COMP 170 Discrete Mathematical Tools for CS
2005 Fall Semester – Practice Assignment # 2
Distributed: Dec 2, 2005

This handout is meant as a *practice* assignment to let you practice the material taught during the last weeks of class. Solutions to this assignment *should not* be handed in.

Many of these problems are taken (modified) from the backs of section 5.7 of the book.

Problem 1: Suppose a student who knows 60% of the material covered in a chapter of a textbook is going to take a five-question objective (each answer is either right or wrong, not multiple choice or true-false) quiz. Let X be the random variable that gives the number of questions the student answers correctly for each quiz in the sample space of all quizzes the instructor could construct.
(a) What is the expected value of the random variable $X - 3$?
(b) What is the expected value of $(X - 3)^2$?
(c) What is the variance of X?

X has a binomial distribution with $n = 5$ and $p = .6$ so $E(X) = np = 3$.

(a) $E(X - 3) = E(x) - 3 = 0$.

(b)

$$E((x-3)^2) = (-3)^2 \cdot .4^5 + (-2)^2 \cdot 5 \cdot .6 \cdot .4^4 + (-1)^2 \cdot 10 \cdot .6^2 \cdot .4^3 + (1)^2 \cdot 5 \cdot .6^4 \cdot .4^2 + 2^2 \cdot (.6)^5 = 1.2$$

(c) $Var(X) = E((x - 3)^2) = 1.2$

Alternatively, we can set X_i to be the indicator random variable as to whether question i is answered correctly ($X_i = 1$) or not ($X_i = 0$). Then $Var(X_i) = (.6) \cdot (.4) = .24$. Since $X = \sum_{i=1}^{5} X_i$ and the X_i are all independent,

$$Var(X) = \sum_{i=1}^{5} Var(X_i) = 5Var(X_1) = 1.2.$$

Problem 2: If the quiz in Problem 1 has 100 questions;
(a) what is the expected number of right answers?
(b) what is the variance of the expected number of right answers?

In this case $X = \sum_{i=1}^{100} X_i$, *where the* X_i *are as defined in the previous problem.*

(a) By linearity of expectation,

$$E(X) = \sum_{i=1}^{100} E(X_i) = 100 \cdot (.6) = 60.$$

(b) Because the X_i *are all independent*

$$Var(X) = \sum_{i=1}^{100} Var(X_i) = 100 \cdot (2.4) = 24.$$

Problem 3: Show that if X and Y are independent and b and c are constant, then $X - b$ and $Y - c$ are independent.
(Note: This simplifies the proof of Theorem 5.29)

Let $X' = X - b$ *and* $Y' = Y - c$. *Then,*

$$\begin{aligned} P((X' = x) \wedge (Y' = y)) &= P((X = x + b) \wedge (Y = y + c)) \\ &= P(X = x + b) \cdot P(Y = y + c) \\ &= P(X' = x) \cdot P(Y' = y) \end{aligned}$$

where the second equality comes from the independence of X *and* Y.

Problem 4: A cup contains three coins; one \$1 coin; one \$2 coin and one \$5 coin. Withdraw two coins, first one and then the second, without replacement.
(a) What is the expected amount of money and variance for the first draw?
(b) For the second draw?
(c) For the sum of both draws?

(a) & (b) Note that, on both the first and second draws, each coin is drawn with the same probability, i.e., $\frac{1}{3}$. *therefore, the expectation for both draws is*

$$\frac{1}{3}(1+2+5) = \frac{8}{3}.$$

The variance of each draw is then

$$\frac{1}{3}\left(\left(1-\frac{8}{3}\right)^2 + \left(2-\frac{8}{3}\right)^2 + \left(5-\frac{8}{3}\right)^2\right) = \frac{26}{9}.$$

(c) For the sum of the two draws first note that, by linearity of expectation the expectation of the sum of the first two draws is $2 \cdot \frac{8}{3}\frac{16}{3}$. *Next note that each combination of two coins is equally likely to be picked, so the variance is*

$$\frac{1}{3}\left(\left(3-\frac{16}{3}\right)^2 + \left(6-\frac{16}{3}\right)^2 + \left(7-\frac{16}{3}\right)^2\right) = \frac{26}{9}$$

This answers the questions. You might have noticed that the variance of the sum of the first two draws *has the same variance as the variance* each draw by itself. *This is* not *a coincidence. Here's why. Let X be the random variable denoting the value of the first draw, Y the value of the second draw and Z the coin left in the cup* after *the first two draws.*

The thing to notice is that X, Y and Z all have the same distribution functions so $Var(X) = Var(Y) = Var(Z)$. That is, for each of them, each coin has probability $\frac{1}{3}$ of being chosen. Now notice that $X + Y + Z = 8$ so $Z = 8 - (X + Y)$.

We now need the following two observations:

(i) if W is a random variable then $Var(-W) = W$ (this can either be proven from scratch or use the result of problem 6).

(ii) if c is a constant, then $Var(c+W) = Var(W)$. Again, this can either be proven from scratch, or you can notice that c, if considered as a random variable *and W are independent of each other and $Var(c) = 0$ so*
$Var(c+W) = Var(c) + Var(W) = Var(W)$.

Combining the two gives

$$Var(c-W) = Var(c+(-W)) = Var(-W) = Var(W).$$

Plugging in $c = 8$ *and* $W = X + Y$ *gives*

$$Var(X) = Var(Z) = Var(8 - (X + Y)) = Var(X + Y).$$

Problem 5: What are the expected number of failures and the variance of the number of failures in n independent trials with probability p of success?
Compare your answers with the corresponding results for successes.

A failure has probability $1 - p$ *of occuring. Let* Y *be the number of failures. Then* Y *is simply the number of* "successes" *in* n *independent trials with probability* $1 - p$ *of success. Thus*

$$E(Y) = n(1-p) \quad \textit{and} \quad Var(Y) = n(1-p)(1-(1-p)) = np(1-p).$$

Let X *be the number of sucesses.*

Then $E(X) = np = n - E(Y)$.

Also, $Var(X) = Var(Y)$,

Problem 6: Let X be a random variable and c a constant number. What is $Var(cX)$ (as a function of $Var(X)$)?

$$\begin{aligned} Var(cx) &= E\left((cX - E(cx))^2\right) \\ &= E\left((cX - cE(x))^2\right) \\ &= E\left(c^2(X - E(x))^2\right) \\ &= c^2 E\left((X - E(x))^2\right) \\ &= c^2 Var(X) \end{aligned}$$

Problem 7: (a) Roll a fair die and let X be the number of dots showing on top. What are $E(X)$ and $Var(X)$?
(b)What are $E(2X)$ and $Var(2X)$?
(c) Now roll another die and let Y be the number of dots showing. What are $E(X+Y)$ and $Var(X+Y)$?

(a) $E(X) = 3.5$

$$\begin{aligned} Var(X) &= \frac{1}{6}[(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2] \\ &= \frac{35}{12} \end{aligned}$$

(b) By Linearity of Expectation

$$E(2x) = 2E(X) = 7.$$

By the result of the previous question

$$Var(2X) = 4Var(X) = 4 \cdot \frac{35}{12} = \frac{35}{3}.$$

(c) By Linearity of Expectation

$$E(X+Y) = E(X) + E(Y) = 3.5 + 3.5 = 7.$$

Since X and Y are independent

$$Var(X+Y) = Var(X) + Var(Y) = 2 \cdot \frac{35}{12} = \frac{35}{6}.$$

Problem 8: Flip four fair coins. let X be the number of heads showing. Now flip four $\frac{1}{3}$-biased coins (that is, they have $P(H) = \frac{1}{3}$) and let Y be the number of heads showing.
(a) What is $E(X+Y)$?
(b) What is $Var(X+Y)$?

X is the number of successes in $n = 4$ independent trials with $p = \frac{1}{2}$. Therefore, by the theorems derived in class

$$E(X) = np = 2 \quad \textit{and} \quad Var(X) = np(1-p) = n \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

Similarly Y is the number of successes in $n = 4$ independent trials with $p = \frac{1}{3}$. Therefore, by the theorems derived in class

$$E(Y) = np = \frac{4}{3} \quad \textit{and} \quad Var(Y) = np(1-p) = n \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{8}{9}$$

(a) By Linearity of Expectation

$$E(X+Y) = E(X) + E(Y) = 2 + \frac{4}{3} = \frac{10}{3}.$$

Since X and Y are independent

$$Var(X+Y) = Var(X) + Var(Y) = 1 + \frac{8}{9} = \frac{17}{9}.$$