Binary Relations

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September 27, 2015

Contents

Binary Relations

Definition 1

1. Let A and B be two sets. A binary relation R from A to B is a subset of $A \times B$. A binary relation on A is a subset of $A \times A$.
2. Given an ordered pair $(x, y) \in A \times B$, we say that x is related to y by R, written $x\ R\ y$, if and only if $(x, y) \in R$.

Example 2

Let $A = \{$Alice, Jim$\}$ and $B = \{$Math, Biology$\}$. Assume that

Student	Course
Alice	Math, Biology
Jim	Math

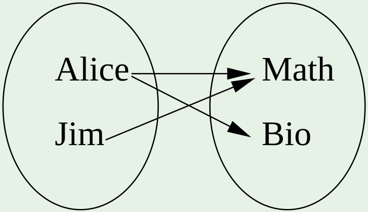
Then the students and modules involved can be described by the following relation: $R = \{$(Alice,Math), (Alice,Biology), (Jim,Math)$\}$.

Ways to describe a Binary Relation (1)

1. In terms of a subset of $A \times B$ (see Example 2).
2. Arrow diagram.

Example 3

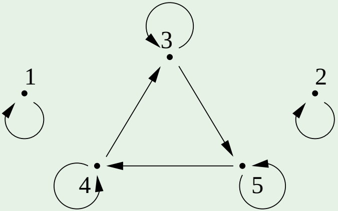
The arrow diagram representation of the binary relation of Example 2 is the following:



Ways to describe a Binary Relation (2)

Example 4

Write the following relation R in terms of ordered pairs.



Solution:

$$A = \{1,2,3,4,5\}$$
$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(3,5),(5,4),(4,3)\}$$

The Inverse Relation

Definition 5

Let R be a relation from X to Y. The inverse of R, denoted by R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y,x) \mid (x,y) \in R\}$$

Example 6

Let $X = \{2,3,4\}$ and $Y = \{3,4,5,6,7\}$. Define

$$R = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}$$

Then

$$R^{-1} = \{(4,2),(6,2),(3,3),(6,3),(4,4)\}$$

The Composition of Relations

Definition 7

Let R_1 be a relation from X to Y, and R_2 be a relation from Y to Z. The composition of R_1 and R_2 is denoted by $R_2 \circ R_1$ or simply $R_2 R_1$, is the relation from X to Z defined as

$$R_2 R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2, \text{ for some } y \in Y.\}$$

Example 8

Let $X = Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Z = \{u, s, t\}$, and let

$$\begin{aligned} R_1 &= \{(1,2), (1,6), (2,4), (3,4), (3,6), (3,8)\}, \\ R_2 &= \{(2,u), (4,s), (4,t), (6,t), (8,u)\} \end{aligned}$$

Then

$$R_2 R_1 = \{(1,u), (1,t), (2,s), (2,t), (3,s), (3,t), (3,u)\}$$

Reflexive Relations

Definition 9

A binary relation R on a set A is reflexive if and only if $(a, a) \in R$ for all $a \in A$.

Example 10

Let $A = \{1, 2, 3, 4\}$ and let
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$. Is R reflexive?

Example 11

Let $\mathbb{R}$ be the set of real numbers and let $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$. Then R is reflexive.

Example 12

$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 0\}$ is not a reflexive relation.

Proof.

$(0, 0) \notin R$. □

Symmetric Relations

Definition 13

A binary relation R on a set A is symmetric if and only if for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$.

Example 14

$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a symmetric relation on R.

Example 15

Let $A = \{1, 2, 3\}$, $R = \{(1, 1), (2, 2)\}$. Then R is symmetric, but not reflexive.

Example 16

$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ is not a symmetric relation on R.

Proof.

$(2, 1) \in R$ but $(1, 2) \notin R$. $\square$

Transitive Relations

Definition 17

A binary relation on R on a set A is <u>transitive</u> if and only if for all $a, b, c \in A,\ (a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.

Example 18

$R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ is a transitive relation on $\mathbb{R}$.

Proof.

Let $(x, y) \in R$ and $(y, z) \in R$. Then $x \leq y$ and $y \leq z$. Hence $x \leq z$ and $(x, z) \in R$. □

Example 19

$R = \{(a, b), (b, a), (a, a)\}$ is not a transitive relation on $\{a, b\}$.

Proof.

$(b, a) \in R$, $(a, b) \in R$, but $(b, b) \notin R$. □

Equivalence Relations (1)

Definition 20

An <u>equivalence relation</u> on a set A is a binary relation R on A which is reflexive, symmetric and transitive.

(a) Many people use $\sim$ to denote an equivalence relation.

(b) To prove that R is an equivalence relation, we need to prove reflexivity, symmetry, and transitivity.

(c) To prove that R is NOT an equivalence relation, we need to prove one of the reflexivity, symmetry, and transitivity does not hold.

Example 21

The relation $\leq$ on the real numbers is not an equivalence relation.

Proof.

The relation $\leq$ is not symmetric. For example, $(3,7) \in R$, but $(7,3) \notin R$. $\square$

Equivalence Relations (2)

Example 22

Let $R = \{(x,y) \mid (x,y) \in \mathbb{Z}^2 \text{ and } 3 \mid (x-y)\}$. Then R is an equivalence relation on $\mathbb{Z}$.

Proof.

Reflexivity: Since 3 divides $x - x = 0$ for any $x \in \mathbb{Z}$, by definition, $(x,x) \in R$.
Symmetry: Let $(x,y) \in R$, we want to prove $(y,x) \in R$.

$$(x,y) \in R \text{ implies } 3 \mid (x-y) \text{ implies } 3 \mid (y-x) \text{ implies } (y,x) \in R.$$

Transitivity: For any $(x,y) \in R$ and $(y,z) \in R$, we now prove $(x,z) \in R$.

$$(x,y) \in R \text{ implies } 3 \mid (x-y) \text{ implies} (x-y) = 3k_1 \text{ for some } k_1$$
$$(y,z) \in R \text{ implies } 3 \mid (y-z) \text{ implies } (y-z) = 3k_2 \text{ for some } k_2$$

Hence $x - z = (x-y) + (y-z) = 3(k_1 + k_2)$. Thus $3 \mid (x-z)$. By definition, $(x,z) \in R$.

□

Partitions on A

Definition 23

A partition of a set A is a collection of disjoint **nonempty** subsets of A whose union is A. These disjoint sets are called cells (or blocks). The cells are said to partition A.

Example 24

Let $A = \{0,1,2,3,4,5,6\}$. Then

1. $\{\{0,1\},\{2,3,4\},\{5,6\}\}$ is a partition of A.
2. $\{\{0\},\{1,2,3,4,5,6\}\}$ is also a partition of A.

Equivalence Relation Induced by a Partition (1)

Let A be a set and let $\{A_1, A_2, \ldots, A_n\}$ be a partition of A. Define a binary relation R by

$$a\ R\ b \text{ iff } a \text{ and } b \in A_i \text{ for some } i$$

This is a binary relation.

Example 25

Let $A = \{1,2,3\}$ and let $\{\{1\},\{2,3\}\}$ be a partition on A. The equivalence relation R induced by this partition is:

$$R = \{(1,1),(2,3),(3,2),(2,2),(3,3)\}$$

Equivalence Relation Induced by a Partition (2)

Let A be a set and let $\{A_1, A_2, \ldots, A_n\}$ be a partition of A. Define a binary relation R by

$$a \, R \, b \text{ iff } a \text{ and } b \in A_i \text{ for some } i$$

Theorem 26

The above R is an equivalence relation defined on A.

Proof.

1. Reflexivity: a and a in the same subset implies that $a \, R \, a$.
2. Symmetry:

$$a \, R \, b \text{ implies } \{a, b\} \subseteq A_i \text{ implies } \{b, a\} \subseteq A_i \text{ implies } b \, R \, a$$

3. Transitivity: Assume that $a \, R \, b$ and $b \, R \, c$. Then a and b are in the same subset A_i, and b and c are in the same subset A_j. If $i \neq j$, then $b \in (A_i \cap A_j)$. This is contrary to $A_i \cap A_j = \emptyset$. Therefore $i = j$ and $a \, R \, c$.

□

Partition Induced by an Equivalence Relation

Definition 27

Let R be an equivalence relation on a set A. The equivalence class containing a, denoted $\overline{a}$, is defined by

$$\overline{a} = \{x \in A \mid x \; R \; a\}.$$

It is straightforward to prove the following.

Theorem 28

Let R be an equivalence relation on a set A. The set $\{\overline{a} \mid a \in A\}$ of distinct equivalence classes forms a partition of A.

Antisymmetric Relations

Definition 29

A binary relation R on a set A is antisymmetric if and only if for all $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ implies that $a = b$.

Example 30

Let S be any set and let $A = P(S)$ be the power set of S, then

$$R = \{(X, Y) \mid X, Y \in P(S),\ X \subseteq Y\}$$

is an antisymmetric relation on $P(S)$.

Warning: "Antisymmetric" $\neq$ "not Symmetric".

Example 31

$R = \{(1,2), (2,3), (3,3), (2,1)\}$ defined on $A = \{1,2,3\}$ is not symmetric, but neither is it antisymmetric.

Partial Orders

Definition 32

1. A partial order R on a set A is a reflexive, antisymmetric, transitive relation on A.
2. A partially ordered set, is a pair (A, R), where R is a partial order on A.

Example 33

The binary relation $\leq$ on the set of real numbers is a partial order.

Proof.

1. Reflexivity: $a \leq a$.
2. Antisymmetry: $a \leq b,\ b \leq a$ implies that $a = b$.
3. Transitivity: $a \leq b,\ b \leq c$ implies that $a \leq c$.

□

Total Orders

Definition 34

Let (A, R) be a partially ordered set. Elements a and b of A are said to be comparable if and only if either $a\ R\ b$ or $b\ R\ a$.

$(P(S), \subseteq)$ is a partially ordered set. Note that $\{a\} \not\subseteq \{b, c\}$, $\{b, c\} \not\subseteq \{a\}$.

Definition 35

Let R be a partial order. If for every $a, b \in A$, either $a\ R\ b$ or $b\ R\ a$, then R is called a total order, and (A, R) is called a totally ordered set.

Example 36

$(\mathbb{R}, \leq)$ is a totally ordered set, and $\leq$ is a total order on $\mathbb{R}$.

The Greatest and Least Elements

Definition 37

1. A greatest element g of a partially ordered set (A, R) is an element such that $a \; R \; g$ for every element $a \in A$.
2. A least element l of a partially ordered set (A, R) is an element such that $l \; R \; a$ for every element $a \in A$.

Example 38

Let S be a set, and let $P(S)$ be the power set. Then $(P(S), \subseteq)$ is a partially ordered set. Then

1. $\emptyset$ is the least element;
2. S is the greatest element.

Remark: Least and greatest elements are comparable with every element.

Question 39

Are the greatest and least elements unique, if they exist?

Comments on Binary Relations

- Binary relations are an important part of discrete mathematics.
- Functions that will be covered in the sequel are special binary relations.
- Also graphs can also be viewed as binary relations.
- Many of the things in our daily life can be viewed as binary relations, for examples, love and hate.

Question 40

Let A be the set of students in this classroom, and let Love be the binary relation defined on A.

1. *Is this relation reflexive?*
2. *Is this relation symmetric?*
3. *Is this relation antisymmetric?*
4. *Is this relation transitive?*