Combinatorics II

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Contents

Combinations (1)

Definition 1

Given a set S of n distinct elements, a subset of size r of S is called an <u>r-combination</u> of the set. The symbol $\binom{n}{r}$, read "n choose r", denotes the number of r-combinations of a set of n distinct elements.

Example 2

Let $S = \{a, b, c\}$. Then S has three 2-combinations:

$$\{a, b\}, \{a, c\}, \{b, c\}$$

Remark

The distinction between r-permutation and r-combination is that the former takes order into consideration, while the latter does not.

For example, the above S in Example 2 has six 2-permutations:

$$ab,\ ba,\ ac,\ ca,\ bc,\ cb$$

Combinations (2)

Proposition 3

The number of r-combinations of a set of n distinct elements, $\binom{n}{r}$*, is given by*

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

where n and r are non-negative integers with $r \leq n$.

Combinations (3): Proof of Proposition 3

We prove this result by making use of the relationship between r-permutations and r-combinations.

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of n distinct elements. All the r-permutations of S can be obtained by completing the following two steps:

1. Find all the $\binom{n}{r}$ r-combinations of S.
2. For each r-combination, $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ obtain all the permutations of the set $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ The number of such permutations is $r!$.

Clearly, every r-permutation of S must be obtained this way.

There are $\binom{n}{r}$ possible r-combinations in step 1 and for each r-combination obtained in step 1, we have $r!$ r-permutations. Then by the Multiplication Rule, there are altogether $(\binom{n}{r} \cdot r)$ r-permutations of S. Hence

$$P(n, r) = \binom{n}{r} \cdot r!$$

and

$$\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)! \cdot r!}$$

Combinations (4)

Example 4

How many committees of 5 people can be chosen from 20 men and 12 women if exactly 3 men must be on each committee?

Solution 5

We must choose 3 men from 20 and then 2 women from 12. The answer is

$$\binom{20}{3} \cdot \binom{12}{2} = 1140 \times 66 = 75240$$

Definition of r-Combinations with Repetitions Allowed

Definition 6

An r-combination with repetition allowed, or multiset of size r, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \dots, x_n\}$, we write an r-combination with repetition allowed as $[x_{i_1}, x_{i_2}, \cdots, x_{i_r}]$, where each x_{i_j} is in X and some of the x_{ij} may equal each other.

Example 7

Let $X = \{1, 2, 3\}$. Then all the 2-combinations with repetition allowed of X are

$$[1,1],\ [2,2],\ [3,3],\ [1,2],\ [1,3],\ [2,3]$$

while X has only three 2-combinations

$$\{1,2\},\ \{1,3\},\ \{2,3\}$$

Vertical-Bar Representation of r-Combinations with Repetition Allowed

Example 8

Let $X = \{1, 2, 3\}$. Then all the 2-combinations with repetition allowed of X are

$$[1,1],\ [2,2],\ [3,3],\ [1,2],\ [1,3],\ [2,3].$$

In this example, $r = 2$ and $n = 3$. Hence, we have $r + n - 1$ positions. The correspondence between

$$[1,1] \leftrightarrow 11||, \qquad [2,2] \leftrightarrow |22|, \qquad [3,3] \leftrightarrow ||33$$

and

$$[1,2] \leftrightarrow 1|2|, \qquad [1,3] \leftrightarrow 1||3, \qquad [2,3] \leftrightarrow |2|3.$$

Vertical-Bar Representation of r-Combinations with Repetition Allowed

Let $X = \{x_1, \cdots, x_n\}$ be a given set of n elements, and let $[x_{i_1}, \cdots, x_{i_r}\}$ be an r-combination with repetition allowed. Let l_i be the number of times that x_i appear in this r-combination. Then the r-combination with repetition allowed can be written as

$$\underbrace{x_1, x_1, \cdots, x_1}_{l_1 \text{ factors}} \underbrace{x_2, x_2, \cdots, x_2}_{l_2 \text{ factors}} \cdots \underbrace{x_n, x_n, \cdots, x_n}_{l_n \text{ factors}}.$$

This can be further written as

$$\underbrace{\times \times \cdots \times}_{l_1 \text{ factors}} | \underbrace{\times \times \cdots \times}_{l_2 \text{ factors}} | \cdots | \underbrace{\times \times \cdots \times}_{l_n \text{ factors}},$$

where each cross represent some x_i. So here we have altogether $r + n - 1$ positions for the $n - 1$ vertical bars and r crosses.

Repetitions: the Formula

Proposition 9

The number of r-combinations with repetition allowed that can be selected from a set of n distinct elements is

$$\binom{r+n-1}{r}$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Proof of Proposition 9

Each r-combination with repetition allowed is obtained in the following way:

1. **Consider $(r+n-1)$ positions:** Choose $(n-1)$ positions and put $(n-1)$ vertical bars into the $(n-1)$ positions.
2. **Write crosses $\times$ in the remaining r position.**
3. **The 3rd step consists of:**
   - The no. of crosses before the 1st vertical bar is the no. of appearances of the 1st element of the set in the r-combination with repetition allowed.
   - The no. of crosses after the last vertical bar is the no. of appearances of the n-th element of the set in the r-combination.
   - The no. of crosses between the $(i-1)$-th and i-th vertical bar is the number of appearances of the i-th element of the set in the r-combination, where $2 \le i \le n$.

In this way, we get all the r-combinations with repetition allowed. So there are

$$\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$$

r-combinations with repetition allowed that can be chosen from a set of n elements.

Repetitions: an Example of the Application of the formula

Example 10

A store sells 30 kinds of balloons. How many different combinations of 24 balloons can be chosen?

Solution 11

This is a problem of 24-combinations with repetition allowed. The total number of 24-combinations with repetition allowed is

$$\binom{24+30-1}{24} = \binom{53}{24}$$

Algebra of Combinatorics (1)

Proposition 12

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof.

By Proposition 3,

$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r!)}$$

So

$$\binom{n}{n-r} = \frac{n!}{(n-r)! \cdot (n-(n-r))!} = \frac{n!}{(n-r)! \cdot r!} = \binom{n}{r}$$

$\square$

Algebra of Combinatorics (2): Pascal's Formula

Proposition 13

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

where n and r are positive integers with $r \leq n$.

Proof: By Proposition 3,

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)! \cdot (n-r+1)!} + \frac{n!}{r! \cdot (n-r)!} \\
&= \frac{n!}{(r-1)! \cdot (n-r+1)!} \cdot \frac{r}{r} + \frac{n!}{r! \cdot (n-r)!} \cdot \frac{n-r+1}{n-r+1} \\
&= \frac{r \cdot n!}{(n-r+1)! \cdot r \cdot (r-1)!} + \frac{n \cdot n! - r \cdot n! + n!}{(n-r+1) \cdot (n-r)! \cdot r!} \\
&= \frac{(n+1)!}{((n+1)-r)! \cdot r!} = \binom{n+1}{r}
\end{aligned}$$

Algebra of Combinatorics (3)

Proposition 14

Given any real numbers a and b and any integer $n \geq 1$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof:

If $n \geq 2$, by the distribution law, $(a+b)^n$ can be expanded into the sum of products of n letters, where each letter is either a or b.
For each $k = 0, 1, \ldots n$, the product

$$a^{n-k} b^k = \underbrace{a \cdot a \cdot a \cdots a}_{n-k \text{ factors}} \cdot \underbrace{b \cdot b \cdot b \cdots b}_{k \text{ factors}}$$

occurs as a term in the sum the same number of orderings of $(n-k)$ a's and k b's. But this number is $\binom{n}{k}$, the number of ways to choose k positions into which to place the b's. Hence the coefficient of the term $a^{n-k}b^k$ is $\binom{n}{k}$.
Thus

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Algebra of Combinatorics (4)

Proposition 15

Let S be a set of n elements and let $P(S)$ be the power set of S. Then $P(S)$ has 2^n elements.

Proof.

$P(S)$ consists of subsets of cardinality 0, 1, 2, ..., n of S. Clearly, S has $\binom{n}{k}$ subsets of cardinality k. Hence $P(S)$ has

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

elements. Let $a=b=1$ in the binomial theorem. Then

$$2^n = \sum_{k=0}^{n}\binom{n}{k}$$

Hence $P(S)$ has 2^n elements. □

Algebra of Combinatorics (5)

Another application of the binomial theorem is the following.

Example 16

Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof.

In the binomial formula described before, let $a = 1$ and $b = -1$. The conclusion then follows. $\square$

Algebra of Combinatorics (6)

Theorem 17 (Vandermonde Identity)

Let m, n, and r be nonnegative integers such that $r \leq m$ and $r \leq n$. Then

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.$$

Proof.

Partition a set S of size $m+n$ into two subsets T and U. To choose r objects from S, one may choose k objects from T and the remaining $r-k$ objects from U.

On the other hand, if one has r objects from S, there must be an integer $0 \leq k \leq r$ such that k of therm are from T and the remaining $r-k$ objects are from U. □

Algebra of Combinatorics (7)

Theorem 18

Let n be a positive integer. Then

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof.

Note that $\binom{n}{k} = \binom{n}{n-k}$. In the Vandermonde Identity, let $m = n$ and $r = n$. Then the desired conclusion follows. $\square$