The Theory of Sets

Cunsheng Ding

HKUST, Hong Kong

September 25, 2015

Contents

Sets and Elements

Definition 1

- A <u>set</u> is a collection of objects. The objects are called <u>elements</u>.
- A set is completely determined by its elements; the order in which the elements are listed is irrelevant.
- The symbols $a \in S$ and $a \notin S$ mean that a is and a is not an element of S respectively.

Example 2

The following are four sets:

1. $S = \{\text{Ann, Bob, Cal}\}$.
2. $A = \{1, 2, 3, \cdots, 100\}$.
3. $B = \{a \geq 2 |\ a \text{ is a prime }\}$.
4. $C = \{2n | n = 0, 1, 2, \cdots\}$.

Two Ways to Describe a Set

1. List all elements, for example, the sets S and A in the following example.
2. Describe all elements, for example, the sets B and C in the following example.

Example 3

The following are four sets:

1. $S = \{\text{Ann, Bob, Cal}\}$.
2. $A = \{1, 2, 3, \cdots, 100\}$.
3. $B = \{a \geq 2 | \ a \text{ is a prime } \}$.
4. $C = \{2n | n = 0, 1, 2, \cdots\}$.

Subsets

Definition 4

Let A and B be sets. A is called a <u>subset</u> of B, written $A \subseteq B$, iff every element of A is also an element of B.

- Two equivalent sayings: A is contained in B and B contains A.
- $A \not\subseteq B$ means there is at least one element $a \in A$, but $a \notin B$.
- A is a <u>proper subset</u> of B means that there is a $b \in B$ such that $b \notin A$.
- Clearly, $A \subseteq A$ for any A.

Example 5

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, the set of integers.
- $\mathbb{N} = \{1, 2, 3, \cdots\}$, the set of natural numbers.
- $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$, the set of rational numbers.
- $A = \{-1, 0, 1\}$.

Then $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, but $A \not\subseteq \mathbb{N}$.

The Empty Set

Definition 6

The empty set is the set that contains no elements, written $\emptyset$.

Proposition 7

$\emptyset \subseteq A$ for any set A.

Example 8

Answer the following questions:

- $\{\emptyset\} = \emptyset$?
- $\{\emptyset\} \in \{\{\emptyset\}\}$?
- $\{\emptyset\} \subseteq \{\{\emptyset\}\}$?

The Equality of Sets

Definition 9

Sets A and B are said equal, if and only if A and B contain the same elements or A and B are both $\emptyset$.

Example 10

$\{1, 2, 1\} = \{2, 1\}$.

Proposition 11

$$A = B \Longleftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Proof.

Suppose now that $A = B$. By definition, A and B have the same set of elements. This means that every element in A is also an element of B, i.e., $A \subseteq B$. Similarly, $B \subseteq A$. $\square$

How to Prove $A = B$?

Step 1: Show that $A \subseteq B$.

Step 2: Show that $B \subseteq A$.

Example 12

Let $A = \{1, 5\}$ and $B = \{x | x \in \mathbb{Z}$ with $x^2 - 6x + 5 = 0\}$. Prove that $A = B$.

Proof.

Step 1: $A \subseteq B$.
Note that $1, 5 \in \mathbb{Z}$ satisfying $x^2 - 6x + 5 = 0$. Hence $1, 5 \in B$ and $A \subseteq B$.

Step 2: $B \subseteq A$.
Assume that $x \in B$. Then $x^2 - 6x + 5 = (x - 1)(x - 6) = 0$. Hence $x = 1$ or $x = 5$. It follows that $x \in A$ and $B \subseteq A$.

Step 3: Combining Steps 1 and 2 proves the equality.

□

Power Sets

Definition 13

Let S be a set. The power set, denoted $P(S)$, is the set consisting of all the subsets of A.

Example 14

Let $S = \{a, b\}$. Then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

The Cartesian Product

Definition 15

Let A and B be sets. The <u>Cartesian product</u> or <u>direct product</u> of A and B is:

$$A \times B = \{(a,b) | a \in A, b \in B\}.$$

- "$A \times B$" reads "A cross B".
- $A \times A$ is denoted A^2. Generally,

$$A^n = A \times A \times \cdots \times A = \{(a_1, a_2, \cdots, a_n) | a_i \in A \text{ for all } i\}.$$

Remark

Usually, $A \times B \neq B \times A$.

The Cartesian Product: Example

Example 16

Let $A = \{a, b\}$ and $B = \{x, y, z\}$. Then

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$$

and

$$B \times A = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}.$$

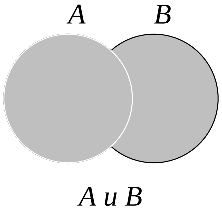
Observe that $A \times B \neq B \times A$.

The Union of Sets (1)

Definition 17

The union of two sets A and B is defined as

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$



Example 18

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Then $A \cup B = \{a, b, c, 1, 2, 3\}$.

The Union of Sets (2)

Proposition 19

Let A and B be any sets. Then we have the following:

- $A \cup A = A$ *for all* A.
- $A \cup \emptyset = A$ *for all* A.
- $A \cup B = B \cup A$ *for all* A *and* B. *(commutative law)*
- $A \cup (B \cup C) = (A \cup B) \cup C$. *(associative law)*

Proof.

To prove the associative law above, one needs to prove

$$A \cup (B \cup C) \subseteq (A \cup B) \cup C \text{ and } (A \cup B) \cup C \subseteq A \cup (B \cup C).$$

If $x \in A \cup (B \cup C)$, then either $x \in A$ or $x \in B \cup C$. Hence, x is an element of at least one of A, B and C. Hence, $x \in (A \cup B) \cup C$. It then follows that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. The other part can be similarly proved. □

The Union of Sets (3)

Due to the associativity of the operation $\cup$, we define

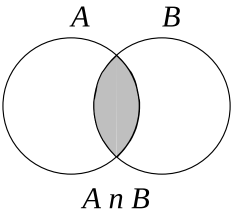
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

The Intersection of Sets (1)

Definition 20

The <u>intersetion</u> of two sets A and B is defined as

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$



Example 21

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 6\}$. Then $A \cap B = \{2, 4, 5\}$.

The Intersection of Sets (2)

Proposition 22

- *$A \cap A = A$ for all A.*
- *$A \cap \emptyset = \emptyset$ for all A.*
- *$A \cap B = B \cap A$ for all A and B. (commutative law)*
- *$A \cap (B \cap C) = (A \cap B) \cap C$. (associative law)*

Proof.

The proofs are left as exercises. $\square$

Due to the associativity of $\cap$, we define

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

The Set Difference and the Complement

Definition 23

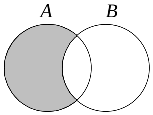
1. The <u>difference</u> of A and B is the set $A \setminus B = \{x | x \in A, x \notin B\}$.
2. The <u>complement</u> of a set A with respect to U is $A^c = \{x | x \in U, x \notin A\}$, where U is some universal set made clear by the context.

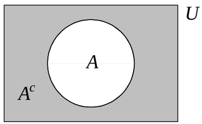
Example 24

The above two operations on sets are illustrated by the following examples:

- $\{a, b, c\} \setminus \{a, b, d\} = \{c\}$.
- $\{1, 2, 3\} \setminus \{3, 4\} = \{1, 2\}$.
- If $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 6\}$, then

$$A^c = \{2, 3, 4, 5\}.$$





Relations among Set Operations

Proposition 25

Distribution Law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proposition 26

Distribution Law

$$(A \cap B)^c = A^c \cup B^c,$$
$$(A \cup B)^c = A^c \cap B^c.$$

Proof.

The proofs of the two propositions are left as exercises. $\square$

The Cardinality

Definition 27

Let S be a set. If there are exactly n distinct elements in S, we called S a finite set and say that n is the cardinality of S, denoted by $|S|$.

Example 28

$$
\begin{aligned}
|\{a, b, c\}| &= 3, \\
|\{1, 2, a, b\}| &= 4, \\
|\{x \in \mathbb{Q} \mid x^2 + 1 = 0\}| &= 0.
\end{aligned}
$$

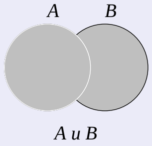
The Inclusion-exclusion Principle (1)

Proposition 29

Let A and B be two finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof.

$|A| + |B|$ counts $|A \cap B$ two times.



$\square$

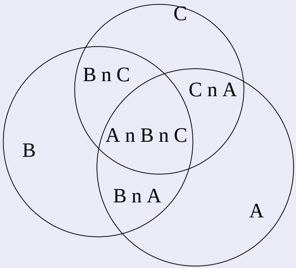
The Inclusion-exclusion Principle (2)

Proposition 30

Let A, B and C be three finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Proof.



$\square$

The Inclusion-exclusion Principle (3)

Proposition 31

Given a finite number of finite sets, $A_1, A_2, \ldots A_n$, we have

$$|\cup_{i=1}^{n} A_i| = \sum_{i} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |\cap_{i=1}^{n} A_i|$$

where the first sum is over all i, the second sum is over all pairs i,j with $i < j$, the third sum is over all triples i, j, k, and so forth.

The above formula can be proved by induction on n. However, we shall not give the proof here.

The Cardinality of the Cartesian Product

The following is called the Multiplication Rule.

Theorem 32

Let A and B be two finite sets. Then

$$|A \times B| = |A| \times |B|.$$

Proof.

A proof will be given later when we introduce the Multiplication Rule in general later. □

The Cardinality of the Power Set

Question 33

Let A be a set with cardinality n. What is the cardinality of the power set $P(A)$?