Recursions

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October 2, 2015

Contents

The First Approach

Sequences were defined in the lecture on mathematical induction. In this and the next lecture we continue our journey to the land of sequences.

The school method

Write the **first few terms** with the expectation that the general pattern will be obvious.

Example 1

Define a sequence by

$$1, 3, 5, 7, \ldots,$$

Comments

This may cause misunderstanding as the unspecified terms may be understood in different ways.

The Second Approach

The formula method

Define a sequence by giving an explicit formula for its i-th term.

Example 2

Define a sequence $(s_i)_{i=0}^{\infty}$ by

$$s_i = \frac{(-1)^i}{i+1} \text{ for all integers } i \geq 0.$$

Comments

- This method defines a sequence in a clear way, and is an ideal method in most cases.
- However, in many cases we may have to define a sequence in other ways, i.e., by "**recurrence relations**".

Recurrence Relations

Definition 3

A <u>recurrence relation</u> for a sequence $(s_i)_{i=0}^{\infty}$ is a formula that relates each term s_i to certain of its predecessors $s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}$ for all $i \geq \ell$, where ℓ is an integer, where ℓ may be a fixed constant or varies according to i.
The **initial conditions** for such a recurrence relation specify the values of $s_0, s_1, s_2, \ldots, s_{\ell-1}$, if ℓ is a fixed integer.

Example 4

Let $(s_i)_{i=0}^{\infty}$ be defined by the recurrence relation

$$s_i = s_{i-1}^2 + s_{i-2} \text{ for all } i \geq 2,$$

with initial condition $s_0 = 1$ and $s_1 = 2$.

Linear Recurrence Relations

Definition 5

A linear recurrence relation with constant coefficients for a sequence $(s_i)_{i=0}^{\infty}$ is a formula that relates each term s_i to certain of its predecessors $s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}$ in the form

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} + d \text{ for all } i \geq \ell, \tag{1}$$

where ℓ is some fixed integer, d is a constant, and c_i's are real constants with $c_\ell \neq 0$.

Example 6

Let $(s_i)_{i=0}^{\infty}$ be defined by $s_i = i$ for all integers $i \geq 0$. Then $s_i = s_{i-1} + 1$ is a linear recurrence relation for the sequence with the initial condition that $s_0 = 0$.

Linear Homogeneous Recurrence Relations

Definition 7

A

linear homogeneous recurrence relation of degree ℓ with constant coefficients for a sequence $(s_i)_{i=0}^{\infty}$ is a formula that relates each term s_i to its predecessors $s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}$ in the form

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell, \tag{2}$$

where ℓ is some fixed integer, and c_i's are real constants with $c_\ell \neq 0$.
The equation

$$x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0 \tag{3}$$

is called the characteristic equation of the linear recursion of (2).

Linear Homogeneous Recurrence Relations

Example 8

Let $(s_i)_{i=0}^{\infty}$ be defined by $s_i = 2i + 1$ for all integers $i \geq 0$. Then the following is a linear homogeneous recurrence relation of degree 2 of this sequence:

$$s_{i+1} = 2s_i - s_{i-1} \text{ for all } i \geq 1$$

with initial conditions $s_0 = 1$ and $s_1 = 3$. The characteristic equation of the linear recursion is given by

$$x^2 - 2x + 1 = 0.$$

Remark

If a sequence $(s_i)_{i=0}^{\infty}$ has a recurrence relation, then it has many recurrence relations.

Problem 9

Find another recurrence relation for the sequence of Example 8.

Linear Homogeneous Recurrence Relations

Example 10

Let $(s_i)_{i=0}^{\infty}$ be defined by $s_i = 2 \times 3^n$ for all integers $i \geq 0$. Then the following is a linear homogeneous recurrence relation of degree 2 of this sequence:

$$s_{i+1} = s_i + 6s_{i-1} \text{ for all } i \geq 1$$

with initial conditions $s_0 = 2$ and $s_1 = 6$. The characteristic equation of this recursion is given by

$$x^2 - x - 6 = 0.$$

Showing a Sequence Satisfying a Recurrence Relation

Problem

Given a sequence defined by a mathematical formula, show that the sequence satisfies a specific recurrence relation.

Comments

1. This may be an easy problem in some cases.
2. But It may be very technical in other cases.

Showing a Sequence Satisfying a Recurrence Relation

Example 11

Let $(s_i)_{i=1}^{\infty}$ be defined by

$$s_i = \frac{1}{i+1}\binom{2i}{i}.$$

Show that this sequence satisfies the recurrence relation $s_i = \frac{4i-2}{i+1}s_{i-1}$ for all integers $i \geq 2$.

Proof.

By definition, $s_{i-1} = \frac{1}{i}\binom{2i-2}{i-1}$. It then follows that

$$\frac{4i-2}{i+1}s_{i-1} = \frac{4i-2}{i+1}\frac{1}{i}\binom{2i-2}{i-1} = \frac{4i-2}{i+1}\frac{1}{i}\frac{(2i-2)!}{(i-1)!(i-1)!} = \frac{1}{i}\binom{2i}{i} = s_i$$

$\square$

Showing a Sequence Satisfying a Recurrence Relation

Example 12

Let $(s_n)_{n=0}^{\infty}$ be defined by

$$s_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \quad n \geq 0.$$

Show that this sequence satisfies the recurrence relation $s_i = s_{i-1} + s_{i-2}$ for all integers $i \geq 2$, with the initial condition that $s_0 = s_1 = 1$.

Proof

Note that $k > \lfloor (n-1)/2 \rfloor$ is equivalent to $k > n-k-1$. By convention, $\binom{m}{\ell} = 0$ for all integers m and ℓ such that $\ell > m$. Hence, we may write s_n as

$$s_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k}, \quad n \geq 1.$$

It then follows that

Showing a Sequence Satisfying a Recurrence Relation

Proof – Continued

$$
\begin{aligned}
s_{n+1} + s_n &= \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-k-1}{k} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \left[\binom{n-k}{k} + \binom{n-k}{k-1} \right] \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} \text{ (by the Pascal formula)} \\
&= \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n-k+1}{k} + \binom{0}{n+1} \\
&= \sum_{k=0}^{n+1} \binom{(n+2)-k-1}{k} = s_{n+2}.
\end{aligned}
$$

Driving a Recurrence Relation for a Given Sequence

Question 1

For a sequence defined by an explicit mathematical formula, is it possible to derive a recurrence relation?

Question 2

For a sequence defined by an explicit mathematical formula, is it possible to derive a linear recurrence relation?

Answer

- Sometimes, these could be done easily. One example is the sequence documented in Example 8.
- Sometimes, these may be very hard, even if they are possible.
- Usually, the answer is No.

Driving a Recurrence Relation for a Given Sequence

Example 13

Let $(s_i)_{i=1}^{\infty}$ be defined by

$$s_i = i^2.$$

Derive a recurrence relation for the sequence.

Classroom solution

Work out a solution together with students.

Driving a Linear Recurrence Relation for a Given Sequence

Example 14

Let $(s_i)_{i=1}^{\infty}$ be defined by

$$s_i = i^2.$$

Derive a linear recurrence relation for the sequence.

Classroom solution

Work out a solution together with students.

Driving a Linear Recurrence Relation for a Given Sequence

Question 3

Let $(s_i)_{i=1}^{\infty}$ *be defined by*

$$s_i = \frac{1}{i+1}\binom{2i}{i}.$$

Can you derive a linear recurrence relation for this sequence?

Answer

- This is the Catalan sequence, and its entries are Catalan numbers.
- The only known recursion is the following:

$$s_n = \sum_{k=0}^{n-1} s_i s_{n-1-i},$$

  which is quadratic, homogeneous, and not of fixed degree.
- No linear recursion formula for this sequence is known.

Solving Recursively Defined Sequences

Problem

Given a recursively defined sequence, find a mathematical formula description of the sequence.

Comments

1. This is a very hard problem in general, but can be handled in some special cases.
2. Some examples will be demonstrated subsequently in this lecture.
3. A few techniques will be dealt with in the next lecture.

The Tower of Hanoi

Definition 15

The sequence $(s_i)_{i=1}^{\infty}$ is defined by $s_i = 2s_{i-1} + 1$ for all $i \geq 2$, with initial condition $s_1 = 1$.

Problem 16

Find a linear homogeneous recurrence relation of the sequence $(s_i)_{i=1}^{\infty}$ above.

Problem 17

Find a mathematical formula for each term s_i.

Solution

Write down a few initial terms. Guess a formula. Use mathematical induction to prove the guessed formula.

The Fibonacci Sequence

Definition 18

The sequence $(F_i)_{i=0}^{\infty}$ is defined by the linear homogeneous recursion

$$F_i = F_{i-1} + F_{i-2} \text{ for all } i \geq 2,$$

with initial condition $F_0 = 0$ and $F_1 = 1$.

Problem 19

Find another mathematical formula for each term F_i.

Remark

An answer was described earlier in this lecture. Another solution to this problem will be provided in the next lecture.

The Arithmetic Sequence

Definition 20

The sequence $(s_i)_{i=0}^{\infty}$ is defined by $s_i = s_{i-1} + d$ for all $i \geq 1$, with initial condition $s_0 = a$, where a and d are constants.

Problem 21

Find a linear homogeneous recursion for the arithmetic sequence.

Problem 22

Find a mathematical formula for each term s_i.

The Geometric Sequence

Definition 23

The sequence $(s_i)_{i=0}^{\infty}$ is defined by the linear homogeneous recursion $s_i = rs_{i-1}$ for all $i \geq 1$, with initial condition $s_0 = a$, where a and r are constants.

Problem 24

Find a mathematical formula for each term s_i.

The Geometric Sequence

The problem

The sequence $(s_i)_{i=0}^{\infty}$ is defined by the linear homogeneous recursion $s_i = rs_{i-1}$ for all $i \geq 1$, with initial condition $s_0 = a$, where a and r are constants. Find a mathematical formula for each term s_i.

Proof.

The first few terms of the sequence is

$$a, ar, ar^2, ar^3, ar^4, \ldots,$$

We guess that $s_i = ar^i$ for all $i \geq 1$. With mathematical induction, one can indeed prove that this is indeed the mathematical formula for the geometric sequence. □