

COMP170

Discrete Mathematical Tools for Computer Science

Inclusion-Exclusion

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Discrete Math for Computer Science

K. Bogart, C. Stein and R.L. Drysdale

Section 5.2, pp. 224-233

Unions and Intersections

- The Probability of a Union of Events
- The Principle of Inclusion and Exclusion for Probability
- The Principle of Inclusion and Exclusion for Counting

The Probability of a Union of Events

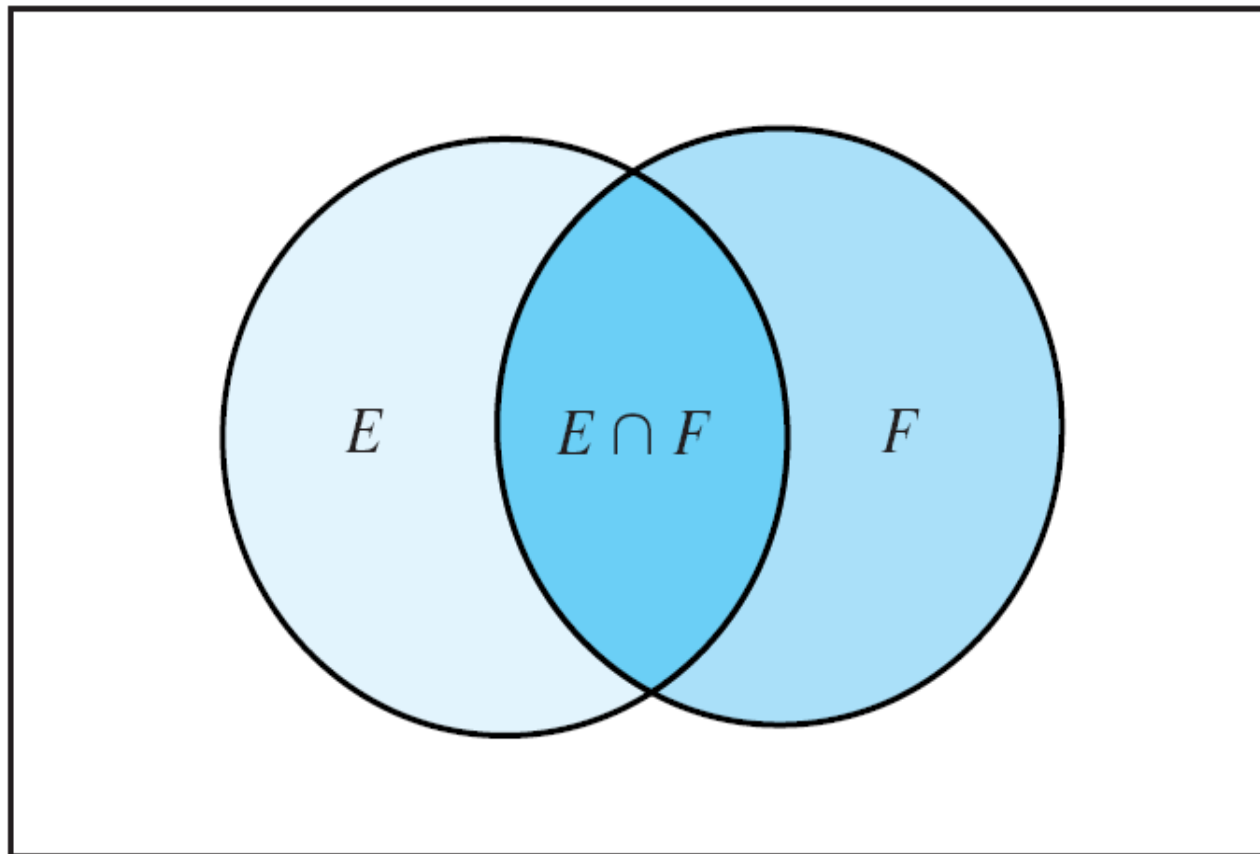
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Venn Diagram

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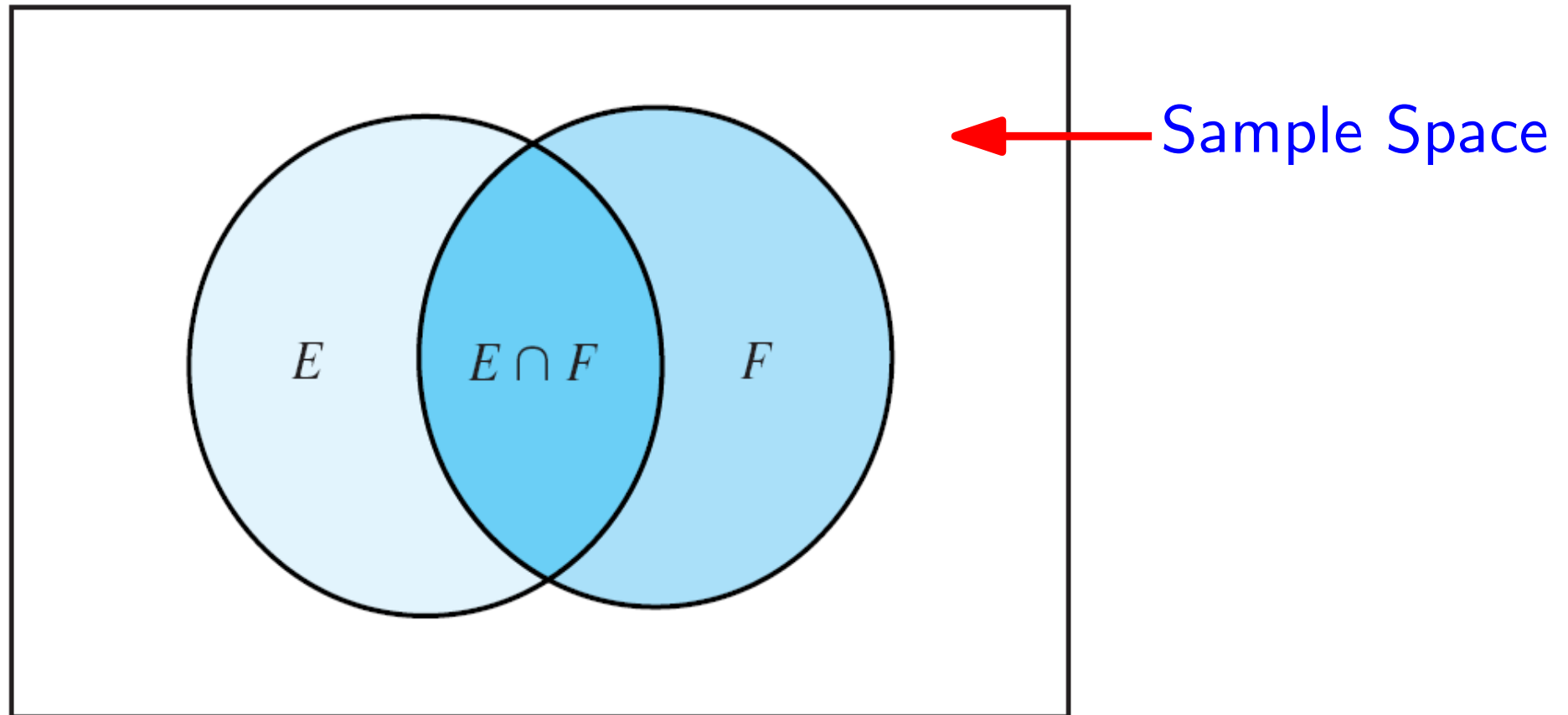
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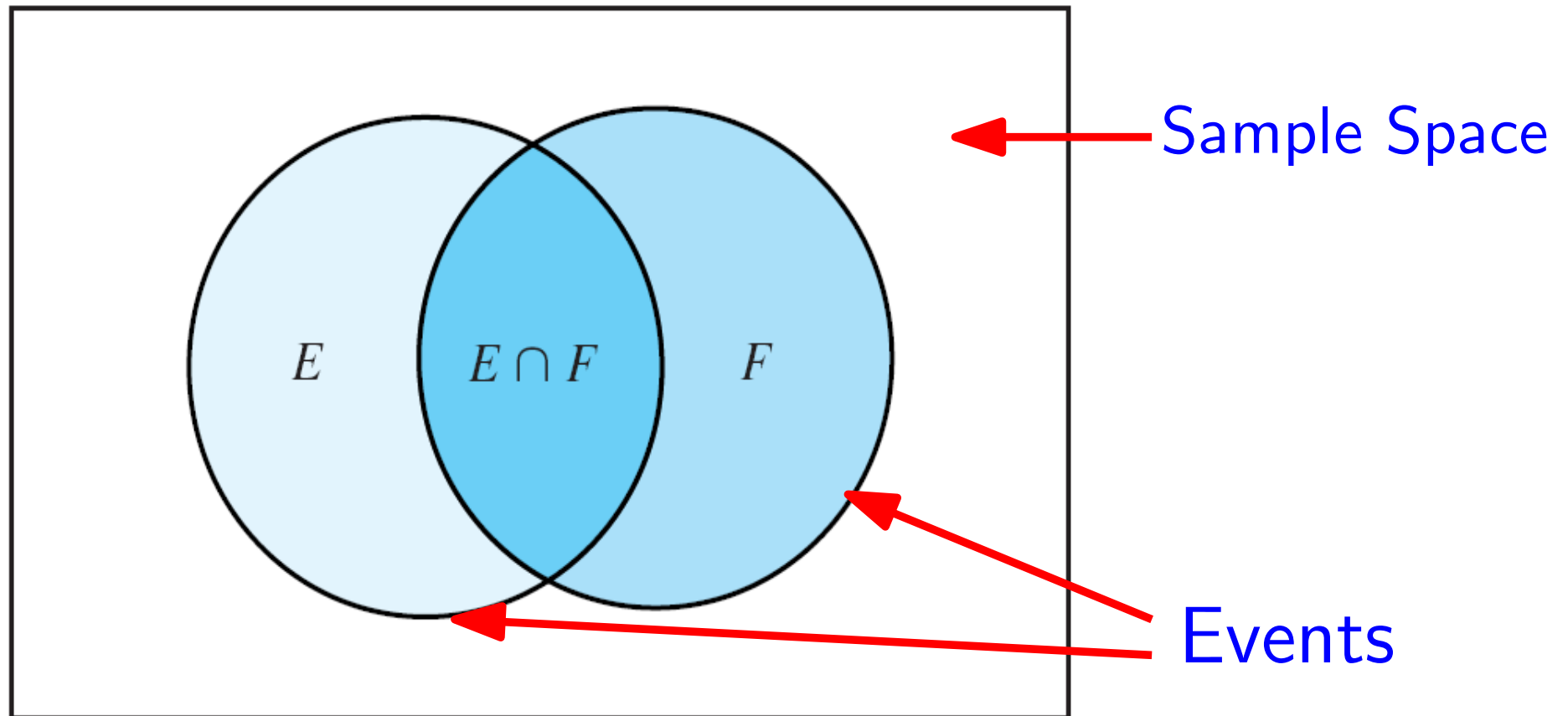
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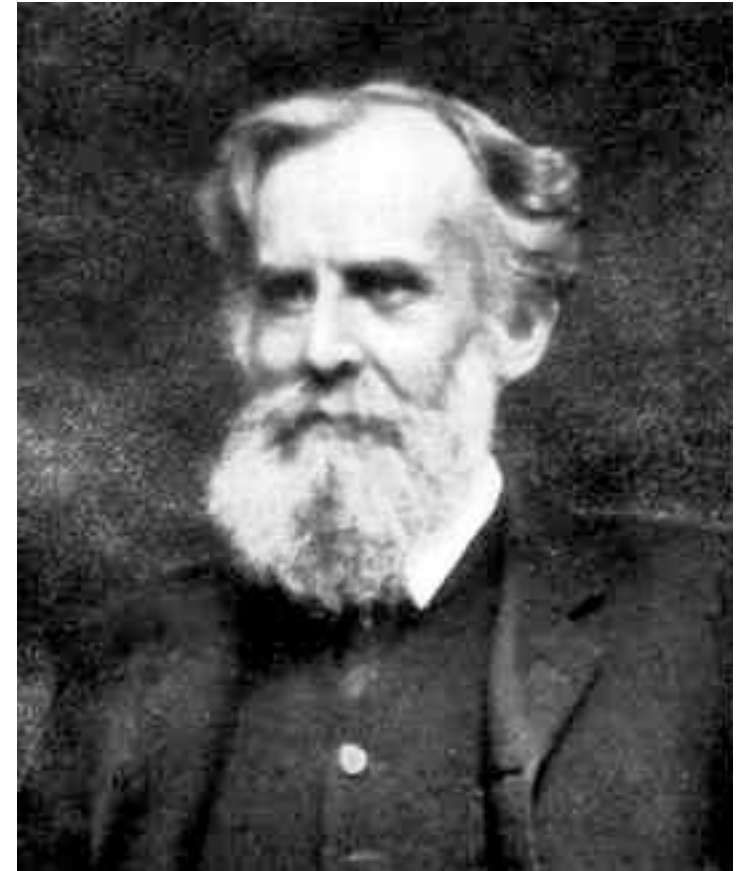
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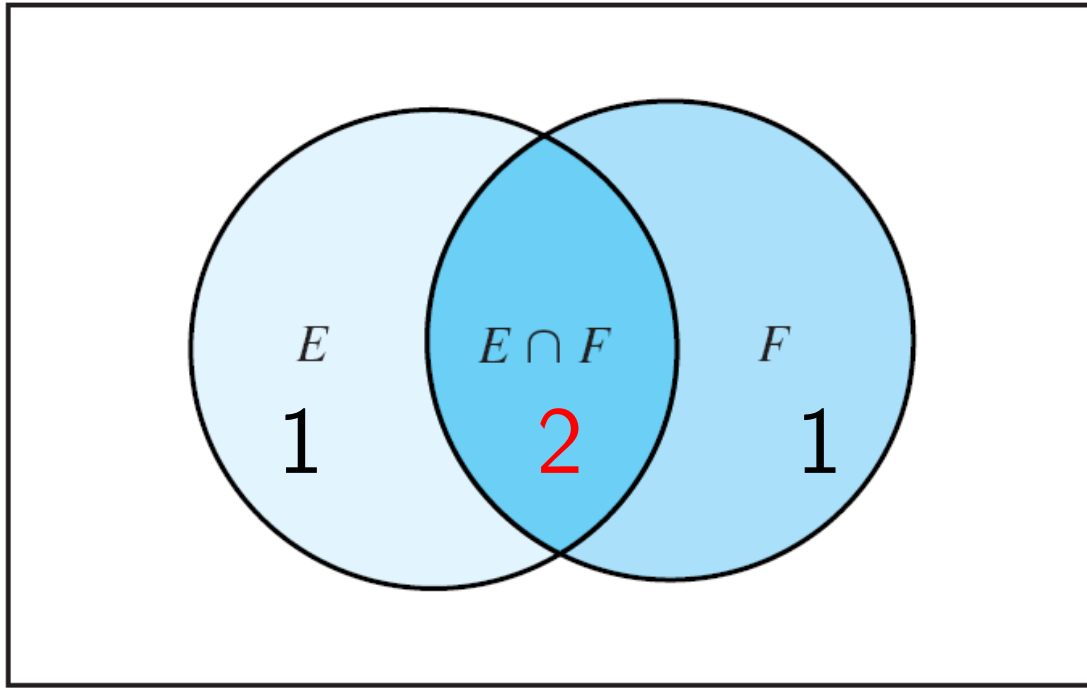
John Venn

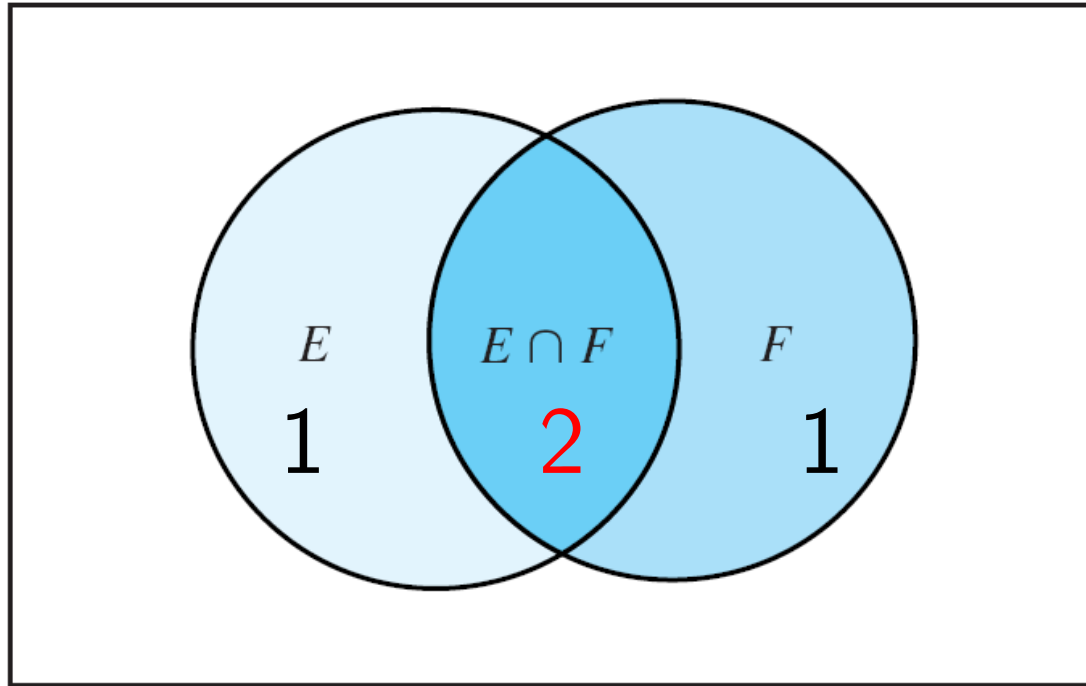
b. 1834, d. 1923

British Mathematician who continued the work of Boole. Although he was not the first person to use diagrams in formal logic, he seems to have been the first to formalize their usage and generalize them.

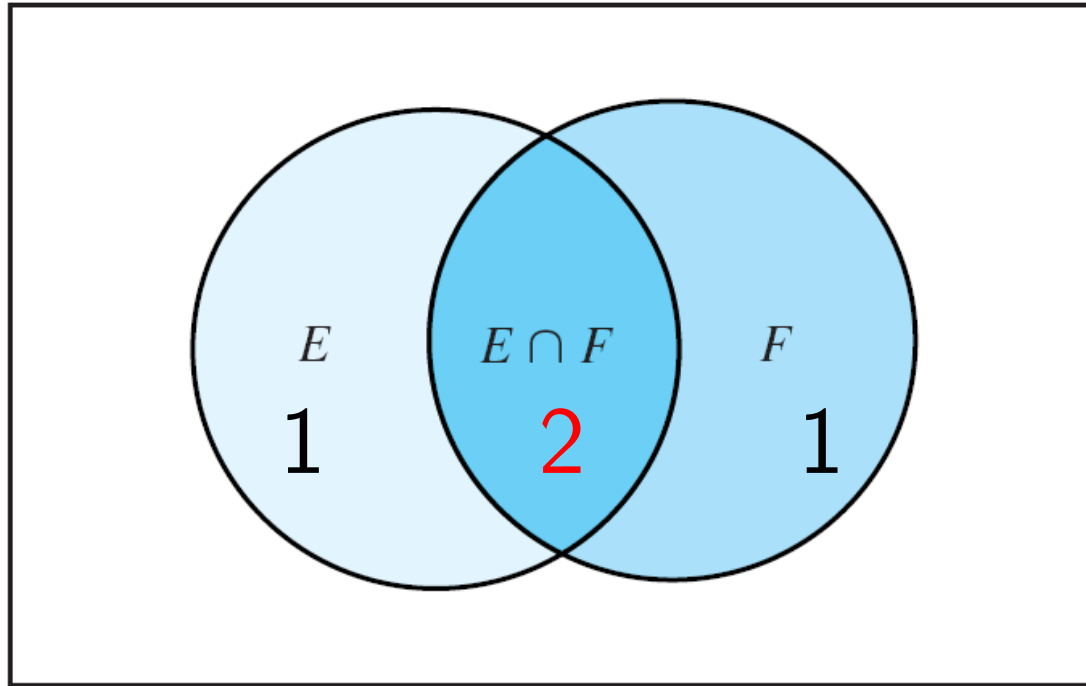


*For more, see the survey of Venn diagrams at
<http://www.combinatorics.org/Surveys/ds5/VennJohnEJC.html>*



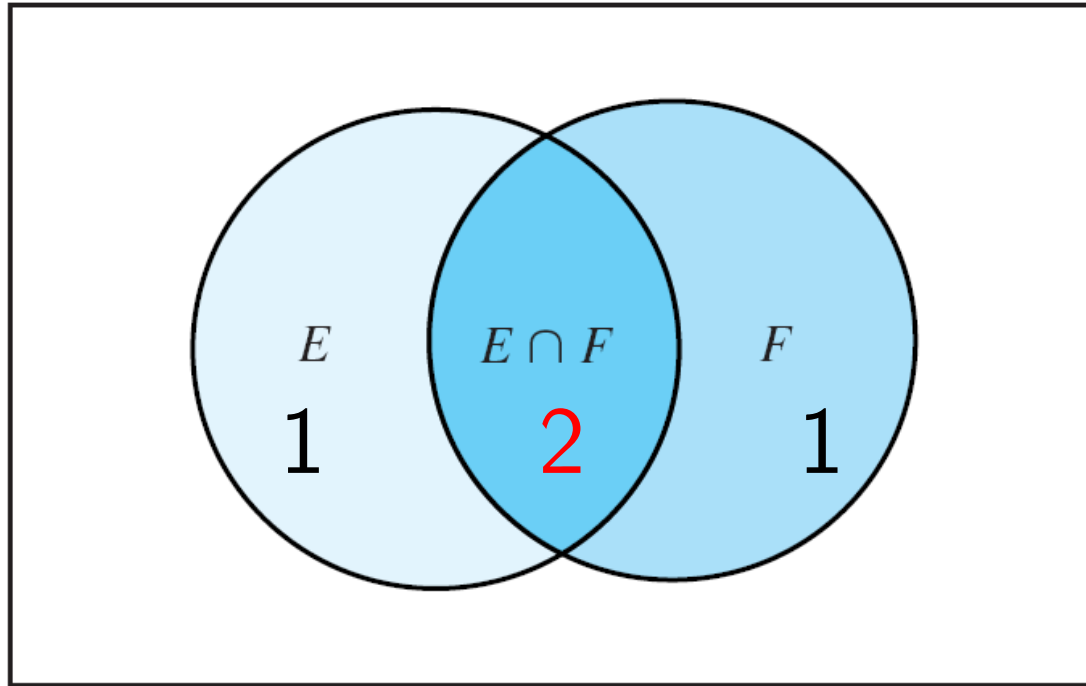


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$$P(E \cup F) = P(E) + P(F) - P(E \cap F) \quad (*)$$

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Probability of even
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$$P(E \cap F) = \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{9}{36}$$

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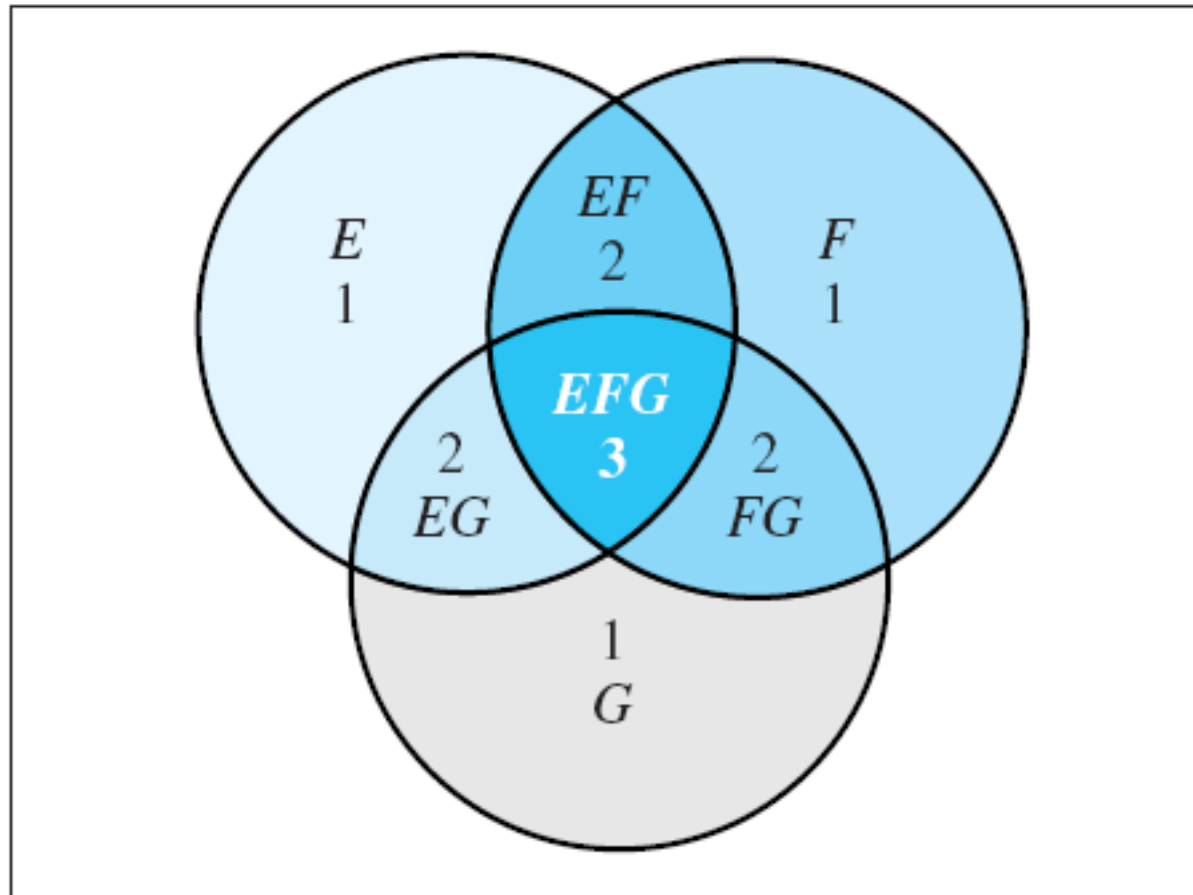
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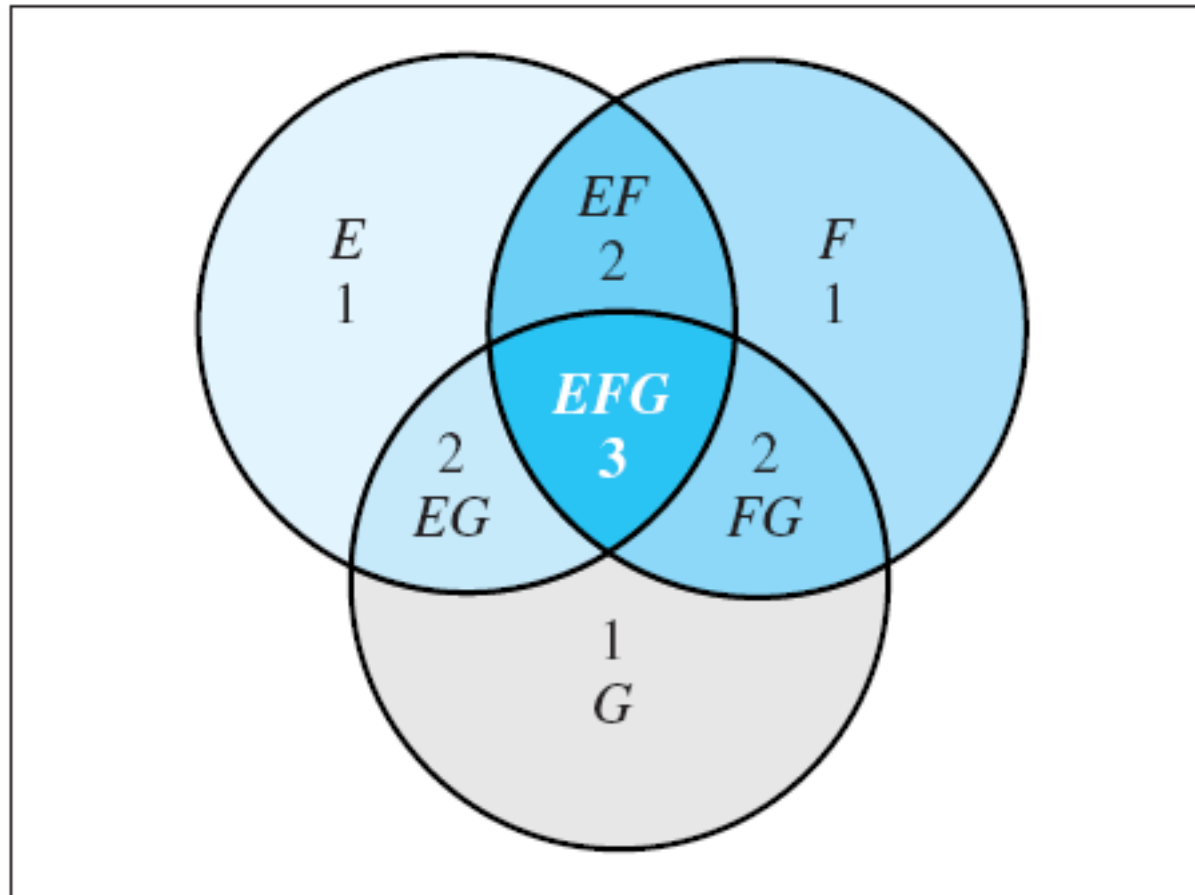
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$$\Rightarrow P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{15}{36} - \frac{9}{36} = \frac{2}{3}$$

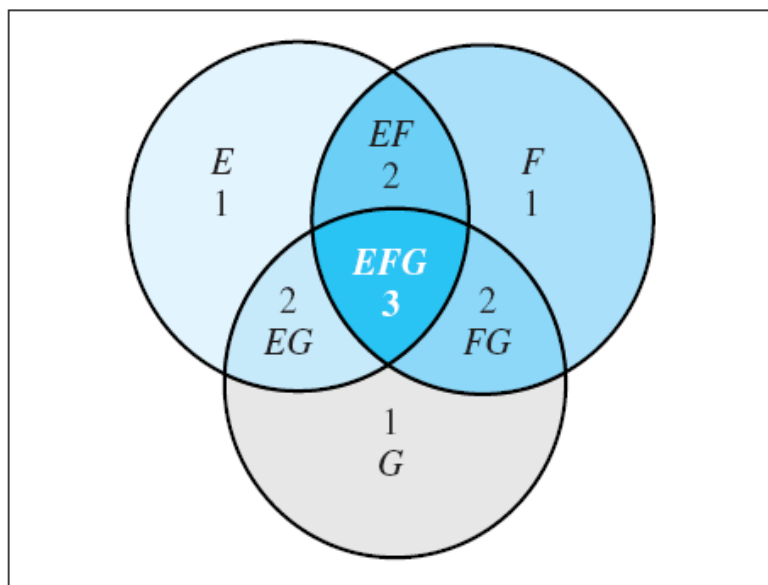
The Union of Three events: $E \cup F \cup G$



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When adding $P(E) + P(F) + P(G)$,
weights of elements in regions $E \cap F$, $F \cap G$, and $E \cap G$
but not $E \cap F \cap G$, are counted exactly twice but
weights of elements in $E \cap F \cap G$,
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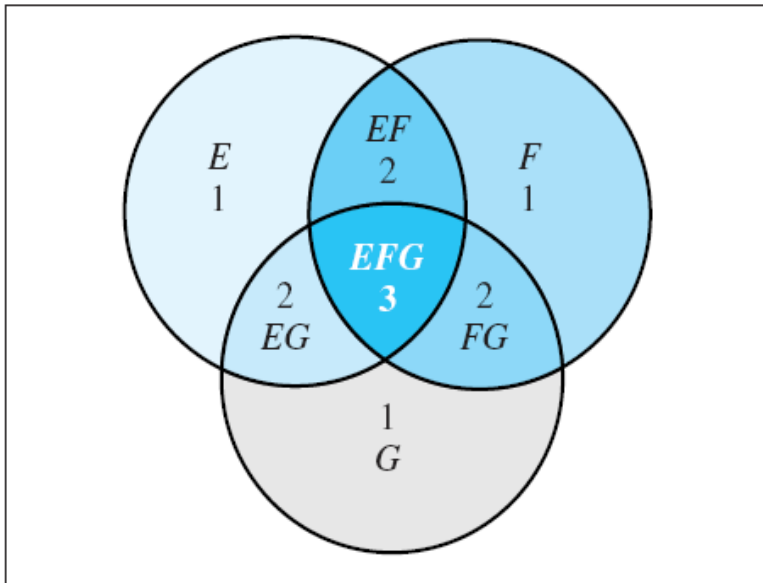
Want to calculate $P(E \cup F \cup G)$.

Start with $P(E) + P(F) + P(G)$.

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Double counts events in EF , EG , FG

Triple counts events in EFG



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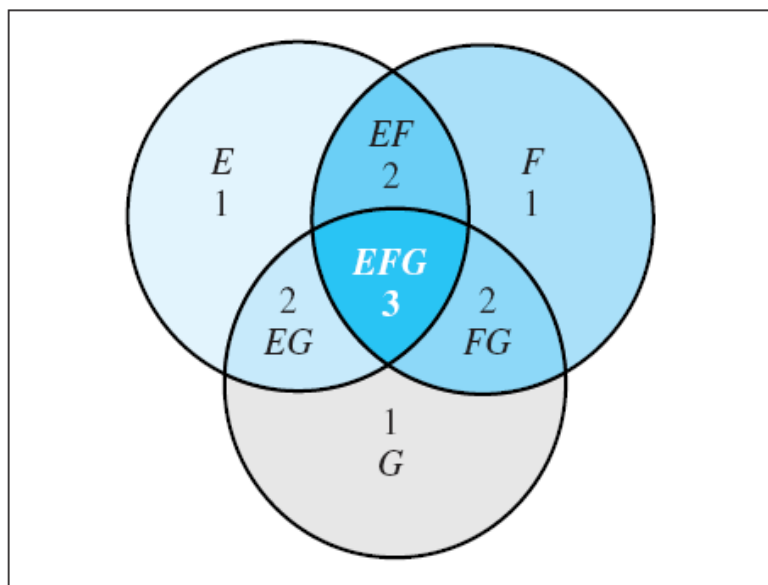
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Subtracting weights of elements of each $E \cap F$, $F \cap G$, and $E \cap G$ doesn't quite work, since this subtracts weights of elements in EF , FG , and EG once (**good**) but also subtracts weights of elements in EFG three times (**bad**).



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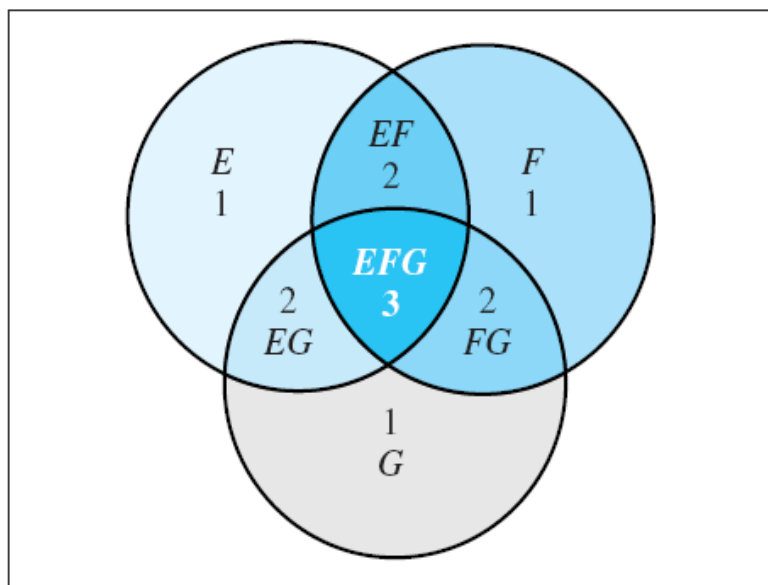
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$$\begin{aligned}
 P(E \cup F \cup G) = & P(E) + P(F) + P(G) \\
 & - P(E \cap F) - P(E \cap G) - P(F \cap G) \\
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We now guess the general formula:

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^n P(E_i \cap E_j) \\ + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n P(E_i \cap E_j \cap E_k) - \dots$$

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More generally:

$$\sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} f(i_1, i_2, \dots, i_k)$$

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Example:

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$$\sum_{\substack{i_1, i_2, i_3: \\ 1 \leq i_1 < i_2 < i_3 \leq 4}} (i_1 + i_2 + i_3) \quad ?$$

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$$\begin{aligned} & (1 + 2 + 3) + (1 + 2 + 4) + (1 + 3 + 4) + (2 + 3 + 4) \\ &= 6 + 7 + 8 + 9 = 30. \end{aligned}$$

Theorem 5.3

(Principle of Inclusion and Exclusion for Probability)

The probability of the union $E_1 \cup E_2 \cup \dots \cup E_n$ of events in a sample space S is given by

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$$

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$$n = 2 \quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

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Base case $n = 2$:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Suppose inductively that for any
family of $n - 1$ sets F_1, F_2, \dots, F_{n-1} ,

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Now assume we have family E_1, E_2, \dots, E_n of n sets.

Set $E = E_1 \cup \dots \cup E_{n-1}$ and $F = E_n$.

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So Far

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First term on RHS is given by i.h.

So Far

$E = E_1 \cup \dots \cup E_{n-1}$ and $F = E_n$.

$$P\left(\bigcup_{i=1}^n E_i\right) = P\left(\bigcup_{i=1}^{n-1} E_i\right) + P(E_n) - P\left(\left(\bigcup_{i=1}^{n-1} E_i\right) \cap E_n\right)$$

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To get 3rd term we note, from distributive law, that

$$\left(\bigcup_{i=1}^{n-1} E_i\right) \cap E_n = \bigcup_{i=1}^{n-1} (E_i \cap E_n)$$

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$$\left(\bigcup_{i=1}^{n-1} E_i\right) \cap E_n = \bigcup_{i=1}^{n-1} (E_i \cap E_n) = \bigcup_{i=1}^{n-1} G_i$$

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We can now use i.h. to evaluate the last term on the RHS.
To do this, we will need to note that (why?)

$$\begin{aligned} & -(-1)^{k+1} P(G_{i_1} \cap G_{i_2} \cap \dots \cap G_{i_k}) \\ & = (-1)^{k+2} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} \cap E_n) \end{aligned}$$

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Applying i.h. once

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n-1}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$$

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$$+ P(E_n)$$

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Applying i.h. once and again

$$\begin{aligned} P\left(\bigcup_{i=1}^n E_i\right) &= \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n-1}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) \\ &\quad + P(E_n) + \sum_{k=1}^{n-1} (-1)^{k+2} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n-1}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} \cap E_n) \end{aligned}$$

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First summation on RHS sums $(-1)^{k+1} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$ over all lists i_1, i_2, \dots, i_k that **do not** contain n .

$$\begin{aligned}
P\left(\bigcup_{i=1}^n E_i\right) &= \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n-1}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) \\
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$P(E_n)$ and second summation together sums $(-1)^{k+1} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$ over all lists i_1, i_2, \dots, i_k that **do** contain n .

Therefore,

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$$

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Thus, by principle of MI, formula holds for all $n > 1$.

Example:

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Example:

There are n students who have the same model and color of backpack. They went to a class and hung their backpacks up on the wall. Someone came along and totally mixed up the backpacks so the students get back random backpacks.

What is the probability that

- (i) Exactly k specified students get their OWN backpacks back?
- (ii) At least one student gets his/her OWN backpack back?
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Problem (iii) is sometimes known as the derangement problem

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$$\begin{aligned} \Rightarrow P(k \text{ given students get their own backpack back}) \\ &= P(\text{for } k \text{ given numbers } x_1, x_2, \dots, x_k, \quad f(x_i) = x_i) \\ &= (n - k)! / n! . \end{aligned}$$

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For later use, set $D_{n,k} = \frac{(n-k)!}{n!}$.

Note

If E_i is event that person i gets correct backpack back.

$$\Rightarrow P(E_i) = D_{n,1} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

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Also note (why?)

$$\begin{aligned} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) \\ &= P(\text{Students } i_1, i_2, \dots, i_k \text{ get their backpacks back}) \\ &= \frac{(n-k)!}{n!} = D_{n,k} \end{aligned}$$

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Probability that at least one person gets his or her own backpack is $P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$.

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Probability that at least one person gets his or her own backpack is $P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$.

Then, by **principle of inclusion and exclusion**, probability that at least one person gets his or her own backpack is

$$\begin{aligned} &P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) \\ &= \sum_{k=1}^5 (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq 5}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}) \quad (*) \end{aligned}$$

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So rewrite RHS of (*) as $\sum_{k=1}^5 (-1)^{k+1} \binom{5}{k} \frac{(5-k)!}{5!}$.

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$$= \sum_{k=1}^5 (-1)^{k+1} \frac{5!}{k!(5-k)!} \frac{(5-k)!}{5!} = \sum_{k=1}^5 (-1)^{k+1} \frac{1}{k!}$$

Probability that at least one person gets his or her own backpack is then

$$\sum_{k=1}^5 (-1)^{k+1} \frac{1}{k!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!}$$

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General case n :

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Probability nobody gets his or her own backpack is 1 minus the probability above, or

$$\sum_{i=2}^n (-1)^i \frac{1}{i!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}.$$

Recall from calculus:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

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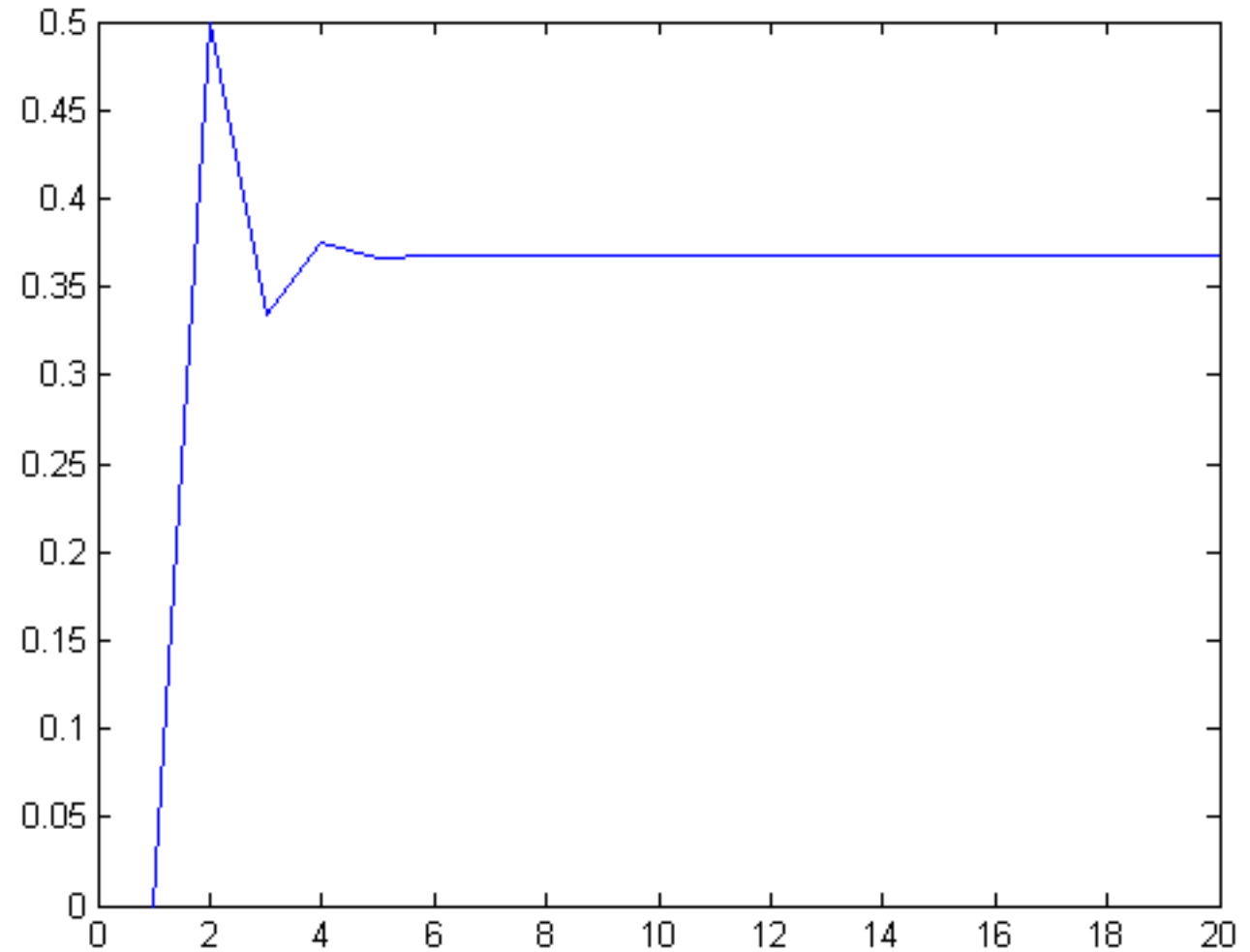
Probability of **no one** getting their backpack back is

$$\sum_{i=0}^n (-1)^i \frac{1}{i!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}.$$

which is approximation to e^{-1} , by substituting -1 for x in the power series and stopping at $i = n$.

$$n \sum_{i=0}^n (-1)^i \frac{1}{i!}$$

1	0.000000000000
2	0.500000000000
3	0.333333333333
4	0.375000000000
5	0.366666666667
6	0.368055555556
7	0.367857142857
8	0.367881944444
9	0.367879188713
10	0.367879464286
11	0.367879439234
12	0.367879441321
13	0.367879441161
14	0.367879441172
15	0.367879441171
16	0.367879441171
17	0.367879441171
18	0.367879441171
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Unions and Intersections

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given k -element subset K of M ?

Using same reasoning as above, number of functions
that map nothing to a given set K of k elements will
be $(m - k)^n$.

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from an n -element set N to an m -element set M ?

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- (b) How many functions from an n -element set N to an m -element set M map nothing to at least one element of M ?

Since there are exactly m^n functions from an n -element set N to an m -element set M

The answer to (b) is, m^n **minus the answer to (a)!**

(b) How many functions
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map nothing to at least one element of M ?

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Principle of inclusion and exclusion for counting:

$$\left| \bigcup_{i=1}^m E_i \right| = \sum_{k=1}^m (-1)^{k+1} \sum_{i_1, i_2, \dots, i_k: 1 \leq i_1 < i_2 < \dots < i_k \leq m} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

Applying this formula to number of functions from N to M that map nothing to at least one element of K gives

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where $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$ is number of functions that map nothing to k -element set $K = \{i_1, i_2, \dots, i_k\}$.

Applying this formula to number of functions from N to M that map nothing to at least one element of K gives

$$\begin{aligned} \left| \bigcup_{i=1}^m E_i \right| &= \sum_{k=1}^m (-1)^{k+1} \sum_{i_1, i_2, \dots, i_k: 1 \leq i_1 < i_2 < \dots < i_k \leq m} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}| \\ &= \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} (m-k)^n \end{aligned}$$

where $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$ is number of functions that map nothing to k -element set $K = \{i_1, i_2, \dots, i_k\}$.

$\binom{m}{k}$ is number of ways to pick subset K

For fixed K , number of these functions is $(m-k)^n$.

Applying this formula to number of functions from N to M that map nothing to at least one element of K gives

$$\begin{aligned} \left| \bigcup_{i=1}^m E_i \right| &= \sum_{k=1}^m (-1)^{k+1} \sum_{i_1, i_2, \dots, i_k: 1 \leq i_1 < i_2 < \dots < i_k \leq m} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}| \\ &= \sum_{k=1}^m (-1)^{k+1} \boxed{\binom{m}{k} (m-k)^n} \end{aligned}$$

where $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$ is number of functions that map nothing to k -element set $K = \{i_1, i_2, \dots, i_k\}$.

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because $\binom{m}{0} = 1$, $(m-0)^n$ is m^n , and $-(-1)^{k+1} = (-1)^k$.

Theorem 5.4:

The number of functions from an n -element set onto an m element set is

$$\sum_{k=0}^m (-1)^k \binom{m}{k} (m-k)^n.$$