B-Trees

Version of October 2, 2014

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  - Each operation on an n-node AVL tree takes $O(\log n)$ time
- This only works, though, long as the entire data structure fits into main memory
- When the data size is too large and data must reside on disk, AVL performance may deteriorate rapidly.

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- Consider a database with 10^9 items (stored on disk)
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- Want a way to substantially reduce number of disk accesses.

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  - This reduces disk accesses and speeds up search significantly

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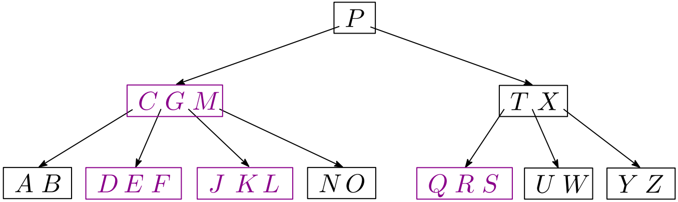

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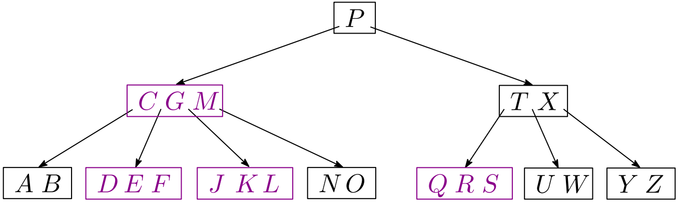

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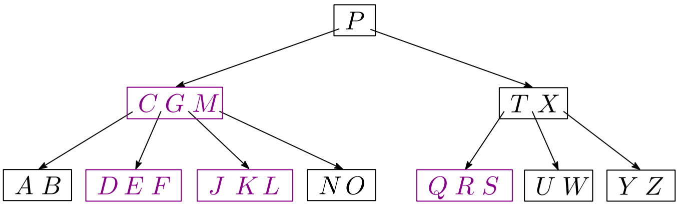

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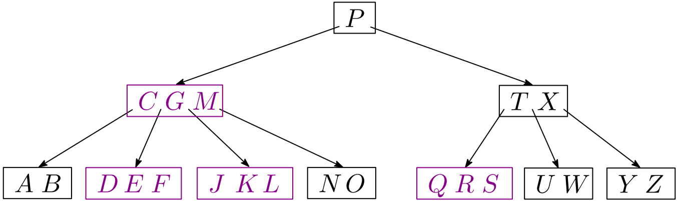
  - Every node has at most 3 keys.
    Every internal node has at most 4 children



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- A node is **full** if it contains exactly $2t - 1$ keys (e.g., nodes colored in the above example)
- We choose t such that an internal node fits in one disk block.

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which implies,

- 1 node at depth 0; 2 nodes at depth 1; $2t$ nodes at depth 2; $2t^2$ nodes at depth 3; ...; $2t^{h-1}$ nodes at depth h

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Thus, for any n-key B-tree of minimum degree $t \geq 2$ and height h

$$\begin{aligned} n &\geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1} = 1 + 2(t-1)\left(\frac{t^h - 1}{t-1}\right) \\ &= 2t^h - 1. \end{aligned}$$

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- Compared with AVL trees, a factor of about $\log_2 t$ is saved in the number of nodes examined for most tree operations.

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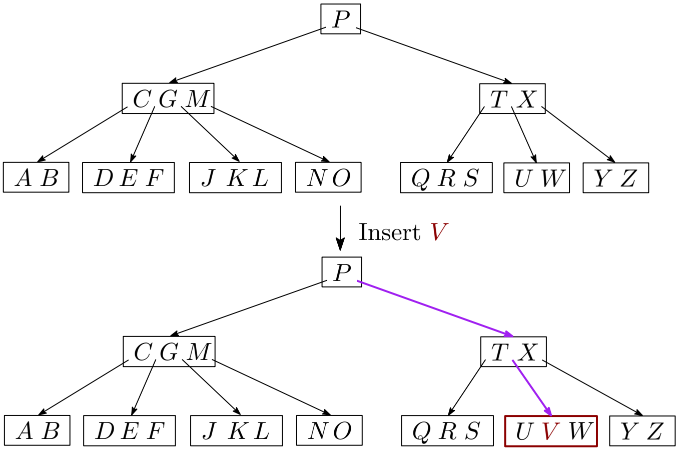

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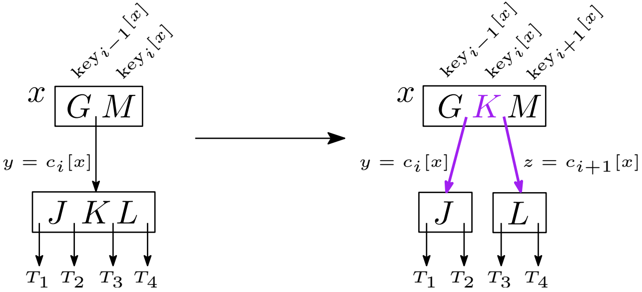

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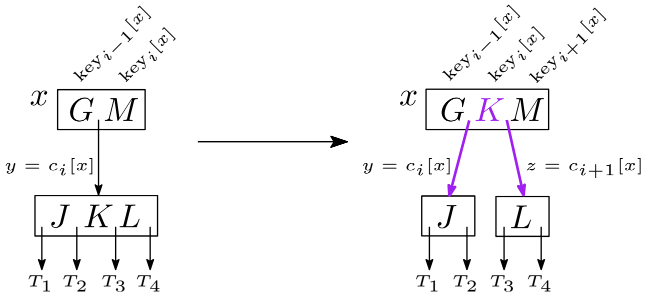

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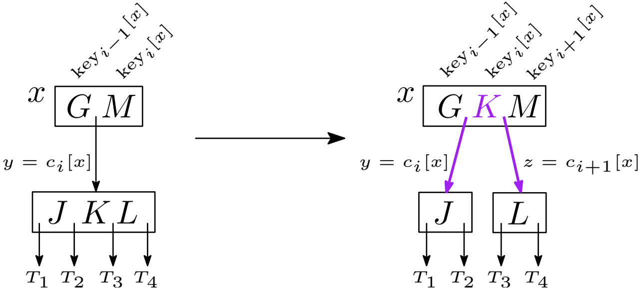

1. split the full node y (with $2t - 1$ keys) around its median key $\text{key}_t[y]$ into two nodes with $t - 1$ keys each
2. move $\text{key}_t[y]$ up into y's parent x to separate the two nodes

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How can we insure that the parent of a full node is not full?

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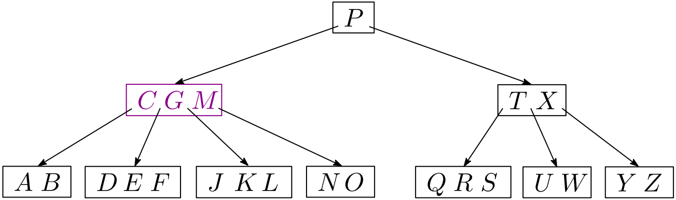
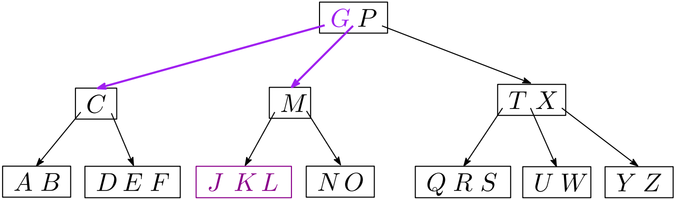
- A key can be inserted into a B-tree in a single pass down the tree from the root to a leaf
- Splitting the root is the only way to increase the height of a B-tree

Insertion: Example

(a) initial tree

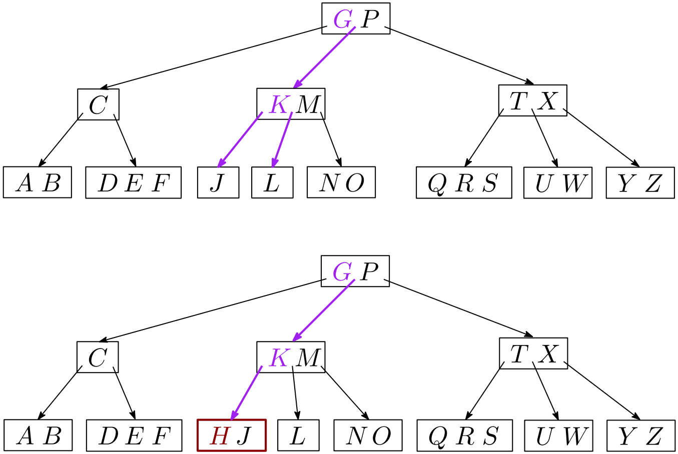




(b) insert H: split the encountered full node

Insertion: Example

(c) insert H: split the encountered full node



(d) insert H: insert into an existing nonfull leaf node

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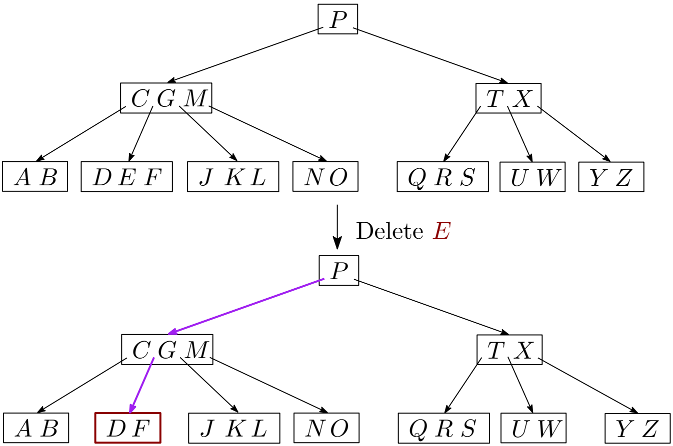

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Remove(x, k) will remove k from subtree rooted at x.. Algorithm walks down tree towards k. Will (for non-root x) first ensure that x contains at least t keys. Then will either remove k or recursively call *Remove*(x', k'), where k' is some key (possibly not k) and x' is the root of subtree of x containing k'.

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Case 2 k is not in the internal node x

Deletion: Comments on root deletion

When viewing the following, note that height of tree remains unchanged *except* in special cases of 1(c) and 2(c) when

- root x contains exactly one key
- root x's two children $c_1[x]$ and $c_2[x]$ each contain exactly $t-1$ keys.

Both 1(c) and 2(c) will then merge $c_1[x]$ $key[x]$ and $c_2[x]$ into one new root node and then proceed to delete k from the new tree rooted at this new node.

We will not explicitly illustrate these cases in the following slides.

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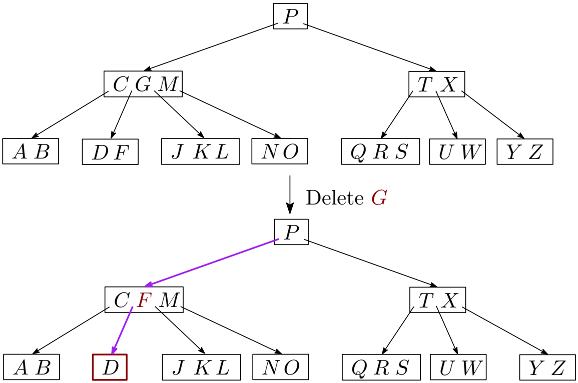

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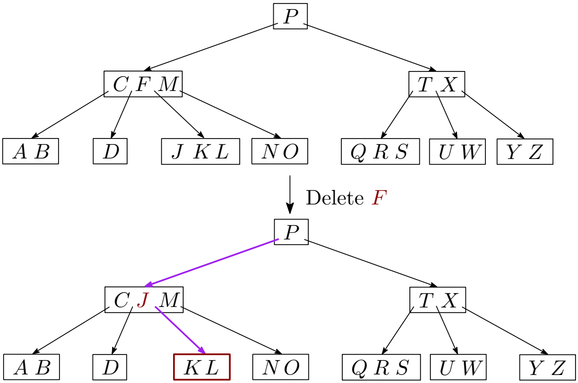

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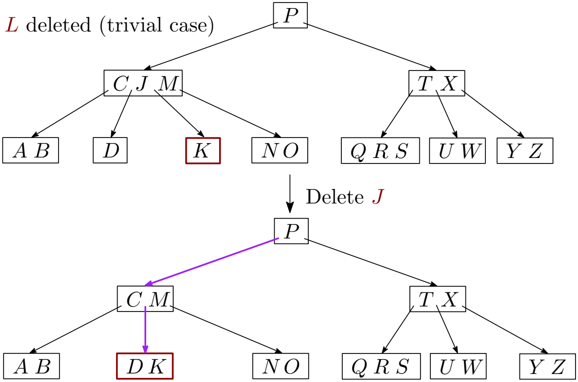

- J is pushed down to make node DJK, from where J is deleted

Deletion: Case 2a

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has $> t-1$ keys then

- Recursively delete k from $c_i[x]$

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b. If $c_i[x]$ has an immediate sibling with at least t keys

Deletion: Case 2b

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t - 1$ keys

b. If $c_i[x]$ has an immediate sibling with at least t keys
   1. give $c_i[x]$ an extra key by moving a key from x down into $c_i[x]$

Deletion: Case 2b

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t-1$ keys

b. If $c_i[x]$ has an immediate sibling with at least t keys
   1. give $c_i[x]$ an extra key by moving a key from x down into $c_i[x]$
   2. move a key from $c_i[x]$'s immediate left or right sibling up into x

Deletion: Case 2b

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t-1$ keys

b. If $c_i[x]$ has an immediate sibling with at least t keys
   1. give $c_i[x]$ an extra key by moving a key from x down into $c_i[x]$
   2. move a key from $c_i[x]$'s immediate left or right sibling up into x
   3. move the appropriate child pointer from the sibling into $c_i[x]$

Deletion: Case 2b

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4. recursively delete k from the appropriate child of x

Deletion: Case 2b

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t-1$ keys

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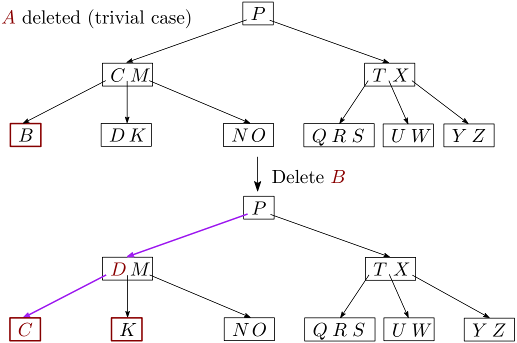

- C is moved to fill B's position, and D is moved to fill C's

Deletion: Case 2c

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t-1$ keys

c. and both of its immediate siblings have $t-1$ keys

Deletion: Case 2c

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t - 1$ keys

c. and both of its immediate siblings have $t - 1$ keys
   1. merge $c_i[x]$ with one sibling

Deletion: Case 2c

Case 2: the key k is not in the internal node x, then determine the root $c_i[x]$ whose subtree contains k. If $c_i[x]$ has only $t-1$ keys

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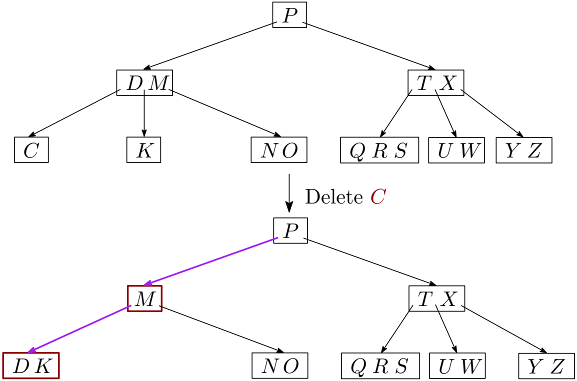

- D is pushed down to get node CDK, from where C is deleted

B-Trees: More

- Saw how to maintain a B-tree using $\log_t n$ "operations"
  - each operation requires constant number of disk reads.
  - could also require many internal memory operations
- For "large" t; useful for storing large databases on disk
  - with each node a disk page

- B-Trees created by Bayer and McCreight at Boeing in 1972
- B^+ tree variant keeps data keys in leaves

- Simplest B-Tree is $(2-3-4)$-tree
  - Balanced tree good for internal memory storage

- Another variation is (a, b)-trees: all non-root nodes have between a and b children
  - Is a B-tree if $b = 2a$.
  - Smallest (non B-Tree) version is $(2, 3)$-trees