Randomized Algorithms: Quicksort and Selection

Version of October 2, 2014

Outline

Outline:

- Quicksort
  - Average-Case Analysis of QuickSort
  - Randomized quicksort
- Selection
  - The selection problem
  - First solution: Selection by sorting
  - Randomized Selection

Quicksort: Review

Quicksort(A, p, r)

```
begin
    if p < r then
        q = Partition(A, p, r);
        Quicksort(A, p, q − 1);
        Quicksort(A, q + 1, r);
    end
end
```

- *Partition(A, p, r)* reorders items in $A[p \dots r]$; items $< A[r]$ are to its left; items $> A[r]$ to its right.
- Showed that if input is a random input (permutation) of n items, then average running time is $O(n \log n)$

Average Case Analysis of Quicksort

- Formally, the average running time can be defined as follows:
  - $\mathcal{I}_n$ is the set of all $n!$ inputs of size n
  - $I \in \mathcal{I}_n$ is any particular size-n input
  - $R(I)$ is the running time of the algorithm on input I
- Then, the average running time over the random inputs is

$$\sum_{I \in \mathcal{I}_n} \Pr(I) R(I) \quad = \quad \frac{1}{n!} \sum_{I \in \mathcal{I}_n} R(I) \quad = \quad O(n \log n)$$

- Only fact that was used was that $A[r]$ was a random item in $A[p \ldots r]$, i.e., the partition item is equally likely to be any item in the subset.

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Randomized-Partition(A, p, r)

Idea:

- In the algorithm Partition(A, p, r), $A[r]$ is always used as the pivot x to partition the array $A[p..r]$
- In the algorithm Randomized-Partition(A, p, r), we randomly choose j, $p \leq j \leq r$, and use $A[j]$ as pivot
- Idea is that if we choose randomly, then the chance that we get unlucky every time is extremely low.

Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r

Randomized-Partition(A, p, r)

begin
  j = random(p, r);
  exchange $A[r]$ and $A[j]$;
  Partition(A, p, r);
end

Randomized-Quicksort Algorithm

We make use of the Randomized-Partition idea to develop a new version of quicksort

Randomized-Quicksort(A, p, r)

begin
  if $p < r$ **then**
    $q =$ Randomized-Partition(A, p, r);
    Randomized-Quicksort$(A, p, q - 1)$;
    Randomized-Quicksort$(A, q + 1, r)$;
  end
end

Running Time of Randomized-Quicksort

Let $I \in \mathcal{I}_n$ be *any* input.

- The running time $R(I)$ depends upon the random choices made by the algorithm in the step

  random(p, r**); exchange** $A[r]$ **and** $A[j]$

- This can be different for different random choices.

- We are actually interested in $E(R(I))$, the *Expected (average) Running Time (ERT)*
  - average now is not over the input, which is fixed
  - average is over the random choices made by the algorithm.

Running Time of Randomized-Quicksort

Let $I \in \mathcal{I}_n$ be *any* input.

Want $E(R(I))$, the *Expected Running Time*, where average is taken over random choices of algorithm.

Suprisingly, we can use almost exactly the same analysis that we used for the average-case analysis of Quicksort. Recall that only facts that we used were

- Item used as a pivot is random among all items
- this statement is true in all subproblems as well.

Those two facts are still valid here, so the expected running time still satisfies

$$C_n = n - 1 + \frac{1}{n} \sum_{1 \le k \le n} (C_{k-1} + C_{n-k})$$

which we already proved was $O(n \log n)$.

Running Time of Randomized-Quicksort

- Just saw that for any fixed input of size n, ERT is $O(n \log n)$
- Randomized Quicksort is a Randomized Algorithm
  - Makes Random choices to determine what algorithm does next
  - When rerun on same input, algorithm can make different choices and have different running times
  - Running time of Randomized Algorithm is worst case ERT over all inputs I. In our case

$$\max_{I \in \mathcal{I}_n} E[R(I)] = O(n \log n)$$

- Contrast with Average Case Analysis
  - When rerun on same input, algorithm *always* does same things, so $R(i)$ is deterministic.
  - Given a probability distribution on inputs, calculate average running time of algorithm over all inputs

$$\sum_{I \in \mathcal{I}_n} \Pr(I) R(I)$$

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The Selection Problem

Definition (Selection Problem)

Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, and an integer i, $1 \leq i \leq n$, find the ith smallest element. When $i = \lceil n/2 \rceil$, this is called the median problem.

Example

Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

Question

How can this problem be solved efficiently?

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First Solution: Selection by Sorting

1. Sort the elements in ascending order with any algorithm of complexity $O(n \log n)$.
2. Return the ith element of the sorted array.

The complexity of this solution is $O(n \log n)$

Question

Can we do better?

Answer: YES, by using Randomized-Partition(A, p, r)!

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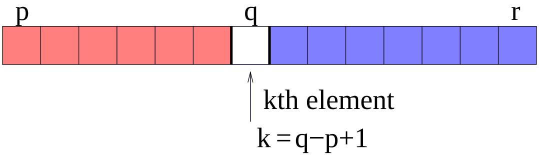
Randomized-Select(A, p, r, i), $1 \leq i \leq r - p + 1$

Problem: Select the ith smallest element in $A[p..r]$, where $1 \leq i \leq r - p + 1$

Solution: Apply Randomized-Partition(A, p, r), getting



1. $i = k$
   - pivot is the solution
2. $i < k$
   - the ith smallest element in $A[p..r]$ must be the ith smallest element in $A[p..q-1]$
3. $i > k$
   - the ith smallest element in $A[p..r]$ must be the $(i-k)$th smallest element in $A[q+1..r]$

If necessary, recursively call the same procedure to the subarray

Randomized-Select(A, p, r, i), $1 \le i \le r - p + 1$

if $p = r$ **then**
  return $A[p]$
end
$q =$ Randomized-Partition(A, p, r) ;
$k = q - p + 1$;
if $i = k$ **then return** $A[q]$;
`// the pivot is the answer`
else if $i < k$ **then**
  return Randomized-Select$(A, p, q - 1, i)$
else
  return Randomized-Select$(A, q + 1, r, i - k)$
end

To find the ith smallest element in $A[1..n]$, call Randomized-Select$(A, 1, n, i)$

Running Time of Randomized-Select($A, 1, n, i$)

Recall that if pivot q is kth item in order, then algorithm is

If $i = k$, stop. If $i < k \Rightarrow A[p..q-1]$. If $i > k \Rightarrow A[q+1..r]$.

Let $m = p - r + 1$.

Note that if $k = p + \lfloor \frac{m}{2} \rfloor$ was always true, this would halve the problem size at every step and the running time would be at most

$$n + \frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \ldots = n\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right) \leq 2n$$

This isn't a realistic analysis because q is chosen randomly, so k is actually random number between $p..r$.

Running Time of Randomized-Select($A, 1, n, i$)

Recall that if pivot q is kth item in order then algorithm is

If $i = k$, stop. If $i < k \Rightarrow A[p..q-1]$. If $i > k \Rightarrow A[q+1..r]$.

Let $m = p - r + 1$.

Suppose that we could *guarantee* that $p + \frac{m}{4} \leq k \leq p + \frac{3}{4}m$.

This would be enough to force linearity because the recursive call would always be to a subproblem of size $\leq \frac{3}{4}m$ and the running time of the entire algorithm would be at most

$$n + \frac{3}{4}n + \left(\frac{3}{4}\right)^2 n + \left(\frac{3}{4}\right)^3 n + \ldots \leq 4n$$

Running Time of Randomized-Select$(A, 1, n, i)$

Set $m = p - r + 1$. We saw that if

$$p + \frac{m}{4} \le k \le p + \frac{3}{4}m$$

then algorithm is linear.

While this is *not* always true, we *can* easily see that

$$\Pr\left(p + \frac{m}{4} \ \le \ k \ \le \ p + \frac{3}{4}m\right) \ge \frac{1}{2}.$$

This means that each stage of the algorithm has probability at least $1/2$ of reducing the problem size by $3/4$.
A careful anlysis will show that this implies an $O(n)$ expected running time.

Running Time of Randomized-Select$(A, 1, n, i)$

More formally, suppose t'th call to the algorithm is $A(p_t, r_t, i_t)$. Let $M_t = r_t - p_t + 1$ be size of array in the subproblem and k_t location of the random pivot in that subarray. Note

- $p_1 = 1,\ r_1 = n,\ M_1 = n$
- $M_{t+1} \leq M_t - 1$
- Total cost of the algorithm is bounded by $\sum_t M_t$
- Set E_t to be event that is true if

$$p_t + \frac{M_t}{4} \leq k_t \leq p_t + \frac{3}{4} M_t,$$

  and false otherwise. Then

  - $\Pr(E_t) \geq 1/2$
  - If E_t occurs then $M_{t+1} \leq \frac{3}{4} M_t$.

Running Time of Randomized-Select$(A, 1, n, i)$

Recall that

$M_1 = n; \quad M_{t+1} \le M_t - 1; \quad \text{If } E_t \quad \Rightarrow \quad M_{t+1} \le \frac{3}{4} M_t.$

Note that E_t is undefined after the algorithm ends, i.e., $M_t \le 1$. For larger t, define E_t by flipping fair coin and setting E_t True if HEAD seen.

Now define M'_t as follows

- $M'_1 = n$
- If $E_t \Rightarrow M'_{t+1} = \frac{3}{4} M'_t$. If (not E_t) $\Rightarrow M'_{t+1} = M'_t$.

Then $\forall t, \quad M_t \le M'_t$.

In particular, since $\sum_t M_t$ bounds the algorithm's runtime, $\sum_t M'_t$ also bounds the algorithm's runtime!

Review of Geometric Random Variables

Consider a p-biased coin, i.e., a coin with with probability p of turning up Heads and $(1-p)$ of Tails.

- Let X be the number of flips until seeing the first Head
- X is a *Geometric Random Variable* with parameter p
- $\Pr(X = i) = (1-p)^{i-1}p$
- $E(X) = \frac{1}{p}$
- In particular, if the coin is fair, i.e., $p = 1/2$, then $E(X) = 2$
- If at every step the coin probability can change,
  BUT the probability of Heads is always $\geq 1/2$,
  then $E(X) \leq 2$.
- In this case we say X is *bounded* by a geometric random variable with $p = 1/2$

Running Time of Randomized-Select$(A, 1, n, i)$

Given sequence of events $E_1, E_2, E_3, \ldots$ with $\forall t$, $\Pr(E_t) \geq 1/2$

- Set $Z_0 = 1$ and Z_i to be the location of the i^{th} true E_t.
- Set $X_i = Z_{i+1} - Z_i$.
  - X_i is time from Z_i until next success so it is bounded by a geometric random variable with $p = 1/2$.
  - $\Rightarrow$ Then $E(X_i) \leq 2$
- Recall $M_1 = n$; If E_t, set $M_{t+1} = \frac{3}{4} M_t$. Else $M_{t+1} = M_t$.

  Then $\sum_t M'_t = \sum_i X_i \left(\frac{3}{4}\right)^i n$ (why)
- By linearity of expectation

$$E\left(\sum_t M'_t)\right) = \sum_i E(X_i)\left(\frac{3}{4}\right)^i n \leq 2n \sum_i \left(\frac{3}{4}\right)^i = 8n$$

QED

Running Time of Randomized-Select$(A, 1, n, i)$

Worst Case:

$$T(n) = n - 1 + T(n-1), T(n) = O(n^2).$$

Expected Running Time:

$$O(n)$$

Expected running time much better than worst case!

Randomized Quicksort vs Randomized Selection

Question

Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

Answer:

- Randomized Selection needs to work on only one of the two subproblems.
- Randomized Quicksort needs to work on both of the two subproblems.

Epilogue

How do we generate a random number?

Dice, coin flipping, roulette wheels, ...

How does a computer generate a random number?

- By hardware: electronic noise, thermal noise, etc. Expensive but "true" random numbers in some sense
- By software: pseudorandom numbers. A long sequence of seemingly random numbers whose pattern is difficult to find
- Pseudorandom numbers are good enough for most applications

Another Analysis of the Running Time of Randomized-Select($A, 1, n, i$)

$T(n)$: upper bound on the expected number of comparisons made by Randomized-Select($A, 1, n, i$) for any i

$T(1) = 0$

For $n > 1$, we get

$T(n) \leq n$	initial partition
$+\sum_{k=1}^{n}\left(\frac{1}{n} \cdot T(\max\{k-1, n-k\})\right)$	recursion, assume the bad case

$$T(n) \leq n + \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} T(k)$$

Which is a complicated recurrence!

We use the *guess & induction* method

Guess:

$$T(n) \leq c\, n, \quad \text{for all } n$$

for some constant c to be figured out later.

Proof that $T(n) \le c\, n$

Induction step: Assume that $T(m) \le c\, m$ for all $m \le n-1$. Then try to show $T(n) \le cn$:

$$\begin{aligned} T(n) &\le n + \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} T(k) \\ &\le n + \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck \\ & \quad \ldots \\ &\le \frac{3c}{4} n + \frac{c}{2} + n \end{aligned}$$

We want $\frac{3c}{4} n + \frac{c}{2} + n \le cn$, or $n \ge \frac{2c}{c-4}$.
If we choose $c \ge 12$. Then the induction step works for $n \ge 3$.
Induction basis: $T(1) \le c \cdot 1$, $T(2) \le c \cdot 2$.
So if we choose $c = \max\{12, T(1), T(2)/2\}$, then the entire proof works.