Graphs II

Cunsheng Ding

HKUST, Hong Kong

October 30, 2015

Contents

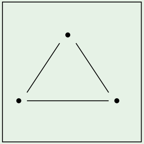
Euler Circuits (1)

Definition 1

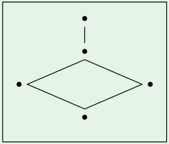
Let G be a graph. An <u>Euler circuit</u> for G is a circuit that contains every vertex and every edge of G.

1. An Euler circuit starts and ends at the same vertex.
2. Every vertex of G is used <u>at least</u> once.
3. Every edge of G is used exactly once.

Example 2



With an Euler Circuit



Without an Euler Circuit

Euler Circuits (2)

Proposition 3

If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof.

Suppose that G is a graph that has an Euler circuit. Let v be any vertex of G.

1. The Euler circuit contain all edges incident on v.
2. Imagine that we are traveling along the Euler circuit. If we travel along an edge to v, we must leave v along another edge.
3. Every edge of G is traversed once in the process (because the Euler circuit uses every edge of G exactly once).

Hence the degree of v is even. □

Question 1

When does a graph has an Euler circuit?

Euler Circuits (3)

Proposition 4

If a graph G is connected and the degree of every vertex is a positive even integer, then G has an Euler circuit.

Proof.

For a proof of this proposition, see p. 650 of the book by Epp. A proof will be presented during a tutorial. □

Combining Propositions 3 and 4, we obtain the following.

Theorem 5

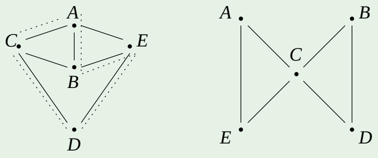
A graph G has an Euler circuit if and only if every vertex of G has a positive even degree.

Hamiltonian Circuits (1)

Question 2

Given a graph G, is it possible to find a circuit for G in which all the vertices of G appear exactly once except the first and last?

Example 6



G_1
(With such a circuit)

G_2
(Withou

Definition 7

Given a graph G, a <u>Hamiltonian circuit</u> for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

Hamiltonian Circuits (2)

Question 3

When does a graph G have a Hamiltonian circuit?

There are some very technical characterisations. We mention only the following sufficient condition whose proof is omitted.

Theorem 8 (Ore's theorem 1960)

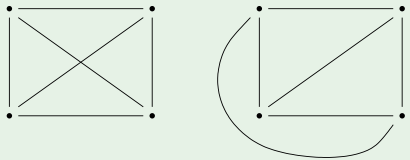
A graph with n vertices ($n \geq 3$) has a Hamiltonian circuit if, for every pair of nonadjacent vertices, the sum of their degrees is n or greater.

Planar Graphs

Definition 9

A graph is planar if it can be drawn in the plane in such a way that no two edges cross.

Example 10



Proposition 11

Let G be a simple planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n - 6$.

Proof.

No proof is given in this course. $\square$

Corollary 12

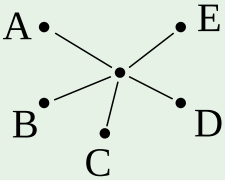
K_5 is not planar.

Trees (1)

Definition 13

A tree is a connected graph without any circuits.

Example 14



Trees (2)

Proposition 15

Let G be a graph. Suppose the degree of each vertex of G is at least 2. Then G contains a circuit.

Proof.

Let v_1 be any vertex of G. Since $deg(v_1) \neq 0$, we may proceed to an adjacent vertex v_2. If $v_2 = v_1$, we have already got a circuit. Otherwise, since $deg(v_2) \geq 2$, we may proceed to an adjacent vertex v_3 along a new edge. If $v_3 = v_1$ or $v_3 = v_2$, we have obtained a circuit.

Generally, suppose $R \geq 3$, and we have determined a path through distinct vertices $v_1, v_2, \ldots, v_R$. Since $deg(v_R) \geq 2$, v_R is adjacent to some vertex other than v_{R-1}. If it is adjacent to one of $v_1, v_2, \ldots, v_{R-1}$, we have a circuit; otherwise, it is adjacent to a new vertex v_{R+1}. Hence $v_1, v_2, \ldots, v_{R+1}$ is a path. Since there are finite many vertices in G, we cannot continue to find new vertices, So eventually, we find a circuit. $\square$

Trees (3)

As a corollary of Proposition 15, we have the following:

Corollary 16

A tree with at least 2 vertices must contain a vertex of degree 1.

Definition 17

Let T be a tree. A vertex of degree 1 in T is called a **terminal vertex** (or a **leaf**), and a vertex of degree at least 2 in T is called an **internal vertex** (or a **branch vertex**)

Trees (4)

Proposition 18

Any tree T with n vertices has $(n-1)$ edges.

By induction on n. When $n=1$, the conclusion is clear. Assume the conclusion is true for $n=R$. We now prove that it is also true for $n=R+1$. Let v be a vertex of degree 1 in T. Remove the single edge that is incident on v. Let T_1 be the remaining graph, where $T_1 = T \setminus \{v\}$. We now prove that T_1 is a tree.

1. T_1 has no circuit, as the original T has no circuit.
2. T_1 is connected.
   Let u and w be any two vertices in T_1. These are also vertices in T. Thus, there is a walk from u to w. By deleting some repeated edges in this walk from u to w, we get a path from u to w. Since you enter and leave each vertex of this path along different edges and $deg(v)=1$, v is not on this path. So this path lies completely in T_1. Hence u and w are connected.

Thus, T_1 is a tree with R vertices, so T_1 has $(R-1)$ edges. It follows that T has $R-1+1=R$ edges.

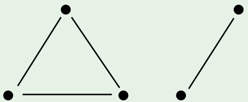
Trees (5)

Proposition 19

If G is a connected graph with n vertices and $(n-1)$ edges, then G is a tree.

A proof of this proposition will be presented in a tutorial.

Example 20



Graph G

G has 5 vertices and 4 edges, but is not a tree because it is **not connected**.

Trees (6)

Corollary 21

A tree with more than one vertex must contain at least two vertices of degree 1.

Proof.

Suppose that T is a tree with n vertices $v_1, \cdots, v_n$ and $n-1$ edges. Then the total degree is $\sum_{i=1}^{n} \deg(v_i) = 2(n-1)$. We already know that some vertex, say v_1, has degree 1. If the remaining $n-1$ vertices each have degree 2 or more, then the sum of all degrees would be at least $1+2(n-1) = 2n-1$, a contradiction. So there must be another vertex other than v_1 with degree 1. $\square$

Rooted Trees (1)

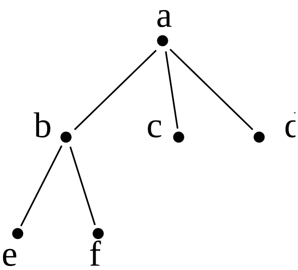
Definition 22

1. A rooted tree is a tree in which one vertex is distinguished from others and is called the root.
2. The level of a vertex is the number of edges along the unique path between it and the root.
3. The height of a rooted tree is the maximum level to any vertex of the tree.
4. Given any internal vertex v of a rooted tree, the children of v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called the parent of w. Two vertices that are both children of the same parent are called siblings.
5. Given vertices v and w, if v lies on the unique path between w and the root, then v is an ancestor of w, and w is a descendent of v.

Rooted Trees (2)

Example 23

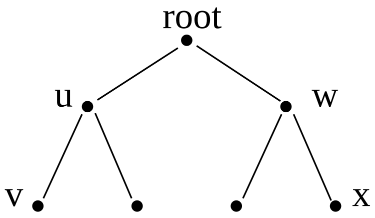
1. a is the root.
2. a, b are internal vertices.
3. c, d, e, f are terminal vertices.
4. The level of b is 1 and the level of e is 2.
5. The height of this rooted tree is 2.
6. a has three children: b, c, d.
7. b is a descendent of a and an ancestor of e.
8. e and f are siblings.



Binary Trees

Definition 24

1. A binary tree is a rooted tree in which every internal vertex has at most two children. Each child in a binary tree is designated either a left child or a right child (but not both), and an internal vertex has at most one left and one right child.
2. A full binary tree is a binary tree in which each internal vertex has exactly two children and all terminal vertices are at the same level.



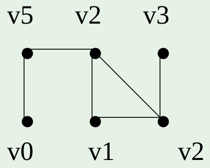
Spanning Trees

Definition 25

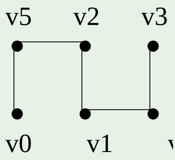
A spanning tree for a graph G is a subgraph of G that contains every vertex of G and is a tree.

Example 26

The following graph G has three spanning trees, and one of them is given on the right-hand side:



G



spanning tree

Spanning Trees

Question 4

Does any connected graph have a spanning tree?

Question 5

If the answer to the question above is positive, how could you construct a spanning tree?

Spanning Trees

Proposition 27

Every connected graph has a spanning tree.

A proof of this proposition will be given in the next few slides.

A Lemma

Lemma 28

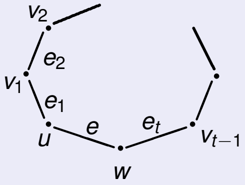
If G is any connected graph, C is any circuit in G, and one of the edges of C is removed from G, then the graph that remains is connected.

Proof.

Let G' be the graph remained. Then

$$V(G') = V(G), \ \ E(G') = E(G) \setminus \{e\}$$

where e is the edge removed. We now prove that G' is connected. The circuit C can be drawn in the following way:



Proof of the Lemma

Proof of Lemma 28.

where the deleted edge e has endpoints u and w. Since C is a circuit, $e_1, e_2, \ldots, e_t$ are not equal to e. For any two vertices x and y in G', there are also vertices in G. Since G is connected, there is a walk from x to y in G. If $u\ e\ w$ appears in this walk $x \ldots y$, replace all $u\ e\ w$ with $u\ e_1\ v_1 \ldots v_{t-1}\ e_t\ w$. If $w\ e\ u$ appears in this walk $x \ldots y$, replace it with $w\ e_t\ v_{t-1} \ldots v_1\ e_1\ u$. In this way, we get another walk from x to y, which does not contain the edge e. So this is a walk from x to y in G', and x and y are connected in G'.
If both uew and weu do not appear in the walk $x \ldots y$, this is a walk in G'. Thus in every case we can find a walk from x to y in G'.

$\square$

Proof of Proposition 27

Proof.

Suppose G is a connected graph. If G is circuit-free, then G is its own spanning tree and we are done. If not, then G has at least one circuit C_1. By lemma 28, the subgraph of G obtained by removing an edge from C_1 is connected. If the subgraph is circuit-free, then it is a spanning tree and we are done. If not, then it contains at least one circuit C_2, and a connected subgraph can be similarly obtained by removing one edge.

Continue in this way, we can remove successive edges from circuits, until eventually we obtain a connected circuit-free subgraph T of G.

Also T contains all vertices of G, as no vertices were removed in this process.

Thus T is a spanning tree for G. $\square$

How to Find a Spanning Tree

Answer

The proof of Proposition 27 is constructive and can be employed to find a spanning tree of a connected graph.

Example 29

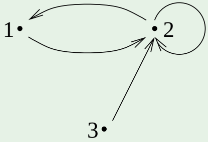
Write down any connected graph on the white board and work out a spanning tree of the given graph.

Digraphs (1)

Definition 30

A digraph (directed graph) is just a graph in which each edge has a direction assigned to it. Such an edge is called a directed edge or arc.

Example 31



Definition 32

1. The indegree of a vertex v is the number of arcs directed into v.
   Example: The indegree of 2 is 3.
2. The outdegree of a vertex v is the number of arcs directed from v.
   Example: The outdegree of 2 is 2.

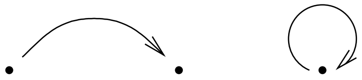
Digraphs (2)

Proposition 33

The sum of the indegrees of the vertices of a digraph equals the sum of the outdegrees of the vertices, this common number being the number of arcs.

Proof.

Every arc contributes 1 to the indegree and outdegree respectively. Hence, the sum of the indegrees of the vertices
= the sum of the outdegrees of the vertices
= the number of arcs □



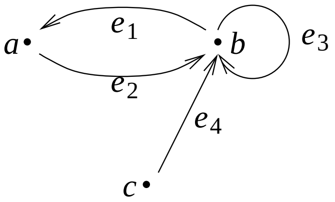
Digraphs (3)

Definition 34

Walks, paths and circuits in digraphs are the same as in graphs, except that the direction of arcs is followed.

Example 35

1. Walk : $c\ e_4\ b\ e_1\ a\ e_2\ b$.
2. Path : $c\ e_4\ b\ e_1\ a\ e_2\ b$.
3. Circuit : $a\ e_2\ b\ e_1\ a$.



Digraphs (4)

Definition 36

A digraph is called strongly connected if and only if there is a walk from any vertex to any other vertex which respects the direction of each arc.

Example 37

G is strongly connected. *H* is not strongly connected, as there is no walk from v_4 to v_1.

$$\begin{tikzcd} v_1 \arrow[r] & v_2 \arrow[d] \\ v_3 \arrow[u] & v_4 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} v_1 \arrow[r] \arrow[d] & v_2 \arrow[d] \\ v_3 \arrow[r] & v_4 \end{tikzcd}$$

$$G \qquad\qquad\qquad H$$

Digraphs (5)

Theorem 38

If a digraph has an Euler circuit, then it is strongly connected, and for every vertex, the indegree equals the outdegree.

Proof.

Let G be a digraph that has an Euler circuit $C : v_0\ e_1\ v_1\ e_2\ \dots\ v_{n-1}\ e_n\ v_0$. Then by definition

1. Every edge appears exactly once in C.
2. Every vertex appears at least once.

Since C is a circuit and every vertex appears at least once in C, there is a walk from any vertex to any other vertex. Hence G is strongly connected.

Let v be any vertex of G. Then all arcs incident on v appear in C. Imagine that we are traveling along the Euler circuit C. If we travel to v along an arc, we must leave v from another arc. Hence each arc directed to v corresponds an arc directed away from v. Hence the indegree equals the outdegree. $\square$

Digraphs (6)

Example 39

The following digraph G is strongly connected, but does not have an Euler circuit, as the indegree of v_1 is 1, but the outdegree is 2.