Graphs I

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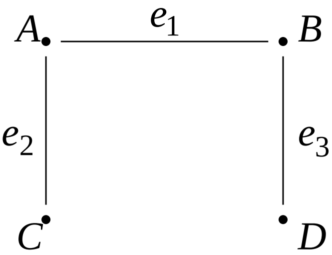
Contents

Basic Definitions (1)

Example 1

Computer network: A company has four computers. The following figure illustrates how they are connected, where A, B, C, D are the four computers and the lines e_1, e_2, e_3 indicate the connections between the computers. In graph theory A, B, C, D are called **vertices**, and e_1, e_2, e_3 are called **edges**.



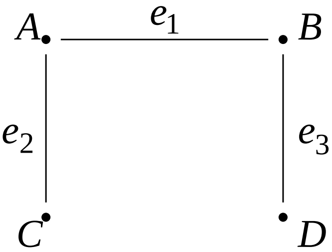
Basic Definitions (2)

Definition 2

A graph G consists of three components $(V(G), E(G), F(G))$, where

1. $V(G)$ is the set of vertices,
2. $E(G)$ the set of edges,
3. $F(G)$ the edge-endpoint function.

Each edge connects two or one vertices, which are called endpoints.



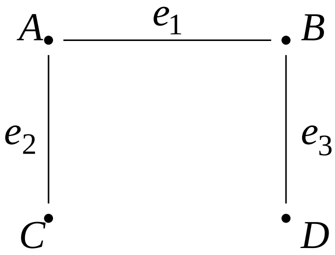
Basic Definitions (3)

Example 3

In Example 1,

1. $V(G) = \{A, B, C, D\}$
2. $E(G) = \{e_1, e_2, e_3\}$
3. $F(G)$ is given in the table:

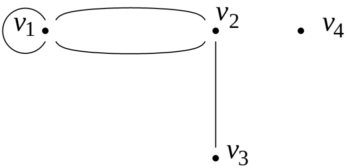
Edge	Endpoints
e_1	$\{A, B\}$
e_2	$\{A, C\}$
e_3	$\{B, D\}$



Basic Definitions (4)

Definition 4

1. An edge with just one endpoint is called a loop, eg. v_1.
2. Two vertices connected by an edge are called adjacent, eg. v_2 and v_3.
3. An edge is said to be incident on each of its endpoints.
4. Two edges incident on the same endpoint are called adjacent.
5. Two edges with the same endpoints are called parallel.
6. A vertex on which no edges are incident is called isolated.
7. A graph with no vertices is called empty.



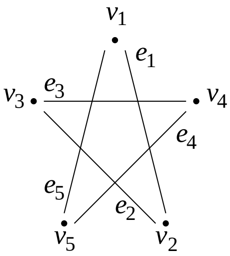
Basic Definitions (5)

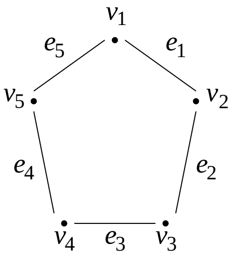
Definition 5

Two graphs are the <u>same</u> if they have the same set of vertices, the same set of edges, and the same edge-endpoint function.

Example 6

The two pictures represent the same graph.





Subgraphs

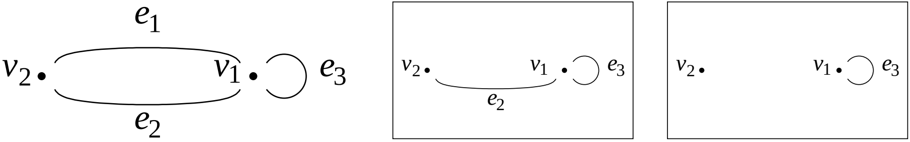
Definition 7

A graph H is said to be a subgraph of a graph G iff

1. every vertex in H is also a vertex in G;
2. every edge in H is also an edge in G; and
3. every edge in H has the same endpoints as in G.

Example 8

Let G the graph on the left-hand side. Then the two on the right-hand side are subgraphs of G.

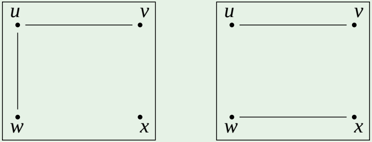


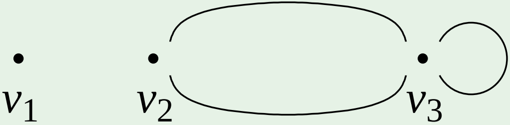
Simple Graphs

Definition 9

A simple graph is a graph without any loops and parallel edges.

Example 10





Question: Which of the three graphs are simple, and which are not simple?

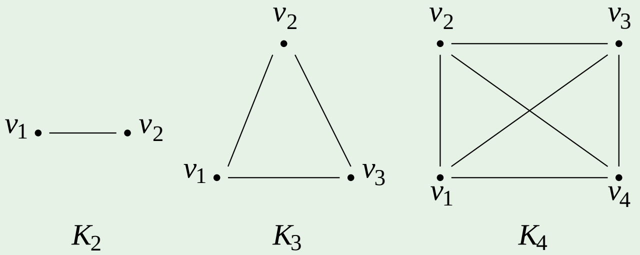
Complete Graphs

Definition 11

A complete graph on n vertices, denoted K_n, is a simple graph with n vertices $v_1, v_2, \ldots, v_n$ whose set of edges contains exactly one edge for each pair of distinct vertices.

Example 12

K_2, K_3, K_4.



Complete Bipartite Graphs

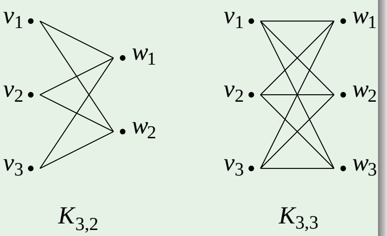
Definition 13

A **complete bipartite graph on** (m, n) **vertices**, denoted $K_{m,n}$, is a simple graph with vertices $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ that satisfies the following properties: for all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

1. there is an edge between each vertex v_i and each vertex w_j.
2. there is not an edge between any vertex v_i to any other vertex v_k.
3. there is not an edge between any vertex w_i to any other vertex w_k.

Example 14

$K_{3,2}$ and $K_{3,3}$.



Isomorphisms of Graphs

Definition 15

Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. G is isomorphic to G' iff there exist one-to-one correspondences

$$\begin{aligned} g &: V(G) \to V(G') \\ h &: E(G) \to E(G') \end{aligned}$$

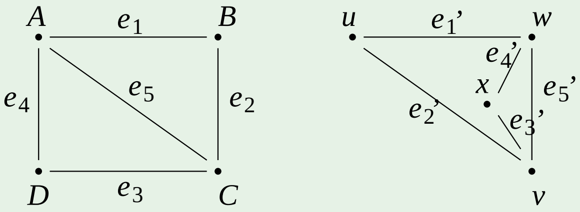
that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$,

$$v \text{ is an endpoint of } e \text{ implies that } g(v) \text{ is an endpoint of } h(e).$$

Isomorphisms of Graphs: an Example

Example 16

Consider the following two graphs:



We define two one-to-one correspondences:

$$\begin{aligned} g &: A \mapsto w,\ B \mapsto u,\ C \mapsto v,\ D \mapsto x \\ h &: e_i \mapsto e_i' \end{aligned}$$

It is easily seen that v is an endpoint of e implies that $g(v)$ is an endpoint of $h(e)$. Hence the above two graphs are isomorphic.

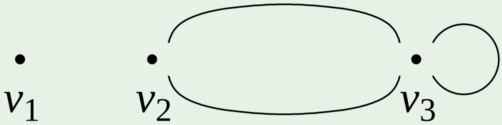
The Degree of Graphs (1)

Definition 17

Let G be a graph and v a vertex of G.

1. The degree of v, denoted by $\deg(v)$, is the number of edges that are incident on v, with a loop being counted twice.
2. The total degree of G, denoted $\deg(G)$, is the sum of the degrees of all the vertices of G.

Example 18



$\deg(v_1) = 0, \quad \deg(v_2) = 2, \quad \deg(v_3) = 4, \quad \deg(G) = 0 + 2 + 4 = 6.$

The Degree of Graphs (2)

Proposition 19

Let G be a graph. Then $\deg(G) = 2 \cdot$ *(number of edges of G).*

Proof.

Recall that $\deg(G) = \sum_i \deg(v_i)$.

Every edge contributes 2 to the total degree.

Case 1: Let e be a loop with endpoint v. Then e contributes 2 to the degree of v.

Case 2: Let e be an edge with two distinct endpoints $\{v_1, v_2\}$. Then e contributes 1 to $\deg(v_1)$ and $\deg(v_2)$.

This proves that every edge contributes exactly 2 to the total degree. The desired conclusion then follows. □

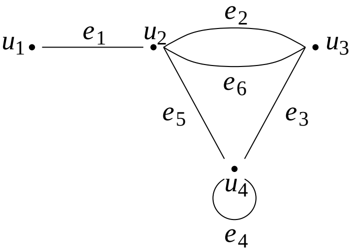
Paths and Circuits (1)

Definition 20

Let G be a graph and let v and w be vertices in G.
A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G:

$$v \; e_1 \; v_1 \; e_2 \; \ldots \; v_{n-1} \; e_n \; w.$$

There is a **trivial walk** from every vertex v to itself, i.e., v. So intuitively a trivial walk on v may be understood as standing at v without moving.
Example: $u_2 \; e_2 \; u_3 \; e_3 \; u_4$ in the graph.



Paths and Circuits (2)

Definition 21

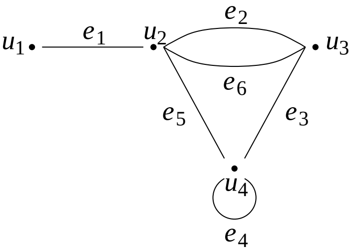
Let G be a graph and let v and w be vertices in G.

A path from v to w is a walk from v to w that does not contain a repeated edge:

$$v\ e_1\ v_1\ e_2\ \ldots\ v_{n-1}\ e_n\ w$$

where $e_i \neq e_j$ for any $i \neq j$.

Example: $u_1\ e_1\ u_2\ e_2\ u_3$ in the graph.



Paths and Circuits (3)

Definition 22

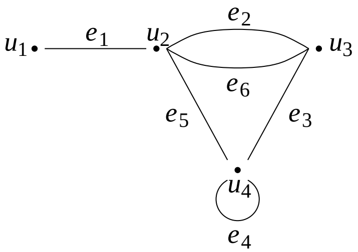
Let G be a graph and let v and w be vertices in G.

A simple path from v to w is a path that does not contain a repeated vertex:

$$v \ e_1 \ v_1 \ e_2 \ \dots \ v_{n-1} \ e_n \ w$$

where $v, v_1, \dots, v_{n-1}$ are pairwise distinct and $e_1, e_2, \dots, e_n$ are also pairwise distinct.

Example: $u_1 \ e_1 \ u_2 \ e_2 \ u_3 \ e_4 \ u_4$ in the graph.



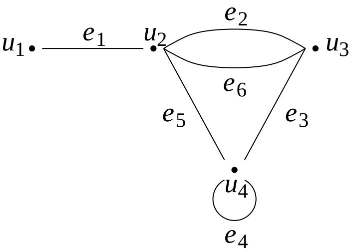
Paths and Circuits (4)

Definition 23

Let G be a graph and let v and w be vertices in G.

A closed walk is a walk that starts and ends at the same vertex.

Example: $u_1\ e_1\ u_2\ e_2\ u_3\ e_6\ u_2\ e_1\ u_1$ in the graph.



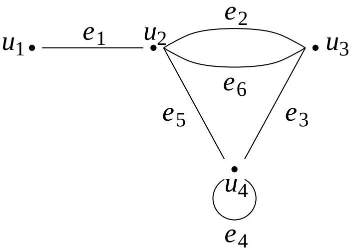
Paths and Circuits (5)

Definition 24

Let G be a graph and let v and w be vertices in G.

A circuit is a closed walk without any repeated edges.

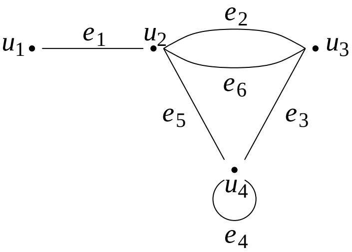
Example: $u_2\ e_2\ u_3\ e_6\ u_2$ in the graph.



Paths and Circuits (6)

Definition 25

Let G be a graph and let v and w be vertices in G.
A simple circuit is a circuit that does not have a repeated vertex except the first and last.
Example: $u_2\ e_2\ u_3\ e_6\ u_2$ in the graph.



A Comparison

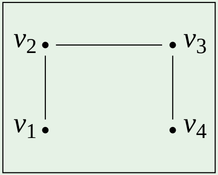
	Repeated Edge?	Repeated Vertex?	Starts and Ends at the same point
Walk	Allowed	Allowed	Allowed
Path	No	Allowed	Allowed
Simple Path	No	No	No
Closed Walk	Allowed	Allowed	Yes
Circuit	No	Allowed	Yes
Simple Circuit	No	First and Last only	Yes

The Connectedness of Graphs

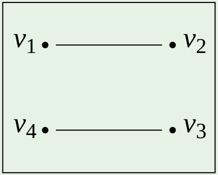
Definition 26

1. Let G be a graph. Two vertices v and w are connected iff there is a walk from v to w.
2. The graph G is connected iff given any two distinct vertices v and w, there is a walk from v to w.

Example 27



Connected



Not Connected

Connected Components

Definition 28

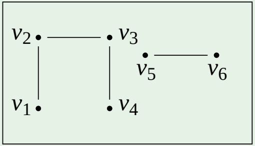
A graph H is a connected component of a graph G iff

1. H is a subgraph G;
2. H is connected;
3. No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

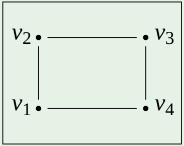
In other words, a connected component is a largest connected subgraph.

Example 29

G has two connected components. G' has only one, i.e. itself.



G



G'

Connected Components and Equivalence Relations

Definition 30

Given a symmetric relation R on a set A, its associated graph G'_R has vertex set A and

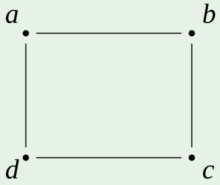
$$\text{For any } u, v \in A, \ \{u, v\} \in E(G'_R) \text{ iff } uRv.$$

Example 31

Let

$$A = \{a, b, c, d\}$$

and $R = \{(a,b),(b,c),(c,d),(d,a),(b,a),(c,b),(d,c),(a,d)\}$. Then G'_R is



Connected Components and Equivalence Relations

Proposition 32

Let R be an equivalence relation on A. Since it is symmetric, we consider its associated graph G'_R. Then the set of vertices of the connected components of G'_R are precisely the equivalence classes of R.

Proof.

Left as an exercise. □

Connected Components and Equivalence Relations

Proposition 33

Given an undirected graph G, we define a relation R'_G on $V(G)$ as follows:

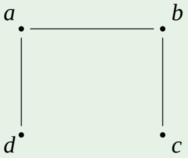
$$\textit{for any } u, v \in V(G),\ \ uR'_G v \textit{ iff } \{u, v\} \in E(G).$$

Then R'_G is a symmetric relation.

Proof: If $u\ R'_G\ v$, then $\{u, v\} \in E(G)$. Hence $\{v, u\}$ is also an edge in $E(G)$. Therefore $v\ R'_G\ u$. This proves symmetry. □

Example 34

Let graph G be as follows. Then
$R'_G = \{(a, b), (b, a), (a, d), (d, a), (b, c), (c, b)\}$.



Connected Components and Equivalence Relations

Proposition 35

Suppose G is a graph. Define a relation R on the set $V(G)$ as follows: "For any $u, v \in V(G)$, u R v iff u and v are connected in G". Then

1. *R is an equivalence relation*
2. *The equivalence classes of R are precisely the set of vertices of the connected components of G.*

Proof.

Left as an exercise. $\square$