Sorting: Lower Bounds and Linear Time

Last Revision: September 11, 2014

Lower Bound for Sorting

- All sorting algorithms we seen so far are based on comparing elements
  - E.g., Insertion sort, Selection sort, Mergesort, Heapsort and Quicksort

- All sorting algorithms we seen so far are based on comparing elements
  - E.g., Insertion sort, Selection sort, Mergesort, Heapsort and Quicksort
- Insertion sort, Selection sort and Quicksort have worst-case running times $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Lower Bound for Sorting

- All sorting algorithms we seen so far are based on comparing elements
  - E.g., Insertion sort, Selection sort, Mergesort, Heapsort and Quicksort
- Insertion sort, Selection sort and Quicksort have worst-case running times $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Question

Can we do better?

Lower Bound for Sorting

- All sorting algorithms we seen so far are based on comparing elements
  - E.g., Insertion sort, Selection sort, Mergesort, Heapsort and Quicksort
- Insertion sort, Selection sort and Quicksort have worst-case running times $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Question

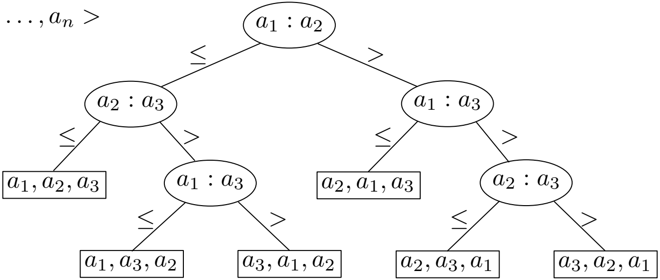
Can we do better?

Goal

We will prove that any comparison-based sorting algorithm has a worst-case running time $\Omega(n \log n)$.

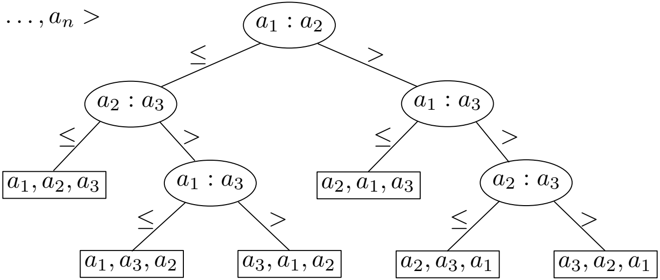
Decision-tree Example

Sort $< a_1, a_2, \ldots, a_n >$



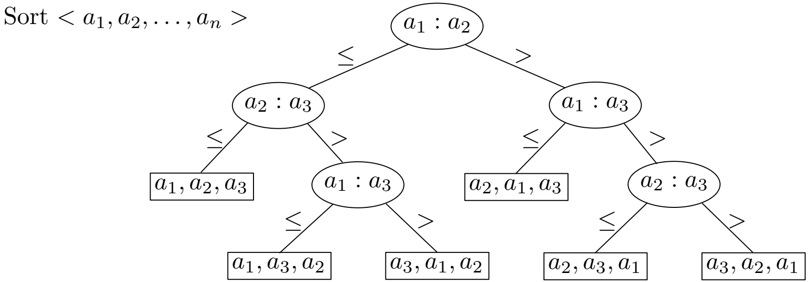
Decision-tree Example

Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$

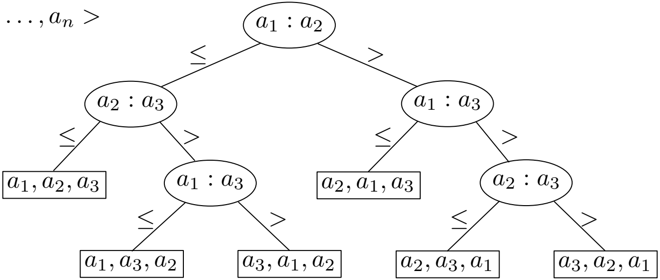
Decision-tree Example



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$

Decision-tree Example

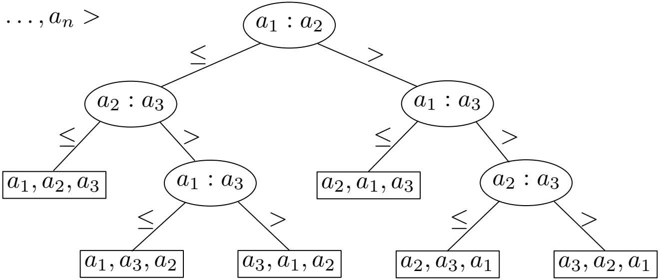
Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$

Decision-tree Example

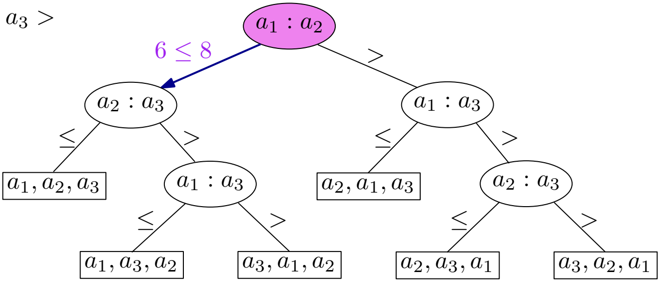
Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

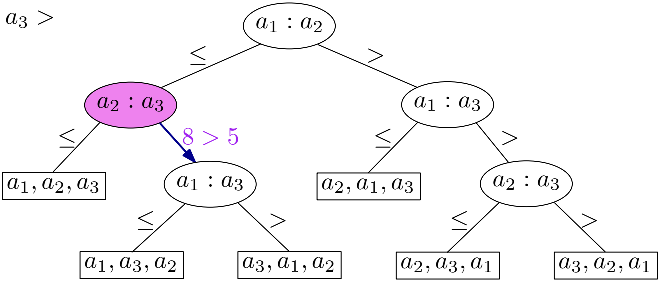
Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

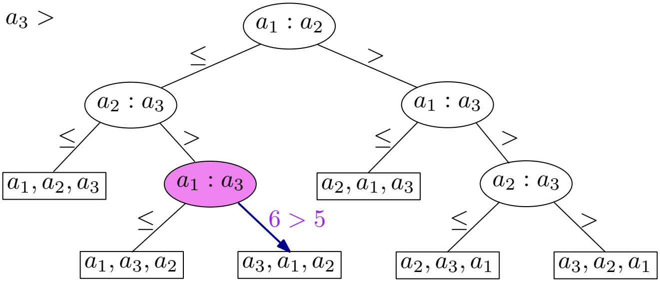
Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
    - The left subtree shows subsequent comparisons if $a_i \leq a_j$
    - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

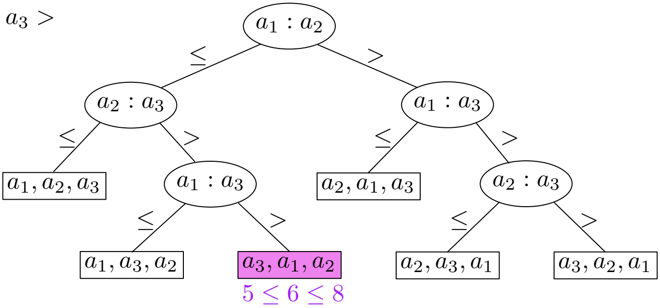
Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Model

A decision tree can model the execution of **any** comparison-based sorting algorithm

Decision-tree Model

A decision tree can model the execution of **any** comparison-based sorting algorithm

- One tree for each input size n

Decision-tree Model

A decision tree can model the execution of **any** comparison-based sorting algorithm

- One tree for each input size n
- Worst-case running time $=$ height of tree

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since each of the $n!$ orderings is a possible answer.

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since each of the $n!$ orderings is a possible answer.
- A binary tree of height h has at most 2^h leaves

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ *comparisons in the worst case.*

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since each of the $n!$ orderings is a possible answer.
- A binary tree of height h has at most 2^h leaves
- Thus, $n! \leq 2^h$
  $\Rightarrow h \geq \log n! = \Omega(n \log n)$ (Stirling's approximation)

$\square$

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since each of the $n!$ orderings is a possible answer.
- A binary tree of height h has at most 2^h leaves
- Thus, $n! \leq 2^h$
  $\Rightarrow h \geq \log n! = \Omega(n \log n)$ (Stirling's approximation)

$\square$

Corollary

Heapsort and merge sort are asymptotically optimal comparison-based sorting algorithms.

Lower Bound for Average Running Time of Sorting

- We just proved that worst case number of comparisons used is $\Omega(n \log n)$
- Suppose that each of the $n!$ input permutations is equally likely. What can be said about the average case running time?

Note: Average is taken by adding up individual running time of algorithm on each possible input and dividing total by $n!$.

- We just proved that worst case number of comparisons used is $\Omega(n \log n)$
- Suppose that each of the $n!$ input permutations is equally likely. What can be said about the average case running time?

Note: Average is taken by adding up individual running time of algorithm on each possible input and dividing total by $n!$.

Theorem

When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.

Lower Bound for Average Running Time of Sorting

Theorem

When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.

Proof.

- The *External Path Length (EPL)* of a tree is the sum over all leaves of the tree, of the length of the paths from the root to the leaves.

Lower Bound for Average Running Time of Sorting

Theorem

When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.

Proof.

- The *External Path Length (EPL)* of a tree is the sum over all leaves of the tree, of the length of the paths from the root to the leaves.
- Average number of comparisons used by a sorting algorithm is EPL of its associated comparison tree divided by $n!$.

Lower Bound for Average Running Time of Sorting

Theorem

When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.

Proof.

- The *External Path Length (EPL)* of a tree is the sum over all leaves of the tree, of the length of the paths from the root to the leaves.
- Average number of comparisons used by a sorting algorithm is EPL of its associated comparison tree divided by $n!$.
- The EPL of a binary tree with m leaves is at least $m \log_2 m + O(m)$.
- The comparison tree has $m = n!$ leaves
  $\Rightarrow$ its external path length is $n! \log_2 n! + O(n!)$
  $\Rightarrow$ average number of comparisons used is $\log_2 n! + O(1)$.

Lower Bound for Average Running Time of Sorting

Theorem

When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.

Proof.

- The *External Path Length (EPL)* of a tree is the sum over all leaves of the tree, of the length of the paths from the root to the leaves.
- Average number of comparisons used by a sorting algorithm is EPL of its associated comparison tree divided by $n!$.
- The EPL of a binary tree with m leaves is at least $m \log_2 m + O(m)$.
- The comparison tree has $m = n!$ leaves
  $\Rightarrow$ its external path length is $n! \log_2 n! + O(n!)$
  $\Rightarrow$ average number of comparisons used is $\log_2 n! + O(1)$.
- We already saw $\log_2 n! = \Omega(n \log n)$.

□

Can we do better?

Are there sorting algorithms which are not comparison-based?
Can they beat the $\Omega(n \log n)$ lower bound?

Can we do better?

Are there sorting algorithms which are not comparison-based?
Can they beat the $\Omega(n \log n)$ lower bound?

- Counting sort
  - Assumes items are in set $\{1, 2, \ldots, k\}$.
  - Is a *stable* sort (defined soon).

Can we do better?

Are there sorting algorithms which are not comparison-based?
Can they beat the $\Omega(n \log n)$ lower bound?

- Counting sort
  - Assumes items are in set $\{1, 2, \ldots, k\}$.
  - Is a *stable* sort (defined soon).
- Radix sort
  - Assumes items are stored in fixed size words using finite alphabet

Counting-sort(A, B, k)

Input: $A[1 \dots n]$, where $A[j] \in \{1, 2, \dots, k\}$
Output: $B[1 \dots n]$, sorted
let $C[1 \dots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
  $C[i] \leftarrow 0$;
end
for $j \leftarrow 1$ **to** n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1$; // $C[i] = |\{\text{key} = i\}|$
end
for $i \leftarrow 2$ **to** k **do**
  $C[i] \leftarrow C[i] + C[i-1]$; // $C[i] = |\{\text{key} \leq i\}|$
end
for $j \leftarrow n$ **to** 1 **do**
  $B[C[A[j]]] \leftarrow A[j]$;
  $C[A[j]] \leftarrow C[A[j]] - 1$;
end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C				

B					

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	0	0	0

B					

for $i \leftarrow 1$ **to** k **do**

  $C[i] \leftarrow 0;$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	0	0	1

B					

for $j \leftarrow 1$ **to** n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1;$ // $C[i] = |\{\text{key} = i\}|$
end

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	1	0	1

B					

for $j \leftarrow 1$ **to** n **do**
$\quad C[A[j]] \leftarrow C[A[j]] + 1$; // $C[i] = |\{\text{key} = i\}|$
end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	1	0	1

B					

for $j \leftarrow 1$ **to** n **do**

$\quad C[A[j]] \leftarrow C[A[j]] + 1;$ // $C[i] = |\{\text{key} = i\}|$

end

Example: Counting Sort

A

1	2	3	4	5
4	2	1	4	2

C

1	2	3	4
1	1	0	2

B

for $j \leftarrow 1$ **to** n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1$; // $C[i] = |\{\text{key} = i\}|$
end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	2	0	2

B					

for $j \leftarrow 1$ **to** n **do**

$\quad C[A[j]] \leftarrow C[A[j]] + 1;$ // $C[i] = |\{\text{key} = i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	2	0	2

B					

C'	1	3	0	2

for $i \leftarrow 2$ **to** k **do**
  $C[i] \leftarrow C[i] + C[i-1]$; // $C[i] = |\{\text{key} \leq i\}|$
end

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	2	0	2

B					

C'	1	3	3	2

for $i \leftarrow 2$ **to** k **do**

$\quad C[i] \leftarrow C[i] + C[i-1]$; // $C[i] = |\{\text{key} \leq i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2
B					

	1	2	3	4
C	1	2	0	2
C'	1	3	3	5

for $i \leftarrow 2$ **to** k **do**
  $C[i] \leftarrow C[i] + C[i-1]$; // $C[i] = |\{\text{key} \leq i\}|$
end

A

1	2	3	4	5
4	2	1	4	2

C

1	2	3	4
1	3	3	5

B

		2		

C'

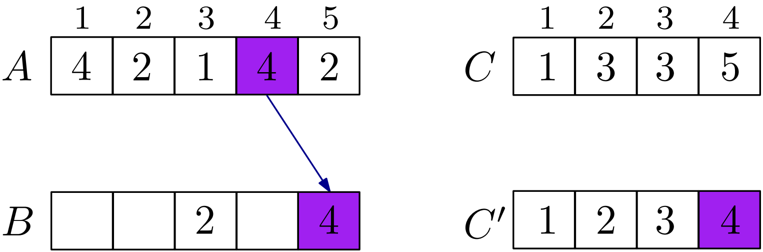
1	2	3	5

for $j \leftarrow n$ **to** 1 **do**

$\quad B[C[A[j]]] \leftarrow A[j];$

$\quad C[A[j]] \leftarrow C[A[j]] - 1;$

end



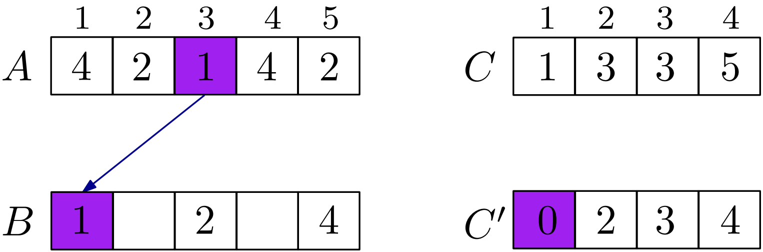
for $j \leftarrow n$ **to** 1 **do**

$\quad B[C[A[j]]] \leftarrow A[j]$;

$\quad C[A[j]] \leftarrow C[A[j]] - 1$;

end

Example: Counting Sort



for $j \leftarrow n$ **to** 1 **do**

$\quad B[C[A[j]]] \leftarrow A[j];$

$\quad C[A[j]] \leftarrow C[A[j]] - 1;$

end

	1	2	3	4	5
A	4	2	1	4	2
B	1	2	2		4

	1	2	3	4
C	1	3	3	5
C'	0	1	3	4

for $j \leftarrow n$ **to** 1 **do**

$\quad B[C[A[j]]] \leftarrow A[j];$

$\quad C[A[j]] \leftarrow C[A[j]] - 1;$

end

Example: Counting Sort

	1	2	3	4	5
A	**4**	2	1	4	2

	1	2	3	4
C	1	3	3	5

B	1	2	2	**4**	4

C'	0	1	3	**3**

for $j \leftarrow n$ **to** 1 **do**

  $B[C[A[j]]] \leftarrow A[j]$;

  $C[A[j]] \leftarrow C[A[j]] - 1$;

end

Analysis

Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$
Output: $B[1 \ldots n]$, sorted
let $C[1 \ldots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
  $C[i] \leftarrow 0$; // $O(k)$
end

Analysis

Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$
Output: $B[1 \ldots n]$, sorted
let $C[1 \ldots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
    $C[i] \leftarrow 0$; // $O(k)$
end
for $j \leftarrow 1$ **to** n **do**
    $C[A[j]] \leftarrow C[A[j]] + 1$; // $O(n)$
end

Analysis

Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$
Output: $B[1 \ldots n]$, sorted
let $C[1 \ldots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
    $C[i] \leftarrow 0$; // $O(k)$
end
for $j \leftarrow 1$ **to** n **do**
    $C[A[j]] \leftarrow C[A[j]] + 1$; // $O(n)$
end
for $i \leftarrow 2$ **to** k **do**
    $C[i] \leftarrow C[i] + C[i-1]$; // $O(k)$
end

Analysis

Input: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$
Output: $B[1 \ldots n]$, sorted
let $C[1 \ldots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
  $C[i] \leftarrow 0$; // $O(k)$
end
for $j \leftarrow 1$ **to** n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1$; // $O(n)$
end
for $i \leftarrow 2$ **to** k **do**
  $C[i] \leftarrow C[i] + C[i-1]$; // $O(k)$
end
for $j \leftarrow n$ **to** 1 **do**
  $B[C[A[j]]] \leftarrow A[j]$;
  $C[A[j]] \leftarrow C[A[j]] - 1$; // $O(n)$
end

Analysis

Input: $A[1 \dots n]$, where $A[j] \in \{1, 2, \dots, k\}$
Output: $B[1 \dots n]$, sorted
let $C[1 \dots k]$ be a new array;
for $i \leftarrow 1$ **to** k **do**
    $C[i] \leftarrow 0$; // $O(k)$
end
for $j \leftarrow 1$ **to** n **do**
    $C[A[j]] \leftarrow C[A[j]] + 1$; // $O(n)$
end
for $i \leftarrow 2$ **to** k **do**
    $C[i] \leftarrow C[i] + C[i-1]$; // $O(k)$
end
for $j \leftarrow n$ **to** 1 **do**
    $B[C[A[j]]] \leftarrow A[j]$;
    $C[A[j]] \leftarrow C[A[j]] - 1$; // $O(n)$
end

Total: $O(n + k)$

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.
- Note that counting sort is *not* a comparison-based sorting algorithm.

Running Time

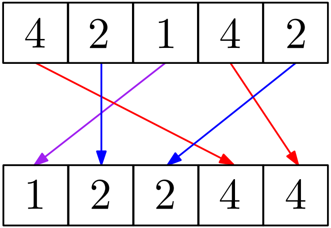
If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.
- Note that counting sort is *not* a comparison-based sorting algorithm.
- In fact, it makes no comparisons at all!

Stable Sorting

Counting sort is a stable sort

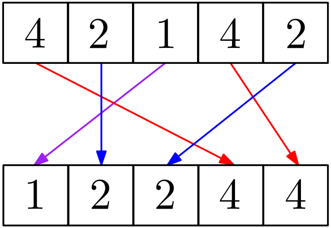
- it preserves the input order among equal elements.



Stable Sorting

Counting sort is a stable sort

- it preserves the input order among equal elements.



Exercise

What other sorts have this property?

Radix Sort

- Sort on *least significant* digit first using stable sort

Radix Sort

- Sort on *least significant* digit first using stable sort

2 3 2 9		2 7 2 0		2 7 2 0		2 3 2 9		2 3 2 9
5 4 5 7		5 3 5 5		2 3 2 9		5 3 5 5		2 7 2 0
3 6 5 7		3 4 3 6		3 4 3 6		3 4 3 6		3 4 3 6
5 8 3 9	→	5 4 5 7	→	5 8 3 9	→	5 4 5 7	→	3 6 5 7
3 4 3 6		3 6 5 7		5 3 5 5		3 6 5 7		5 3 5 5
2 7 2 0		2 3 2 9		5 4 5 7		2 7 2 0		5 4 5 7
5 3 5 5		5 8 3 9		3 6 5 7		5 8 3 9		5 8 3 9

Induction on digit position

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit i

2	7	2	0			2	3	2	9
2	3	2	9			5	3	5	5
3	4	3	6			3	4	3	6
5	8	3	9	→		5	4	5	7
5	3	5	5			3	6	5	7
5	4	5	7			2	7	2	0
3	6	5	7			5	8	3	9

Radix Sort: Correctness

Induction on digit position

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit i
  - Two numbers that differ on digit i are correctly sorted by their low-order i digits

2 7 2 0	2 3 2 9
2 3 2 9	5 3 5 5
3 4 3 6	3 4 3 6
5 8 3 9	5 4 5 7
5 3 5 5	3 6 5 7
5 4 5 7	2 7 2 0
3 6 5 7	5 8 3 9

Radix Sort: Correctness

Induction on digit position

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit i
  - Two numbers that differ on digit i are correctly sorted by their low-order i digits
  - Two numbers equal on digit i are put in the same order as the input $\Rightarrow$ correctly sorted by their low-order i digits

2 7 2 0	2 3 2 9
2 3 2 9	5 3 5 5
3 4 3 6	3 4 3 6
5 8 3 9	5 4 5 7
5 3 5 5	3 6 5 7
5 4 5 7	2 7 2 0
3 6 5 7	5 8 3 9

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- $b \log n$ bits for each number

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ *time if the stable sort it uses takes* $O(n+k)$ *time.*

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- $b \log n$ bits for each number
- each number can be viewed as having $O(b)$ digits of $\log n$ bits each

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- $b \log n$ bits for each number
- each number can be viewed as having $O(b)$ digits of $\log n$ bits each
- running time is $O(d(n+k)) = O(b(n+2^{\log n})) = O(bn)$

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- $b \log n$ bits for each number
- each number can be viewed as having $O(b)$ digits of $\log n$ bits each
- running time is $O(d(n+k)) = O(b(n + 2^{\log n})) = O(bn)$
- since b is a constant, the running time is $O(n)$.