Proof by Smallest Counterexample

Definitions:

- $\log_2(n)$ is x such that $2^x = n$.
  $\lfloor \log_2(n) \rfloor$ is the unique i s.t. $2^i \leq n < 2^{i+1}$

  e.g. $\lfloor \log_2(2) \rfloor = 1$, $\lfloor \log_2(3) \rfloor = 1$, $\lfloor \log_2(4) \rfloor = 2$
  $\lfloor \log_2(31) \rfloor = 4$, $\lfloor \log_2(32) \rfloor = 5$, $\lfloor \log_2(33) \rfloor = 5$

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- Prime factorization of n is the representation of n as multiplication of a list of primes.
  e.g. $12 = 2 \times 2 \times 3$, $6! = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5$

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- Define $SIZE(n)$ to be the number of prime factors in prime factorization of n.
  e.g. $SIZE(12) = 3$, $SIZE(6!) = SIZE(720) = 7$

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Theorem:
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i.e. There is a smallest integer m s.t. $P(m)$ is false.

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$\le \lfloor \log_2(m) \rfloor$ — Contradiction!

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$\Rightarrow$ No prime is a factor of n .
$\Rightarrow$ n is a prime greater than m. Contradiction!