Illustration of the Proof of Lemma 5.28

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If X and Y are independent random variables on sample space S with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

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In these sides, we illustrate the proof of Lemma 5.28 with an example.

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$$P(X = 4) = 1/3 \qquad P(Y = 4) = 1/4$$

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$$P(X = 2) = 1/3 \qquad P(Y = 2) = 1/4 \quad \Rightarrow$$
$$P(X = 4) = 1/3 \qquad P(Y = 4) = 1/4$$

$$E(X) = 7/3$$
$$E(Y) = 2$$
$$E(X)(EY) = 14/3$$

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$$\begin{array}{lll} P(X=1)=1/3 & P(Y=1)=1/2 & \\ P(X=2)=1/3 & P(Y=2)=1/4 & \\ P(X=4)=1/3 & P(Y=4)=1/4 & \end{array} \Rightarrow \begin{array}{l} E(X)=7/3 \\ E(Y)=2 \\ E(X)(EY)=14/3 \end{array}$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$.

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$$P(X=2)=1/3 \quad P(Y=2)=1/4 \quad \Rightarrow \quad E(Y)=2$$
$$P(X=4)=1/3 \quad P(Y=4)=1/4 \qquad E(X)(EY)=14/3$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$.

$$P(Z=1) = P(X=1 \ \wedge Y=1) = \tfrac{1}{6}$$

$$P(Z=2) = P(X=1 \ \wedge \ Y=2)+P(X=2 \ \wedge \ Y=1) = \tfrac{1}{12} + \tfrac{1}{6} = \tfrac{1}{4}$$

$$P(Z=4) = P(X=1 \ \wedge \ Y=4)+P(X=4 \ \wedge \ Y=1) \\ +P(X=2 \ \wedge \ Y=2) = \tfrac{1}{12} + \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{1}{3}$$

$$P(Z=8) = P(X=2 \ \wedge \ Y=4)+P(X=4 \ \wedge \ Y=2) = \tfrac{1}{12} + \tfrac{1}{12} = \tfrac{1}{6}$$

$$P(Z=16) = P(X=4 \ \wedge Y=4) = \tfrac{1}{12}$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$. So

$$\begin{aligned} E(XY) &= E(Z) \\ &= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\ &\qquad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\ &= \frac{14}{3} \\ &= E(X) \cdot E(Y) \end{aligned}$$

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On the next page, we mimic the proof of Lemma 5.28, using these X, Y. Reading the proof with this example in mind, might make the proof more understandable.

$E(X)E(Y)$

$$E(X)E(Y) = \sum_{x \in \{1,2,4\}} xP(X = x) \sum_{y \in \{1,2,4\}} yP(Y = y)$$

$$
\begin{aligned}
E(X)E(Y) &= \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y) \\
&= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)
\end{aligned}
$$

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$$= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x,y \in \{1,2,4\} \\ xy = z}} P(X = x)P(Y = y)$$

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$\Updownarrow$ Ind of X, Y

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$$= \sum_{z\in\{1,2,4,8,16\}} zP(Z=z) = E(Z) = E(XY)$$

$$\boxed{E(X)E(Y)} = \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y)$$

$$= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)$$

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