COMP170
Discrete Mathematical Tools for Computer Science

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Basic Counting

Version 2.0: Last updated, May 13, 2007

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 1.1, pp. 1-8

Counting

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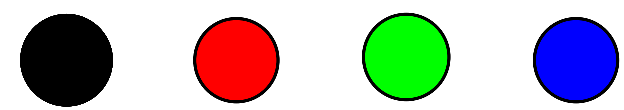
How many different ways are there to choose 2 balls from 

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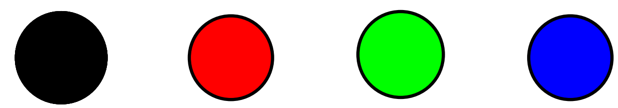
How many different ways are there to choose 2 balls from  $\boxed{6}$

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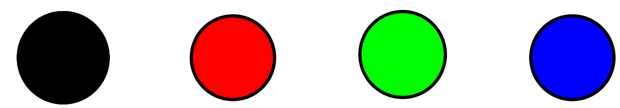
How many different ways are there to choose 2 balls from  $\boxed{6}$

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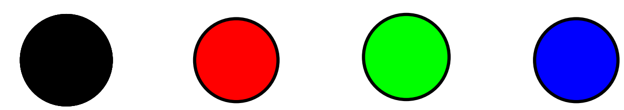
How many different ways are there to choose 2 balls from  $\boxed{6}$

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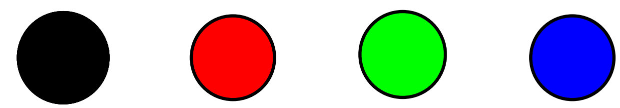
How many different ways are there to choose 2 balls from  $\boxed{6}$

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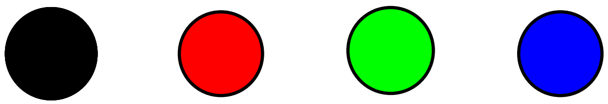
How many different ways are there to choose 2 balls from 

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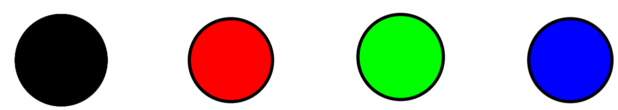

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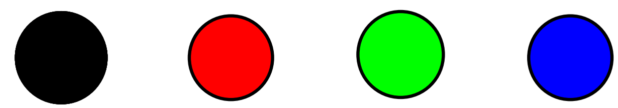

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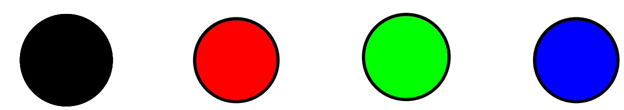
How many different ways are there to choose 2 balls from  $\boxed{6}$

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Might still be able to list all

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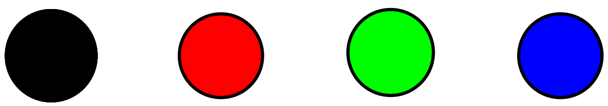
How many different ways are there to choose 2 balls from  $\boxed{6}$

How many different ways are there to choose 2 students from a class of 5 students? $\boxed{10}$

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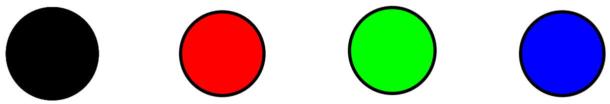

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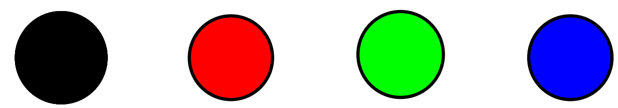
How many different ways are there to choose 2 balls from  $\boxed{6}$

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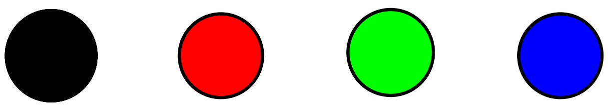
How many different ways are there to choose 2 balls from  $\boxed{6}$

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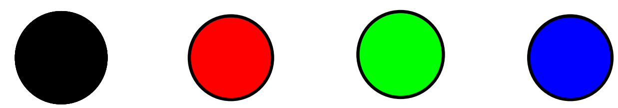
How many different ways are there to choose 2 balls from  $\boxed{6}$

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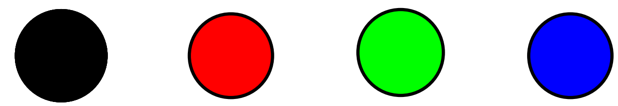

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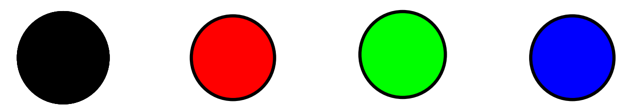
How many different ways are there to choose 2 balls from 

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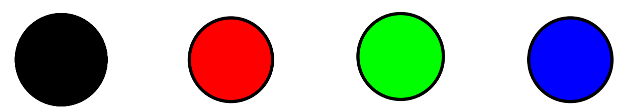
How many different ways are there to choose 2 balls from  $\boxed{6}$

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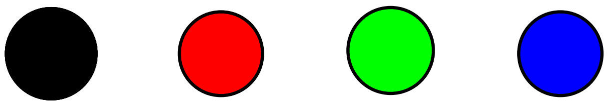

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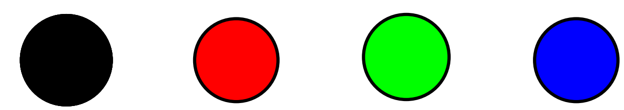
How many different ways are there to choose 2 balls from  $\boxed{6}$

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Hong Kong Mark 6!

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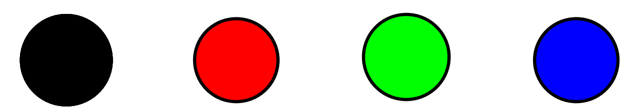
How many different ways are there to choose 2 balls from  $\boxed{6}$

How many different ways are there to choose 6 numbers out of $1 \ldots 49$? $\boxed{13{,}983{,}816}$

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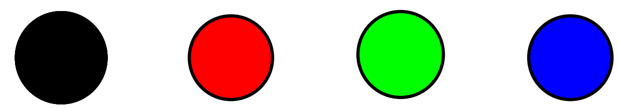
How many different ways are there to choose 2 balls from 

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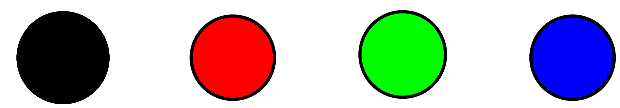
How many different ways are there to choose 2 balls from  $\boxed{12}$

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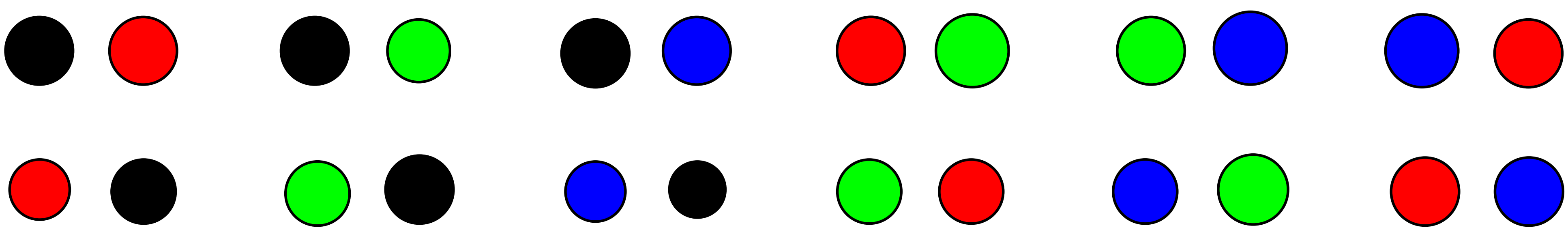

In Computer Science we often need to count objects.

Sometimes it's the number of steps
a computer program takes

This lets us compare runtimes
of different programs.

Sometimes, it's the number of objects
of a particular type, e.g., passwords
containing between 6-10 characters

This lets us evaluate security.
The more passwords available,
the lower the chance that
someone can guess a password

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- The Sum Principle and set notation

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- Two-Element Subsets

The Sum Principle

Start with an exercise illustrating the sum principle.

Consider the following loop from **selection-sort**, (comp171), which sorts a list of items

```
for i = 1 to n-1
    for j = i+1 to n
        if (A[i] > A[j])
            exchange A[i] and A[j]
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If you've never programmed before **Don't worry!**

This is *Pseudocode*; You will learn more in the tutorial

The Sum Principle

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Consider the following loop from **selection-sort**, (comp171), which sorts a list of items

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(1) for i = 1 to n-1
(2)   for j = i+1 to n
(3)     if (A[i] > A[j])
(4)       exchange A[i] and A[j]
```

How many times is the comparison

A[i] $>$ **A[j]**

made in line 3?

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(1) for i = 1 to n-1
(2)   for j = i+1 to n
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Lines 2–4 are executed $n-1$ times, once for each i between 1 and $n-1$.

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Lines 2–4 are executed $n-1$ times,
once for each i between 1 and $n-1$.

First time,	$n-1$ comparisons.
Second time,	$n-2$ comparisons.
ith time,	$n-i$ comparisons.
$(n-1)$st time,	1 comparison.

Thus, total number of comparisons is

$$(n-1)+(n-2)+\cdots+1.$$

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Took a difficult problem:
Counting *all* comparisons made by code

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```

Took a difficult problem:
Counting *all* comparisons
made by code

comparisons when $i = 1$:	$n-1$
comparisons when $i = 2$:	$n-2$
$\vdots$	
comparisons when $i = t$:	$n-t$
$\vdots$	
comparisons when $i = n-1$:	1

Split into *simpler* parts
Easier to count in each part
Add parts together to get

$$1 + 2 + 3 + \ldots + (n-1)$$

We showed how to *partition* a
large *set* of comparisons into
the *union* of smaller *mutually disjoint* sets.

We then derived our result using a general principle, the
(Sum Principle)
The size of a union of a
family of mutually disjoint finite sets
is the sum of sizes of the sets.

The next few slides are devoted
to defining all of the *red* terms.

Sets

Sets

Definition: *A set is a collection of objects*

Examples:

- The set of all men in this class
- The set of all women in this class
- The set of all students in this class surnamed *Ng*
- The set of all departments in the Engineering School
  $S = \{$ COMP, ECE, MechE, CivilE, ChemE, IELM $\}$

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Notation: Sets are usually denoted as $S = \{a, b, c\}$

In this case, set S contains *elements* or *objects* a, b and c

Definition: Two sets are **disjoint** if they have no elements in common.

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Examples:

- $S_1 = \{a, b, c\}$, $S_2 = \{a, e, f\}, S_3 = \{d, e, f\}$
- $S_4 =$ The set of all men in this class
- $S_5 =$ The set of all women in this class
- $S_6 =$ The set of all students in this class surnamed *Wong*
- $S_7 =$ The set of all students in this class surnamed *Chan*

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Which sets are disjoint?

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is a family of **mutually disjoint sets**,
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$S_1 = \{a, b, c\}$, $S_2 = \{d, e, f\}$, $S_3 = \{g, h, i\}$,
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Which families are (not) mutually disjoint?

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Which families are (not) mutually disjoint?

- S_1, S_2, S_3, S_4 are mutually disjoint
- S_1, S_2, S_3, S_5 are mutually disjoint
- S_1, S_2, S_3, S_4, S_5 are not mutually disjoint

Definition: The size, $|S|$, of set S is the number of different items in S

Example:

If $S_1 = \{a, b, c\}$, $S_2 = \{a, b, c, d\}$,

Then $|S_1| = 3$, $|S_2| = 4$,

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Note: An item can either be in or not in a set.
It can not be in a set more than once.

So, $S_3 = \{a, b, c, a\}$, denotes exactly the same set as $S_1 = \{a, b, c\}$, , i.e., $S_3 = S_1$ and $|S_3| = |S_1| = 3$

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- $S_1 \cup S_2 = \{a, b, c, d, e, f\}$
- $S_1 \cup S_3 = \{a, b, c, f, g, h\}$
- $S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e, f, g, h\}$
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The S_i are the *blocks* of the partition

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Example (cont):
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Equivalently
If $S_1, S_2, \ldots, S_m$ are mutually disjoints sets, then

$$|S_1 \cup S_2 \cup \cdots \cup S_m| = |S_1| + |S_2| + \cdots + |S_m|.$$

Example:

$$S_1 = \{a, b, c\},\ \ S_2 = \{d, e, f\},\ \ S_3 = \{g, h\},$$

$$S = S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e, f, g, h\}$$

$$|S| = 8 = 3 + 3 + 2 = |S_1| + |S_2| + |S_3|$$

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To avoid the dots, we sometimes write

$$\left| \bigcup_{i=1}^{m} S_i \right| = \sum_{i=1}^{m} |S_i|.$$

So Far

We counted comparisons in

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(1) for i = 1 to n-1
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In set notation, let S_t be the set of all comparisons $A[i] > A[j]$ made when $i = t$

Then the S_t are mutually disjoint with $|S_t| = n - t$

From sum principle, set $S = \bigcup_{i=1}^{n-1} S_i$ of all comparisons has size

$$|S| = \sum_{i=1}^{n-1} |S_i| = (n-1) + (n-2) + \ldots + 2 + 1$$

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The value of abstraction is that recognizing the abstract elements of a problem often helps us solve subsequent problems.

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By reading from right to left instead of left to right we observe that

$$\sum_{i=1}^{n-1}(n-i)=\sum_{i=1}^{n-1} i.$$

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Sidenote : Gauss (1777-1855) was one of the most brilliant mathematicians in history. This particular trick was supposedly discovered by him during his first year of school (age 7).

An alternative derivation

We already saw that $\sum_{i=1}^{n-1} i = \sum_{i=1}^{n-1} (n-i)$ so

$$\begin{aligned} 2\sum_{i=1}^{n-1} i &= \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} (n-i) \\ &= \sum_{i=1}^{n-1} [i + (n-i)] \\ &= \sum_{i=1}^{n-1} n = n(n-1) \end{aligned}$$

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How many multiplications (in terms of r, m, n) does this pseudo-code carry out in total among all iterations of line 5?

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Therefore, this program segment makes
n multiplications m times $\Rightarrow$ nm multiplications.

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This yields the

(Product Principle) The size of the union of m disjoint sets, each of size n, is mn.

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By *product principle,* set of *all* multiplications has size rmn.
Then, program carries out, in total, rmn multiplications

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Example:

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(1) for i = 1 to n-1
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Before, we counted total no. of comparisons by partitioning set of comps into disjoint subsets and then using the *sum principle* to derive $n(n-1)/2$. We will now see a different way of counting the same thing

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In general, for each pair of numbers i and j, compare $A[\mathbf{i}]$ and $A[\mathbf{j}]$ exactly once.

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In general, for each pair of numbers i and j, compare $A[\mathbf{i}]$ and $A[\mathbf{j}]$ exactly once.

Thus, number of comparisons is the same as number of two-element subsets of $\{1, 2, \ldots, n\}$.

In how many ways can we choose an ordered pair from $\{1, 2, \ldots, n\}$?

Example: In *ordered pair*, $(2, 5)$ is different from $(5, 2)$.

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Each pair $\{a, b\}$ of distinct elements of $\{1, 2, \ldots, n\}$ can be ordered in two ways, (a, b) and (b, a).

So, there are twice as many ordered pairs as two-element sets.

In how many ways can we choose two elements from $\{1, 2, \ldots, n\}$?

Number of ordered pairs is $n(n-1)$,
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$$\boxed{\binom{n}{2} = \frac{n(n-1)}{2} = 1 + 2 + \cdots + (n-1)}$$

This is the end of Lecture 1.

In it we learnt some basic counting techniques (using set abstraction) and applied them to counting the number of comparisons in *selection-sort* and the number of ways to choose 2 items out of n.

These both turned out to be equal to

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Please see section 1.1 of the book for more details