$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic proof

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}.$$

$$= \binom{n}{k}.$$

Proof using Bijection

Set: $S = \{1, 2, \cdots, n\}$

X : Collection of k-element subsets

Y: Collection of $n-k$-element subsets.

Theorem 1.2 : $|X| = \binom{n}{k}$ $\quad |X| = |Y|$?

$|Y| = \binom{n}{n-k}$

$f : X \longrightarrow Y$

$A \longrightarrow S \setminus A$ Complement of A

$(\bar{A})$

k $n-k$

Example: $S = \{1, 2, 3, 4, 5\}$

A	$\bar{A}$
$\{1\}$	$\{2, 3, 4, 5\}$
$\{1, 3\}$	$\{2, 4, 5\}$.
S	ϕ, $\{\}$
ϕ	S

---

f is one-to-one:

$A \neq B \implies \bar{A} \neq \bar{B}$

EX $k=2$

$A = \{1, 2\}$ $\bar{A} = \{3, 4, 5\}$

$B = \{1, 3\}$ $\bar{B} = \{2, 4, 5\}$

f is onto:

For any $B \in Y$, Exist $A \in X$

s.t. $\bar{A} = B$

$k=2$

$n-k=3$

$B = \{1, 3, 5\}$

$A = \{2, 4\}$

$\bar{A} = \{1, 3, 5\} = B$

EX

Therefore f is a bijection.

$\Rightarrow \quad |X| = |Y|$

$\Rightarrow \quad \binom{n}{k} = \binom{n}{n-k}$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$n = 4$:

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

$$= 2^4 = 16$$

Proof: $N = \{1, 2, \cdots, n\}$ $|S_i| = \binom{n}{i}$

S_i: collection of i-element subsets of N

P: collection of all subsets of N

$\{S_0, S_1, \cdots, S_n\}$ is a partition of P

Sum principle:

$$|P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Theorem 1.2

Q: $|P| = 2^n$?

$L = L_1 L_2 \cdots L_n \qquad L_i \in \{0, 1\}$

$\mathcal{L}$ = set of all such lists

$|\mathcal{L}| = 2^n$.

Q: $|P| = |\mathcal{L}|$?

Use bijection principle

$f : \mathcal{L} \longrightarrow P$

$L_1 L_2 \cdots L_n \longrightarrow S = \{ i \mid L_i = 1 \}$

Ex:

$10101 \rightarrow \{1, 3, 5\}$

$11101 \rightarrow \{1, 2, 3, 5\}$

$00000 \rightarrow \{\}$

$11111 = \{1, 2, 3, 4, 5\}$

f is one-to-one:

EX $\quad 10101 \rightarrow \{1, 3, 5\}$

$\quad 11101 \rightarrow \{1, 2, 3, 5\}$

f is onto:

Ex $\quad \{2, 4\} \in P$

$\quad 01010 \in \mathcal{L}$

$\quad 01010 \longrightarrow \{2, 4\}$

Therefore f is a bijection.

$\Rightarrow |\mathcal{L}| = |P|$

$\Rightarrow |P| = 2^n$

$$\Rightarrow \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

proved

Pascal's Triangle

Ex:

$$\binom{7}{1} = 7 = 1+6 = \binom{6}{0} + \binom{6}{1}$$

$$\binom{7}{2} = 21 = 6+15 = \binom{6}{1} + \binom{6}{2}$$

$$\binom{7}{3} = 35 = 15+20 = \binom{6}{2} + \binom{6}{3}$$

.....

In general

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$0 < k < n$$

Algebraic proof

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= \frac{k\,(n-1)!}{k\,(k-1)!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-1-k)!\,(n-k)}$$

$$= \frac{(n-1)!\,k}{k!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,(k+n-k)}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

Proof by sum principle

Ex: $n = 5 \quad k = 2$

Need to prove: $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

$S = \{A, B, C, D, E\}$

S_1 : collection of 2-subsets of S

$|S_1| = \binom{5}{2}$

$S_1 = \{ \{A, B\}, \{A, C\}, \{A, D\}, \underline{\{A, E\}}$
$\{B, C\}, \{B, D\}, \underline{\{B, E\}}, \{C, D\}$
$\underline{\{C, E\}}\ \underline{\{D, E\}} \}$

splits into:

S_2 : those contain E

S_3 : those not contain E

$\{S_2, S_3\}$: partition of S_1

$\Rightarrow |S_1| = |S_2| + |S_3|$

S_2 : Choose two elements from S,

one of which must be E

$\Leftrightarrow$ Choose one element from $\{A, B, C, D\}$

$\Rightarrow |S_2| = \binom{4}{1}$

S_3 : Choose two elements from S,

but cannot chose E

$\Leftrightarrow$ Choose two elements from $\{A, B, C, D\}$

$\Rightarrow |S_3| = \binom{4}{2}$

$$\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}.$$

proof completed.

General case

$S = \{x_1, x_2, \cdots, x_n\}$

S_1 : collection k-subsets of S

$|S_1| = \binom{n}{k}$

---

split S_1 into

S_2: those contain x_n

S_3: those not contain x_n

$\Rightarrow |S_1| = |S_2| + |S_3|$

S_2 : Choose k elements from S, one of which must be x_n

$\Leftrightarrow$ choose $k-1$ elements from $\{x_1, x_2, \cdots, x_{n-1}\}$

$\Rightarrow |S_2| = \binom{n-1}{k-1}$

S_3 : choose k-elements from S,

but cannot choose X_n

$\Longleftrightarrow$ choose k-elements from

$\{X_1, X_2, \cdots, X_{n-1}\}$

$\Rightarrow |S_3| = \binom{n-1}{k}$

Putting together:

$$\binom{n}{k} = |S_1| = |S_2| + |S_3|$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}$$

proof completed.

Expanding $(x+y)^n$

$(x+y)(x+y)$

$= xx + xy + yx + yy$ $\qquad 2^2$ terms

$\underline{(x+y)(x+y)}(x+y)$

$= [xx + xy + yx + yy](x+y)$

$= xxx + \underline{xyx} + \underline{yxx} + yyx$

$+ \underline{xxy} + xyy + yxy + yyy$ $\qquad 2^3$ terms

$3x^2y$

$=$ # of monomial terms with 2 x's & 1 y

$3xy^2$

$=$ # of monomial terms with 1 x & 2 y's

Expanding $(x+y)^n$ result in

2^n monomial terms

Binomial Theorem

# of terms in the expansion with n x's is $\underline{\binom{n}{0}}$

# of terms in the expansion with $n-1$ x's & 1 y is $\underline{\binom{n}{1}}$

# of terms in the expansion with $n-i$ x's & i y's is $\underline{\binom{n}{i}}$