COMP170
Discrete Mathematical Tools for Computer Science

Intro to Induction

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.1, pp. 127-142

4.1 Mathematical Induction

- Smallest Counterexamples
- The Principle of Mathematical Induction
- Strong Induction
- Induction in General

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- We conclude by distinguishing between the weak principle of mathematical induction and the strong principle of mathematical induction

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- This will lead us to transform the indirect proof technique of proof by counterexample to direct proof technique. This direct proof technique will be induction
- We conclude by distinguishing between the weak principle of mathematical induction and the strong principle of mathematical induction
  - Note that the strong principle can actually be derived from the weak principle. The difference between them has less to do with the power of the techniques, than with proof format

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$1 \quad 2 \quad 3 \quad 4 \quad 5$

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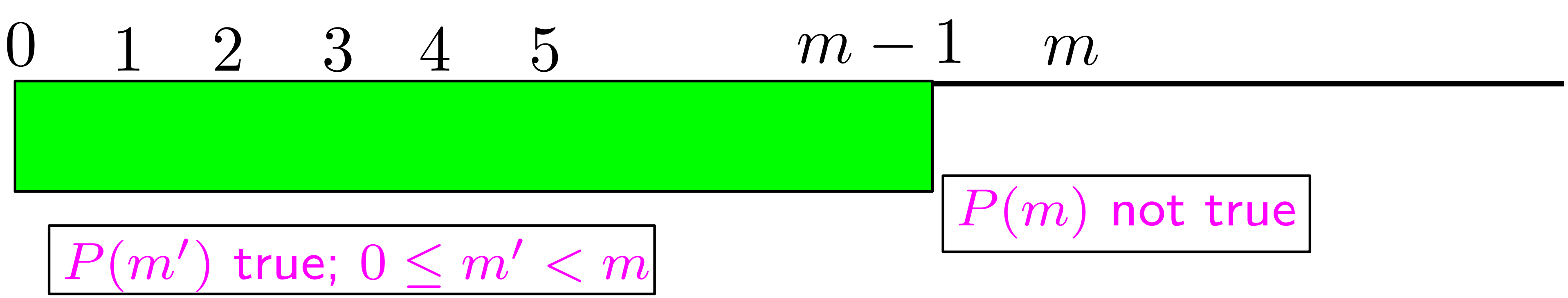

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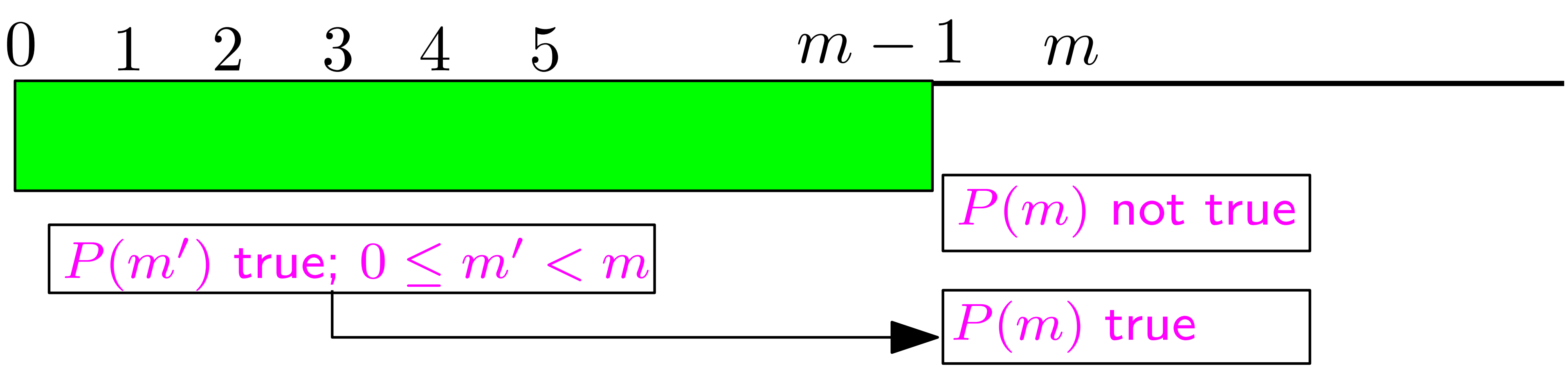

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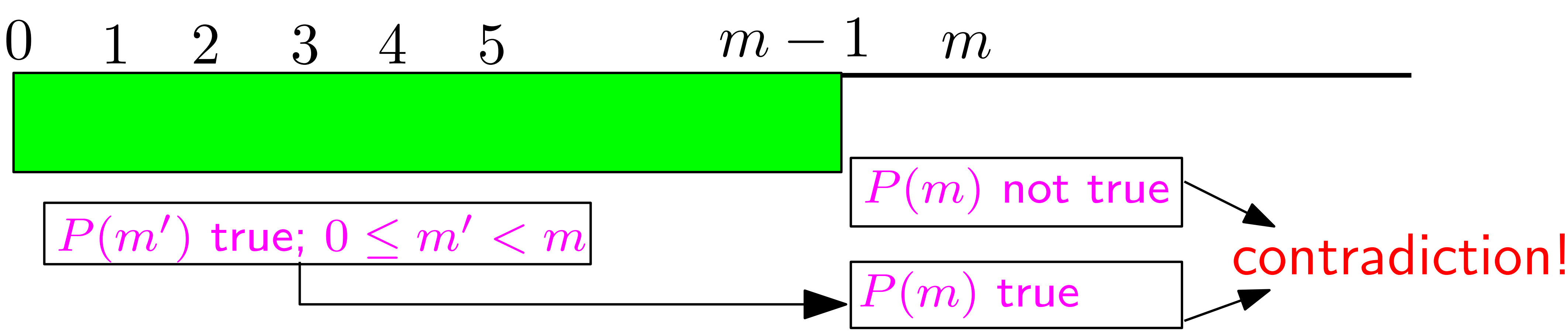

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Use proof by s.c. to show that, $\forall n \in N$, (non-negative ints)

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- Hence, (*) holds for all positive integers n.

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The crucial step was proving that

$$p(n-1) \Rightarrow p(n)$$

where $p(n)$ is the statement $1+2+\ldots+n = \frac{n(n+1)}{2}$.

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(*) will let us draw a contradiction to $2^{n+1} \not\geq n^2 + 2$.

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This should permit us to directly derive $p(n)$ for every n!

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Principle 4.1 (The Weak Principle of Mathematical Induction)
(a) If the statement $p(b)$ is True, and
(b) the statement $p(n-1) \Rightarrow p(n)$ is True for all $n > b$,

then $p(n)$ is True for all integers $n \geq b$.

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We have therefore just proven that, for $n > 0$, $p(n-1) \Rightarrow p(n)$

Proof by induction that $\forall n \geq 0,\ 2^{n+1} \geq n^2 + 2$:

Let $p(n)$ be statement that $2^{n+1} \geq n^2 + 2$

(i) Note that for $n = 0$, $2^{0+1} = 2 \geq 2 = 0^2 + 2$. $p(0)$

(ii) Suppose that $n > 0$ and that (*) $2^n \geq (n-1)^2 + 2$.

Then
$$
\begin{aligned}
2^{n+1} = 2 \cdot 2^n &\geq 2(n-1)^2 + 4 \\
&= (n^2 + 2) + (n^2 - 4n + 4) \\
&= n^2 + 2 + (n-2)^2 \\
&\geq n^2 + 2.
\end{aligned}
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We have therefore just proven that, for $n > 0$, $p(n-1) \Rightarrow p(n)$

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The final sentence of the proof is called the **Inductive Conclusion**.

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Thus, by the principle of mathematical induction,
$2^n > n^2$ for all $n \geq 5$.

4.1 Mathematical Induction

- Smallest Counterexamples
- The Principle of Mathematical Induction
- Strong Induction
- Induction in General

Strong Induction

Recall that when we used Proof by smallest counterexample in Euclid's Division Theorem we actually

> chose a smallest counterexample m to the EDT property $p(n)$, and observed that $m - n$ was a non-negative integer less than m. Therefore $p(m - n)$ had to be true. This in turn implied that $p(m)$ was true, yielding a contradiction.

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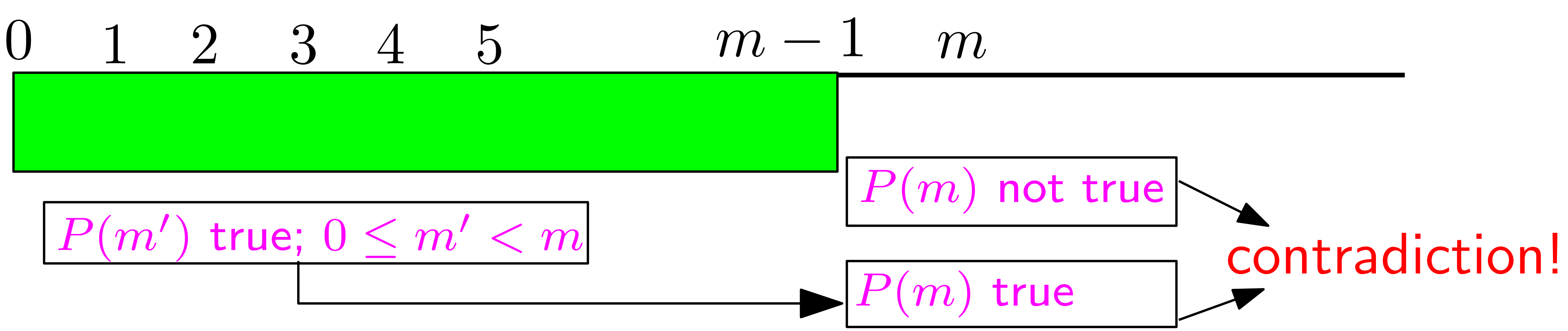
Note that the contradiction came from $p(m-n)$ and not from $p(m-1)$. We strongly used the fact that $p(i)$ was True for all $i < m$ and not just for $i = m-1$.

Proof by smallest counterexample that
statement $P(n)$ is true for all $n = 0, 1, 2 \ldots$ works by

(i) Assuming that a non-zero counterexample exists, i.e.,
There is some $n > 0$ for which $P(n)$ is not true

(ii) Letting $m \geq 0$ be *smallest* value for which $P(m)$ is not true

(iii) Then use fact that $P(m')$ is true for all $0 \leq m' < m$
to show that $P(m)$ is true,
contradicting original choice of m.
$\Rightarrow P(n)$ true for **all** $n = 0, 1, 2, \ldots$



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4. We then use this assumption to derive a proof of $q(k)$, thus generating our contradiction.

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- Iterating gives us a proof of $q(n)$ for every n
- This is another form of the principle of mathematical induction.

Principle 4.2

(The Strong Principle of Mathematical Induction)

(a) If the statement $p(b)$ is True and

(b) for all $n > b$, the statement

$$p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$$

is True

then $p(n)$ is True for all integers $n \geq b$.

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- n is therefore a power of a prime number or the product of powers of prime numbers
- Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime number or a product of powers of prime numbers.

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In reality, they are equivalent to each other in that the weak form is a special case of the strong form and the strong form can be derived from the weak form.

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3. We conclude on the basis of the principle of mathematical induction that $p(n)$ is true for all integers $n \geq b$

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This is usually where hard work, creativity and insights are most needed