Dynamic Programming: The Rod Cutting Problem

Version of October 26, 2016

THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Introduction

- Dynamic Programming (DP) bears similarities to Divide and Conquer (D&C)
  - Both partition a problem into smaller subproblems and build solution of larger problems from solutions of smaller problems.
  - In D&C, work top-down. Know exact smaller problems that need to be solved to solve larger problem.
  - In D, (usually) work bottom-up. Solve *all* smaller size problems and build larger problem solutions from them.
    - In DP, many large subproblems reuse solution to same smaller problem.
  - DP often used for optimization problems
  - Problems have many solutions; we want the *best* one
- Main idea of DP
  1. Analyze the structure of an optimal solution
  2. Recursively define the value of an optimal solution
  3. Compute the value of an optimal solution (usually bottom-up)

Rod Cutting

- Input: We are given a rod of length n and a table of prices p_i for $i = 1, \ldots, n$; p_i is the price of a rod of length i.
- Goal: to determine the maximum revenue r_n, obtainable by cutting up the rod and selling the pieces
- Example: $n = 4$ and $p_1 = 1, p_2 = 5, p_3 = 8, p_4 = 9$
  - If we do not cut the rod, we can earn $p_4 = 9$
  - If we cut it into 4 pieces of length 1, we earn $4 \cdot p_1 = 4$
  - If we cut it into 2 pieces of length 1 & a piece of length 2, we earn $2 \cdot p_1 + p_2 = 9$
  - If we cut it into 2 pieces of length 2, we can earn $2 \cdot p_2 = \mathbf{10}$
  - There are more options, but the maximum revenue is **10**
- In general, rod of length n can be cut in 2^{n-1} different ways, since we can choose cutting, or not cutting, at all distances i $(1 \leq i \leq n-1)$ from the left end

Optimal Solution

- We can calculate the maximum revenue r_n in terms of optimal revenues for shorter rods

$$r_n = \max(p_n,\ r_1 + r_{n-1},\ r_2 + r_{n-2},\ \ldots,\ r_{n-1} + r_1)$$

  - p_n if we do not cut at all
  - $r_1 + r_{n-1}$ if we take the sum of optimal revenues for 1 and $n-1$
  - $r_2 + r_{n-2}$ if we take the sum of optimal revenues for 2 and $n-2$
  - ...

- Another approach. Set $r_0 = 0$ and

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

  - Cut a piece of length i, with remainder of length $n-i$
  - Only the remainder, and not the first piece, may be further divided

Recursive Top-down Implementation

Cut-Rod(p, n)

```
if n = 0 then
    return 0;
end
q = -∞;
for i = 1 to n do
    q = max(q, p[i] + Cut-Rod(p, n - i));
end
return q;
```

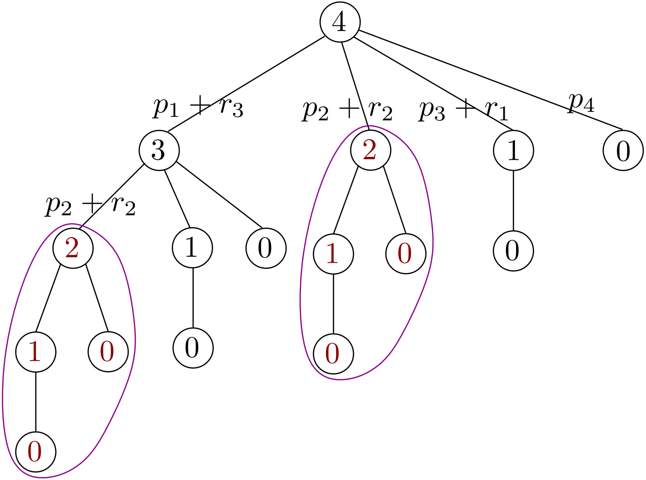
Algorithm Time

- $T(n)$: the total number of calls made to Cut-Rod when called with rod length n

$$T(n) = \begin{cases} 1 + \sum_{0 \le j \le n-1} T(j), & \text{if } n > 0, \\ 1, & \text{if } n = 0. \end{cases}$$

- Induction $\Rightarrow T(n) = 2^n$

- Algorithm calls same subproblem many times



Concept of DP

- After solving a *subproblem*, store the solution
  - Next time you encounter same subproblem, lookup the solution, instead of solving it again
  - Uses space to save time
- Two main methodologies: top-down and bottom-up
  - Corresponding algorithms have the same asymptotic cost, but bottom-up is usually faster in practice
- Main idea of bottom-up DP
  - Don't wait until until subproblem is encountered.
  - Sort the subproblems by size; solve smallest subproblems first
  - Combine solutions of small subproblems to solve larger ones

- p_i are the problem inputs.
- r_i is max profit from cutting rod of length i.
- Goal is to calculate r_n
- r_i defined by
  - $r_1 = 1$ and $r_n = \max_{1 \le i \le n}(p_i + r_{n-i})$
- Iteratively fill in r_i table by calculating $r_1, r_2, r_3, \ldots$
- r_n **is final solution**

i	1	2	3	4	... ...	n
$\mathbf{r_i}$	p_1				... ...	

DP Bottom-up Implementation

Bottom-Up-Cut-Rod(p, n)

```
r[0] = 0; // Array r[0...n] stores the computed optimal values
for j = 1 to n do
    // Consider problems in increasing order of size
    q = -∞;
    for i = 1 to j do
        // To solve a problem of size j, we need to consider all
           decompositions into i and j - i
        q = max(q, p[i] + r[j - i]);
    end
    r[j] = q;
end
return r[n];
```

- Cost: $O(n^2)$
  - The outer loop computes $r[1], r[2], \ldots, r[n]$ in this order
  - To compute $r[j]$, the inner loop uses all values $r[0], r[1], \ldots, r[j-1]$ (i.e., $r[j-i]$ for $1 \le i \le j$)

Outputting the Cutting

- Algorithm only *computes* r_i. It does not output the cutting.
- Easy fix
  - When calculating $r_j = \max_{1 \le i \le j}(p_i + r_{j-i})$
    store value of i that achieved this max in new array $s[j]$.
  - This j is the size of last piece in the optimal cutting.
- After algorithm is finished, can reconstruct optimal cutting by unrolling the s_j.

Extended-Bottom-Up-Cut-Rod(p, n)

```
// Array s[0...n] stores the optimal size of the first piece to
   cut off
r[0] = 0; // Array r[0...n] stores the computed optimal values
for j = 1 to n do
    q = -∞;
    for i = 1 to j do
        // Solve problem of size j
        if q < p[i] + r[j - i] then
            q = p[i] + r[j - i];
            s[j] = i; // Store the size of the first piece
        end
    end
    r[j] = q;
end
while n > 0 do
    // Print sizes of pieces
    Print s[n];
    n = n - s[n];
end
```