

COMP170

Discrete Mathematical Tools for Computer Science

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Basic Counting

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*Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 1.1, pp. 1-8*

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Counting

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What's the big deal?

Counting is easy, isn't it?

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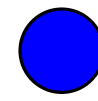
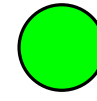
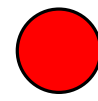
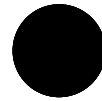
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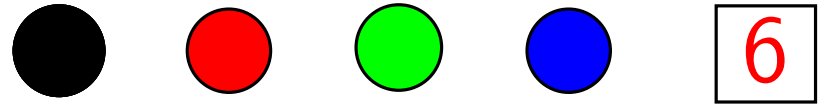


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Counting

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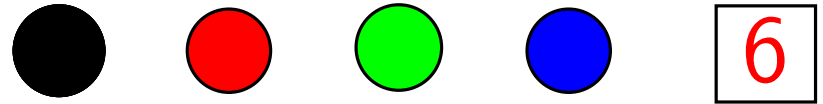
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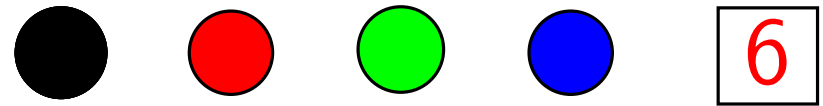


How many different ways are there to choose 2 students from a class of 4 students?

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How many different ways are there to choose 2 students from a class of 4 students?

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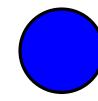
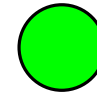
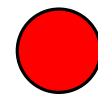
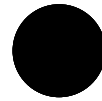
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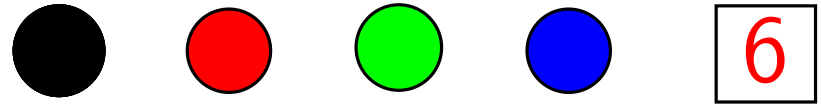


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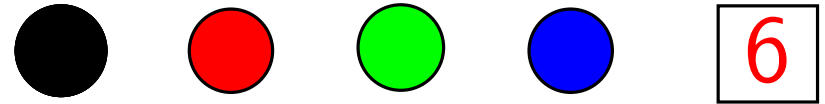
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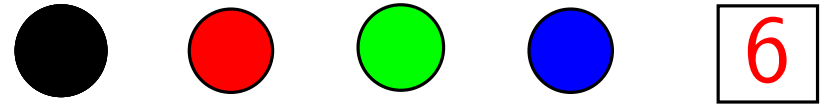
How many different ways are there to choose 2 students from a class of 5 students?

Might still be able to list all

Counting

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How many different ways are there to choose 2 students from a class of 5 students?

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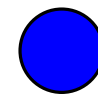
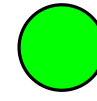
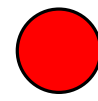
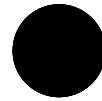
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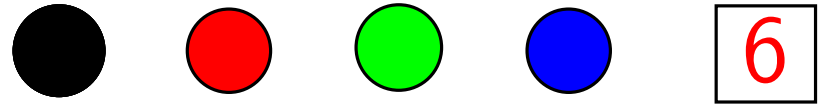


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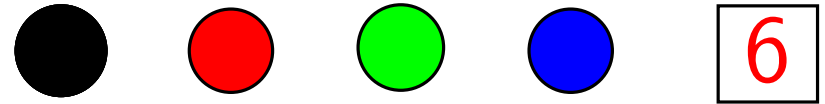
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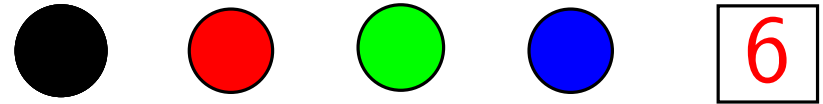
How many different ways are there to choose 2 students from a class of 100 students?

Too many to list

Counting

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How many different ways are there to choose 2 balls from



How many different ways are there to choose 2 students from a class of 100 students?

4950

Too many to list

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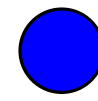
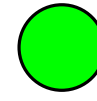
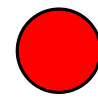
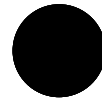
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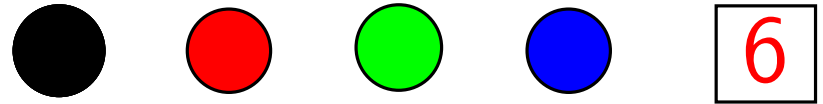


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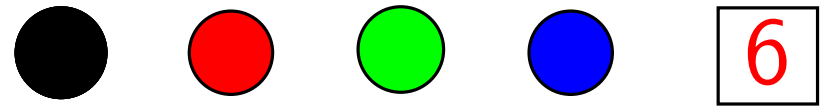
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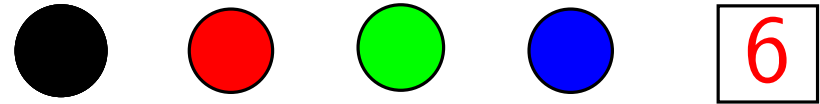
How many different ways are there to choose 6 numbers out of 1...49?

Hong Kong Mark 6!

Counting

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How many different ways are there to choose 2 balls from



How many different ways are there to choose 6 numbers out of 1...49?

13,983,816

Hong Kong Mark 6!

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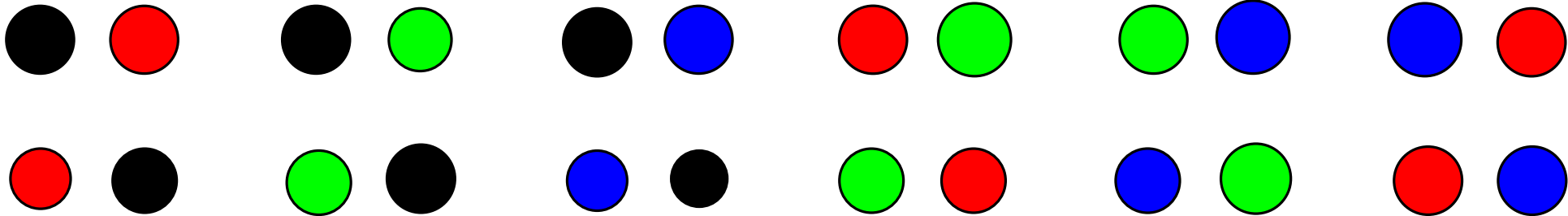
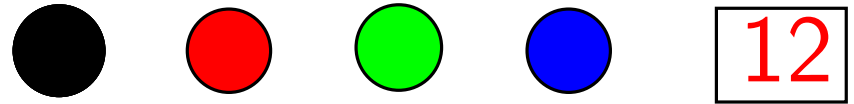
How many different ways are
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when order counts



Counting

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In Computer Science we often need to count objects.

Sometimes it's the number of steps
a computer program takes

This lets us compare runtimes
of different programs.

Sometimes, it's the number of objects
of a particular type, e.g., passwords
containing between 6-10 characters

This lets us evaluate security.

The more passwords available,
the lower the chance that
someone can guess a password

1.1 Basic Counting

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- The Sum Principle and set notation

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- Abstraction

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- The Product Principle
- Two-Element Subsets

The Sum Principle

Start with an exercise illustrating the sum principle.

Consider the following loop from **selection-sort**,
(comp171), which sorts a list of items

```
(1) for i = 1 to n-1
(2)   for j = i+1 to n
(3)     if (A[i] > A[j])
(4)       exchange A[i] and A[j]
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```

If you've never programmed before **Don't worry!**

This is *Pseudocode*; You will learn more in the tutorial

The Sum Principle

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(2)   for j = i+1 to n
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```

How many times is the comparison
A[i] > A[j]
made in line 3?

```
(1) for i = 1 to n-1
(2)     for j = i+1 to n
(3)         if (A[i] > A[j])
(4)             exchange A[i] and A[j]
```

```
(1) for  $i = 1$  to  $n-1$   
  (2)   for  $j = i+1$  to  $n$   
    (3)     if ( $A[i] > A[j]$ )  
      (4)       exchange  $A[i]$  and  $A[j]$ 
```



Lines (2–4) are executed $n - 1$ times,
once for each i between 1 and $n - 1$.

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```

Lines 2–4 are executed $n - 1$ times,
once for each i between 1 and $n - 1$.

First time, $n - 1$ comparisons.

Second time, $n - 2$ comparisons.

i th time, $n - i$ comparisons.

$(n - 1)$ st time, 1 comparison.

Thus, total number of comparisons is

$$(n - 1) + (n - 2) + \cdots + 1.$$

What did we really do?

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Took a difficult problem:
Counting *all* comparisons
made by code

What did we really do?

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```



comparisons when $i = 1$:	$n - 1$
comparisons when $i = 2$:	$n - 2$
\vdots	
comparisons when $i = t$:	$n - t$
\vdots	
comparisons when $i = n - 1$:	1

Took a difficult problem:
Counting *all* comparisons
made by code



Split into *simpler* parts
Easier to count in each part
Add parts together to get
 $1 + 2 + 3 + \dots + (n - 1)$

We showed how to *partition* a large *set* of comparisons into the *union* of smaller *mutually disjoint* sets.

We then derived our result using a general principle, the **(Sum Principle)**

The size of a union of a family of mutually disjoint finite sets is the sum of sizes of the sets.

The next few slides are devoted to defining all of the *red* terms.

Sets

Sets

Definition: *A set is a collection of objects*

Examples:

- The set of all men in this class
- The set of all women in this class
- The set of all students in this class surnamed Ng
- The set of all departments in the Engineering School
 $S = \{ \text{COMP, ECE, MechE, Civile, ChemE, IELM} \}$

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Notation: Sets are usually denoted as $S = \{a, b, c\}$

In this case, set S contains *elements* or *objects* a , b and c

Definition: Two sets are **disjoint**
if they have no elements in common.

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Examples:

- $S_1 = \{a, b, c\}$, $S_2 = \{a, e, f\}$, $S_3 = \{d, e, f\}$
- $S_4 =$ The set of all men in this class
- $S_5 =$ The set of all women in this class
- $S_6 =$ The set of all students in this class surnamed *Wong*
- $S_7 =$ The set of all students in this class surnamed *Chan*

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Which sets are disjoint?

Definition: A set of sets $\{S_1, \dots, S_m\}$
is a family of **mutually disjoint sets**,
if **every** pair of sets S_i, S_j in the family are disjoint.

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Example:

$$S_1 = \{a, b, c\}, S_2 = \{d, e, f\}, S_3 = \{g, h, i\}, \\ S_4 = \{j, k, l\}, S_5 = \{k, l, m\}$$

Which families are (**not**) mutually disjoint?

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Which families are (**not**) mutually disjoint?

- S_1, S_2, S_3, S_4 are mutually disjoint
- S_1, S_2, S_3, S_5 are mutually disjoint
- S_1, S_2, S_3, S_4, S_5 are **not** mutually disjoint

Definition: The size, $|S|$, of set S
is the number of different items in S

Example:

If $S_1 = \{a, b, c\}$, $S_2 = \{a, b, c, d\}$,

Then $|S_1| = 3$, $|S_2| = 4$,

Definition: The size, $|S|$, of set S
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Example:

If $S_1 = \{a, b, c\}$, $S_2 = \{a, b, c, d\}$,

Then $|S_1| = 3$, $|S_2| = 4$,

Note: An item can either be in or not in a set.

It can not be in a set more than once.

So, $S_3 = \{a, b, c, a\}$, denotes exactly the same set as
 $S_1 = \{a, b, c\}$, , i.e., $S_3 = S_1$ and $|S_3| = |S_1| = 3$

Definition: The **union** of sets S_1, S_2, \dots, S_m is the set S that contains exactly all of the objects that appear in at least one of the S_i

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- $S_1 \cup S_2 = \{a, b, c, d, e, f\}$
- $S_1 \cup S_3 = \{a, b, c, f, g, h\}$
- $S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e, f, g, h\}$
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Definition:

S_1, S_2, \dots, S_m are a **partition** of $S = S_1 \cup S_2 \cup \dots \cup S_m$ if S_1, S_2, \dots, S_m are a family of mutually disjoint sets

The S_i are the **blocks** of the partition

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Example (cont):

$$S = \{a, b, c, d, e, f, g, h\} = S_1 \cup S_2 \cup S_3 = S_1 \cup S_2 \cup S_4$$

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Is S_1, S_2, S_3 a partition of S ?

☐

Is S_1, S_2, S_4 a partition of S ?

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Example (cont):

$$S = \{a, b, c, d, e, f, g, h\} = S_1 \cup S_2 \cup S_3 = S_1 \cup S_2 \cup S_4$$

Is S_1, S_2, S_3 a partition of S ?

☐

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(Sum Principle)

The size of the union of a family of mutually disjoint finite sets is the sum of sizes of the individual sets.

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Equivalently

If S_1, S_2, \dots, S_m are mutually disjoint sets, then

$$|S_1 \cup S_2 \cup \dots \cup S_m| = |S_1| + |S_2| + \dots + |S_m|.$$

Example:

$$S_1 = \{a, b, c\}, \quad S_2 = \{d, e, f\}, \quad S_3 = \{g, h\},$$

$$S = S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e, f, g, h\}$$

$$|S| = 8 = 3 + 3 + 2 = |S_1| + |S_2| + |S_3|$$

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To avoid the dots, we sometimes write

$$\left| \bigcup_{i=1}^m S_i \right| = \sum_{i=1}^m |S_i|.$$

So Far

We counted
comparisons in

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(1) for i = 1 to n-1
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by noting that there are $n - t$ comps when $i = t$. Summing over t gives a total of $(n - 1) + (n - 2) + \dots + 2 + 1$ comps.

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In set notation, let S_t be the set of
all comparisons $A[i] > A[j]$ made when $i = t$

Then the S_t are mutually disjoint with $|S_t| = n - t$

From sum principle, set $S = \bigcup_{i=1}^{n-1} S_i$ of all comparisons has size

$$|S| = \sum_{i=1}^{n-1} |S_i| = (n - 1) + (n - 2) + \dots + 2 + 1$$

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The value of abstraction is that recognizing the abstract elements of a problem often helps us solve subsequent problems.

Summing Consecutive Integers

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By reading from right to left instead of left to right we observe that

$$\sum_{i=1}^{n-1} (n - i) = \sum_{i=1}^{n-1} i.$$

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← Gauss on old
German 10DM bill

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Sidenote : Gauss (1777-1855) was one of the most brilliant mathematicians in history. This particular trick was supposedly discovered by him during his first year of school (age 7).

An alternative derivation

We already saw that $\sum_{i=1}^{n-1} i = \sum_{i=1}^{n-1} (n - i)$ so

$$\begin{aligned} 2 \sum_{i=1}^{n-1} i &= \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} (n - i) \\ &= \sum_{i=1}^{n-1} [i + (n - i)] \\ &= \sum_{i=1}^{n-1} n = n(n - 1) \end{aligned}$$

The Product Principle

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(1) for i = 1 to r
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How many multiplications (in terms of r , m , n) does this pseudo-code carry out in total among all iterations of line 5?

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Therefore, this program segment makes

n multiplications m times $\Rightarrow nm$ multiplications.

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Algorithm performs a certain *set* of multiplications.

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set S_2 of multiplications performed when $j = 2$,

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and each set S_j contains exactly n multiplications

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This yields the

(Product Principle) The size of the union of m disjoint sets, each of size n , is mn .

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By *product principle*, set of *all* multiplications has size rmn .

Then, program carries out, in total,
 rmn multiplications

Two-Element Subsets

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Example:

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(1) for i = 1 to n-1
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Before, we counted total no. of comparisons by partitioning set of comps into disjoint subsets and then using the *sum principle* to derive $n(n-1)/2$. We will now see a different way of counting the same thing

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In general, for each pair of numbers i and j ,
compare $A[i]$ and $A[j]$ **exactly** once.

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In general, for each pair of numbers i and j ,
compare $A[i]$ and $A[j]$ **exactly** once.

Thus, number of comparisons is the same as
number of two-element subsets of $\{1, 2, \dots, n\}$.

In how many ways can we choose an
ordered pair from $\{1, 2, \dots, n\}$?

Example: In *ordered pair*, $(2, 5)$ is different from $(5, 2)$.

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Each pair $\{a, b\}$ of distinct elements of $\{1, 2, \dots, n\}$ can be ordered in two ways, (a, b) and (b, a) .

So, there are twice as many ordered pairs as two-element sets.

In how many ways can we choose
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Number of ordered pairs is $n(n - 1)$,
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$$\binom{n}{2} = \frac{n(n - 1)}{2} = 1 + 2 + \dots + (n - 1)$$

This is the end of Lecture 1.

In it we learnt some basic counting techniques (using set abstraction) and applied them to counting the number of comparisons in *selection-sort* and the number of ways to choose 2 items out of n .

These both turned out to be equal to

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Please see section 1.1 of the book for more details