COMP 170 – Fall 2006
Midterm 1 Solution

Q1. Suppose that we have a student hall with 9 rooms labelled $A, B, \ldots, I$.

How many ways are there to assign 10 students to the 9 hall rooms so that every student is assigned and no hall room is empty?

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Once the pair is chosen, there are $9!$ ways of assigining the students to the rooms.

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Once the pair is chosen, there are $9!$ ways of assigining the students to the rooms.

So the answer is

$$\binom{10}{2} 9! = 16329600$$

Q2. Suppose you have 5 standard six-sided dice; one red, one green, one yellow, one black, one orange. Each die has the numbers 1-6 on its sides. Roll all of the dice. An "outcome" of a roll is a number for each die.

An outcome contains a
(i) *two-of-a-kind* if there are at least two dice showing the same number;
(ii) *three-of-a-kind* if there are at least three dice showing the same number;
(iii) *four-of-a-kind* if there are at least four dice showing the same number;
(iv) *five-of-a-kind* if all five dice show the same number.

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6

(c) How many possible outcomes contain a *four-of-a-kind* but no *five-of-a-kind*?

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Each such outcome is uniquely determined by
(i) the value of the four-of-a-kind,
(ii) the color of the die that is not part of the four-of-a-kind and
(iii) the value of that die. So the answer is

$$6 \cdot 5 \cdot 5 = 150.$$

(d) How many possible outcomes contain a *three-of-a-kind* but no *four-of-a-kind*?

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Each such outcome is uniquely determined by
(i) the value of the three-of-a-kind,
(ii) the colors of the die that are **not** part of the three-of-a-kind and
(iii) the values of those dice. So the answer is

$$6 \cdot \binom{5}{2} \cdot 5^2 = 1500.$$

(e) How many possible outcomes contain a *three-of-a-kind* AND a *two-of-a-kind* that have different dice values but no *five-of-a-kind*?

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(i) the value of the three-of-a-kind,
(ii) the colors of the die that are **not** part of the three-of-a-kind and
(iii) the values of those dice. So the answer is

$$6 \cdot \binom{5}{2} \cdot 5 = 300.$$

Q3. Is the following formula correct for all $n \geq 1$?

$$\sum_{i=0}^{n} 3^i \binom{n}{i} = 2^n \sum_{i=0}^{n} \binom{n}{i}.$$

If yes, prove it. If no, give a value of n for which the formula is incorrect.

Q3. Is the following formula correct for all $n \geq 1$?

$$\sum_{i=0}^{n} 3^i \binom{n}{i} = 2^n \sum_{i=0}^{n} \binom{n}{i}.$$

If yes, prove it. If no, give a value of n for which the formula is incorrect.

The proof is that the both equal 4^n.
This can be derived using two applications of the binomial theorem.

$$4^n = (3+1)^n = \sum_{i=0}^{n} 3^i 1^{n-i} \binom{n}{i}$$

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Alternatively, the second equality can be derived as

$$4^n = (2+2)^n = \sum_{i=0}^{n} 2^i 2^{n-i} \binom{n}{i} = 2^n \sum_{i=0}^{n} \binom{n}{i}$$

Q4. A *permutation* of the letters A, B, C, D, E, F, G, H is a list (string) of the eight letters in some order.

A permutation *contains* the substring ABC if the letters appear in the given order. For example, permutations $DEFABCGH$ and $HGABCFED$ contain the substring ABC but $ADBECFGH$ doesn't.

(a) How many *permutations* of the letters A, B, C, D, E, F, G, H contain the substring ABC?

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To solve this problem consider substring ABC' as an indivisible item and each of the other individual 5 letters as a separate indivisible item.

Then the answer is the number of ways of writing a permutation of 6 items (the five letters and ABC) which is

$$6! = 720.$$

(b) How many *permutations* of the letters A, B, C, D, E, F, G, H contain A before B and B before C?

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Such a permutation is uniquely defined by
(i) choosing the locations of the three letters A,B,C (once the locations are chosen we know that A must go in the first, B in the 2nd and 'C in the 3rd) and
(ii) the locations of the five remaining letters in the 5 remaining spaces.
This is therefore

$$\binom{8}{3} \cdot 5! = \frac{8!}{3!} = 6720.$$

Q5. An *anagram* is a distinct ordering of the letters. For example,

the word eat has six anagrams

‘‘eat”, “eta”, “ate”, “aet”, “tea” , “tae”,

while the word ‘ ‘too” has only three anagrams

“too”, “oto” and “oot”.

(a) How many anagrams does the word "mammal" have?

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This is the number of ways to label 6 items so that
(i) 3 are labeled ’m’,
(ii) 2 are labelled ’a’ and
(iii) 1 is labelled ’l’
which is

$$\binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = \frac{6!}{3! \cdot 2! \cdot 1!} = 60$$

(b) How many anagrams does the word "mississippi" have?

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This is the number of ways to label 11 items so that
(i) 1 is labeled 'm',
(ii) 4 are labelled 'i',
(iii) 4 are labelled 's' and
(iv) 2 are labelled 'p'
which is

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$$

Q6. Give a combinatorial proof of the identity, for all $n \geq 9$,

$$\binom{n}{2}\binom{n-2}{3}\binom{n-5}{4} = \binom{n}{4}\binom{n-4}{3}\binom{n-7}{2}.$$

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"Consider the problem of how to color n items so that 2 are red, 3 are green, 4 are blue and the remaining $n-9$ are yellow."

The left hand side of the inequality obviously counts this.

“Old problem was how to color n items so that 2 are red, 3 are green, 4 are blue and the remaining $n-9$ are yellow.

Notice that the problem is the same if we change the order in which we ask the question.
“Consider the problem of how to color n items so that 4 are blue, 3 are green, 2 are red and the remaining $n-9$ are yellow.”

This is what the right hand side of the equation is counting, so the two sides are the same.

Q7. Consider the following statement

$$gcd(j,k) = gcd(k-j,j).$$

Is this statement always true for k,j with $k \geq j \geq 0$? Either *prove* that it is true for all k,j with $k \geq j \geq 0$, or give values for k,j with $k \geq j \geq 0$ such that $gcd(j,k) \neq gcd(k-j,j)$.

We will show that

d is a common divisor of j and k

if and only if

d is a common divisor of $k - j$ and j.

This shows that

the set of common divisors of j and k

is exactly the same as

the set of common divisors of $k - j$ and j.

This immediately implies that the

greatest common divisor of j and k

is the same as the

greatest common divisor of $k - j$ and j.

To prove the *if* direction
suppose that d is a common divisor of j and k.
Then, for some a and b

$$j = a \cdot d \quad \text{and} \quad k = b \cdot d$$

so

$$k - j = (b - a)d$$

and
d is a divisor of $k - j$.
Thus d is a common divisor of $k - j$ and j.

To prove the *only if* direction
suppose that d is a common divisor of $k - j$ and j.

Then, for some c and a

$$j = a \cdot d \quad \text{and} \quad k - j = c \cdot d$$

so

$$k = (k - j) + j = (c + a)d$$

and
d is a divisor of k.
Thus d is a common divisor of k and j.

Q8. (a) Consider the equation

$$3 \cdot_{100} x = 13$$

Does this equation have a solution x in Z_{100}?

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(b) Consider the equation

$$15 \cdot_{100} x = 13$$

Does this equation have a solution x in Z_{100}?

(a) Yes.

To start, first notice that $3 \cdot (-33) + 1 \cdot 100 = 1.$

(This can either be derived using the extended GCD algorithm or just by observation.)

This tells us that the multiplicative inverse of 3 in Z_{100} is $-33 \bmod 100 = 67.$

So, the answer is

$$x = 67 \cdot_{100} 13 = 871 \bmod 100 = 71.$$

As a reality check, note that

$$3 \cdot_{100} 71 = 213 \bmod 100 = 13$$

so $x = 71$ really does solve our problem.

(b) No.
This can be proven by contradiction.

Suppose there was some solution x.
Then $15 \cdot_{100} x = 13$ so there is some q such that $15 \cdot x = q \cdot 100 + 13$ or

$$15 \cdot x - q \cdot 100 = 13.$$

But the left hand side of this equation is divisble by 5 while the right hand side is not, leading to a contradiction.

(b) No.
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But the left hand side of this equation is divisible by 5 while the right hand side is not, leading to a contradiction.

Note: the fact that $gcd(15, 100) = 5 \neq 1$ *does* **not** *prove that* $15 \cdot_{100} x = 13$ *has no answer.*