COMP170 Discrete Mathematical Tools for Computer Science

Binomial Coefficients

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Pascal's Triangle

- Pascal's Triangle
- A Proof using the Sum Principle

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- The Binomial Theorem

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- The Binomial Theorem
- Labeling and Trinomial Coefficients

•
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
 is the number of k -element subsets of an n -element set.

- $\binom{n}{0} = 1$ only one set of size 0.
- $\binom{n}{n} = 1$ only one set of size n.
- $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

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Use Sum Principle

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Let $L = L_1 L_2 \dots L_n$ be a list of size n from $\{0, 1\}$ If $\mathcal{L} = \text{set of all such lists} \Rightarrow |\mathcal{L}| = 2^n$

There is a *bijection* (next page) between $\mathcal L$ and P so $|P|=2^n$ and we are done.

Define the following function $f:\mathcal{L}\to P$ If $L\in\mathcal{L}$ then f(L) is the set $S\subseteq\{1,,2,\ldots,n\}$ defined by $i\in S \iff L_i=1$

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Ex:
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 $f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset.$

Note: L is sometimes called the incidence vector or membership vector associated with L

$$P = \left\{ \begin{array}{cccc} \{1\} & \{1,2\} \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{2\} & \{1,4\} \{2,3\} & \{1,2,4\} \\ \{3\} & \{2,4\} \{3,4\} & \{1,3,4\} \\ \{4\} & & \{2,3,4\} \end{array} \right\}$$

$$P = \begin{cases} \{1\} & \{1,2\} \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{2\} & \{1,4\} \{2,3\} & \{1,2,4\} \\ \{3\} & \{2,4\} \{3,4\} & \{1,3,4\} \\ \{2,3,4\} & \{2,3,4\} & S_2, \end{cases}$$

$$P = \{S_0, S_1, S_2, S_3, S_4 \}$$

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$$|S_0| = {4 \choose 0}, |S_1| = {4 \choose 1}, |S_2| = {4 \choose 2}, |S_3| = {4 \choose 3}, |S_4| = {4 \choose 4}$$

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$$|P| = |S_0| + |S_1| + |S_2| + |S_3| + |S_4|$$

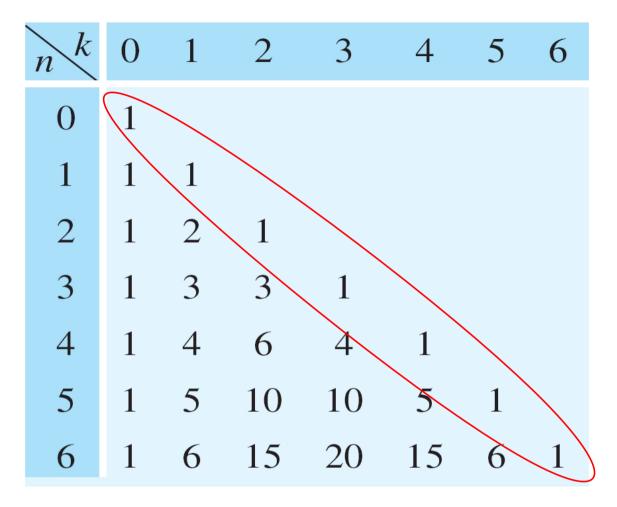
$$= {4 \choose 0} + {4 \choose 1} + {4 \choose 2} + {4 \choose 3} + {4 \choose 4}$$

$$= 2^4 = 16$$

n^{k}	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

n^{k}			2				
0 1 2 3 4 5 6	$\sqrt{1}$						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
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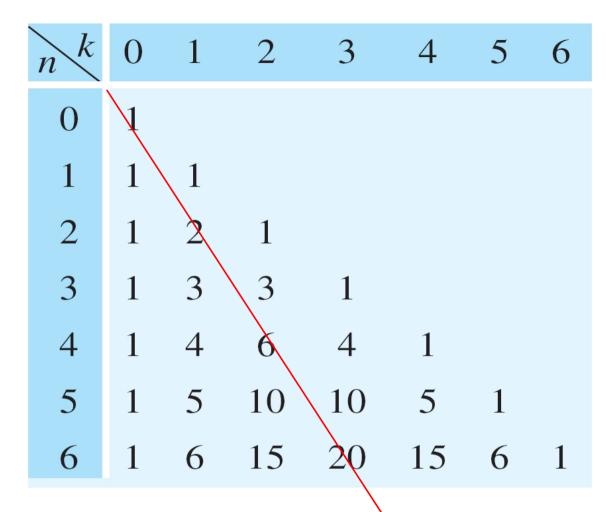
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0	1						
1	1	1					
2	1	2	1				
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Each row increases at first and then decreases.

(will see why in homework)



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Sum of items on n^{th} row is 2^n

Pascal's Triangle

Pascal's Triangle

Take the table

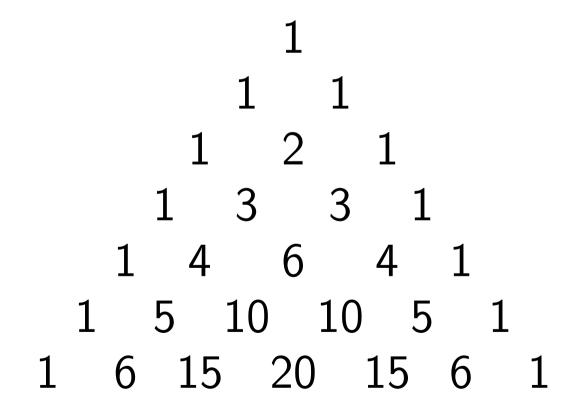
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0	1						
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4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle



What is the next row in the table?

```
1 2 1
     3 3 1
   4 6 4
  5
     10 10 5
   15 20 15 6 1
7 21 35 35 21 7 1
```

What is the next row in the table?

1 2 1 1 3 3 1 4 6 4 5 10 10 5 6 15 20 15 6 1 7 21 35 35 21 7 1

Pascal relationship

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

1 2 1 4 6 4 5 10 10 5 1 6) 15 20 15 6 1 21 35 35 21 7 1

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4 6 4 5 10 10 5 1 15) 20 15 6 1 35 35 21 7 1

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4 6 4 5 10 10 5 20) 15 6 1 **35 35 21 7 1**

Pascal relationship

4 6 4 5 10 10 5 6 15 1 7 21 35

Pascal relationship

1 2 1 4 6 4 5 10 10 5 1 6 15 20 (15) 1 7 21 35 35

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1 2 1 4 6 4 5 10 10 5 1 6 15 20 15 1 7 21 35 35 21

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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's relationship says that, for 0 < k < n,

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A purely *algebraic* proof (manipulating formulas) is possible.

In discrete mathematics, though, we prefer to derive intuitive explanations. In this case, that would involve interpreting Pascal's relationship as a statement describing *relationships among sets*.

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

we know $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

Therefore, each term (left and right) in

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

represents the number of subsets of a particular size chosen from an appropriately sized set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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Try to use sum principle to explain relationship among these three terms.

Example: n = 5, k = 2

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Set S_1 of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts. S_2 the 2-subsets that contain E and S_3 , the set of 2-subsets that do not contain E.

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 S_2 is equivalent to choosing 1 item out of $\{A,B,C,D\}$: $|S_2|={4 \choose 1}$

 S_3 chooses 2 items out of $\{A,B,C,D\}$: $|S_3|=\binom{4}{2}$

Sum Principle:
$$\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}$$

If n and k are integers satisfying 0 < k < n, then

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To apply sum principle, partition S_1 into S_2 and S_3 .

Let S_2 be set of k-element subsets that contain x_n .

Let S_3 be set of k-element subsets that don't contain x_n

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 $|S_3| = \binom{n-1}{k}$ since this is just how to choose a k-element subset from a (n-1) size set

 $|S_2|=\binom{n-1}{k-1}$ since this is just how to choose a (k-1)-element subset from a (n-1) size set

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 $|S_2|=\binom{n-1}{k-1}$ since this is just how to choose a (k-1)-element subset from a (n-1) size set

$$\Rightarrow \binom{n}{k} = |S_1| = |S_2| + |S_3| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Blaise Pascal

- Born 1623; Died 1662
- French Mathematician
- A Founder of Probability Theory
- Inventor of one of the first (the 2nd?) mechanical calculating machines
- Pascal Programming Language named for him



$$(x+y) = \begin{pmatrix} 1\\0 \end{pmatrix} x + \begin{pmatrix} 1\\1 \end{pmatrix} y$$

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$$(x+y)^{3} = x^{3} + 3x^{2}y + 3xy^{2} + y^{3}$$
$$= {3 \choose 0}x^{3} + {3 \choose 1}x^{2}y + {3 \choose 2}xy^{2} + {3 \choose 3}y^{3}$$

The Binomial Theorem

Number of k-element subsets of an n-element set is called a **binomial coefficient** because of its role in the algebraic expansion of a binomial $(x + y)^n$.

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Theorem 1.4 (Binomial Theorem)

For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

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or, in summation notation,

$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

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$$(x+y)(x+y)(x+y) = [x(x+y) + y(x+y)](x+y) = (xx + yx + xy + yy)(x+y) = (xx + yx + xy + yy)x + (xx + yx + xy + yy)y = xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.$$

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= $(xx + yx + xy + yy)(x + y)$
= $(xx + yx + xy + yy)x + (xx + yx + xy + yy)y$
= $xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy$.

Each monomial term in the final result is of form $x^{3-i}y^i$ and is the product of – one blue, one red, and one green.

For each color we can choose either an x or a y.

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Each *monomial* term in the final result is of form $x^{3-i}y^i$ and is the product of – one blue, one red, and one green.

For each color we can choose either an x or a y.

Coefficient of $x^{3-i}y^i$ is # of ways of choosing i y's from three colors $= \binom{3}{i}$

Example:
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Aternatively, can think of the monomial as *lists* where each item of the list is either x or y.

```
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```

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Each monomial is a length-n list of x's and y's.

In each list, the *i*th entry comes from the *i*th binomial factor.

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Therefore, the coefficient of $x^{n-k}y^k$, is $\binom{n}{k}$.

By applying the binomial theorem

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What is $(2 + y)^4$?

By applying the binomial theorem

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What is
$$(2+y)^4$$
?
 $(2+y)^4 = 16 + 32y + 24y^2 + 8y^3 + y^4$.

By applying the binomial theorem

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Now set x = y = 1. This gives

$$2^{n} = (1+1)^{n} = \sum_{i=0}^{n} \binom{n}{i}$$

Labelling and Trinomial Coefficients

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There are $\binom{n}{k}$ ways to choose the items with red labels. The other n-k items will then get the green labels So this is just $\binom{n}{k}$

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There are $\binom{n}{k_1}$ ways to choose the red items

There are then $\binom{n-k_1}{k_2}$ ways to choose the green items from the remaining $n-k_1$.

The remaining k_3 items get labelled orange

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Using the product principle the total number of labellings is

$$\binom{n}{k_1} \binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$

When $k_1 + k_2 + k_3 = n$, we call

$$\frac{n!}{k_1!k_2!k_3!}$$

a trinomial coefficient and denote it as

$$\begin{pmatrix} n \\ k_1 & k_2 & k_3 \end{pmatrix}$$

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Note that this slightly modifies the notation for binomial coefficients. If we really wanted the notation to be consistent (which we don't) we could write the binomial coefficient $\binom{n}{k}$ as

$$\begin{pmatrix} n \\ k & (n-k) \end{pmatrix}$$

We really just saw that the Trinomial Coefficient

$$\begin{pmatrix} n \\ k_1 & k_2 & k_3 \end{pmatrix}$$

is the number of ways to partition a set of size n into three subsets (where order of the subsets does not count) of sizes k_1 , k_2 and k_3 .

Example:

$$(x + y + z)(x + y + z)(x + y + z)(x + y + z) = xxxx + xxxy + xxxz + xxyx + \dots + zzzy + zzzz.$$

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After opening the parentheses and multiplying, there will be, in total, $3^4 = 81$ different monomial terms (lists)

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