COMP170
Discrete Mathematical Tools for Computer Science

Dealing with floors and ceilings in divide-and-conquer recurrences

Version 2: Last updated, Dec 8, 2005

We have seen that when n is a power of 2.

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases} \qquad (*)$$

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Note that, when n is not a power of 2, a D&C recurrence will split n into $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Eq (*) then becomes

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases} \quad (**)$$

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When n is a power of 2 then (**) is defined by (*).

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Let $m = 2^{i+1}$ be the smallest power of $2 \geq n$. Since the interval $[n, 2n-1]$ contains a power of 2 we have $m < 2n$. So,

$$\begin{aligned} T(n) &\leq T(m) \\ &= m(1 + \log_2 m) \\ &\leq 2n(1 + \log_2 2n) \\ &= 2n(2 + \log_2 n) \end{aligned}$$

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This gives us an *upper bound*.

On the other hand, $m/2 = 2^i \leq n < m$. So,

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\begin{aligned}
T(n) &\geq T(\frac{m}{2}) \\
&= \frac{m}{2}(1 + \log_2 \frac{m}{2}) \\
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This gives us a *lower bound*.

We have just seen that if $T(n)$ is defined by

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

then (assuming that Theorem 1 is true)

$$\frac{n}{2}\log_2 n \;\leq\; T(n) \;\leq\; 2n(2+\log_2 n)$$

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So, getting rid of the condition that n be a power of 2 and adding the floors and ceilings didn't really change much. The approach we have seen can, with a bit more work, be made into a general technique for getting rid of floors and ceilings

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We will actually prove the *stronger* statement

Theorem 2

For any positive integer n, $T(n) < T(n+1)$

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Theorem 2
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Proof: (by strong induction)
Basis: $T(2) = 2 * T(1) + 2 = 4 > T(1)$.

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Hypothesis: Let $n > 2$.
Assume that for all $m < n$, $T(m) < T(m+1)$.

Step: There are two possibilities for n:

(i) n is even: Then, for some $m < n$, $n = 2m$,

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We just saw that in both cases, n even and n odd, the Hypothesis implies that $T(n) < T(n+1)$. We have therefore proven Theorem 2.

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We are now finished since this immediately implies (why?)

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