An Introduction to Hashing

(Following CLRS)

COMP 3711H - HKUST
Version of 9/17/2016
M. J. Golin

Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends

Introduction

Known: A set $U = \{, 1, 2, \ldots, u - 1\}$ of the universe of possible keys that could exist.

Goal: To maintain a dictionary that permits the following operation on keys

- **Search(x):** Find the record with key x or report that it does not exist
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Would like $O(1)$ (average) time per operation.

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For now, assume uniform hashing, that, every key is equally likely to hash to any of the m slots,

$$\forall x, i, \quad \Pr\left(h(x) = i\right) = \frac{1}{m}.$$

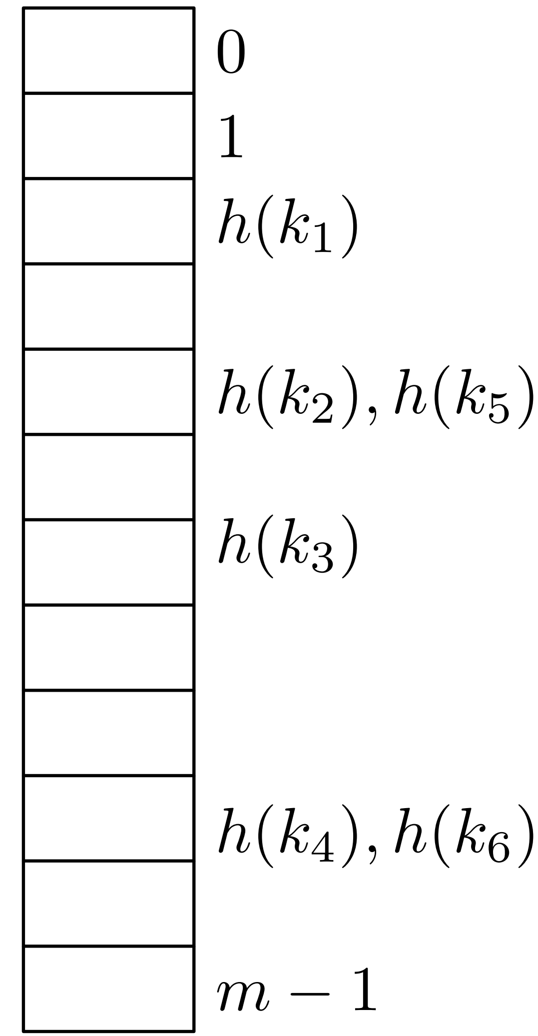
Introduction (iii)

$h : U \to \{0, 1 \ldots, m-1\}$

h maps the set of keys into a "small" table.
Key k is stored in table slot $h(k)$.

Fnding key k is then just a matter of going to table location $h(k)$.

Problem is that, since m is small, many keys might be mapped to same slot, creating collision.



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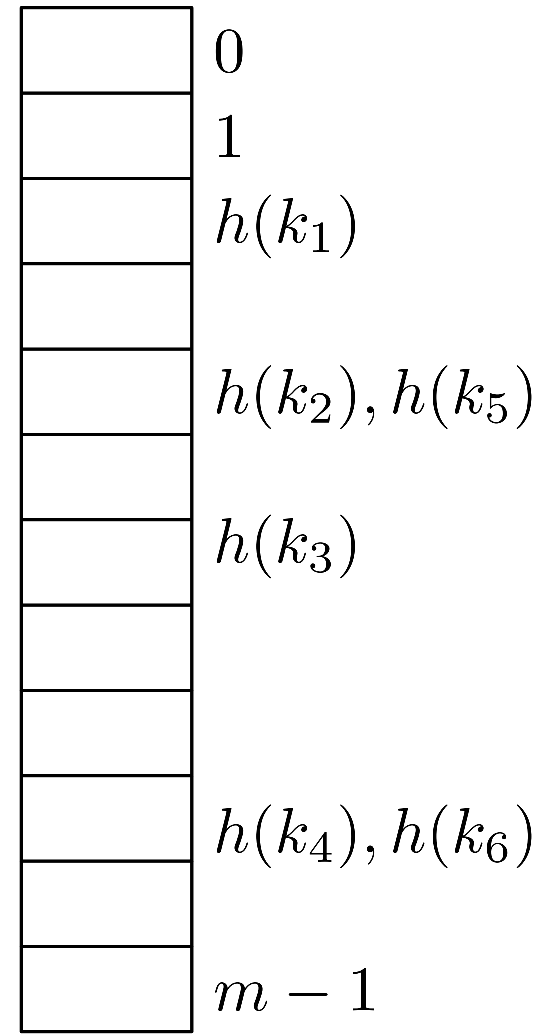

Two major approaches to addresing collisions:
(1) Chaining
(2) Open Addressing

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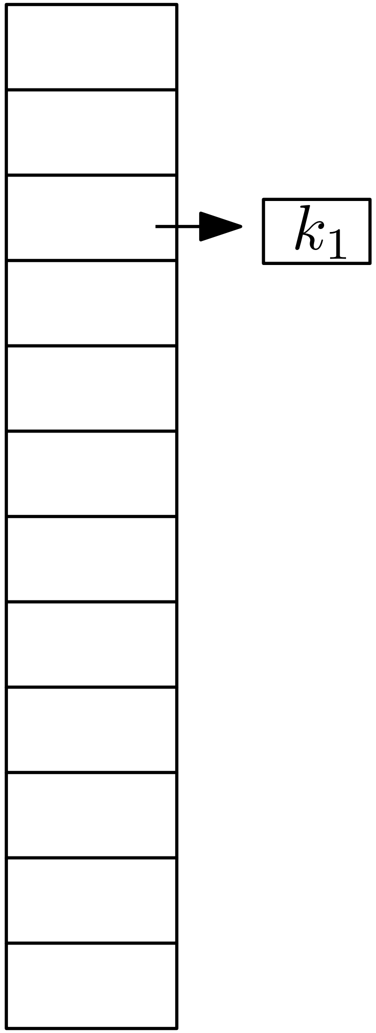

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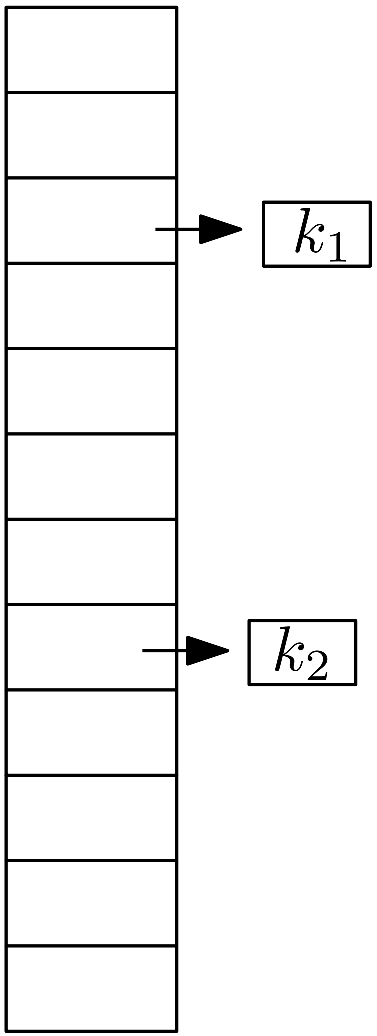

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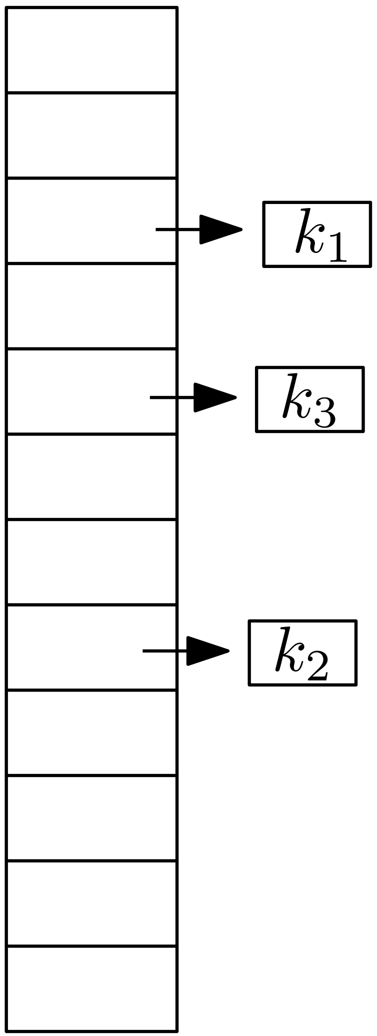

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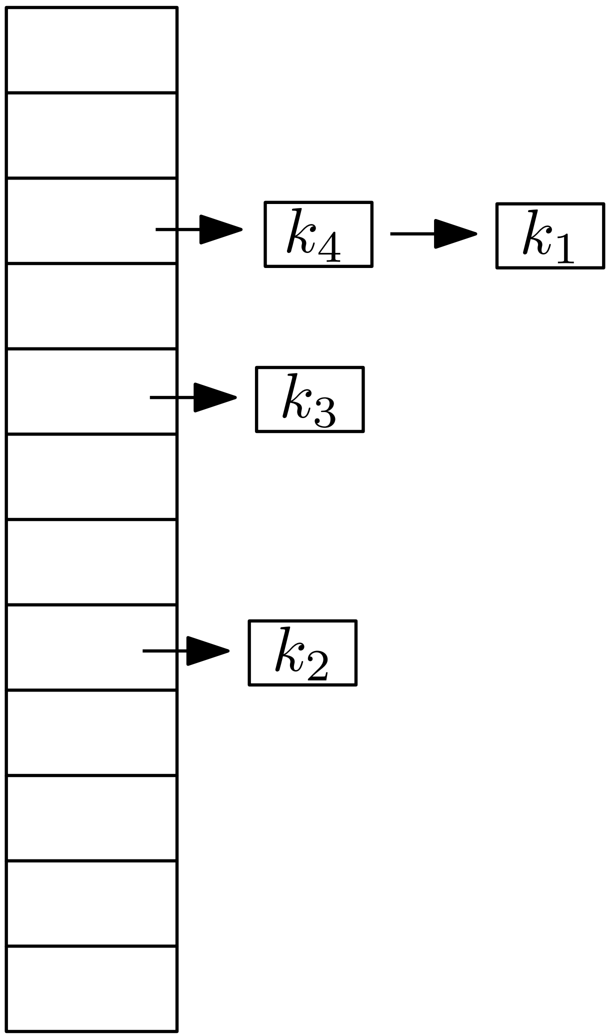

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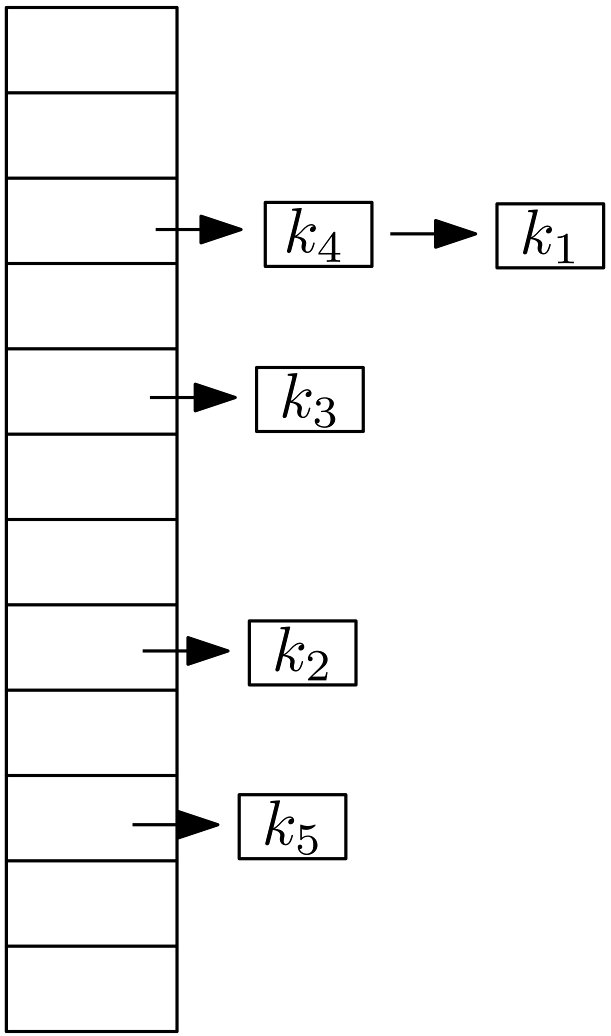

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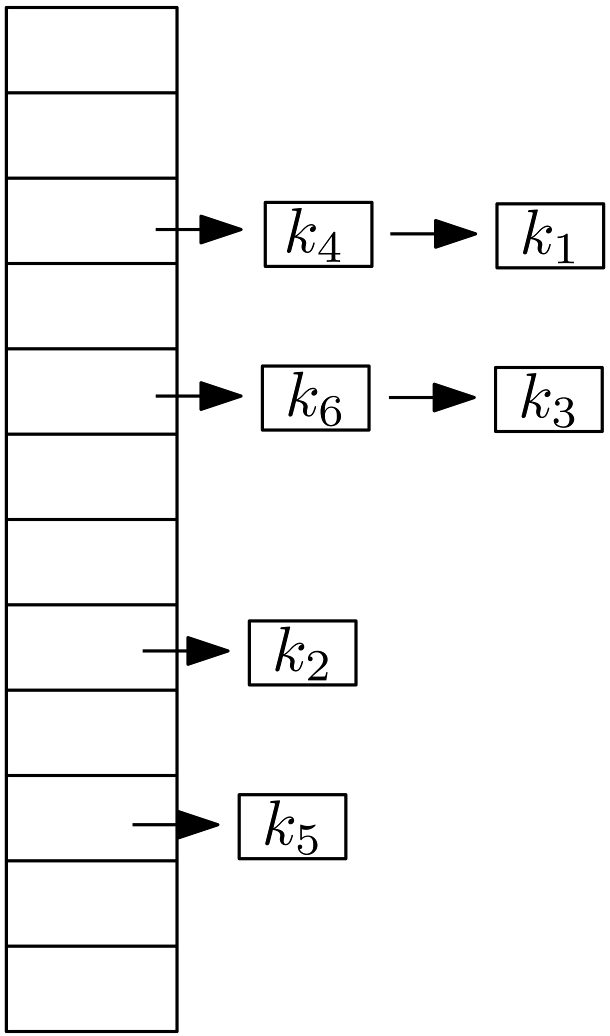

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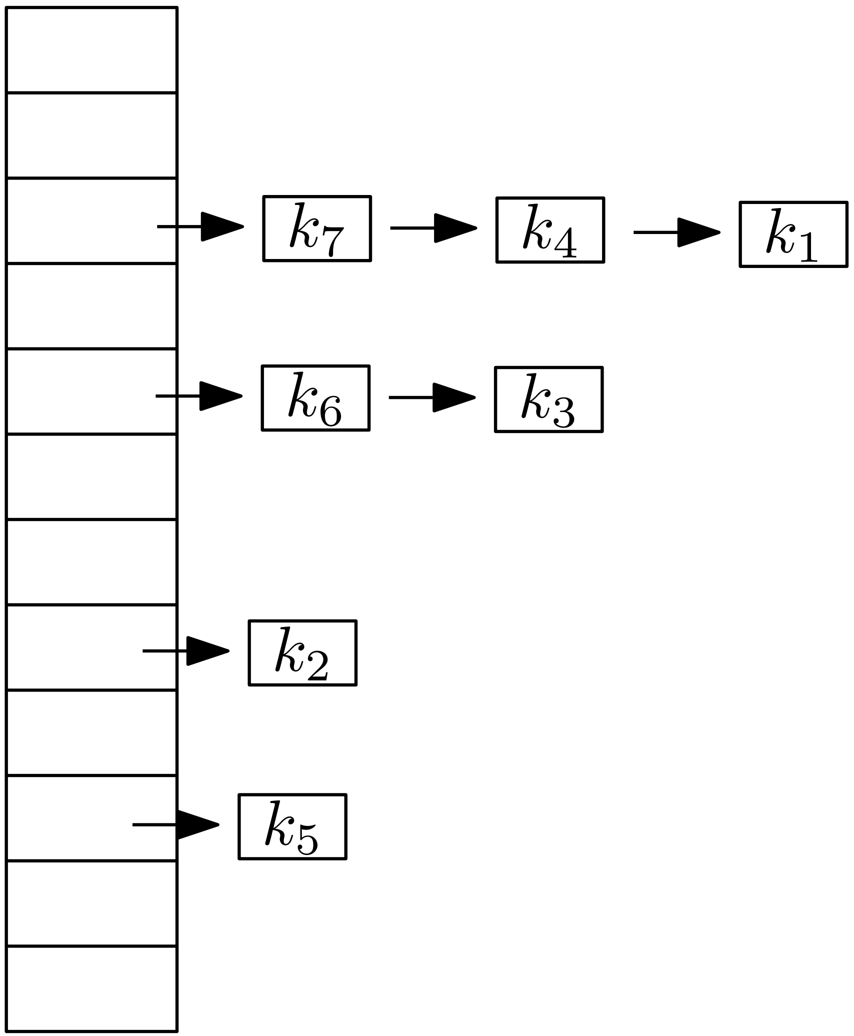

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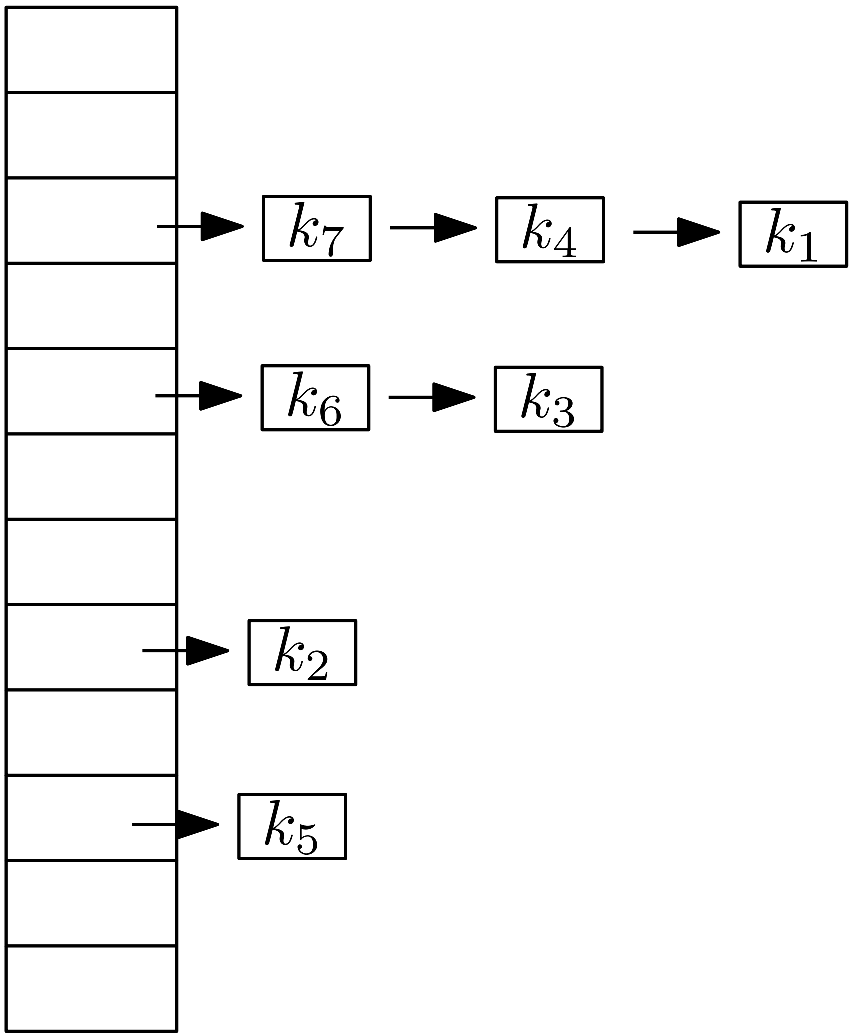

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Insert(x):Insert x into front of list for slot $h(x)$

Delete(x): Delete x from list for slot $h(x)$, if it's there.
Use doubly linked lists

Search(x): Search for x in list for $h(x)$

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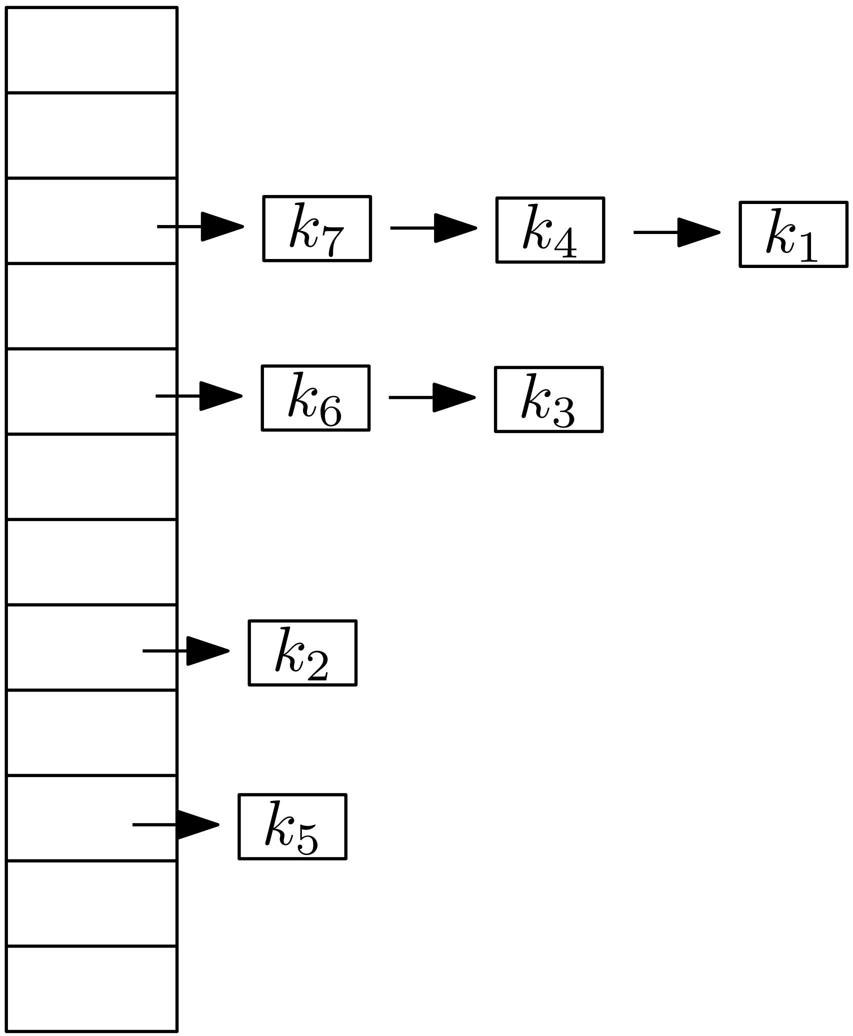

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Insert(x):Insert x into front of list for slot $h(x)$ $O(1)$

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Search(x): Search for x in list for $h(x)$ $O(\textit{length of list})$

Chaining: Unsuccessful Search

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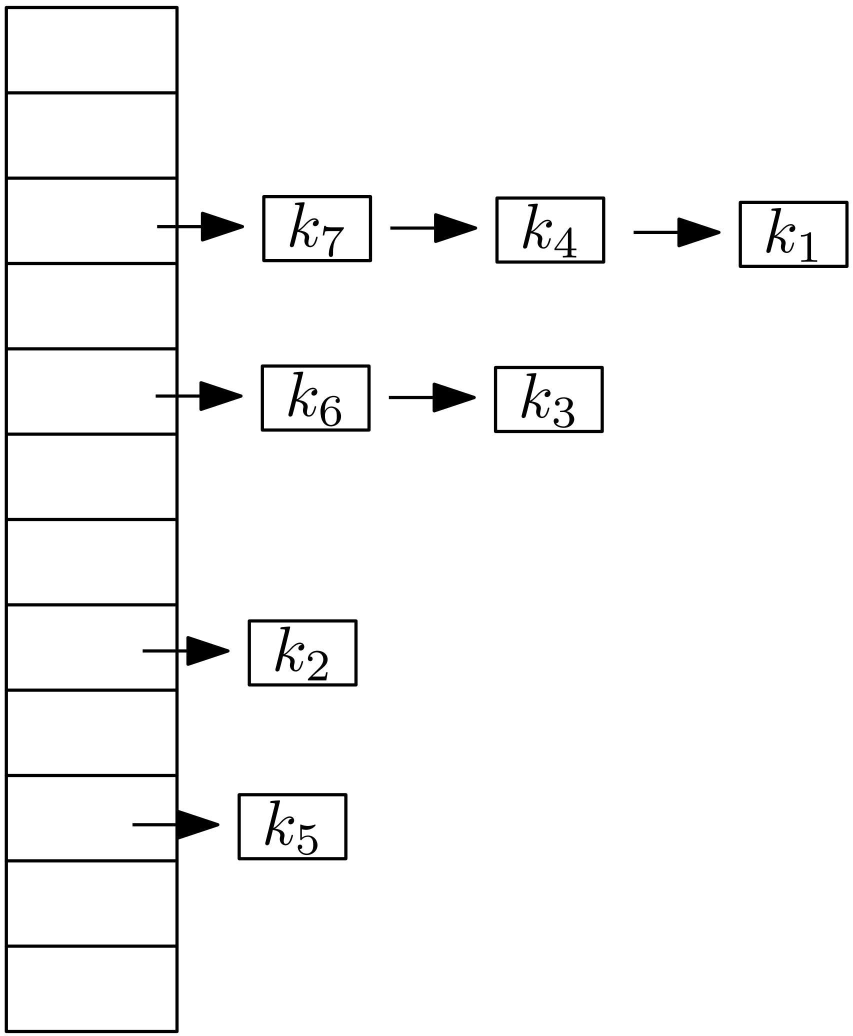

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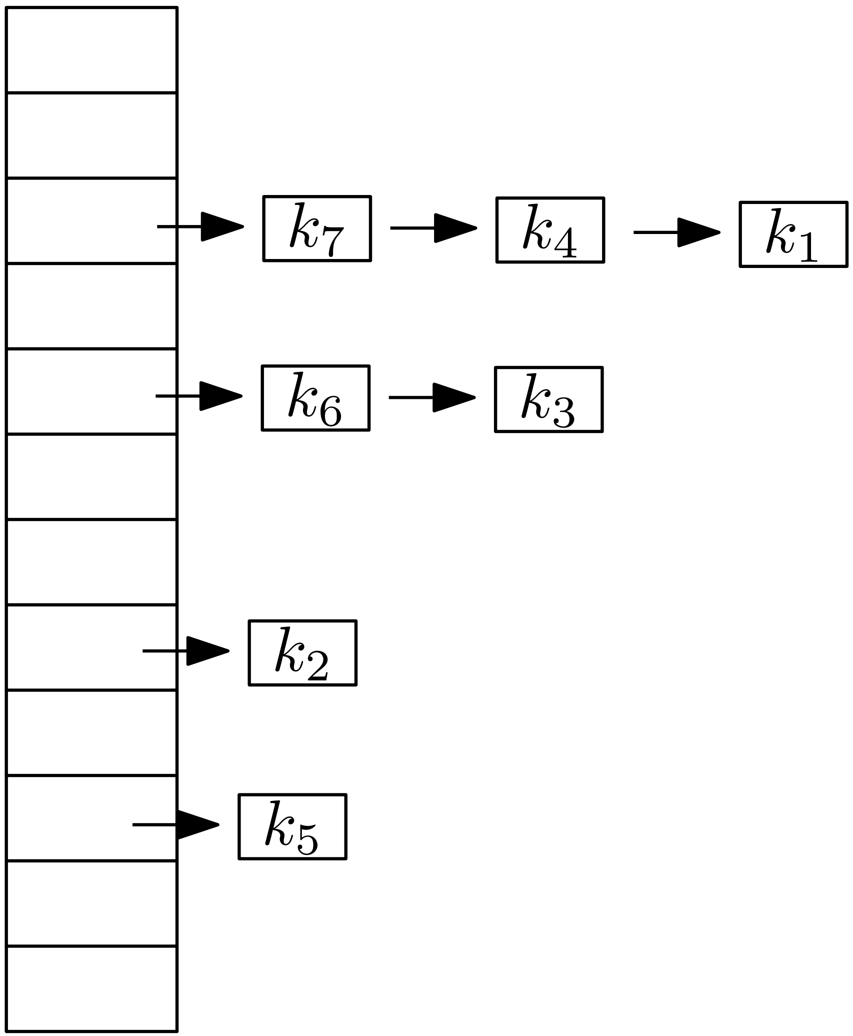

Search(x): Search for x in list for $h(x)$ O(*length of list*)

Recall load factor $\alpha = \frac{n}{m}$.

This is *average # items per list*.

Unsucessful search for x not in table will require searching entire list for $h(x)$.

Worst case length is $O(1)$.
Average case length is $O(\alpha)$.

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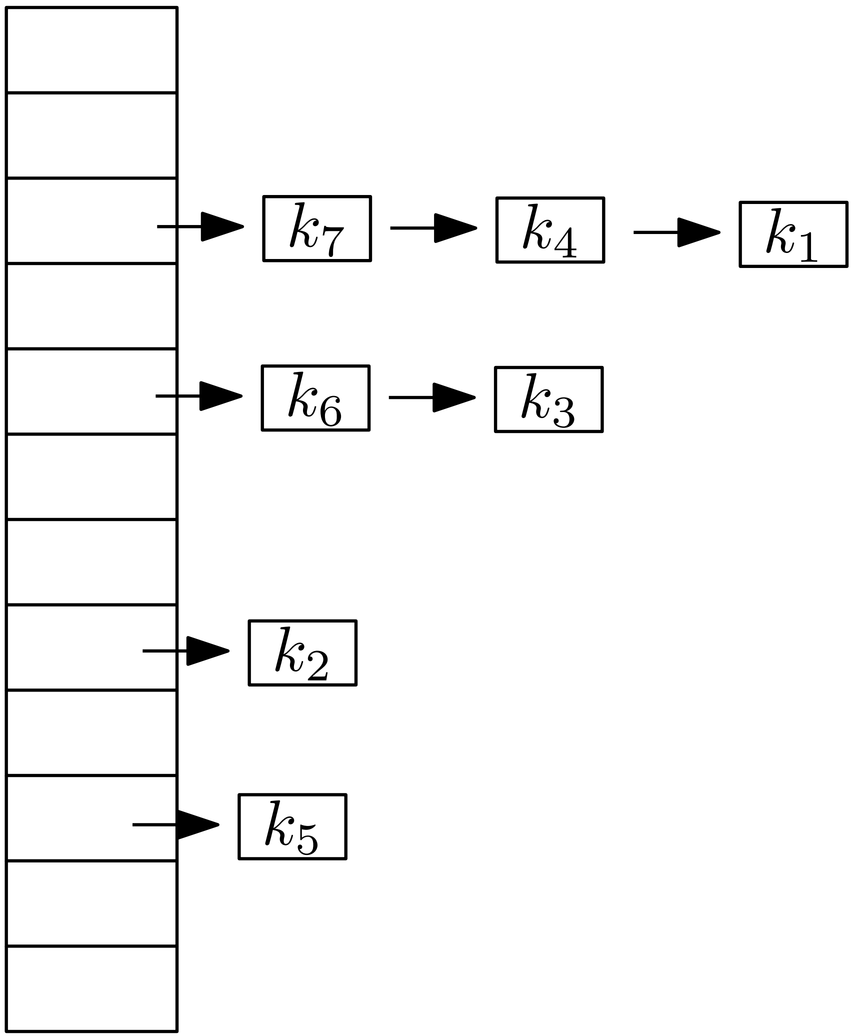

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This is *average # items per list*.

Unsucessful search for x not in table will require searching entire list for $h(x)$.

Worst case length is $O(1)$.
Average case length is $O(\alpha)$.

Average Unsucessful Search time is

$O(1 + \alpha)$

where 1 is amount of time to calculate $h(x)$.

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For Successful Search for x:

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If x is i'th item inserted

$\Rightarrow n - i$ items inserted after x

$\Rightarrow \alpha - \frac{i}{m} = \frac{n-i}{m}$ items inserted on average into $h(x)$ after x

x is equally likely (with prob $1/n$) to be i'th inserted item.

Average $\#$ of items ahead of x in list $h(x)$ is

$$\frac{1}{n}\sum_{i=1}^{n}\left(\alpha - \frac{i}{m}\right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}$$

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Adding 1 unit of time to calculate $h(x)$

Average cost of successful search is $\Theta(1+\alpha)$.

Chaining

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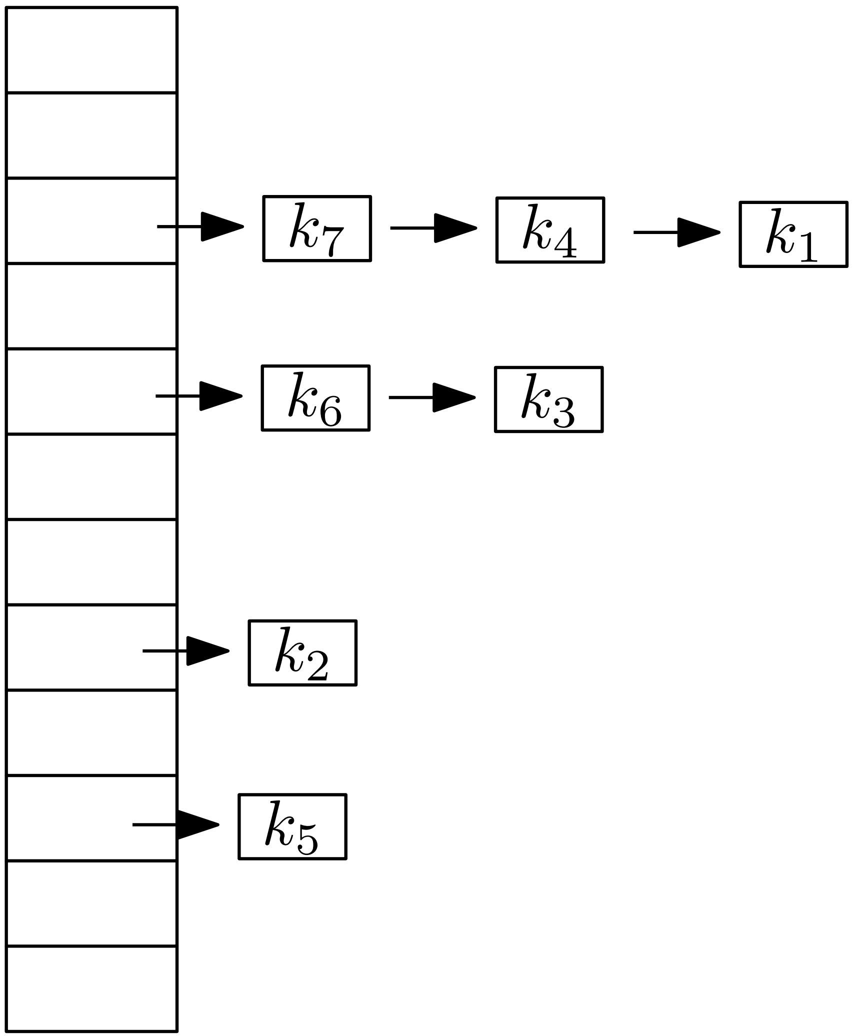

Search(x): Search for x in list for $h(x)$ $O($*length of list*$)$

Both Successful and Unsuccessful Search require $O(1+\alpha)$ time on average

where $\alpha = \frac{n}{m}$ is the load factor

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Open Addressing

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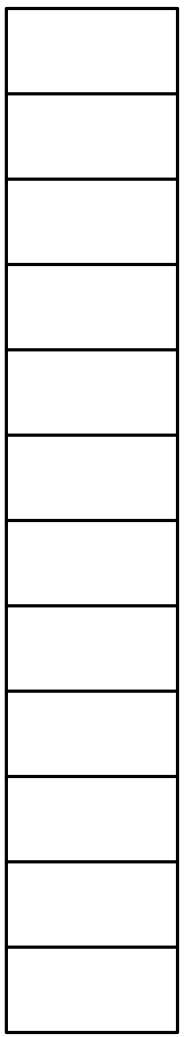

- No lists. All keys stored in hash table itself.
- For insertion, *probe* hash table until empty slot for insertion is found.
- *Probe Sequence* is part of hash function.
- Hash function is now

$$h : U \times \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$$

- Probe sequence for x is,

$$h(x, 0),\ h(x, 1), \ldots,\ h(x, m-1)$$

  which is a permutation of $\{0, 1, \ldots, m\}$

- For search(x), *probe* hash table using probe sequence for $h(x)$ until either x or empty slot for insertion is found.

Open Addressing: Linear Probing

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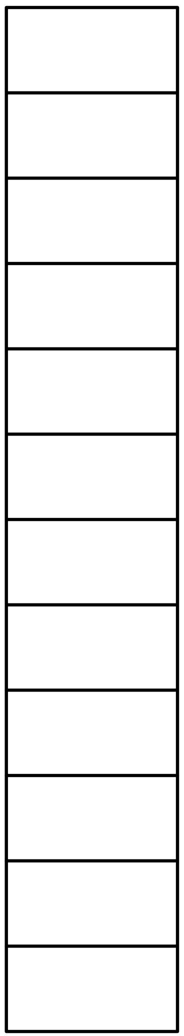

- Hash Function is $h(x, i) = \big(h'(x) + i\big) \bmod m$ where $g'(x)$ is original hash function.
- Insert: Attempts insertion at $h'(x)$, then $h'(x) + 1$, $h'(x) + 2$, etc,. (wrapping around to 0 after reaching end of table) until empty slot is found and x inserted there.
- Search(x): Examines probe sequence until it finds x or an empty slot.
  If empty slot is found then x wasn't previously inserted and search unsuccessful
- Deletion: More complicated.

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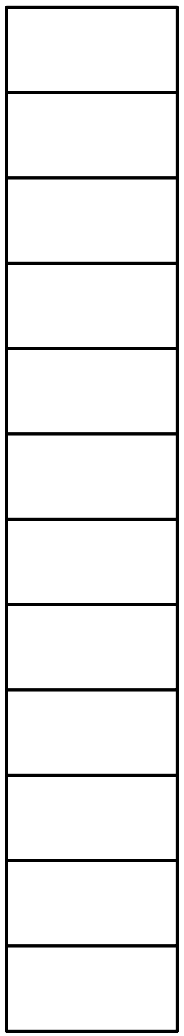

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- Search(x): Examines probe sequence until it finds x or an empty slot.
  If empty slot is found then x wasn't previously inserted and search unsuccessful
- Deletion: More complicated.
  Can't actually delete item and reset slot as 'empty"
  That would mess up Search(x).
  Can mark *slot as (used but) deleted.*
  Deletion in open addressing does cause difficulties.
  Better to use chaining.

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

As example, let $h(x) = x \bmod m$ with $m = 12$.

Open Addressing: Linear Probing

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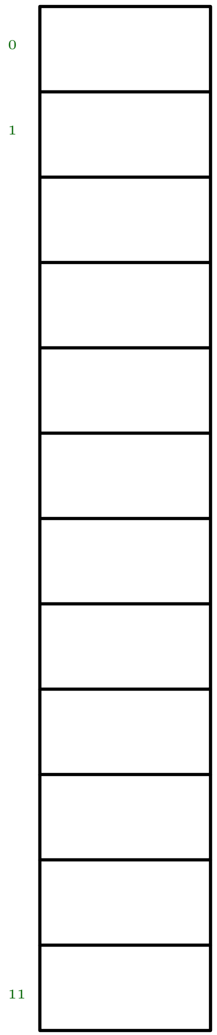

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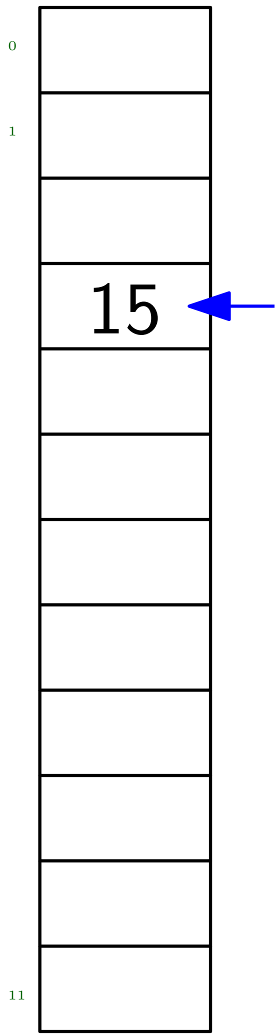

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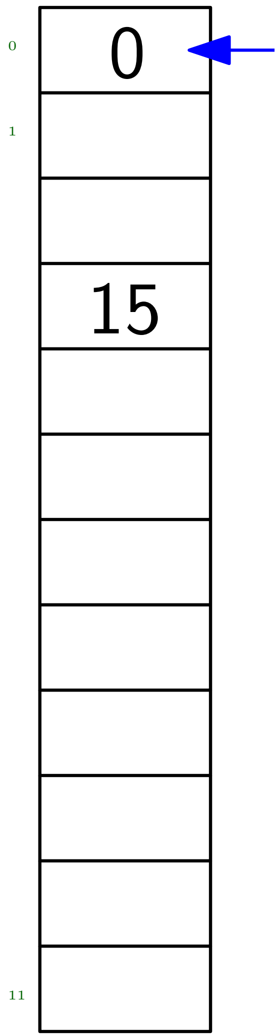

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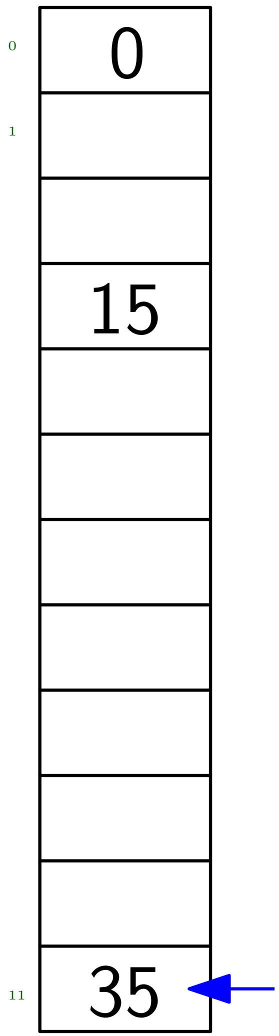

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0	0
1	
	15
11	35

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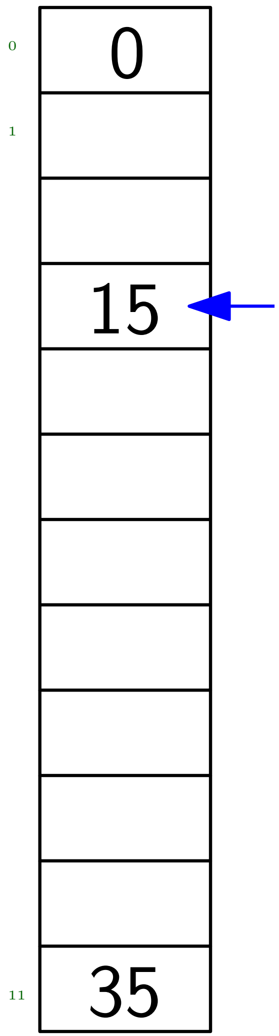

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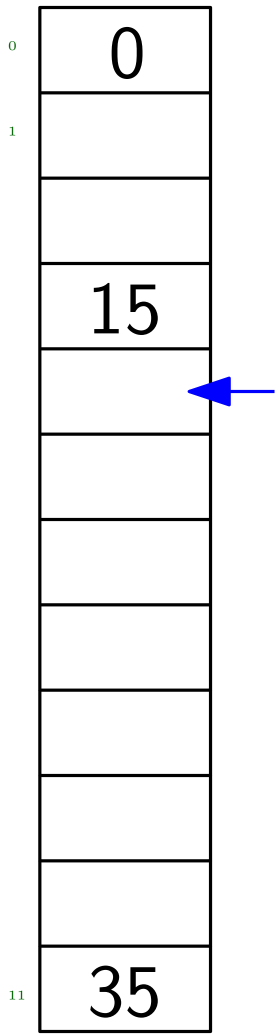

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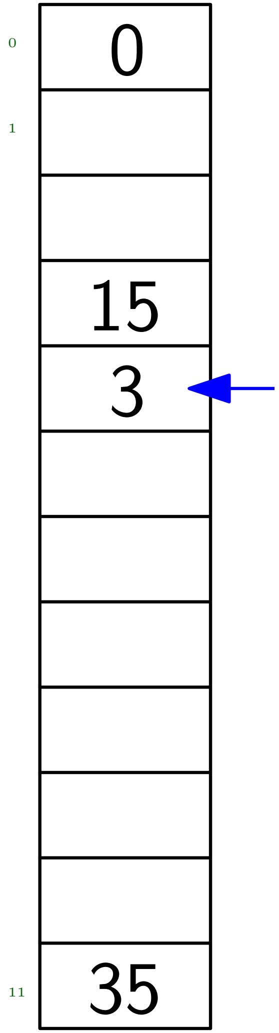

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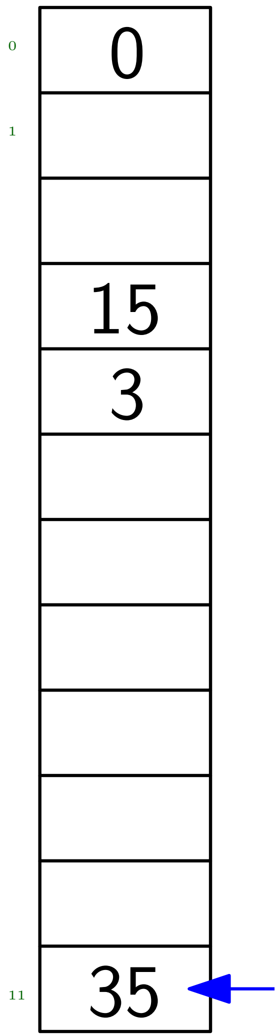

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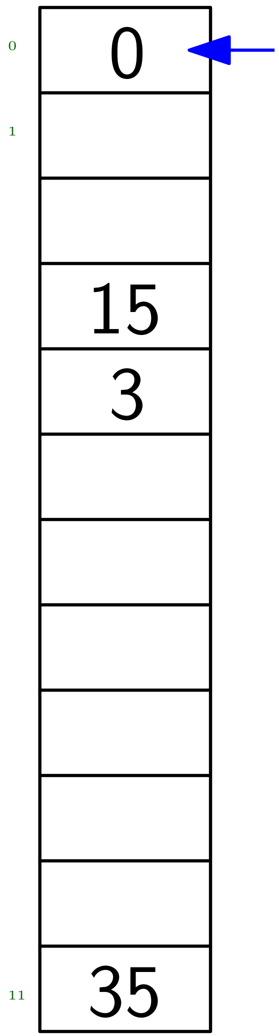

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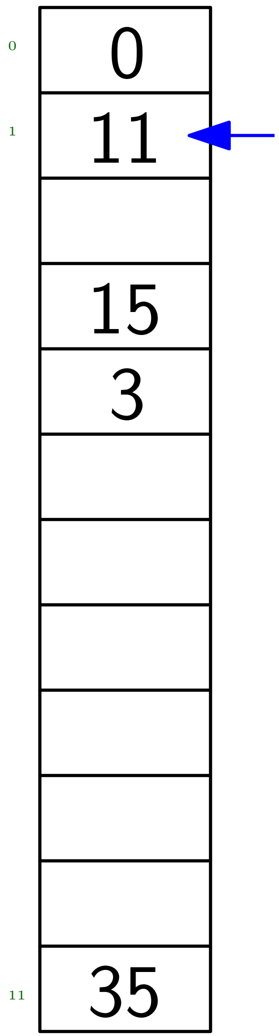

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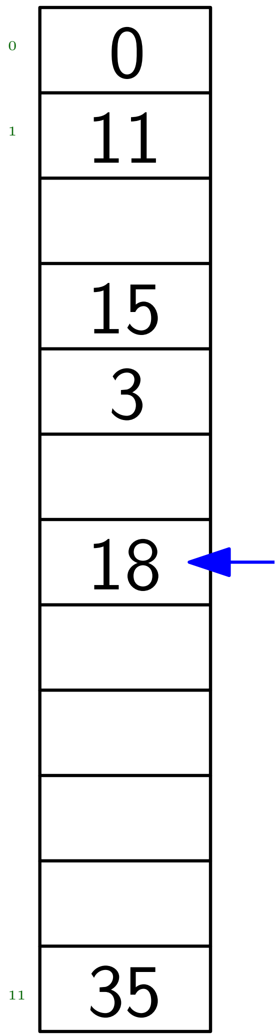

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0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash function

Insert(15)

Insert(0)

Insert(35)

Insert(3)

Insert(11)

Insert(18)

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

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Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

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Search(11)

Open Addressing: Linear Probing

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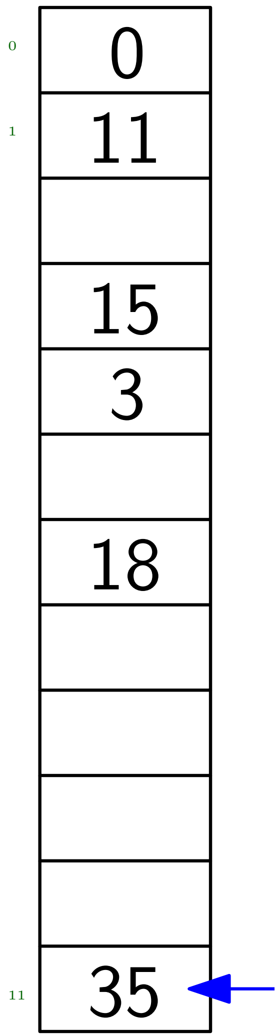

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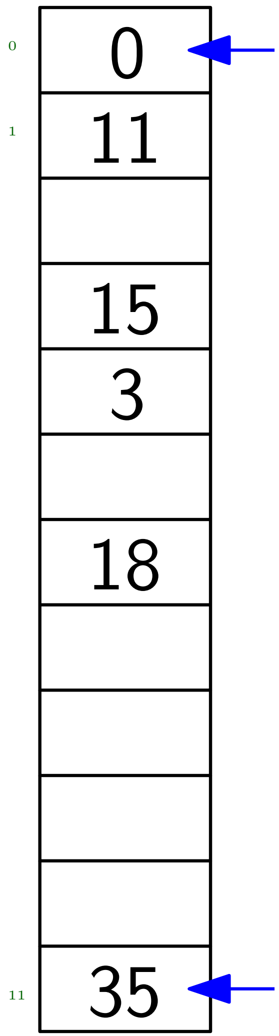

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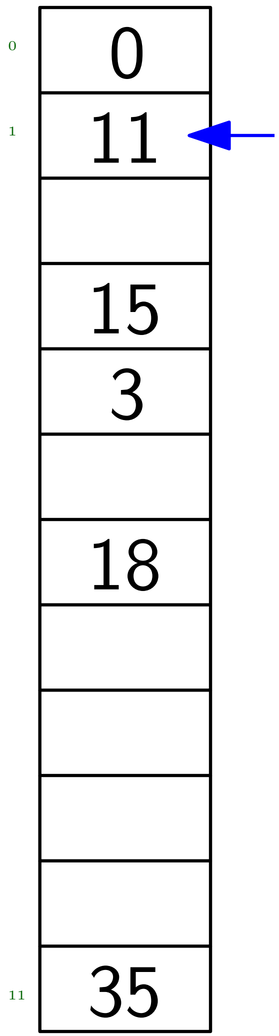

As example, let $h(x) = x \bmod m$ with $m = 12$.

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Search(11) $\qquad$ Exists

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

0	0
1	11
	15
	3
	18
11	35

As example, let $h(x) = x \bmod m$ with $m = 12$.

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Search(11) Exists

Search(3)

Open Addressing: Linear Probing

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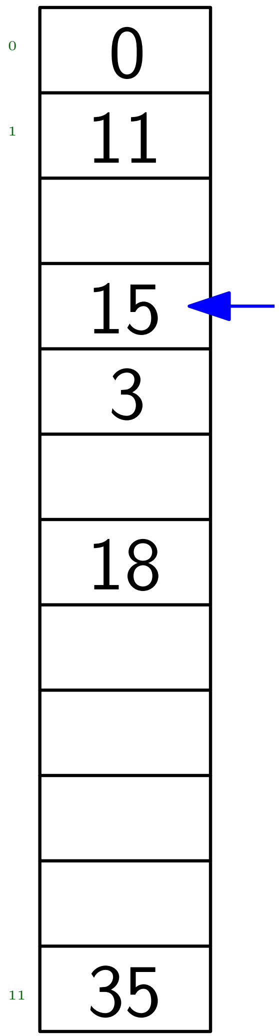

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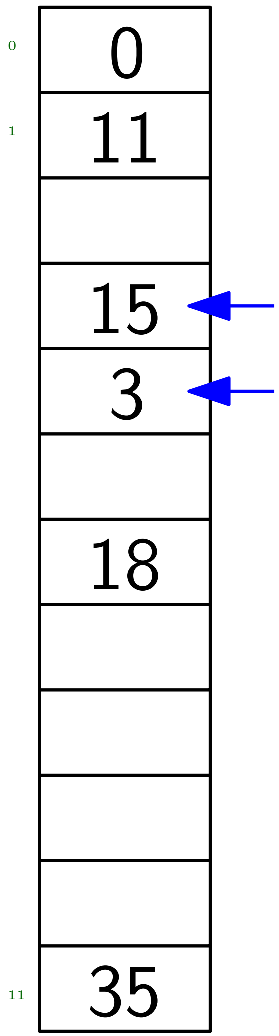

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Search(11) $\qquad$ Exists

Search(3) $\qquad$ Exists

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0	0
1	11
	15
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As example, let $h(x) = x \bmod m$ with $m = 12$.

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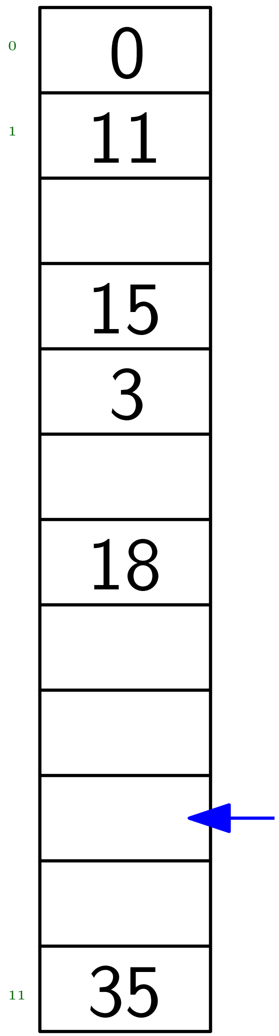
Search(11) Exists

Search(3) Exists

Search(9)

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



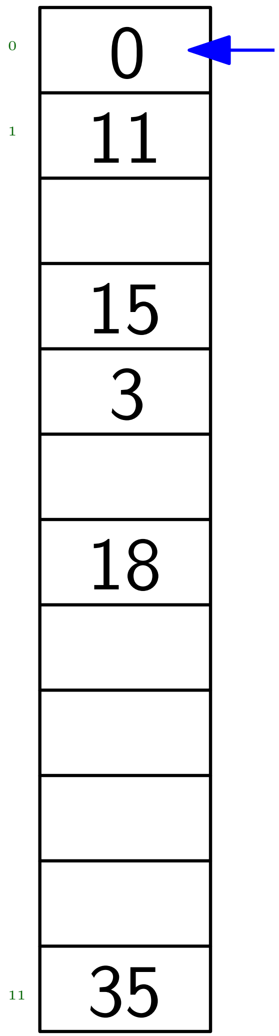
As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash functionn

Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



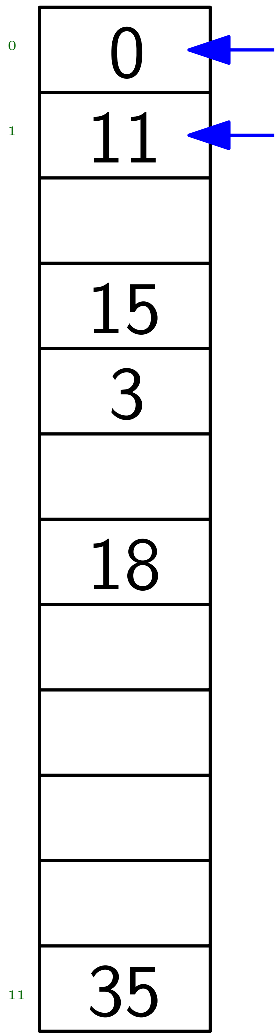
As example, let $h(x) = x \bmod m$ with $m = 12$.

Only for illustration. This is a BAD hash functionn

Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\quad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



As example, let $h(x) = x \bmod m$ with $m = 12$.

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Search(11) Exists

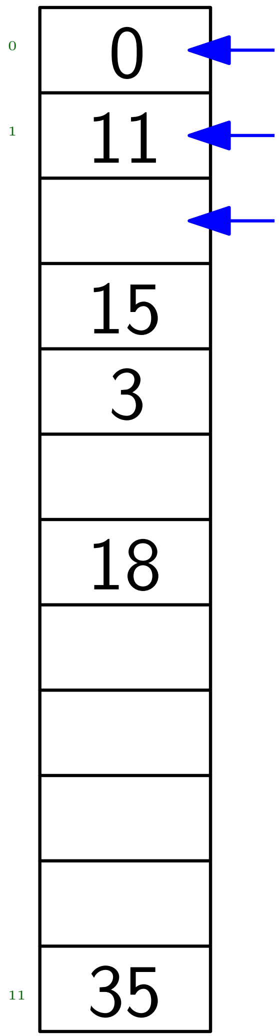
Search(3) Exists

Search(9) Does not exist

Search(24)

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



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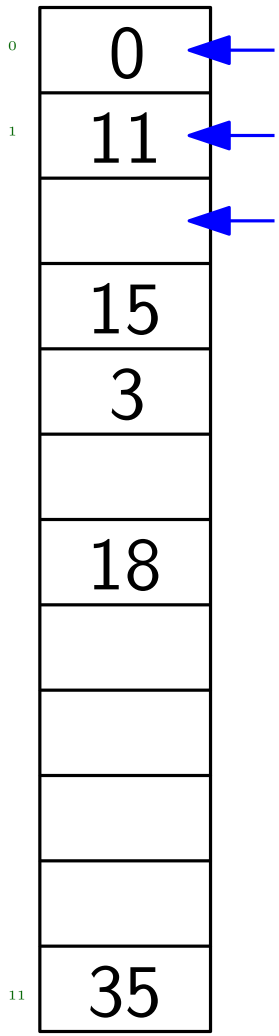
Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	

Open Addressing: Linear Probing

$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$



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Search(11)	Exists
Search(3)	Exists
Search(9)	Does not exist
Search(24)	Does not exist

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$h' : U \to \{0, 1 \ldots, m-1\}$ $\qquad$ $h(x) = \big(h'(x) + 1\big) \bmod m$

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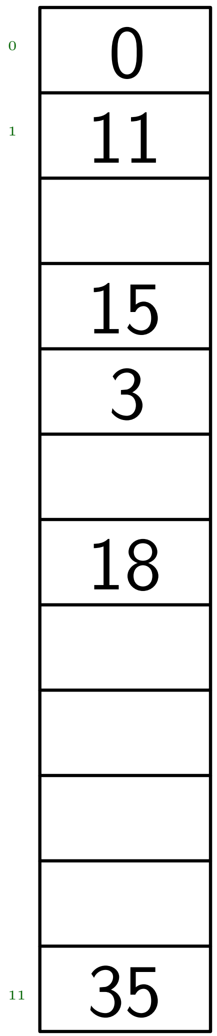

As example, let $h(x) = x \bmod m$ with $m = 12$.

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Easy to code but suffers from primary clustering.

Long runs build up, increasing average search time

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1	11
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As example, let $h(x) = x \bmod m$ with $m = 12$.

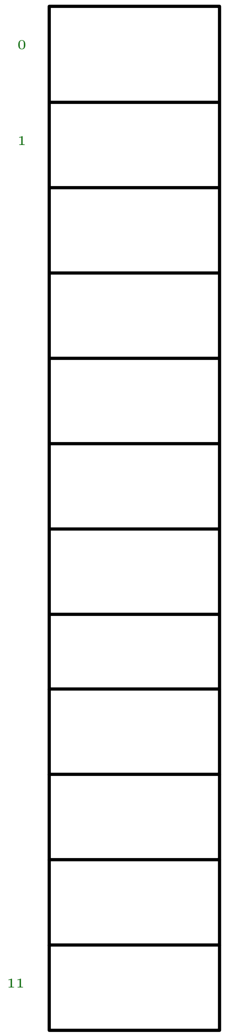
Only for illustration. This is a BAD hash functionn

Easy to code but suffers from primary clustering.
Long runs build up, increasing average search time

One fix is to change probe sequence to no longer be linear.

Open Addressing: Quadratic Probing

$h' : U \to \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x, i) = \left(h'(x) + c_1 + c_2 i^2\right) \bmod m$ where $h'(x)$ is original hash function and c_1, c_2 fixed constants.

- As example we will set $h'(x) = x \bmod 12$, $c_1 = 0$ and $c_2 = 1$ so $h(x, i) = \left(x + i^2\right) \bmod 12$

Open Addressing: Quadratic Probing

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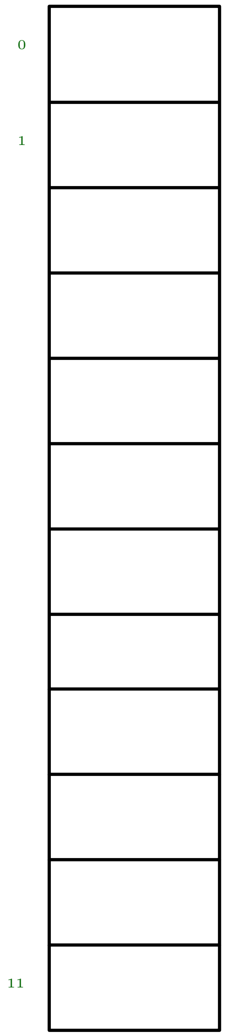

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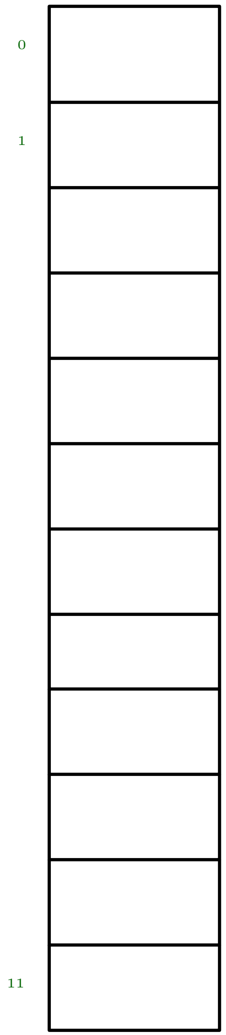

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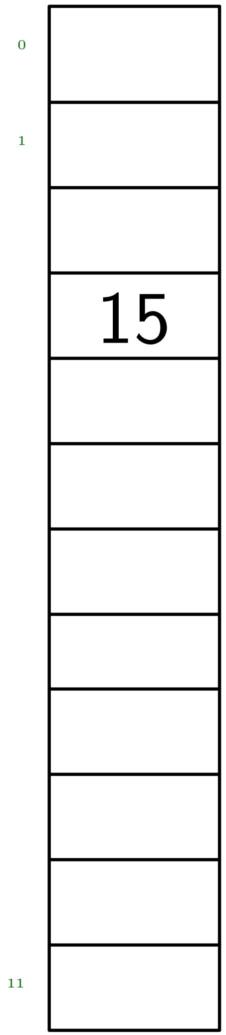
Open Addressing: Quadratic Probing

$h' : U \to \{0, 1 \ldots, m-1\}$



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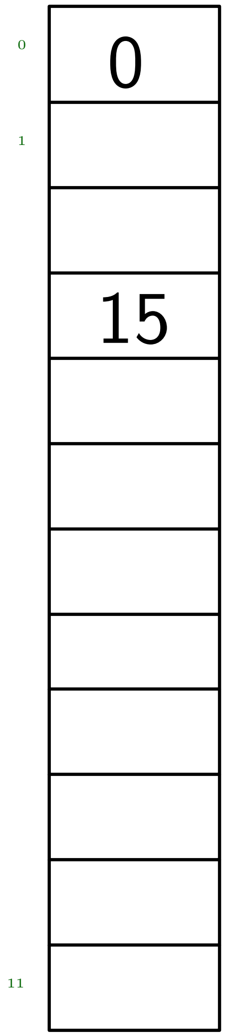

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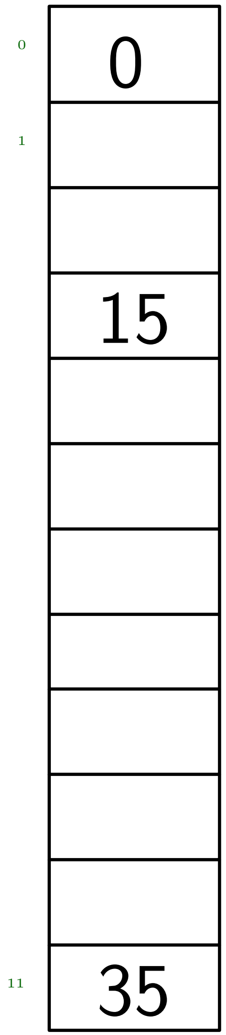

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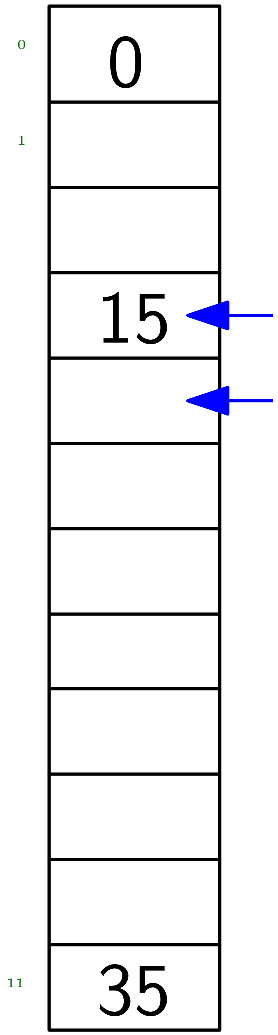

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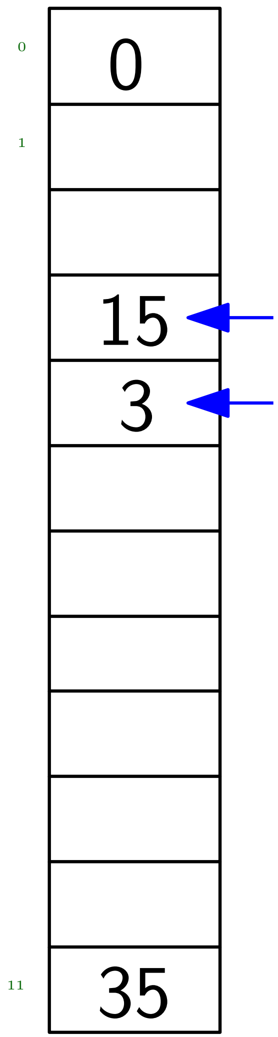

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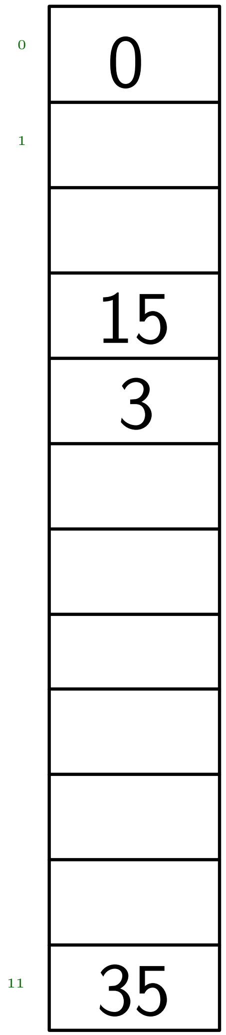

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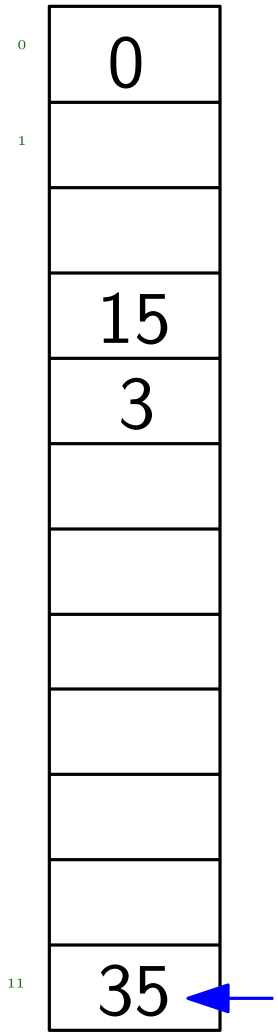

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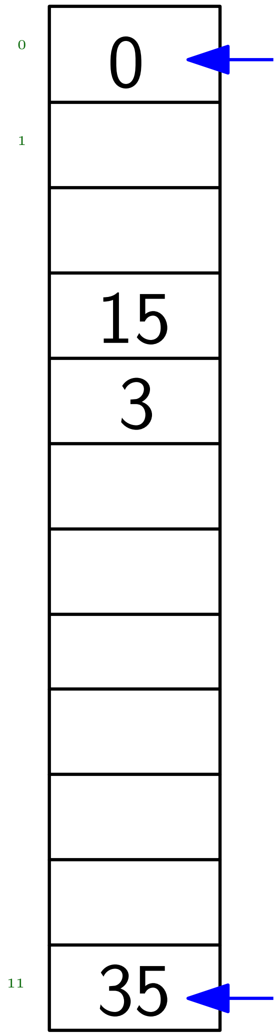

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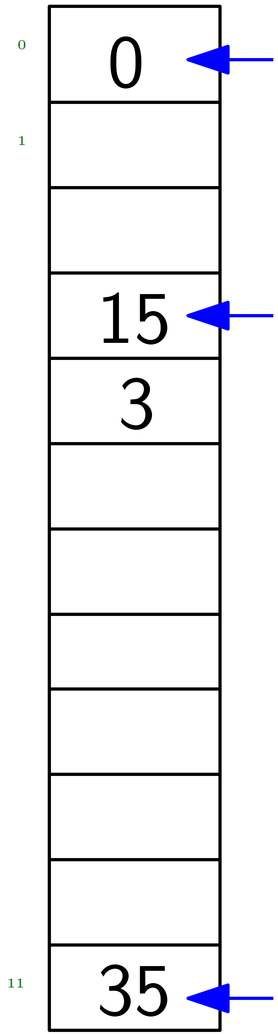

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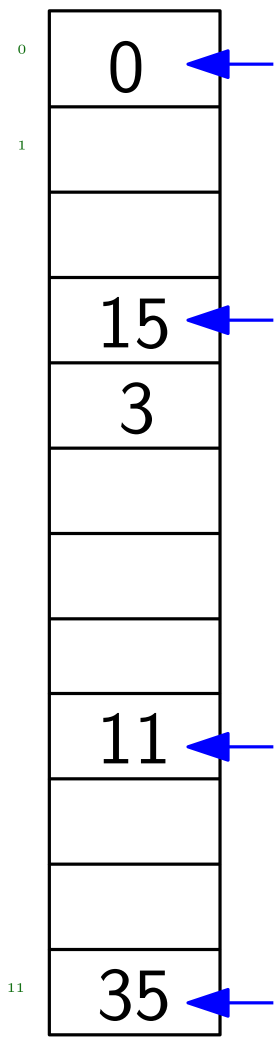

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0	0
1	
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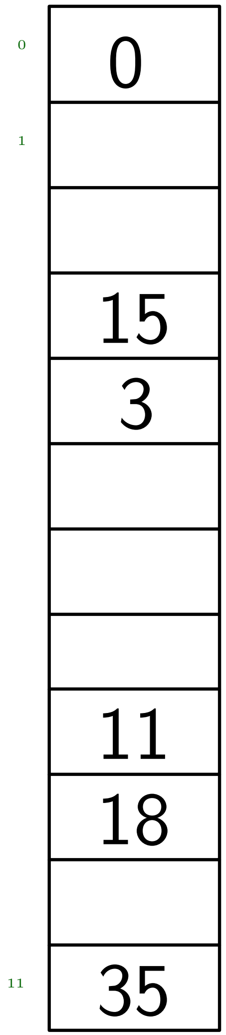

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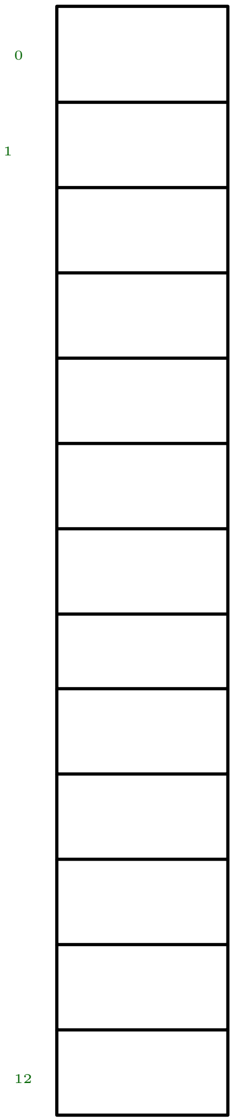
Open Addressing: Double Hashing

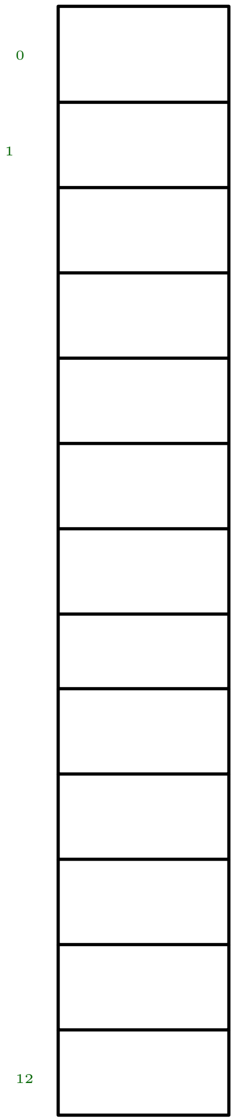
$h' : U \to \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x,i) = \big(h_1(x) + i h_2(x)\big) \bmod m$
- $h_1(x)$ and $h_2(x)$ are auxillary hash functions
- Note that (unlike before) probe sequence depends upon x
- In order for probe sequence to check entire table, must have $h_2(x)$ be relatively prime to m, e.g.,
  - m a power of 2; $h_2(x)$ always odd
  - m prime; $h_2(x)$ always less than m.

Open Addressing: Double Hashing

$h' : U \to \{0, 1 \ldots, m-1\}$



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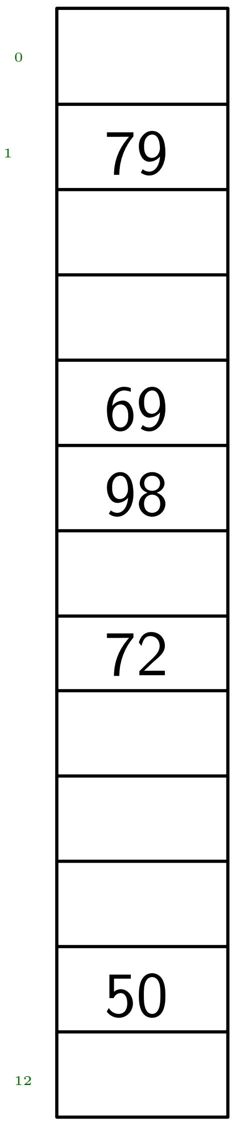
  - m prime; $h_2(x)$ always less than m.

Example: $m = 13$
$h_1(x) = k \bmod m$
$h_2(x) = 1 + (k \bmod 11)$

Open Addressing: Double Hashing

$h' : U \to \{0, 1 \ldots, m-1\}$



- Hash Function is $h(x,i) = \big(h_1(x) + ih_2(x)\big) \bmod m$
- $h_1(x)$ and $h_2(x)$ are auxillary hash functions
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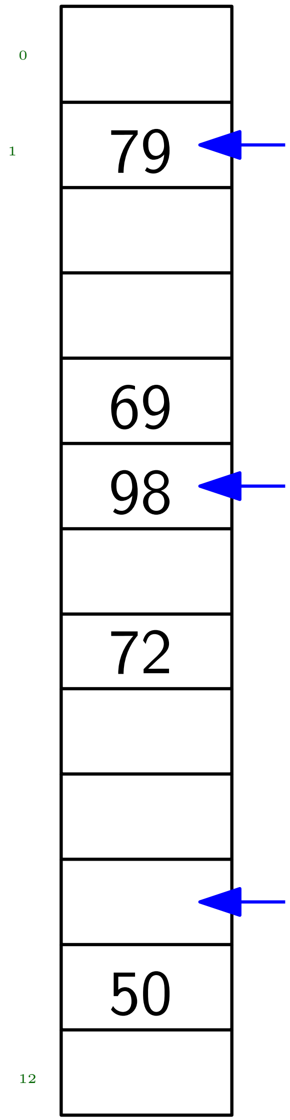

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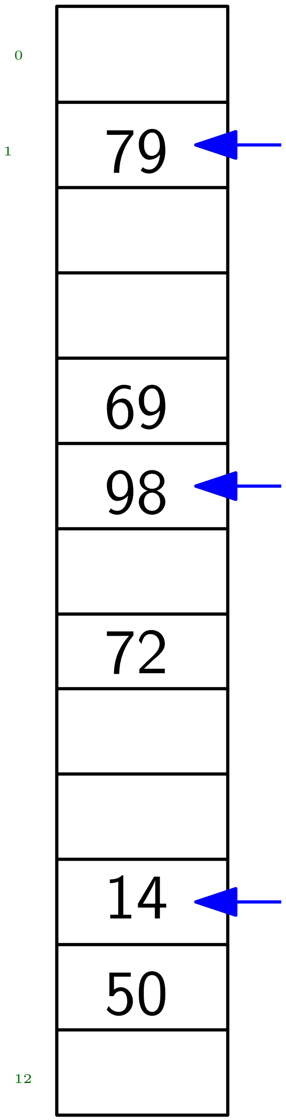

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Open Addressing: Analysis Results

We have seen 3 different open addressing collision resolution methods:

- Linear Probing: $h(x, i) = (h'(x) + 1) \bmod m$
- Quadratic Probing $h(x, i) = (h'(x) + c_1 + c_2 x^2) \bmod m$
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For analysis, we often assume uniform hashing.
This states that the probe sequence

$$h(x,0),\ h(x,1), h(x,2),\ \ldots,\ h(x,m)$$

is equally likely to be any of the $m!$ permutations of $1, 2, \ldots, m$.

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Uniform Hashing is not actually realizable.
The more random our probe sequence, though, the closer actual behavior is to theory.

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Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends

Hash Functions & Universal Hashing

Returning to chained hashing, notice that our analysis assumed that the hashed keys were equally distributed among the slots.

- If all keys hashed to same slot, performance would be very bad.
- If the hash function $h(x)$ is given in advance and $n << U$, very easy to construct bad case in which all keys map to the same slot.

- We sidestep this issue by choosing a *random hash function*
- More specifically, we will have a *collection* of hash functions $\mathcal{H}$

- Given any set of keys, we will choose a random hash function $h \in \mathcal{H}$ and then hash using $h(x)$.

- *On average*, the n set of keys will be hashed so that each slot will get $O(n/m) = O(\alpha)$ keys.

- Our $O(1+\alpha)$ sucessful/unsuccessful search times for chained hashing will then hold on average.

- One class $\mathcal{H}$ of hash functions having this property are the *Universal* ones; they permit *Universal Hashing*

Universal Hashing

- Let $\mathcal{H}$ be a set of hash functions, such that each $h \in \mathcal{H}$ maps
  $h : U \to \{0, 1, \ldots, m-1\}$
- $\mathcal{H}$ is Universal if, for every two different keys k, ℓ, the number of hash functions in $\mathcal{H}$ that map k, ℓ to the same slot is at most $|\mathcal{H}|/m$, i.e.,

$$\forall k \neq \ell \in U, \quad |\{h \in \mathcal{H} \,:\, h(k) = h(\ell)\}| \leq \frac{|\mathcal{H}|}{m}.$$

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Let $k_1, k_2, \ldots, k_n$ be the n keys. Let i be any fixed index.
Then, for $j \neq i$, if $h \in \mathcal{H}$ is chosen uniformly at random,
$\Pr(h(k_i) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}$.

From linearity of expectation, if $h \in \mathcal{H}$ is chosen uniformly at random, average $\#$ of other keys mapping to the same slot as k_i is then

$$\sum_{j \neq i; 1 \leq j \leq n} \Pr(h(k_i) = h(k_j)) \leq \frac{n-1}{m} < \alpha$$

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Similarly, if k is not one of the n keys then, for all j,
$\Pr(h(k) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}$ and average # of keys mapping to same slot as k is $\sum_{j=1}^{n} \Pr(h(k) = h(k_j)) \leq n/m = \alpha$

Construction of Universal Hash Functions

- Choose prime $p > U$
- Set $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ and $Z_p = \{0, 1, 2, 3, \ldots, p-1\}$
- Define

$$\forall a \in Z_p^*,\ b \in Z_p, \quad h_{a,b}(x) = \Big((ax + b) \bmod p\Big) \bmod m$$

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Proof: Need to show that for all $k \neq \ell$, number of pairs (a, b) with $h_{a,b}(k) = h_{a,b}(\ell)$ is $\leq p(p-1)/m$

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(1) Let $k \neq \ell \in U$. For given $(a, b) \in Z_p^* \times Z_p$ set

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(2) Every different (a,b) pair generates a unique (r,s) pair

For a given (r,s) pair we can solve

$$a = (r-s)(k-\ell)^{-1} \bmod p, \quad b = (r-ak) \bmod p.$$

where $(k-\ell)^{-1}$ is the multiplicative inverse base p. Since, for fixed p,k,ℓ, we must have $r \neq s$, there are are $p(p-1)$ (r,s) pairs. Since there are also $p(p-1)$ (a,b) pairs, there is a one-one correspondence between them, with every (a,b) pair generating a diffferent (r,s).

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$\boxed{\Rightarrow \mathcal{H} \text{ is Universal}}$

Universal Hashing: Wrap Up

- Just saw that the set of Hash functions

$$\mathcal{H} = \left\{h_{a,b} \,:\, a \in Z_p^*,\, b \in Z_p\right\}$$

  is *Universal*

- This implies that for *any* set of n keys $K = \{k_1, k_2, \ldots, k_n\}$, an effective way of storing the keys is to
  - Choose a random pair (a, b) uniformly at random from the $p(p-1)$ pairs in $Z_p^* \times Z_p$
  - Hash the items in K using hash function $h_{a,b}$

- Because $\mathcal{H}$ is Universal, average time for storing the data will be $O(n\alpha)$ where $\alpha = n/m$ is the load factor

- Average time for doing a search will be $(1 + \alpha)$

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Odds & Ends

- Hashing first recognized as a technique in the 1950's
- Comes from English word implying *chop and mix*
- Many different types of hashing for dictionary storage out there. This introduction only scratch d the surface
- A *Cryptographic Hash Funtion* is a hash function that is *almost* impossible to invert efficiently, i.e, given $h(x)$ very dfifficult to find x
  - Almost by necessity requires that function h distributes keys pretty "randomly" over $0, 1, 2, \ldots, m$. If not true, then would have first step towards guessing value of x that produces $h(x)$.
  - Example: Password protection. System password file only stores h(password) and not the password itself.
    * When user logs in and types password p, system checks $h(p)$ against file.
    * If an attacker steals the file it wouldn't be helpful, since attacker can't invert hashed password to get original one