COMP170
Discrete Mathematical Tools for Computer Science

Quantifiers

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 3.2, pp. 104-114

3.2 Variables and Quantifiers

- Variables and Universes
- Quantifiers
- Standard Notation for Quantification
- Statements about Variables
- Proving Quantified Statements True or False
- Negation of Quantified Statements
- Implicit Quantification

Variables and Universes

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In statements such as $m^2 > m$, variable m is *not constrained*.
Unconstrained variables are called *free variables*.

Each possible value of a free variable gives a new statement.
The Truth or Falsehood of this new statement, is determined by substituting in the new value for the variable.

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Two main points:

- Clearly state the universe
- A statement about a variable can be True for some values of a variable and False for others.

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- While $m^2 > m$ is True for values such as $m = -3$ or $m = 9$ it is False for $m = 0$ or $m = 1$.

- Thus, it is not True that $m^2 > m$ for every integer m, so (*) is False

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- A phrase like for every integer m that converts a symbolic statement about potentially any member of our universe into a statement about the universe is called a **quantifier**

- A quantifier that asserts a statement about a variable is true for every value of the variable in its universe, is called a **universal quantifier**.

Examples of universal quantifiers

The statement

(**) For every integer m, $2m$ is even

is True.

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$$(**) \qquad \text{For every real number } m,\ 2m \text{ is even}$$

is False.

The statement

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  - Example for (***): set $m = 2$

- What would you have to do to show that a statement about one variable with an existential quantifier is False?

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Examples: Use Z for universe of all integers

- For all integers n, $n^2 \geq n$ becomes $\forall n \in Z \ (n^2 \geq n)$
- There exists an integer n such that $n^2 \not\geq n$ becomes $\exists n \in Z \ (n^2 \not\geq n)$

Using the given notation let's rewrite the part of Euclid's division theorem (Theorem 2.12) that states

> For every positive integer n and every nonnegative integer m, there are integers q and r, with $0 \leq r < n$, such that $m = qn + r$

Let Z^+ be the positive integers and N the nonnegative integers.

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$$\forall n \in Z^+ \ \ (\forall m \in N \ \ (\exists q \in N \ \ (\exists r \in N$$

$$((r \ < \ n) \wedge (m \ = \ qn + r)) \ \ )))$$

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Let $p(m, n, q, r)$ denote $m = nq + r$ with $0 \leq r < n$.

Leave out references to universes to clearly see the order in which the quantifiers occur.

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$$\forall n \ (\forall m \ (\exists q \ (\exists r \ p(m, n, q, r) \ )))$$

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a') $\forall x \in R \ (\ (x > 0) \ \Rightarrow \ (x > 1) \ )$

b') $\exists x \in R \ (\ (x > 0) \ \wedge \ (x > 1) \ )$

Theorem 3.2:
Let U_1, U_2 be two universes with $U_1 \subseteq U_2$.
Suppose that $q(x)$ is a statement such that

$$(*) \qquad U_1 = \{x \in U_2 \,|\, q(x) \text{ is true}\}.$$

Then, if $p(x)$ is a statement about U_2, it may also be interpreted as a statement about U_1, and

a. $\forall x \in U_1\ (p(x))$ is equiv. to $\forall x \in U_2\ (q(x) \Rightarrow p(x))$, and

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Proof:

By (*), $q(x)$ must be True for all $x \in U_1$ and
False for all $x \in U_2$ but $x \notin U_1$.

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For each of the following, state T or F and explain why.

a) $\forall x \in R^+ \ (x > 1)$

b) $\exists x \in R^+ \ (x > 1)$

c) $\forall x \in R \ (\exists y \in R \ (y > x))$

d) $\forall x \in R \ (\forall y \in R \ (y > x))$

e) $\exists x \in R \ (\ (x \geq 0) \ \wedge \ \forall y \in R^+ \ (y > x))$

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T. Let $x = 0$.

Principle 3.2
(The Meaning of Quantified Statements)

- The statement $\exists x \in U\ (p(x))$ is True
  if there exists at least one value of $x \in U$
  for which the statement $p(x)$ is True.

- The statement $\exists x \in U\ (p(x))$ is False
  if there is no $x \in U$ for which $p(x)$ is True.

- The statement $\forall x \in U\ (p(x))$ is True
  if $p(x)$ is True for every value of $x \in U$.

- The statement $\forall x \in U\ (p(x))$ is False
  if $p(x)$ is False for at least one value of $x \in U$.

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Thus, the negation of our for all ($\forall$) statement is a there exists ($\exists$) statement.

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Proof:

$p(x)$	$\neg p(x)$	$\forall x \in U(p(x))$	$\neg\forall x \in U(p(x))$	$\exists x \in U(\neg p(x))$
always true	always false	true	false	false
not always true	not always false	false	true	true

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Example: Let $p(x)$ be the statement $x^2 > 0$. Then

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Negating both statements gives

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Now, setting $q(x) = \neg p(x)$ gives

$$\forall x \in U(\neg p(x)) \quad \text{and} \quad \neg\exists x \in U(p(x)) \quad \text{are equivalent,}$$

and proves the corollary.

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Example:

Let $p(x)$ be $2x$ is odd.

Then $\neg p(x)$ is $2x$ is even.

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The corollary then says that

$$\neg\exists x \in Z \ (2x \text{ is odd})$$

is equivalent to

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Our original statement really means that

For every two even integers, m, n, $m + n$ is even

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In symbols

$$\forall m \in Z \ (\forall n \in Z \ (\ (p(m) \wedge p(n)) \Rightarrow p(m+n) \ ) \ )$$