COMP170
Discrete Mathematical Tools for Computer Science

Lecture 12 *Version 4: Last updated, Nov 3, 2005*

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.3, pp. 157-167

Growth Rates of Solutions to Recurrences

- Divide and Conquer Algorithms
- Recursion Trees
- Three Different Behaviors

Divide and Conquer Algorithms

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$$T(n) = \begin{cases} a & \text{if } n = b \\ c \cdot T(n-1) + d & \text{if } n > b \end{cases}$$

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These corresponded to the analyses of recursive algorithms in which a problem of size n is solved by recursively solving a problem(s) of size $n-1$.

We will now look at recurrences that arise from recursive algorithms in which problems of size n are solved by recursively solving problems of size n/m, for some fixed m. These recurrences will be in the form

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$$T(n) = \begin{cases} \text{something given} & \text{if } n \le b \\ c \cdot T(n/m) + d & \text{if } n > b \end{cases}$$

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Our first example will be binary search.

Someone has a chosen a number x between 1 and n.

We need to discover x.

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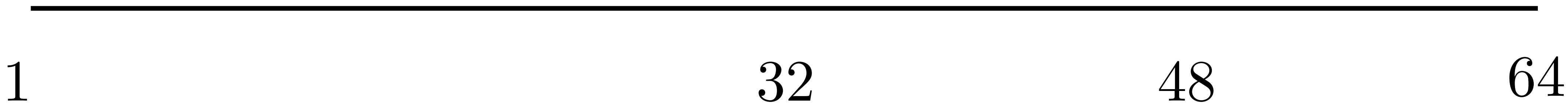
- Is x greater than k?
- Is x equal to k?

Our strategy will be to always ask greater than questions,
at each step halving our search range,
until the range only contains one number,
when we ask a final equal to question

Binary Search Example

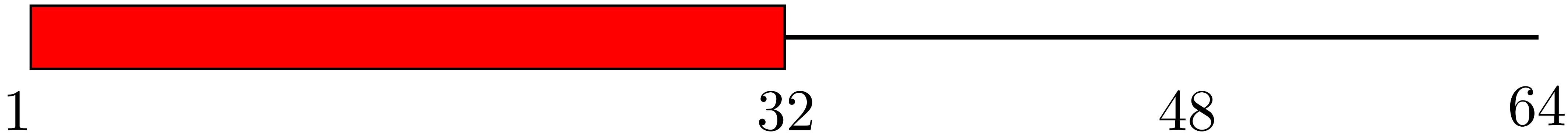
Binary Search Example



Binary Search Example

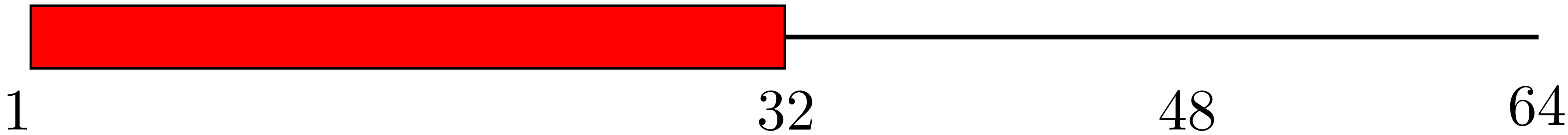
Is $x > 32$?

Binary Search Example



Is $x > 32$? Answer: Yes

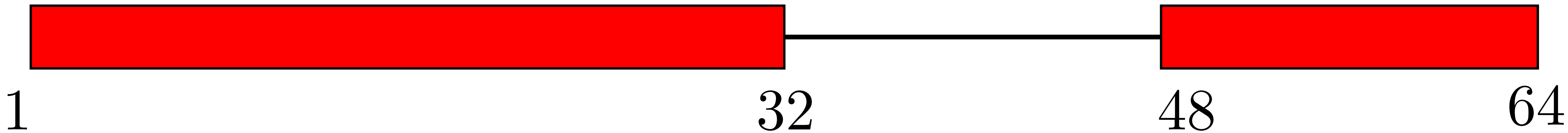
Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$?

Binary Search Example



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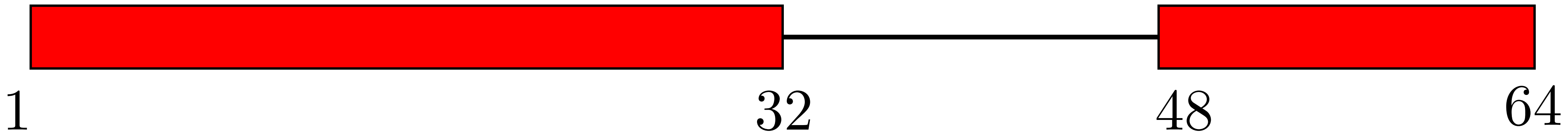
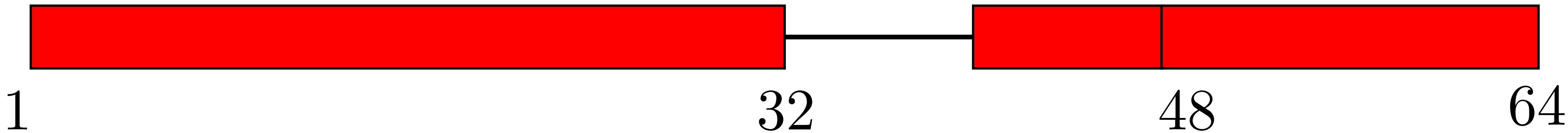


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$?

Binary Search Example



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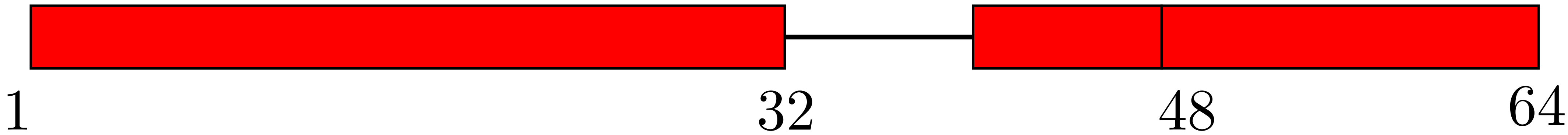

Is $x > 32$? Answer: Yes
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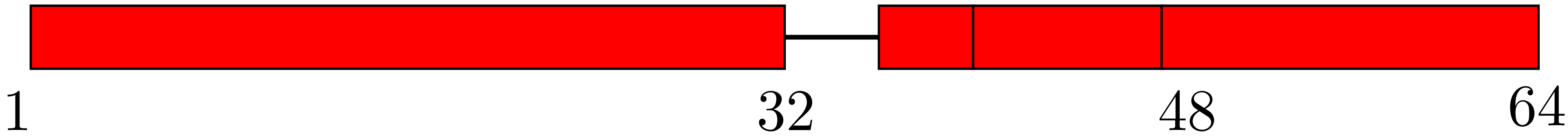

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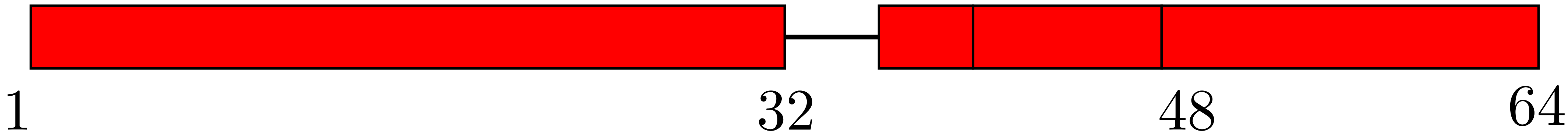

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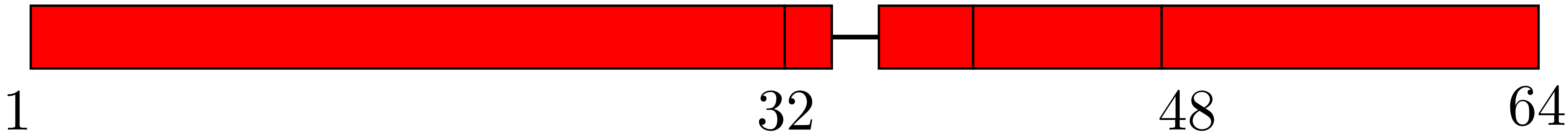

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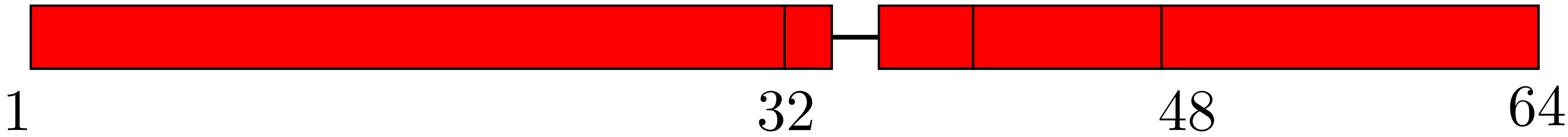

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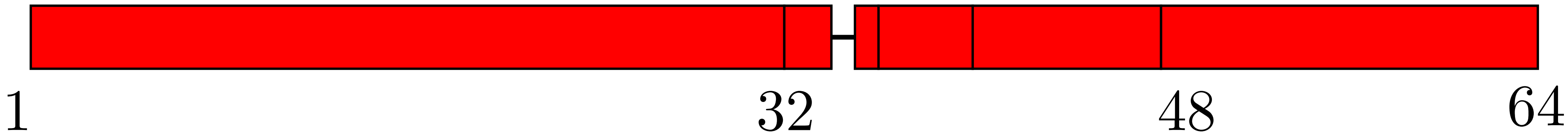

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Is $x > 34$?	Answer: Yes
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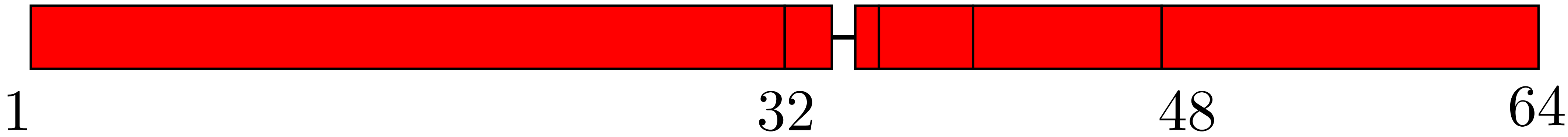

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Is $x = 35$?

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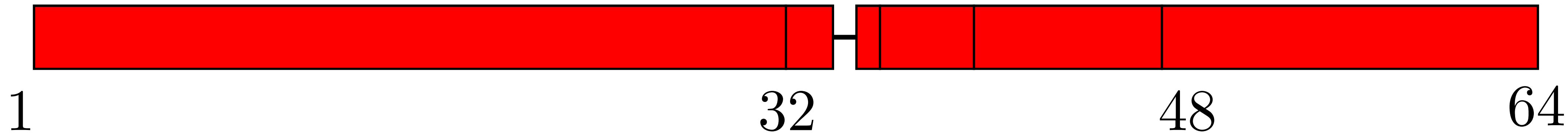

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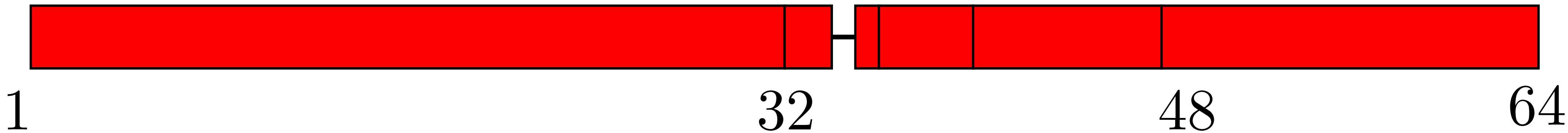

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Method: Each guess reduces the problem to one in which the range is only **half** as big.

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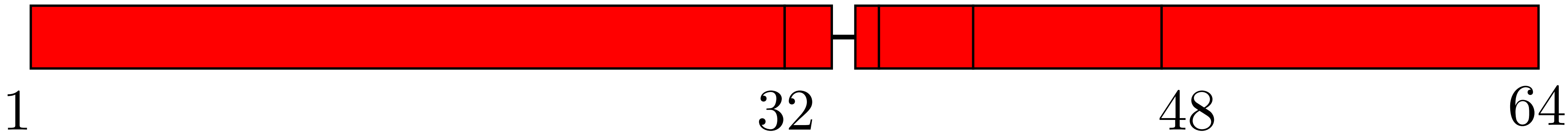

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Method: Each guess reduces the problem to one in which the range is only **half** as big.

This divides the original problem into one that is only half as big; we can now (recursively) conquer this smaller problem.

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Assume: n is power of 2. Give recurrence for $T(n)$.

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Note: Our derivation that, when n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

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was actually, implicitly, an inductive proof. This is similar to what we saw with the tower of Hanoi recurrence. We did not write out all the formal steps of the inductive proof, though.

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Also, why should a greater than query
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More realistic time modeling is

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$$T(n) = \begin{cases} T(\lceil n/2 \rceil) + C_1 & \text{if } n \geq 2, \\ C_2 & \text{if } n = 1, \end{cases}$$

where C_1 and C_2 are constants.

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(*) $$T(n) = \begin{cases} T(\lceil n/2 \rceil) + C_1 & \text{if } n \geq 2, \\ C_2 & \text{if } n = 1, \end{cases}$$

In order to avoid complications we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).

(**) $$T(n) = \begin{cases} T(n/2) + 1 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

In practice, the solution of (*) will be very close to the solution of (**) (this can be proven mathematically) so, as in this class, we can restrict ourselves to (**) without losing much.

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To solve some problem of size n, we
(ii) solve 3 subproblems of size $n-1$ and
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We will start off by examining the recurrence

$$(*) \qquad T(n) = 2T\left(\frac{n}{2}\right) + n$$

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In your later "analysis of algorithms" class (COMP271), you will see that this is exactly how Mergesort, one of the most famous sorting algorithms, works.

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We will now see how to "solve" (*), first by algebraically iterating the recurrence, and then by using a recusion tree (which is a visual method for iterating the recurrence).

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$$= 8T\left(\frac{n}{8}\right) + 3n$$

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$$= 2^{(\log_2 n)} T\left(\frac{n}{2^{(\log_2 n)}}\right) + (\log_2 n)\, n$$

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# of Probl	Probl Size	Work per Probl	Work per Level
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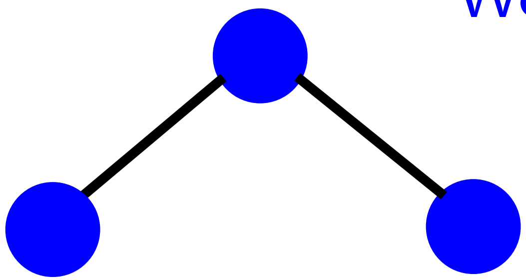
2	$n/2$		



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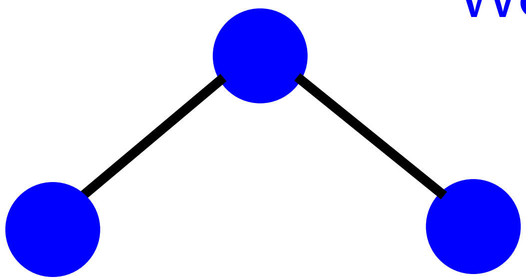

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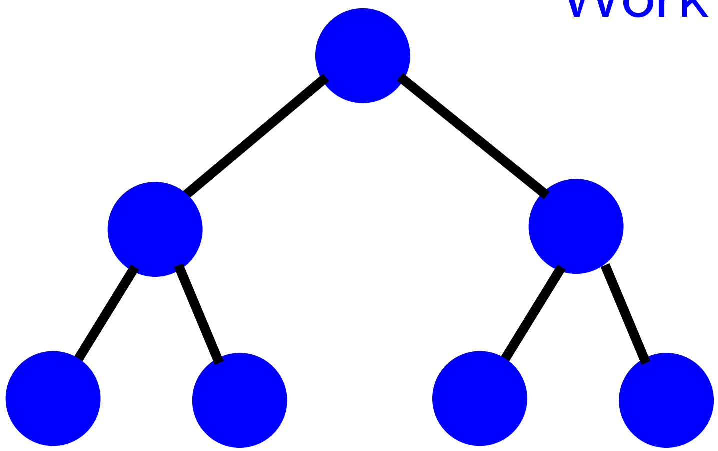
2	$n/2$	$n/2$	$2(n/2) = n$

Example: Visually "iterating" the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n = 4T\left(\frac{n}{4}\right) + 2n$$

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
2	$n/2$	$n/2$	$2(n/2) = n$
$2^2 = 4$	$n/4$		



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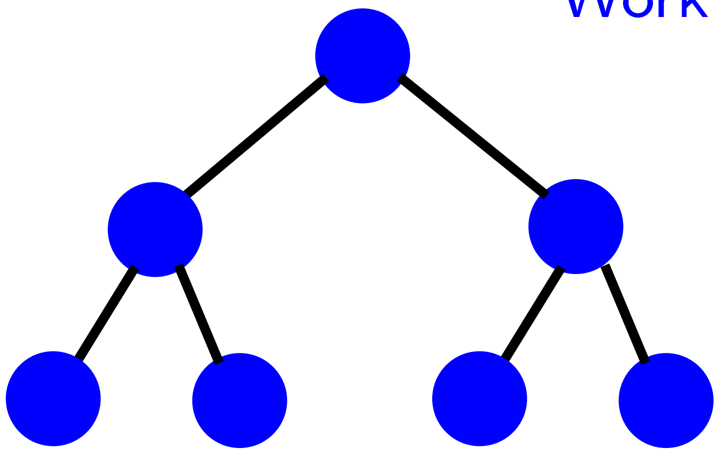

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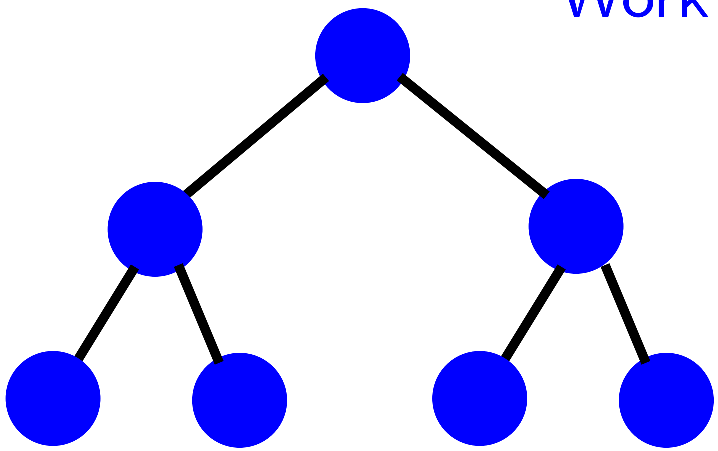
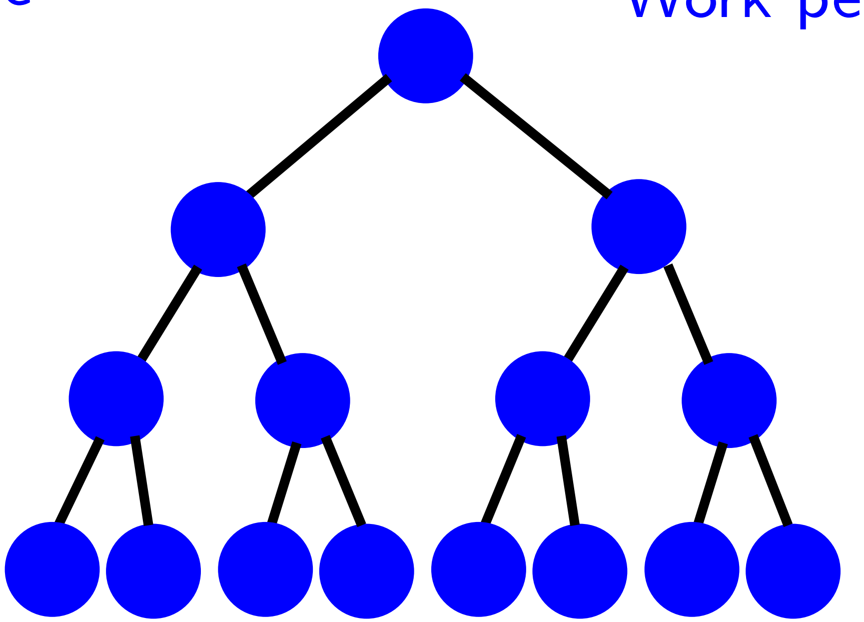


Example: Visually "iterating" the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n \;= 4T\left(\frac{n}{4}\right) + 2n \;= 8T\left(\frac{n}{4}\right) + 3n.$$

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
2	$n/2$	$n/2$	$2(n/2) = n$
$2^2 = 4$	$n/4$	$n/4$	$4(n/4) = n$
$2^3 = 8$	$n/8$		



Recursion Tree Diagram

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We also need to figure out how many levels there are in the recursion tree.

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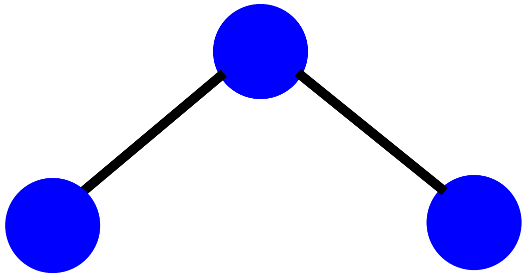
1	n	n	n
2	$n/2$		



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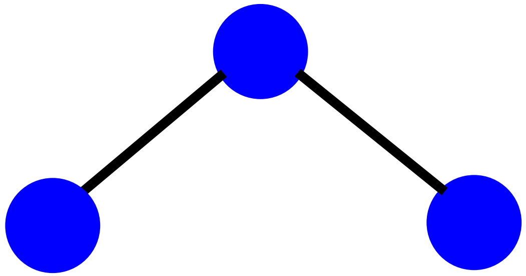

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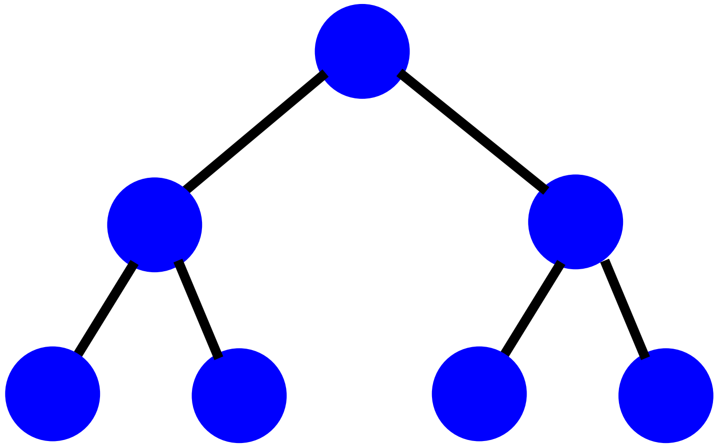

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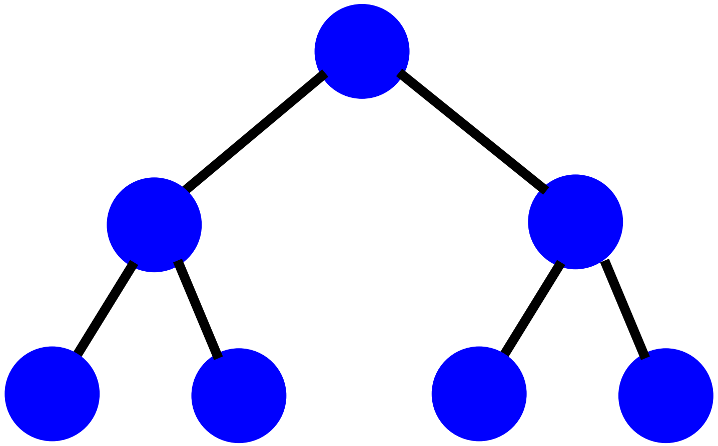

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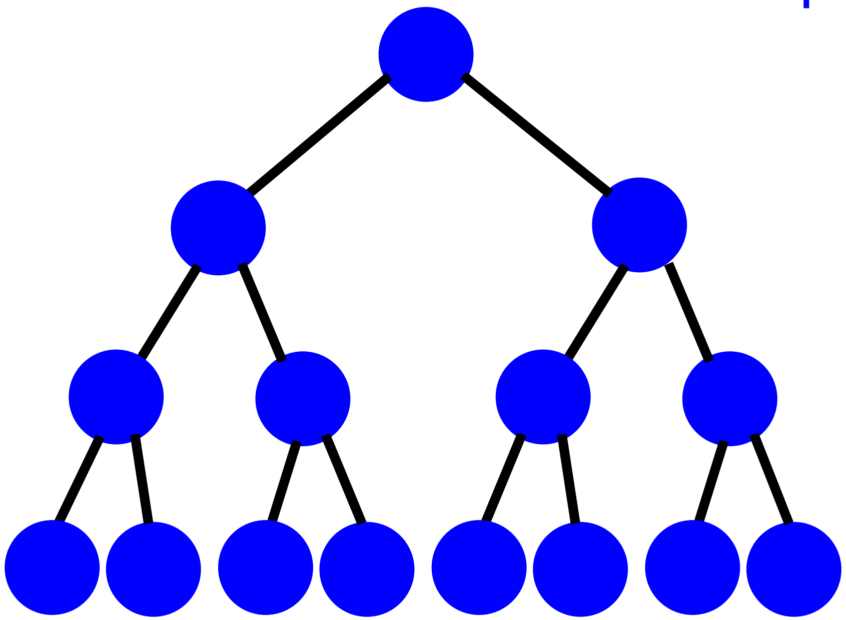

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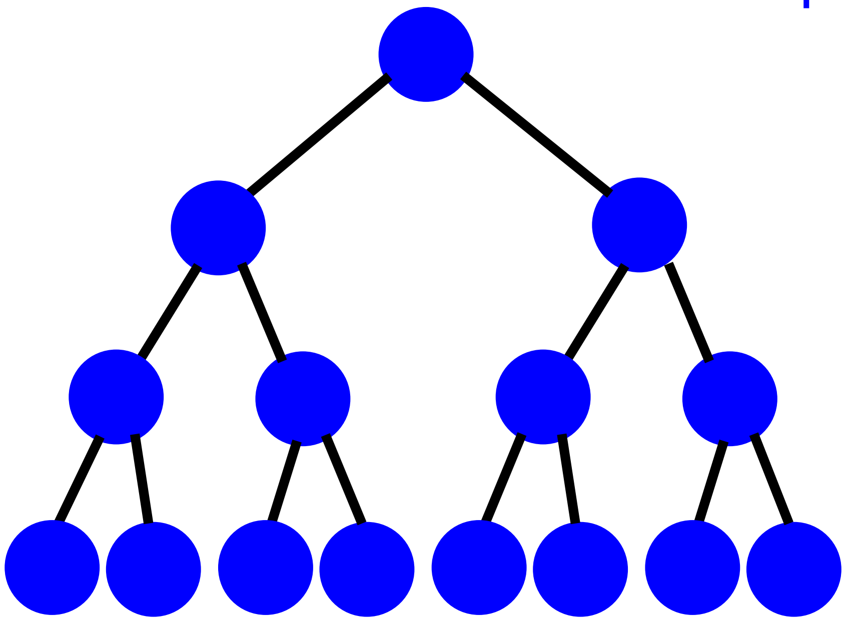

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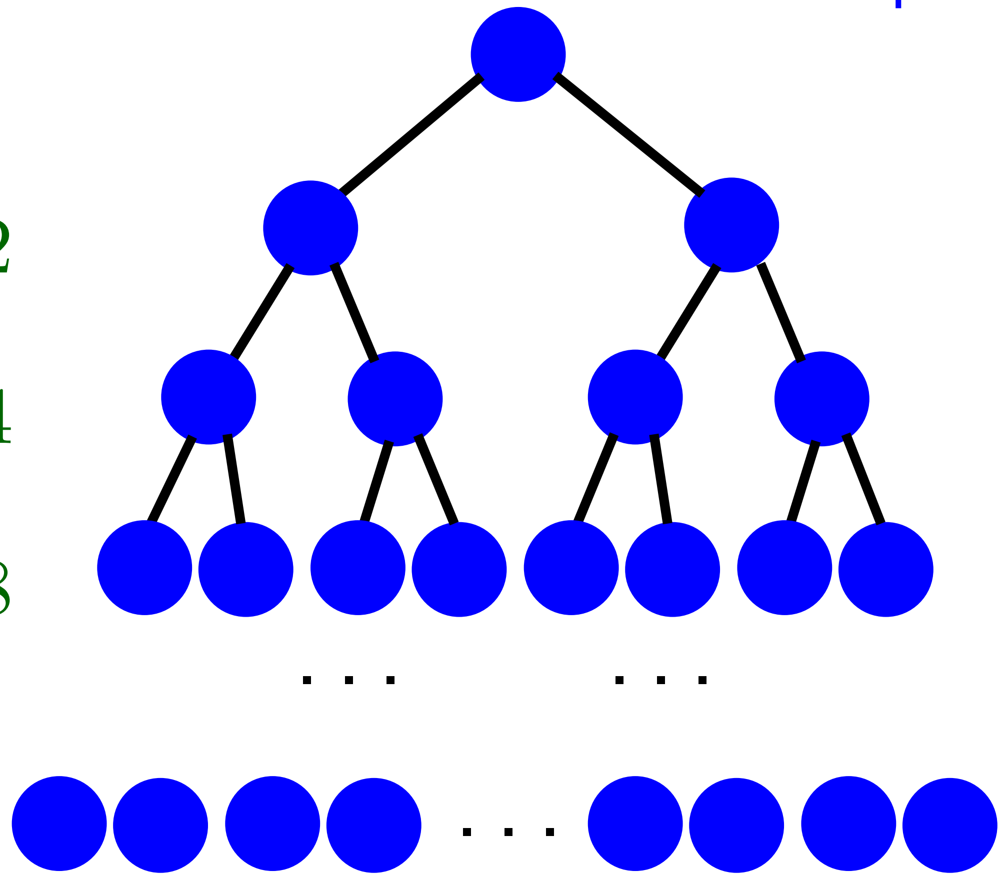

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$2^3 = 8$	$n/8$	$n/8$	$8(n/8) = n$
$2^{\log n} = n$	1		

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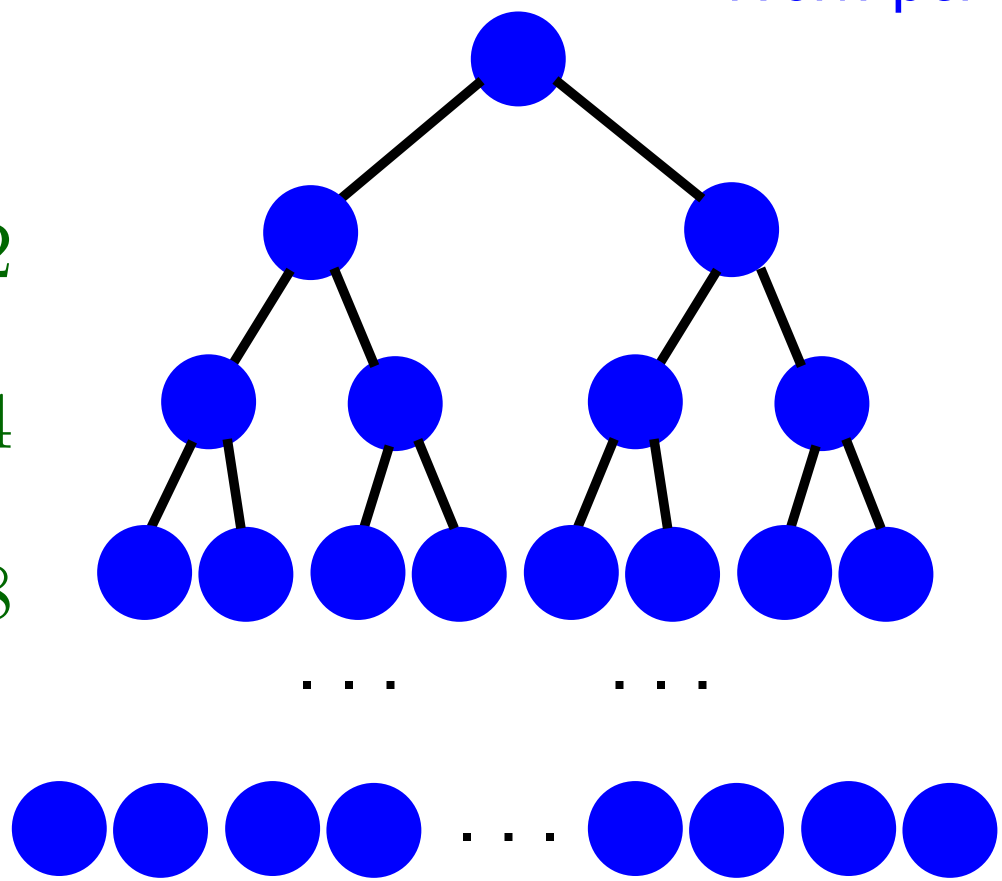

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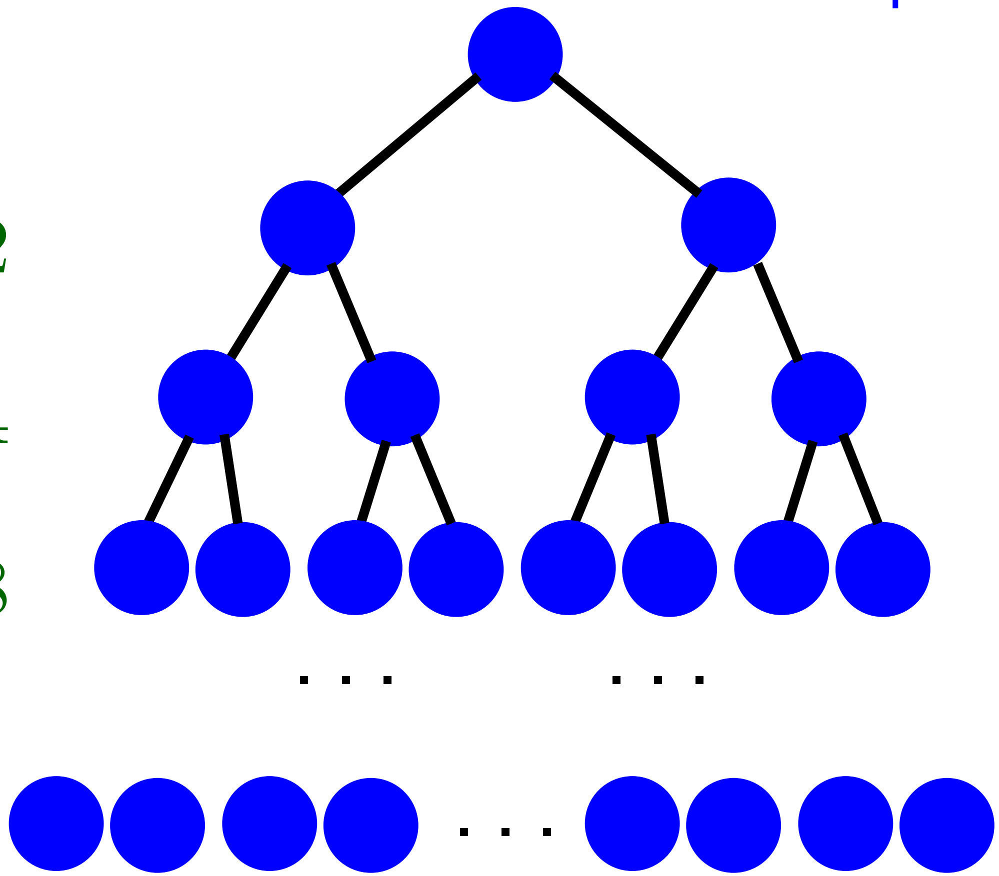

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$2^{\log n} = n$	1	$T(1)$	$nT(1)$

$(1 + \log_2 n)$ levels $\Rightarrow$ total work $= n \log_2 n + nT(1)$

We just derived that the solution to

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n \geq 2, \\ T(1) & \text{if } n = 1. \end{cases}$$

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We derived this in two different, but similar, ways; first by algebraically iterating the recurrence. Second by drawing the recursion tree.

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We derived this in two different, but similar, ways; first by algebraically iterating the recurrence. Second by drawing the recursion tree.

Note: Technically, we still need to use induction to prove *that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work (the* $\boxed{\ldots}$ *in the algebraic iteration and the recursion tree, are really hiding an inductive step).*

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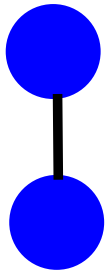
1	n	1	1
1	$n/2$		



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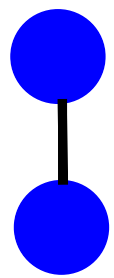

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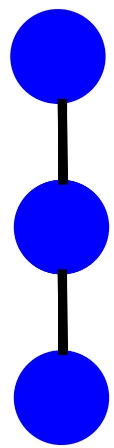
1	n	1	1
1	$n/2$	1	1
1	$n/4$		



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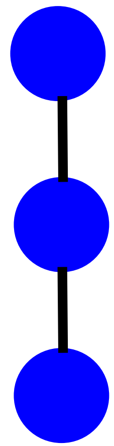

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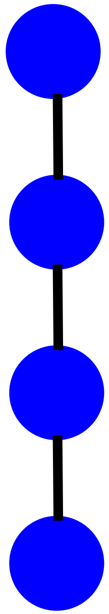

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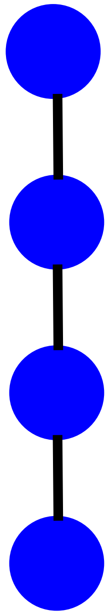

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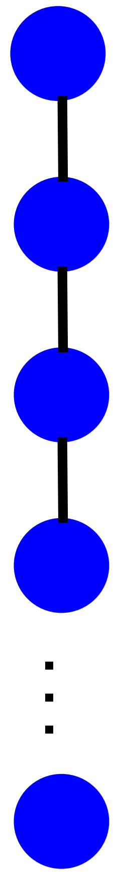

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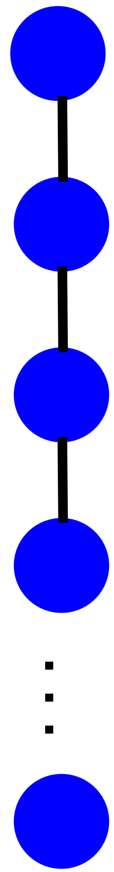

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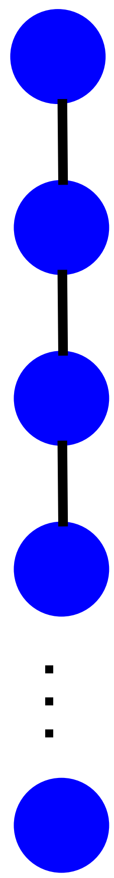

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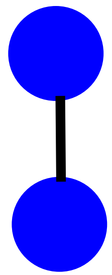

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1	n	n	n
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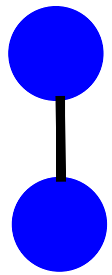

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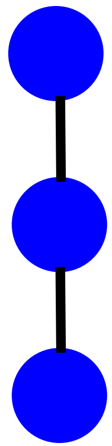

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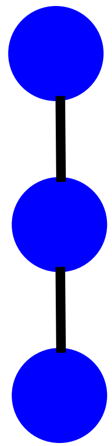

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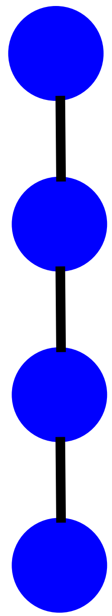

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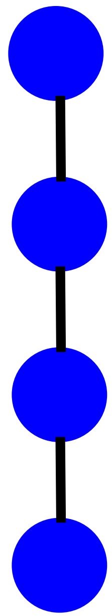

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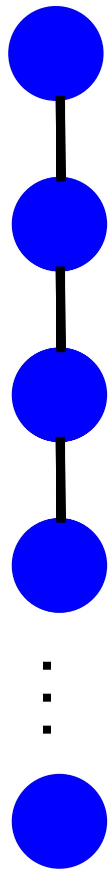

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1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$		
1	1		

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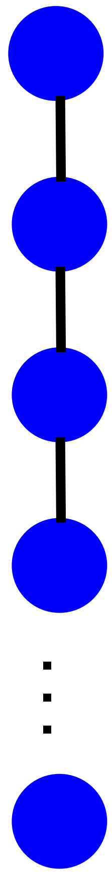
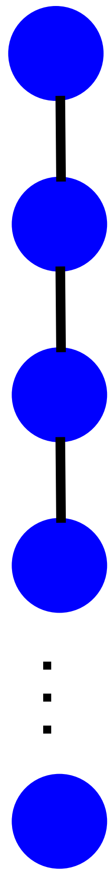
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1



Example 3

$$T(n) = \begin{cases} T(n/2) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$



# of Probl	Probl Size	Work per Probl	Work per Level
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1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1

$(1 + \log_2 n)$ levels. Total work $= n + \frac{n}{2} + \frac{n}{4} + \cdots + 2 + 1$

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Total amount of work:

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\begin{aligned}
n + \frac{n}{2} + \frac{n}{4} + \cdots + 2 + 1 \\
&= n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{\log n}\right) \\
&= n \sum_{i=0}^{\log_2 n} \left(\frac{1}{2}\right)^{i}
\end{aligned}
$$

Theorem 4.4 tells us that the value of the geometric series is $O(1)$ (in fact it is ≤ 2) so, the total amount of work done is $O(n)$.

Example 4

$$T(n) = \begin{cases} 3T(n/3) + n & \text{if } n \geq 3, \\ 1 & \text{if } n < 3. \end{cases}$$

Example 4

assume n *is power of* 3

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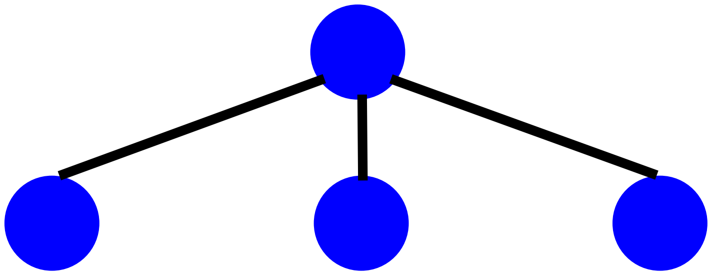

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$		

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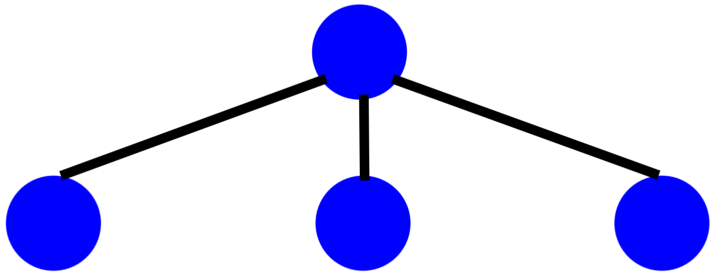

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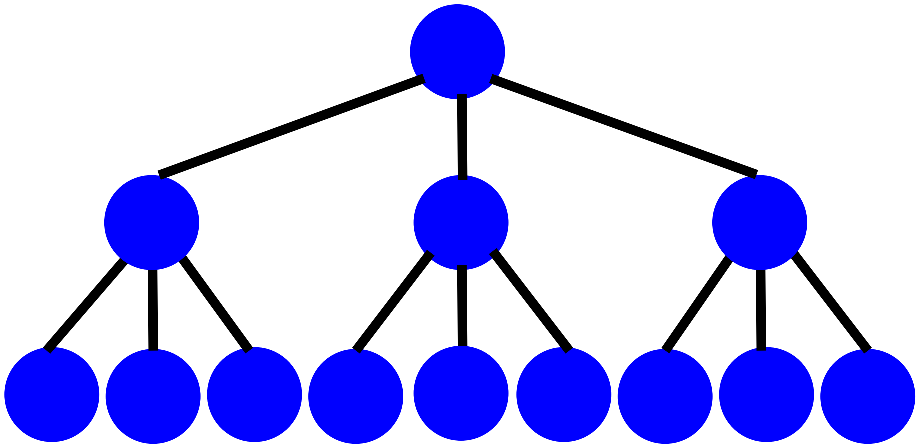

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$		

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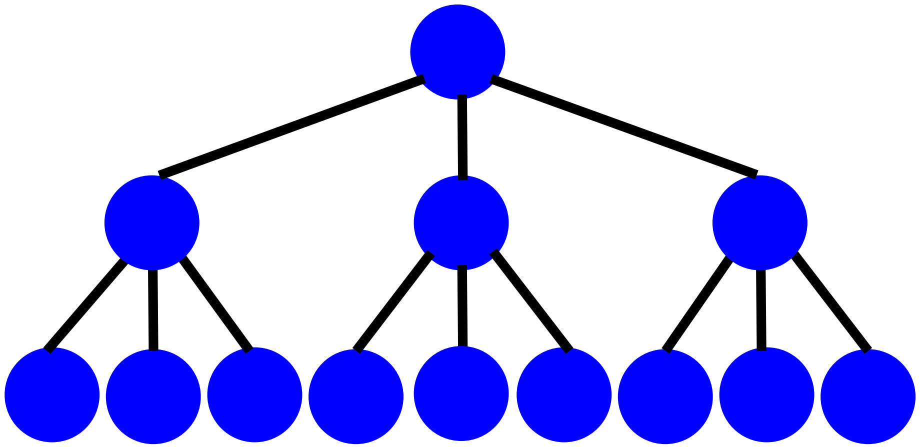

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
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$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$

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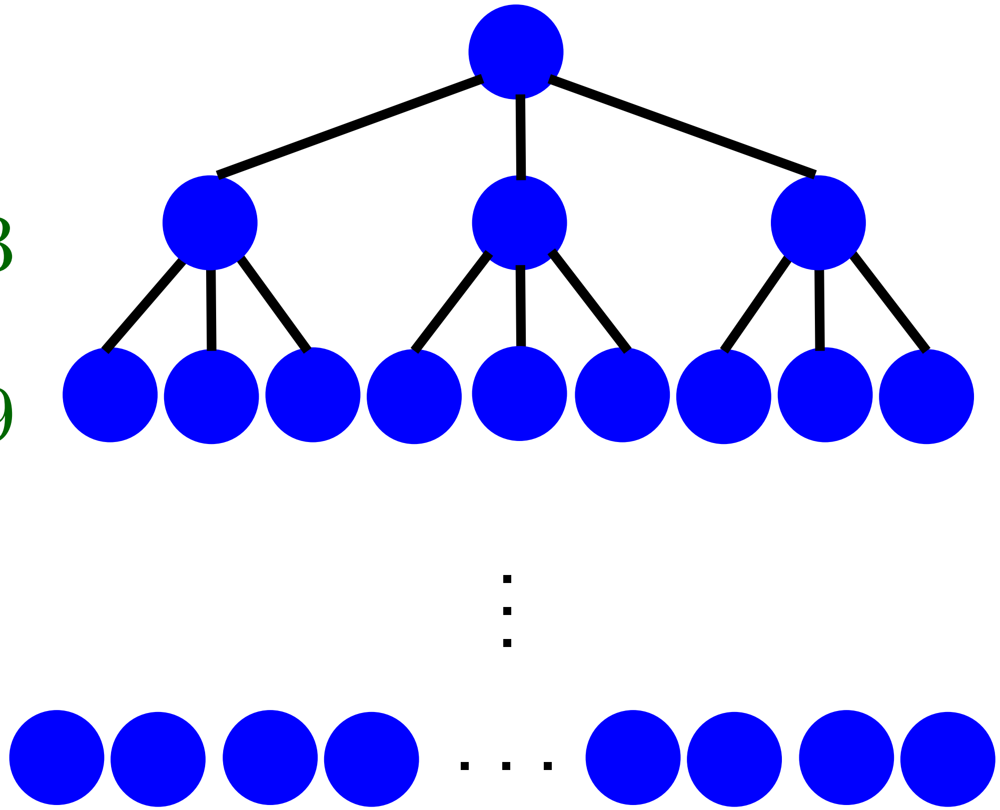

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
$\vdots$	$\vdots$		
$3^{\log_3 n} = n$	1		

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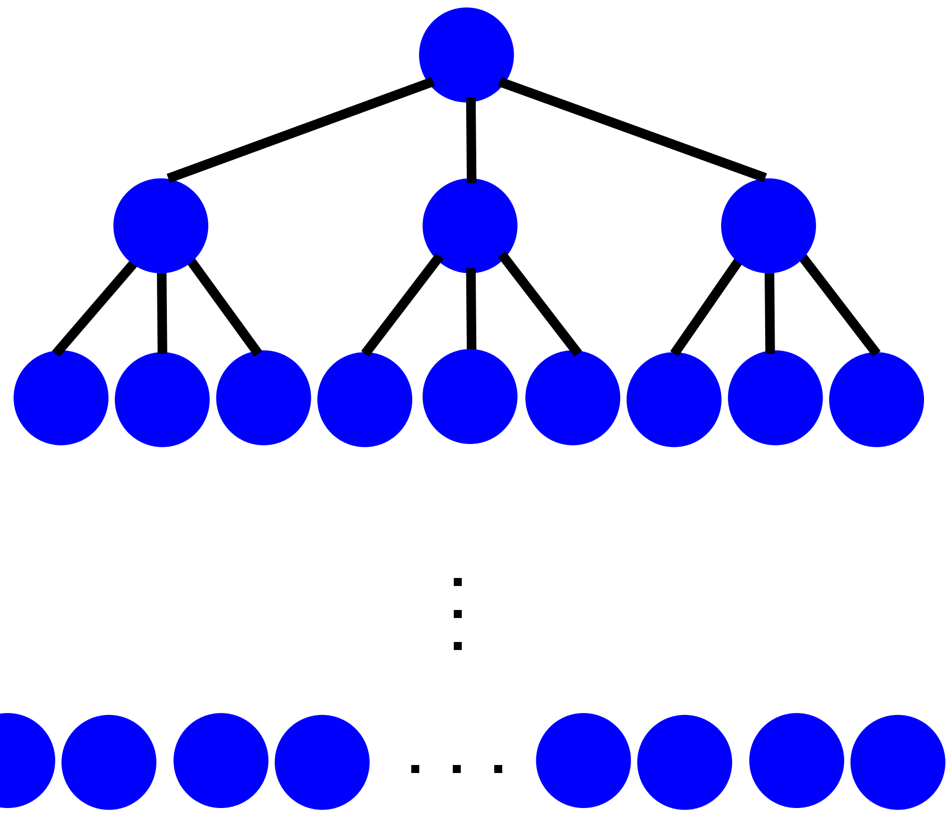
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{\log_3 n} = n$	1	1	$n(1) = n$



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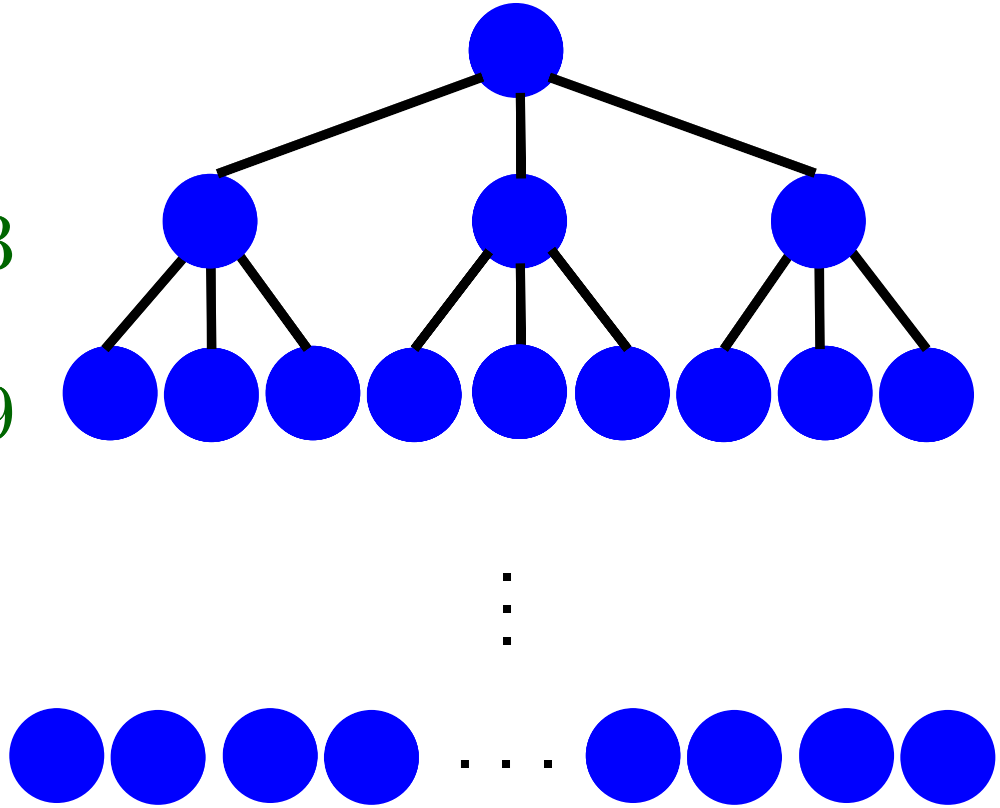

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$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
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$3^{\log_3 n} = n$	1	1	$n(1) = n$

$(1 + \log_3 n)$ levels $\Rightarrow$ total work $= n(1 + \log_3 n)$

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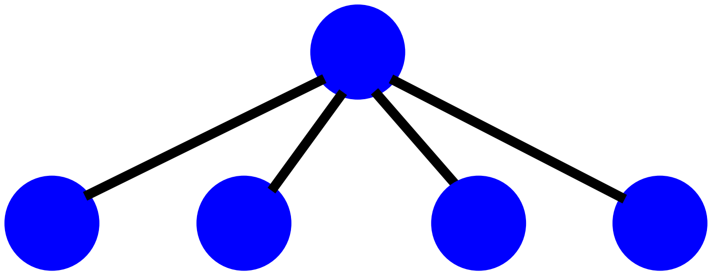
1	n	n	n
4	$n/2$		



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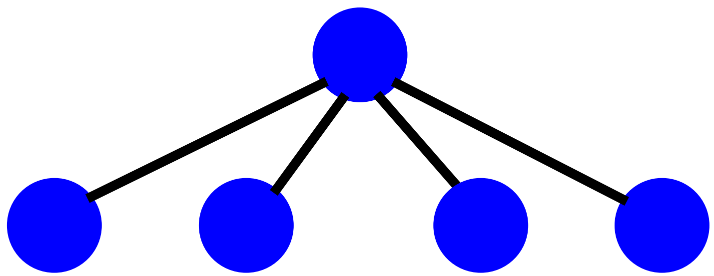
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$



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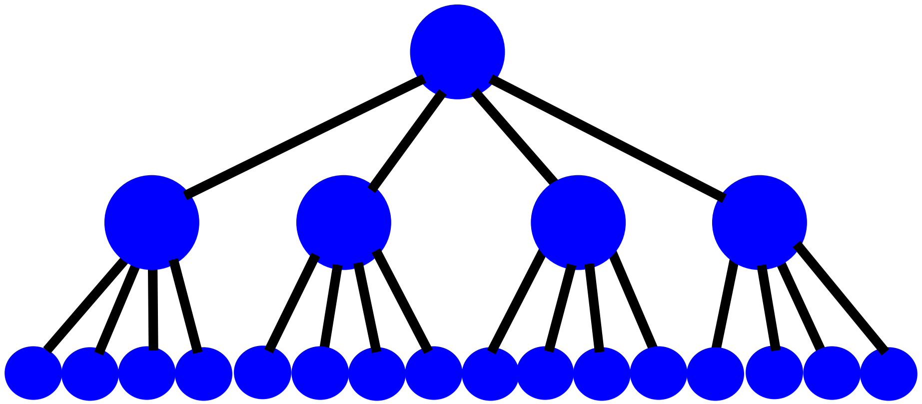
$4^2 = 16$	$n/4$		



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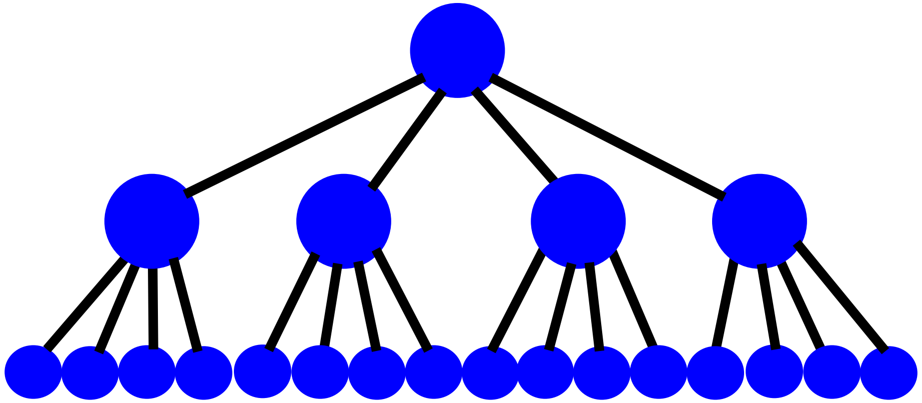

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1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$	$n/4$	$16(n/4) = 4n$

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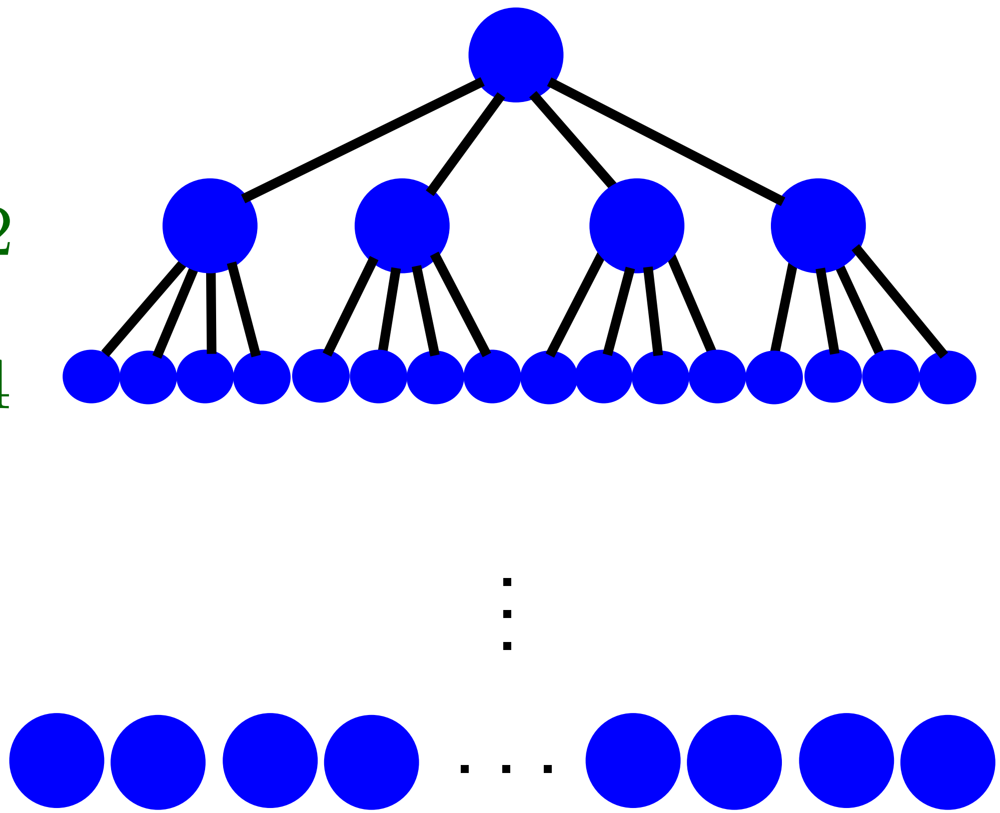

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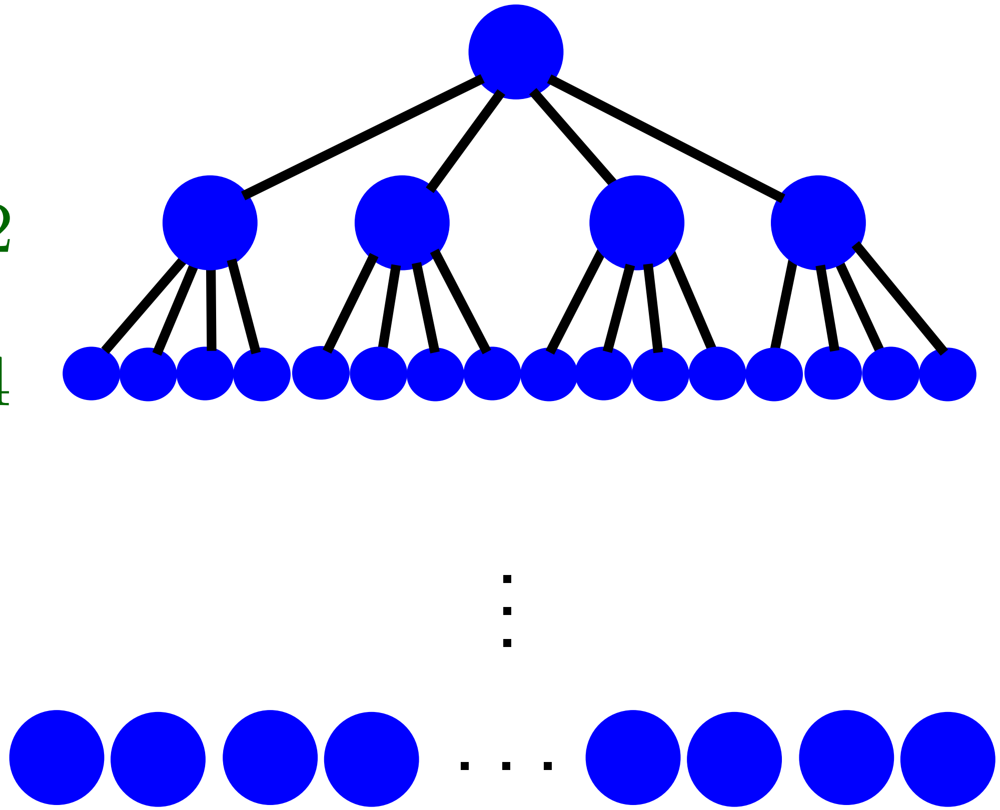

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$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4^{\log_2 n} = n^2$	1	1	$n^2 \cdot 1 = n^2$

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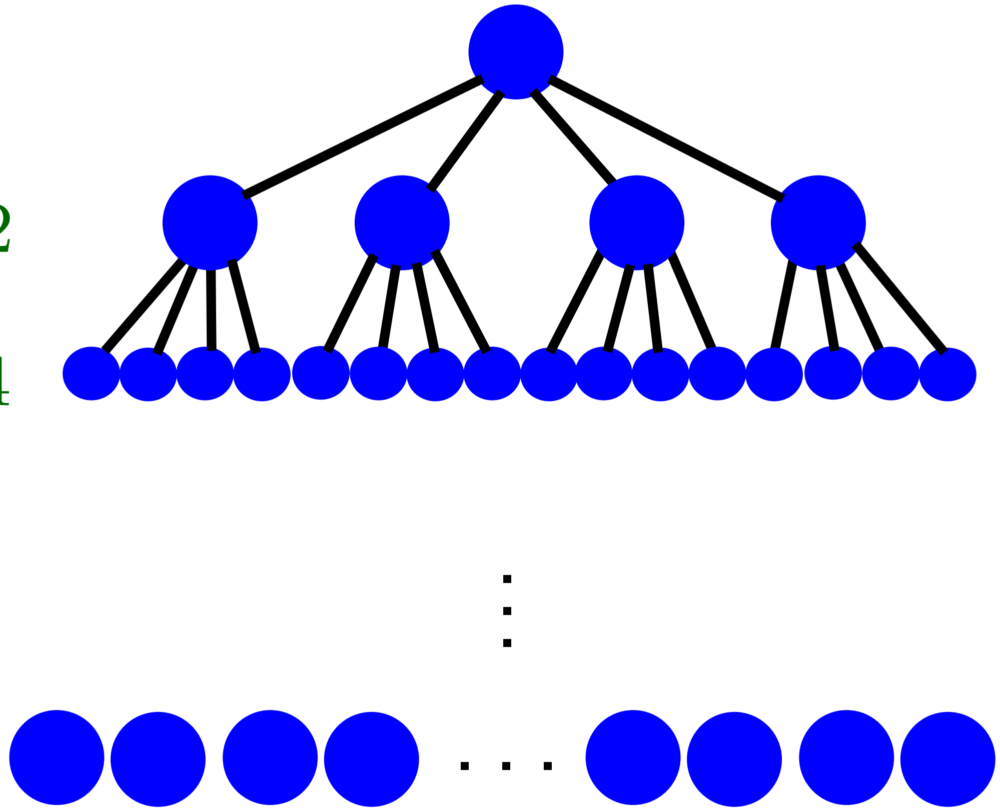

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&= 2n^2 - n
\end{aligned}
$$

Growth Rates of Solutions to Recurrences

- Divide and Conquer Algorithms
- Recursion Trees
- Three Different Behaviors

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- all three trees have depth $1 + \log_2 n$

- in each case, the size of each subproblem is
  half the size of the parent problem

- differ in the amount of work done per level

Lemma 4.7:
Suppose that we have a recurrence of the form

$$T(n) = aT\left(\frac{n}{2}\right) + n,$$

where a is a positive integer and $T(1)$ is nonnegative.
Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.

2. If $a = 2$, then $T(n) = \Theta(n \log n)$.

3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

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We already proved Case 1 when $a = 1$ in Example 2.

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We will now prove Case 3.

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Summing over the $1 + \log_2 n$ levels, we get

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Summing over the $1 + \log_2 n$ levels, we get

$$\underbrace{a^{\log_2 n} T(1)}_{\text{Work at bottom level}} \quad + \quad \underbrace{n \sum_{i=0}^{(\log_2 n)-1} \left(\frac{a}{2}\right)^i}_{\text{Work on non-bottom levels}}.$$

Total work is

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{(\log_2 n)-1} \left(\frac{a}{2}\right)^i.$$

This sum is a geometric series.

Because $a/2 \neq 1$, Theorem 4.4 tells us that the sum will be big Θ of the largest term.

Because $a > 2$, the largest term in this case is clearly the last one, namely, $n(a/2)^{(\log_2 n)-1}$.

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so the total work done is

$$a^{\log_2 n} T(1)$$

n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{(\log_2 n)-1} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a} \cdot \frac{n \cdot a^{\log_2 n}}{n} = \frac{2}{a} \cdot a^{\log_2 n}$$

Now notice that

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

so the total work done is

$$\underbrace{a^{\log_2 n} T(1)}_{\Theta\left(n^{\log_2 a}\right)}$$

n times the largest term in the geometric series is

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and we are done!

As an example of Case 3 consider

$$T(n) = \begin{cases} 4T(n/2) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

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$a = 4$ so the Theorem says that

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This matches with the exact answer of $2n^2 - n$, which we already derived in Example 5.