COMP170
Discrete Mathematical Tools for Computer Science

Intro to Logic

Version 1.1: Last updated, October 28, 2006

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 3.1, pp. 91-101

3.1 Equivalence and Implication

- Equivalence of Statements
- Truth Tables
- DeMorgan's Laws
- Implication
- If and Only If

Equivalence of Statements

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Code is same except for line 1

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Are they equivalent? Let's rewrite using

$s \sim$ (i+j $\leq$ p+q) $\quad t \sim$ (i$\leq$ p) $\quad u \sim$ (j$>$ q)

$v \sim$ (List[i] $\leq$ List2[j])

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logical connectives

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- Left and right parentheses (,)

logical connectives

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$\boxed{\text{(1) } w \text{ and } (u \text{ or } v)}$

$\boxed{\text{(1') } (w \text{ and } u) \text{ or } (w \text{ and } v)}$

Bigger compound statements are built out of smaller ones

(1) w and (u or v)

$\downarrow$

(1) $w \wedge (u \vee v)$

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(1') $(w \wedge u) \vee (w \wedge v)$

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Or something as complicated as

$$(s \oplus t) \wedge (\neg u \vee (s \wedge t)) \wedge \neg(s \oplus (t \vee u))$$

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Or something as complicated as

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We will always use parentheses to make our statements unambiguous. The one exception will be $\neg$, which we will often write without parentheses.

$\neg$ is always combined with the statement immediately to its right

e.g., $\neg u \vee (s \wedge t)$ is $(\neg u) \vee (s \wedge t)$ and not $\neg(u \vee (s \wedge t))$.

This is same rule used for negative numbers in algebraic expressions.

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How can we calculate whether a statement such as

$$\boxed{(1)\ w \wedge (u \vee v)}$$

is True or False or, even more, whether it is equivalent to another statement such as $\boxed{(1')\ (w \wedge u) \vee (w \wedge v)}$

3.1 Equivalence and Implication

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- DeMorgan's Laws
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- A Truth table works by first listing all of the possible combinations of values of the truth values T/F of the variables used by the compound statement
- It then evaluates the truth values of the smaller compound statements, building up to evaluating the truth values of the *topmost* compound statement

- $s \wedge t$ is True iff *both* s and t are True

AND

s	t	$s \wedge t$
T	T	T
T	F	F
F	T	F
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AND		
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OR		
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XOR

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NOT

s	$\neg s$
T	F
F	T

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Truth tables for our original programs

(1) $w \wedge (u \vee v)$

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(1') $(w \wedge u) \vee (w \wedge v)$

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The Same!

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Examples:

a) $w \wedge (u \vee v)$ and $(w \wedge u) \vee (w \wedge v)$ are equivalent.

We showed this on the previous page using truth tables

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a) $w \wedge (u \vee v)$ and $(w \wedge u) \vee (w \wedge v)$ are equivalent.

We showed this on the previous page using truth tables

b) $(w \wedge v) \vee u$ and $(w \vee v) \wedge u$ are not equivalent

Set $w = T$, $v = T$, $u = F$.

The left statement is True and the right one is False

Lemma 3.1: "Distributive Law"

The statements

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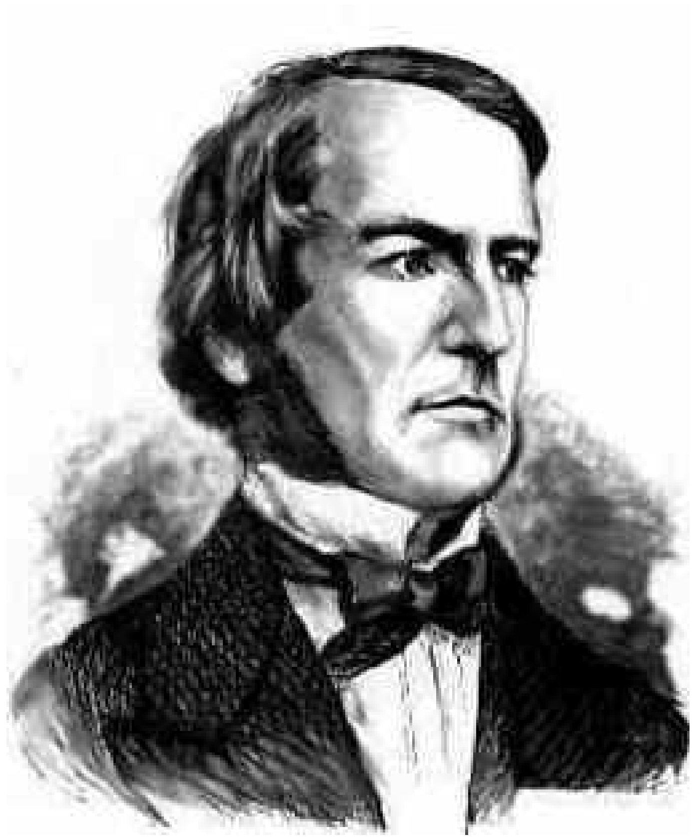
Lemma 3.X1 "Associative Laws"
$(w \wedge u) \wedge v$ is equivalent to $w \wedge (u \wedge v)$
and
$(w \vee u) \vee v$ is equivalent to $w \vee (u \vee v)$

George Boole

English Mathematician

b. 1815, d. 1864



The Inventor of Boolean Algebra

(Truth Tables are an example of B.A.)

Although Boole's work was not originally perceived as particularly interesting, even by other mathematicians, he is now seen as one of the founders of the field of Computer Science.

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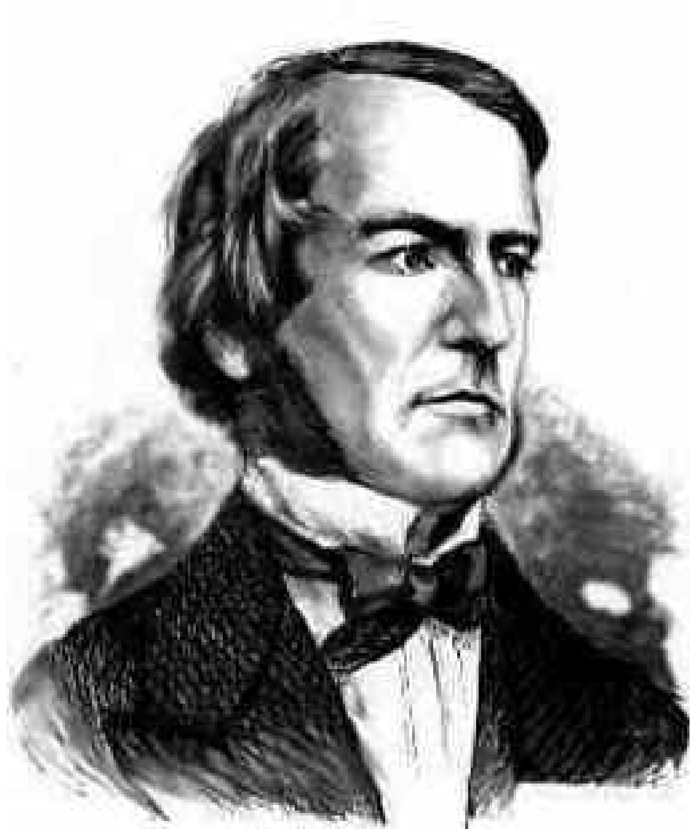

The Inventor of Boolean Algebra

(Truth Tables are an example of B.A.)

Although Boole's work was not originally perceived as particularly interesting, even by other mathematicians, he is now seen as one of the founders of the field of Computer Science.

See *http://en.wikipedia.org/wiki/George_Boole* for more details

3.1 Equivalence and Implication

- Equivalence of Statements
- Truth Tables
- DeMorgan's Laws
- Implication
- If and Only If

DeMorgan's Laws

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DeMorgan's Laws say that

(i) $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$,

and that (ii) $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.

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T	T
T	F
F	T
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T	T	T
T	F	T
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F	T	T	F
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T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
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T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
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$p \oplus q$ (the exclusive or of p and q)

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T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
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T	T	F	T	T	F	F
T	F	T	T	F	T	T
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If p is a prime, then $a^{p-1} \bmod p = 1$ for each nonzero $a \in Z_p$.

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If p is a prime, then a^{p-1} mod $p = 1$ for each nonzero $a \in Z_p$.

It combines two different statements:

- $s \sim$ (p is a prime), and
- $t \sim$ (a^{p-1} mod $p = 1$ for each nonzero $a \in Z_p$).

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Fermat's Little Theorem then becomes

$$s \Rightarrow t$$

In $s \Rightarrow t$, statement s is the **hypothesis** of the implication statement t is the **conclusion** of the implication.

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Note that English is not a very precise language. In English, the following four phrases all usually mean the same thing. In other words, they are all defined by the same truth table:

- s implies t.
- if s then t.
- t if s.
- s only if t.

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If and Only If "$\Leftrightarrow$"

If and Only If "$\Longleftrightarrow$"

s if and only if t.

If and Only If "$\Leftrightarrow$"

s if and only if t.

We parse this as

s if t and s only if t.

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We denote the statement "s if and only if t" by

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Statements of the form $s \Rightarrow t$ and $s \Leftrightarrow t$ are called **conditional statements**; the connectives $\Rightarrow$ and $\Leftrightarrow$ are called **conditional connectives**.

“Conditional” Truth Tables

IMPLIES

s	t	$s \Rightarrow t$
T	T	T
T	F	F
F	T	T
F	F	T

IF AND ONLY IF

s	t	$s \Leftrightarrow t$
T	T	T
T	F	F
F	T	F
F	F	T

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Suppose a classmate holds an ordinary playing card (with its back to you) and says,

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When is your classmate telling the truth?

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Suppose a classmate holds an ordinary playing card (with its back to you) and says,

"If this card is a heart, then it is a queen."

When is your classmate telling the truth?

- The card is a heart and a queen.
- The card is a heart and a king.
- The card is a diamond and a queen.
- The card is a diamond and a king.

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When is your classmate telling the truth?

	Truth	Lie
• The card is a heart and a queen.	$\surd$	
• The card is a heart and a king.		$\surd$
• The card is a diamond and a queen.	?	No
• The card is a diamond and a king.	?	No

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The Principle of the Excluded Middle:

A statement is true exactly when it is not false.

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• The card is a diamond and a king.	~~?~~ √	No

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