Illustration of the Proof of Lemma 5.28

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In class, we proved that the expectation of the product of two independent random variables, is the product of their expectations. Formally

Lemma 5.28
If X and Y are independent random variables on sample space S with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

In these sides, we illustrate the proof of Lemma 5.28 with an example.

Suppose that we have two independent random variables, X, Y that each can take on the values $1, 2, 4$, but with different probability weights.

$$P(X=1)=1/3 \quad P(Y=1)=1/2 \qquad E(X)=7/3$$
$$P(X=2)=1/3 \quad P(Y=2)=1/4 \quad \Rightarrow \quad E(Y)=2$$
$$P(X=4)=1/3 \quad P(Y=4)=1/4 \qquad E(X)(EY)=14/3$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$.

$$P(Z=1) = P(X=1 \ \wedge Y=1) = \tfrac{1}{6}$$

$$P(Z=2) = P(X=1 \ \wedge \ Y=2)+P(X=2 \ \wedge \ Y=1) = \tfrac{1}{12}+\tfrac{1}{6}=\tfrac{1}{4}$$

$$P(Z=4) = P(X=1 \ \wedge \ Y=4)+P(X=4 \ \wedge \ Y=1) +P(X=2 \ \wedge \ Y=2) = \tfrac{1}{12}+\tfrac{1}{6}+\tfrac{1}{12}=\tfrac{1}{3}$$

$$P(Z=8) = P(X=2 \ \wedge \ Y=4)+P(X=4 \ \wedge \ Y=2) = \tfrac{1}{12}+\tfrac{1}{12}=\tfrac{1}{6}$$

$$P(Z=16) = P(X=4 \ \wedge Y=4) = \tfrac{1}{12}$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$. So

$$
\begin{aligned}
E(XY) &= E(Z) \\
&= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\
&\qquad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\
&= \frac{14}{3} \\
&= E(X) \cdot E(Y)
\end{aligned}
$$

On the next page, we mimic the proof of Lemma 5.28, using these X, Y. Reading the proof with this example in mind, might make the proof more understandable.

$$\boxed{E(X)E(Y)} = \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y)$$

$$= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)$$

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P(X=x)P(Y=y)$$

Ind of X, Y

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P\big((X=x)\wedge(Y=y)\big)$$

$$= \sum_{z\in\{1,2,4,8,16\}} zP(Z=z) = E(Z) \boxed{= E(XY)}$$