COMP170
Discrete Mathematical Tools for Computer Science
The RSA Algorithm

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Sections 2.3, 2.4, pp. 72-86

2.3 The RSA Cryptosystem

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- Assorted Tools and Definitions

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Consider multiplication in Z_7

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For every nonzero $a \in Z_7$, the function $f_a(x) = x \cdot_7 a$ is one-to-one and therefore a permutation of $Z_7 - \{0\}$, i.e., every row is a permutation.

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Proof: Suppose $f_a(x)$ is not 1-to-1. Then there are $x \neq y$ with $f_a(x) = f_a(y)$. Since p is prime, Corollary 2.17 tells us that there is $a^{-1} \in Z_p$ s.t. $a \cdot_p a^{-1} = 1$.

Multiplying the two sides by a^{-1}gives

$$\begin{aligned} x = (x \cdot_p a) \cdot_p a^{-1} &= f_a(x) \cdot_p a^{-1} \\ &= f_a(y) \cdot_p a^{-1} = (y \cdot_p a) \cdot_p a^{-1} = y \end{aligned}$$

Contradiction!

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Contradiction! $\Rightarrow$ Then $f_a(x)$ is 1-to-1

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- For public-key cryptography,
  the public encoding function, P_B, needs to be one-way.
  The secret decoding function, S_B, is actually
  an efficient way of calculating the inverse of P_B.
  This efficient way is only available to the
  "owner" who constructed P_B.

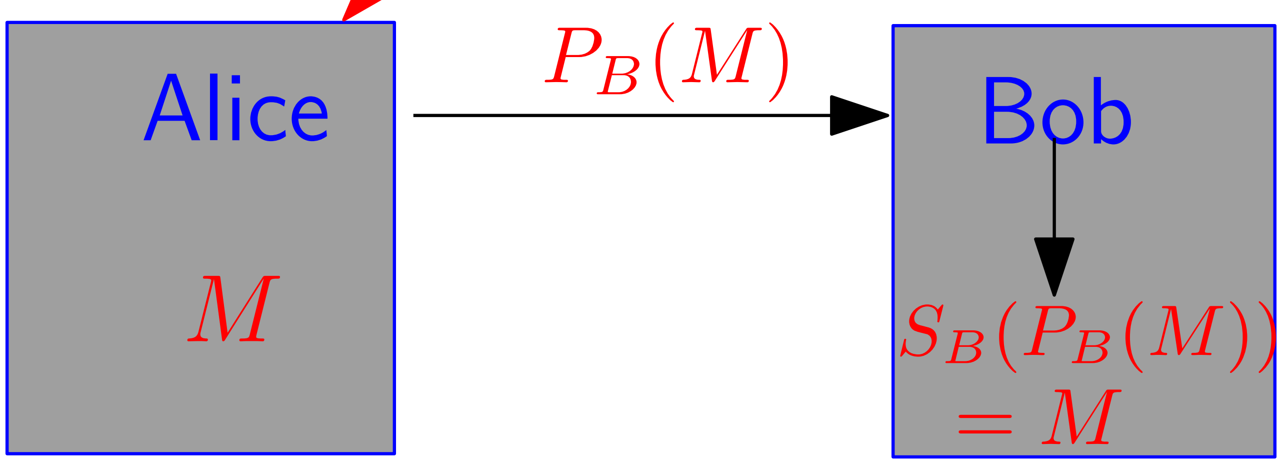
Recall the Public-Key Setup

i) Alice wants to send M to Bob

ii) In public directory, Alice looks up Bob's Public Key, P_B

iii) Alice sends $P_B(M)$ to Bob

iv) Bob uses his Secret Key, S_B to decrypt $M = S_B(P_B(M))$



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By Lemma 2.3, if $a \in Z_n$, then

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$$a^j \text{ mod } n \;=\; \underbrace{a \cdot_n a \cdot_n \cdots \cdot_n a}_{j \text{ factors}}.$$

a^j mod n is the product in Z_n of j factors, each equal to a.

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From Lemma 2.3 and exponentiation for integers, we have

Lemma 2.19:
For any $a \in Z_n$ and any nonnegative integers i, j,

a) $$(a^i \bmod n) \cdot_n (a^j \bmod n) = a^{i+j} \bmod n$$

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Examples:

$$\begin{array}{lll} 3^2 & = & 9 \\ 3^4 & = & 81 \\ 3^6 & = & 729 \\ 3^8 & = & 6561 \end{array} \qquad \begin{array}{lll} 3^2 \bmod 7 & = & 2 \\ 3^4 \bmod 7 & = & 4 \\ 3^6 \bmod 7 & = & 1 \\ 3^8 \bmod 7 & = & 2 \end{array}$$

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a) $$1 = \left(3^2 \bmod 7\right) \cdot_7 \left(3^4 \bmod 7\right) = 3^6 \bmod 7$$

b) $$2 = 16 \bmod 7 = \left(3^4 \bmod 7\right)^2 \bmod 7 = 3^8 \bmod 7$$

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x	x^0	x^1	x^2	x^3	x^4	x^5	x^6
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For *every* $x \in Z_7$, the sequence starts *cycling*.
In particular, for every $x \in Z_7$, we have
$x^0 = 1 = x^6 = x^{7-1}$.

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Proof:
Direct application of Lemma 2.3, because if we replace a with $a \bmod p$, then Theorem 2.21 applies.

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Example: $a = 5,\ p = 7, m = 15$

$$\Rightarrow \qquad a^{15} \bmod 7 = a^{(2\cdot 6+3)} \bmod 7 = a^3 \bmod 7 = 6$$

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It took mathematicians more than 300 years to "rediscover" a proof for this (if you believe that Fermat ever had one). Andrew Wiles finally managed to prove this in 1994.

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RSA are the initials of three Computer Scientists, Ron Rivest, Adi Shamir and Len Adleman, who discovered their algorithm when they were working together at MIT in 1977.

It is now known that Cifford Cocks, a mathematician working for Government Communications Headquarters (GCHQ), the secret coding agency in Britan, independently discovered this earlier, in 1973, but did not publish his work. This fact was not known until certain secret British documents were declassified in 1997.

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Bob's RSA Key Choice Algorithm

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(6) Keep d as secret key

Alice-send-message-to-Bob(x) $(0 \leq x < n)$

(1) Alice does:
(2) Read public directory for Bob's public keys e and n.
(3) Compute $y = x^e \bmod n$
(4) Send y to Bob

(5) Bob does:
(6) Receive y from Alice
(7) Compute $z = y^d \bmod n$, using secret key d
(8) Read z

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To show that the RSA cryptosystem works — that is, that it allows us to correcty decode encoded messages — we must show that $x = z$, i.e., for all x, $0 \le x < n$,

$$x = (x^e \bmod n)^d \bmod n = x^{ed} \bmod n$$

Story so far: We have (*)

Want to prove that,

if $0 \leq x < n$

$$x = x^{ed} \bmod n$$

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Plan

(1) Proving that $x \bmod p = x^{ed} \bmod p$ for all x

(2) Proving that $x \bmod q = x^{ed} \bmod q$ for all x

(3) Showing that, if $0 \le x < n$, (1) + (2) imply

$$x = x^{ed} \pmod{n}$$

Story so far: We have (*)

Want to prove

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$ed \bmod T = 1$ so there is some k such that $ed = 1 + kT$ and

$$\begin{aligned} x^{ed} \bmod p &= x^{1+k(q-1)(p-1)} \bmod p \\ &= x\left(x^{k(q-1)}\right)^{p-1} \bmod p \end{aligned}$$

There are two possible cases

(a) $x^{k(q-1)}$ is a multiple of p

(b) $x^{k(q-1)}$ is not a multiple of p

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(a) If $x^{k(q-1)}$ is a multiple of p

$\Rightarrow$ since p is prime, x is also a multiple of p.

$\Rightarrow$ $x^{ed} \bmod p = 0 = x \bmod p$

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Fermat's little thm:
If $p \not| a$ then
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We have therefore just finished proving that, for all x

$$x^{ed} \bmod p = x \bmod p$$

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Exact same proof as (1)

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Quick review of prime number properties

If p and q are both prime numbers and *both* divide z
then pq divides z

Example:

$p = 3$, $q = 11$, $z = 99$

$3, 11$ both divide 99 $\quad\Rightarrow\quad$ $33 = pq$ also divides 99

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Note that if p, q are *not* prime this is not necessarily true

Example:

$p = 6,\ q = 15,\ z = 60$

$6, 15$ both divide 60 but $90 = pq$ does not divide 60

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Want to prove that,

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Know that $x^{ed} \bmod p = x \bmod p$ and $x^{ed} \bmod q = x \bmod q$

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Then $\begin{array}{l} v = x^{ed} - x = ip \\ v = x^{ed} - x = jq \end{array}$ for some integers i, j

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p, q prime

$n = pq$

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e s.t. $gcd(e, T) = 1$

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Then $x^{ed} = kn + x$ for some k. Since $0 \leq x < n$,

$$x^{ed} \bmod n = x$$

We just saw that if

Alice-send-message-to-Bob(x) $(0 \leq x < n)$

(1) Alice does:
(2) Read public directory for Bob's public keys e and n.
(3) Compute $y = x^e \bmod n$
(4) Send y to Bob

(5) Bob does:
(6) Receive y from Alice
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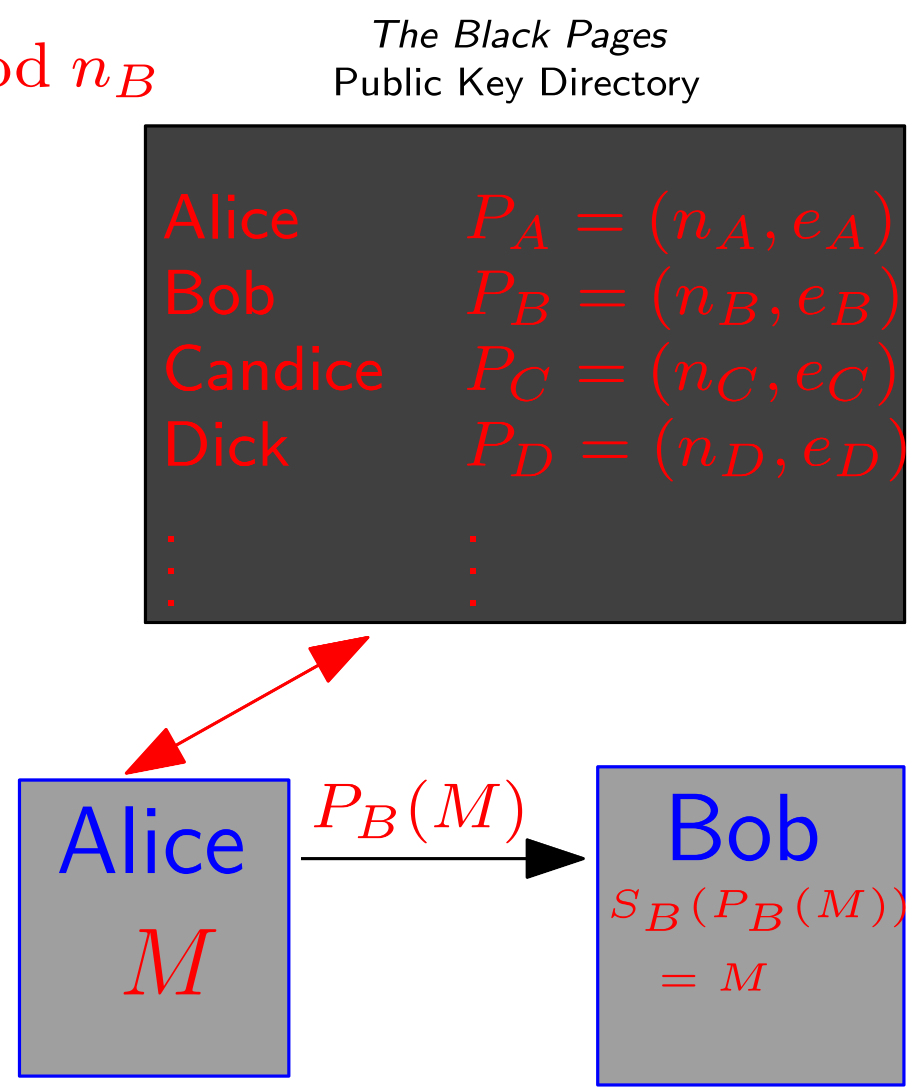
$$\Rightarrow \quad z = x^{ed} \bmod n = x$$

Theorem 2.23 (Rivest, Shamir, and Adleman)
The RSA procedure for encoding and decoding messages works correctly.

$P_B(M) = M^{e_B} \bmod n_B \quad S_B(Y) = Y^{d_B} \bmod n_B$



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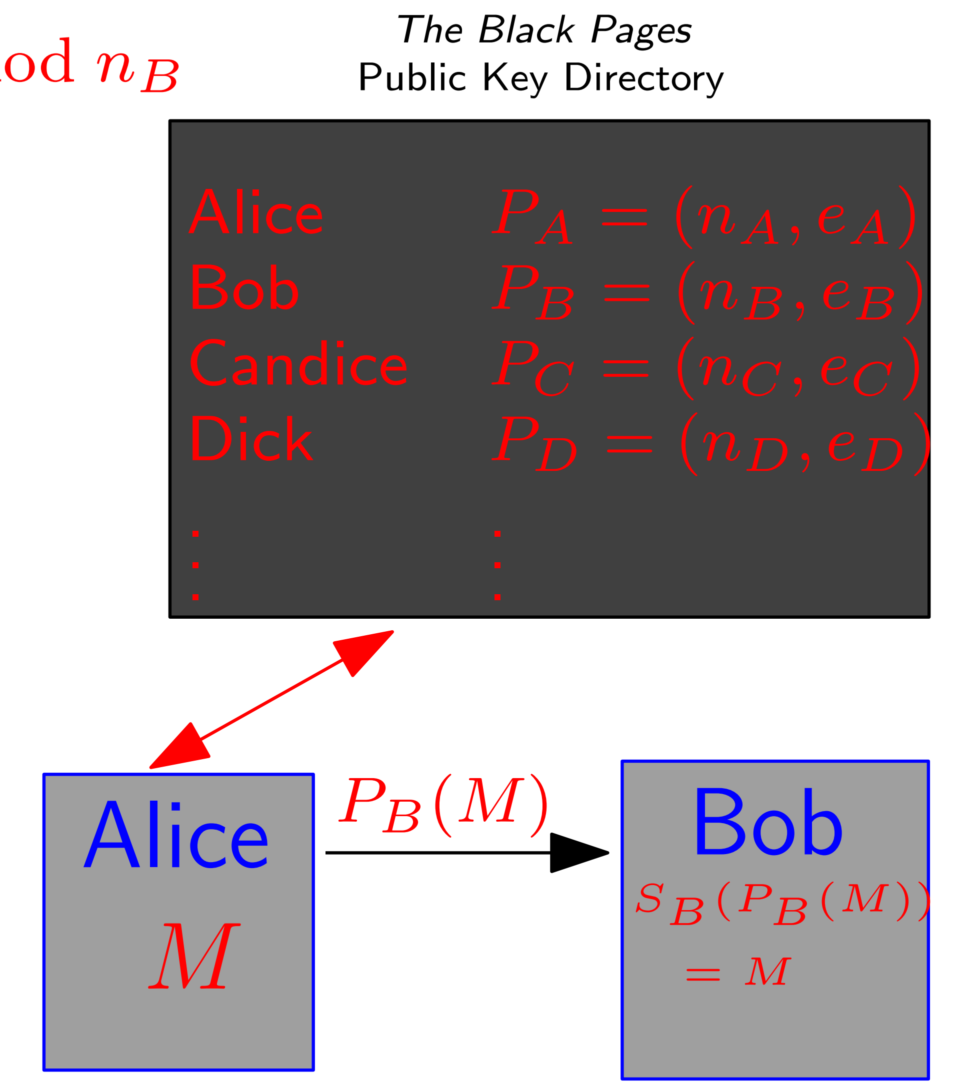

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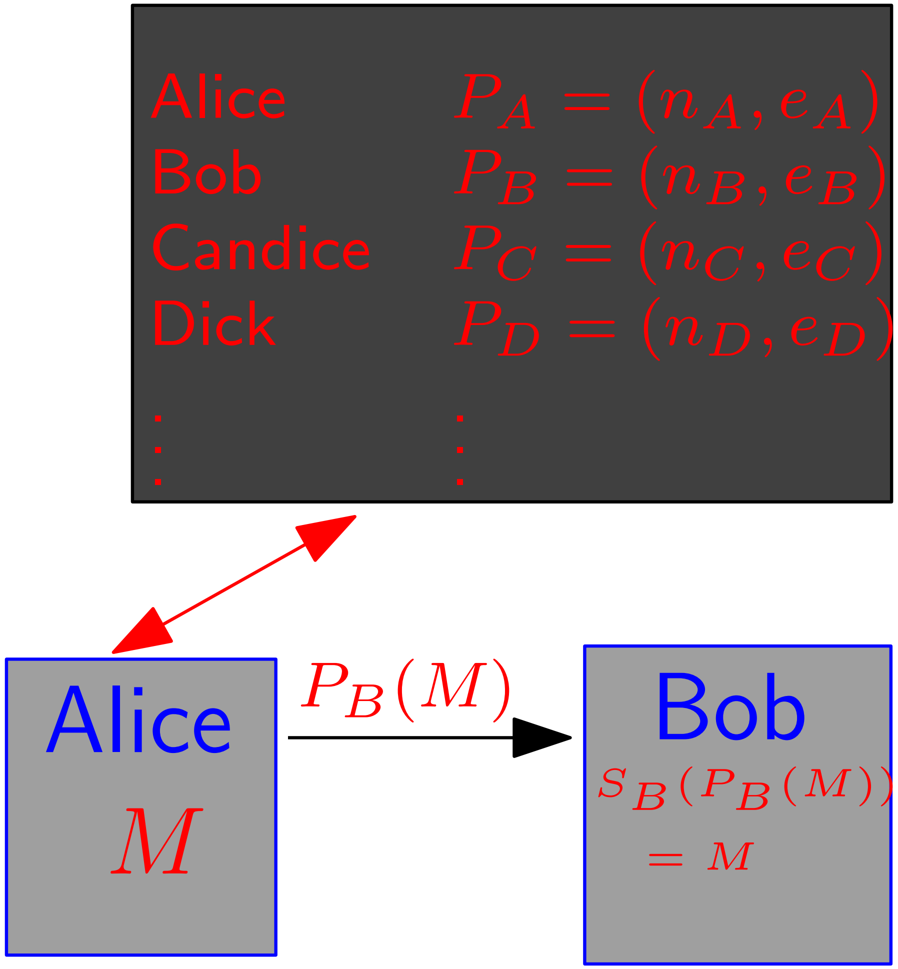
The Black Pages
Public Key Directory



Why is this secret?

We claim that someone (adversary) who knows the public information n, e and $M^e \bmod n$ can not figure out M.

- To show that the RSA cryptosystem is secure, we must argue that an adversary (eavesdropper) who knows n, e, and $M^e \bmod n$, but does not know p, q or d, can not easily compute M.

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  No!!. Nobody knows how to factor quickly!

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- Most e-commerce and computer security is based on RSA or similar schemes
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- So, if you could figure out a quick factoring scheme, you could break most modern computer security
- *Note: Although nobody knows how to factor quickly we don't have any proof that factoring* **must** *be slow. It's possible that there's a fast factorization algorithm out there that no one has found yet . . . .*

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Examples: For the given $n = p \cdot q = 55$, $e = 7$, $d = 23$

for $x = 12$:
$12^7 \bmod 55 = 35831808 \bmod 55 = 23$ and
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for $x = 22$:
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2.3 The RSA Cryptosystem

- Assorted Tools and Definitions
- Exponentiation mod n
- The Rules of Exponents
- Fermat's Little Theorem
- The RSA Cryptosystem
- Practical Aspects of Exponentiation mod n
- The Chinese Remainder Theorem

Practical Aspects of Exponentiation $\bmod n$

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Suppose you want to calculate $a^e \bmod n$

Sizes to right not unusual in RSA

a – 150 digits

e – 10^{120}, 121 digits

n – 150 digits

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  No! Too many iterations.

  Sun would "burn out" before we finished!

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- 3rd try: Use *Repeated Squaring*

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Sizes to right not unusual in RSA

a – 150 digits

e – 10^{120}, 121 digits

n – 150 digits

- 3rd try: Use *Repeated Squaring*

Idea: $a^{50} = a^{32} \cdot a^{16} \cdot a^2$ so we could calculate a^{50} by

Practical Aspects of Exponentiation $\bmod n$

Suppose you want to calculate $a^e \bmod n$

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Idea: $a^{50} = a^{32} \cdot a^{16} \cdot a^2$ so we could calculate a^{50} by first using 5 muls to find

$$
\begin{aligned}
a^2 &= a \cdot a \\
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a^8 &= a^4 \cdot a^4 \\
a^{16} &= a^8 \cdot a^8 \\
a^{32} &= a^{16} \cdot a^{16}
\end{aligned}
$$

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and then another 2 muls to get

$$\boxed{a^{50} = a^{32} \cdot a^{16} \cdot a^2}$$

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Much better than 49 muls needed by iterative method!

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Setting $I_i = a^{2^i} \bmod n$ and

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I_3 &= (I_2 \cdot I_2) \bmod n \\
I_4 &= (I_3 \cdot I_3) \bmod n \\
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Note: No factor is ever $\geq n$

Repeated squaring to evaluate $a^e \bmod n$

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- Calculate binary representation of e: $e_s e_{s-1} \cdots e_2 e_1 e_0$
  $e = \sum_{i=0}^{s} e_i 2^i$ and $s \leq \log_2 n$
  Example: $50 = 110010$

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- Now find $k_1, k_2, \ldots, k_t$ so that $e = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$.
  The k_i are just the locations of the 1s in the bin rep of e
  Example: $50 = 2^1 + 2^4 + 2^5$ so $(k_1, k_2, k_3) = (1, 4, 5)$

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- Calculate $I_0 = a$, $I_1 = (I_0)^2 \bmod n$, $I_2 = (I_1)^2 \bmod n$, $I_3 = (I_2)^2 \bmod n$, $\ldots$
  where $I_i = (I_{i-1})^2 \bmod n$ for $i = 1, 2, 3, \ldots, n$

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  where $I_i = (I_{i-1})^2 \bmod n$ for $i = 1, 2, 3, \ldots, n$

- $a^e \bmod n = (I_{k_1} I_{k_2} \cdots I_{k_t}) \bmod n$ so we can calculate this using $t-1$ multiplications where no factor is ever $\ge n$.

- How many multiplications and mods does this procedure use to calculate $a^e \bmod n$?

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- Note that if e has binary representation $e_s e_{s-1} \cdots e_2 e_1 e_0$ then it performs s multiplications and mods in the *repeated squaring* part, and, at most, another s multiplications and mods in the second part.

  Since $s \sim \log_2 e$ this means it performs at most around $2 \log_2 e \leq 2 \log_2 n$ of these operations.

  Compare this to the $e - 1$ operations it would require if we did naive exponentiation without repeated squaring.

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- To put this in perspective, consider $e = 10^{120}$. This number is so big that, at current computer speeds, we would not be able to finish running the naive algorithm **before the sun died**. On the other hand, $2 \log_2 e = 240 \log_2 10 \sim 796$ so we could run the repeated squaring algorithm in **just a few seconds**!

Comment: This idea of designing *efficient* programs for solving problems and then analyzing their running times is something that you will see a lot more of in Data Structures and The Design and Analysis of Algorithms

2.3 The RSA Cryptosystem

- Assorted Tools and Definitions
- Exponentiation mod n
- The Rules of Exponents
- Fermat's Little Theorem
- The RSA Cryptosystem
- Practical Aspects of Exponentiation mod n
- The Chinese Remainder Theorem

The Chinese Remainder Theorem

While proving the correctness of RSA, we proved the following:

If (i) $0 \le x < n = pq$,
(ii) $x^{ed} \bmod p = x \bmod p$ and
(iii) $x^{ed} \bmod q = x \bmod q$

$$\Rightarrow \quad x^{ed} \pmod{n} = x$$

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$$\Rightarrow \quad x^{ed} \pmod{n} = x$$

This turns out to be a special case of a general rule:

The Chinese Remainder Theorem

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For each $x \in Z_{15}$, write
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Is x uniquely determined
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Is x uniquely determined by these values?

x	x mod 3	x mod 5
0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
5	2	0
6	0	1
7	1	2
8	2	3
9	0	4
10	1	0
11	2	1
12	0	2
13	1	3
14	2	4

The Chinese Remainder Theorem

For each $x \in Z_{15}$, write $x \bmod 3$ and $x \bmod 5$.

Is x uniquely determined by these values? Yes!

Each $x \in Z_{15}$ has a different $x \bmod 3$, $x \bmod 5$ pair.

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Each $x \in Z_{15}$ has a different $x \bmod 3$, $x \bmod 5$ pair.

Thus, the function $f(x) = (x \bmod 3,\ x \bmod 5)$ from Z_{15} to the 15 pairs (i, j) with $0 \le i < 3$ and $0 \le j < 5$ is one-to-one.

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0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
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$\Rightarrow$ x is uniquely determined by its pair of remainders.

x	x mod 3	x mod 5
0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
5	2	0
6	0	1
7	1	2
8	2	3
9	0	4
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Theorem 2.24 (Chinese Remainder Theorem)
If m and n are relatively prime integers, then the equations $x \bmod m = a \in Z_m$ and $x \bmod n = b \in Z_n$ have one and only one solution for an integer x between 0 and $mn - 1$.

<table><tr><td>

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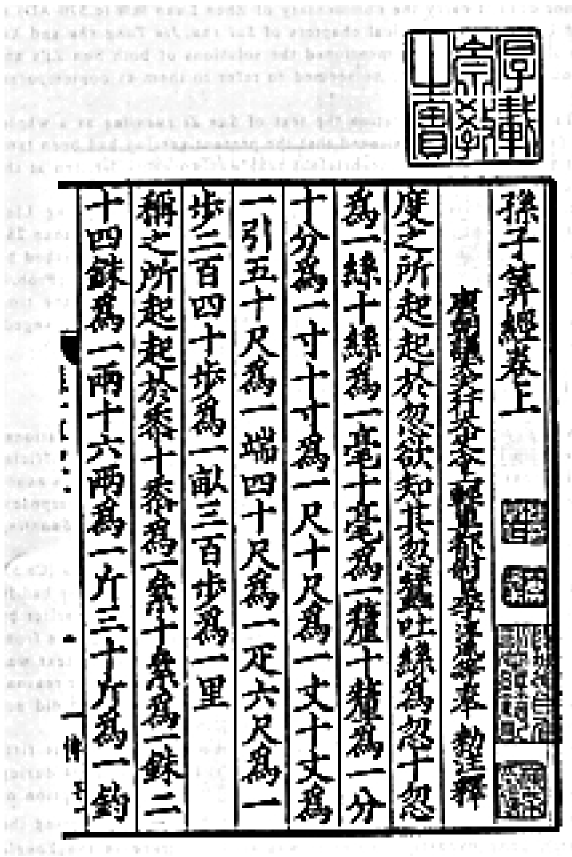
</td></tr></table>

Why is this called the Chinese Remainder Theorem?

The earliest reference known is from the Sun Tzu Suan Ching (also known as Sunzi Suanjing) written in approximately the late third century by Sun Zi. Problem 26 in the third volume of the Sun Tzu Suan Ching offers the earliest recorded description of the Chinese Remainder Problem.



孫子算經卷上

度之所起起於忽欲知其忽蠶吐絲為忽十忽
為一絲十絲為一毫十毫為一氂十氂為一分
十分為一寸十寸為一尺十尺為一丈十丈為
一引五十尺為一端四十尺為一疋六尺為一
步二百四十步為一畝三百步為一里
稱之所起起於黍十黍為一絫十絫為一銖二
十四銖為一兩十六兩為一斤三十斤為一鈞

see *http://www.math.sfu.ca/histmath/China/3rdCenturyBC/CRP1.html* for more details

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Proof: Let $f(x) = (x \bmod m,\ x \bmod n)$

$f : \{0, 1, 2, \ldots, mn-1\} \rightarrow$ the pairs (a, b) : $0 \leq a < m$ and $0 \leq b < n$

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To prove the theorem we must show that f is a bijection.
f is mapping a set of size mn to a set of size mn, so,
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y might not be $< mn$ but we can set $x = y \bmod (mn)$ and get
$x < mn$ and (why?) $x \bmod m = a$ and $x \bmod n = b$

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Example: $m = 6$, $n = 11$, $a = 3$, $b = 7$

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- $\overline{m} = 2$ and $\overline{n} = 5$ since

$$6 \cdot 2 \bmod 11 \;=\; 12 \bmod 11 \;=\; 1$$
$$11 \cdot 5 \bmod 6 \;=\; 55 \bmod 6 \;=\; 1$$

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If m and n are relatively prime integers, then the equations $x \bmod m = a \in Z_m$ and $x \bmod n = b \in Z_n$ have one and only one solution for an integer x between 0 and $mn - 1$.

Example: $m = 6$, $n = 11$, $a = 3$, $b = 7$

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- Reality Check: $\boxed{\begin{array}{c} 51 \bmod 6 = 3 \\ 51 \bmod 11 = 7 \end{array}}$

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- Our proof was a constructive proof.

  We not only showed that the theorem was correct, but we did so by giving a procedure to construct an x satisfying the statement of the theorem.