COMP170
Discrete Mathematical Tools for Computer Science

Binomial Coefficients

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Discrete Math for Computer Science
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1.3 Binomial Coefficients

- Pascal's Triangle
- A Proof using the Sum Principle
- The Binomial Theorem
- Labeling and Trinomial Coefficients

Some properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of k-element subsets of an n-element set.

- $\binom{n}{0} = 1$ only one set of size 0.

- $\binom{n}{n} = 1$ only one set of size n.

- $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

Some properties of Binomial Coefficients (cont)

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Use Sum Principle

Let P = set of all subsets of {1,2,. . . ,n}

S_i = set of all i subsets of {1,2,. . . ,n}

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

If $\mathcal{L}$ = set of all such lists $\Rightarrow$ $|\mathcal{L}| = 2^n$

There is a *bijection* (next page) between $\mathcal{L}$ and P so

$|P| = 2^n$ and we are done.

Let $P =$ set of all subsets of $\{$1,2,. . . ,n$\}$

Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

and $\mathcal{L} =$ set of all such lists

Define the following function $f : \mathcal{L} \rightarrow P$

If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, , 2, \ldots, n\}$ defined by

$$i \in S \;\Leftrightarrow\; L_i = 1$$

f is a *bijection* between $\mathcal{L}$ and P (why?) so $|\mathcal{L}| = |P|$

Ex: $n = 5$

$f(10101) = \{1, 3, 5\},\; f(11101) = \{1, 2, 3, 5\},\; f(00000) = \emptyset.$

Note: L *is sometimes called the* incidence vector *or* membership vector *associated with* L

Example: $n = 4,\ S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{ccccc} & \{1\} & \{1,2\}\ \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\}\ \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\}\ \{3,4\} & \{1,3,4\} & \\ & \{4\} & & \{2,3,4\} & \end{array} \right\}$$

Example: $n = 4,\ S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{c|c|cc|c|c} & \{1\} & \{1,2\} & \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\} & \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\} & \{3,4\} & \{1,3,4\} & \\ & \{4\} & & & \{2,3,4\} & \end{array} \right\}$$

$$P = \{S_0,\ S_1,\ S_2,\ S_3,\ S_4\}$$

Example: $n = 4,\ S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{ccccc} & \{1\} & \{1,2\}\,\{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\}\,\{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\}\,\{3,4\} & \{1,3,4\} & \\ & \{4\} & & \{2,3,4\} & \end{array} \right\}$$

$$|S_0| = \binom{4}{0},\ |S_1| = \binom{4}{1},\ |S_2| = \binom{4}{2},\ |S_3| = \binom{4}{3},\ |S_4| = \binom{4}{4}$$

Example: $n = 4$, $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{ccccc} & \{1\} & \{1,2\}\ \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\}\ \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\}\ \{3,4\} & \{1,3,4\} & \\ & \{4\} & & \{2,3,4\} & \end{array} \right\}$$

$$\begin{aligned} |P| &= |S_0| + |S_1| + |S_2| + |S_3| + |S_4| \\ &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\ &= 2^4 = 16 \end{aligned}$$

Binomial Coefficients

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Each row begins with a 1 because $\binom{n}{0} = 1$

Each row ends with a 1 because $\binom{n}{n} = 1$.

Each row increases at first and then decreases.
(will see why in homework)

Second half of each row is the reverse of the first half.

Sum of items on n^{th} row is 2^n

Pascal's Triangle

Take the table

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal relationship

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's relationship says that, for $0 < k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

A purely *algebraic* proof (manipulating formulas) is possible.

In discrete mathematics, though, we prefer to derive intuitive explanations. In this case, that would involve interpreting Pascal's relationship as a statement describing *relationships among sets*.

A Proof Using the Sum Principle

From Theorem 1.2 and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

we know $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

Therefore, each term (left and right) in

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

represents the number of subsets of a particular size chosen from an appropriately sized set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Number of k-subsets of an n-element set.

Number of $(k-1)$-subsets of an $(n-1)$-element set.

Number of k-subsets of an $(n-1)$-element set.

Try to use sum principle to explain relationship among these three terms.

Example: $n = 5$, $k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider $S = \{A, B, C, D, E\}$.

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts.

S_2 the 2-subsets that contain E and

S_3, the set of 2-subsets that do not contain E.

$$S_1 = \{\{A,B\},\{A,C\},\{A,D\},\{A,E\},\{B,C\},\{B,D\},\{B,E\},\{C,D\},\{C,E\},\{D,E\}\}.$$

S_2 is equivalent to choosing 1 item out of $\{A, B, C, D\}$: $|S_2| = \binom{4}{1}$

S_3 chooses 2 items out of $\{A, B, C, D\}$: $|S_3| = \binom{4}{2}$

Sum Principle: $\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}$

Theorem 1.3

If n and k are integers satisfying $0 < k < n$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof: Apply sum principle.

Partition set of k-element subsets of an n-element set into *disjoint union* of two other disjoint sets.

Suppose $S = \{x_1, x_2, \ldots, x_n\}$.

Let S_1 be set of all k-element subsets. $|S_1| = \binom{n}{k}$

To apply sum principle, partition S_1 into S_2 and S_3.

Let S_2 be set of k-element subsets that contain x_n.

Let S_3 be set of k-element subsets that don't contain x_n

$|S_3| = \binom{n-1}{k}$ since this is just how to choose a k-element subset from a $(n-1)$ size set

$|S_2| = \binom{n-1}{k-1}$ since this is just how to choose a $(k-1)$-element subset from a $(n-1)$ size set

$$\Rightarrow \binom{n}{k} = |S_1| = |S_2| + |S_3| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Blaise Pascal

- Born 1623; Died 1662
- French Mathematician
- A Founder of Probability Theory
- Inventor of one of the first (the 2nd?) mechanical calculating machines
- Pascal Programming Language named for him

The Binomial Theorem

$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}x^1y^1 + \binom{2}{2}y^2$$

$$\begin{aligned}(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\end{aligned}$$

The Binomial Theorem

Number of k-element subsets of an n-element set is called a **binomial coefficient** because of its role in the algebraic expansion of a binomial $(x+y)^n$.

Theorem 1.4 (Binomial Theorem)
For any integer $n \geq 0$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

or, in summation notation,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Example: $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

First,

$$(x+y)(x+y)(x+y)$$
$$= [x(x+y) + y(x+y)]\,(x+y)$$
$$= (xx + yx + xy + yy)(x+y)$$
$$= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y$$
$$= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.$$

Each *monomial* term in the final result is of form $x^{3-i}y^i$ and is the product of – one blue, one red, and one green.

For each color we can choose either an x or a y.

Coefficient of $x^{3-i}y^i$ is

\# of ways of choosing i y's from three colors

$= \binom{3}{i}$

Example: $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

First,

$$(x+y)(x+y)(x+y)$$
$$= [x(x+y) + y(x+y)]\,(x+y)$$
$$= (xx + yx + xy + yy)(x+y)$$
$$= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y$$
$$= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.$$

Aternatively, can think of the monomial as *lists* where each item of the list is either x or y.

Coefficient of $x^{3-i}y^i$ is

$\#$ of lists containing i y's (and $(3-i)$ x's)

$= \binom{3}{i}$

Proof of the Binomial Theorem

Use same explanation as in example just seen:

When multiplying three factors of $x+y$, we get a sum of eight products (monomials, lists)

Each factor $x+y$ doubles the number of monomials

Thus, product of n binomials, $(x+y)^n$, is sum of 2^n monomials

Each monomial is a length-n list of x's and y's.

In each list, the ith entry comes from the ith binomial factor.

A list that becomes $x^{n-k}y^k$ *(after applying commutative law)* will have a y in k places and an x in the remaining $(n-k)$ places.

Number of lists that have a y in k places is thus the number of ways to select k binomial factors to contribute a y to our list.

Number of ways to select k binomial factors from n binomial factors is simply $\binom{n}{k}$.

Therefore, the coefficient of $x^{n-k}y^k$, is $\binom{n}{k}$.

What is $(x+1)^4$?

By applying the binomial theorem

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

What is $(2+y)^4$?

$$(2+y)^4 = 16 + 32y + 24y^2 + 8y^3 + y^4.$$

What is $(x+y)^4$?

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

At the beginning of this lesson we gave a combinatorial proof that

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n.}$$

Here's another, simple, algebraic, proof.
We just saw that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Now set $x = y = 1$. This gives

$$\boxed{2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i}}$$

Labelling and Trinomial Coefficients

- Suppose we have k labels of one kind, e.g., red and $n-k$ labels of another, e.g., green. In how many different ways can we apply these labels to n objects?

- Show that if we have k_1 labels of one kind, e.g., red, k_2 labels of a second kind, e.g., green, and $k_3 = n - k_1 - k_2$ labels of a third kind, e.g., orange, then there are $\frac{n!}{k_1!k_2!k_3!}$ ways to apply these labels to n objects

- What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$

- Suppose we have k labels of one kind, e.g., red and $n-k$ labels of another, e.g., green. In how many different ways can we apply these labels to n objects?

There are $\binom{n}{k}$ ways to choose the items with red labels.
The other $n-k$ items will then get the green labels
So this is just $\binom{n}{k}$

- Show that if we have k_1 labels of one kind, e.g., red, k_2 labels of a second kind, e.g., green, and $k_3 = n - k_1 - k_2$ labels of a third kind, e.g., orange, then there are $\frac{n!}{k_1!k_2!k_3!}$ ways to apply these labels to n objects

There are $\binom{n}{k_1}$ ways to choose the red items
There are then $\binom{n-k_1}{k_2}$ ways to choose the green items
from the remaining $n - k_1$.
The remaining k_3 items get labelled orange

Using the *product principle* the total number of labellings is

$$\begin{aligned}\binom{n}{k_1}\binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!}\,\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}\end{aligned}$$

When $k_1 + k_2 + k_3 = n$, we call

$$\frac{n!}{k_1!k_2!k_3!}$$

a *trinomial coefficient* and denote it as

$$\binom{n}{k_1 \quad k_2 \quad k_3}$$

Note that this slightly modifies the notation for binomial coefficients. If we really wanted the notation to be consistent (which we don't) we could write the binomial coefficient $\binom{n}{k}$ as

$$\binom{n}{k \quad (n-k)}$$

We really just saw that the Trinomial Coefficient

$$\binom{n}{k_1\ k_2\ k_3}$$

is the number of ways to partition a set of size n into three subsets (where order of the subsets does not count) of sizes k_1, k_2 and k_3.

- What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$

Example:

$$(x+y+z)(x+y+z)(x+y+z)(x+y+z)$$
$$= xxxx + xxxy + xxxz + xxyx + \cdots + zzzy + zzzz.$$

After opening the parentheses and multiplying,
there will be, in total, $3^4 = 81$ different monomial terms (lists)

Each term, (after rewriting using commutativity),
is in the form $x^{k_1}y^{k_2}z^{k_3}$ where $k_1+k_2+k_3=4$

The coefficient of $x^{k_1}y^{k_2}z^{k_3}$ is exactly the number of ways of writing a list of size 4 with k_1 x's, k_2 y's, and k_3 z's such that $k_1+k_2+k_3=4$, which is $\binom{4}{k_1 \quad k_2 \quad k_3}$

- What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$

After opening the parentheses and multiplying,
there will be, in total, 3^n different monomial terms (lists) .

Each term, (after rewriting using commutativity),
is in the form $x^{k_1}y^{k_2}z^{k_3}$ where $k_1+k_2+k_3=n$

The coefficient of $x^{k_1}y^{k_2}z^{k_3}$ is exactly the number of ways of writing a list of size n with k_1 x's, k_2 y's, and k_3 z's such that $k_1+k_2+k_3=n$, which is

$$\binom{n}{k_1 \quad k_2 \quad k_3}$$