COMP170
Discrete Mathematical Tools for Computer Science

More Counting

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 1.2, pp. 9-19

1.2 Counting Lists, Permutations, and Subsets

- Using the Sum and Product Principles
- Lists and Functions
- The Bijection Principle
- k-Element Permutations of a Set
- k-Element Subsets of a Set
  Binomial Coefficients

Some Simple Examples

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Note; This is really the *product principle*
What are the sets being used?

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$$52 \times 52 \times 52 \times 52 \times 52 \times 52 = 52^6$$

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Each L_i has 26 possibilities and
each D_i has 10 possibilities
so the total number of car plates is

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4 = 6,760,000$$

We have just seen examples of

Product Principle, Version 2

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S, and

2. for each $j > 1$
   and each choice of the first $j-1$ elements of a list in S,
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$$\boxed{L_1} \quad \boxed{L_2} \quad \boxed{L_3} \quad \cdots \quad \boxed{L_m}$$

$$i_1 \times i_2 \times i_3 \times \cdots \times i_m$$

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So *sum principle* might help us.

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$\Rightarrow$ total number of passwords is

$$52^4 + 52^5 + 52^6 + 52^7 + 52^8.$$

Lists and Functions

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Example: $MORE$ and $ROME$ are
two different 4-letter lists from $T = \{A, B, \ldots, Z\}$

$\{M, O, R, E\}$ and $\{R, O, M, E\}$ are
the same subset of T.

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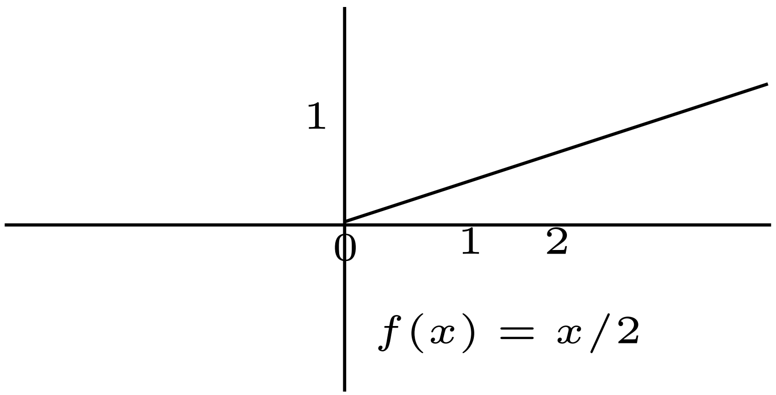

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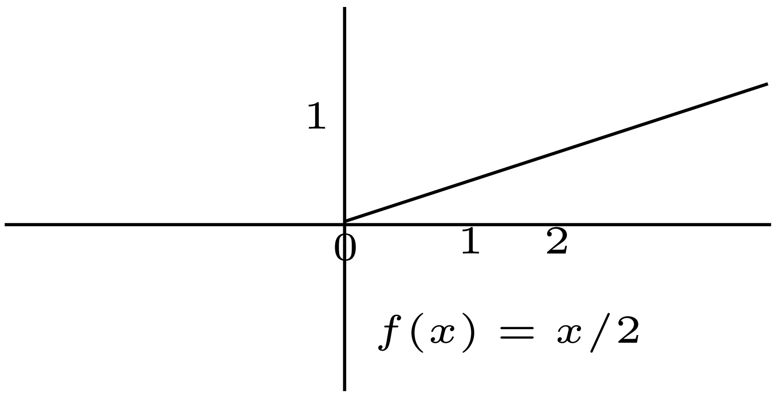

In continuous math, e.g., basic calculus and algebra, you often have formulas, e.g., $f(x) = x/2$ that describe the function. In discrete math, where we often deal with finite sets, we can frequently state the full function explicitly: e.g.,

$$f(1) = \text{Sam}, \ f(2) = \text{Mary}, \ f(3) = \text{Sarah}$$

Lists and Functions (cont)

We often write a function from S to T as

$$f : S \rightarrow T$$

S is the domain of f; T is the range of f.

Lists and Functions (cont)

A k-element List, $L = L_1 L_2 \ldots L_k$ from set T can now be defined as a function

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Example: 4-element lists from $\{A, \ldots, Z\}$

$MORE$ is the function

$$f(1) = M,\ f(2) = O,\ f(3) = R,\ f(4) = E$$

$ROME$ is the function

$$f(1) = R,\ f(2) = O,\ f(3) = M,\ f(4) = E$$

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Set of *all* functions from $\{1,2\}$ to $\{a,b\}$ is just the set of 2-element lists from $\{a,b\}$ which we already saw, by the product principle, has size $2 \times 2 = 4$.

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In French
sur $=$ "on"

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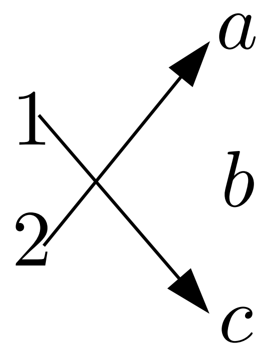

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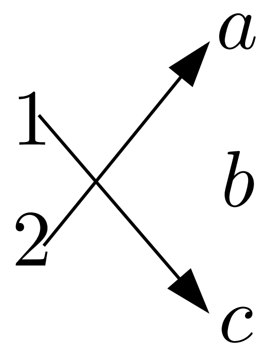

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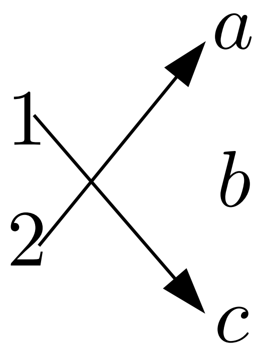

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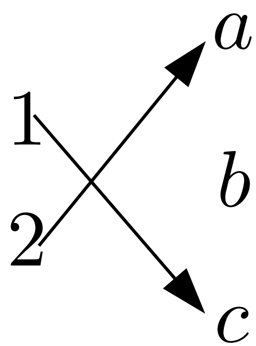

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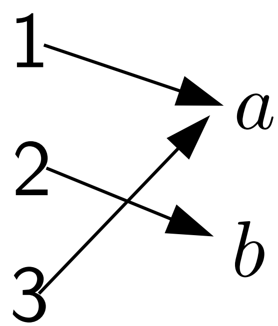

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A bijection from a set onto itself is called a **permutation**

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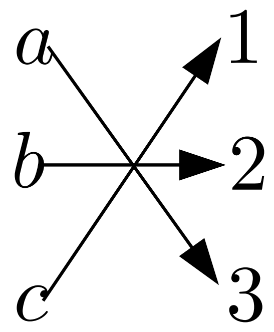

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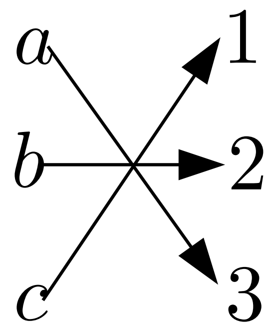

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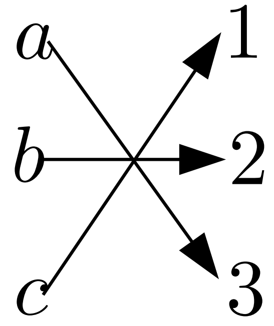

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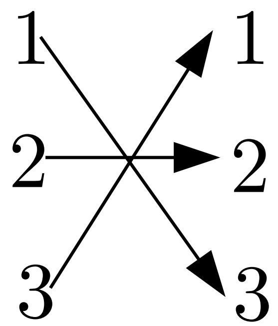

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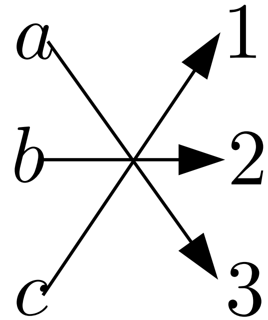
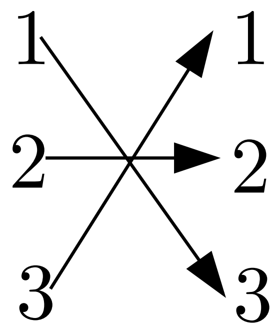


In a bijection
exactly one arrow
leaves each item on the left
and
exactly one arrow
arrives at each item on the right
so the left and right sides
must have the same size



The Bijection Principle

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The following loop is a part of program to determine the number of triangles formed by n points in the plane.

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The Bijection Principle

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(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

Among all iterations of line 5 in the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)

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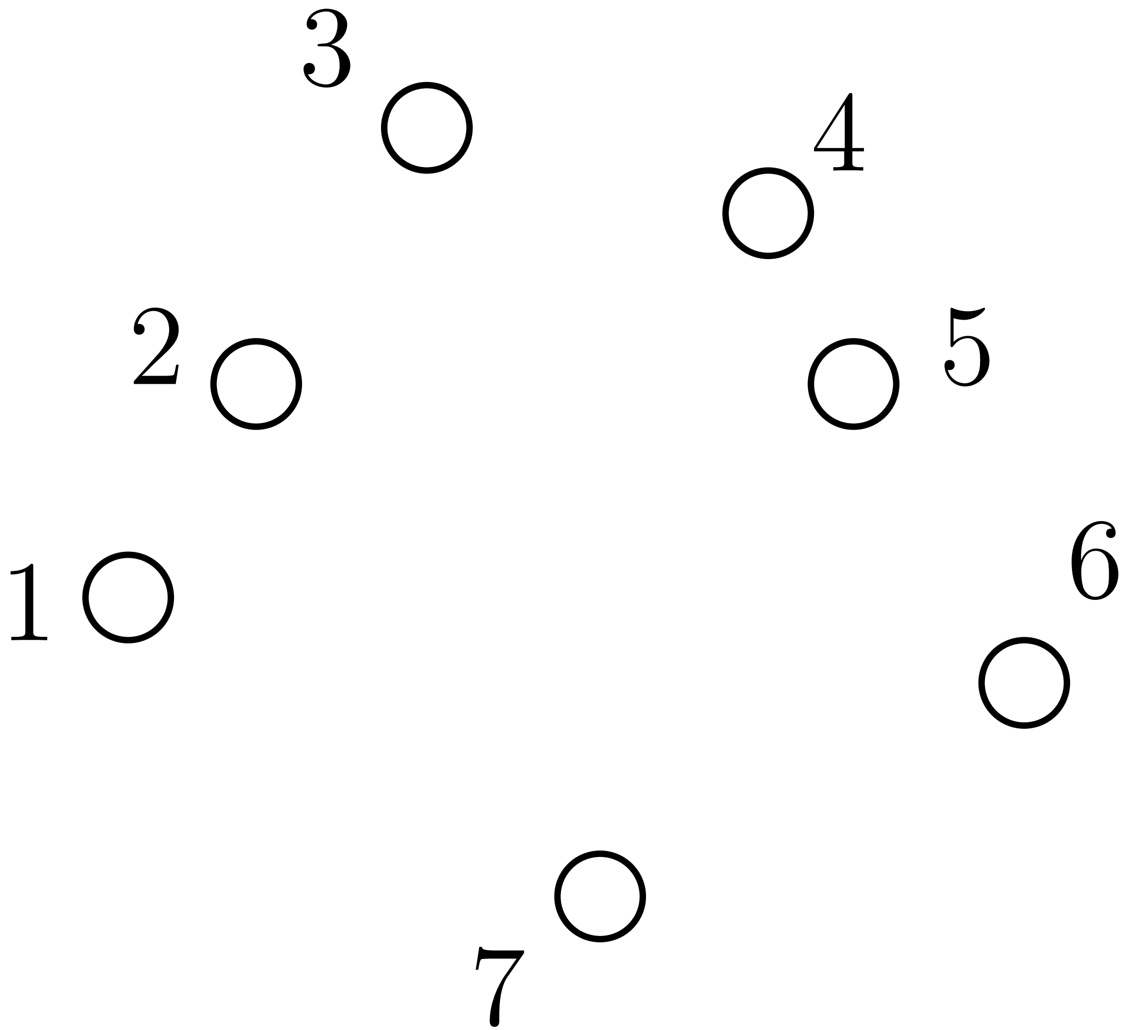

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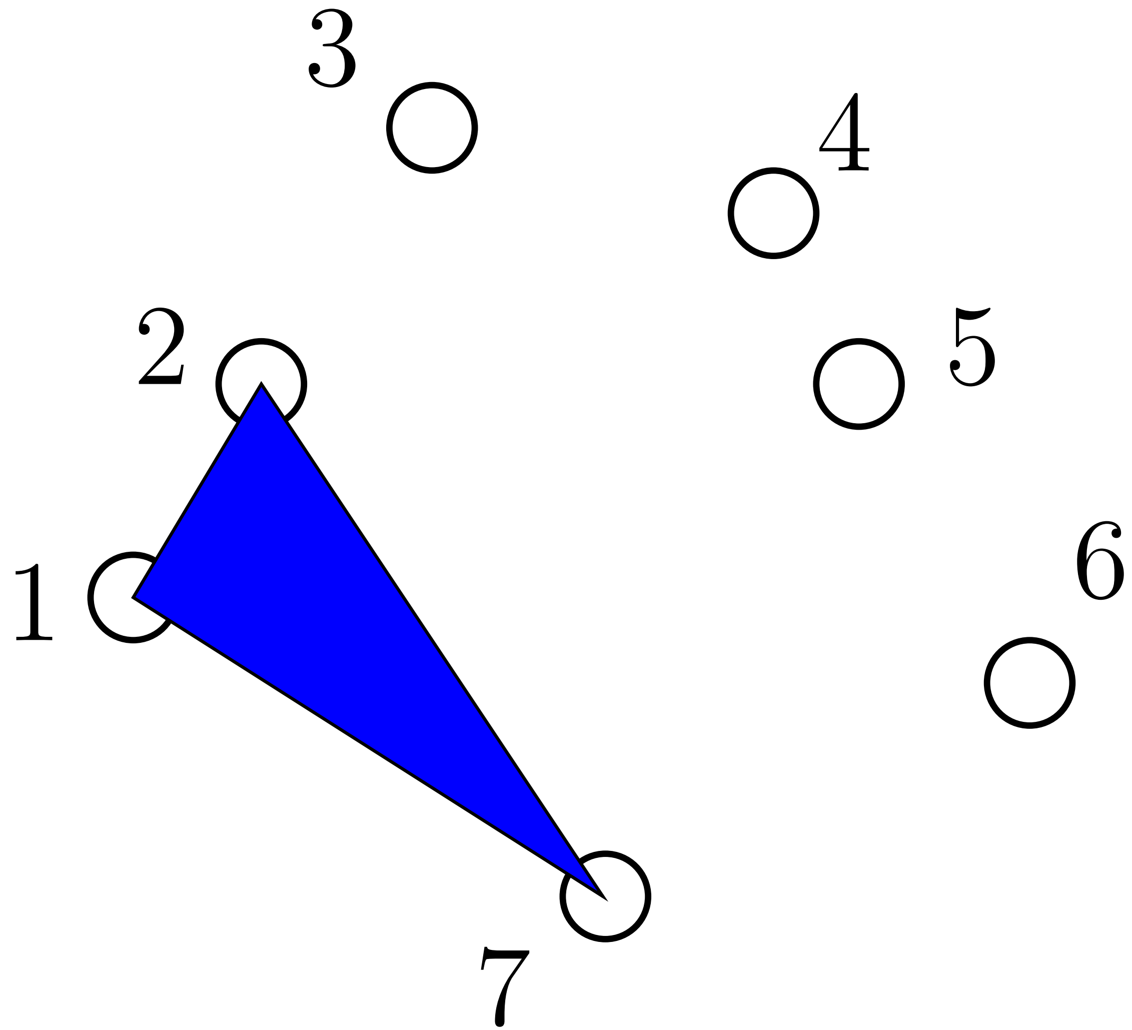

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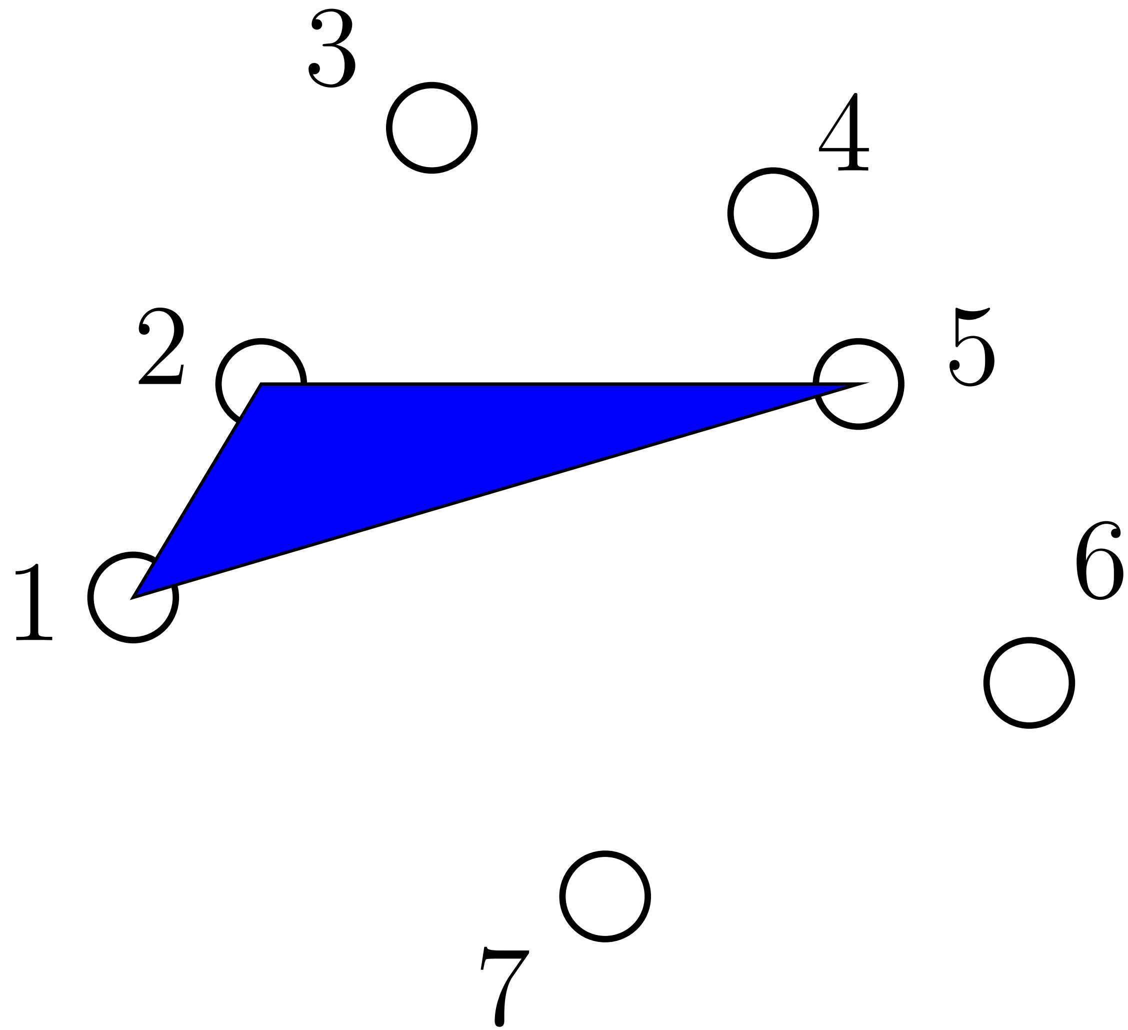
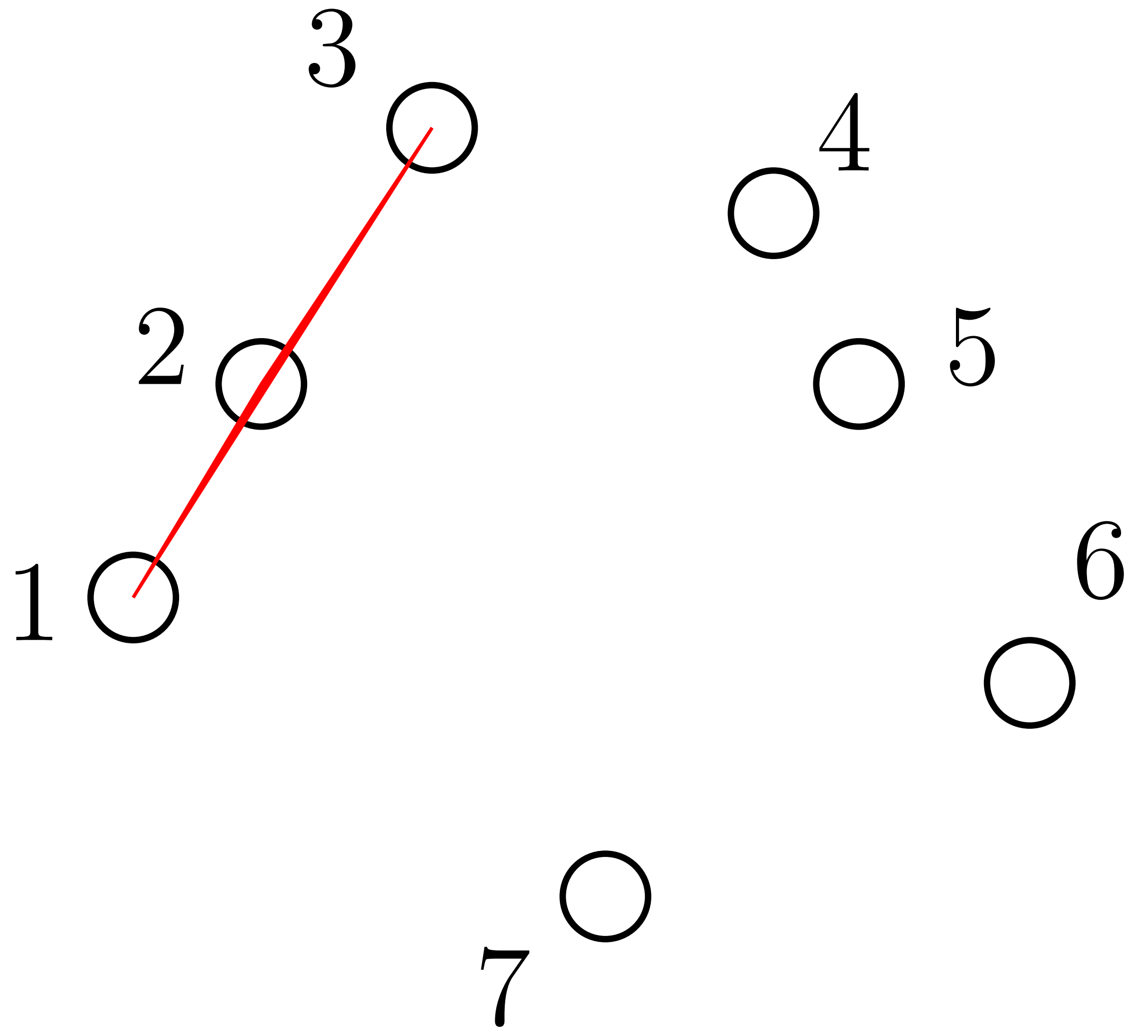


$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

$1 - 2 - 3$: no

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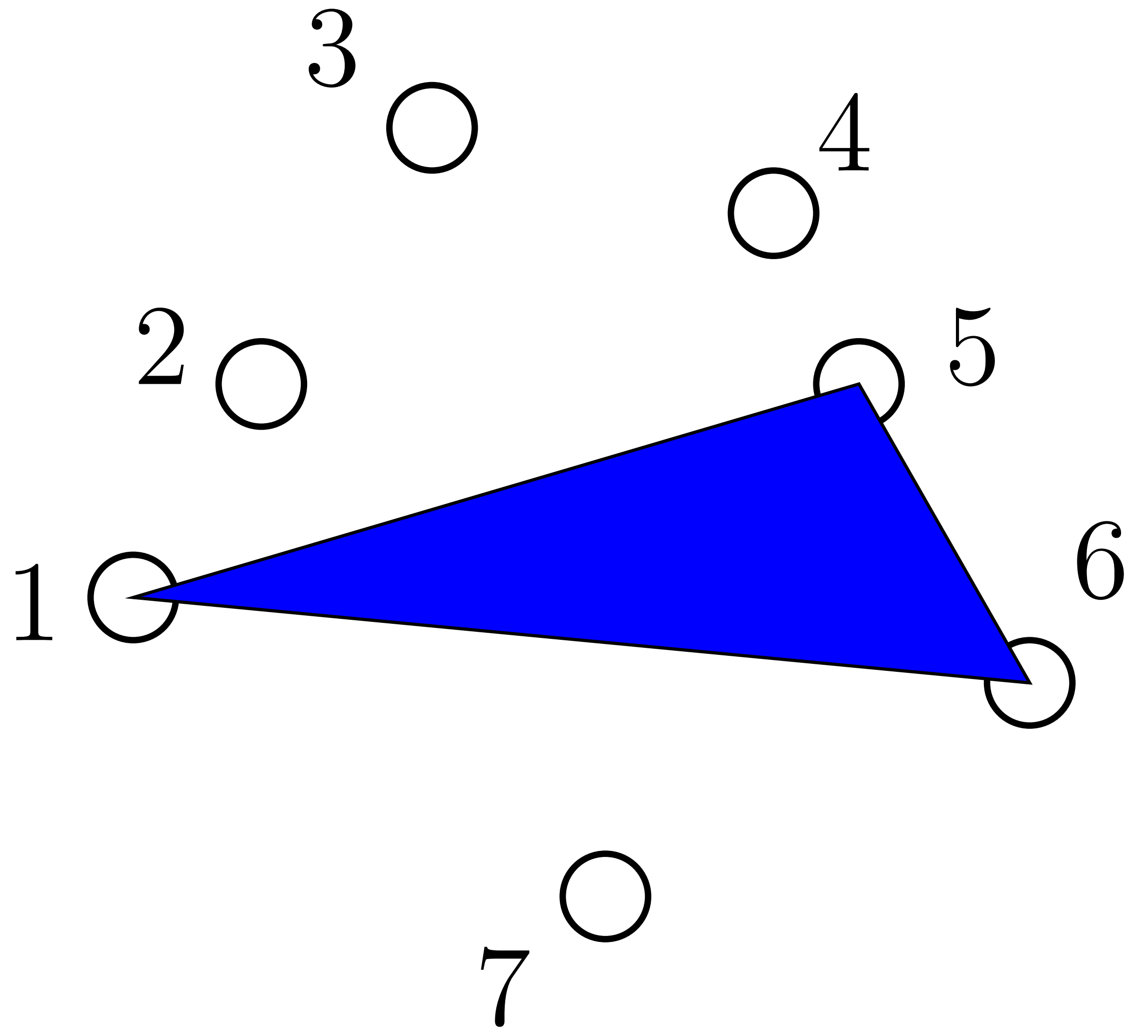

$1 - 2 - 7$: yes

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$1 - 2 - 3$: no

$1 - 5 - 6$: yes

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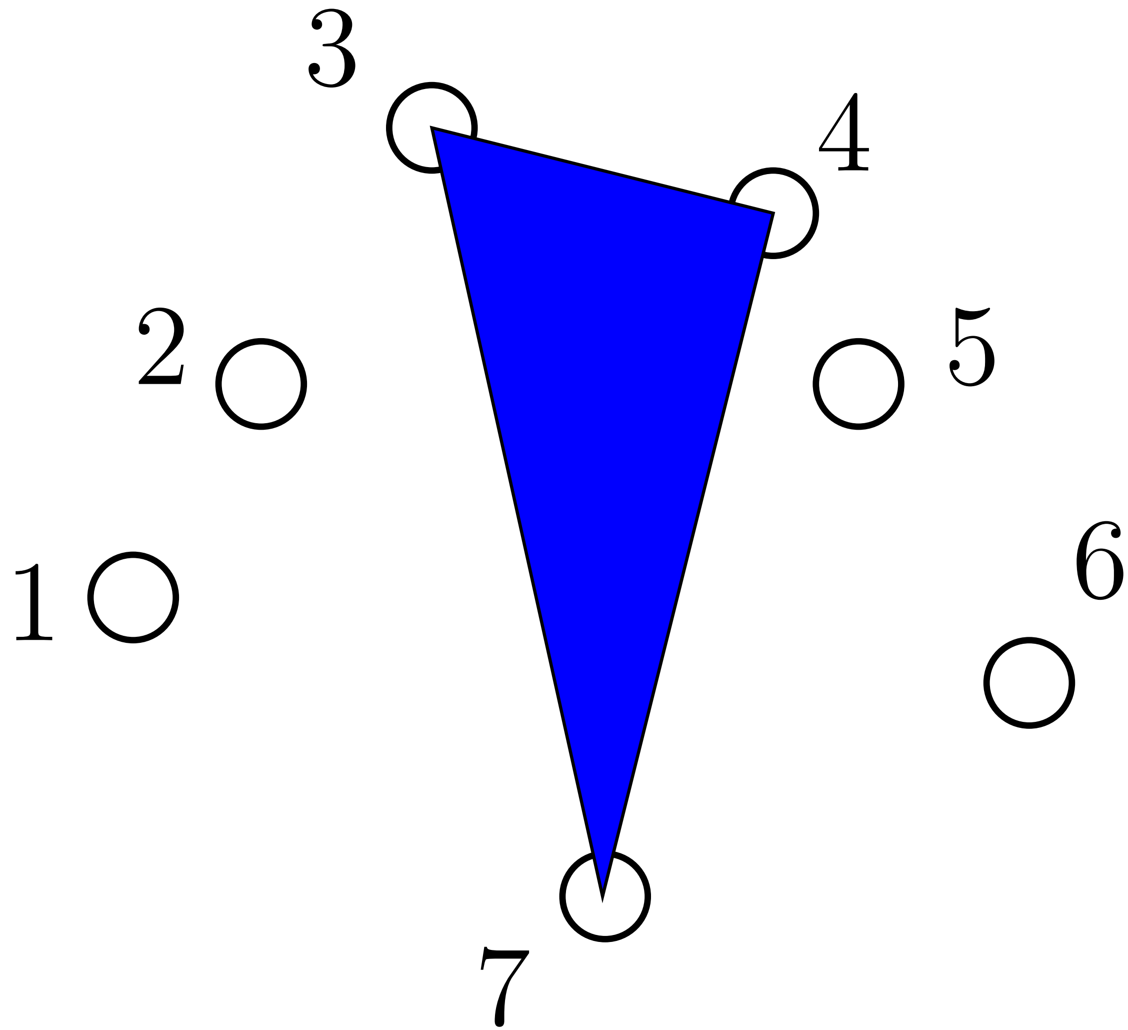

$1-2-7$: yes

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$1-2-3$: no

$1-5-6$: yes

$3-4-7$: yes

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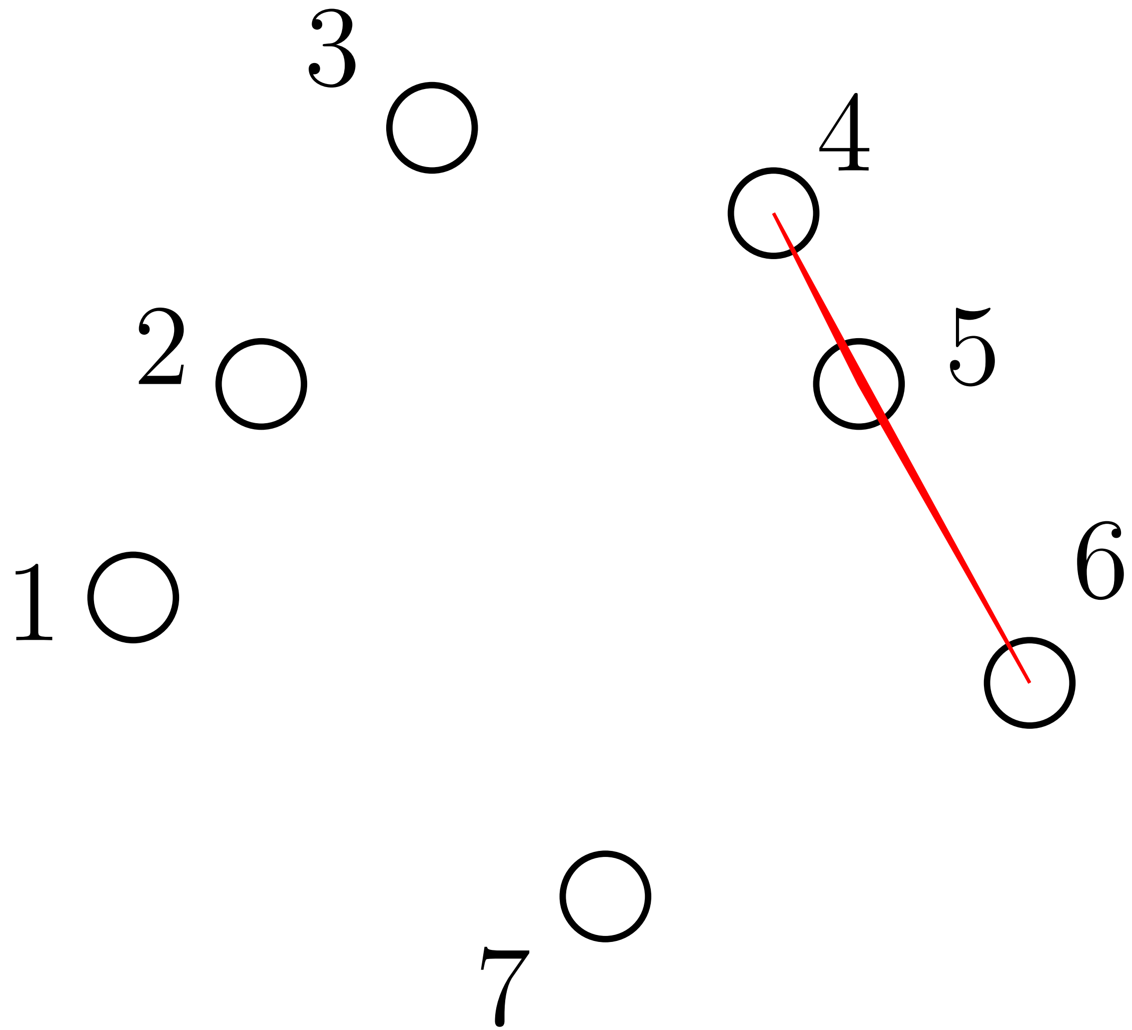

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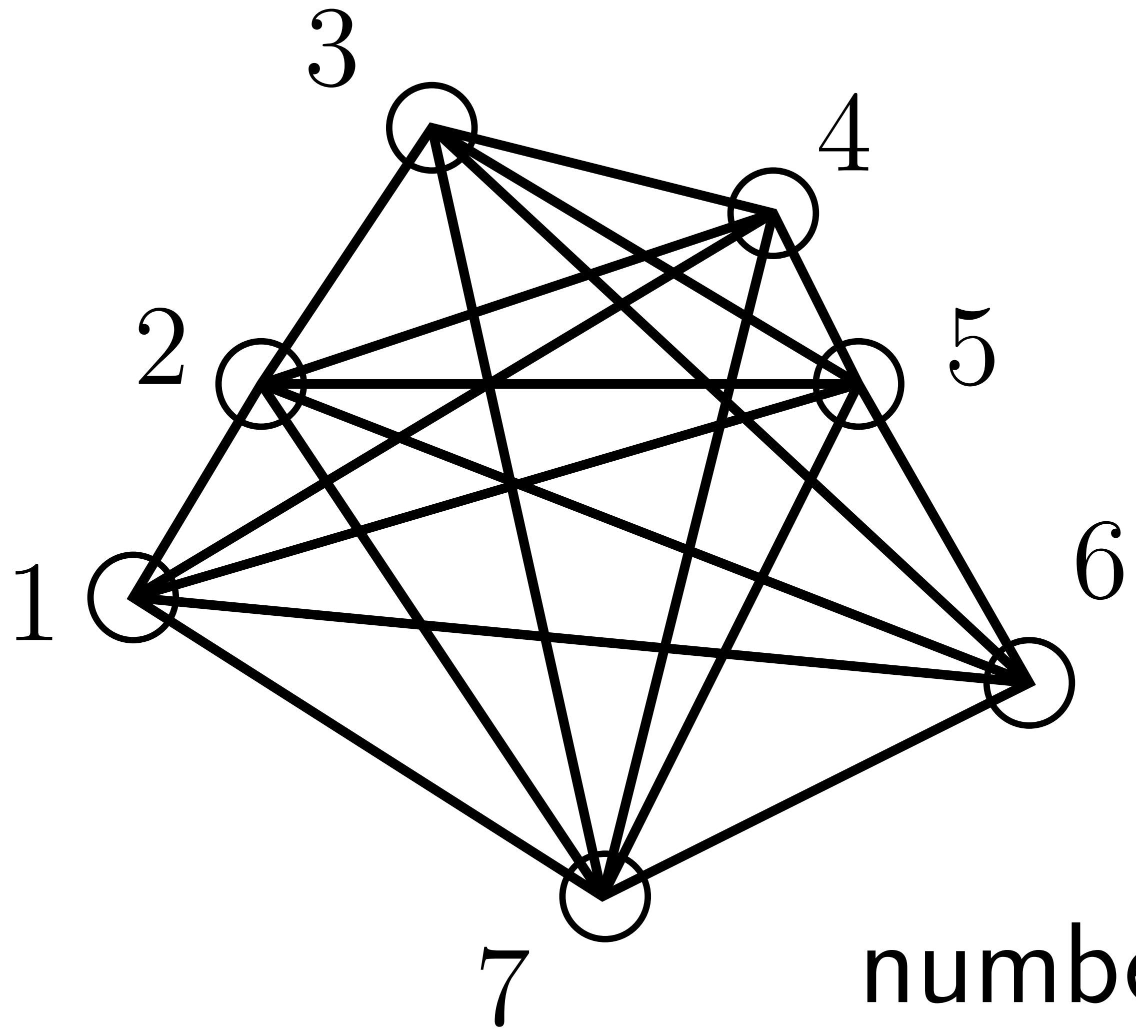

$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

$1 - 2 - 3$: no

$1 - 5 - 6$: yes

$3 - 4 - 7$: yes

$4 - 5 - 6$: no

number of triangles: 33

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A loop embedded in a loop

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A loop embedded in a loop embedded in another loop.

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Third loop begins with $k = j + 1$ and k increases up to n.

Thus each triple i, j, k with $i < j < k$ is examined exactly once.

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Thus, compute number of increasing triples!

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  f is a bijection because
  f is one-to-one
  if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$
  f is onto
  if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
  where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

- We've already seen something very similar.
  The number of

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- We actually already saw that $|X| = |Y| = \binom{n}{2}$

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- Currently, we started with the problem of counting the $\#$ of increasing triples and changed it to the problem of counting the $\#$ of 3-element sets from $\{1, 2, \ldots, n\}$

- We will now see how to count the $\#$ of k-element permutations of $\{1, 2, \ldots, n\}$. From this we will derive how to count $\#$ of k-element sets.

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- Note that the case of $k = n$ is special;
  An n-element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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  - To start, how many 3-element permutations of $\{1, 2, \ldots, n\}$ are there?

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3 -element permutations of $\{1, 2, 3, 4\}$

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243$$
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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a **lexicographic ordering** and is used quite often.

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$$\Rightarrow \boxed{\frac{n(n-1)(n-2)}{6} = \binom{n}{3}}$$

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Thus, by version 2 of product principle, there are $n(n-1)\cdots(n-k+1)$ ways to choose a k-element permutation.

We just saw that there are

$$n(n-1)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

ways to choose a k-element permutation.

In the special case of $k = n$ (a permutation) this reduces to

$$n(n-1)\cdots 3\cdot 2\cdot 1 = n!$$

So there are $n!$ different permutations of a set of size n.

Handy notation suggested by Donald E. Knuth, is $n^{\underline{k}}$, the k**th falling factorial power of** n, defined as

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Theorem 1.1

The #of k-element permutations of an n-element set is

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Donald Ervin Knuth (born 1938) is *Professor Emeritus of the Art of Computer Programming* at Stanford University. He is most famous for his 3+-volume set, *The Art of Computer Programming* but he's done many other things, including designing the system, TeX, with which these notes were typeset.

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- Let $\binom{n}{k}$ denote the # of k-element subsets
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- However, the number of k-element permutations of a k-element set is $k!$ (Theorem 1.1 with $n = k$).

- Thus, by sum principle,

$$n^{\underline{k}} = (\#\text{ of blocks}) \times (\#\text{ in each block}) = \binom{n}{k} k!.$$

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For integers n and k with $0 \leq k \leq n$, the number of k-element subsets of an n-element set is

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

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This will work if we define $0! = 1$.

Note: Both cases $k = 0$ and $k = n$ use fact that $0! = 1$.

Binomial Coefficients

The number of ways of choosing a
k-element subset from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This term is called a binomial coefficient.
Be aware that there are other, alternative, notations for the same thing, occasionally used, e.g., $C(n, k)$ or nCk