Topological Sort

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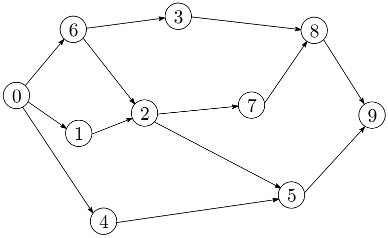
THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Directed Graph

In a directed graph, we distinguish between edge (u, v) and edge (v, u)



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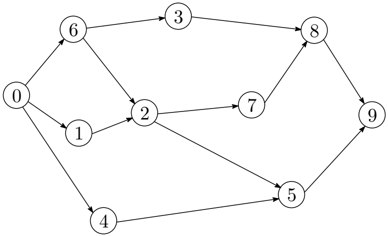

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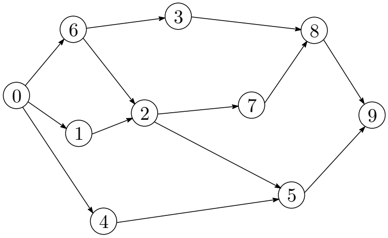

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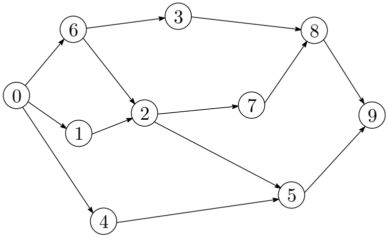

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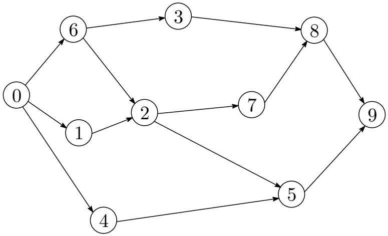

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$$\sum_{v \in V} \text{out-degree}(v) = \sum_{v \in V} \text{in-degree}(v) = |E|$$

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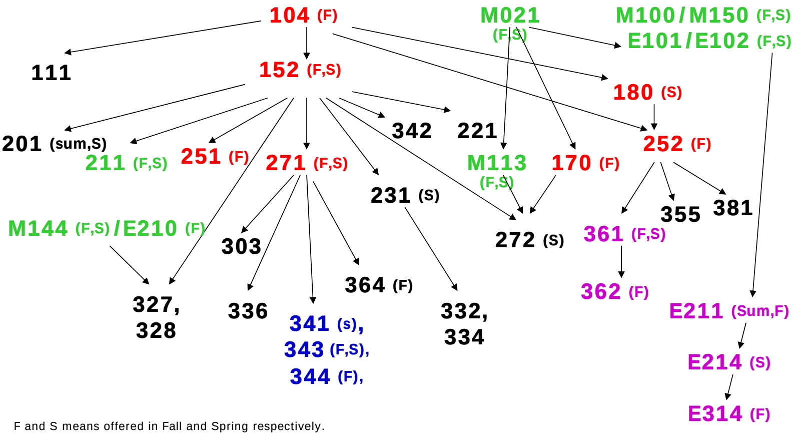
- Clearly, for the system not to hang, the graph must be acyclic
  - It must be a directed acyclic graph (or DAG)

Course dependence chart 09/10

Red: COMP/CSIE Core
Green: COMP/CSIE Required
Purple: CSIE (NW) Required
Blue: CSIE (MC) Required



F and S means offered in Fall and Spring respectively.
Course offering schedule shown here is for reference only;
the actual offering schedule may vary slightly from year to year.

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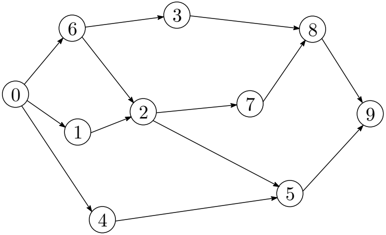

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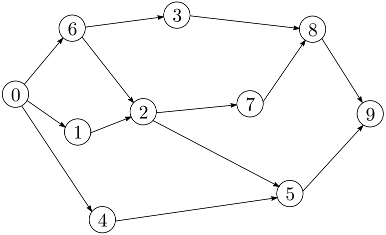

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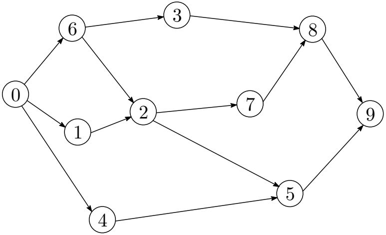

- Topological ordering may not be unique as there are many "equal" elements!
- E.G., there are several topological orderings
  - 0, 6, 1, 4, 3, 2, 5, 7, 8, 9
  - 0, 4, 1, 6, 2, 5, 3, 7, 8, 9
  - ...

Topological Sort Algorithm

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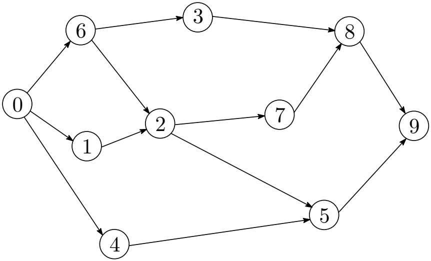
  3. If graph is not empty, goto step 1.
- Correctness
  - At every stage, current graph is a DAG (why?)
  - Because current graph is always a DAG, algorithm can always output some vertex. So algorithm outputs all vertices.
  - Suppose order output was **not** a topological order. Then there is some edge (u, v) such that v appears before u in the order. This is impossible, though, because v can not be output until edge (u, v) is removed!

Topological Sort Algorithm

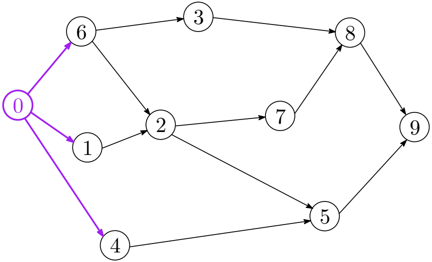
Topological_sort(G)

```
Initialize Q to be an empty queue;
foreach u in V do
    if in-degree(u) = 0 then
        // Find all starting vertices
        Enqueue(Q, u);
    end
end
while Q is not empty do
    u = Dequeue(Q);
    Output u;
    foreach v in Adj(u) do
        // remove u's outgoing edges
        in-degree(v) = in-degree(v) − 1;
        if in-degree(v) = 0 then
            Enqueue(Q, v);
        end
    end
end
```

Example



$Q = \{\}$



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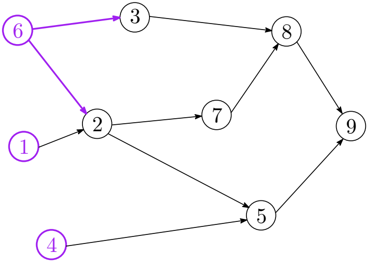
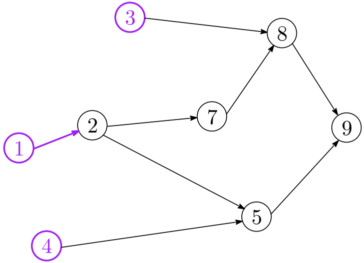


$Q = \{6, 1, 4\}$

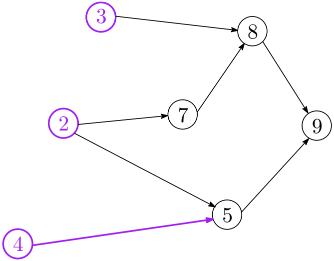
Output: 0



Q= {1, 4, 3}

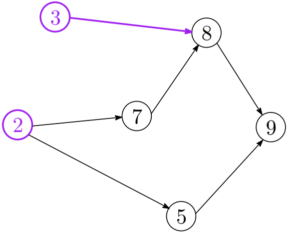
Output: 0, 6

Example



$Q = \{4, 3, 2\}$

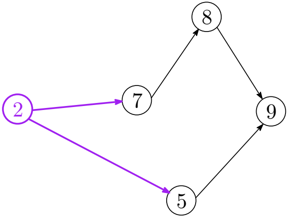
Output: 0, 6, 1



$Q = \{3, 2\}$

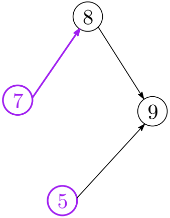
Output: 0, 6, 1, 4

Example



$Q = \{2\}$

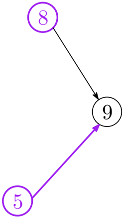
Output: 0, 6, 1, 4, 3



$Q = \{7, 5\}$

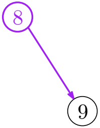
Output: 0, 6, 1, 4, 3, 2

Example



$Q = \{5, 8\}$

Output: 0, 6, 1, 4, 3, 2, 7

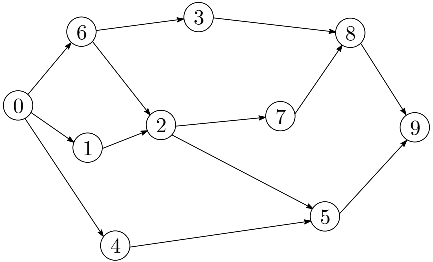


$Q = \{8\}$

Output: 0, 6, 1, 4, 3, 2, 7, 5

$Q = \{9\}$

Output: 0, 6, 1, 4, 3, 2, 7, 5, 8



$Q = \{\}$

Output: 0, 6, 1, 4, 3, 2, 7, 5, 8, 9

Done!

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Question

Can we use DFS to implement topological sort?