Quicksort

Revision of September 11, 2014

THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Outline

Reference: Chapter 7 of CLRS

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- Partitions
- Quicksort
- Analysis of Quicksort

Partition

Given: An array of numbers
Partition: Rearrange the array $A[p..r]$ in place into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

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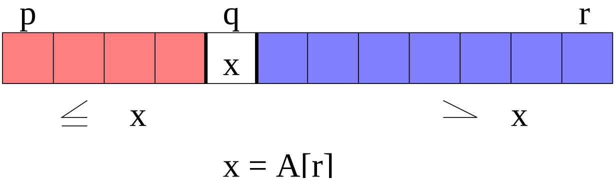

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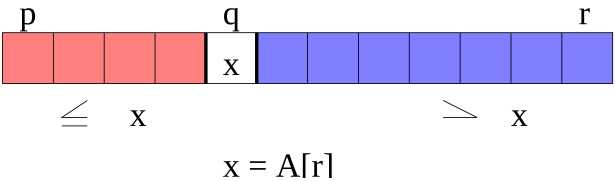

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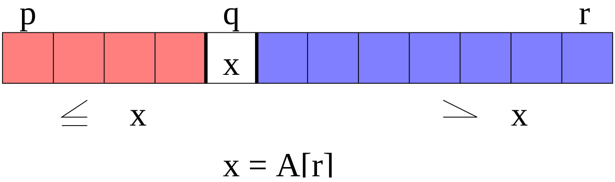

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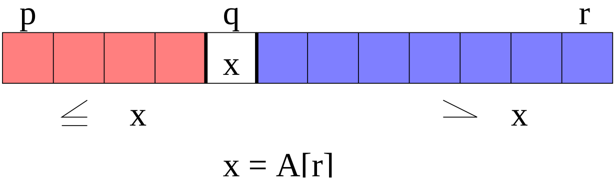

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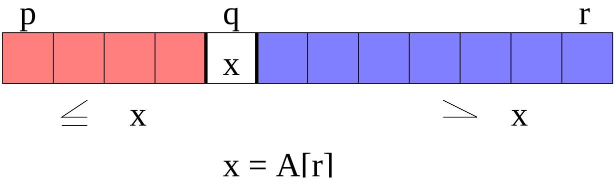

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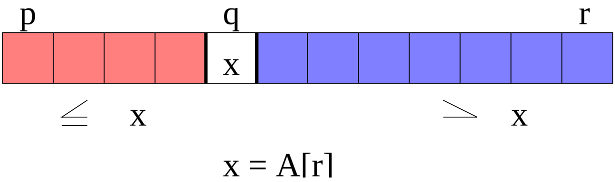

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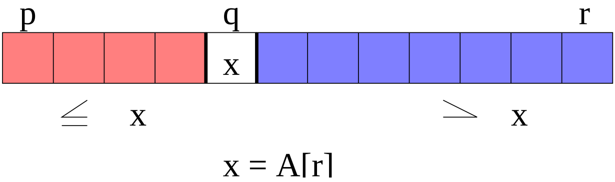

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  - items smaller than x into first $R - 1$ locations
  - x into location $p + R - 1$
  - items larger than x into last $r - R$ locations

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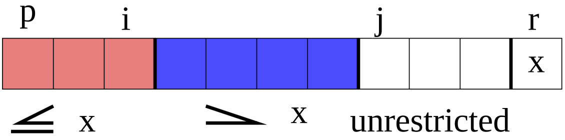
  - items smaller than x into first $R - 1$ locations
  - x into location $p + R - 1$
  - items larger than x into last $r - R$ locations
- $O(r - p)$ time but needs extra space.

Partition(A, p, r) without extra memory

Use $A[r]$ as the pivot, and grow partition from left to right.

- i will be largest index of processed item $\leq x$.
- j will be smallest index of unprocessed item.



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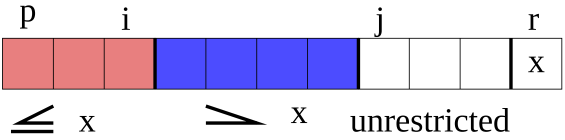

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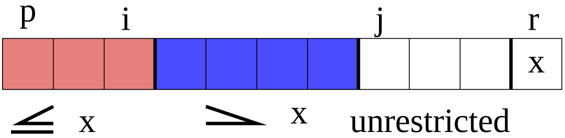

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2. Increase j by 1 each time to find a place for $A[j]$
   At the same time increase i when necessary

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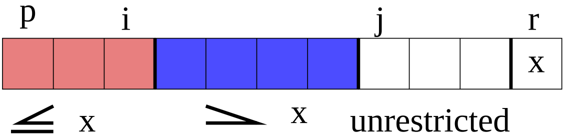

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3. Stops when $j = r$

One Iteration of the Procedure Partition

Increase j by 1 each time to find a place for $A[j]$

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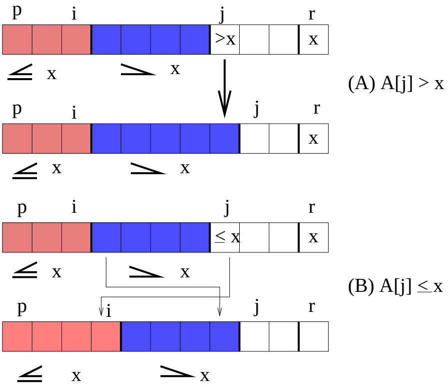

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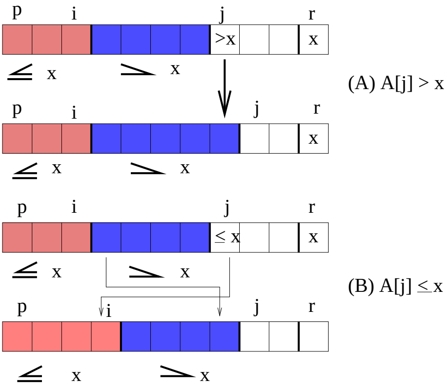

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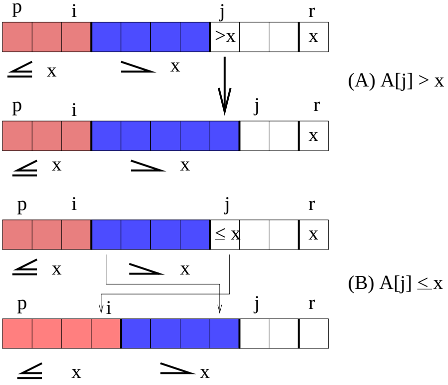

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2. $i = i + 1$.

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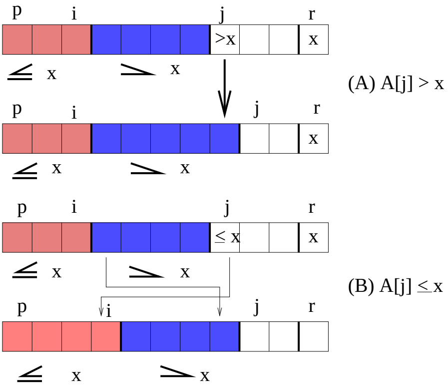

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2. $i = i + 1$. $A[i] \leftrightarrow A[j]$.

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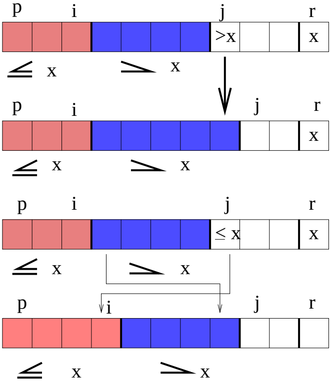

(A) A[j] > x

(B) A[j] ≤ x

1. Only increase j by 1
2. $i = i + 1$. $A[i] \leftrightarrow A[j]$. $j = j + 1$

Example: The Operation of Partition(A, p, r)

i	p, j							r
	2	8	7	1	3	5	6	4

(1)

p, i	j						r
2	8	7	1	3	5	6	4

(2)

p, i		j					r
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(3)

p, i			j				r
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(4)

p	i			j			r
2	1	7	8	3	5	6	4

(5)

p		i			j		r
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(6)

p		i				j	r
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(7)

p		i					j, r
2	1	3	8	7	5	6	4

(8)

p		i					j, r
2	1	3	4	7	5	6	8

(9)

Partition(A, p, r)

begin

$x = A[r]$; `// A[r] is the pivot element`

The Partition(A, p, r) Algorithm

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  $i = p - 1$;

  for $j = p$ *to* $r - 1$ **do**

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            i = i + 1;
            exchange A[i] and A[j];
        end
    end
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                              ; // put pivot in position
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    return i + 1 // q = i + 1
end
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    for j = p to r − 1 do
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Running Time of Partition(A, p, r)

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Quicksort

Quicksort(A, p, r)

```
begin
  if p < r then
    q = Partition(A, p, r);
    Quicksort(A,          );
    Quicksort(A,          );
  end
end
```

Quicksort(A, p, r)

begin
    if $p < r$ **then**
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```

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$

Outline:

- Partition
- Quicksort
- Average Case Analysis of Quicksort

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Recurrence: $T(n) =$

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\end{aligned}
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What inputs give worst case performance?

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We will analyze average case running time.

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1. Worst-case doesn't make sense: for any given input, the worst case is very unlikely to happen
2. Use Average Case Analysis
3. Assume every possible input permutation of the n items are equally likely.
4. $n!$ permutations so each one has probability $\frac{1}{n!}$ of ocurring
5. If S_n is set of all permutations, $\sigma \in S_n$ is a possible input permutation, then average running time is

$$\frac{1}{n!} \sum_{\sigma \in S_n} C(\sigma)$$

Average Case Analysis

Let **A** be the set of items in $A[p..r]$ and σ a random permutation of **A**.

1. $A[r]$ is equally likely to be any item in **A**.
2. After running the partition algorithm on $A[p..r]$, the input to the new left and right subproblems are again random permutations (need to argue why).

Recall that if X is a random variable and $E_1, E_2, \ldots, E_n$ are events that partition the probability space then we can write the expectation of X in terms of the Expectation of X conditioned on E_i. That is

$$E(X) = \sum_i E(X|E_i)\Pr(E_i).$$

Average Case Analysis

Assume that the input to is a random permutation of N items.

- Let C_N be the average amount of work performed on the input
- $C_0 = C_1 = 0$.
- Partition requires $N - 1$ comparisons
- Each item has probability $1/N$ of being pivot.
- If Item k is pivot, the two remaining subproblems require $C_{k-1} + C_{N-k}$ average time

$$
\begin{aligned}
C_N &= N - 1 + \frac{1}{N} \sum_{1 \le k \le N} (C_{k-1} + C_{N-k}) \\
&= N - 1 + \frac{2}{N} \sum_{1 \le k \le N} C_{k-1}
\end{aligned}
$$

Average Case Analysis

Multiplying both sides of previous equation by N and then rewriting the equation for $N-1$ yields

$$NC_N = N(N-1) + 2\sum_{1\le k\le N} C_{k-1}, \qquad (N-1)C_{N-1} = (N-1)(N-2) + 2\sum_{1\le k\le N-1} C_{k-1}$$

Subtracting the 2nd from the 1st and simplifying yields

$$NC_N = (N+1)C_{N-1} + 2N - 2$$

Dividing both sides by $N(N+1)$ gives

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1} - \frac{2}{N(N+1)}.$$

Telescoping the recurrence down to $N = 3$ and recalling that $C_1 = 0$ yields

$$\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} - \frac{2}{N(N+1)} \\
&= \frac{C_{N-2}}{N-1} + \left(\frac{2}{N} - \frac{2}{(N-1)N}\right) + \left(\frac{2}{N+1} - \frac{2}{N(N+1)}\right) \\
&= \dots \\
&= \frac{C_1}{2} + 2\sum_{i=3}^{N} \frac{1}{i+1} - \sum_{i=3}^{N} \frac{2}{i(i+1)} \\
&= 2H_{N+1} - 2H_3 + O(1) = 2H_N + O(1)
\end{aligned}$$

where $H_N = \sum_{i=1}^{N} 1/i$ and we are using the fact that $\sum_{i=1}^{\infty} 1/i(i=1)$ is bounded.

Average Case Analysis

We just saw that

$$\frac{C_N}{N+1} = 2H_N + O(1).$$

H_N is called the Nth *Harmonic number* and it is well known that

$$H_n = \ln n + O(1).$$

So, we have just proven that the average number of operations performed running Quicksort on a random permutation of N items is

$$C_N = 2(N+1)H_N + O(N) = 2N \ln N + O(N).$$

Odds and Ends

- Quicksort is a divide and conquer algorithm.
- The Quicksort code can be *tuned*
  - When N is small, call Insertion Sort rather than Quicksort (on very small N, Insertion sort is faster.
  - Instead of using last item $A[r]$ as pivot, set pivot to be median of first, last and middle item. (Why should this help?)
- *qsort* under UNIX was an extremely popular sorting routine for decades. It was a finely tuned version of Quicksort
- Quicksort was published by Tony Hoare in the Communications of the ACM **4**(7), 1961.