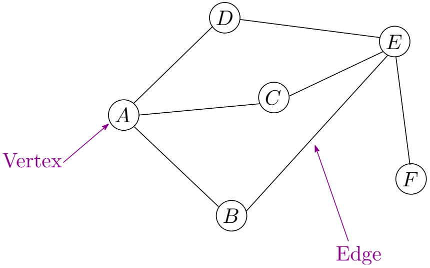
Introduction to Graph Algorithms

Version of October 11, 2014

Graphs

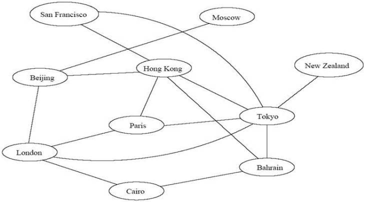
- Extremely useful tool in modeling problems
- Consist of:
  - Vertices
  - Edges



Vertices can be considered as "sites" or locations.

Edges represent connections.

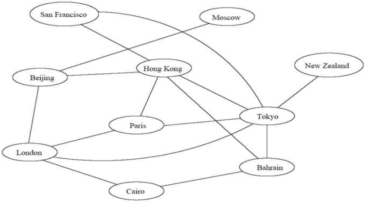
Graph Application



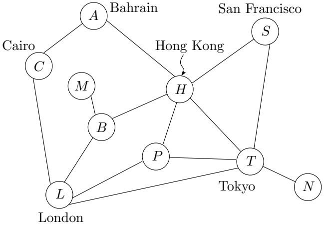
Air flight system

- Each vertex represents a city
- Each edge represents a direct flight between two cities
- A query on direct flight = a query on whether an edge exists
- A query on how to get to a location = does a path exist from A to B
- We can even associate costs to edges (weighted graphs), then ask "what is the cheapest path from A to B"

Graph Application



original graph



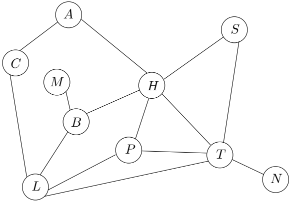
simplified graph

- Each vertex represents a city
- Each edge represents a direct flight between two cities
- A query on direct flight = a query on whether an edge exists
- A query on how to get to a location = does a path exist from A to B
- We can even associate costs/time to edges (weighted graphs), then ask "what is the cheapest/fastest path from A to B"

Why Graph Algorithms?

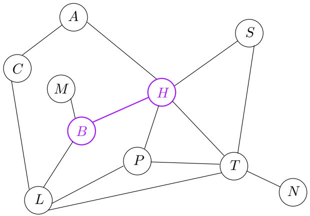
- Graphs are a ubiquitous data structure in computer science
  - Networks: LAN, the Internet, wireless networks
  - Logistics: transportation, supply chain management
  - Relationship between objects: online dating, social networks (Facebook!)
- Hundreds of interesting computational problems defined on graphs
- We will sample a few basic ones

- A graph $G = (V, E)$ consists of
  - a set of vertices V, $|V| = n$, and
  - a set of edges E, $|E| = m$
- Each edge is a pair of (u, v), where u, v belongs to V



$$V = \{A, B, C, H, L, M, N, P, S, T\}$$
$$E = \{(A, C), (A, H), \ldots, (H, P), \ldots\}$$

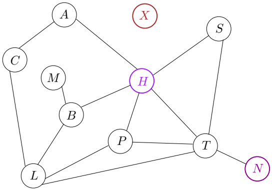
- For directed graph, we distinguish between edge (u, v) and edge (v, u); for undirected graph, no such distinction is made.



- Each edge has two endpoints
  - H and B are the endpoints of (H, B)
- An edge joins its endpoints
  - (H, B) joins H and B
- Two vertices are adjacent (or neighbors) if they are joined by an edge.
  - H and B are adjacent
  - H is a neighbor of B
- If vertex v is an endpoint of edge e, then the edge e is said to be incident on v. Also, the vertex v is said to be incident on e
  - (H, B) is incident on H and B
  - H and B are incident on (H, B)

The Degree of a Vertex

The degree of a vertex v (degree(v)) in a graph is the number of edges incident on it.



- Vertex H has degree 5
- Vertex N has degree 1
- Vertex X has degree 0
  (It is called an isolated vertex)

Lemma

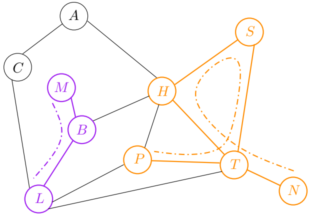
$$\sum_{v \in V} degree(v) = 2|E|.$$

Proof.

An edge $e = (u, v)$ in a graph contributes one to degree(u) and contributes one to degree(v). $\square$

A path in a graph is a sequence $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ of vertices such that $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$

- There is a path from v_0 to v_k
- Length of a path = # of edges on the path
- Path contains the vertices $v_0, v_1, \ldots, v_k$ and the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$
- For any $0 \le i \le j \le k$, $\langle v_i, v_{i+1}, \ldots, v_j \rangle$ is its subpath
- If there is a path p from u to v, v is said to be reachable from u
- A path is simple if all vertices in the path are distinct

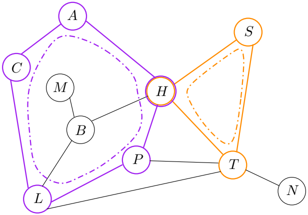


- $\langle L, B, M \rangle$ is a path
  - length is 2
  - $\langle B, M \rangle$ is its subpath
  - M is reachable from L
  - a simple path
- $\langle N, T, H, S, T, P \rangle$ is a path
  - length is 5
  - $\langle T, H, S \rangle$ is its subpath
  - P is reachable from N
  - not a simple path

Cycle

A path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ forms a cycle if $v_0 = v_k$ and all edges on the path are distinct

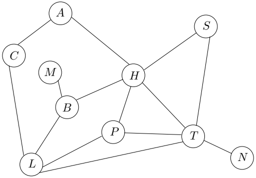
- A cycle is simple if $v_1, v_2, \ldots, v_k$ are distinct
- A graph with no cycles is acyclic



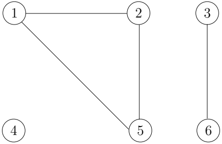
- $\langle T, S, H, T \rangle$ is a simple cycle
- $\langle A, C, L, P, H, A \rangle$ is a simple cycle

- Two vertices are connected if there is a path between them
- A graph is connected if every pair of vertices is connected; otherwise, the graph is disconnected
- The connected components of a graph are the equivalence classes of vertices under the "is reachable from" relation

- connected graph
- one connected component $\{A, B, C, H, L, M, N, P, S, T\}$

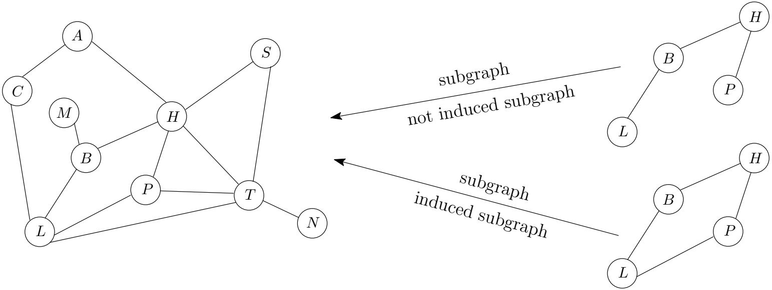


- disconnected graph
- 3 connected components
  - $\{1, 2, 5\}$
  - $\{3, 6\}$
  - $\{4\}$

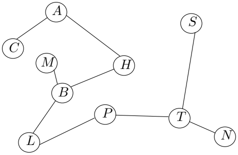


Subgraph

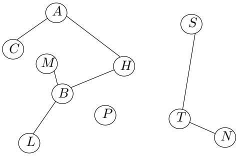
- Graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$
- G' is an induced subgraph of G if G' is a subgraph of G and every edge of G connecting vertices of G' is an edge of G'.



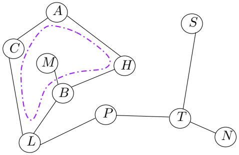
- A tree is a connected, acyclic, undirected graph
- If an undirected graph is acyclic but possibly disconnected, it is a forest



A tree



A forest



neither a tree nor a forest

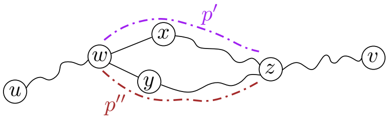
Properties of Trees

Let $G = (V, E)$ be an undirected graph. The following statements are equivalent.

1. G is a tree
2. Any two vertices in G are connected by a unique simple path
3. G is connected, but if any edge is removed from E, the resulting graph is disconnected
4. G is connected, and $|E| = |V| - 1$
5. G is acyclic, and $|E| = |V| - 1$
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle

Proof

(1) G is a tree
$\Rightarrow$ (2) Any two vertices in G are connected by a unique simple path



- Proof by contradiction
- Suppose that vertices u and v are connected by two distinct simple paths p_1 and p_2, as shown in the above figure
  - p_1 and p_2 first diverge at vertex w
  - p_1 and p_2 first reconverge at vertex z
  - p' is the subpath of p_1 from w through x to z
  - p'' is the subpath of p_2 from w through y to z
  - The path obtained by concatenating p' and the reverse of p'' is a cycle, which yields the contradiction!

Proof

(2) Any two vertices in G are connected by a unique simple path $\Rightarrow$ (3) G is connected, but if any edge is removed from E, the resulting graph is disconnected

- If any two vertices in G are connected by a unique simple path, then G is connected
- Let (u, v) be any edge in E
- This edge is a path from u to v, and so it must be the unique path from u to v
- If (u, v) is deleted from G, there is no path from u to v, and hence its removal disconnects G

Proof

(3) G is connected, but if any edge is removed from E, the resulting graph is disconnected
$\Rightarrow$ (4) G is connected, and $|E| = |V| - 1$

- By assumption, the graph G is connected
- Prove $|E| = |V| - 1$ by induction
  - Base ($n = 1$): A connected graph with one vertex has zero edge
  - Suppose that G has $n \geq 2$ vertices and that all graphs satisfying (3) with fewer than n vertices also satisfy $|E| = |V| - 1$
  - Removing an arbitrary edge from G separates the graph into 2 connected components
  - Each component satisfies (3), or else G would not satisfy (3)
  - Thus, by induction, the number of edges in 2 components combined is $|V| - 2$
  - Adding in the removed edge yields $|E| = |V| - 1$

Proof

(4) G is connected, and $|E| = |V| - 1$
$\Rightarrow$ (5) G is acyclic, and $|E| = |V| - 1$

- Suppose that G is connected and that $|E| = |V| - 1$
- Suppose that G has a cycle containing k vertices $v_1, v_2, \ldots, v_k$, and without loss of generality assume that this cycle is simple
- Let $G_k = (V_k, E_k)$ be the subgraph of G consisting of the cycle
- Note that $|V_k| = |E_k| = k$
- If $k < |V|$, there must be a vertex $v_{k+1} \in V - V_k$ that is adjacent to some vertex $v_i \in V_k$, since G is connected
- Define $G_{k+1} = (V_{k+1}, E_{k+1})$ to be the subgraph of G with $V_{k+1} = V_k \cup \{v_{k+1}\}$ and $E_{k+1} = E_k \cup \{(v_i, v_{k+1})\}$
- Note that $|V_{k+1}| = |E_{k+1}| = k + 1$
- If $k + 1 < |V|$, we can continue, defining G_{k+2} in the same manner, and so forth, until we obtain $G_n = (V_n, E_n)$, where $n = |V|$, $V_n = V$, and $|E_n| = |V_n| = |V|$
- Since G_n is a subgraph of G, we have $E_n \subseteq E$, and hence $|E| \geq |V|$, which contradicts the assumption that $|E| = |V| - 1$

Proof

(5) G is acyclic, and $|E| = |V| - 1$
$\Rightarrow$ (6) G is acyclic, but if any edge is added to E, the resulting graph contains a cycle

- Suppose that G is acyclic and that $|E| = |V| - 1$
- Let k be the number of connected components of G
- Each connected component is a free tree by definition, and since (1) implies (5), the sum of all edges in all connected components of G is $|V| - k$
- Consequently, we must have $k = 1$, and G is in fact a tree (That is, (1) holds)
- Since (1) implies (2), any two vertices in G are connected by a unique simple path
- Thus, adding any edge to G creates a cycle

Proof

(6) G is acyclic, but if any edge is added to E, the resulting graph contains a cycle
$\Rightarrow$ (1) G is a tree

- Suppose that G is acyclic but that if any edge is added to E, a cycle is created
- We must show that G is connected
- Let u and v be arbitrary vertices in G
- If u and v are not already adjacent, adding the edge (u, v) creates a cycle in which all edges but (u, v) belong to G
- Thus, there is a path from u to v, and since u and v were chosen arbitrarily, G is connected