COMP170
Discrete Mathematical Tools for Computer Science

More on *"time until first success"*

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Example 1

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Then, by linearity of expectation,

$$\boxed{E(X) = E(X_1) + E(X_2) + \ldots + E(X_6) = 6 \cdot 6 = 36.}$$

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i	1	2	3	4	5	6
N_i	2	5	3	6	1	4
X_i	1	2	2	4	5	7

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$i = 1$: $X_1 = 1$ so $E(X_1) = 1$.

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$i = 1$: $\quad X_1 = 1$ so $E(X_1) = 1$.

$i = 2:$ Once N_1 is chosen, X_2 is the number of times we need to throw the die until we see something that is *not* N_1. Since being "*not* N_1" occurs with probability $\frac{5}{6}$, X_2 is geometric with $p = \frac{5}{6}$ so $E(X_2) = \frac{1}{p} = \frac{6}{5}$.

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$i = 3:$ Similarly, once N_1, N_2 are chosen, X_3 is the number of times we need to throw the die until we see something that is *not* N_1, N_2. Since being "*not* N_1, N_2" occurs with probability $\frac{4}{6}$, X_3 is geometric with $p = \frac{4}{6}$ so $E(X_3) = \frac{1}{p} = \frac{6}{4}$.

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General i In the general case, once $N_1, N_2, \ldots.N_{i-1}$ are chosen, X_i is the number of times we need to throw the die until we see something that is *not* $N_1, N_2.\ldots, N_{i-1}$. Since being "*not* $N_1, N_2, \ldots, N_{i-1}$" occurs with probability $\frac{6-(i-1)}{6}$, X_i is geometric with $p = \frac{6-(i-1)}{6}$ so $E(X_i) = \frac{1}{p} = \frac{6}{6-(i-1)}$.

$$\Rightarrow E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6)$$

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$$\Rightarrow E(X) = E(X_1)+E(X_2)+E(X_3)+E(X_4)+E(X_5)+E(X_6)$$

$$= \frac{6}{6} + \frac{6}{5} + \ldots + \frac{6}{1} \quad = 6\sum_{i=1}^{6}\frac{1}{i} \quad = 6 \cdot \frac{49}{20} = \frac{147}{10}.$$

Compare this to previous problem in which we needed $6 \cdot 6$ flips on average, to see the numbers in order.

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$X_1 = 1$.
For $i > 1:\ X_i =$ time needed to receive ith new coupon after having received $(i-1)$st new coupon.

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$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = n \sum_{i=1}^{n} \frac{1}{i}$$

We just showed that

$$E(X) = n \sum_{i=1}^{n} \frac{1}{i}$$

$H_n = \sum_{i=1}^{n} 1/i$ has a special name. It is called the n^{th} **harmonic number**.

It is also known that $\forall n \, |H_n - \ln n| \leq 2$. So H_n grows like $\ln n$ and

$$E(X) \text{ grows like } n \ln n.$$