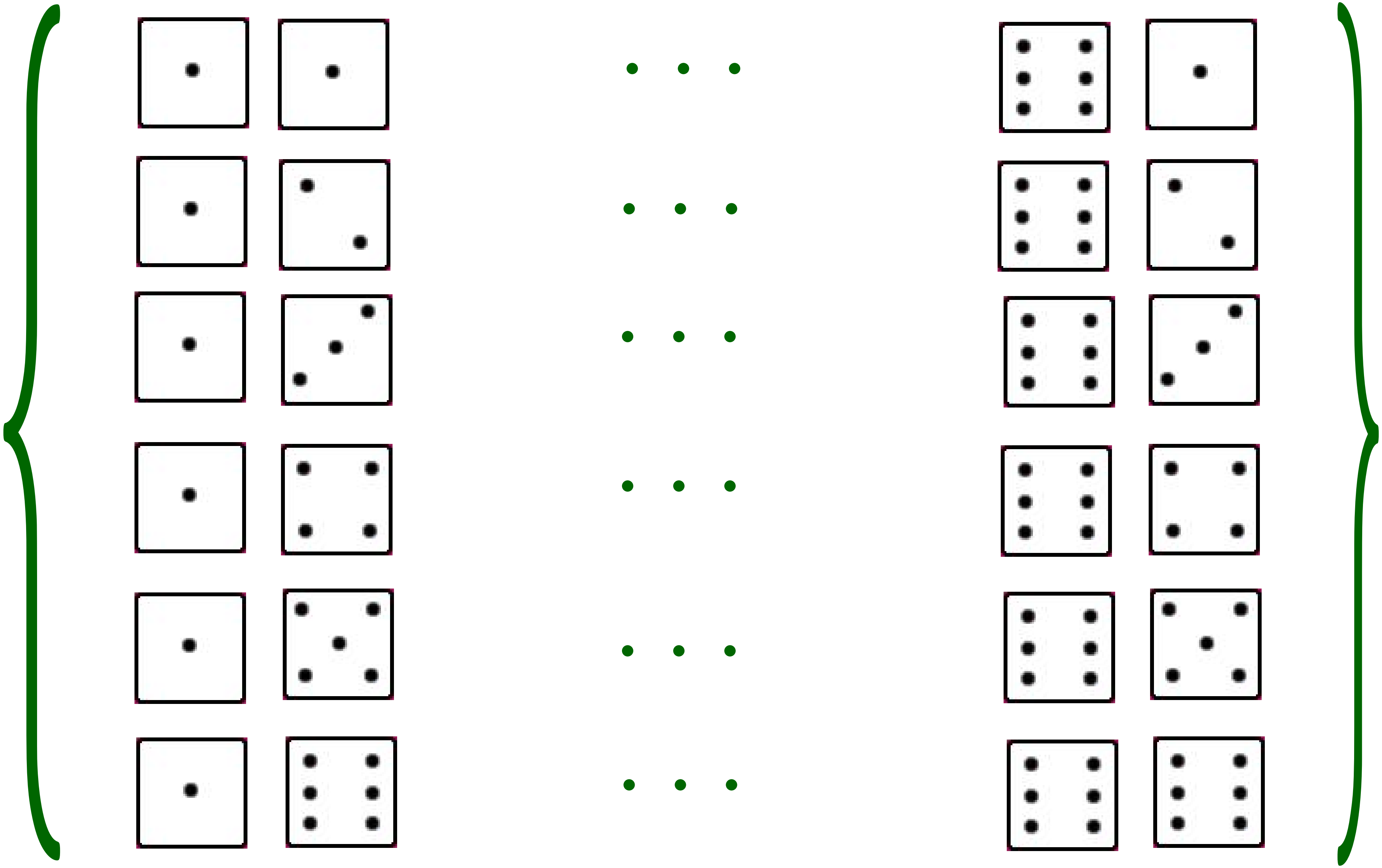

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Throw a two-sided (Heads, H, and Tails, T) coin, until you see a Head. This sample space contains an *infinite* number of outcomes.

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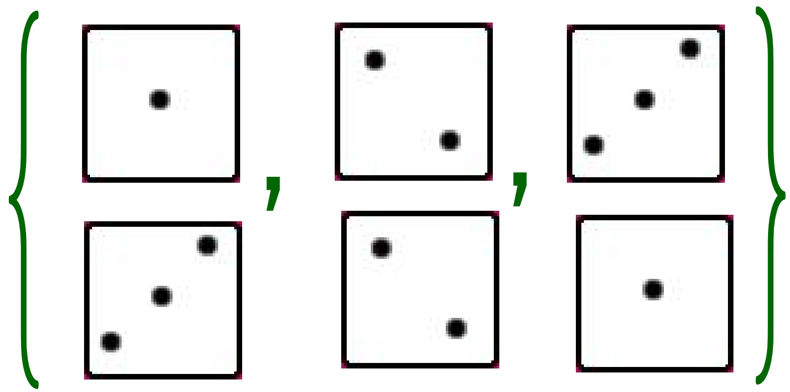
2) Event of the dice adding up to 4: 

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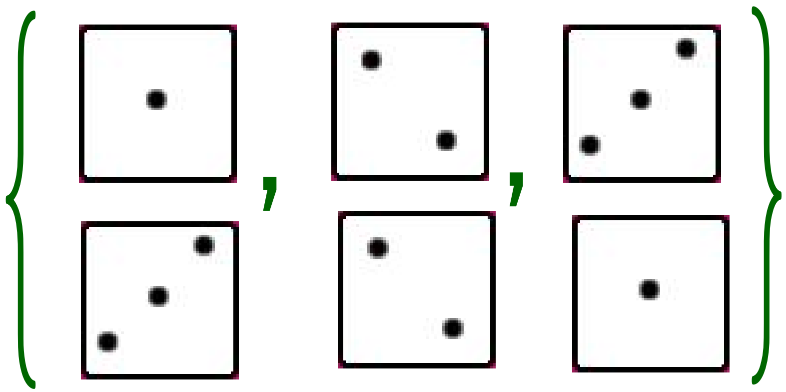
2) Event of the dice adding up to 4: 

3) Event of a Head occurring in the first 3 flips: $\{\texttt{H}, \texttt{TH}, \texttt{TTH}\}$.

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$$P(E) = \sum_{x : x \in E} P(x)$$

read: "The probability of event E *is the sum, over all* x *such that* x *is in* E*, of* $P(x)$*."*

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A function P satisfying these rules is called a **probability distribution** *or a* **probability measure**.

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Suppose that each sequence of T and F is equally likely. We then want to assign each outcome the same *(uniform)* probability weight. Since the sum of the weights must add up to 1, we assign each of the 8 outcomes a weight of $\frac{1}{8}$.

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The event "the first answer is true"
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$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Example 1b:

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$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Example 1c:

Now suppose that the professor has a bias towards true answers and a T is more likely than a F. Let s assume that a T has a probability of $2/3$ and a F has a probability of $1/3$. Then, we will see later, we will get the following different weights on the *same* sample space as before

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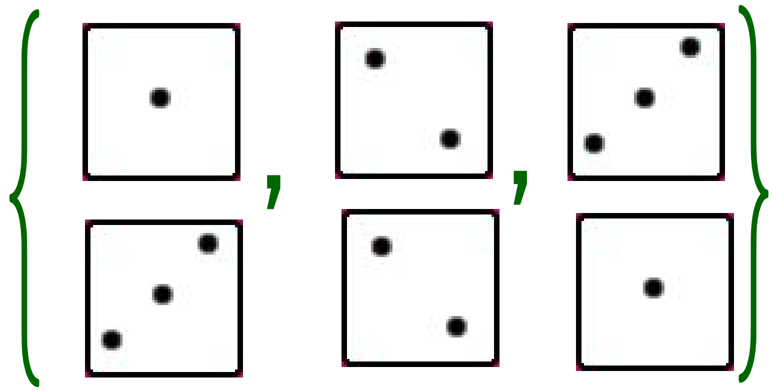
The event of the dice adding up to 4 is: 

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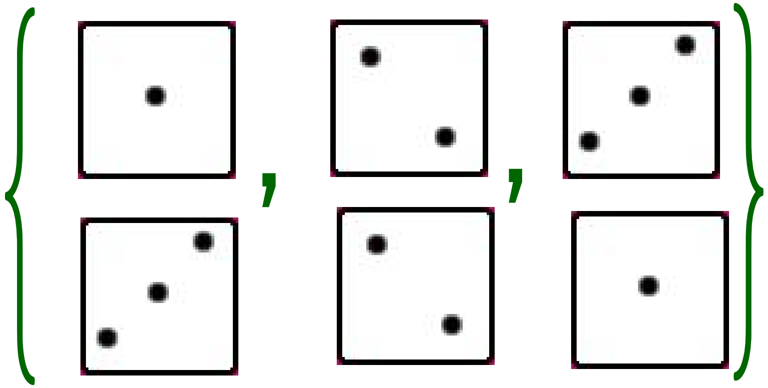
The event of the dice adding up to 4 is: 

There are three outcomes in this event, so its probability is

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}$$

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When throwing a coin until the first H is seen,
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Note that this is a legal probability distribution, since

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

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Note that $S = A \cup B$ (or $B = S - A$) and that $P(A) + P(B) = 1$.

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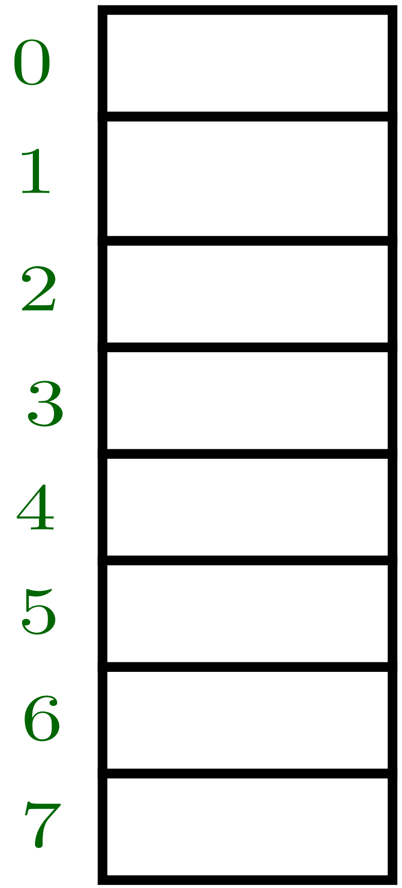

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Example: $3, 12, 15, 8, 11, 5$

is outcome $(3, 4, 7, 0, 3, 5)$

0	8
1	
2	
3	3 11
4	12
5	5
6	
7	15

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Weight function: Assuming that hash function is "random" then every n-tuple is equally likely. So, every n-tuple should have (the same) weight $1/m^n$.

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$$\Rightarrow \boxed{P(A) = \frac{20 \cdot 19 \cdot 18}{20^3} = .855}$$

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n	Probability of No Collisions
1	1
2	.95
3	.855
4	.72675
5	.5814
6	.43605
7	.305235
8	.19840275
9	.11904165
10	.065472908
11	.032736454
12	.014731404
13	.005892562
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$$\Rightarrow p = \frac{1}{|S|}$$

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$$P(E) = |E|p = |E|(1/|S|) = |E|/|S|.$$

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Use Theorem 5.2

$\Rightarrow$ probability is $4/8 = 1/2$ by Theorem 5.2.

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Theorem 5.2 doesn't apply to this distribution. For example, let E be the event that the outcome is not positive.
Then $E = \{0\}$ but

$$P(E) = \frac{1}{8} \neq \frac{1}{4} = \frac{|E|}{|S|}$$

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3 ways to get 2 H's. $P(2)$ should be 3 times $P(0)$

Where does this strange distribution come from?

Consider $\{0, 1, 2, 3\}$ as the sample space for flipping fair coin three times and counting number of H.

Determine appropriate weights $P(0), P(1), P(2), P(3)$.

{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}.

1 way to get no H.

3 ways to get 1 H. $P(1)$ should be 3 times $P(0)$

3 ways to get 2 H's. $P(2)$ should be 3 times $P(0)$

1 way to get 3 H's. $P(3)$ should be equal $P(0)$

We need

$$P(1) = 3P(0) \quad P(2) = 3P(0) \quad P(3) = P(0)$$

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$$\begin{aligned} 1 &= P(0) + P(1) + P(2) + P(3). \\ &= P(0) + 3P(0) + 3P(0) + P(0) = 8P(0) \end{aligned}$$

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The **unique** solution:

$$\boxed{P(0) = \frac{1}{8}, \quad P(1) = \frac{3}{8}, \quad P(2) = \frac{3}{8}, \quad P(3) = \frac{1}{8}}$$

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which is exactly the non-uniform distribution we just saw.