COMP170
Discrete Mathematical Tools for Computer Science

Recursion, Recurrences and Induction

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.2, pp. 143-153

Recursion, Recurrences and Induction

- Recursion
- Recurrences
- Iterating a Recurrence
- Geometric Series
- First-Order Linear Recurrences

Recursion

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- A classic example of this is the Towers of Hanoi problem

Towers of Hanoi

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Towers of Hanoi

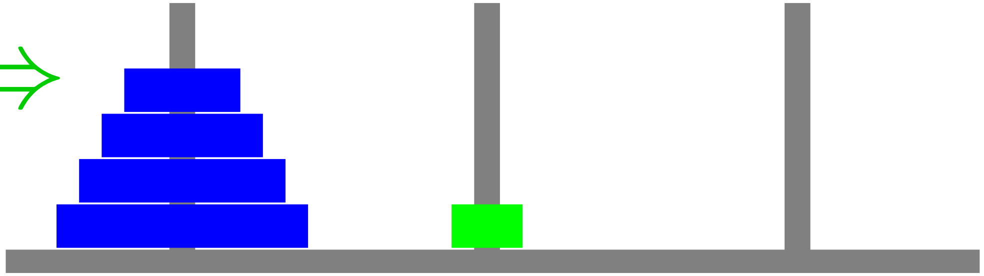
- 3 pegs; n disks of different sizes.
- A legal move takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- Problem: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

Towers of Hanoi

legal move

$\Rightarrow$



Towers of Hanoi

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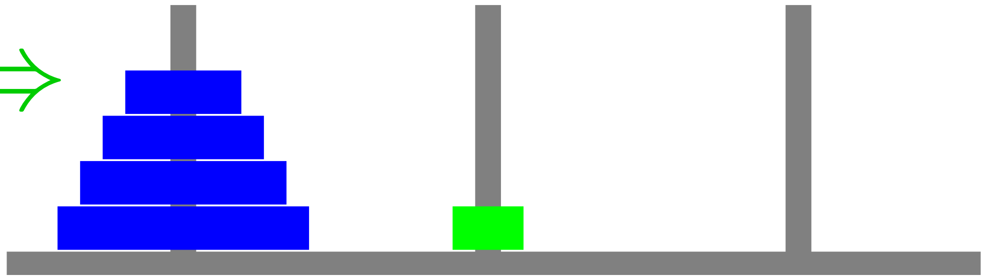

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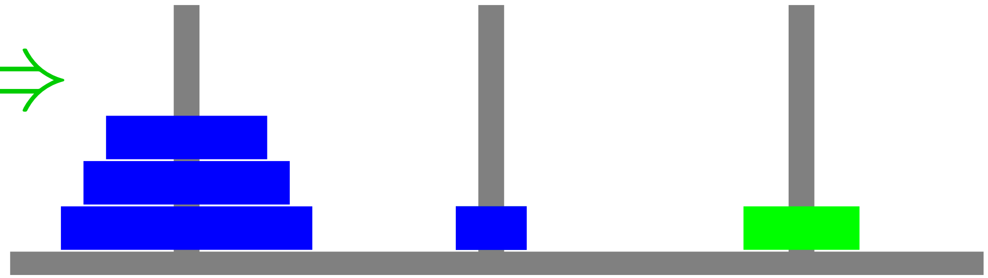

Towers of Hanoi

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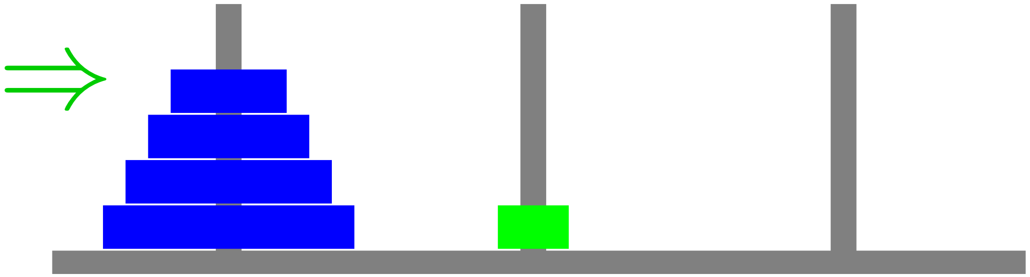
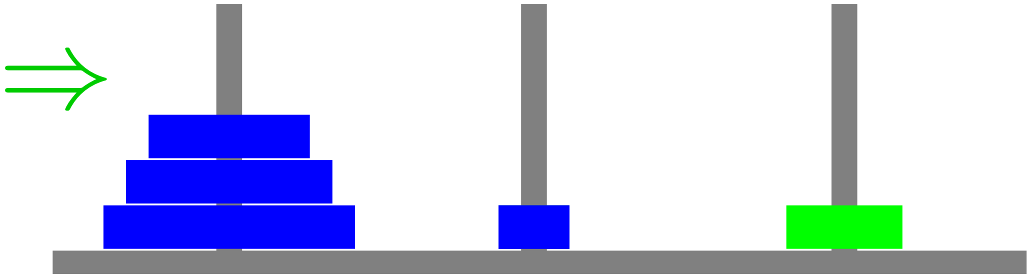
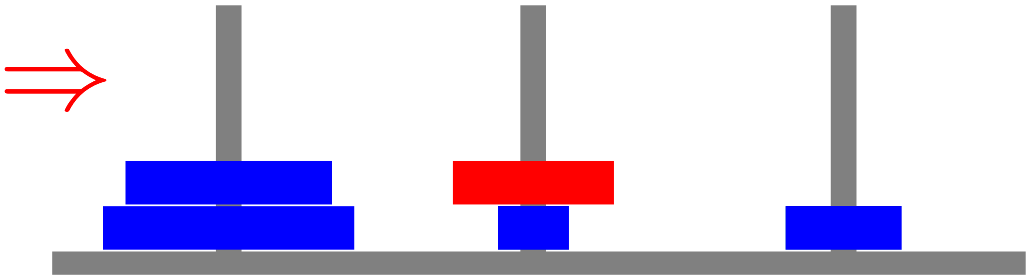


legal move



not legal



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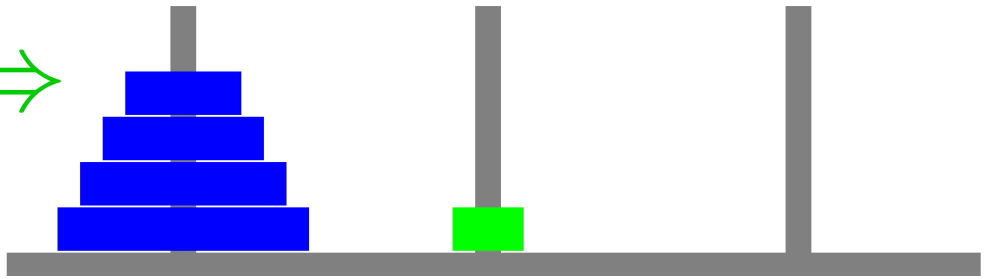

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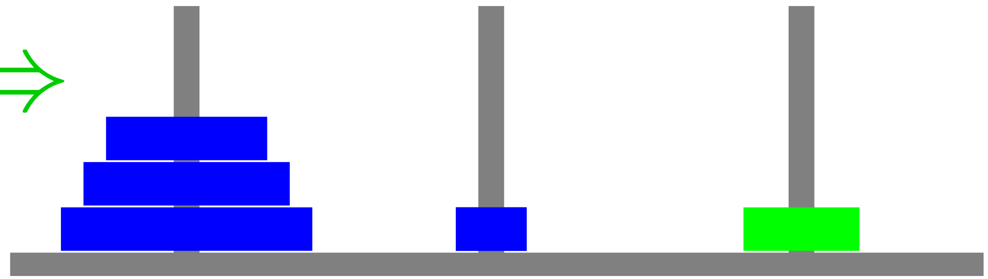

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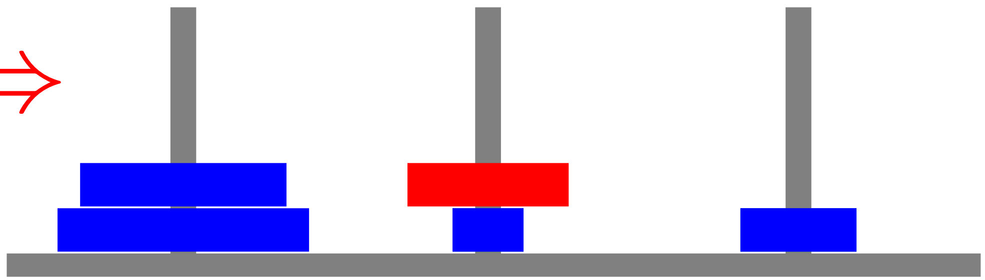

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Towers of Hanoi

Problem

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Start with n disks
on leftmost peg

Towers of Hanoi

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Start with n disks
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using only
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Towers of Hanoi

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move all disks to rightmost peg.

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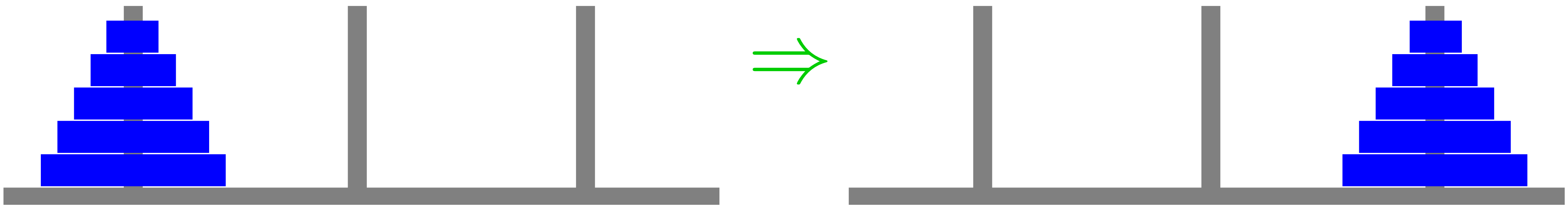

Given $i, j \in \{1,2,3\}$ let $\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$

i.e., $\overline{\{1,2\}} = 3$, $\overline{\{1,3\}} = 2$, $\overline{\{2,3\}} = 1$.

Towers of Hanoi

General Solution

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Recursion Base:

If $n = 1$ moving one disk from i to j is easy.

Just move it.

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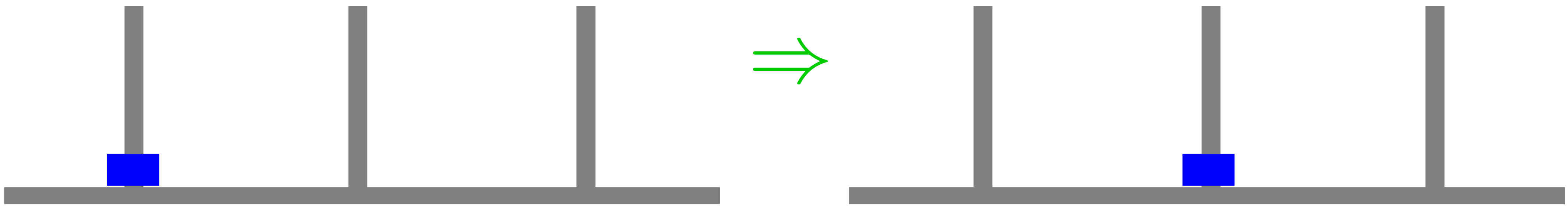

Towers of Hanoi

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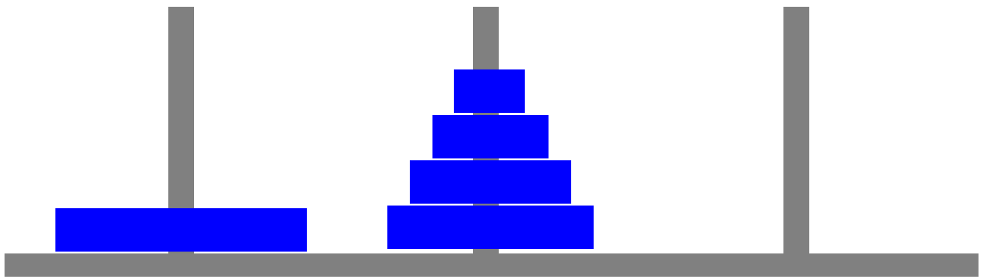

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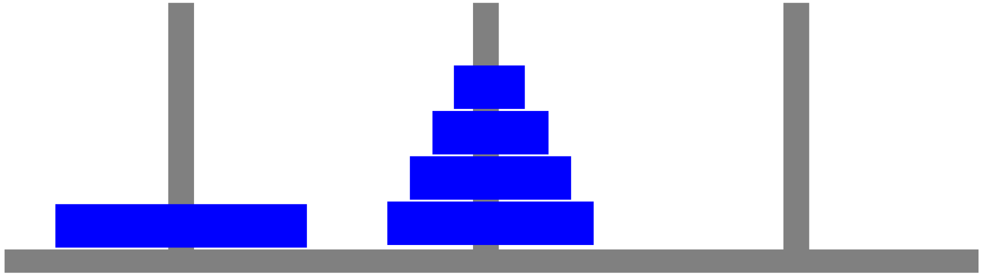

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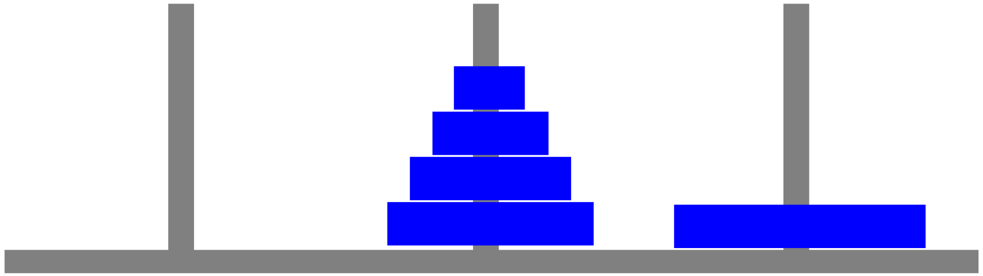

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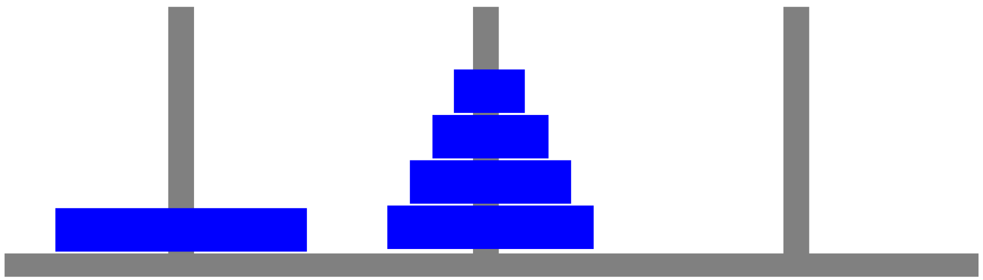

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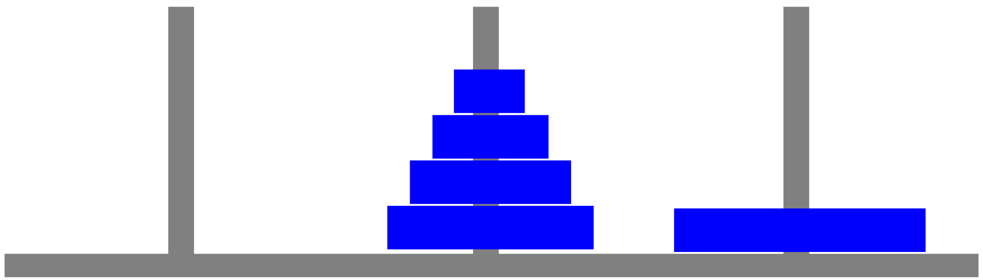

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3)

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from $\overline{\{i, j\}}$ to j.

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

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- $p(n-1) \Rightarrow p(n)$ is "recursion" statement that if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Running Time

$M(n)$ is number of disk moves needed for n disks

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  - Later, we'll see how to solve without guessing

Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2M(n-1)+1 & \text{otherwise.} \end{cases}$$

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The second time was to derive the closed form solution

$$M(n) = 2^n - 1$$

of the recurrence

Francois Edouard Anatole Lucas

b. 1842, d. 1891

French mathematician.
Best known for his results in number theory.

He is also famous for being a creator of mathematical puzzles, among the most well-known of which is the Tower of Hanoi puzzle (1883).

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A **recurrence equation** or **recurrence** for a function defined on the set of integers greater than or equal to some number b is one that tells us how to compute the nth value from some or all the first $(n-1)$ values.

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Towers of Hanoi

$$F(n) = \begin{cases} 1 & \text{if } n = 0, 1, \\ F(n-1) + F(n-2) & \text{otherwise.} \end{cases}$$

Fibonacci Numbers

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Consider the eight subsets of $\{1, 2, 3\}$:

$$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

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This suggests that the recurrence for the number of subsets of an n-element set ($\{1, 2, ..., n\}$) is

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

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Thus, if $n > 0$, then $S(n) = 2S(n-1)$.

We already observed that $\emptyset$ has only one subset (itself), so $S(0) = 1$ and we have proved the correctness of the recurrence.

If

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then, $S(n) = 2^n$ for all $n \geq 0$.

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ii) If the statement is true for $n - 1$ then $S(n-1) = 2^{n-1}$ so

If

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then, $S(n) = 2^n$ for all $n \geq 0$.

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$$S(n) = 2S(n-1) = 2 \cdot 2^{n-1} = 2^n$$

and we are done!

Example 3:

When paying off a loan with initial amount A and monthly payment M at an interest rate of p percent, the total amount $T(n)$ of the loan still due after n months is computed by adding $p/12$ percent to the amount due after $n-1$ months and then subtracting the monthly payment M.

Convert this description into a recurrence for the amount owed after n months.

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$T(n) = (1 + \frac{0.01p}{12}) \cdot T(n-1) - M,$ with $T(0) = A$.

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We will now see a general tool for deriving closed form solution to these type of recurrence relations

Recursion, Recurrences and Induction

- Recursion
- Recurrences
- Iterating a Recurrence
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Iterating a Recurrence

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Can you generalize this to find a closed form solution to
$T(n) = rT(n-1) + a$?

Note that $T(n) = rT(n-1) + a$, implies that,

$\forall i < n$, $T(n-i) = rT((n-i)-1)) + a$.

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$$T(n) = r^nT(0) + a\sum_{i=0}^{n-1} r^i = \boxed{r^nb + a\sum_{i=0}^{n-1} r^i.}$$

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This could lead to the same guess

$$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

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$\sum_{i=0}^{n-1} r^i$ is a **finite geometric series with common ratio** r.

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Note: We will see another proof of this soon.

Theorem 4.1

If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

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So, the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1-r^{n-1}}{1-r}.$$

Then we have $T(n) = rT(n-1) + a$

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Therefore, by the principle of mathematical induction, our formula holds for all integers $n \geq 0$.

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Example:

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$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

Plugging $r = 3$, $a = 2$, $b = 5$ into the formula, gives

$$T(n) = 3^n \cdot 5 + 2\frac{1 - 3^n}{1 - 3} = 3^n \cdot 6 - 1$$

Corollary 4.2: The formula for the sum of a geometric series with $r \neq 1$ is

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Then $T(n) = rT(n-1) + 1$.
Applying Theorem 4.1 with $b = 0$ and $a = 1$ gives

$$T(n) = \frac{1 - r^n}{1 - r}$$

Lemma 4.3: Let $r \neq 1$ be a positive value independent of n. Let $t(n)$ be the largest term in the geometric series

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Then the value of the geometric series is $O(t(n))$.

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(i) is easy because in this case

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In case (ii), $r > 1$, $t(n) = r^{n-1}$ and

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Thus, $\sum_{i=0}^{n-1} r^i = O\left(r^{n-1}\right) = O(t(n))$.

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  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

$$T(n) = f(n)T(n-1) + g(n)$$

When $f(n)$ is a constant, say r, the general solution is almost as easy to write as in Theorem 4.1.

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When $f(n)$ is a constant, say r, the general solution is almost as easy to write as in Theorem 4.1. Iterating the recurrence gives

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&\vdots \\
&= r^nT(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}$$

Theorem 4.5 For any positive constants a and r, and any function g defined on the nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0, \\ a & \text{if } n = 0, \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i). \qquad (*)$$

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Proof by induction:

Because the sum (*) has no terms when $n = 0$, the formula gives $T(0) = a$ and, so, is valid when $n = 0$.

We now assume that n is positive and
$T(n-1) = r^{n-1}a + \sum_{i=1}^{n-1} r^{(n-1)-i} g(i)$.

Using the definition of the recurrence and the inductive hypothesis, we get that

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$$T(n) = rT(n-1) + g(n)$$

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&= r^n a + \sum_{i=1}^{n-1} r^{(n-1)+1-i} g(i) + g(n) \\
&= r^n a + \sum_{i=1}^{n-1} r^{n-i} g(i) + g(n)
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&= r^n a + \sum_{i=1}^{n-1} r^{n-i} g(i) + g(n) \\
&= r^n a + \sum_{i=1}^{n} r^{n-i} g(i).
\end{aligned}
$$

Therefore, by the principle of mathematical induction, the solution to the recurrence is given by (*) for all nonnegative integers n.

Example: Solve $T(n) = 4T(n-1) + 2^n$ with $T(0) = 6$.

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Using Theorem 4.5

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Using Theorem 4.5

$$\begin{aligned} T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\ &= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \end{aligned}$$

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T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + 4^n \cdot \frac{1}{2} \cdot \sum_{i=0}^{n-1} (\frac{1}{2})^i
\end{aligned}
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Example: Solve $T(n) = 4T(n-1) + 2^n$ with $T(0) = 6$.

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&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + 4^n \cdot \frac{1}{2} \cdot \sum_{i=0}^{n-1} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - (\frac{1}{2})^n) \cdot 4^n
\end{aligned}
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&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + 4^n \cdot \frac{1}{2} \cdot \sum_{i=0}^{n-1} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - (\frac{1}{2})^n) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n.
\end{aligned}
$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

Using Theorem 4.5

$$T(n) = 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i$$

$$= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i}$$

We now need the following well known theorem
(can be proven by induction or see book for another proof)

Theorem 4.6

For any real number $x \neq 1$,

$$\sum_{i=1}^{n} i x^i = \frac{n x^{n+2} - (n+1) x^{n+1} + x}{(1-x)^2}.$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

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$$\begin{aligned} T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\ &= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \end{aligned}$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

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$$\begin{aligned}
T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\tfrac{3}{2}(n+1)3^{-(n+1)} - \tfrac{3}{4}3^{-(n+1)} + \tfrac{3}{4}\right)
\end{aligned}$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

Using Theorem 4.5

$$
\begin{aligned}
T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\tfrac{3}{2}(n+1)3^{-(n+1)} - \tfrac{3}{4}3^{-(n+1)} + \tfrac{3}{4}\right) \\
&= \tfrac{43}{4}3^n - \tfrac{n+1}{2} - \tfrac{1}{4}
\end{aligned}
$$