Perceptrons

COMP4211

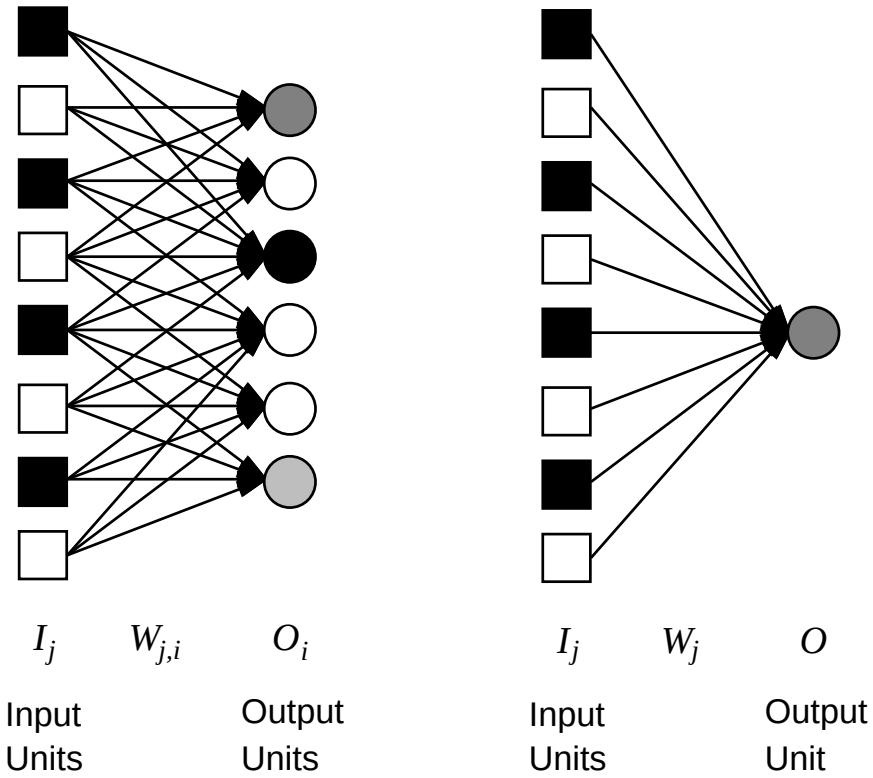
THE DEPARTMENT OF
COMPUTER SCIENCE & ENGINEERING
計算機科學及工程學系

Outline

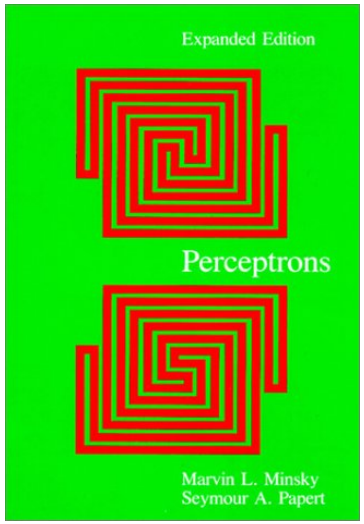
Introduction

Perceptron

a feed-forward network with only one layer of adjustable (learnable) weights connected to one or more threshold units (as output units)



Perceptron...



- written by Marvin Minsky and Seymour Papert and published in 1969
- introduced by Frank Rosenblatt in 1957

Model

input: $I_1, I_2, \ldots, I_n$

- signals from the other neurons

weights: $w_1, w_2, \ldots, w_n$

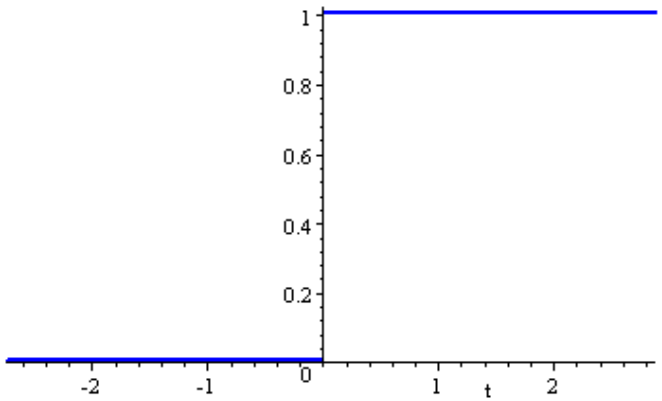
- can be negative

activation function:

- relating the input and output

$$O = \text{step}(\sum_{j=1}^{n} w_j I_j - \theta)$$

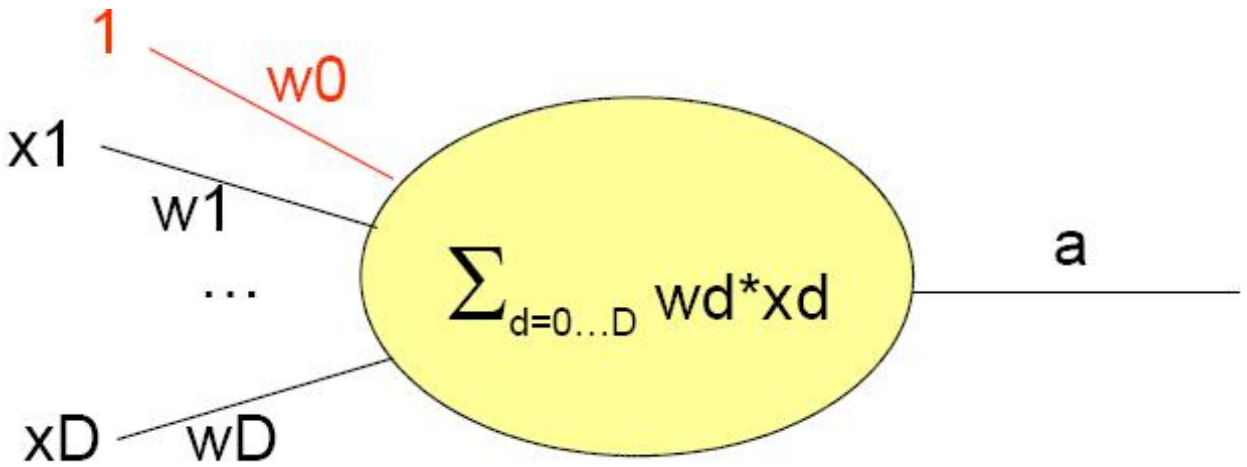
$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad \text{(step function)}$$



Model...

$$O = \text{step}(\sum_{j=1}^{n} w_j I_j - \theta)$$

Or, with $w_0 = -\theta$ and $I_0 = 1$

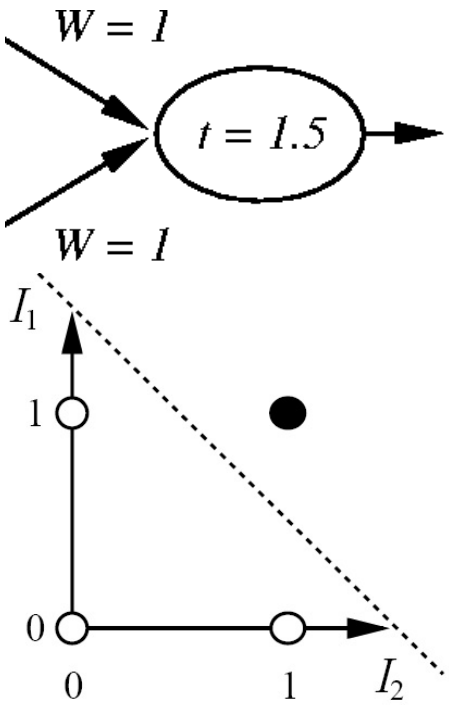


$$O = \text{step}\left(\sum_{j=0}^{n} w_j I_j\right)$$

What Boolean Functions can Perceptrons Represent?

Can it Represent AND?

AND?

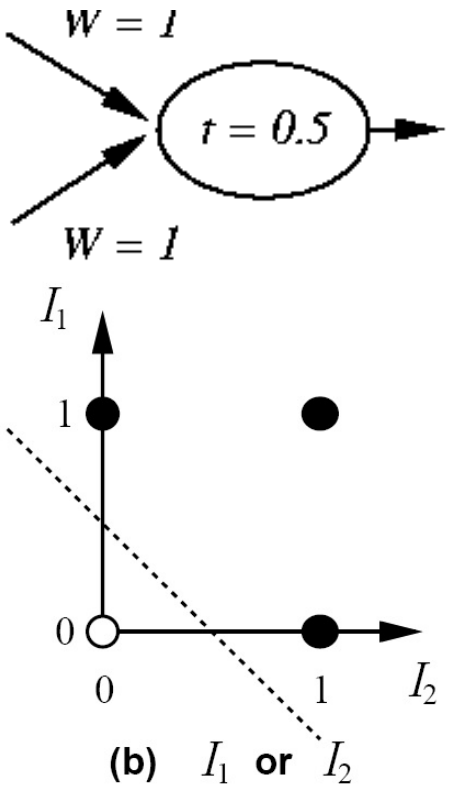


(a) I_1 **and** I_2

- Perceptron output: $O = \text{step}(\sum_{j=0}^{n} w_j I_j)$
  - decision boundary: $\sum_{j=0}^{n} w_j I_j = 0$

Can it Represent OR?

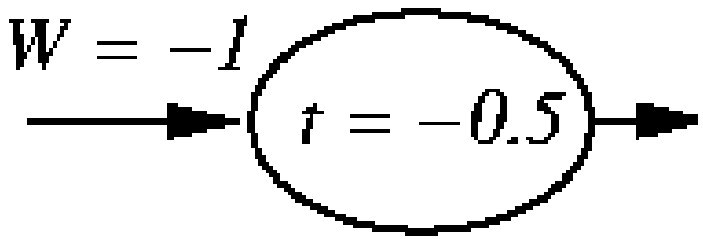
OR?



(b) I_1 or I_2

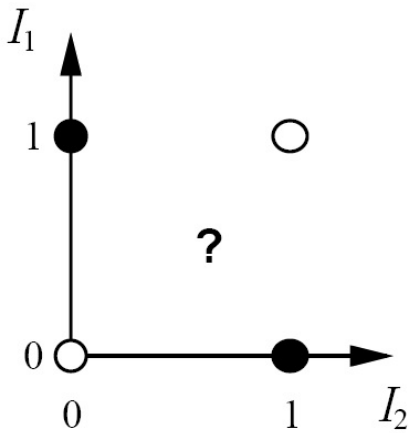
Can it Represent NOT?

NOT?



Can it Represent XOR?

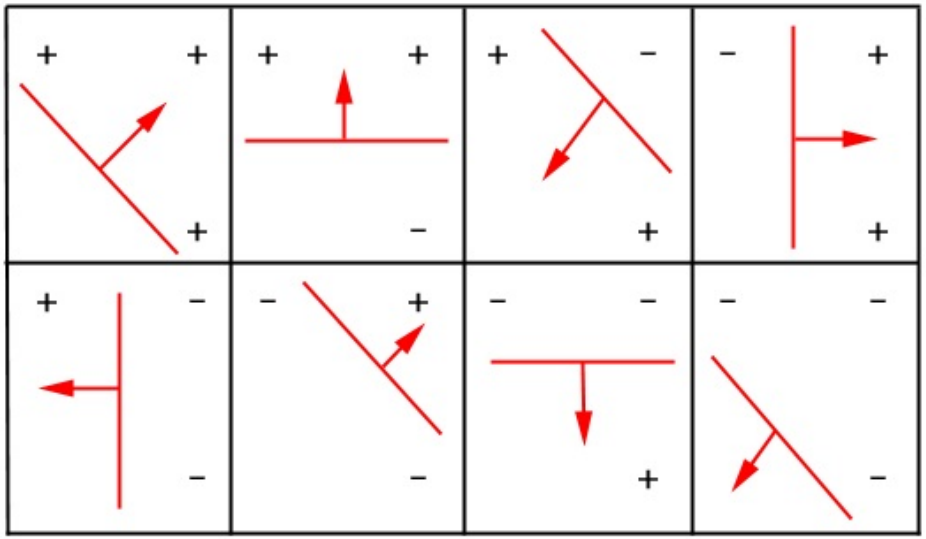
XOR?



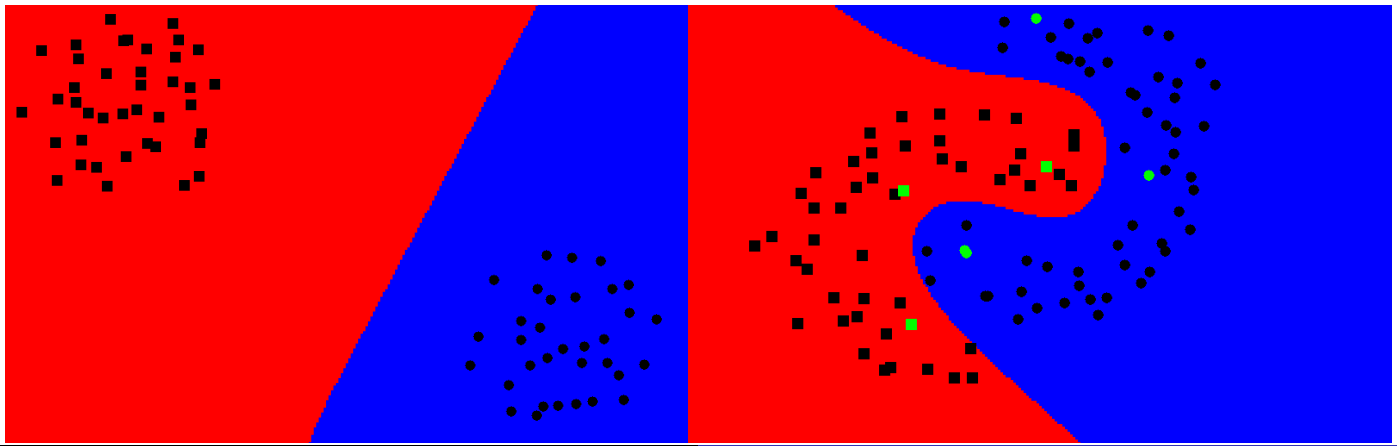
(c) I_1 **xor** I_2

Linearly Separable Functions

- a function can be represented by a single perceptron if and only if the function is linearly separable



Three points in a plane shattered by a half-space.



What Boolean Functions can Neural Networks Represent?

Theorem

With more layers of sufficiently many perceptrons, any Boolean function can be represented

- any Boolean function can be represented in either DNF (disjunctive normal form) or CNF (conjunctive normal form)
- CNF: conjunction of disjuncts

$$(a \vee c) \wedge (b \vee c)$$
$$(a \vee b) \wedge (\neg b \vee c \vee \neg d) \wedge (d \vee \neg e)$$

- DNF: disjunction of conjuncts

$$(a \wedge c) \vee (b \wedge c)$$
$$(a \wedge \neg b \wedge \neg c) \vee (\neg d \wedge e \wedge f)$$

How to Learn Linearly Separable Functions

How to Find the Weights?

Learning Linearly Separable Functions

supervised learning $\Rightarrow$ training examples

Example

apples

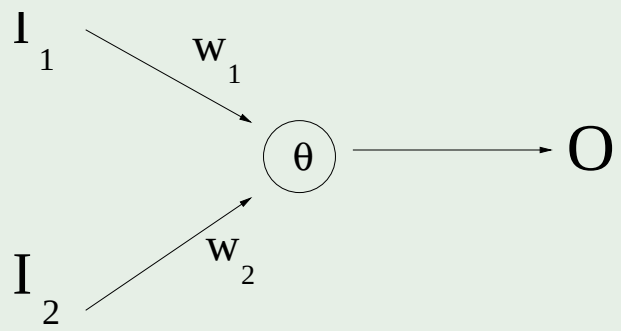
not apples

Learning Linearly Separable Functions...

- convert into features

Example

	I_1	I_2	T
e_1 :	5	1	0
e_2 :	2	1	0
e_3 :	1	1	1
e_4 :	3	3	1
e_5 :	4	2	0
e_6 :	2	3	1



$$O = \text{step}(w_0 + w_1 I_1 + w_2 I_2)$$

Basic Algorithm

- start with some initial values for the weights
- use the perceptron to classify training examples
- modify the weights when errors occur

function NEURAL-NETWORK-LEARNING(*examples*) **returns** *network*

network ← a network with randomly assigned weights
repeat
  for each *e* **in** *examples* **do**
    O ← NEURAL-NETWORK-OUTPUT(*network*, *e*)
    T ← the observed output values from *e*
    update the weights in *network* based on *e*, **O**, and **T**
  end
until all examples correctly predicted or stopping criterion is reached
return *network*

- an iterative algorithm

How to Update the Weights?

idea

if the observed output (T) is different from the predicted one (O), then make small adjustments in the weights to reduce the difference

- if the error $T - O$ is positive, then we need to increase O
- if the error is negative, then we need to decrease O
- each input unit contributes $w_j I_j$ to the total input, so if I_j is positive, an increase in w_j will tend to increase O
- if I_j is negative, an increase in w_j will tend to decrease O

weight update rule

$$w_j \leftarrow w_j + \alpha I_j (T - O), \quad \forall j$$

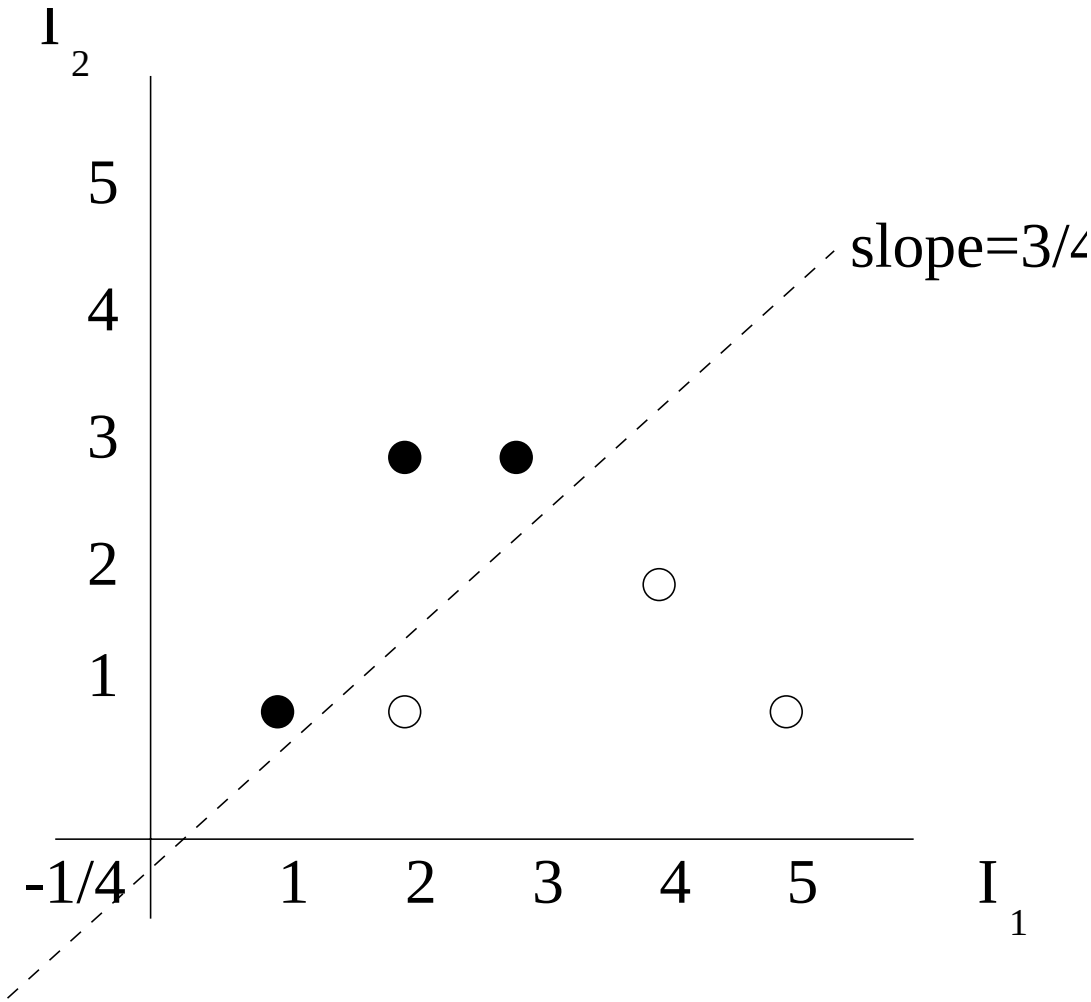
- α: learning rate
- demo (hrefhttps://www.youtube.com/watch?v=vGwemZhPlsA)

Trace of The Learning Process

Take $\alpha = 1$ and the initial weight values to be 0

Iteration	w^{old}	I	T	O	T = O?	w^{new}
1	(0 0 0)	(1 5 1)	0	1	no	(-1 -5 -1)
2	(-1 -5 -1)	(1 2 1)	0	0	yes	(-1 -5 -1)
3	(-1 -5 -1)	(1 1 1)	1	0	no	(0 -4 0)
4	(0 -4 0)	(1 3 3)	1	0	no	(1 -1 3)
5	(1 -1 3)	(1 4 2)	0	1	no	(0 -5 1)
6	(0 -5 1)	(1 2 3)	1	0	no	(1 -3 4)
7	(1 -3 4)	(1 5 1)	0	0	yes	(1 -3 4)
8	(1 -3 4)	(1 2 1)	0	0	yes	(1 -3 4)
9	(1 -3 4)	(1 1 1)	1	1	yes	(1 -3 4)
10	(1 -3 4)	(1 3 3)	1	1	yes	(1 -3 4)
11	(1 -3 4)	(1 4 2)	0	0	yes	(1 -3 4)
12	(1 -3 4)	(1 2 3)	1	1	yes	(1 -3 4)

Geometric Interpretation of Solution



Perceptron Convergence Theorem

if the training examples are linearly separable, then applying the perceptron weight updating rule can

- always converge to some solution (i.e. a set of weights)
- in a finite number of steps for any initial choice of weights

what if the examples are not linearly separable?

- perceptron may ...