COMP170 – Fall 2007
Midterm 1 Review

Question 1

n guests are arranged seats in a row facing the audience

a) How many different ways are there to seat the n guests at n seats?

Question 1

n guests are arranged seats in a row facing the audience

a) How many different ways are there to seat the n guests at n seats?

$$n!$$

Note that the order is important since the guests could be listed from left to right from the audience's perspective

Question 1

n guests are arranged seats in a row facing the audience

b) Let $n > 2$. How many ways to seat n guests if two specific guests will not sit next to each other?

Question 1

n guests are arranged seats in a row facing the audience

b) Let $n > 2$. How many ways to seat n guests if two specific guests will not sit next to each other?

$\#$ ways *not sitting together* $= n! -$ $\#$ ways sitting together

Question 1

n guests are arranged seats in a row facing the audience

b) Let $n > 2$. How many ways to seat n guests if two specific guests will not sit next to each other?

$\#$ ways *not sitting together* $= n! -$ $\#$ ways sitting together

Treating the two people as one indivisible group, there are $(n-1)!$ different ways of seating the guests.
Within the two-person group, there are 2 ways to sit them.

$\#$ ways sitting together $= 2(n-1)!$
$\#$ ways *not* sitting together $= n! - 2(n-1)! = (n-2)(n-1)!$

Question 1

n guests are arranged seats in a row facing the audience

c) $n > 10$. Guests include 5 couples. For each couple, husband must sit with wife. How many seatings are there?

Question 1

n guests are arranged seats in a row facing the audience

c) $n > 10$. Guests include 5 couples. For each couple, husband must sit with wife. How many seatings are there?

Treat each couple as an indivisible group, and others as individual groups.
There are $(n-5)!$ ways to seat $n-5$ groups.

For each couple (group),
there are 2 ways to seat husband and wife.

ways to seat guests $= 2^5(n-5)!$

Question 2

$$Z_n = \{0, ..., n-1\}$$

a) How many 5-element subsets of Z_{10} contain at least one element in Z_3?

Question 2

$Z_n = \{0, ..., n-1\}$

a) How many 5-element subsets of Z_{10} contain at least one element in Z_3?

\# 5-element sets of $Z_{10} = \binom{10}{5}$
\# 5-element sets of Z_{10} *not* containing elements of $Z_3 = \binom{7}{5}$

\# 5-element sets of Z_{10} containing *at least one* element of Z_3

$$= \binom{10}{5} - \binom{7}{5}$$

Question 2

b) How many 5-element subsets of Z_{10} contain two odd and three even numbers?

Question 2

b) How many 5-element subsets of Z_{10} contain two odd and three even numbers?

$$Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$\binom{5}{2}$ ways to choose 2 numbers from $\{1, 3, 5, 7, 9\}$
$\binom{5}{3}$ ways to choose 3 numbers from $\{0, 2, 4, 6, 8\}$

By the product principle, we have

$$\binom{5}{2} \cdot \binom{5}{3}$$

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Let $O_n = \{1, 3, .5, ..., n\}$

$E_n = \{0, 2, 4, ..., n-1\}$

$f(k) = n - k$ defines a bijection between O_n and E_n

Since $\binom{n}{k} = \binom{n}{n-k}$, we have

$$\sum_{k \in O_n} \binom{n}{k} = \sum_{k \in O_n} \binom{n}{n-k} = \sum_{k \in E_n} \binom{n}{k}$$

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Alternatively:

$$2^n = (1+1)^n = \sum_{0 \le i \le n} \binom{n}{i} \quad \text{and} \quad 0 = (1-1)^n = \sum_{0 \le i \le n} (-1)^i \binom{n}{i}$$

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Alternatively:

$$2^n = (1+1)^n = \sum_{0 \leq i \leq n} \binom{n}{i} \quad \text{and} \quad 0 = (1-1)^n = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i}$$

$$\Rightarrow \qquad 2 \sum_{i \in E_n} \binom{n}{i} = \sum_{0 \leq i \leq n} \binom{n}{i} + \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} = 2^n$$

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Alternatively:

$$2^n = (1+1)^n = \sum_{0 \le i \le n} \binom{n}{i} \quad \text{and} \quad 0 = (1-1)^n = \sum_{0 \le i \le n} (-1)^i \binom{n}{i}$$

$$\Rightarrow \qquad 2 \sum_{i \in E_n} \binom{n}{i} = \sum_{0 \le i \le n} \binom{n}{i} + \sum_{0 \le i \le n} (-1)^i \binom{n}{i} = 2^n$$

$$\Rightarrow \qquad \sum_{i \in E_n} \binom{n}{i} = 2^{n-1} = \sum_{i \in O_n} \binom{n}{i}$$

Question 2

c) Let n be positive and odd. Show that # of even-sized subsets of Z_n equals # of odd-sized subsets of Z_n

Alternatively:

$$2^n = (1+1)^n = \sum_{0 \le i \le n} \binom{n}{i} \quad \text{and} \quad 0 = (1-1)^n = \sum_{0 \le i \le n} (-1)^i \binom{n}{i}$$

$$\Rightarrow \qquad 2 \sum_{i \in E_n} \binom{n}{i} = \sum_{0 \le i \le n} \binom{n}{i} + \sum_{0 \le i \le n} (-1)^i \binom{n}{i} = 2^n$$

$$\Rightarrow \qquad \sum_{i \in E_n} \binom{n}{i} = 2^{n-1} = \sum_{i \in O_n} \binom{n}{i}$$

Note: This works for all n, not just even n.

Question 3

Our office door has a lock whose keypad contains only the 8 digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. For technical reasons, legal key-codes should satisfy the following three requirements:

(i) They are 4 digits long with all of the digits being different.

(ii) They must end with an even digit.

(iii) They cannot have their first digit equal to 0.

For example, 1350 and 5432 are legal key-codes, while 1150, 1357 and 0536 are not legal key-codes.
How many legal key-codes are there?

Question 3

We split the set of legal key-codes into two sets; those that end with a 0 and those that don't.

Those that end with a 0 have 7 possible first digits, 6 possible second ones and 5 possible third ones. So the total number is $7 \cdot 6 \cdot 5 = 210.$

Those that do not end with a 0 have 3 possible last digits, 6 possible first ones, 6 possible second ones and 5 possible third ones. So the total number is $3 \cdot 6 \cdot 6 \cdot 5 = 540.$

Adding the two gives $210 + 540 = 750.$

Question 4

a) How many surjections are there from S_7 to S_6?

Question 4

a) How many surjections are there from S_7 to S_6?

2 elements in S_7 map to the same element in S_6
Remaining 5 elements in S_7 map to the remaining 5 elements in S_6.

View the 2 elements as a group, and other elements as individual groups. Then we map 6 groups from S_7 to 6 elements in S_6.

ways to choose 2 elements from $S_7 = \binom{7}{2} = 21$.
ways to map the groups from S_7 to $S_6 = 6! = 720$.

By product principle, we have $\binom{7}{2}\, 6! = 15{,}120$.

Question 4

b) How many surjections are there from S_8 to S_6?

Separate the surjections into 2 types:

Type 1

- Map 3 elements in S_8 to the same element in S_6
- Map remaining 5 elements in S_8 to remaning 5 elements in S_6.

Type 2

- Map a pair of elements in S_8 to the same element in S_6
- Map another pair in S_8 to the same element in S_6.
- Map remaining 4 elements in S_8 to remaining 4 elements in S_6.

Question 4

b) How many surjections are there from S_8 to S_6?

Count *Type 1*

$\#$ ways to choose 3 elements in $S_8 = \binom{8}{3}$.

Treat the 3 elements as 1 group, other elements as individual groups. Then we map 6 groups in S_8 to 6 elements in S_6.

$\#$ ways to map 6 groups to 6 elements $= 6!$

So $\#$ *Type 1* surjections is $\binom{8}{3} \cdot 6!$

Question 4

b) How many surjections are there from S_8 to S_6?

Count *Type 2*

$\#$ ways to choose 2 pairs in $S_8 = \frac{1}{2}\binom{8}{2} \cdot \binom{6}{2}$.

Treat the each pair as 1 group, other elements as individual groups. Then we map 6 groups in S_8 to 6 elements in S_6.

$\#$ ways to map 6 groups to 6 elements $= 6!$

So $\#$ *Type 2* surjections is $\frac{1}{2}\binom{8}{2} \cdot \binom{6}{2} \cdot 6!$

Question 4

b) How many surjections are there from S_8 to S_6?

Count *Type 2*

Note: Many students forgot this $\frac{1}{2}$

ways to choose 2 pairs in $S_8 = \frac{1}{2}\binom{8}{2} \cdot \binom{6}{2}$.

Treat the each pair as 1 group, other elements as individual groups. Then we map 6 groups in S_8 to 6 elements in S_6.

ways to map 6 groups to 6 elements $= 6!$

So # *Type 2* surjections is $\frac{1}{2}\binom{8}{2} \cdot \binom{6}{2} \cdot 6!$

Question 4

b) How many surjections are there from S_8 to S_6?

The 2 types of surjections form a partition of the set of all surjections from S_8 to S_6.

By the sum principle, number of surjections from S_8 to S_6 is

$$\# \textit{ Type 1} + \# \textit{ Type 2}$$

$$= \binom{8}{3} \cdot 6! + \frac{1}{2}\binom{8}{2}\binom{6}{2} \cdot 6!$$

$$= 266 \cdot 6!$$

$$= 191,520$$

Question 5

Consider a Cartesian Coordinate system.

a) Each step we move either 1 unit to the right or 1 unit down. How many different paths are there from (0,0) to (20,-10)?

Question 5

Consider a Cartesian Coordinate system.

a) Each step we move either 1 unit to the right or 1 unit down. How many different paths are there from (0,0) to (20,-10)?

Every path consists of 30 steps - 20 “right steps” and 10 “down steps”. So # different paths is

$$\binom{30}{10} = \binom{30}{20}$$

Question 5

b) We start from (0,0) walk 10 steps without a predetermined destination. Each step we move 1 unit to the right or 1 unit down. How many different paths of 10 steps are there?

Question 5

b) We start from (0,0) walk 10 steps without a predetermined destination. Each step we move 1 unit to the right or 1 unit down. How many different paths of 10 steps are there?

Each step, we have 2 choices - to the right or down. So # different 10-step paths is

$$2 \times 2 \times ... \times 2 = 2^{10}$$

Question 5

c) We start from (0,0) and walk k steps without a predetermined destination. Each step we move 1 unit either up, down, left or right. How many different paths of k steps are there?

Question 5

c) We start from (0,0) and walk k steps without a predetermined destination. Each step we move 1 unit either up, down, left or right. How many different paths of k steps are there?

Each step, we have 4 choices - left, right, up or down. So $\#$ different k-step paths is

$$4 \times 4 \times ... \times 4 = 4^k$$

Question 6

We want to divide 12 students into three groups.

a) If the three groups are of sizes 3, 4 and 5, how many different groupings are there?

Question 6

We want to divide 12 students into three groups.

a) If the three groups are of sizes 3, 4 and 5, how many different groupings are there?

Since the groups are of *different* sizes, this is equivalent to # ways of labeling 12 objects with three different colors. The answer is the trinomial coefficient

$$\binom{12}{3\ 4\ 5} = \frac{12!}{3!\,4!\,5!}$$

Question 6

We want to divide 12 students into three groups.

b) If the three groups are of the same size (4 people each), how many different groupings are there?

Question 6

We want to divide 12 students into three groups.

b) If the three groups are of the same size (4 people each), how many different groupings are there?

Since the groups are of the *same* sizes, we need to be aware of double counting. For each grouping, there are 3! possible ways of rearranging the given groups. So the answer is

$$\frac{1}{3!}\binom{12}{4\;\;4\;\;4} = \frac{12!}{3!\,4!\,4!\,4!}$$

Question 6

We want to divide 12 students into three groups.

b) If the three groups are of the same size (4 people each), how many different groupings are there?

Since the groups are of the *same* sizes, we need to be aware of double counting. For each grouping, there are 3! possible ways of rearranging the given groups. So the answer is

$$\frac{1}{3!}\binom{12}{4\ 4\ 4} = \frac{12!}{3!\,4!\,4!\,4!}$$

Note: Many students forgot the $\frac{1}{3!}$

Question 7

Consider the following modular equation

$$a \cdot_n \bar{x} = b,$$

where n is a positive integer and $a, b, \bar{x} \in Z_n$. Given n, a, b, the equation has a solution if we can find $\bar{x} \in Z_n$ that satisfies the equation.

Question 7

Consider the following modular equation

$$a \cdot_n \bar{x} = b,$$

where n is a positive integer and $a, b, \bar{x} \in Z_n$. Given n, a, b, the equation has a solution if we can find $\bar{x} \in Z_n$ that satisfies the equation.

a) One way to check whether a has a multiplicative inverse in Z_n is to try to find integers x, y satisfying some equation that involves x, y, a, n. What is the equation?

Question 7

Consider the following modular equation

$$a \cdot_n \bar{x} = b,$$

where n is a positive integer and $a, b, \bar{x} \in Z_n$. Given n, a, b, the equation has a solution if we can find $\bar{x} \in Z_n$ that satisfies the equation.

a) One way to check whether a has a multiplicative inverse in Z_n is to try to find integers x, y satisfying some equation that involves x, y, a, n. What is the equation?

$$ax + ny = 1$$

Question 7

b) Suppose a has a multiplicative inverse in Z_n. Consider the equation that you wrote down in part (a). Are the integers x, y that satisfy that equation unique?

Question 7

b) Suppose a has a multiplicative inverse in Z_n. Consider the equation that you wrote down in part (a). Are the integers x, y that satisfy that equation unique?

Not unique.
Let $n = 12$ and $a = 5$.
Then the two pairs $x = -7, y = 3$ and $x = -19, y = 8$ both satisfy the equation

$$ax + ny = 5x + 12y = 1.$$

Question 7

b) Suppose a has a multiplicative inverse in Z_n. Consider the equation that you wrote down in part (a). Are the integers x, y that satisfy that equation unique?

Not unique.
Let $n = 12$ and $a = 5$.
Then the two pairs $x = -7, y = 3$ and $x = -19, y = 8$ both satisfy the equation

$$ax + ny = 5x + 12y = 1.$$

Note: The question was about uniqueness of integers x, y not uniqueness of x y in Z_n.

Question 7

c) Find a solution $\bar{x}$ in Z_{12} for the following equation:

$$5 \cdot_{12} \bar{x} = 8.$$

Is the solution unique?

Question 7

c) Find a solution $\bar{x}$ in Z_{12} for the following equation:

$$5 \cdot_{12} \bar{x} = 8.$$

Is the solution unique?

We proved in class that if a has a multiplicative inverse a' in Z_n, then the *unique* solution to the equation is $\bar{x} = a' \cdot_n b$.

If $ax + ny = 1$ then $a' = x \bmod n$. From (b),

$$a' = -7 \bmod 12 = -19 \bmod 12 = 5.$$

So the unique solution is $\bar{x} = 5 \cdot_{12} 8 = 4.$

Question 8

a) Does there exist an x in Z_{99} that solves

$$123 \cdot_{99} x = 5?$$

If yes, give the value of x. If no, prove the fact.

Question 8

a) Does there exist an x in Z_{99} that solves

$$123 \cdot_{99} x = 5?$$

If yes, give the value of x. If no, prove the fact.

No. Suppose there was a solution x. Then there is some integer q such that

$$123\,x = 99\,q + 5$$

$$123\,x - 99\,q = 5.$$

But the left hand side of this equation is divisible by 3 and the right hand side is not. Contradiction. So such an x does not exist.

Question 8

b) Does there exist an x in Z_{100} that solves

$$123 \cdot_{100} x = 5?$$

If yes, give the value of x. If no, prove the fact.

Question 8

b) Does there exist an x in Z_{100} that solves

$$123 \cdot_{100} x = 5?$$

If yes, give the value of x. If no, prove the fact.

Yes. $x = 35$. Using extended GCD algorithm, we have

$$123 \cdot (-13) + 100 \cdot 16 = 1.$$

So $87 = (-13) \bmod 100$ is the multiplicative inverse of 123 in Z_{100}. So the unique solution to the problem is

$$87 \cdot_{100} 5 = (87 \cdot 5) \bmod 100 = 435 \bmod 100 = 35.$$