Combinatorial Auction: A Survey (Part II)

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Outline

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Iterative Auctions

Types of Iterative Auction

Two types of iterative auction (with hybrids possible):

- *Quantity-setting*: In each round, bidders submit prices on various allocations. Auctioneer then makes a provisional allocation.
- *Price-setting*: In each round, auctioneer set the price and bidders announce which bundle they want.

Advantages of iterative auction over single-rounded auctions:

- Save bidders from specifying the bids for every bundles in advance.
- Adaptable in dynamic environments where bidders and objects arrive and depart at different times
- When bidders have private information that is relevant to other bidders, such auctions allow that information to be revealed

Iterative Auctions
Primal-Dual Algorithms

Price-setting and quantity-setting auctions are "dual" to one another.

Example English Auction and its "dual" (Klemperer, 2002)

Price-setting auctions correspond to primal-dual algorithms of CAP.

- Auction interpretation for the decomposition algorithm for linear programming (Dantzig, 1963)
- A collection of dual based algorithms for the class of linear network optimization algorithms (Bertsekas, 1991)
- Auction interpretations of algorithms for optimization problems (Mas-Collel et al., 1995, Chapter 17H): Dual variables $\Leftrightarrow$ prices, updates on their values $\Leftrightarrow$ current best responses

Iterative Auctions

Duality in Integer Programming

- SPP: Maximize $\sum_{j\in V} c_j x_j$ subject to $\sum_{j\in V} a_{ij}x_j \leq 1\ \forall i \in M$
- (Superadditive) dual of SPP: Find a superadditive, non-decreasing function $F\colon \mathbb{R}^m \to \mathbb{R}$ which does the following:

$$\text{Minimize } F(\mathbf{1}) \qquad \text{s.t. } F(a_j) \geq c_j\ \forall j \in V, \quad F(\mathbf{0}) = 0$$

  where a_j is the j-th column of the constraint matrix A.
- If the feasible region of the SPP is integral, the dual function F will be linear, i.e. $F(u) = \sum_i y_i u_i\ \forall u \in \mathbb{R}$. The dual becomes:

$$\text{Minimize } \sum_i y_i \qquad \text{s.t. } \sum_i a_{ij} y_i \geq c_j\ \forall j \in V, \quad y_i \geq 0\ \forall i \in M$$

Iterative Auctions
Duality in Integer Programming

- Optimal allocation given by a solution to the CAP can be supported by prices of individual objects.
- Optimal objective-function values of SPP and its dual coincide.

Theorem

If x is an optimal solution to SPP and F is an optimal solution to the superadditive dual then $(F(a_j) - c_j)\, x_j = 0\ \forall j$. *(Nemhauser and Wosley, 1988)*

- Solving the dual problem is as hard as solving the original problem.
- By solving the LP dual, the optimal value can help to search for an optimal solution to the original primal integer program.

Iterative Auctions
Lagrangian Relaxation

- Basic idea: "Relax" some constraint by moving them into the objective function with a penalty term.
- Infeasible solutions to SPP are allowed, but penalized in the objective function in proportion to the amount of infeasibility.
- Z_{LP} = optimal objective-function value to the LP relaxaion of SPP.
- Consider the following relaxed program:

$$Z(\lambda) = \max \sum_{j \in V} c_j x_j + \sum_{i \in M} \lambda_i \left(1 - \sum_{j \in V} a_{ij} x_j \right)$$
$$\text{s.t. } 0 \leq x_j \leq 1 \ \forall j \in V$$

Iterative Auctions

Lagrangian Relaxation

- It is easy to compute $Z(\lambda)$ because

$$\sum_{j \in V} c_j x_j + \sum_{i \in M} \lambda_i \left(1 - \sum_{j \in V} a_{ij} x_j\right) = \sum_{j \in V} \left(c_j - \sum_{i \in M} \lambda_i a_{ij}\right) x_j + \sum_{i \in M} \lambda_i$$

- To find $Z(\lambda)$, set $x_j = 1$ if $c_j - \sum_{i \in M} \lambda_i a_{ij} > 0$ and 0 otherwise.
- $Z(\lambda)$ is piecewise linear and convex.
- From the duality theorem,

Theorem

$$Z_{LP} = \min_{\lambda \geq 0} Z(\lambda)$$

Iterative Auctions
Lagrangian Relaxation

Finding the λ that minimize $Z(\lambda)$ by the *subgradient algorithm*:

- Let λ^t be the value of the Lagrange multiplier λ at iteration t.
- Choose any subgradient of $Z(\lambda)$ and call it s_t.
- Take $\lambda^{t+1} = \lambda^t + \theta_t s_t$, where $\theta_t > 0$ is the step size.
- If x_t is the optimal solution associated with $Z(\lambda^t)$,

$$\lambda^{t+1} = \lambda^t + \theta_t(Ax_t - \mathbf{1}).$$

- For an appropriate choice of step size at each iteration, this procedure can be shown to converge to the optimal solution.
- Ygge (1999) describes some heuristics for determining the multipliers for the winner determination.

Iterative Auctions

Lagrangian Relaxation - Auction Interpretation

Auction interpretation:

- Auctioneer choose a price vector λ for the individual objects.
- Bidders state which objects are acceptable to them at that price.
- Auctioneer tentatively assign objects according to the bid, randomly in case of ties, and in case of conflict, use the subgradient algorithm to adjust the prices and repeat the process.
- This is the similar to the *simutaneous ascending auction* (Milgrom, 1995), where bidders bid on individual items and bids must be increased by a specified minumum from one round to next.
- On the other hand, *Adaptive user selection mechanism* (Banks et al., 1989) is asynchronous in that bids on subsets can be submitted at any time.

Iterative Auctions
Lagrangian Relaxation - Examples

Examples

- De Martini et al. (1999): Hybrids of SAA and AUSM, easier to connect to the Lagrangean framework.
- Wurman and Wellman (2000): Allows bids on subsets, but use anonymous, non-linear prices to "direct" the auction.
- Kelly and Steinberg (2000): First phase use SAA, second phase use an AUSM-like mechanism, and bidders suggest the assignments.
- *i*Bundle (Parkes, 1999) allows bidders to bid on combinations of items using non-linear price.

Iterative Auctions
Column Generation

- Each variable “generate” a column in the constraint matrix.
- Column generation make use of two things:
  - Optimal solution is found only using a subset of columns / variables.
  - Optimization problems can be solved by finding a non-basic column / variable that has a reduced cost of appropriate sign.
- Brief implementation:
  1. Auctioneer chooses an extreme point solution to the CAP.
  2. Each bidder, based on their valuation, proposes a column / variable / subset to enter the basis.
  3. Auctioneer gathers up the proposed columns, form an initial basis, and find a revenue-maximizing allocation.
  4. Bidders may add new columns to the new basis.
  5. Repear 3 and 4 until an extreme point solution that no bidder wishes to modify.

Iterative Auctions
Column Generation

- No need to transmit/process long list of subsets and bids.
- Bidder may challenge an allocation if that increase the revenue to the seller.
- If this leads to a non-integral solution, it is embedded into a branch-and-cut/price scheme to produce an integer solution.
- Ellipsoid method solves the fractional CAP in polynomial time and generates polynomially-bounded number of columns. Therefore, if the fractional CAP is integral, it can be solved in polynomial time.

Example

- Bidders bid for a subtree of a tree containing a marked edge.
- CAP2 can solve the problem and the constraint matrix is perfect.
- This is a maximum-spanning-tree problem.

Iterative Auctions

Cuts and Nonlinear prices

- Suppose $b(\{1,2\}) = b(\{2,3\}) = b(\{3,4\}) = b(\{4,5\}) = b(\{1,5,6\}) = 2$, $b(\{6\}) = 1$, $b(S) = 0$ for other bundles. Formulation under CAP2:

$$
\begin{array}{llllllll}
\text{max} & 2x_{12} + 2x_{23} + 2x_{34} + 2x_{45} + 2x_{156} + x_6 & & & & & & \\
\text{s.t.} & x_{12} + & & & & x_{156} & & \leq 1 \\
 & x_{12} + & x_{23} & & & & & \leq 1 \\
 & & x_{23} + & x_{34} & & & & \leq 1 \\
 & & & x_{34} + & x_{45} & & & \leq 1 \\
 & & & & x_{45} + & x_{156} & & \leq 1 \\
 & & & & & x_{156} + & x_6 & \leq 1 \\
 & x_{12}, & x_{23}, & x_{34}, & x_{45}, & x_{156}, & x_6 & \geq 0
\end{array}
$$

- The optimal *fractional* solution is all variables equal $1/2$, and the optimal dual variables are $y_1 = \cdots = y_5 = 1/2$, $y_6 = 1$.

Iterative Auctions
Cuts and Nonlinear prices

- However, this optimal solution does not satisfy the inequality

$$x_{12} + x_{23} + x_{34} + x_{45} + x_{156} \leq 2$$

  but every feasible *integer* solution satisfy it.
- If this *cut* is appended to the formulation, the optimal solution is integral ($x_{12} = x_{34} = x_6 = 1$).
- Optimal dual solution: $y_1 = y_5 = y_6 = 0$, $y_2 = y_3 = y_4 = 1$, $\mu = 1$.
- Note that the pricing function is superadditive.

Iterative Auctions

Cuts and Nonlinear prices

- Cuts can be derived in two ways:
    - Combinatorical reasoning (Padnerg, 1973, 1975, 1979; Cornuejols and Sassano 1989; Sassano, 1989)
    - Algebraic technique introduced by Ralph Gomory
- Gomory method generates a cut involving only basic variables in the current extreme point.
- The new inequality will be a nonnegative linear combinations of current basic rows, all coefficients are non-negative.

Iterative Auctions
Extended Formulations

- $\Pi = \{\pi \mid \pi$ is a partition of $M\}$.
- $z_\pi = 1$ if partition π is selected.

$$\begin{aligned}
\text{Maximize} \quad & \sum_{j \in N} \sum_{S \subseteq M} b_j(S)\, y_j(S) \\
\text{subject to} \quad & \sum_{S \subseteq M} y_j(S) \leq 1 \;\forall j \in N \\
& \sum_{j \in N} y_j(S) \leq \sum_{\pi \ni S} z_\pi \;\forall S \subseteq M \\
& \sum_{\pi \in \Pi} z_\pi \leq 1
\end{aligned}$$

Call this formulation CAP3.

Iterative Auctions

Extended Formulations

- CAP3 is stronger than CAP1: For each $i \in M$,

$$\sum_{S \ni i} \sum_{j \in N} y_j(S) \leq \sum_{S \ni i} \sum_{\pi \in S} z_\pi \leq 1$$

- However, CAP3 still admits fractional extreme points. (Bikhchandani and Ostroy, 2001)
- The dual of linear relaxation of CAP3 is:

$$\begin{aligned} \text{min} \ & \sum_{j \in N} s_j + \mu \\ \text{s.t.} \ & s_j \geq b_j(S) - p_S \ \forall j \in N, S \subseteq M, \\ & \mu \geq \sum_{S \in \pi} p_S \ \forall \pi \in \Pi, \ s_j, p_S, \mu \geq 0 \end{aligned}$$

- Interpretation: minimizing the bidders' surplus plus μ.

Incentive Issues
Bids and Valuations

- Suppose three bidders bid on two objects $\{x,\, y\}$, and their *values* are:

$$v_1(x,y) = 100, v_1(x) = v_1(y) = 0,$$
$$v_2(x) = v_2(y) = 75, v_2(x,y) = 0,$$
$$v_3(x) = v_3(y) = 40, v_3(x,y) = 0.$$

- Note that bidders 2 or 3 could lower their bids (assuming the other does not) and they can still win the auction.
- Auction mechanisms should give bidders the incentive to reveal their valuation truthfully.
- Model of bidders' preferences:
  - Let $\{v_j(S)\}_{S \subseteq M}$ be the *valuation* of bidder $j \in N$.
  - Each v_j is an independent draw from a known distribution over a compact convex set, and it is known only to bidder j himself.

Incentive Issues
Economic Efficiency

- An auction is economically efficient if the allocation solves:

$$V = \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S)\, y_j(S)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y_j(S) \leq 1 \; \forall i \in M,$$

$$\sum_{S \subseteq M} y_j(S) \leq 1 \; \forall j \in N$$

- Notice that this is just CAP1 with b_j replaced by v_j.
- The optimal objective-function value is an upper bound on the revenue if no bidder bids above their valuation.
- Since bidders' valuations are private information, auctioner must solve the optimization problem without knowing the objective function.

Incentive Issues
VCG Scheme

The *Vickrey-Clarke-Groves* scheme maximize the revenue to the seller:

1. Agent j report v_j. Given the rule of the auction, it is a weakly dominant strategy to bid truthfully.
2. The seller choose the allocation that solves:

$$V = \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S)\, y_j(S)$$

$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N} y_j(S) \leq 1 \; \forall i \in M,$$

$$\sum_{S \subseteq M} y_j(S) \leq 1 \; \forall j \in N$$

Denote this optimal allocation by y^*.

Incentive Issues
VCG Scheme

3. To compute the payment that each bidder must make let, for each $k \in N$,

$$V_{-k} = \max \sum_{j \in N \setminus \{k\}} \sum_{S \subseteq M} v_j(S)\, y_j(S)$$
$$\text{s.t.} \sum_{S \ni i} \sum_{j \in N \setminus \{k\}} y_j(S) \leq 1 \ \forall i \in M,$$
$$\sum_{S \subseteq M} y_j(S) \leq 1 \ \forall j \in N$$

Denote this optimal allocation with bidder k excluded by y^k.

Incentive Issues
VCG Scheme

4. The payment bidder k makes is:

$$V_{-k} - \left[V - \sum_{S \subseteq M} v_k(S)\, y_k^*(S) \right]$$

Bidder k pays the difference of "welfare" of the other bidders without him and the welfare of others when he is included in the allocation. Notice that the payment made by each bidder to the auctioneer is nonnegative.

Incentive Issues

VCG Scheme

- If a seller adopt the VCG scheme her total revene is:

$$V + \sum_{k \in N} (V_{-k} - V)$$

- If there were a large number of agents, no single agent can have a significant effect i.e. on average, V is very close to V_{-k}. Total revenue would be close to V.
- In Monderer and Tennenholtz (2000), it is shown under this model that the VCG scheme generates a revenue for the seller that is asymptotically close to the revenue from the optimal auction.

Incentive Issues

Problems of the VCG Scheme

- VCG scheme is in general impractical to implement if N is large.
- Replacing y^* and y^k with approximately optimal solutions does not preserve incentive compatibility in general.
  - Solve the embedded optimization problems using greedy algorithm and shows that it is not incentive-compatible (Lehmann et al., 1999)
  - If each bidder is restricted that he could value only one subset, it would be incentive compatable.
- Even the incentive compatibility is relaxed, there are other problems in approximate solution of the optimizaton problem:
  - Many solutions within a specific tolerance
  - Payments are very sensitive to the choice of solution
  - Choice of approximate solution can have a significant impact on the payments made by bidders

Incentive Issues
Applications of VCG Scheme

- VCG scheme can be appled to iterative auctions. It is a Nash equilibrium for bidders to bid truthfully each round.
- Ascending auction for indivisible heterogeneous objects (Ausubel, 2000)
- Finding the efficient allocation can be formulated as a linear program such that the dual variables corresponds to Vickrey payments (Bikhchandani and Ostroy, 2001; Bikhchandani et al., 2002)
- VCG scheme for 3 bidders and 2 objects (Isaac and James, 1998)
- Iterative auction for the sale of multiple units of homogeneous goods. (Kagel and Levin, 2001)
- VCG scheme is vulnerable to collusion (Hobbs et al., 2000)

Incentive Issues
Revenue Maximization

An auction that maximizes the auctioneer's revenue (optimal auction) follows a *direct-revelation* mechanism:

1. Auctioneer announce how he will allocate objects and the payments he will extract as a function of announced value functions
2. Bidders announce their value functions

Models:

- Each bidder assume the other bidders' value functions are independent draws from a known distribution p over a *finite* set V.
- Probability that $\mathbf{v} \in V^n$ is realized is $\prod_{j=1}^{n} p(v_j)$, denoted as $p(\mathbf{v})$.
- *Allocation rule*: A mapping from $\mathbf{v} = (v_1, \ldots, v_n)$ to an allocation $A(\mathbf{v})$, which is an integer solution to CAP1.
- *Payment rule*: A mapping from $\mathbf{v} = (v_j, \mathbf{v}_{-j})$ to the payment $P = (P_1, \ldots, P_n)$ that bidders make.

Incentive Issues
Revenue Maximization

By the revelation principle (Myerson, 1981), optimal auctions satisfy:

Incentive compatibility Expected payoff to a truthful bidder $\geq$ Expected payoff of misreporting bidder

Individual rationality Expected payoff to a truthful player ≥ 0.

$$\max_{A,P} \sum_{\mathbf{v}\in V^n} p(\mathbf{v}) \left[\sum_{j\in N} P_j(\mathbf{v})\right]$$

s.t.
$$\sum_{\mathbf{v}^{-j}\in V^{n-1}} p(\mathbf{v}^{-j})[v_j(A(v_j,\mathbf{v}^{-j})) - P_j(v_j,\mathbf{v}^{-j})]$$
$$\geq \sum_{\mathbf{v}^{-j}\in V^{n-1}} p(\mathbf{v}^{-j})[v_j(A(u,\mathbf{v}^{-j})) - P_j(u,\mathbf{v}^{-j})] \; \forall u \neq v_j, j \in N$$
$$\sum_{\mathbf{v}^{-j}\in V^{n-1}} p(\mathbf{v}^{-j})[v_j(A(v_j,\mathbf{v}^{-j})) - P_j(v_j,\mathbf{v}^{-j})] \geq 0 \; \forall v_j \in V, j \in N$$

Incentive Issues

Revenue Maximization

Fix a choice for allocation rule A, and if agent j reports u, let

- w_A^j = Expected utility agent j receive, given his actual value is v_j,
- ρ^j = Expected payment agent j must make

$$w_A^j(u|v_j) = \sum_{\mathbf{v}^{-j} \in V^{n-1}} p(\mathbf{v}^{-j})\, v_j(A(u, \mathbf{v}^{-j}))$$

$$\rho^j(u) = \sum_{\mathbf{v}^{-j} \in V^{n-1}} p(\mathbf{v}^{-j})\, P_j(u, \mathbf{v}^{-j})$$

- Since the probability that agent j has value function v is independent of j, $w_A^j(u|v_j) = w_A^{j'}(u|v_{j'})$ and $\rho^j(u) = \rho^{j'}(u)$ for all j, $j' \in N$.

Incentive Issues
Revenue Maximization

The optimization problem becomes:

$$\max\ |N| \sum_{v \in V} p(v)\, \rho(v)$$

$$\text{s.t.}\ \ \rho(v) - \rho(u) \le w_A(v|v) - w_A(u|v)\ \forall v, u \in V, j \in N$$

$$\rho(v) \le w_A(v|v)\ \forall v \in V, j \in N$$

It is the dual to the following network-flow problem:

- For each (v, u), there is an directed edge from v to u with length $w_A(v|v) - w_A(u|v)$.
- Each individual rationality constraint introduce a source node.
- Find the shortest-path tree, one for each agent, rooted from the source node corresponding to that agent.

By duality theorem, for each allocation A, we can determine whether an incentive-compatable payment rule exists.

Incentive Issues

Revenue Maximization

- The design of the optimal auction is extremely sensative to the bidder's value functions and the distribution of the value functions. (Rochet and Stole, 2001)
- Levin (1997) identifies the optimal auction under a restrictive setting.
- Krishna and Rosenthal (1996) compared the revenue between different auction schemes under a simplified model of preferences.
- Cybernomics, Inc. (2000) compares a particular simutaneous multi-round auction with a particular multi-round combinatorial auction.

Summary

- Introduced iterated auctions and duality.
- Pointing out "classical" result that applys directly to the problem of designing combinatorial auctions.
- Emphasize the connections between the duality theory of optimization problems and the design of auctions.