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Competitive analysis of incentive compatible on-line auctions☆

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Abstract

This paper studies auctions in a setting where the different bidders arrive at different times and the auction mechanism is required to make decisions about each bid *as it is received*. Such settings occur in computerized auctions of computational resources as well as in other settings. We call such auctions, *on-line auctions*.

We first characterize exactly on-line auctions that are *incentive compatible*, i.e. where rational bidders are always motivated to bid their true valuation. We then embark on a *competitive worst-case* analysis of incentive compatible on-line auctions. We obtain several results, the cleanest of which is an incentive compatible on-line auction for a large number of identical items. This auction has an optimal competitive ratio, both in terms of seller's revenue and in terms of the total social efficiency obtained.

1. Introduction

Auctions are a commonly used tool for selling goods in cases where a true competitive market does not exist. In the typical case multiple buyers aim to buy some good from a single seller, and the seller wishes to sell the good for the highest possible price. Many types of auctions have been considered in the literature, and an elegant theory has evolved. For an introduction see e.g. the textbook [20].

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In recent years auctions have found new applications in trade that is performed on computer networks and especially over the Internet. Such applications include electronic commerce [5,23], computational and network resource allocation [24,25,18,16,22,29, 10,31], trade between software agents [26,9,27], and more. With these new applications new questions arise.

This paper studies auctions in a setting where the different bidders arrive at different times and the auction mechanism is required to make decisions about each bid *as it is received*. This is in contrast to the traditional assumption (in theory and in practice) that the auction organizer must receive all the bids before determining the allocation. The traditional assumption implicitly assumes that all participants (including the auctioneer) are willing to wait for some amount of time (until all bids are gathered) before performing any trade. We argue that in many settings, especially computerized ones, players will not be willing to wait a long time for the allocation decision.

An example of such a setting is bandwidth allocation on a communication link. Consider a fixed communication link in some computer network. In cases where the demand for communication over this link exceeds the link's bandwidth, a popular approach for allocating the limited bandwidth is by auctioning it among all the possible uses [18,10,19,14]. However, in such a setting one would expect the requests (bids) for bandwidth to arrive at different times—each request needing an immediate answer. Similar situations arise in the allocation of other resources such as CPU time or cache space.

In our model, k identical items are sold in an auction. Each bidder has a (privately known) valuation for each quantity of the goods, where the marginal valuations of the bidders are non-increasing. The bidder learns this valuation at a certain time and must make a bid at that time. The auction mechanism must decide, as the bid is received (and before seeing future bids), how many items to allocate to this bidder and at what price. We term such an auction *on-line*. (We also consider more general variants where the valuations as well as bids may be time-dependent—all our results extend to the general variants).

Our main concern in this paper is with the incentive compatibility—also called truthfulness or strategy-proofness—of the auction. An auction is called incentive compatible if participants are rationally motivated to reveal the truth about their valuations. Specifically, in game-theoretic terms, if the truth is a dominating strategy. This is a departure from the field of on-line algorithms (see [7,12]) which does not address any game-theoretic issues but only algorithmic ones.

In his seminal paper, Vickrey [30] argued for the importance of incentive compatibility and first analyzed the incentive compatible second price auction. The main motivation is to free the bidders from strategic considerations. It has been argued [26,9,27,28,17] that this is especially important in computerized settings.

Our first result in this paper is a full characterization of incentive compatible on-line auctions: We define an on-line auction as "based on supply curves" if before receiving the ith bid, $b_i(j)$ for $1 \leqslant j \leqslant k$, it fixes some function (supply curve) $p_i(j)$ based on previous bids, and,

1. The quantity q_i sold to bidder i is the quantity q that maximizes the sum $\sum_{j=1}^{q}(b_i(j) - p_i(j))$ (i.e. the bidder's utility).
2. The price paid by agent i is $\sum_{j=1}^{q_i} p_i(j)$.

Theorem 1. *A deterministic on-line auction is incentive compatible if and only if it is based on supply curves.*

We then employ a *worst case* analysis of on-line auctions. This is in the spirit of computer science and in sharp contrast to the usual Bayesian (average case) analysis employed in auction theory (as well as in other economic situations). The overriding reason that worst case analysis is the almost universal choice in computer science is that it turns out that "real world" distributions in computational settings are almost always very different from any theoretical distributions assumed in an average case analysis. In such cases an average case analysis is worthless while a worst-case analysis does provide performance guarantees. This is not the place for a lengthy discussion of the merits of worst case analysis compared with those of average case analysis. The interested reader may refer e.g. to the introductory textbook [4, pp. 9–10]. We strongly feel that as auction theory is increasingly applied to computational settings, the importance of worst case analysis increases.

Specifically, we assume in this paper that bidders' valuations all belong to some range $[\underline{p}, \bar{p}]$ (without assuming any probability distribution), where $\underline{p}$ is also the seller's reservation price, i.e. each item is worth $\underline{p}$ to him. We compare the performance of an incentive compatible on-line auction to the performance of the standard *off-line* Vickrey auction—for one item, this is the sealed bid second price auction, or, equivalently, the popular open cry English auction with small bid increments (all exact definitions appear in the paper body). This auction is incentive compatible and obtains optimal *social efficiency*, i.e. maximizes the sum of all players' valuations of the items they receive.

Similarly to the definition used in on-line analysis of algorithms, we focus on the, so-called, competitive ratio: An on-line auction is called *c-competitive with respect to the revenue* (relative to the Vickrey auction) if for *every* sequence of valuations of bidders it obtains a revenue that is at least $1/c$ of the revenue obtained by the Vickrey auction for these valuations (where the Vickrey auction knows all bids in advance). Similarly we define *c-competitive with respect to the social efficiency*. We note the dissimilarity between the economic meaning of the term "competitive" and its meaning in computer science, which is the one used here.

The tightest set of results is obtained when the number k of items is large, so it can be treated as a continuum. This can be viewed as the case of one divisible good, i.e. it can be divided to any number of small fractions. For this case we are able to find the optimal on-line incentive compatible auction. We define the *competitive on-line auction* by using the function suggested in [8] to construct the supply curves. Extending the results of [8] for on-line continuous one way trading, we prove the following upper and lower bounds. Let $\phi = \bar{p}/\underline{p}$, and let the constant c denote the solution of the equation $c = \ln((\phi - 1)/(c - 1))$. It can be shown that $c = \Theta(\ln \phi)$.

Theorem 2. *The competitive on-line auction is c-competitive with respect to the revenue as well as with respect to the social efficiency of the off-line Vickrey auction.*

No other on-line auction has a better competitive ratio either with respect to the revenue or with respect to the social efficiency.

For the case of a smaller values of k we obtain the following results. For one good the best competitive on-line auction achieves a competitive ratio of $\sqrt{\phi}$. For other values of k, we show a deterministic lower bound of $\phi^{1/(k+1)}$ and a deterministic upper bound of $k \cdot \phi^{1/(k+1)}$. We observe also that if *randomized* auctions are allowed then a better competitive ratio can be obtained. By using the supply curves of the previous theorem for probabilistic choices, a competitive ratio of c can be obtained (for any number of goods), where $c = \Theta(\ln \phi)$ is as before. In this case, the on-line revenue is also competitive with respect to the *optimal social efficiency*.

It should be noted that the competitive ratio is obtained in the worst case; in the average case the ratio is typically much better. As a demonstration, we also provide a "normal" Bayesian analysis of our competitive on-line auction for the divisible good, in the case of uniformly distributed valuations in the interval $[\underline{p}, \bar{p}]$. For example, for two bidders whose valuations are uniformly distributed in $[1, 2]$ this on-line auction achieves expected revenue of 1.31... as compared to 1.33.. for the Vickrey auction.

While, to the best of our knowledge, competitive worst case analysis of online auctions was not studied before, we note that competitive analysis of auctions have also been employed by Goldberg et al. and Fiat et al. [13,11], where they analyze the (off-line) unlimited supply case, and design revenue maximizing randomized auctions. Other works have adopted the model of on-line auctions: [1,2] describe on-line auctions for unlimited supply. Blum [3] describe on-line double auctions (where the auction mechanism is actually a market that matches buyers to sellers).

This paper is organized as follows. Section 2 describes our model and gives a full characterization of incentive compatible on-line auctions. In Section 3 we describe the competitive on-line auction. Section 4 outlines an extended on-line model, and Section 5 gives a distributional analysis of the competitive auction.

2. On-line auctions

2.1. The model

2.1.1. The goods

We consider an auction of k identical indivisible goods to a set of players. We distinguish the case of a very large k that can be treated as a continuum, viewing this case as auctioning one divisible good.

2.1.2. Players' valuations and utilities

Each player has some positive benefit (valuation) from receiving some quantity of the goods. This valuation is known only to the player himself. We denote the marginal valuation of player i as $v_i(j)$, for $1 \leqslant j \leqslant k$, i.e. $v_i(j)$ is the additional value gained from the jth good. Thus, his total valuation for q goods is $\sum_{j=1}^{q} v_i(j)$. We assume that all players have downward sloping marginal valuation functions, i.e.

$\forall i,j$: $v_i(j+1) \leqslant v_i(j)$.[1] When player i receives q goods and pays for them a total payment of P_i his utility is $U_i(q,P_i) = \sum_{j=1}^{q} v_i(j) - P_i$. We assume that each player aims to maximize his utility.

2.1.3. The on-line game and players' strategies

The on-line game has the following structure. Initially, the set of players is unknown to the auctioneer, and none of the players knows his valuation. At some point in time, t_i, player i determines his valuation and must declare his bid at that time. We focus on direct revelation mechanisms, in which the player declares his marginal valuation function. Thus, the bid is some non-increasing function $b_i(\cdot)$ of the form $b_i : [1\ldots k] \rightarrow \mathscr{R}$. Of-course, a player may be motivated to lie, declaring some $b_i(\cdot) \neq v_i(\cdot)$, in order to increase his utility. The auctioneer must answer the bid immediately, before opening the next bid. In his answer, he determines the quantity to be sold and the total price to be paid for it. We assume that if a player does not receive any positive quantity then his total payment is zero.[2] The game ends when the auctioneer sells all the goods or when the last player announces his bid.

2.1.4. Incentive compatibility

We study truthful implementations in dominant strategies, which we refer to as incentive compatible mechanisms. A strategy (bid) $b_i(q)$ of player i is called dominant if for every other bid $\tilde{b}_i(q)$ and for every sequence of past and future bids of the other players, $U_i(q_i,P_i) \geqslant U_i(\tilde{q}_i,\tilde{P}_i)$, where $q_i,\tilde{q}_i$ are the quantities sold to player i when declaring $b_i(q),\tilde{b}_i(q)$, respectively, and $P_i,\tilde{P}_i$ are the total payments charged for each quantity. In other words, for every bid sequence of the other players the utility of player i is maximized by choosing the specific declaration $b_i(q)$. A direct revelation mechanism is incentive compatible if for every valuation $v_i(\cdot)$, declaring the true valuation is a dominant strategy. Such mechanisms are also called strategy-proof, or truthful. For a more detailed discussion see e.g. [20].

Remark 1. This model explicitly limits the strategy space of the players, excluding any time considerations, i.e. player i must declare his bid at time t_i, and is not allowed to return. In Section 4 we remove this limitations and show that all our results still hold for an extended model with time considerations.

Remark 2. It is also possible to consider a *partially on-line* model, in which the set of players is known in advance to the auctioneer (but the valuation sequence is still revealed on-line). Although this approach weakens the on-line power, it is more close to regular game theory settings.

[1] This assumption is common in economics, and is assumed, e.g. in Vickrey's original paper. Without it the Vickrey multi-unit auction is not efficient, and in fact finding an optimal allocation is NP-complete.

[2] This normalization ensures both participation constraints and no budget deficit.

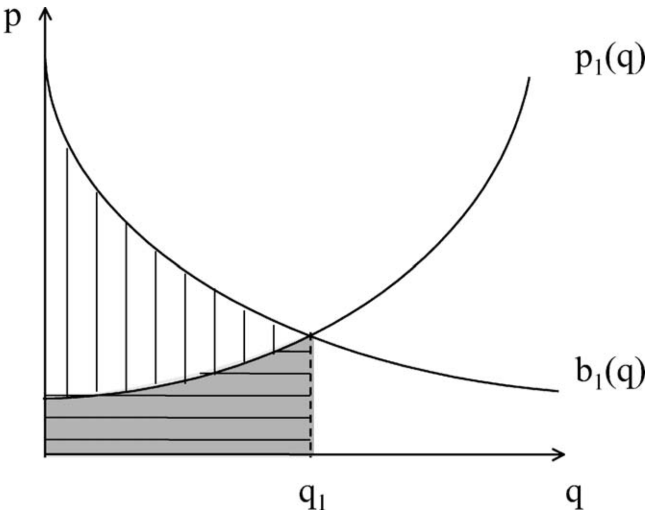

Fig. 1. An example of supply curves based auction.

2.2. *Supply curves for on-line auctions*

We now characterize incentive compatible on-line auctions, as auctions that are *based on supply curves*. Intuitively, such auctions determine the prices for bidder i independently of i's bid, and then sell to i the quantity that maximizes his utility under these prices:

Definition 1 (Supply curves). An on-line auction is called "based on supply curves" if before receiving the i'th bid it fixes a function (supply curve) $p_i(q)$ based on previous bids,[3] and,

1. The quantity q_i sold to bidder i is the quantity q that maximizes the sum $\sum_{j=1}^{q}(b_i(j) - p_i(j))$, i.e. the bidder's utility (ties may be broken arbitrarily).
2. The price paid by bidder i is $\sum_{j=1}^{q_i} p_i(j)$.

A simpler form of supply curves, which we will use below, is when each supply curve $p_i(q)$ is non-decreasing. For such a supply curve, the quantity q_i becomes the largest quantity q such that $b_i(q) \geqslant p_i(q)$. For this simple form, the case of a divisible good is defined similarly: the supply curve $p_i(q)$ is any non-decreasing real function, the quantity q_i is determined as before, and the price becomes $\int_0^{q_i} p_i(q)\,\mathrm{d}q$. If both $b_i(q)$, $p_i(q)$ are continuous then q_i is the unique solution to $b_i(q) = p_i(q)$.

For example, Fig. 1 illustrates a non-decreasing supply curve $p_1(q)$ and a bid $b_1(q)$. According to Definition 1, the quantity received by the player equals q_1, and the total price paid is the area below the supply curve, marked by the horizontal lines. The

[3] Here, the supply curves are determined deterministically. A possible extension of this definition, when allowing randomization, is described in Section 3.2.

player's valuation of the quantity q_1 is the area below $b_1(q)$, and, thus, the resulting utility of the player is the area between the marginal valuation and the supply curve, marked by the vertical lines. This is the entire surplus, in economic terms. After the sale, the auction continues to the next player, presenting some new supply curve $p_2(q)$.

Theorem 1. *A deterministic on-line auction is incentive compatible if and only if it is based on supply curves.*

Proof. We prove the two directions of the theorem by the following two lemmas.

Lemma 1. *An on-line auction that is based on supply curves is incentive compatible.*

Proof. The utility of player i from receiving some quantity q is $U_i(q)=\sum_{j=1}^{q}(v_i(j)-p_i(j))$ (his valuation of the total quantity minus his total payment). Let $b_i(q)\neq v_i(q)$ be some bid and suppose the quantity sold for this bid is $\tilde{q}_i$, and for the truthful bid is q_i. Then it is the case that $U_i(q_i)\geqslant U_i(\tilde{q}_i)$, since this is explicitly verified in the first condition of the supply curves definition (when the bid is truthful then the term maximized there equals $U_i(q)$). Thus the claim follows. □

Lemma 2. *Any deterministic incentive compatible on-line auction is based on supply curves.*[4]

Proof. Fix any deterministic incentive compatible on-line auction A. We first argue that the total payment of player i is determined uniquely by the quantity sold to him (and by previous bids): Otherwise, there are two different bids $v(q),\tilde{v}(q)$ such that the quantity sold when declaring each one of them is the same but the total price paid is different. Let P be the total price when declaring $v(q)$ and $\tilde{P}$ the total price when declaring $\tilde{v}(q)$, and w.l.o.g suppose $P<\tilde{P}$. Thus a player with valuation $\tilde{v}(q)$ will increase his utility by declaring $v(q)$ since he will receive the same quantity and will pay a lower total payment, which is a contradiction since A is incentive compatible.

Therefore, denote by $P_i(q)$, for $1\leqslant q\leqslant k$, the total payment of player i when receiving a quantity q.[5] We claim that i must be allocated the quantity q_i that maximizes $U_i(q)=\sum_{j=1}^{q} v_i(j)-P_i(q)$. Otherwise, let $b(\cdot)$ be some bid for which A sells the quantity q_i to i. Then, if A sells a quantity $\tilde{q}_i\neq q_i$ for the truthful bid $v_i(\cdot)$, player i will increase his utility by declaring $b(\cdot)$ instead, which contradicts incentive compatibility.

To conclude the argument, we claim that $p_i(q)=P_i(q)-P_i(q-1)$ (for $q>0$) is the supply curve according to Definition 1. Indeed, since $P_i(q)=P_i(0)+\sum_{j=1}^{q} p_i(j)$, and $P_i(0)=0$, it follows from the argument above that the total quantity sold is the

[4] Notice, however, that the supply curves structure may be implicit in the formal auction description.

[5] Technically, if it is not possible for i to receive some quantity q, i.e. no bid of his results in selling q units to him, then $P_i(q)$ is undefined, or is equal to "infinity".

quantity q_i that maximizes $U_i(q)=\sum_{j=1}^{q} v_i(j)-P_i(q)=\sum_{j=1}^{q} v_i(j)-\sum_{j=1}^{q} p_i(q)$, and the total price paid is $P_i(q_i)=\sum_{j=1}^{q_i} p_i(j)$. □

From the above two lemmas the theorem follows. □

Remark. Allowing any (non-increasing) marginal valuation functions may increase significantly the complexity of presenting the valuation function to the auctioneer. This problem can be solved by using a modified auction that, instead of receiving valuation functions, presents the (current) supply curve to all (interested) players. In this case each bid is simply a price–quantity coordinate on the supply curve. From the same considerations of incentive compatibility from Lemma 1, declaring the truth (i.e. the maximal quantity according to Definition 1) is dominant. We note that the supply curves we give below can be presented easily.

An interesting special form of players' valuations is fixed marginal valuations. In this case, the marginal valuations are restricted to the form $v_i(q)=v_i$ for all i,q. This case is also useful since we use it for the lower bound we give below. For this case, it is possible to characterize the supply curves more precisely (this holds for the continuous case as well, with a similar proof).

Lemma 3. *Assume that all marginal valuations are of the form $v_i(q)=v_i$. Then any incentive compatible on-line auction is based on non-decreasing supply curves.*

Proof. Fix some incentive compatible on-line auction A. According to Theorem 1, A is based on supply curves. Consider the sale to the ith player. Denote A's total price function by $P_i(q)$, and A's allocation rule by $q_i(v)$, i.e. A sells a quantity $q_i(v)$ for a bid v (since the marginal valuation is fixed, each bid is simply a single value).

We first argue that $q_i(v)$ is non-decreasing. Otherwise, suppose there are two bids $\tilde{v}>v$ such that $q_i(\tilde{v})<q_i(v)$. Denote $q_i(v)=q,\ q_i(\tilde{v})=\tilde{q}$. Since the auction is incentive compatible, $\tilde{v}\cdot\tilde{q}-P_i(\tilde{q})\geqslant\tilde{v}\cdot q-P_i(q)$, and therefore $P_i(q)-P_i(\tilde{q})\geqslant\tilde{v}(q-\tilde{q})>v(q-\tilde{q})$. Thus $v\cdot\tilde{q}-P_i(\tilde{q})>v\cdot q-P_i(q)$ and according to the supply curves definition, A must sell a quantity of $\tilde{q}$ for a bid value of v, in contradiction. Now define

$$p_i(q)=\inf\{v\mid q_i(v)\geqslant q\},$$

i.e. A sells at least q for any bid $v>p_i(q)$. Since $q_i(v)$ is non-decreasing then $p_i(q)$ is non-decreasing as well. We claim that A is based on $p_i(q)$. In other words, for every bid v, if A sells a quantity q then $P_i(q)=\sum_{j=1}^{q} p_i(j)$.

To see this, we argue that for any $l\geqslant 1$ and $q\geqslant l$ such that $p_i(q)=\cdots=p_i(q-l+1)>p_i(q-l)$, it is the case that $P_i(q)-P_i(q-l)=l\cdot p_i(q)$ (to clarify this and what follows, consider first the simpler case where $p_i(\cdot)$ is strictly increasing—then l always equals 1 and the claim becomes that $P_i(q)-P_i(q-1)=p_i(q)$). Denote $x=(P_i(q)-P_i(q-l))/l$, and suppose by contradiction that $x\neq p_i(q)$. If $x<p_i(q)$ then a bidder with marginal valuation $\tilde{v}$ such that $x,\ p_i(q-l)<\tilde{v}<p_i(q)$ will increase his utility by declaring $p_i(q)$ instead—he will now receive q units instead of $q-l$ (this follows from the definition of $p_i(\cdot)$), and will pay for each additional unit x, which is less than

$\tilde{v}$, his value for this unit. In other words, his utility will change by $l(\tilde{v}-x)>0$, since he will receive additional l units, and his additional payment will be $P_i(q)-P_i(q-l)=l\cdot x$. In a similar manner, if $x>p_i(q)$ then a bidder with marginal valuation $x>\tilde{v}\geqslant p_i(q)$ will increase his utility by declaring $p_i(q-l)$ instead, since his utility will change by $l(x-\tilde{v})>0$.[6]

We can now conclude that, if i bids v and receives q, then he pays $\sum_{j=1}^{q} p(j)$. To see this, notice that from the above, the sum $\sum_{j=1}^{q} p(j)$ becomes a telescopic sum that reduces to $P_i(q)-P_i(0)=P_i(q)$, as needed. □

In general, there is no specific relation between the different supply curves of an auction. However, a useful structure of supply curves, which we use in Section 3 below, is when all the supply curves are derived from some global supply curve, as follows:

Definition 2 (A global supply curve). An on-line auction is called "based on a global supply curve $p(q)$" if it is based on supply curves and if $p_i(q)=p(q+\sum_{j=1}^{i-1} q_j)$, where q_j is the quantity sold to the jth bidder.

In other words, the ith supply curve is a left shift of the $(i-1)$st supply curve by q_{i-1}. Thus, the ith bidder receives the quantity according to the first supply curve $p_1(q)$ minus the quantity that was sold previously.

3. Competitive analysis

In this section we describe on-line auctions with worst-case performance guarantees, i.e. the on-line performance for every valuation sequence is not too far from the off-line performance for the same sequence. We first define our performance measure (revenue and social efficiency) and the exact meaning of a performance guarantee (competitiveness).

For the worst-case analysis, we assume that all marginal valuations are taken from some known interval $[\underline{p}, \bar{p}]$, without assuming any distribution on them. We assume that $\underline{p}>0$, and that it is also the reservation price of the auctioneer, i.e. the auctioneer has a value of $\underline{p}$ for any unit he did not sell.[7]

Definition 3 (Revenue and social efficiency). The *revenue* of an auction A for a valuation sequence σ, denoted as $R_A(\sigma)$, is the resulting utility of the auctioneer, i.e. the total payment he received plus his valuation of the quantity he did not sell. More

[6] To be completely exact, all this holds only for q's that can be received, i.e. q s.t. there exists v with $q_i(v)=q$, but those are the only q's we need to worry about.

[7] This may be his manufacturing or shipping cost (that he may save for unsold units), an option to sell the units for a very low price, etc.

specifically, let q_i be the quantity sold to the i'th player in σ and P_i be the total price paid by the i'th player, then:

$$R_A(\sigma) = \sum_i P_i + \underline{p}\left(k - \sum_i q_i\right).$$

The *social efficiency* of an auction A for a valuation sequence σ, denoted as $E_A(\sigma)$, is the sum of the resulting utilities of all players, including the auctioneer. This is also equal to the sum of all the players' valuations of the quantity they possess (including the auctioneer) i.e.:

$$E_A(\sigma) = \sum_i \sum_{j=1}^{q_i} v_i(j) + \underline{p}\left(k - \sum_i q_i\right).$$

We compare the revenue and the social efficiency obtained by on-line auctions to those obtained by the off-line Vickrey auction [30]:

Definition 4 (The Vickrey auction). In the Vickrey auction, each player declares his (supposedly) marginal valuation function. The allocation chosen is the one that maximizes the social efficiency (according to players' declarations). The price charged from player i for the quantity q_i he receives is the worth of this additional quantity to the other players, i.e. the additional value of the other players when dividing q_i optimally among them. Formally, denote by E_{-i} the optimal social efficiency when player i is missing, and by E the actual optimal social efficiency. Then, the price that i pays is $E_{-i} - (E - v_i(q_i))$.

For example, if there is only one indivisible good, this auction becomes the well known second price auction, where the highest bidder wins and pays the second highest offer. This is approximately equivalent to the popular English auction, where increasing bids are announced until no bidder wishes to make any further higher bid [30]—as the bid increments become smaller, the price paid by the winner becomes closer to the second highest value.

The use of the Vickrey auction as our benchmark is not only due to its popularity but also since it is incentive compatible. Such a non-Bayesian equilibrium is required for the worst-case analysis we desire. In any case, the Vickrey auction is always optimal in terms of the social efficiency. While the revenue is not necessarily optimal in a Bayesian setting, the revenue equivalence theorem [21] states that other auctions with equivalent outcomes extract the same revenue.

We compare our on-line auction to the Vickrey auction in the following worst-case sense:

Definition 5 (Competitiveness). An on-line auction A is c-competitive with respect to the revenue if for every valuation sequence σ, $R_A(\sigma) \geqslant R_{\text{vic}}(\sigma)/c$. Similarly, A is c-competitive with respect to the social efficiency if for every valuation sequence σ, $E_A(\sigma) \geqslant E_{\text{vic}}(\sigma)/c$.

3.1. A divisible good

We first focus on the case of a divisible good, i.e. a good that can be divided to any number of small fractions (we assume w.l.o.g that we have *one* divisible good). We describe a global supply curve that is $\Theta(\log(\bar{p}/\underline{p}))$-competitive with respect to both the revenue and the social efficiency. For this purpose we use results of [8] for on-line continuous one way trading. [8]

Let c be the unique solution to the equation:

$$c = \ln \frac{(\bar{p}/\underline{p}) - 1}{c - 1}. \tag{1}$$

It can be shown that $c = \Theta(\ln(\bar{p}/\underline{p}))$. For example, if $(\bar{p}/\underline{p}) = 2$ then $c = 1.28$, and if $(\bar{p}/\underline{p}) = 8$ then $c = 1.97$ [8].

Definition 6 (The competitive on-line auction). Define the *competitive supply curve* by

$$p(x) = \underline{p}(1 + (c - 1)e^{cx}). \tag{2}$$

The *competitive on-line auction* has the competitive supply curve as its global supply curve.

In order to use the results of [8], we need to derive the following two functions from the global supply curve $p(x)$. Let $q(x) = p^{-1}(x)$ (the inverse function of $p(x)$) and let $r(x) = \int_0^{q(x)} p(y)\,\mathrm{d}y$. [9] In our context, these functions can be interpreted as follows: $q(x)$ is the total quantity sold by the competitive on-line auction when the last bid intersects the last supply curve at price x, and $r(x)$ is the total payment charged by the auction for such a sequence. El-Yaniv et al. [8] analyzes these functions (separately from their context to the supply curve), and shows that:

Lemma 4 (El-Yaniv et al. [8]). *The functions $q(x), r(x)$ preserve the following conditions*:
1. $\forall x \leqslant c \cdot \underline{p}$: $q(x) = 0$, $r(x) = 0$,
2. $\forall x > c \cdot \underline{p}$: $r(x) + \underline{p} \cdot (1 - q(x)) = x/c$, *and*
3. $q(\bar{p}) = 1$,

where c is as defined in Eq. (1).

The paper [8] also states the minimality of the constant c in the following sense (this lemma is implicit in [8]):

[8] In this on-line model, a trader needs to convert dollars to yen. The exchange rate is unpredicted, and is determined by an adversary. El-Yaniv et al. [8] give several algorithms to compete in such an environment. Here, we construct a global supply curve from a specific function developed in [8] for the purpose of describing the trader conversion behavior.

[9] The paper [8] uses $r(x) = \int_0^x y q'(y)\,\mathrm{d}y$. It can be verified that both terms are equal.

Lemma 5 (El-Yaniv et al. [8]). *For any constant $\tilde{c} < c$, there is no function $\tilde{q}(x)$ such that*

$$\forall x \in [\underline{p}, \bar{p}], \quad \tilde{r}(x) + \underline{p} \cdot (1 - \tilde{q}(x)) \geqslant x/\tilde{c},$$

where $\tilde{r}(x) = \int_0^{\tilde{q}(x)} \tilde{p}(x)\,\mathrm{d}x$ and $\tilde{p}(x) = \tilde{q}^{-1}(x)$ is the inverse function of $\tilde{q}(x)$.

Theorem 2. *The competitive on-line auction is c-competitive with respect to the revenue and the social efficiency.*

Proof. We prove the following lemmas:

Lemma 6. *For any sequence of valuations σ, $R_{\text{cola}}(\sigma) \geqslant R_{\text{vic}}(\sigma)/c$, where "cola" is the competitive on-line auction and "vic" is the Vickrey auction.*

Proof. Fix some valuation sequence σ. For a player i let q_i be the quantity he received, and denote $p_i = p_i(q_i)$. Let m be the last player that received a positive quantity q_m. For all i and $q > q_i$, $b_i(q) \leqslant p_i$. It also follows that for all i, $p_i \leqslant p_{i+1}$ since $p(q)$ is non-decreasing. Thus, every player values any additional quantity (to that he already received), Δq, by no more than $p_m \cdot \Delta q$. The price that the Vickrey auction determines for the quantity q_i^* it allocates to player i is the highest valuation of the other players for the additional quantity $\Delta q = q_i^*$ divided among them. There is at least one player, say i', such that $q_{i'}^* \geqslant q_{i'}$ since the Vickrey auction allocates the entire quantity. Thus $b_{i'}(q_{i'}^*) \leqslant b_{i'}(q_{i'}) \leqslant p_m$. Since the Vickrey auction is efficient it follows that for all j and $q > q_j^*$, $b_j(q) \leqslant p_m$ (otherwise, if for some j this does not hold, then it is possible to increase the Vickrey efficiency by shifting some quantity from i' to j). Thus, every player values any quantity addition Δq by no more than $p_m \cdot \Delta q$ in the Vickrey auction as well, and therefore $R_{\text{vic}}(\sigma) \leqslant \sum_i (p_m \cdot q_i^*) \leqslant p_m$. According to condition 2 of Lemma 4, the on-line revenue is p_m/c, and the lemma follows. $\square$

Since the Vickrey auction obtains an optimal social efficiency, we need to prove the following:

Lemma 7. *For any sequence of valuations σ, $E_{\text{cola}}(\sigma) \geqslant E_{\text{opt}}(\sigma)/c$, where $E_{\text{opt}}(\sigma)$ is the optimal social efficiency for σ.*

Proof. Fix some valuation sequence σ and denote q_i, p_i, and m as in the previous lemma. Consider a new sequence σ^* as follows:

$$b_i^*(q) = \begin{cases} p_i, & q \leqslant q_i, \\ b_i(q), & \text{otherwise,} \end{cases}$$

i.e. player i has fixed marginal valuation up to $q = q_i$ and then as before. The on-line allocation for this sequence does not change since $b_i^*(q)$ intersects the supply curve at $p(q_i)$. Since $b_i(q) \leqslant p_i \leqslant p_m$ for all i, it follows that $E_{\text{opt}}(\sigma^*) \leqslant p_m$. It is also true that

$E_{\text{cola}}(\sigma^*) \geqslant R_{\text{cola}}(\sigma^*) = p_m/c$ (the equality is due to condition 2 of Lemma 4). Thus, $E_{\text{cola}}(\sigma^*) \geqslant E_{\text{opt}}(\sigma^*)/c$.

Now consider moving from σ to σ^* in m steps. In each step i, if $b_i(q) > b_i^*(q)$ at some points, then $b_i(q)$ is decreased to $b_i^*(q)$. Let σ^i be σ after i such modifications. The on-line auction allocates to i the entire quantity whose value decreased, and thus $E_{\text{cola}}(\sigma^i) - E_{\text{cola}}(\sigma^{i+1}) \geqslant E_{\text{opt}}(\sigma^i) - E_{\text{opt}}(\sigma^{i+1})$, i.e. the on-line efficiency decrease is greater than the off-line decrease since it is the maximal possible. From this it follows that $E_{\text{cola}}(\sigma) - E_{\text{cola}}(\sigma^*) \geqslant E_{\text{opt}}(\sigma) - E_{\text{opt}}(\sigma^*)$, and we get:

$$\begin{aligned} E_{\text{cola}}(\sigma) &\geqslant (E_{\text{opt}}(\sigma) - E_{\text{opt}}(\sigma^*))/c + E_{\text{cola}}(\sigma^*) \\ &\geqslant (E_{\text{opt}}(\sigma) - E_{\text{opt}}(\sigma^*))/c + E_{\text{opt}}(\sigma^*)/c = E_{\text{opt}}(\sigma)/c. \qquad \Box \end{aligned}$$

From the above two lemmas the theorem follows. □

A natural question to ask is whether the on-line revenue is competitive with respect to some higher revenue criteria. As it turns out, it can be shown that for the special case of fixed marginal valuations, the on-line revenue is c-competitive with respect to the *optimal efficiency* (i.e. with respect to the off-line auction that extracts the total surplus—clearly this is the best revenue we can hope for since no player will pay more than his value). This follows basically from the following argument: Let p_i denote the y-coordinate of the intersection point of the ith bid with the supply curve (as in the proof above). Since players have fixed marginal valuations v_i, it follows that $v_i = p_i$. Let m be the last player that received a positive quantity. Since for all i, $p_i \leqslant p_{i+1}$, it follows that $v_m \geqslant v_i$ for any other player i. Therefore, the optimal social efficiency is $v_m \cdot 1$, by allocating the entire quantity to player m. On the other hand, the online revenue is p_m/c (as shown in the proof above), which proves the claim.

In contrast, for general valuations, the on-line revenue is significantly lower than the optimal efficiency in cases where the Vickrey revenue is significantly lower than the optimal efficiency. For example, consider the following scenario of two players. Let p^* be some price and q^* be the quantity such that $p^* = p(q^*)$). The first player has a fixed marginal valuation of $\bar{p}$ up to q^*, and the second player has a fixed marginal valuation of p^*. The optimal efficiency for this scenario is $q^* \cdot \bar{p} + (1 - q^*) \cdot p^*$. In the on-line auction, player 1 will receive a quantity of q^*, since this is the maximal quantity for which his valuation is higher then the supply curve. Player 2 will receive nothing, since the second supply curve is higher than p^* (as the auction is based on a global supply curve). Therefore, the on-line revenue is at most $q^* \cdot p^* + (1 - q^*) \cdot \underline{p}$. Thus, for example, when setting $p^* = \sqrt{\underline{p} \cdot \bar{p}}$ then the optimal efficiency to on-line revenue ratio is larger than $\sqrt{\bar{p}/\underline{p}}$. It is interesting to observe that, if the arrival order of the players is reversed, then the on-line revenue increases significantly to $\bar{p}/c$ (although the Vickrey revenue remains the same).

If the players' valuations are drawn independently from a known probability distribution, then the Vickrey auction with an appropriate reservation price[10] is known

[10] A threshold price—no sale is performed for a lower price.

to have optimal revenue for several special cases (e.g. when each player has unit demand) [21]. We note that our auction can be modified to be competitive with respect to the Vickrey auction with reservation price by simply taking, as the supply curve, the maximum between the original supply curve and the reservation price (at each point).

We now show that the competitive ratio of the Competitive On-Line Auction is the best we can expect:

Theorem 3. *Any incentive compatible on-line auction must have a competitive ratio of at least c with respect to both the revenue and the social efficiency, where c is the solution to Eq. (1).*

Proof. We prove the claim for the special case of fixed marginal valuations (in other words, even if the adversary is restricted to use only fixed marginal valuations the claim holds). For this case we can assume w.l.o.g (according to Lemma 3) that A is based on non-decreasing supply curves. We also assume w.l.o.g that $\underline{p} = 1$, and denote $\bar{p} = \phi$. Let f_n be the nth root of ϕ, i.e. $f_n^n = \phi$, and $c_n = c/(f_n^2)$.

The following lemma assumes only the more restricted partially on-line model, in which the number of players, n, is known in advance. For this case, it lower bounds the competitive ratio of any on-line auction by c_n, thus also implying that knowing the number n in advance may help significantly only for small values of n.

Lemma 8. *No on-line auction with n bidders achieves efficiency that is better than c_n competitive with respect to the revenue of the Vickrey auction.*

Proof. Assume we have a better than c_n competitive auction, we will build a function $\tilde{q}(x)$ satisfying the condition of Lemma 5 with a constant $\tilde{c} < c$, a contradiction.

Consider the behavior of the on-line auction on the sequence of bids of the n bidders: $p_1 = f_n, p_2 = f_n^2, \ldots, p_n = \phi$. Let q_i be the quantity allocated to bidder i. For all x in the range $1 \leqslant x \leqslant \phi$, define $\tilde{q}(x)$ as $\sum_{j=1}^{i} q_j$, where i is such that $p_{i-1} \leqslant x < p_i$ (for completeness, denote $p_0 = 1$, $p_{n+1} = \infty$). The function $\tilde{r}(x)$ (as defined in Lemma 5) is now $\tilde{r}(x) = \sum_{j=1}^{i} q_j p_j$ for the same i. [11]

Now, for each i, consider the sequence of bids where the first i bids are $p_1, \ldots, p_i$, but the other $n-i$ bids are simply 1. The revenue of the Vickrey auction in this case is $p_{i-1} = p_i/f_n$. The efficiency of the on-line auction is given by $\tilde{r}(p_i) + (1 - \tilde{q}(p_i))$. Since we assumed better than c_n competitiveness, we have $\tilde{r}(p_i) + (1 - \tilde{q}(p_i)) > p_i/(c_n f_n) = p_i f_n/c$. It follows that for every x, if we let i be such that $p_{i-1} \leqslant x < p_i$, then we have $\tilde{r}(x) + (1 - \tilde{q}(x)) = \tilde{r}(p_i) + (1 - \tilde{q}(p_i)) > p_i f_n/c \geqslant (x/f_n) f_n/c = x/c$. This is exactly the condition of Lemma 5, completing the contradiction. □

[11] Since $q(x)$ is not a one to one function there is no inverse function $p^{-1}(x)$, but it can be verified that the function $p(x) = p_i$ for i such that $q_{i-1} < x \leqslant q_i$ is the appropriate function to use for the definition of $r(x)$.

From this lemma it follows that for every on-line auction A with n players there is a valuations sequence σ such that $R_A(\sigma) \leqslant E_A(\sigma) \leqslant R_{\text{vic}}(\sigma)/c_n \leqslant E_{\text{vic}}(\sigma)/c_n$. Therefore A is no less than c_n-competitive with respect to both the revenue and the efficiency. Since c_n approaches c as n grows to infinity, the theorem follows. □

3.2. A randomized auction for k indivisible goods

We now discuss the discrete case. First we show that, by using randomization, it is possible to obtain an *expected* revenue and social efficiency that are c-competitive (where c is as before), *both* with respect to the optimal social efficiency. Thus, randomization enables us to improve the performance with respect to the revenue.

When allowing randomization, the definition of supply curves should be altered so that each supply curve may be chosen randomly according to some distribution (an auction that is based on such supply curves is incentive compatible in a strong sense, as detailed below). As noted in [6], the function $q(x) = p^{-1}(x)$ (i.e. the inverse function of the competitive supply curve $p(x)$ of Eq. (2)) may be viewed as a cumulative distribution function in the interval $[\underline{p}, \bar{p}]$, i.e., if we choose x randomly using $q(\cdot)$, we have that for any fixed $v \in [\underline{p}, \bar{p}]$, $Pr(x \leqslant v) = q(v)$ (note that $q(\underline{p}) = 0$, $q(\bar{p}) = 1$, and that $q(\cdot)$ is non-decreasing).

Definition 7. The randomized on-line auction: before receiving any bids, the auction first chooses some fixed price p_{on} randomly by using the cumulative distribution $q(\cdot)$. The supply curve is then simply $p(x) = p_{\text{on}}$, i.e. the auction sells, to each player, all the goods with value of at least p_{on} to him, with price p_{on} for each good (until all the goods are sold).

This auction is incentive compatible in the following strong sense: for *any* result of the randomized choice, a player will maximize his utility by declaring his true valuation. This is because the randomized choice actually determines a supply curve independently of his bid. We note that it is possible to consider a weaker notion of incentive compatibility, in which a player will maximize his *expected* utility (with respect to the distribution of the randomized choice) by declaring his true valuation.

The following theorem shows that this auction is c-competitive with respect to its *expected* revenue and social efficiency. That is, in some cases the on-line revenue and efficiency will not be within a factor of $1/c$ of the optimal efficiency. But, for *any* particular valuation sequence, the expected on-line revenue and efficiency is within a factor of $1/c$ of the optimal efficiency.

Theorem 4. *For any sequence of valuations σ, the expected revenue of the randomized auction is at least $1/c$ times the optimal efficiency, i.e. $E(R_{\text{on}}(\sigma)) \geqslant E_{\text{opt}}(\sigma)/c$.*

Proof. Suppose OPT allocates the k goods to players with valuations $v_1, \ldots, v_k$ (possibly several goods to the same player) such that $v_i \geqslant v_{i+1}$. For convenience assume $v_0 = \bar{p}$, $v_{k+1} = \underline{p}$. Let p_{on} be the actual price determined by the on-line auction. For the specific i such that $v_{i+1} \leqslant p_{\text{on}} \leqslant v_i$, the on-line auction sells at least i goods, and thus its revenue is at least $i \cdot p_{\text{on}} + \underline{p}(k-i)$. Denote the density function that the auction uses by $f(x) = \mathrm{d}[q(x)]/\mathrm{d}x$. Therefore we have:

$$E(R_{\text{on}} | v_{i+1} \leqslant p_{\text{on}} \leqslant v_i) \geqslant \int_{v_{i+1}}^{v_i} i \cdot x \cdot \frac{f(x)}{Pr(v_{i+1} \leqslant p_{\text{on}} \leqslant v_i)} \,\mathrm{d}x + \underline{p}(k-i)$$

and thus

$$E(R_{\text{on}}) \geqslant \sum_{i=0}^{k} \left[i \cdot \int_{v_{i+1}}^{v_i} x f(x) \,\mathrm{d}x + \underline{p}(k-i) \cdot Pr(v_{i+1} \leqslant p_{\text{on}} \leqslant v_i) \right]$$

$$= \sum_{i=0}^{k} \left[i \cdot \int_{v_{i+1}}^{v_i} x f(x) \,\mathrm{d}x + \underline{p}(k-i)(q(v_i) - q(v_{i+1})) \right].$$

Changing the summation order of the right-hand side we get

$$E(R_{\text{on}}) \geqslant \sum_{i=1}^{k} \left[\int_{\underline{p}}^{v_i} x f(x) \,\mathrm{d}x \right] + \sum_{i=1}^{k} [\underline{p}(q(v_0) - q(v_i))]$$

$$= \sum_{i=1}^{k} \left[\int_{\underline{p}}^{v_i} x f(x) \,\mathrm{d}x + \underline{p}(1 - q(v_i)) \right] = \sum_{i=1}^{k} \frac{v_i}{c} = E_{\text{opt}}(\sigma)/c,$$

where the last equality follows from the specific character of the function $q(x)$, as stated in Lemma 4, and thus the claim follows. □

3.3. A deterministic auction for k indivisible goods

We next examine the deterministic case. First consider the case of $k = 1$. It follows from Theorem 1 that for this case the on-line auction must fix some reservation price p_i for the ith player, i.e. the good is sold to the ith player for price p_i if $v_i(1) > p_i$. This is similar to the search algorithm of [6], where it is shown that a reservation price of $\sqrt{\bar{p} \cdot \underline{p}}$ is $\sqrt{\phi}$-competitive. It is not hard to verify that this is optimal.

The general case for any $k \geqslant 1$ may be handled similarly:

Definition 8 (The discrete on-line auction). The discrete on-line auction is based on the following global supply curve:

$$p(j) = \underline{p} \cdot \phi^{j/(k+1)}, \quad \text{for } j = 1, \ldots, k. \tag{3}$$

The following theorems state the competitiveness of this auction, and give a lower bound for this case. We give the proofs in the appendix.

Theorem 5. *The discrete on-line auction is $k \cdot \phi^{1/(k+1)}$-competitive with respect to the revenue and to the social efficiency. When $k \geqslant 2 \cdot \ln \phi$ then the discrete on-line auction is also $2 \cdot e \cdot (\ln(\phi) + 1)$-competitive with respect to the revenue and to the social efficiency.*

Theorem 6. *Any incentive compatible on-line auction of k goods has a competitive ratio of at least* $m = \max\{\phi^{1/(k+1)}, c\}$ *with respect to the revenue and to the efficiency, where c is as defined in Eq.* (1).

Remark. When considering a partially on-line model, in which the number of players, n, is known in advance, this lower bound weakens, becoming dependent in n. For example, consider the following auction of one good to two players: The price for the first bidder is $\underline{p} \cdot \phi^{2/3}$, while the price for the second bidder is $\underline{p} \cdot \phi^{1/3}$. It is easy to verify that this is $\phi^{1/3}$-competitive with respect to the revenue. As can be seen from the lower bound proof, as long as $n < k+2$ then a similar improvement (with respect to the revenue) is possible.

4. Model extensions

We now discuss some natural extensions to our on-line model, incorporating the following time considerations:

1. *Delayed bidding*: Player i learns his valuation at time t_i, and his strategy space allows placing his bid at any time $t \geqslant t_i$.
2. *Split bidding*: Player i's strategy space allows placing several bids at any time $t_{i_1}, \ldots, t_{i_l} \geqslant t_i$.
3. The Players' valuations may be time-dependent (in a non-increasing way). Specifically, player i's valuation is given by $v_i(q,t)$, where $v_i(\cdot)$ is non-increasing both in q and in t. $v_i(q,t)$ is player i's marginal valuation of the qth good at time t.

A truthful bid is still considered as bidding the true valuation exactly once, at time t_i.

We note that even under any of these extensions, when the supply curves are non-decreasing over time there is no possible gain for a player from delaying his bid. Clearly, a non-decreasing global supply curve holds this property. Thus we conclude:

Theorem 7. *Any on-line auction that is based on a non-decreasing global supply curve is incentive compatible even in any of these extensions.*

Thus all our auctions remain truthful. Their competitiveness also remains, since the off-line Vickrey allocation is not affected by the on-line assumptions. The lower bounds we have shown still obviously remain true. In fact it turns out that they even generalize to partially on-line auctions (where the number of players is known in advance).

5. Revenue analysis for the uniform distribution

We compare the expected revenue of the competitive on-line auction to the expected revenue of the Vickrey off-line auction for a divisible good in the special case of fixed marginal valuations uniformly distributed in $[\underline{p}, \bar{p}]$. This is a simple example that demonstrates that the on-line revenue is similar to the Vickrey revenue in some cases.

Table 1
On-line and Vickrey revenue in the average case

	On-line revenue	Vickrey revenue
$\bar{p}=1.5,\ n=2$	1.15	1.17
$\bar{p}=3,\ n=2$	1.60	1.67
$\bar{p}=10,\ n=2$	3.33	4.00
$\bar{p}=2,\ n=2$	1.31	1.33
$\bar{p}=2,\ n=3$	1.37	1.50
$\bar{p}=2,\ n=100$	1.56	1.98

Table 1 compares the revenue of the on-line auction to the revenue of the Vickrey auction for several values of n and $\bar{p}$, where $\underline{p}=1$. The computation details are given in the appendix. From the table, we see that for small values of n and $\bar{p}$ the on-line revenue is close to the Vickrey revenue. When n increases, the Vickrey to on-line revenue ratio approaches c, the competitive ratio.

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Appendix A.

A.1. Proof of Theorem 5

Theorem 5. *The discrete on-line auction is $k \cdot \phi^{1/(k+1)}$-competitive with respect to the revenue and to the social efficiency. When $k \geqslant 2 \cdot \ln \phi$ then the discrete on-line auction is also $2 \cdot e \cdot (\ln(\phi) + 1)$-competitive with respect to the revenue and to the social efficiency.*

Proof. Fix some scenario and suppose that the on-line auction sold q goods. We first prove the claim with respect to the revenue. Since the on-line auction sold q goods, the valuation of one additional good of any player is at most $p(q+1) = \underline{p} \cdot \phi^{(q+1)/(k+1)}$. Therefore the Vickrey auction may charge a unit price of at most $p(q+1)$, thus the Vickrey to on-line revenue ratio is at most:

$$\frac{k \cdot p(q+1)}{\sum_{j=1}^{q} p(j) + (k-q) \cdot \underline{p}} = \frac{k \cdot \phi^{(q+1)/(k+1)}}{\sum_{j=1}^{q} \phi^{j/(k+1)} + (k-q)}$$

$$\leqslant \frac{k \cdot \phi^{(k+1)/(k+1)}}{\sum_{j=1}^{k} \phi^{j/(k+1)}},$$

where the inequality follows from the fact that for any q, $0 \leqslant q \leqslant k-1$,

$$\frac{k \cdot \phi^{(q+1)/(k+1)}}{\sum_{j=1}^{q} \phi^{j/(k+1)} + (k-q)} \leqslant \frac{k \cdot \phi^{(q+2)/(k+1)}}{\sum_{j=1}^{q+1} \phi^{j/(k+1)} + (k-q-1)}$$

(since $k-q \geqslant 1+\frac{k-q-1}{\phi^{1/(k+1)}}$). The first part of the claim follows since $\frac{k \cdot \phi^{(k+1)/(k+1)}}{\sum_{j=1}^{k} \phi^{j/(k+1)}} \leqslant \frac{k \cdot \phi^{(k+1)/(k+1)}}{\phi^{k/(k+1)}} = k \cdot \phi^{1/(k+1)}$. For the second part of the claim, let $l^* = (k+1)/\ln\phi - 1$. If $\ln\phi < 1$ then $2 \cdot e \cdot (\ln(\phi)+1) > e > \phi$ and the claim is trivial since any auction is ϕ-competitive. Otherwise $\ln\phi \geqslant 1$, and $1 \leqslant l^* \leqslant k$ since $k \geqslant 2 \cdot \ln\phi$. We claim that:

$$\sum_{j=1}^{k} \phi^{j/(k+1)} \geqslant l^* \cdot \phi^{(k+1-l^*)/(k+1)}.$$

Clearly this is true for any integer l. Let the right-hand function be $f(l)$. It receives its maximum for l^*+1 and it is increasing in $[l^*, l^*+1]$. Thus for some integer $x \in [l^*, l^*+1]$ it holds that $\sum_{j=1}^{k} \phi^{j/(k+1)} \geqslant f(x) \geqslant f(l^*)$. Thus:

$$\frac{k \cdot \phi}{\sum_{j=1}^{k} \phi^{j/(k+1)}} \leqslant \frac{k \cdot \phi}{l^* \cdot \phi^{(k+1-l^*)/(k+1)}} = \frac{k}{l^*} \cdot \phi^{(l^*)/(k+1)}$$

$$= \frac{k \cdot \ln\phi}{k+1-\ln\phi} \cdot \phi^{1/\ln\phi - 1/(k+1)} \leqslant 2 \cdot e \cdot \ln\phi,$$

where the last inequality follows from the fact that $k \geqslant 2 \cdot \ln\phi$, and $\phi^{1/\ln\phi} = e$.

This proves the claim for the revenue. Now consider the efficiency case. Given a valuation sequence $\sigma = (b_i(q))$, consider a new sequence $\sigma^* = (b_i^*(q))$ built in a similar manner to that of Lemma 7, i.e. player i has fixed marginal valuation up to $q = q_i$ and then as before. By similar arguments to those of Lemma 7, the off-line to on-line efficiency ratio of the new sequence is an upper bound to the ratio of the original sequence (since the off-line efficiency decrease is no more than the on-line decrease). Additionally, the on-line allocation for the two scenarios is identical. Let $\sum_i q_i = q \leqslant k$. Since $b_i^*(q) \leqslant p(q+1)$ then $E_{\text{opt}}(\sigma^*) \leqslant k \cdot p(q+1)$. Clearly $E_{\text{on}}(\sigma^*) \geqslant R_{\text{on}}(\sigma^*) \geqslant \sum_{j=1}^{q} p(j)$, where "on" is the discrete on-line auction. Thus

$$\frac{E_{\text{opt}}(\sigma^*)}{E_{\text{on}}(\sigma^*)} \leqslant \frac{k \cdot p(q+1)}{\sum_{j=1}^{q} p(j)} \leqslant 2 \cdot e \cdot \ln\phi,$$

where the last inequality was shown above for the revenue claim. □

A.2. Proof of Theorem 6

Theorem 6. *Any incentive compatible on-line auction of k goods has a competitive ratio of at least* $m = \max\{\phi^{1/(k+1)}, c\}$ *with respect to the revenue and to the efficiency, where c is as defined in Eq.* (1).

Proof. We prove the claim for the special case of fixed marginal valuations and assume, according to Lemma 3, that A is based on non-decreasing supply curves. We prove each lower bound separately:

Lemma 9. *Any incentive compatible on-line auction of k goods has a competitive ratio of at least $\phi^{1/(k+1)}$ with respect to the revenue and to the efficiency.*

Proof. Fix some incentive compatible on-line auction A. Consider the behavior of A for the sequence of players: $\phi^{1/(k+1)}, \phi^{1/(k+1)}, \phi^{2/(k+1)}, \phi^{2/(k+1)}, \ldots, \phi, \phi$ (i.e. $2(k+1)$ players). Let q be the first i such that both players with valuation $\phi^{i/(k+1)}$ does not receive any positive quantity (there is such q since there are k goods and $k+1$ pairs of players). Denote by σ the above sequence with only the first $2(q+1)$ players. Vickrey's revenue is $k \cdot \phi^{q/(k+1)}$, while A's efficiency is at most $k \cdot \phi^{(q-1)/(k+1)}$. Thus $R_A(\sigma) \leqslant E_A(\sigma) \leqslant R_{\text{vic}}(\sigma)/(\phi^{1/(k+1)}) = E_{\text{vic}}(\sigma)/(\phi^{1/(k+1)})$, and the claim follows. □

Lemma 10. *Any incentive compatible on-line auction has a competitive ratio of at least c with respect to the revenue and to the efficiency.*

Proof. The claim follows from the fact that for the special case of fixed marginal valuations the result of the Vickrey auction for the indivisible case is the same as for the divisible case (i.e. a single player receives all the good(s) and pays the second price). Thus, if there was an on-line auction for k indivisible goods with a competitive ratio $\tilde{c} < c$ it can be used for the divisible case (i.e. allocating quantity multiples of $1/k$), achieving the same competitive ratio $\tilde{c}$. This is in contradiction to Theorem 3, since the lower bound there holds for the special case of fixed marginal valuations. □

A.3. The expected revenue analysis

We consider the special case of fixed marginal valuations uniformly distributed in $[a,b]$ (for $a>0$). Let $f(x) = 1/(b-a), F(x) = (x-a)/(b-a)$ be the distribution function and the cumulative distribution function, respectively, and assume that the players' utilities are independent. For this case, the revenue of the on-line auction is determined by the maximal marginal utility. Its distribution function for n players is $g_n(x) = nf(x)(F(x))^{n-1}$. Let $c = c(b/a)$ be the appropriate competitive ratio. Then,

$$\int g_n(x)\,\frac{x}{c}\,\mathrm{d}x = \frac{n}{c(b-a)^n}\int (x-a)^{n-1} x\,\mathrm{d}x$$

$$= \frac{n}{c(b-a)^n}\left(\frac{(x-a)^{n+1}}{n+1} + a\,\frac{(x-a)^n}{n}\right)$$

$$E(R_{\text{on}}) = \int_a^{ca} g_n(x) a \,\mathrm{d}x + \int_{ca}^{b} g_n(x) \frac{x}{c} \,\mathrm{d}x$$

$$= \int_a^{b} g_n(x) \frac{x}{c} \,\mathrm{d}x + \int_a^{ca} g_n(x) \left(a - \frac{x}{c}\right) \mathrm{d}x$$

$$= \frac{n}{n+1} \frac{b}{c} + \frac{1}{n+1} \frac{a}{c} + \varepsilon.$$

The ε addition is relatively small, e.g. for $a=1,\ b=2,\ n=2$ it is lower than 0.006.

The distribution function for the second maximal price is

$$h_n(x) = 2f(x) \binom{n}{n-2} (F(x))^{n-2}(1-F(x))$$

$$= n(n-1)(F(x))^{n-2}(1-F(x)),$$

$$\int h_n(x) x \,\mathrm{d}x = \frac{n(n-1)}{(b-a)^{n-1}} \left(\int (x-a)^{n-2} x \,\mathrm{d}x - \frac{1}{b-a} \int (x-a)^{n-1} x \,\mathrm{d}x \right),$$

which is solved in a similar manner to the previous integral, thus the expected revenue of the Vickrey auction (see also [15, p. 57]) is:

$$E(R_{\text{vic}}) = \int_a^b h_n(x) x \,\mathrm{d}x = \frac{n-1}{n+1} b + \frac{2}{n+1} a.$$

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