How Bad is Selfish Routing

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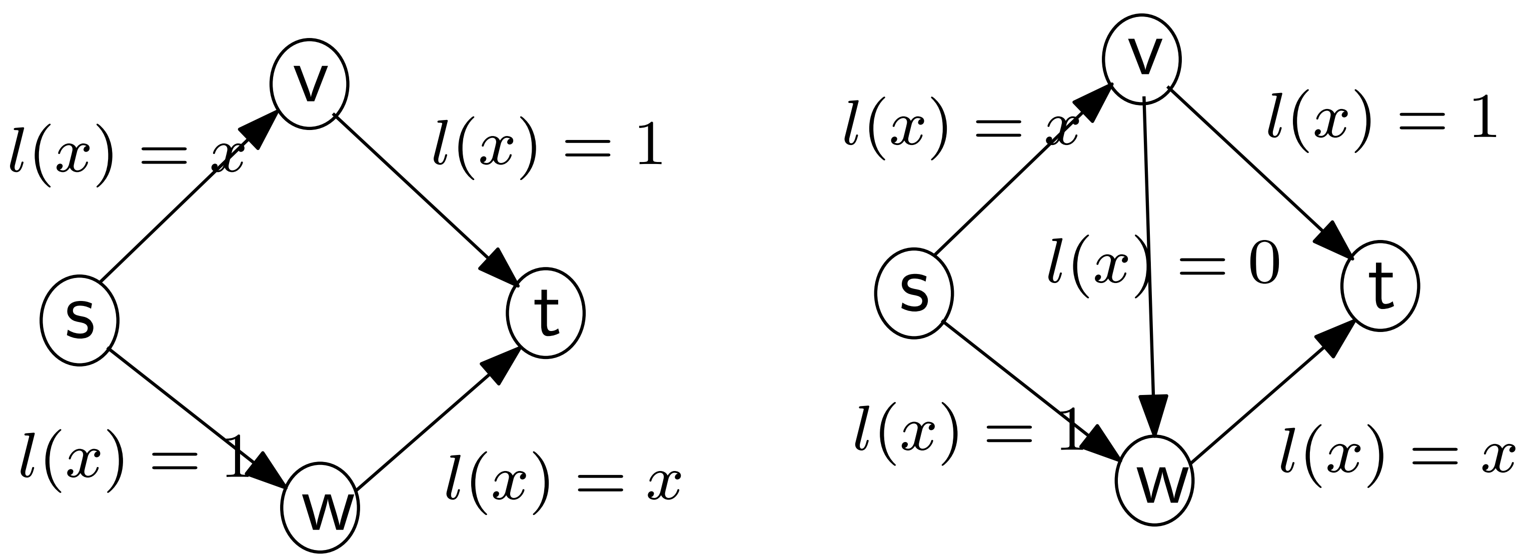
Problem Formulation: Traffic Model

- Given the rate of traffic between each pair of nodes in a network, find an assignment of traffic to minimize the total latency.
- On each edge, the latency is load dependent
- Each player controls a negligible fraction of the overall traffic.



Braess's Paradox

Formal Model

- Graph $G = (V, E)$ and k source-destination pairs $\{s_i, t_i\}$
- $\mathcal{P}_i$ denotes the set of (simple) $s_i - t_i$ paths, and
- $\mathcal{P} = \cup_i \mathcal{P}_i$
- A flow is a function:

$$f \; : \mathcal{P} \rightarrow \mathcal{R}^+$$

- A flow is feasible if :

$$\sum_{P \in \mathcal{P}_i} f_P = r_i$$

- Each edge has a nonnegative, differentiable, nondecreasing latency function $l_e(\cdot)$

Cost for Flows

- Let (G, r, l) be an instance , and f is a flow.

$$f_e = \sum_{P: e \in P} f_P$$

- Latency of a path P

$$l_P(f) = \sum_{e \in P} l_e(f_e)$$

- Cost of a flow f:

$$C(f) = \sum_{P \in \mathcal{P}} l_P(f) f_P = \sum_{e \in E} l_e(f_e) f_e$$

- Players are small flows behave "greedily" and "selfishly"

  There are infinite number of players, each carry a negligible amout of flow.

Flows at Nash Equilibrium

- **Definition (Nash Equilibrium):**

  A flow f is feasible for instance (G, r, l) is at Nash Equilibrium if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in \mathcal{P}_i$, and $\delta \in [0, f_{P_1}]$, we have $l_{P_1}(f) \le l_{P_2}(\tilde{f})$, where

$$\tilde{f}_P = \begin{cases} f_P - \delta & \text{if } P = P_1 \\ f_P + \delta & \text{if } P = P_2 \\ f_P & \text{if } P \notin \{P_1, P_2\} \end{cases}$$

- Lemma: A flow f feasible for instance (G, r, l) is at Nash Equilibrium if and only if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \le l_{P_2}(f)$.

Optimal Flows via Convex Programming

- NonLinear Programming Formulation

$$\text{Min} \sum_{e \in E} c_e(f_e)$$

subject to:

$$\begin{aligned}
&\sum_{P \in \mathcal{P}_i} f_P = r_i && \forall i \in \{1, \ldots, k\} \\
&f_e = \sum_{P \in \mathcal{P} : e \in P} f_P && \forall e \in E \\
&f_P \geq 0 && \forall P \in \mathcal{P}
\end{aligned}$$

Characteristic of Optimal Flows

Let c'_e be the derivative $\frac{d}{dx}c_e(x)$

$$c'_P(f) = \sum_{e \in P} c'_e(f_e)$$

- Lemma: A flow f is optimal for a convex program of the previous form if and only if for every $i \in \{1, \ldots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $c'_{P_1}(f) \leq c'_{P_2}(f)$.

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- **Lemma:** A flow f feasible for instance (G, r, l) is at Nash Equilibrium if and only if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq l_{P_2}(f)$.

$$C(f) = \sum_{i=1}^{k} L_i(f) r_i$$

Nash Equilibrium and Optimal Flow

Marginal cost function:

$$l_e^*(f_e) = (l_e(f_e)f_e)' = l_e(f_e) + l_e'(f_e)f_e$$

- Corollary: Let (G, r, l) be an instance in which $x \cdot l_e(x)$ is a convex function for each edge e, with marginal cost functions l_e^*. Then a flow f feasible for (G, r, l) is optimal if and only if it is at Nash equilibrium for the instance (G, r, l^*)

Nash Equilibrium and Optimal Flow (cont')

- Lemma: An instance (G, r, l) with continuous, nondecreasing latency functions admits a feasible flow at Nash equilibrium. Moreover, if $f, \tilde{f}$ are flows at Nash equilibrium, then $C(f) = C(\tilde{f})$.

Proof: Set $h_e(x) = \int_0^x l_e(t)dt$

Min $\sum_{e \in E} h_e(f_e)$

$$\sum_{P \in \mathcal{P}_i} f_P = r_i \qquad \forall i \in \{1, \ldots, k\}$$

$$f_e = \sum_{P \in \mathcal{P}: e \in P} f_P \qquad \forall e \in E$$

$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

Note, $h'_e(x) = l_e(x)$

"Unique" Nash Equilibrium

- Lemma: An instance (G, r, l) with continuous, nondecreasing latency functions admits a feasible flow at Nash equilibrium. Moreover, if $f, \tilde{f}$ are flows at Nash equilibrium, then $C(f) = C(\tilde{f})$.

Proof (cont') :

Set $h_e(x) = \int_0^x l_e(t)dt$

Min $\sum_{e \in E} h_e(f_e)$

If $f_e \neq \tilde{f}_e$, the function $h_e(x)$ must be linear and l_e is a constant function

This implies $l_e(f_e) = l_e(\tilde{f}_e)$.

$C(f) = \sum_{i=1}^{k} L_i(f) r_i = C(\tilde{f})$.

Nontrivial Upper Bound for Price of Anarchy

For instance (G, r, l), let f^* be an optimal flow and f be a flow at Nash equilibrium.

$$\rho = \rho(G, r, l) = \frac{C(f)}{C(f^*)}$$

Corollary: Suppose the instance (G, r, l) and the constant $\alpha \geq 1$ satisfy:

$$x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t)dt$$

$$\rho(G, r, l) \leq \alpha$$

Nontrivial Upper Bound for Price of Anarchy (cont')

Corollary: Suppose the instance (G, r, l) and the constant $\alpha \geq 1$ satisfy:

$$x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t)dt$$

$$\rho(G, r, l) \leq \alpha$$

Proof:

$$\begin{aligned} C(f) &= \sum_{e \in E} l_e(f_e) f_e \\ &\leq \alpha \sum_{e \in E} \int_0^{f_e} l_e(t)dt \\ &\leq \alpha \sum_{e \in E} \int_0^{f_e^*} l_e(t)dt \\ &\leq \alpha \sum_{e \in E} l_e(f_e^*) f_e^* \\ &= \alpha \cdot C(f^*) \end{aligned}$$

N.E optimizes this objective function.

Upper Bound for Polynomial Latency Function

Corollary: Suppose the instance (G, r, l) has the latency functions:

$$l_e(x) = \sum_{i=0}^{p} a_{e,i} x^i \qquad a_{e,i} \geq 0$$

$$\rho(G, r, l) \leq p + 1$$

Remarks: It is not tight.

$l_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$ $\rho \leq 2$

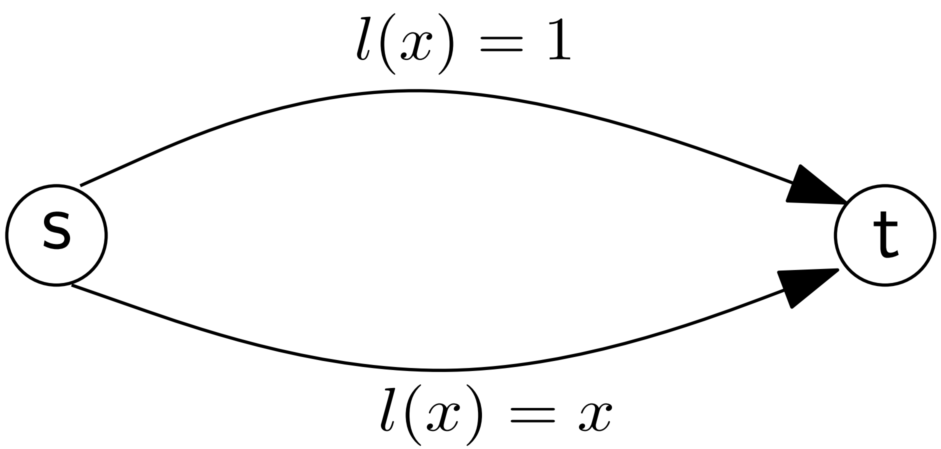
Tight Bound: $\rho \leq 4/3$

For higher degree polynomial latency functions:

$$\rho = O\left(\frac{p}{\ln p}\right)$$

A Bicriteria Result for General Latency Functions

Negative Result: :



$\rho = 4/3$

If $l(x) = x^p$: Optimal flows assgins $(p+1)^{-1/p}$ on the lower link, which has a total latency:

$$1 - p(p+1)^{-(p+1)/p} \longrightarrow 0$$

$$\rho \longrightarrow \infty$$

Augment Analysis for General Latency Function

- **Theorem:** If f is a flow at Nash equilibrium for (G, r, l) and f^* is feasible for $(G, 2r, l)$, then $C(f) \leq C(f^*)$

Let

$$\bar{l}_e(x) = \begin{cases} l_e(f_e) & \text{if } x \leq f_e \\ l_e(x) & \text{if } x \geq f_e \end{cases}$$

$$\begin{aligned} \sum_e \bar{l}_e(f_e^*) f_e^* - C(f^*) &= \sum_{e \in E} f_e^* (\bar{l}_e(f_e^*) - l_e(f_e^*)) \\ &\leq \sum_{e \in E} l_e(f_e) f_e \\ &= C(f) \end{aligned}$$

$$\bar{l}_P(f^*) \geq \bar{l}_P(f_0) \geq L_i(f)$$

$$\begin{aligned} \sum_e \bar{l}_P(f^*) f_P^* &\geq \sum_i \sum_{P \in \mathcal{P}_i} L_i(f) f_P^* \\ &= \sum_i 2 L_i(f) r_i \\ &= 2C(f) \end{aligned}$$

Worst-Case Ratio with Linear Latency Fuctions

$l_e = a_e x + b_e$ with $a_e, b_e \geq 0$

$l_e^* = 2a_e x + b_e$

- Lemma: If (G, r, l) be an instance with edge latency functions $l_e(x) = a_e x + b_e$ for each edge $e \in E$. Then

  (a) a flow f is at Nash equilibrium in G if and only if for $P, P' \in \mathcal{P}_i$ with $f_P > 0$,

  $$\sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e$$

  (b) a flow f^* is (globally) Optimal in G if and only if for $P, P' \in \mathcal{P}_i$ with $f_P^* > 0$,

  $$\sum_{e \in P} 2a_e f_e^* + b_e \leq \sum_{e \in P'} 2a_e f_e^* + b_e$$

Worst-Case Ratio with Linear Latency Fuctions (cont')

- Lemma: Suppose (G, r, l) has linear latency functions and f is a flow at Nash equilibrium. Then

  (a) The flow $f/2$ is optimal for $(G, r/2, l)$

  (b) the marginal cost of increasing the flow on a path P for $f/2$ equals the latency of P for f

$$l_P^*(f/2) = l_P(f)$$

Creating optimal flow in two steps: (f is at Nash equilibrium)

(1) Send a flow optimal for instance $(G, r/2, l)$. $C(f)/4$

(2) Augment to one optimal for instance (G, r, l). $C(f)/2$

Augment Cost for Linear Latency Functions

- **Lemma:** (G, r, l) has linear latency functions and f^* is an optimal flow. Let $L_i^*(f^*)$ be the minimum marginal cost for $s_i - t_i$ paths. For any $\delta > 0$, a feasible flow f for $(G, (1+\delta)r, l)$:

$$C(f) \geq C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i$$

$x \cdot l_e(x) = a_e x^2 + b_e$ is convex.

$$l_e(f_e) f_e \geq l_e(f_e^*) f^* + (f_e - f^*) l_e^*(f_e^*)$$

Augment Cost for Linear Latency Functions

- Proof:

$$
\begin{aligned}
C(f) &= \sum_{e \in E} l_e(f_e) f_e \\
&\geq \sum_{e \in E} l_e(f_e^*) f_e^* + \sum_{e \in E} (f_e - f_e^*) l_e^*(f_e^*) \\
&= C(f^*) + \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} l_P^*(f^*)(f_P - f_P^*) \\
&\geq C(f^*) + \sum_{i=1}^{k} L_i^*(f^*) \sum_{P \in \mathcal{P}_i} (f_P - f_P^*) \\
&= C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i
\end{aligned}
$$

Worst-Case Ratio with Linear Latency Fuctions (cont')

- **Lemma:** If (G, r, l) has linear latency functions, then $\rho(G, r, l) \leq 4/3$

Proof: Let f be a flow at N.E. $f/2$ is optimal for $(G, r/2, l)$. Moreover, $L_i^*(f/2) = L_i(f)$.

$$
\begin{aligned}
C(f^*) &\geq C(f/2) + \sum_{i=1}^{k} L_i^*(f/2)\frac{r_i}{2} \\
&= C(f/2) + \frac{1}{2}\sum_{i=1}^{k} L_i(f) r_i \\
&= C(f/2) + \frac{1}{2}C(f) \\
&\geq \frac{3}{4}C(f)
\end{aligned}
$$

$$
\begin{aligned}
C(f/2) &= \frac{1}{4}a_e f_e^2 + \frac{1}{2}b_e f_e \\
&\geq \frac{1}{4}\sum_e (a_e f_e^2 + b_e f_e) \\
&= \frac{1}{4}C(f)
\end{aligned}
$$

Extensions:

- Approximate Nash Equilibrium:

  If f is at ϵ N.E, and f^* is feasible for $(G, 2r, l)$, then $C(f) \leq \frac{1+\epsilon}{1-\epsilon} C(f^*)$.

- Finite Agents: Splittable Flow

$$C(f) \leq C(f^*).$$

- Finite Agents: Unsplittable Flow

  If for some $\alpha < 2$, $l_e(x + r_i) \leq \alpha \cdot l_e(x)$, $x \in [0, \sum_{j \neq i} r_j]$

$$C(f) \leq \frac{\alpha}{2-\alpha} C(f^*).$$