COMP 170 Discrete Mathematical Tools for CS
2006 Fall Semester – Written Assignment # 7
Distributed: Nov 14, 2006 – Due: Nov 21, 2006 *at end of class*

The top of your submission should contain (i) your name, (ii) your student ID #, (iii) your email address and (iv) your tutorial section.

Please write clearly and briefly. For all questions you should also provide a short explanation as to *how* you derived the solution. A solution that consists of just a number will be counted as wrong.

2nd Note: Please follow the guidelines on doing your own work and avoiding plagiarism given on the class home page. Don't forget to *acknowledge* individuals who assisted you, or sources where you found solutions.

3rd Note: Most of these problems are taken (some modified) from sections 4.1, 4.2 and 4.3 of the textbook.

4th Note: Your assignment can either be submitted at the end of your Tuesday lecture session or before 5PM in the collection bin in front of room 4213A.

Problem 1: Prove that every integer greater than 7 is a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.
(Hint: first prove the *three* base cases of $n = 8, 9, 10$ and then prove the inductive step assuming that $n > 10$.)

Problem 2: Find the error in the following "proof" that all positive integers n are equal: *Let $p(n)$ be the statement that all numbers in an n-element set of positive integers are equal. Then p(1) is true. Now assume $p(n-1)$ is true, and let N be the set of the first n integers. Let N' be the set of the first $n-1$ integers, and let N'' be the set of the last $n-1$ integers. By $p(n-1)$, all members of N' are equal, and all members of N'' are equal. Thus, the first $n-1$ elements of N are equal and the last $n-1$ elements of N are equal, and so all elements of N are equal. Therefore, all positive integers are equal.*

Problem 3: The Fibonacci numbers are defined by $F(0) = F(1) = 1$ and $\forall n \geq 2$, $F(n) = F(n-1) + F(n-2)$. The closed form solution to the Fibonacci numbers is

$$F(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Use strong induction to prove that this formula is correct.

Problem 4: Assuming that $r \neq 1$, solve the recurrence $T(n) = rT(n-1) + n$, with $T(0) = 1$, by giving a closed form solution. Use induction to prove that this solution is correct.

Problem 5: (a) Draw a recursion tree diagram for

$$T(n) = \begin{cases} 4T(n/2) + n^2 & \text{if } n \geq 2, \\ 3 & \text{if } n = 1. \end{cases}$$

Use it to find the exact solution to the recurrence. Assume that n is a power of 2.

(b) Now, prove the correctness of your result by induction. (That is, prove that your result is correct for all $n = 2^i$; your induction should be on i.)

Problem 6: Draw a recursion tree diagram for

$$T(n) = \begin{cases} 3T(n/2) + n & \text{if } n \geq 2, \\ 2 & \text{if } n = 1. \end{cases}$$

Use it to find the exact solution to the recurrence. Assume that n is a power of 2.

Problem 7: Prove by induction that the solution to a recurrence of the form

$$T(n) = \begin{cases} 9T(n/3) + n & \text{if } n \geq 3, \\ d & \text{if } n = 1. \end{cases}$$

is $O(n^2)$. Note that this means that you must prove that the statement is true for every d. You should assume that n is a power of 3.