COMP170
Discrete Mathematical Tools for Computer Science

Lecture 19 *Version 1: Last updated, Nov 30 2005*

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 6.1, pp. 309-320

Graphs

- Basic Definitions
- The Degree of a Vertex
- Connectivity
- Cycles
- Trees

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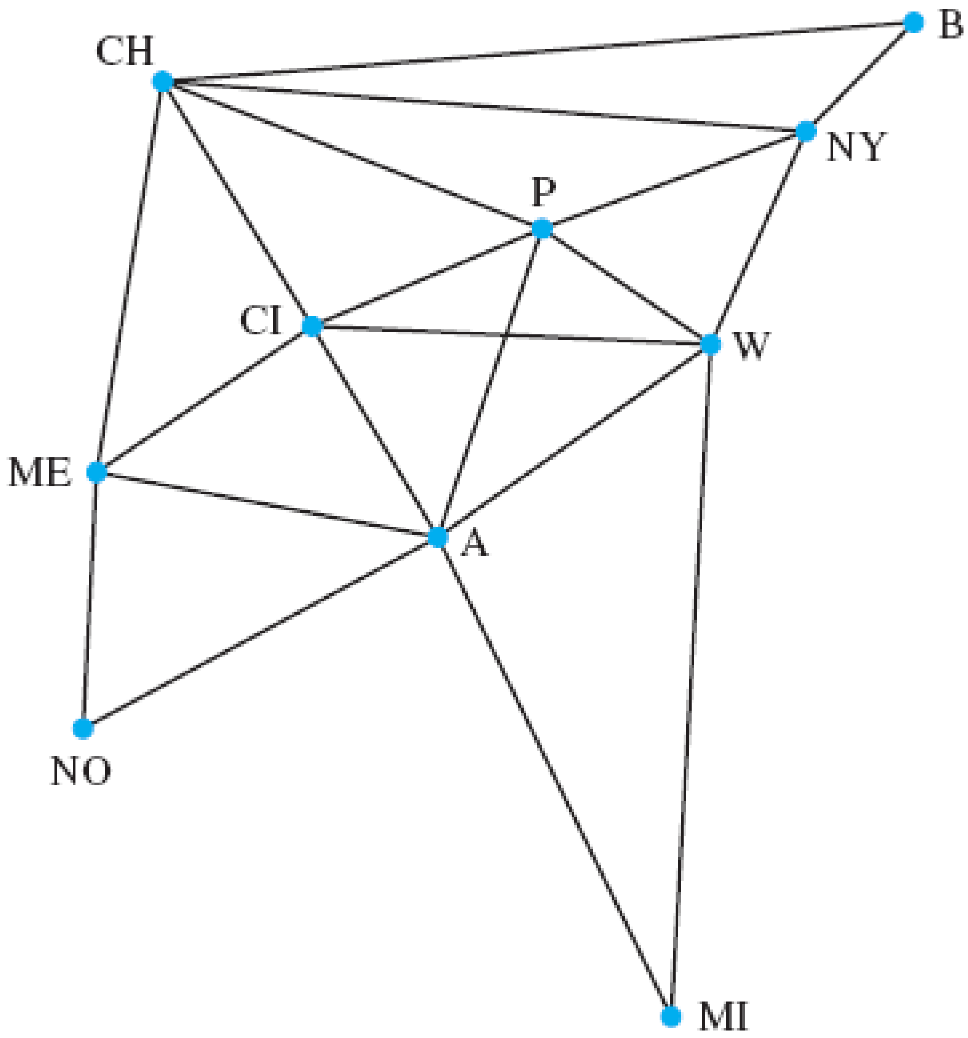
Example

Map of some cities in eastern US.
with communication lines existing
between certain pairs of these cities.


B
CH
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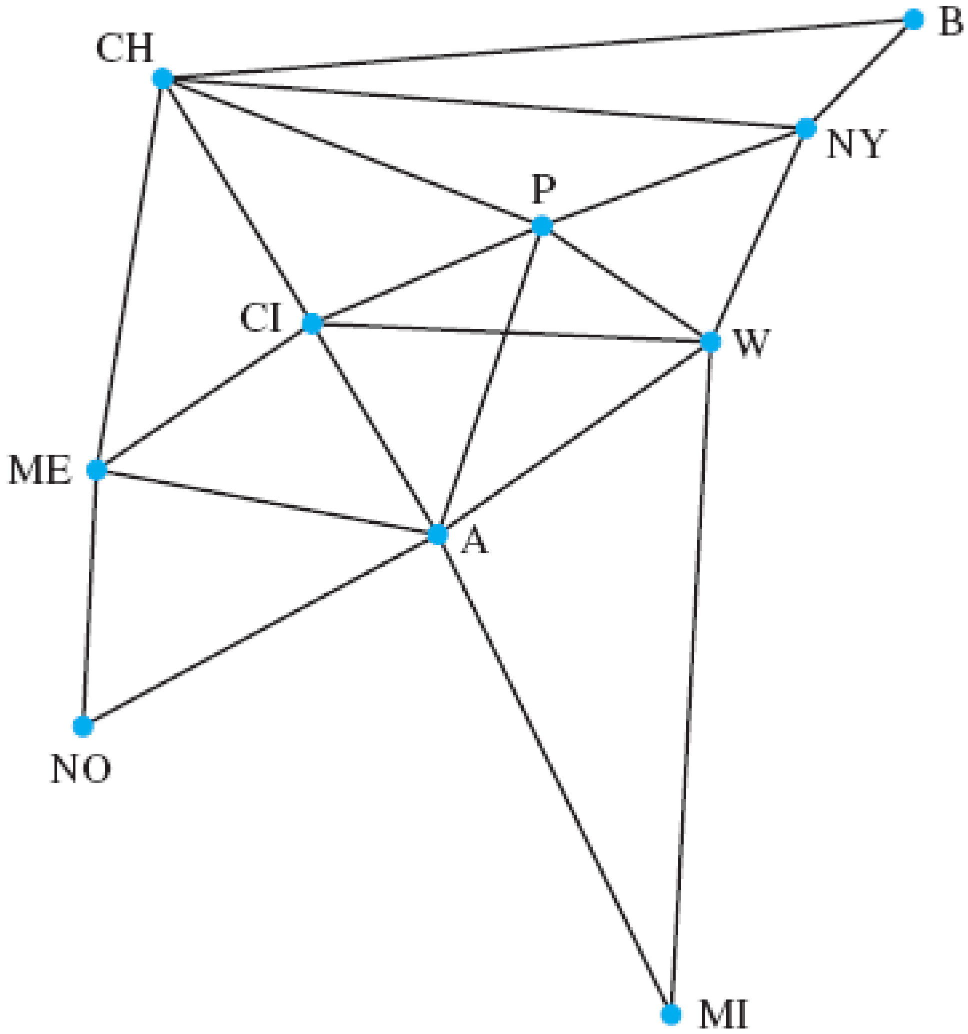
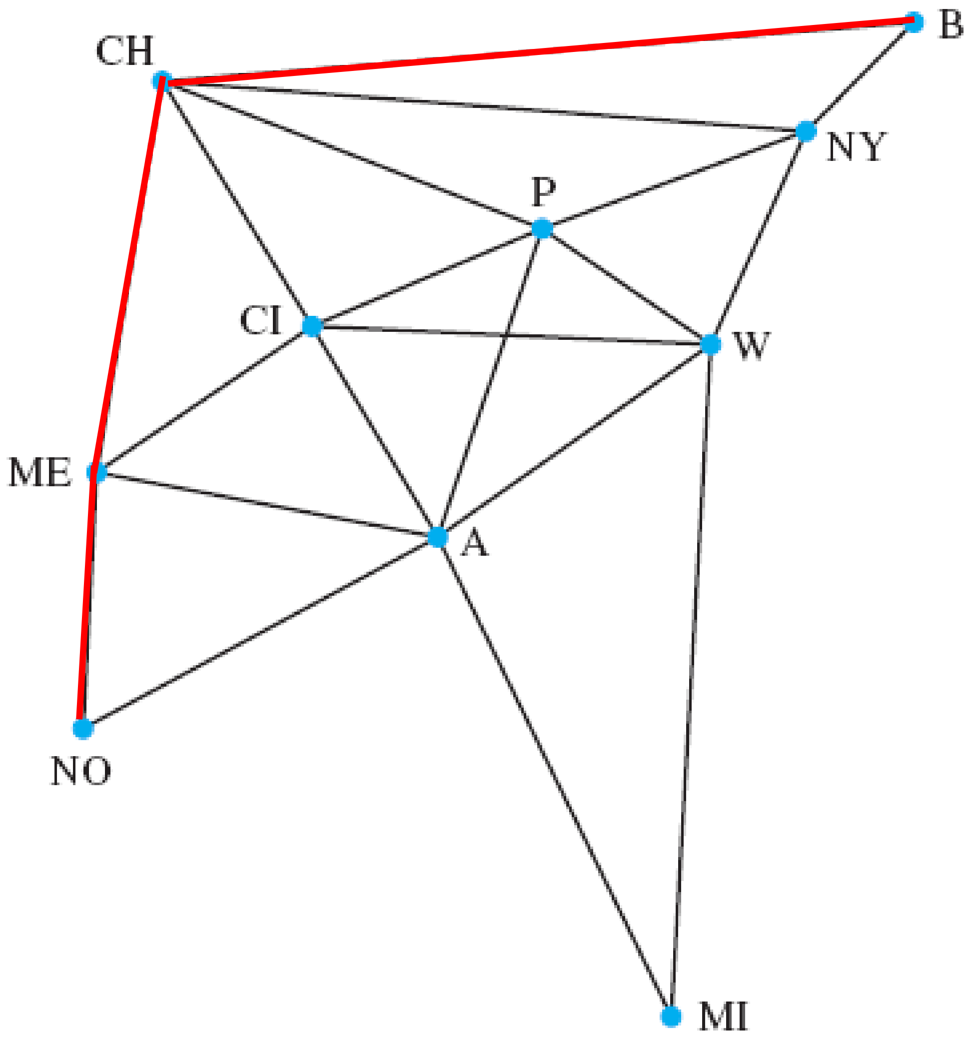


What is the minimum number of links needed to send a message from B to NO?



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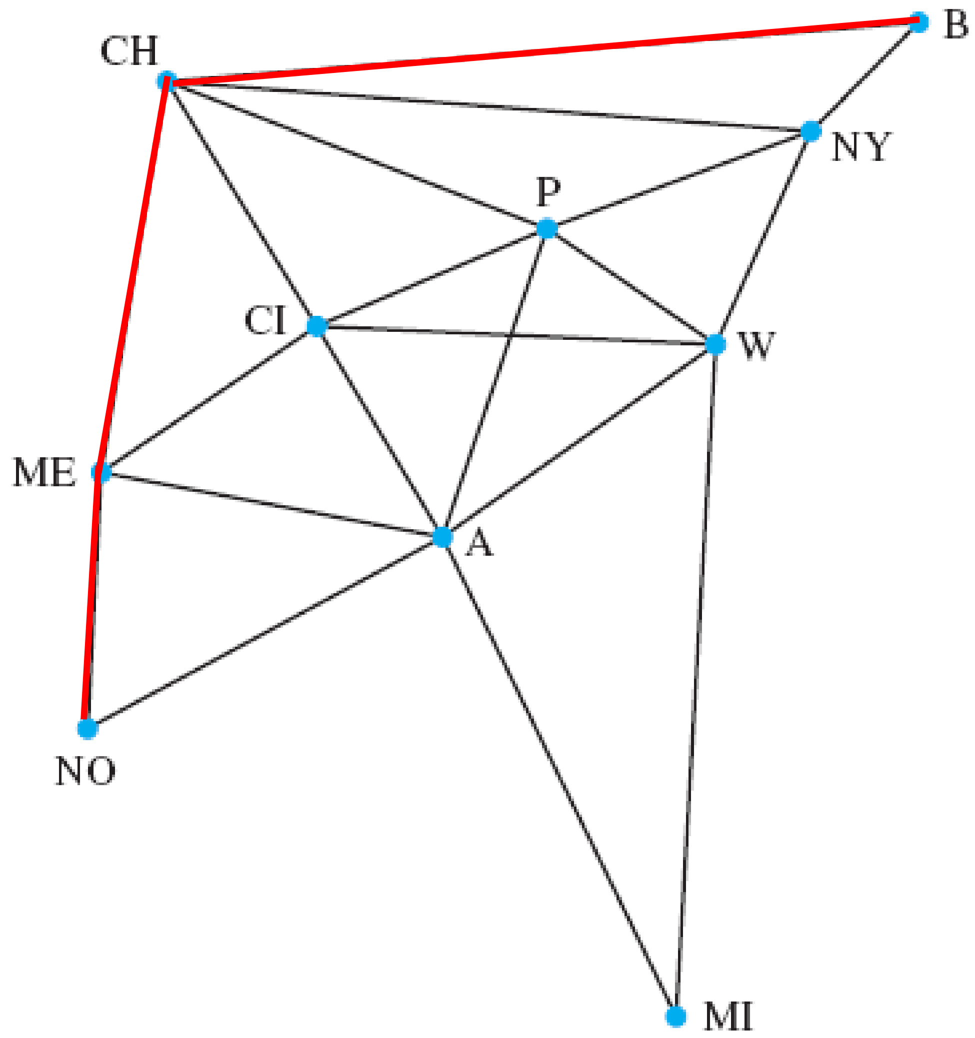

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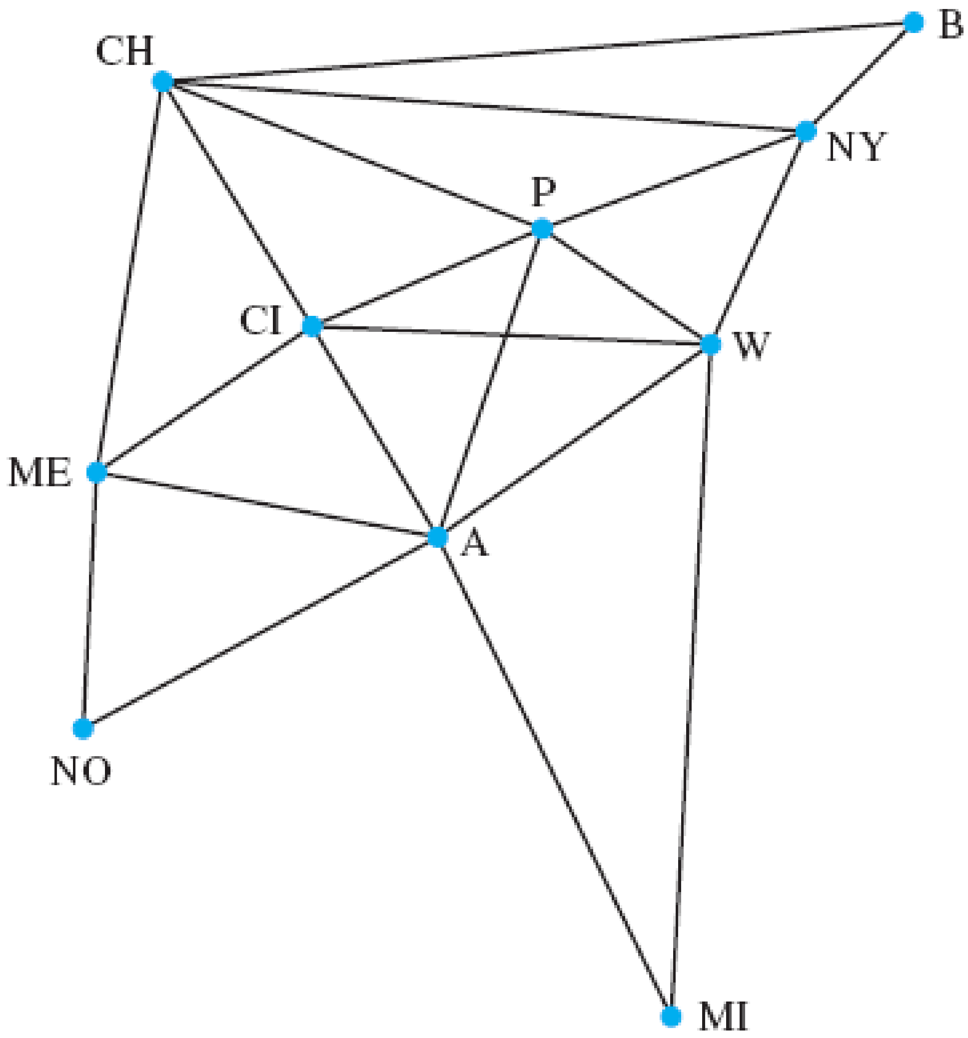

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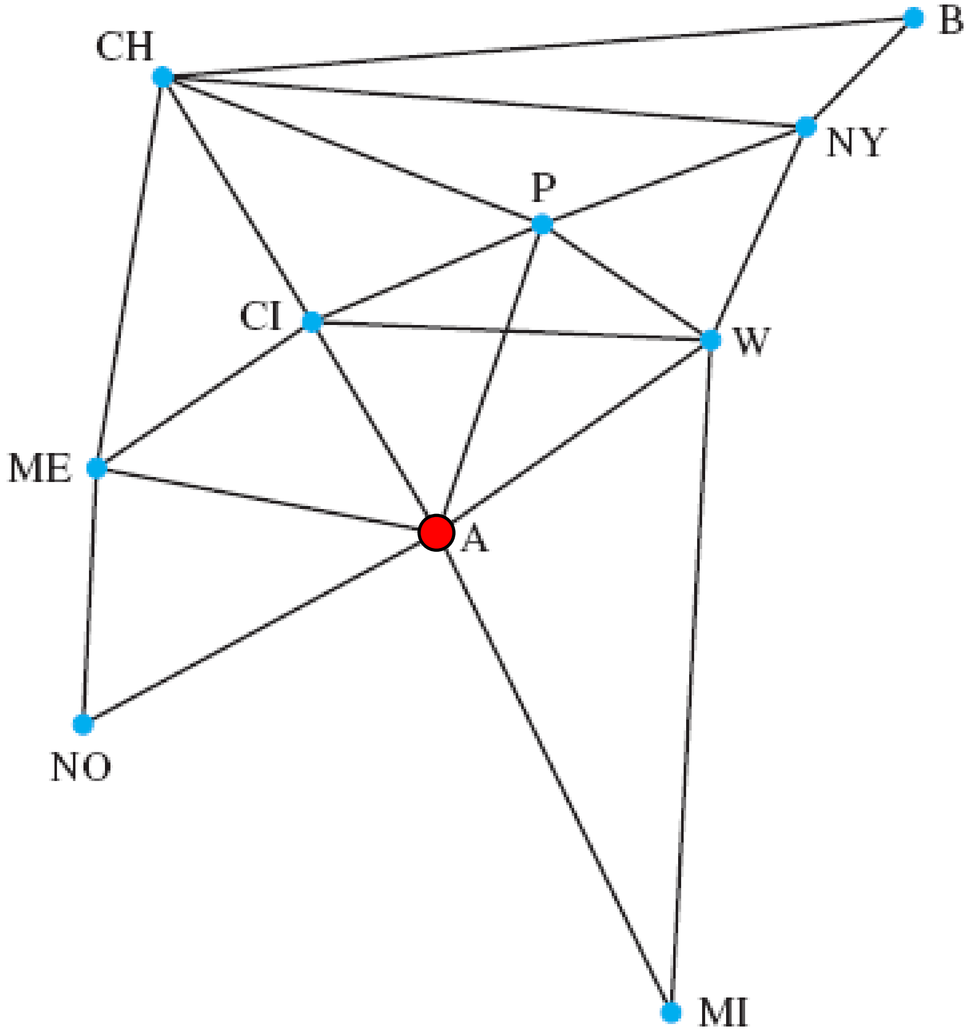

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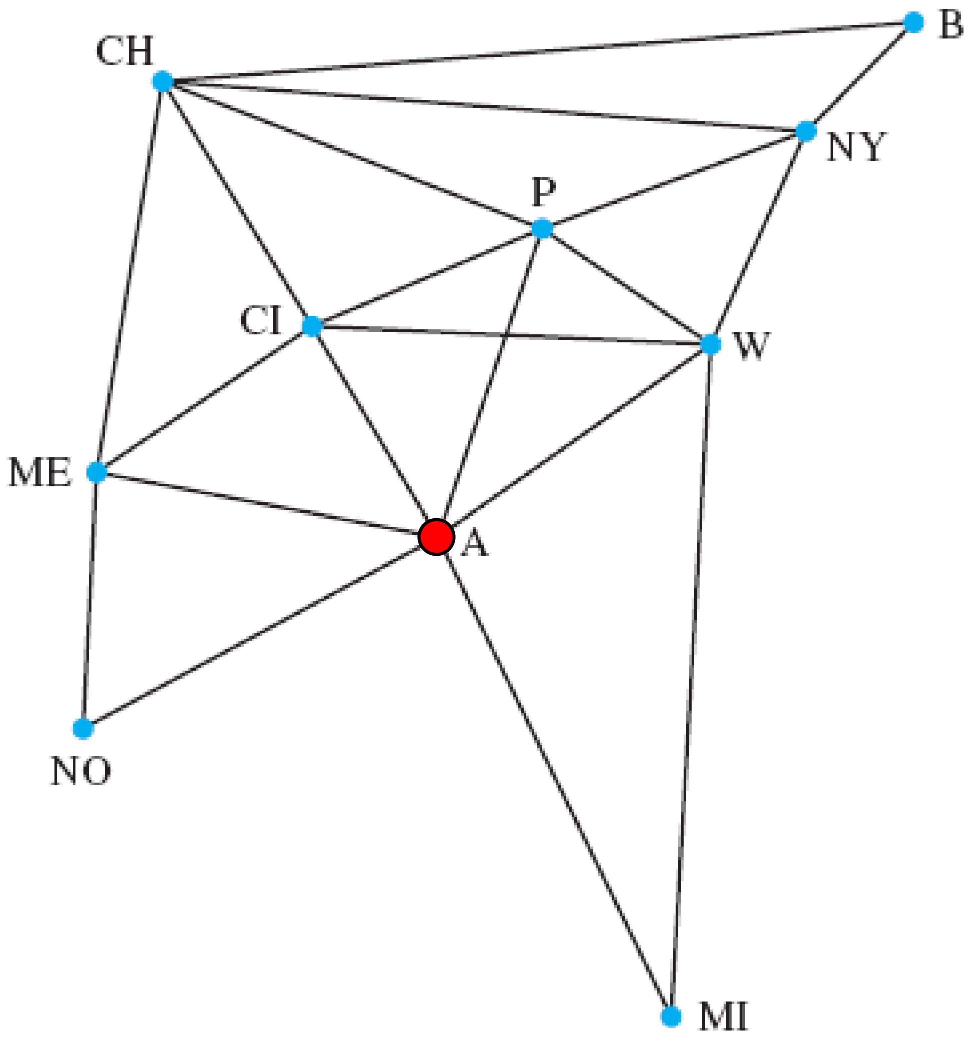

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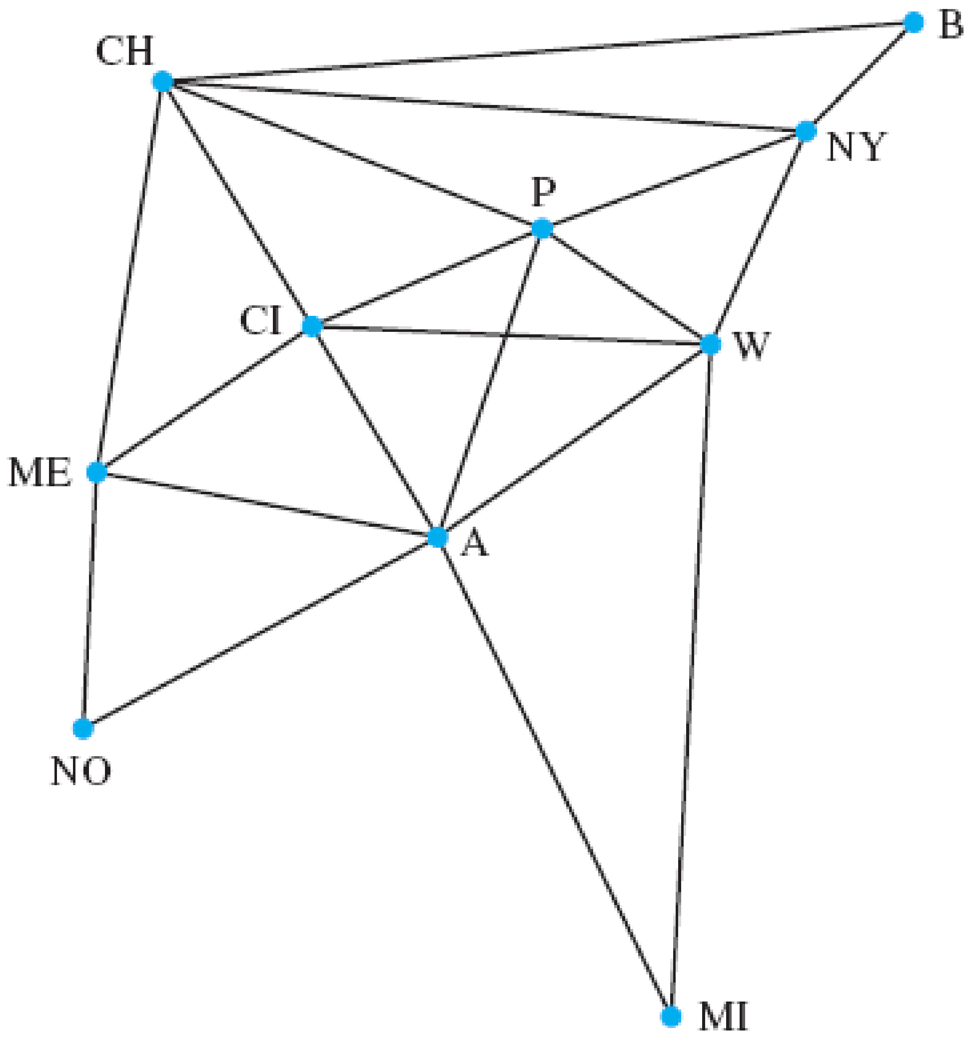

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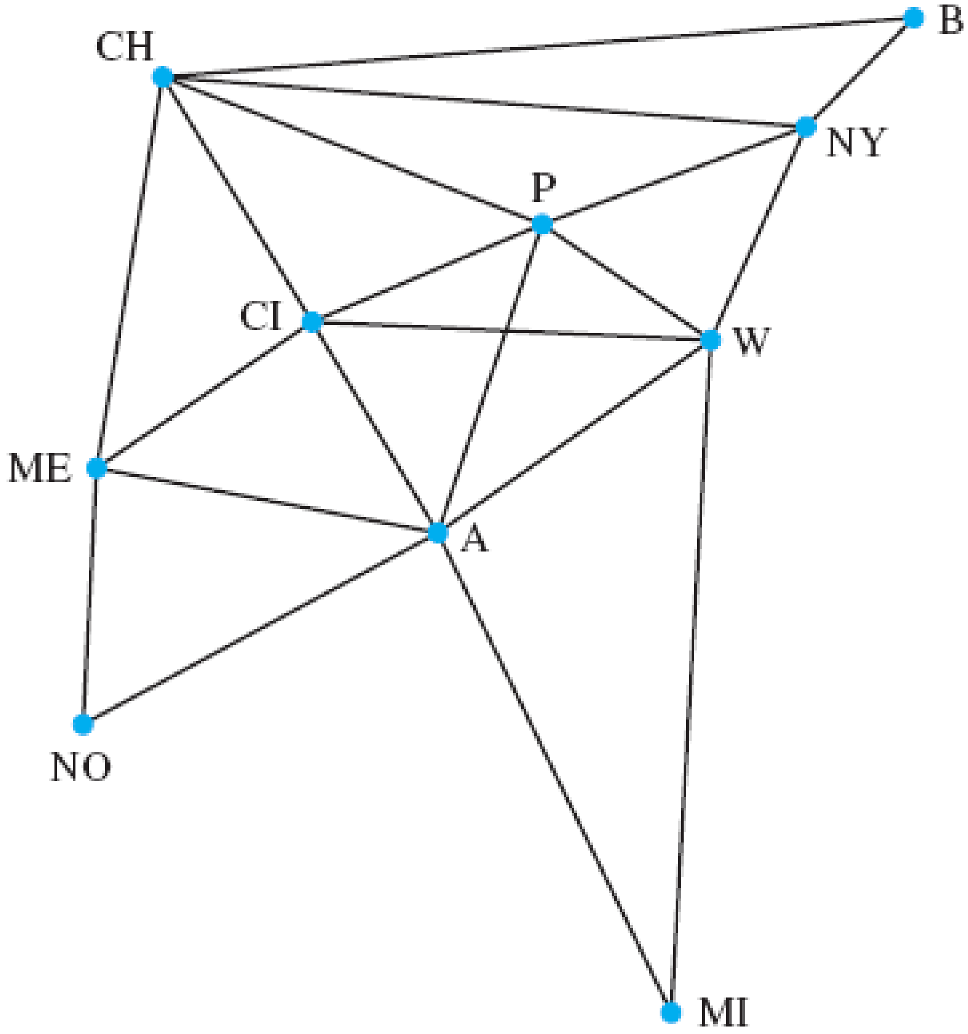

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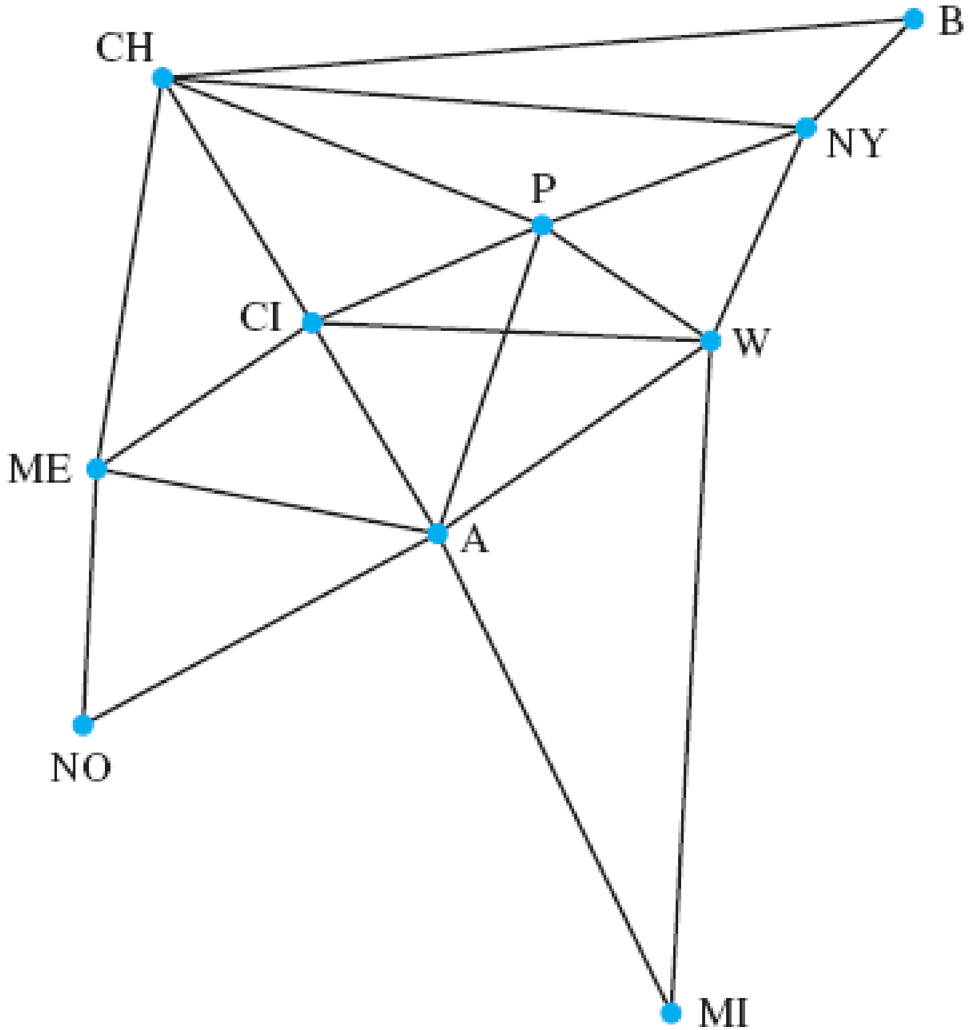

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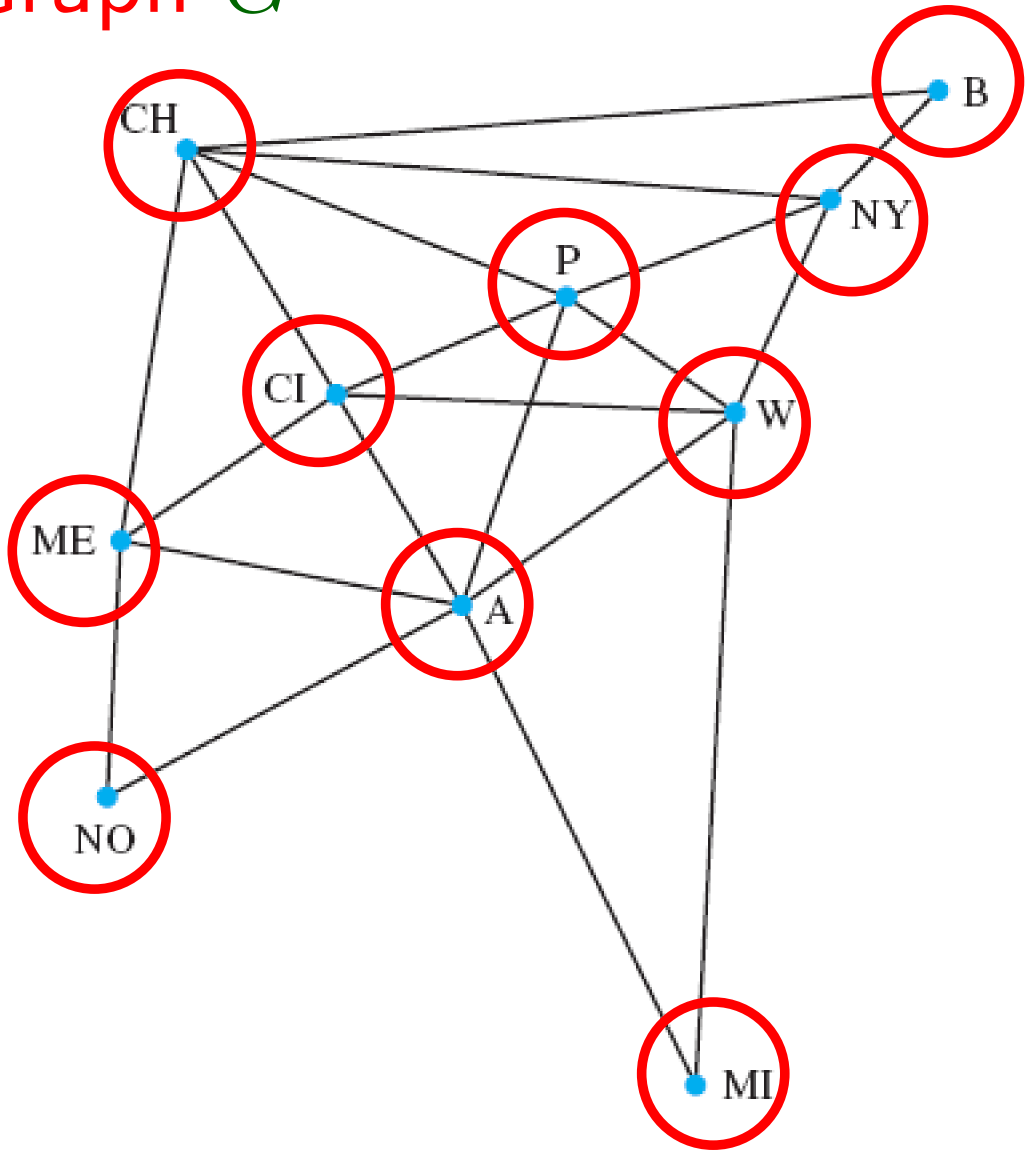

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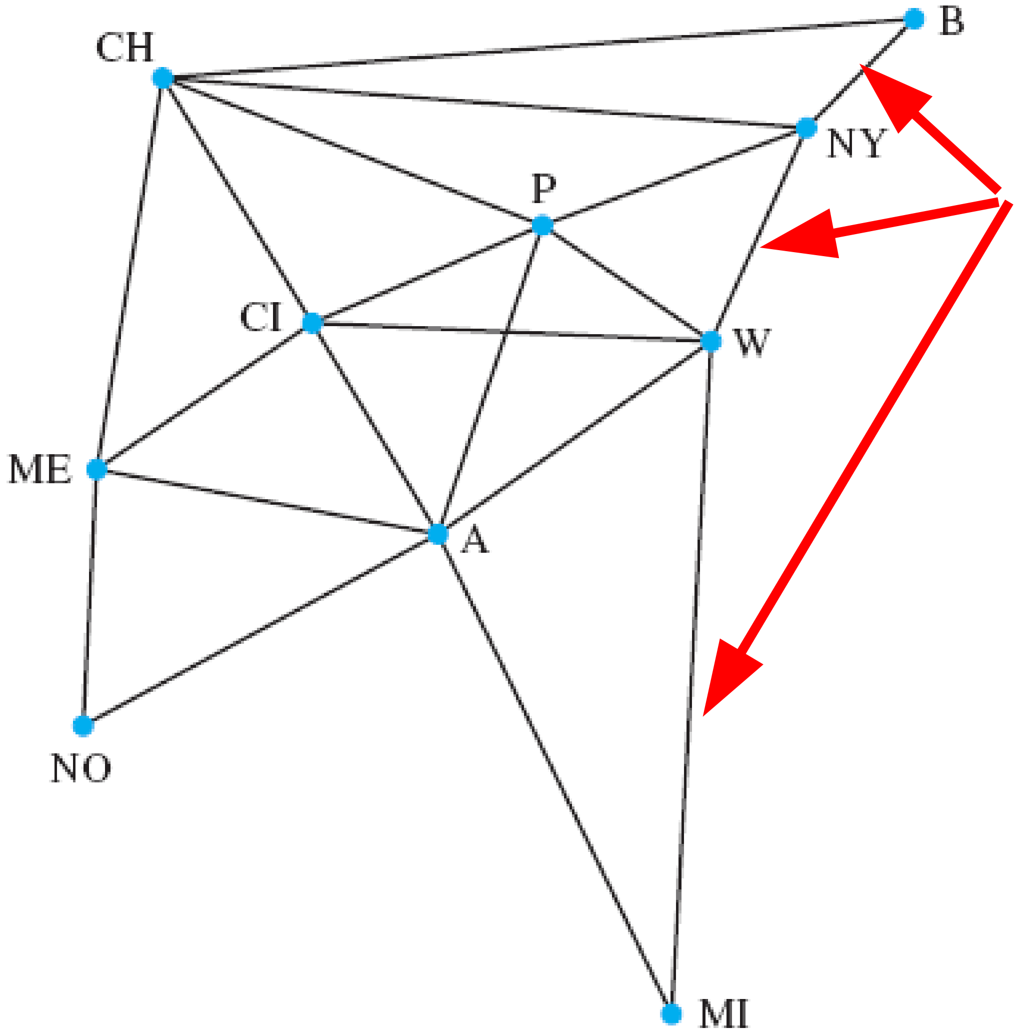

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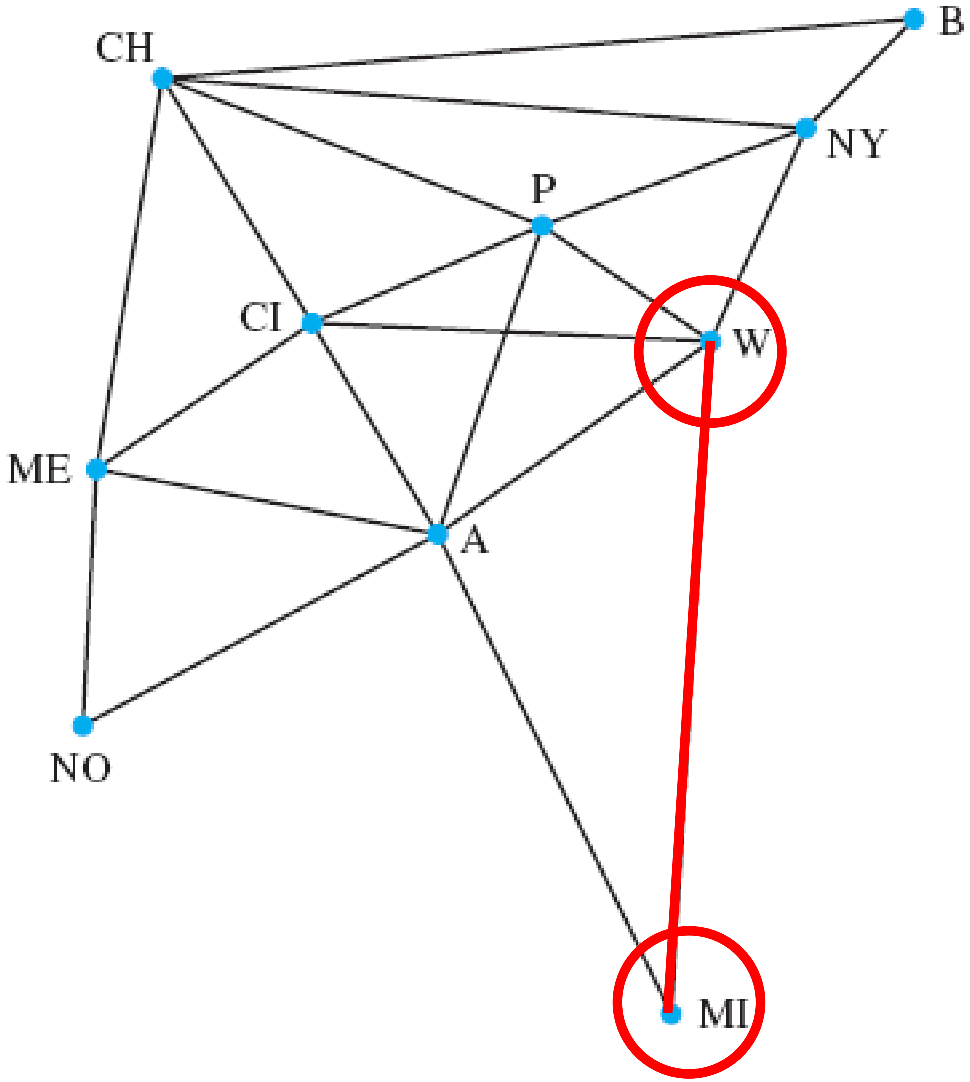

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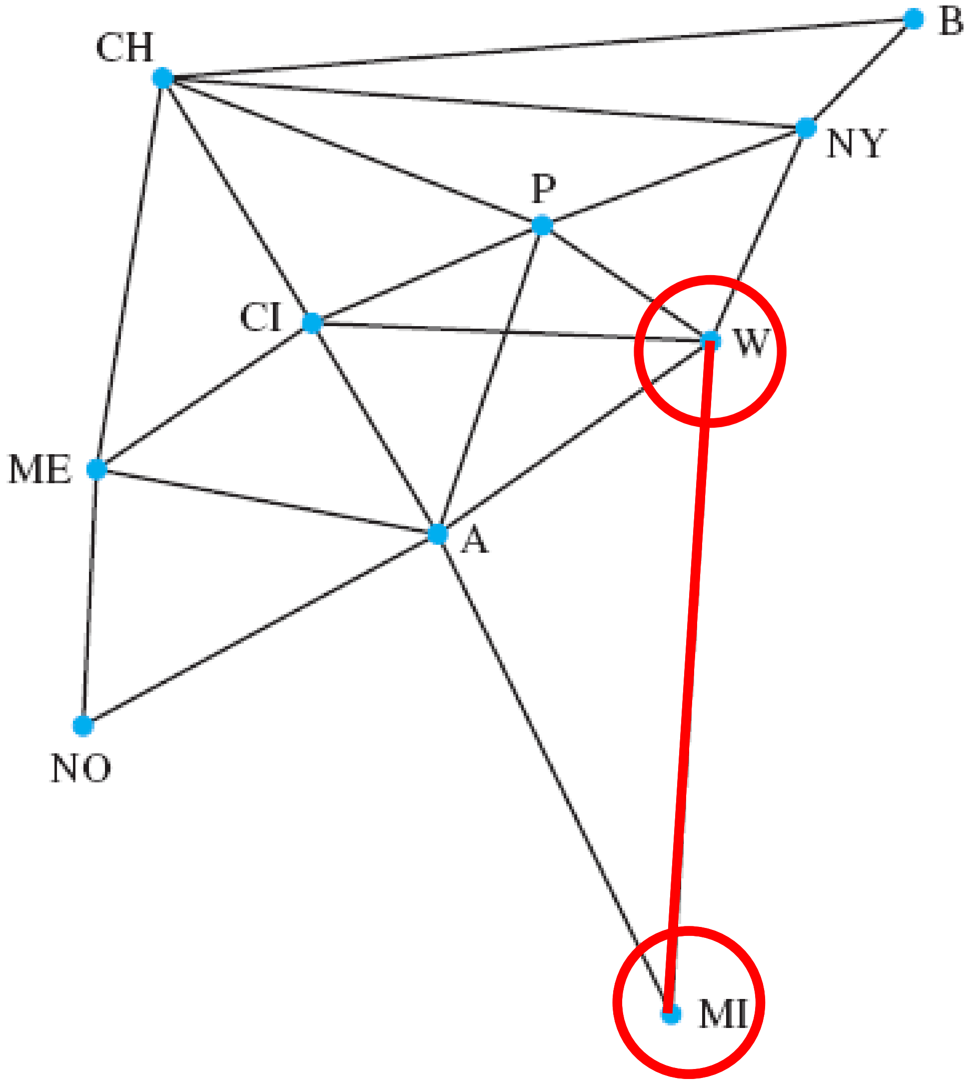

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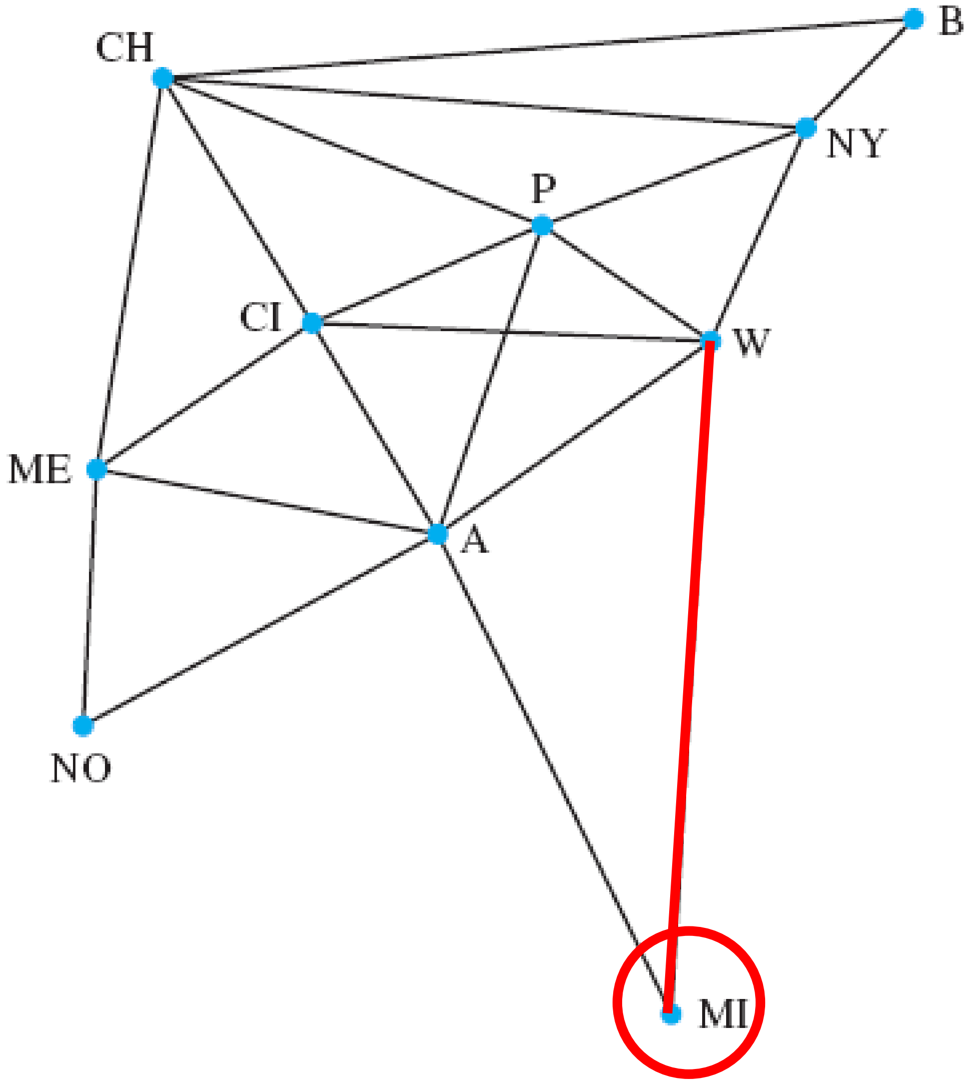

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When a vertex is an endpoint of an edge, we say that the edge and the vertex are **incident** to each other.

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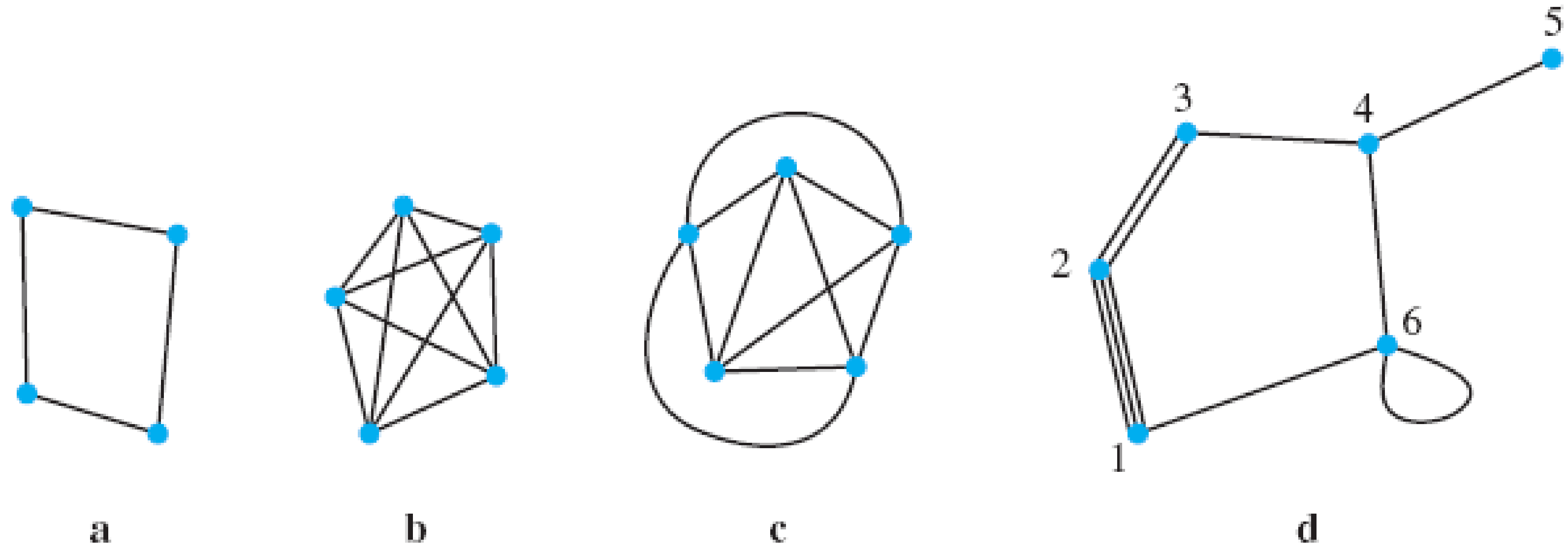
How Google models the Internet!

More Graphs:



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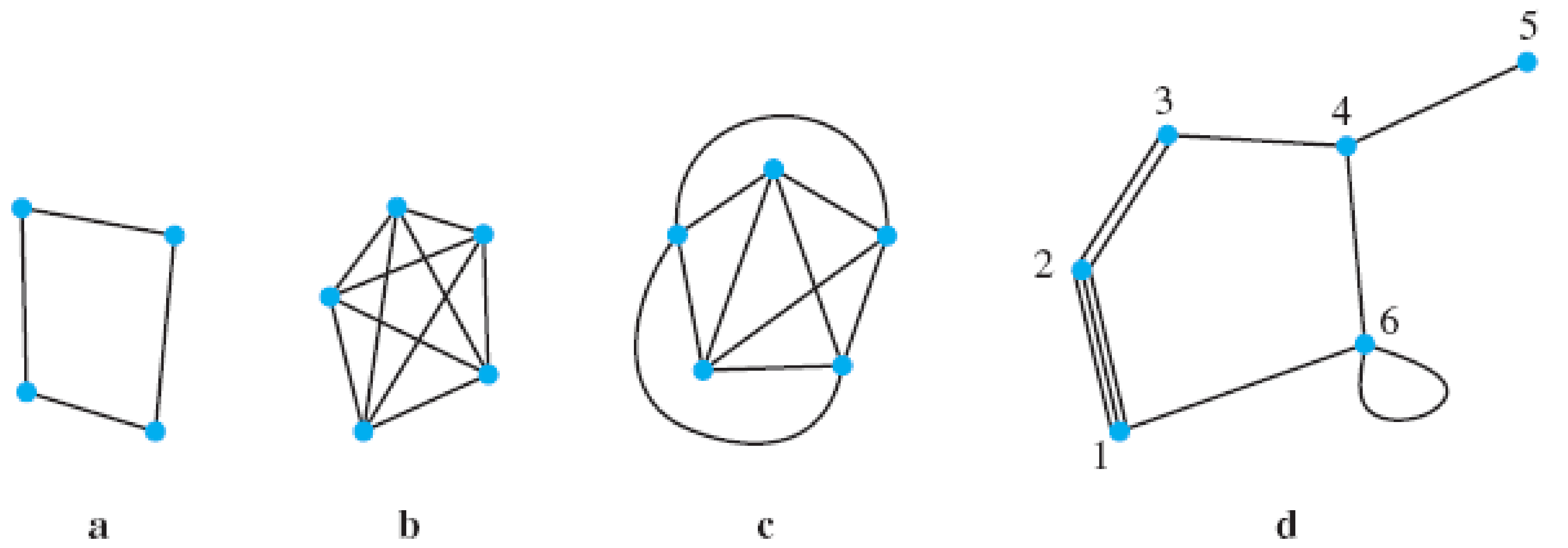

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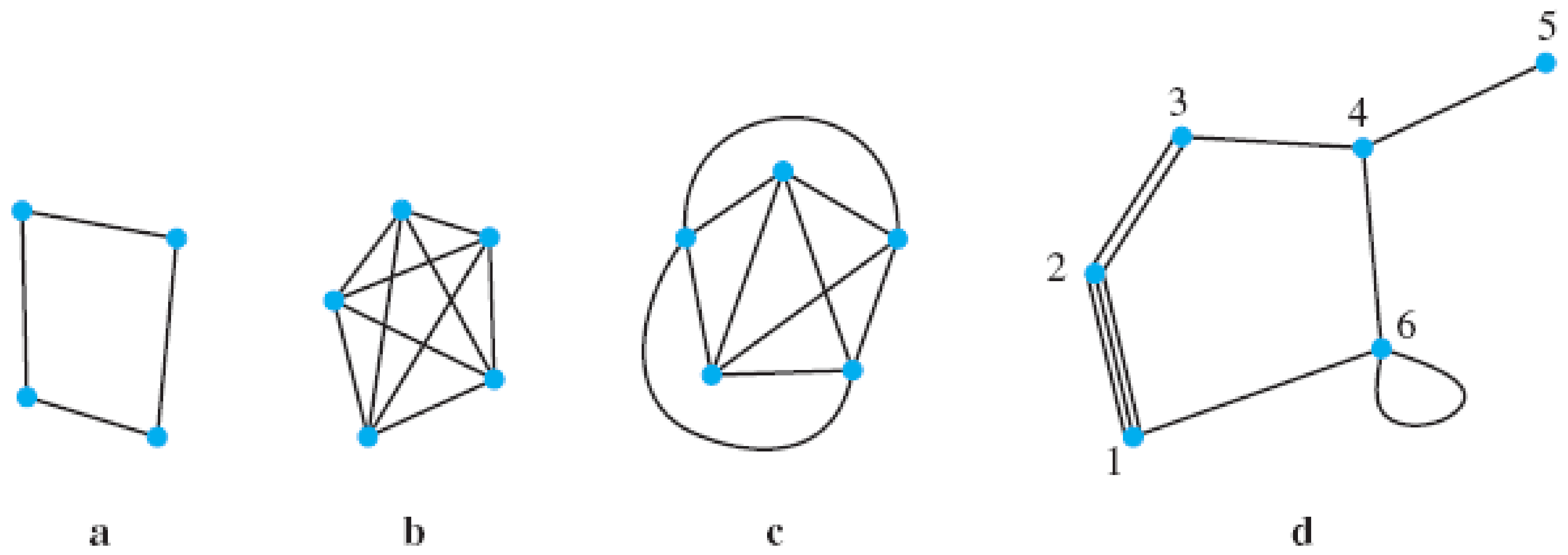

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- **Complete Graph** K_n (b, c): graph with n vertices that has an edge between each pair of vertices.

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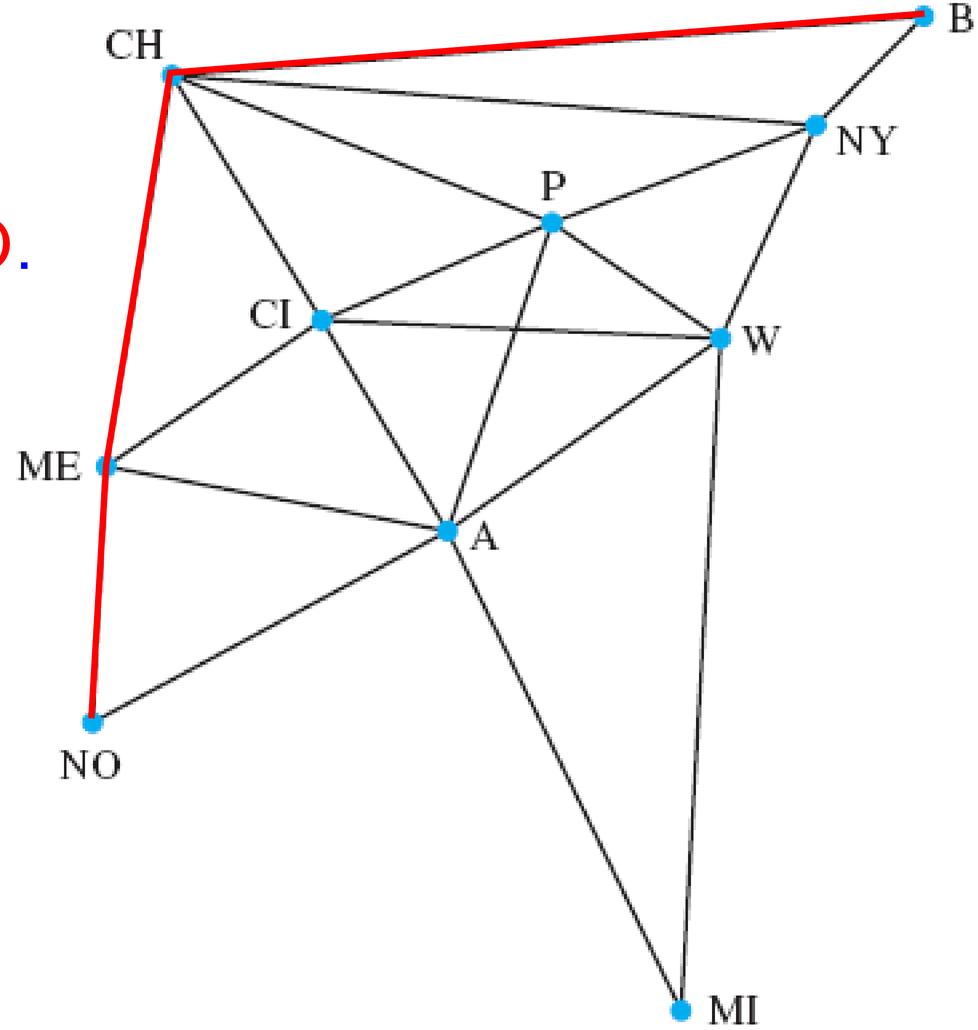
Length of a path = # of edges on path

Example

Path from Boston to New Orleans is

B{B,CH}CH{CH,ME}ME{ME,NO}NO.



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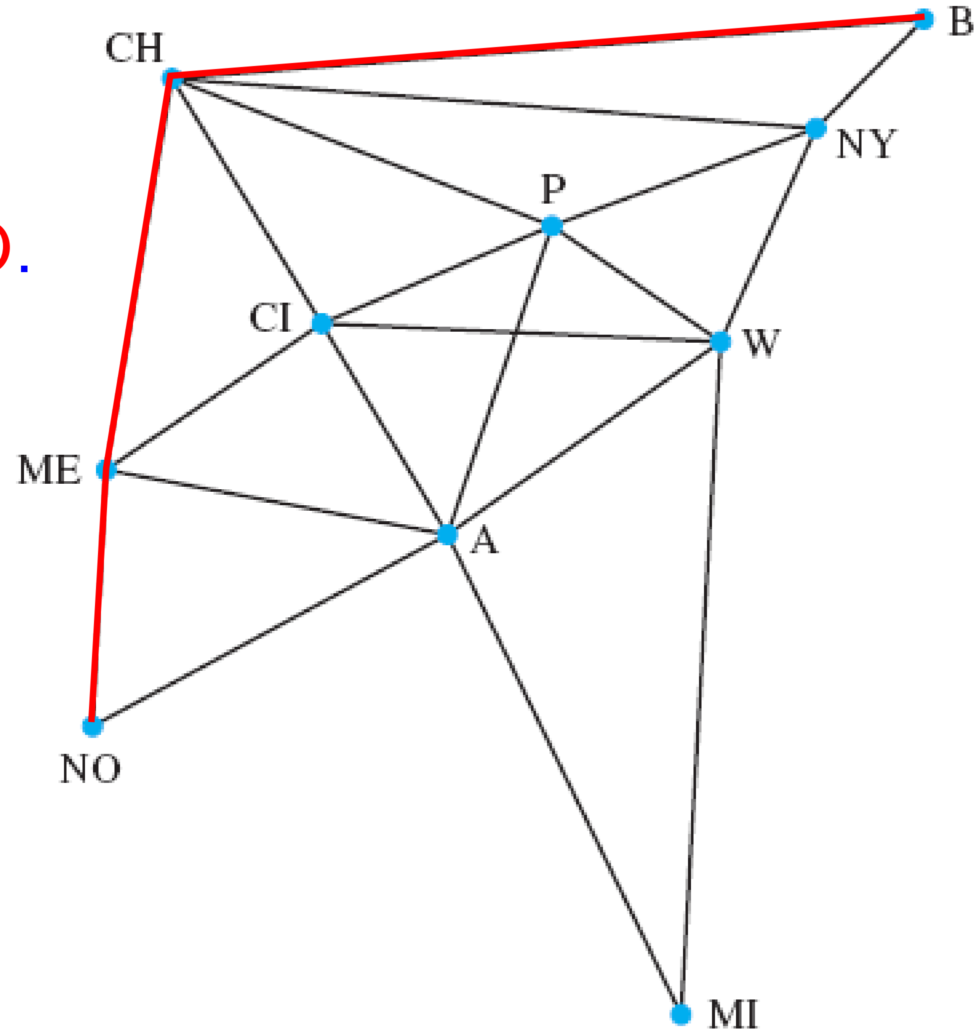

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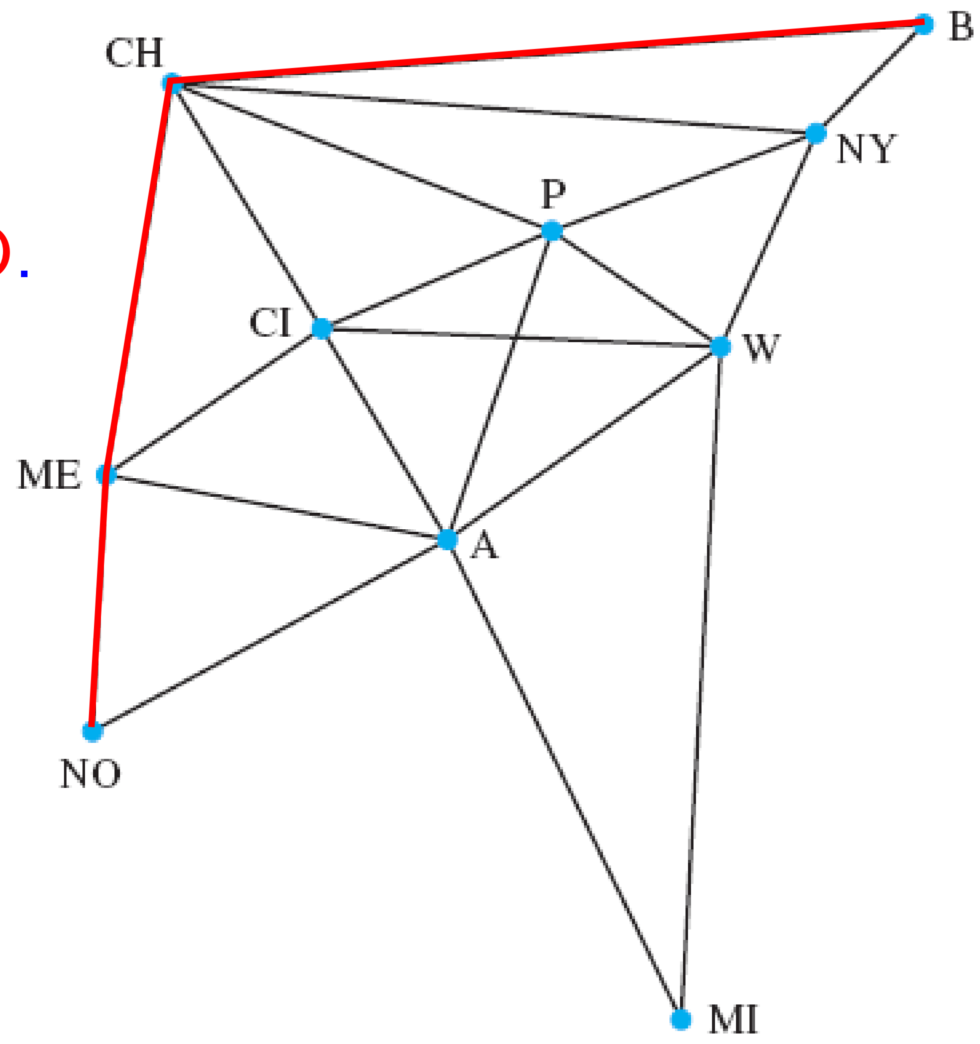
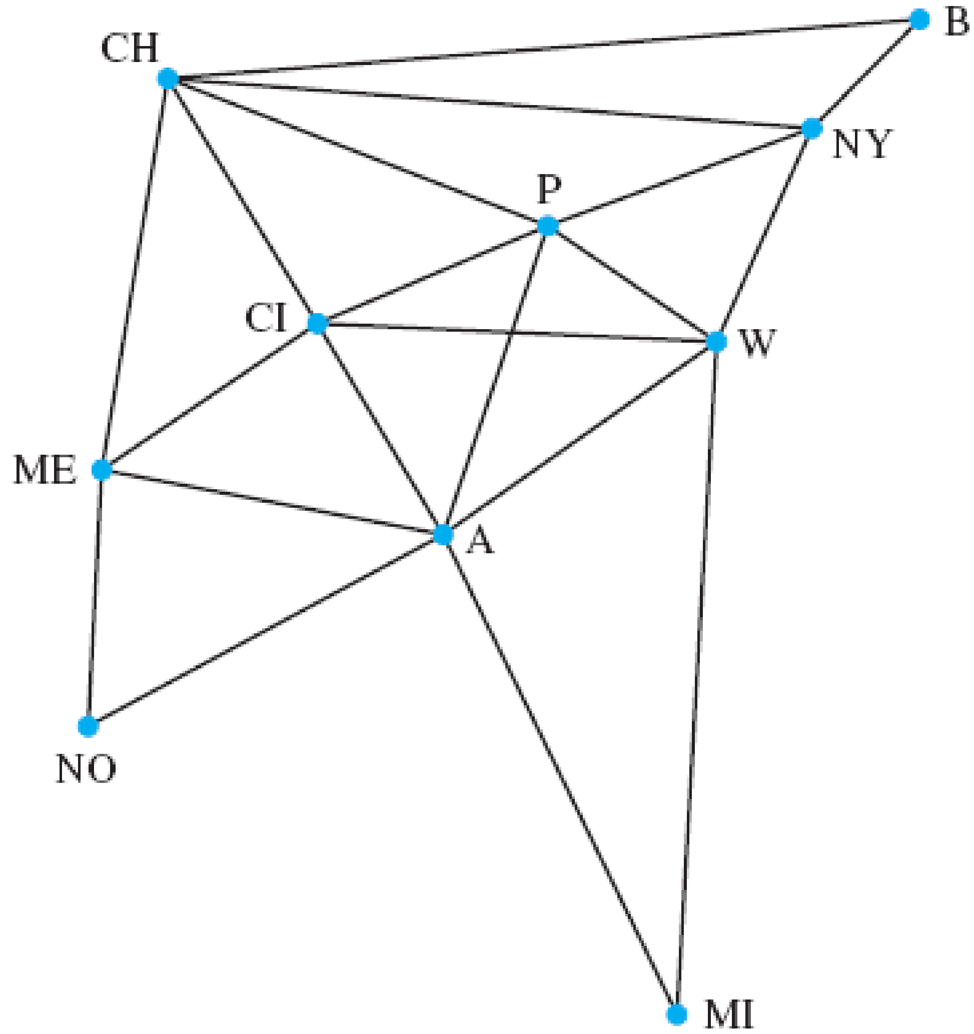
This path has length 3.





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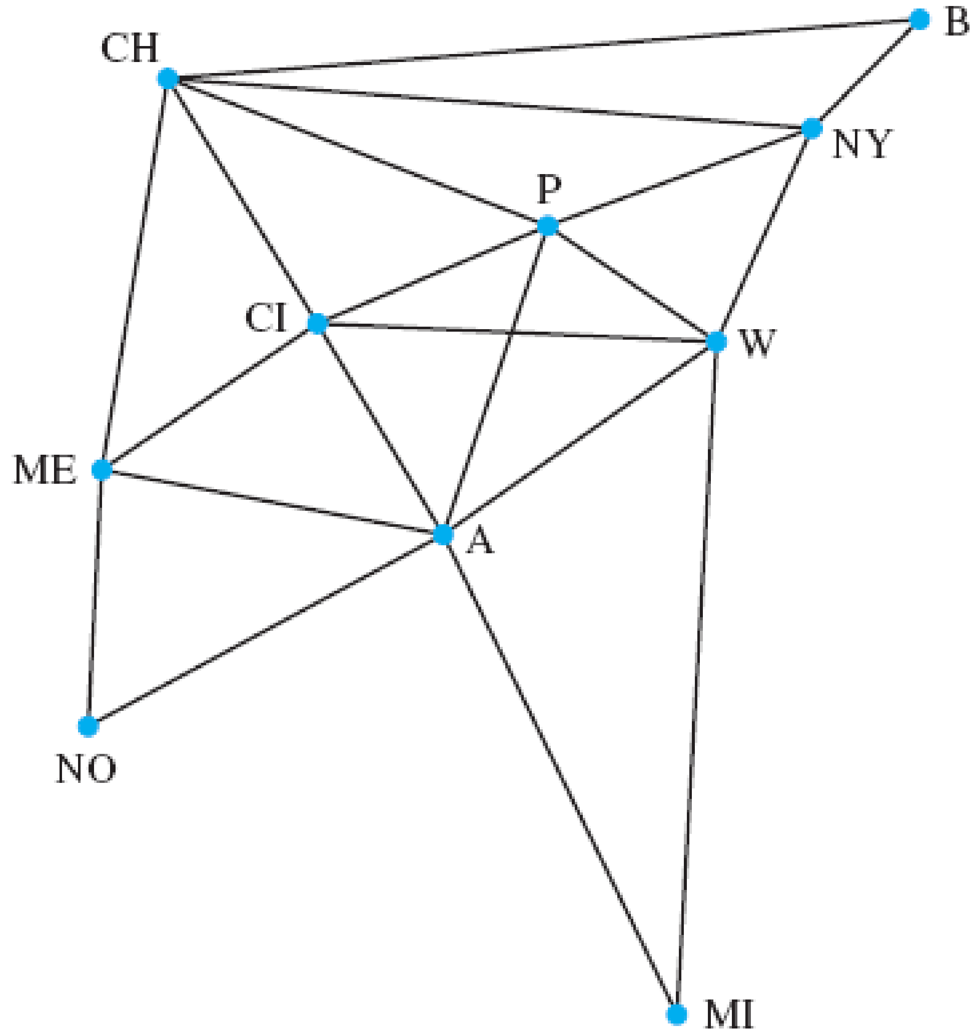
The distance between two vertices is the length of the shortest path between them.



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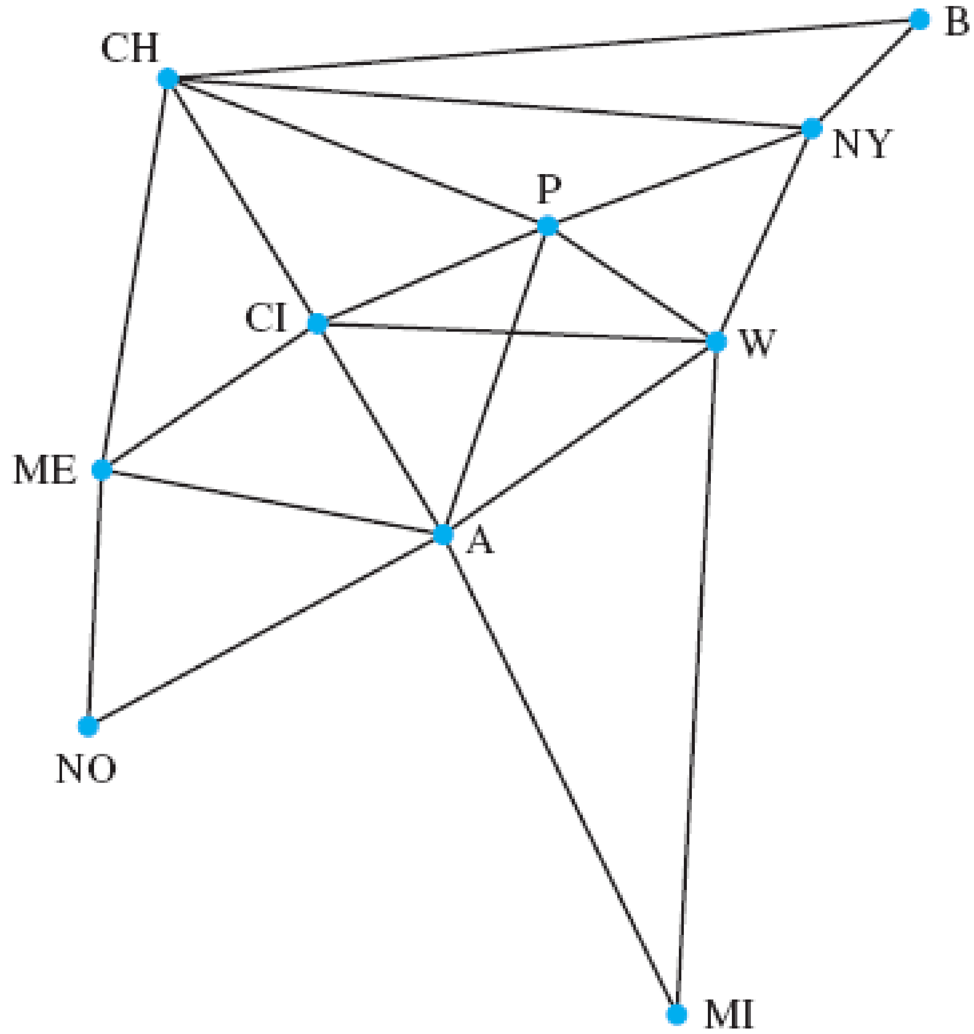

Examples:

$$\begin{aligned} \text{dist(CI, W)} &= 1 \\ \text{dist(CI, B)} &= 2 \\ \text{dist(CI, NO)} &= 2 \end{aligned}$$

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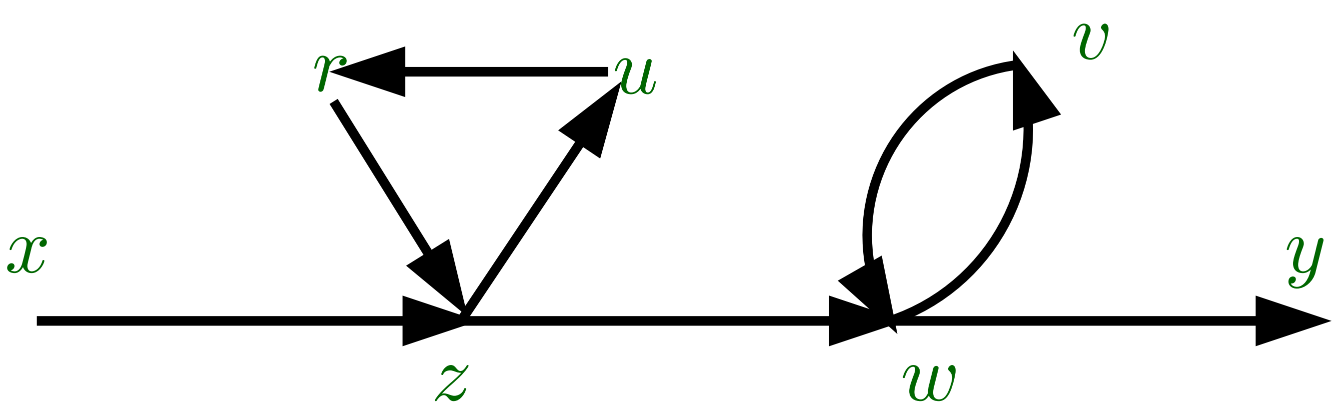

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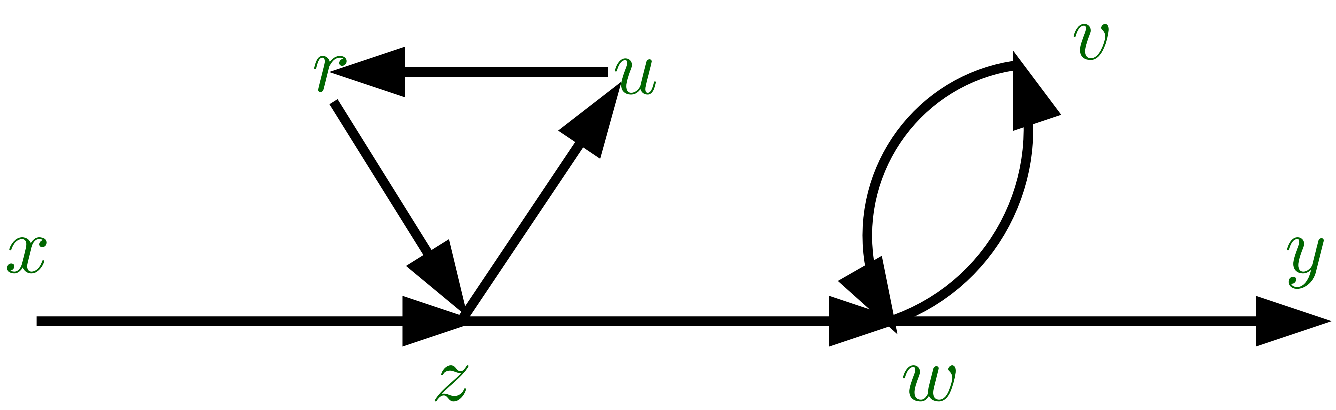

Walk from x to y

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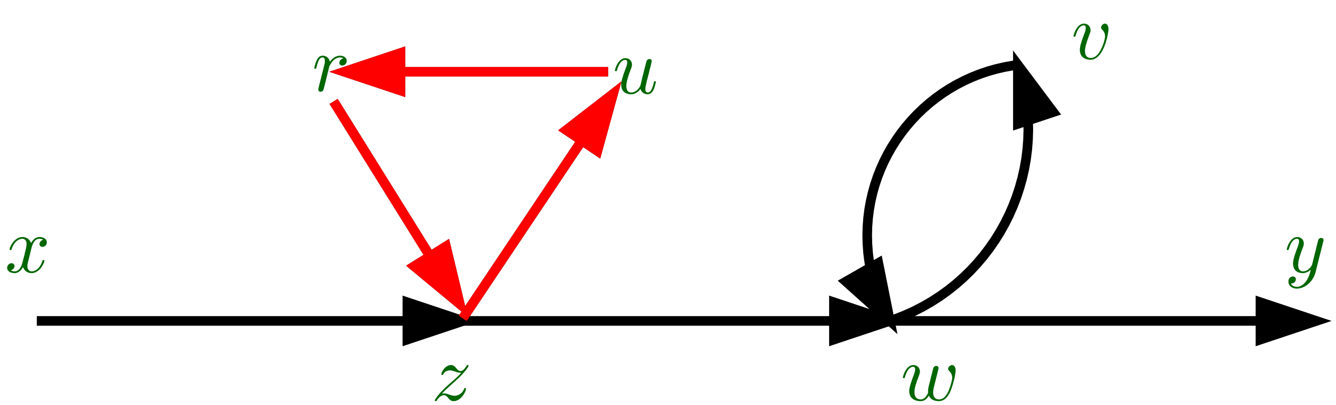

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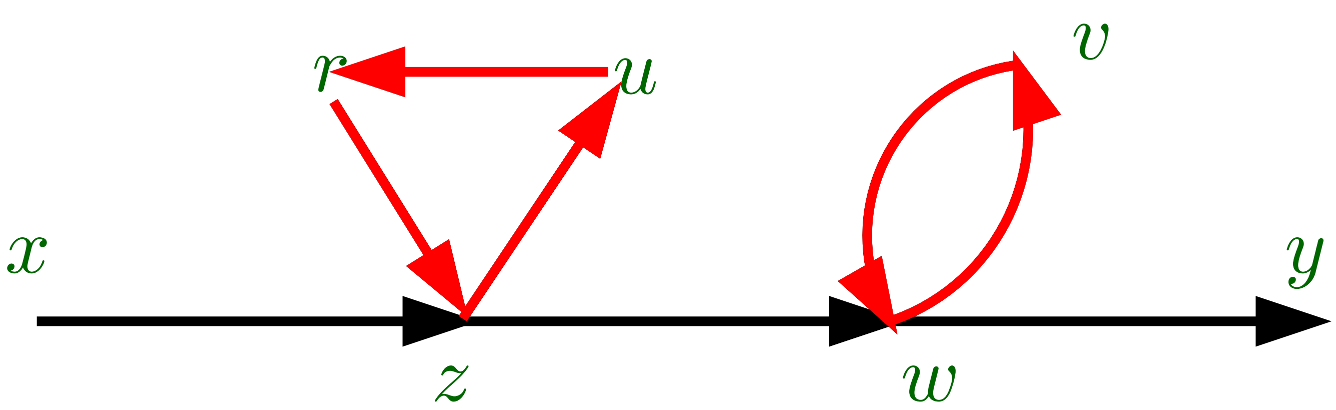

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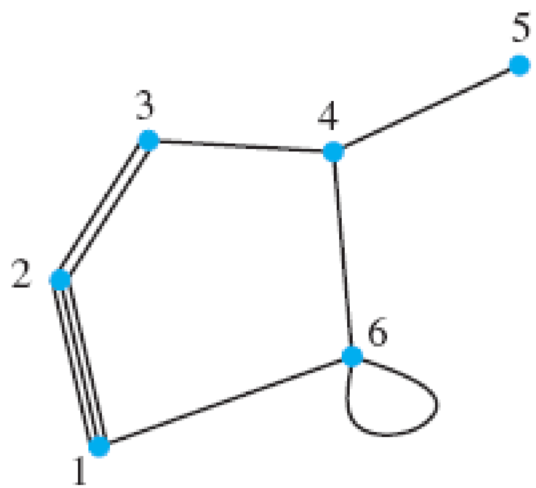

Example: Vertex 2 has degree 5, vertex 6 has degree 4 and vertex 4 has degree 3.

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Inductive Hypothesis:

Suppose that $m > 0$ and that the theorem is true whenever a graph has fewer than m edges.

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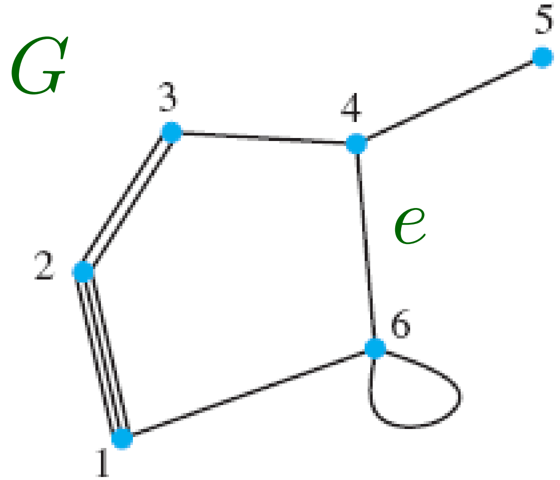

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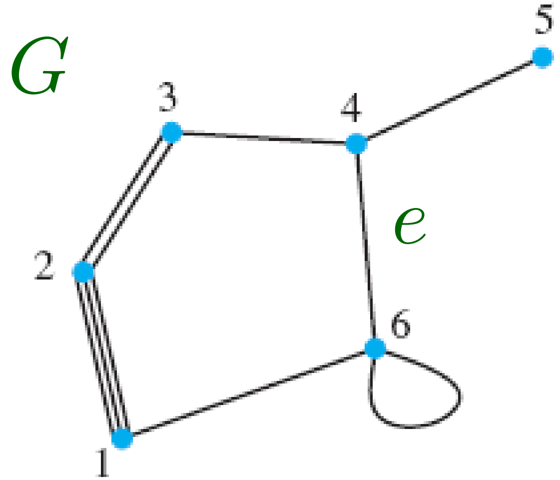

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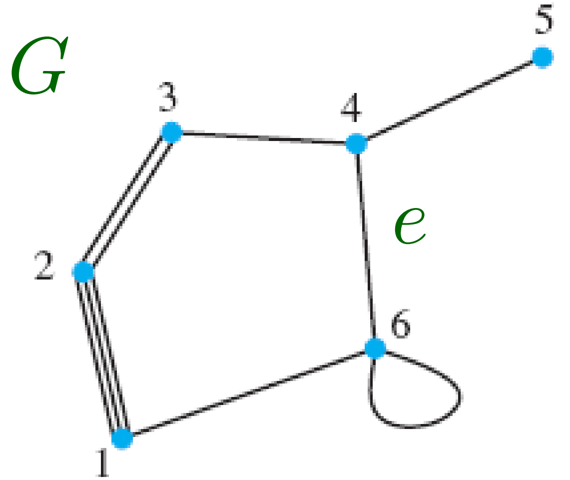

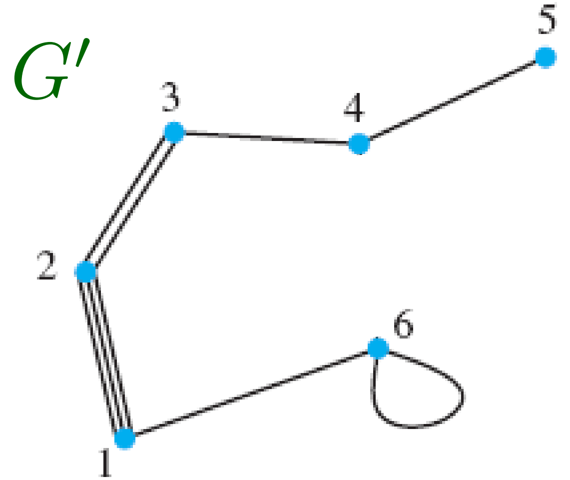

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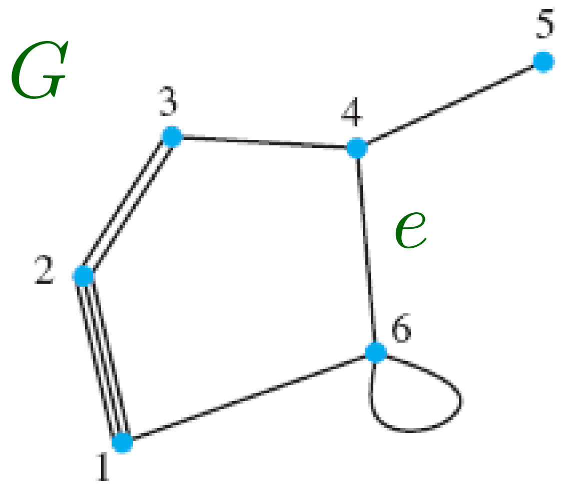

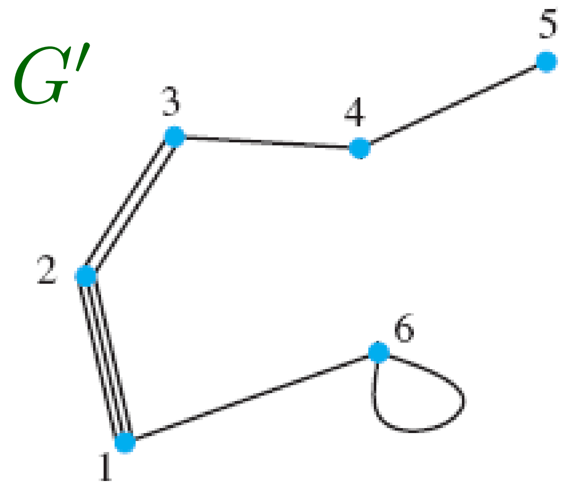

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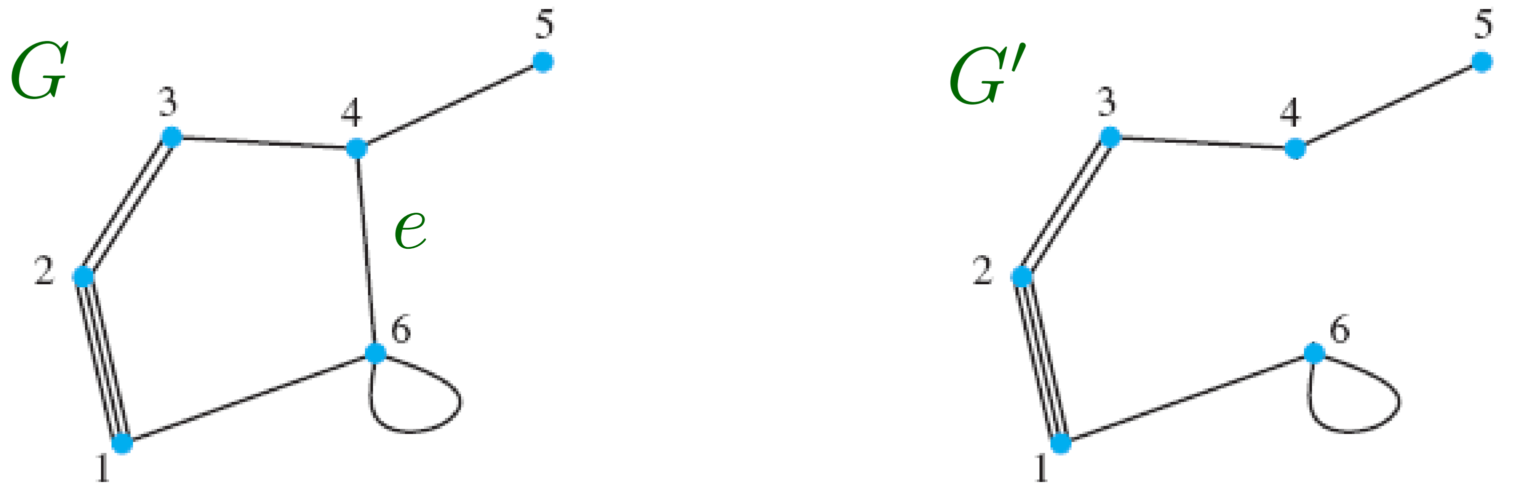

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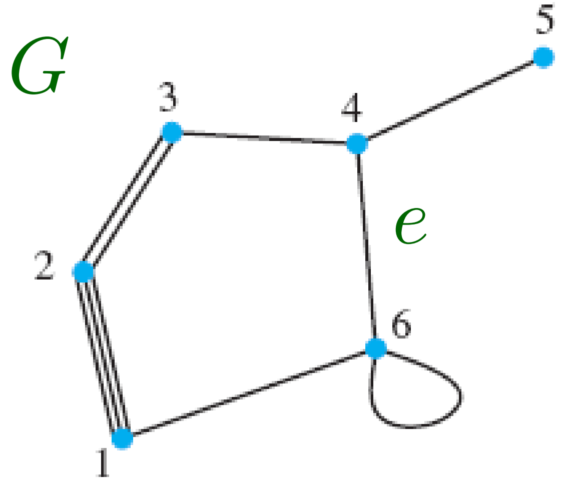

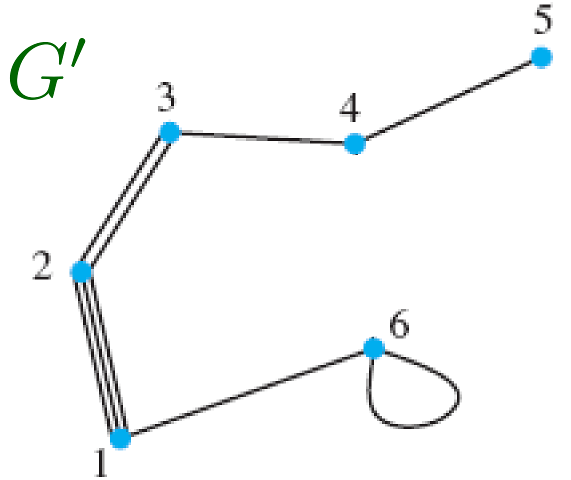

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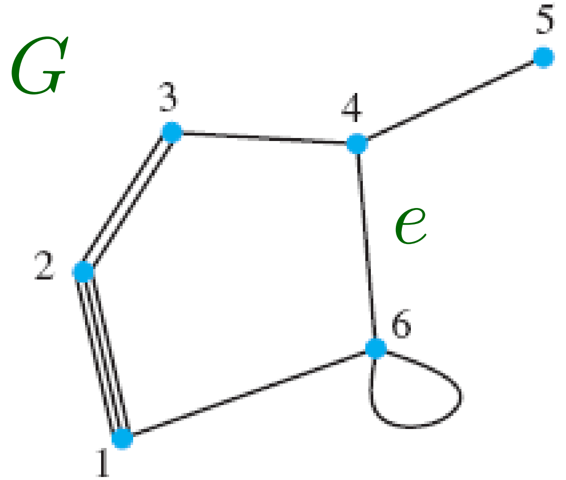

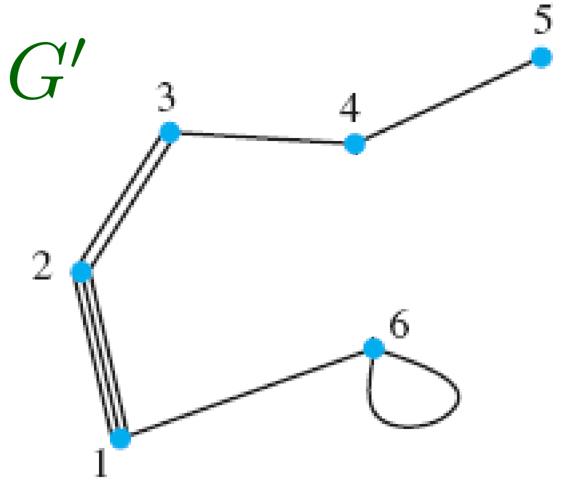

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Therefore, sum of degrees of vertices in G is $(2m-2)+2 = 2m$.

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Therefore, sum of degrees of vertices in G is $(2m-2)+2 = 2m$.

$\Rightarrow$ truth of theorem for graphs with $m-1$ edges implies truth of theorem for graphs with m edges.

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Therefore, by principle of mathematical induction, theorem is true for a graph with any finite number of edges.

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- Basic Definitions
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Connectivity

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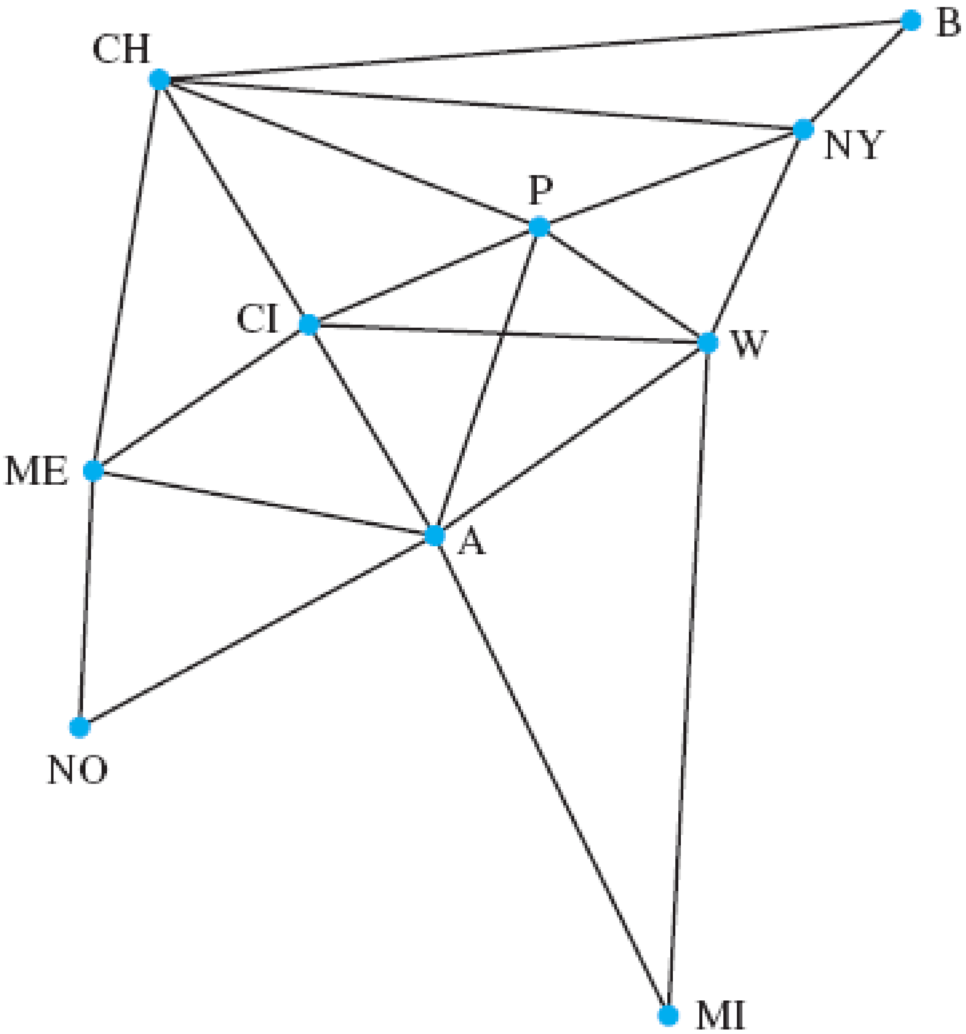

Company decides to lease only minimum number of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is **minimum** number of lines it needs to lease?

Choosing 10 edges?

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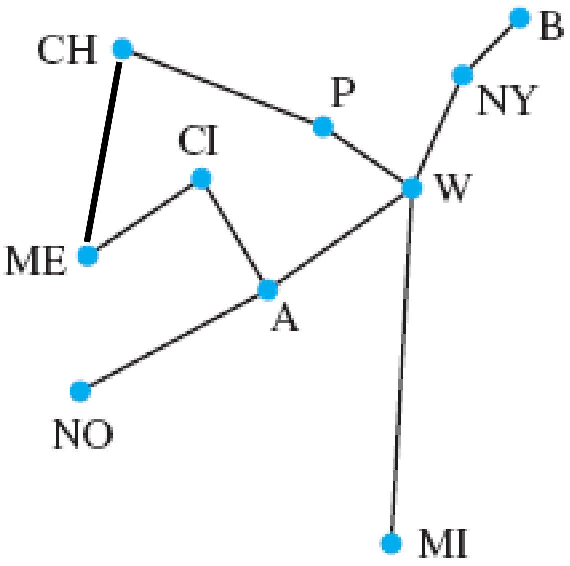

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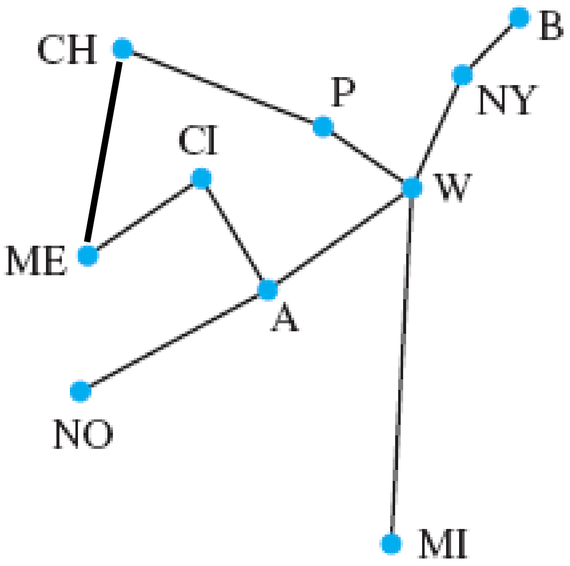

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Could throw away edge CI,A, and still have a solution.

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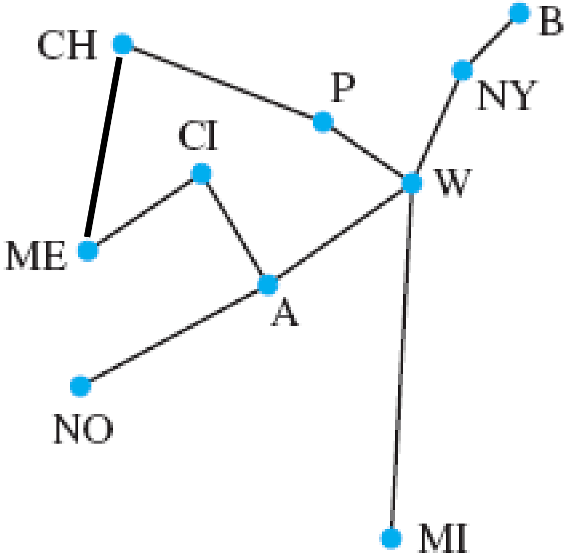

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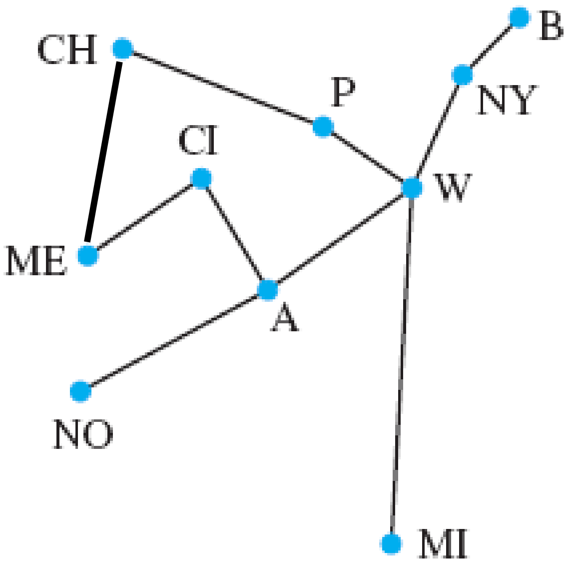

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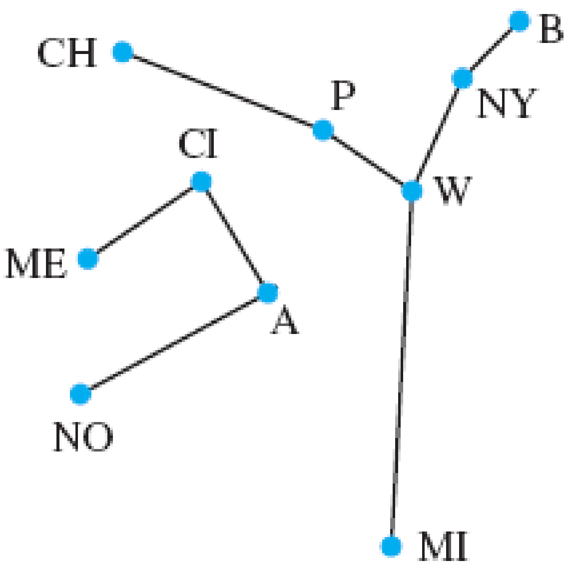

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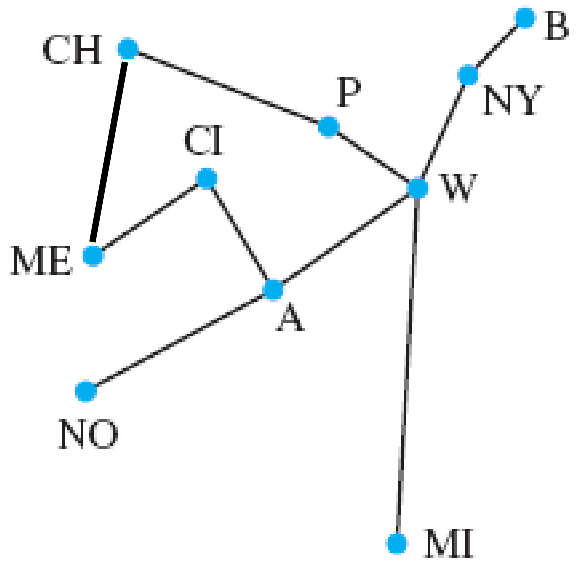

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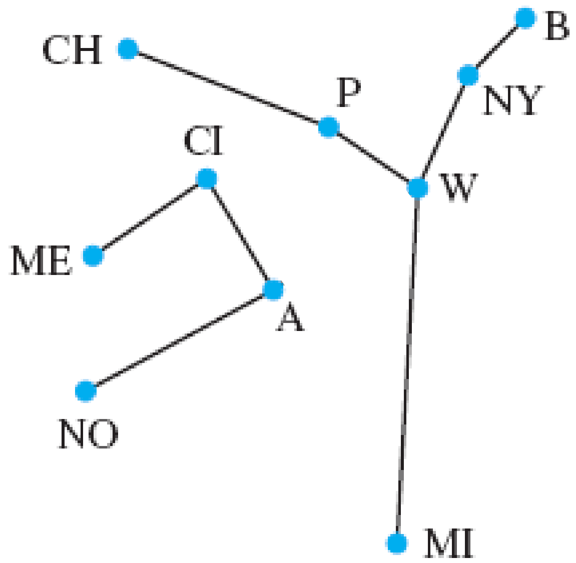

Not enough.

There is no path from, e.g., NO to B.

Choosing 9 edges:

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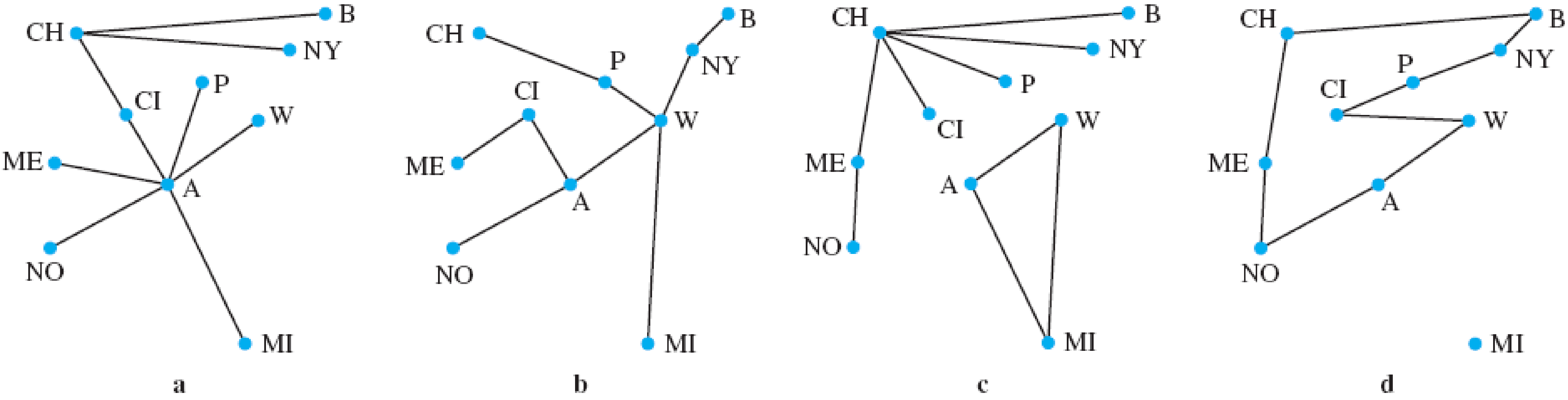

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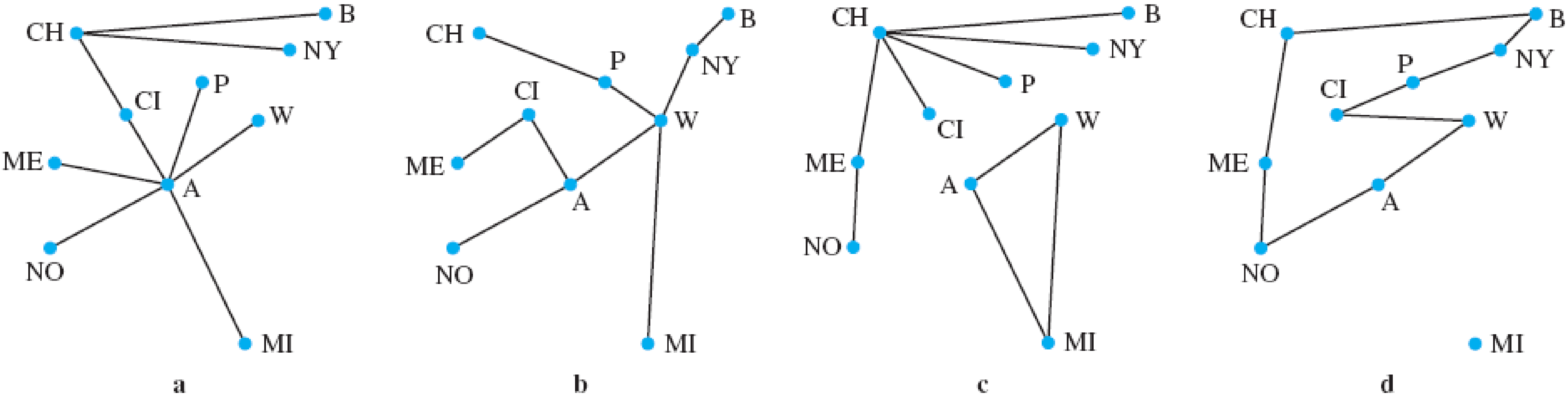

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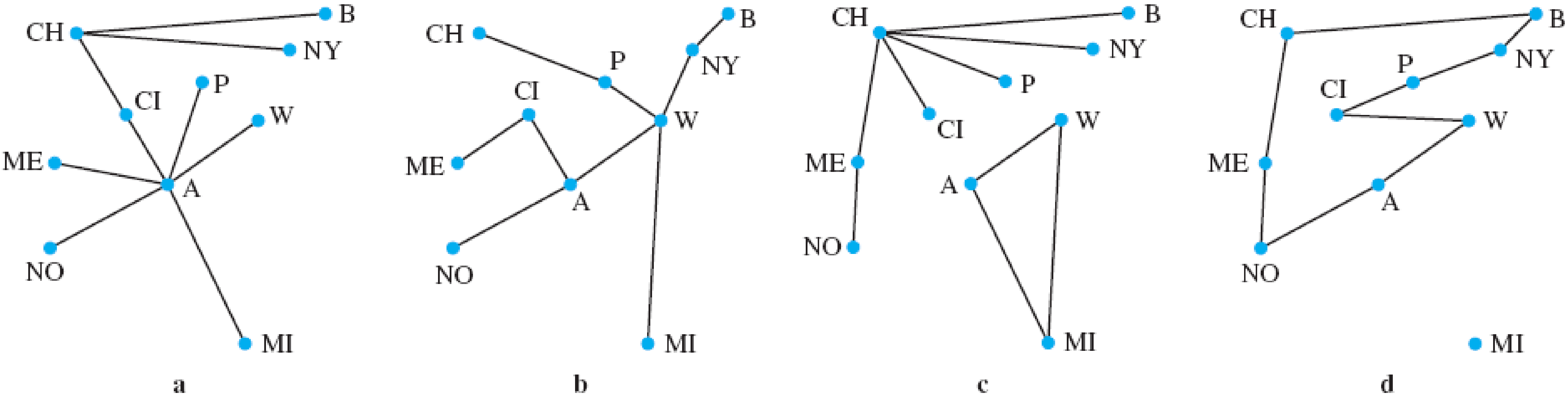

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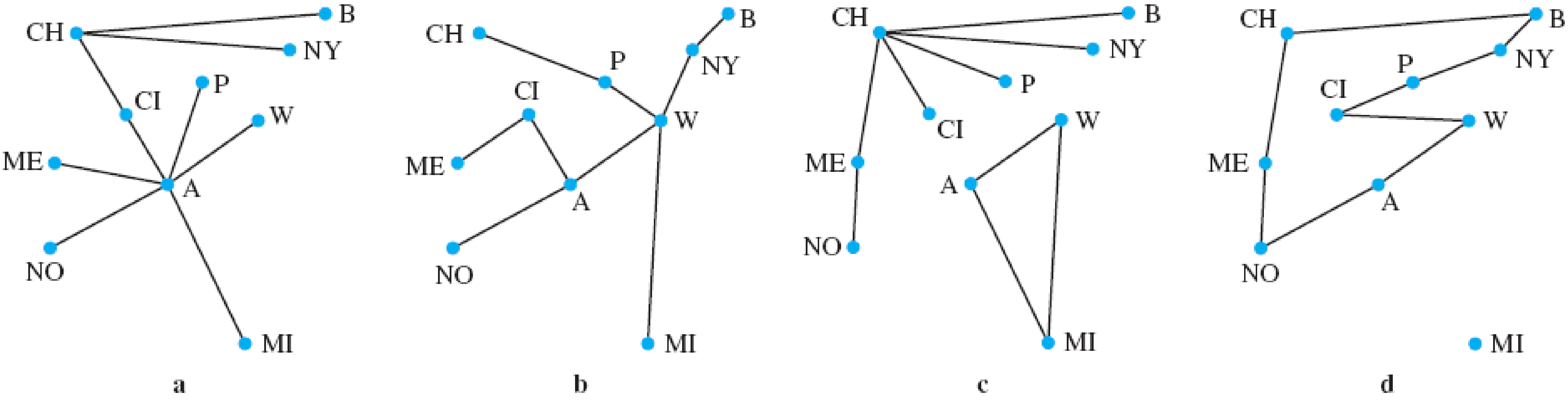

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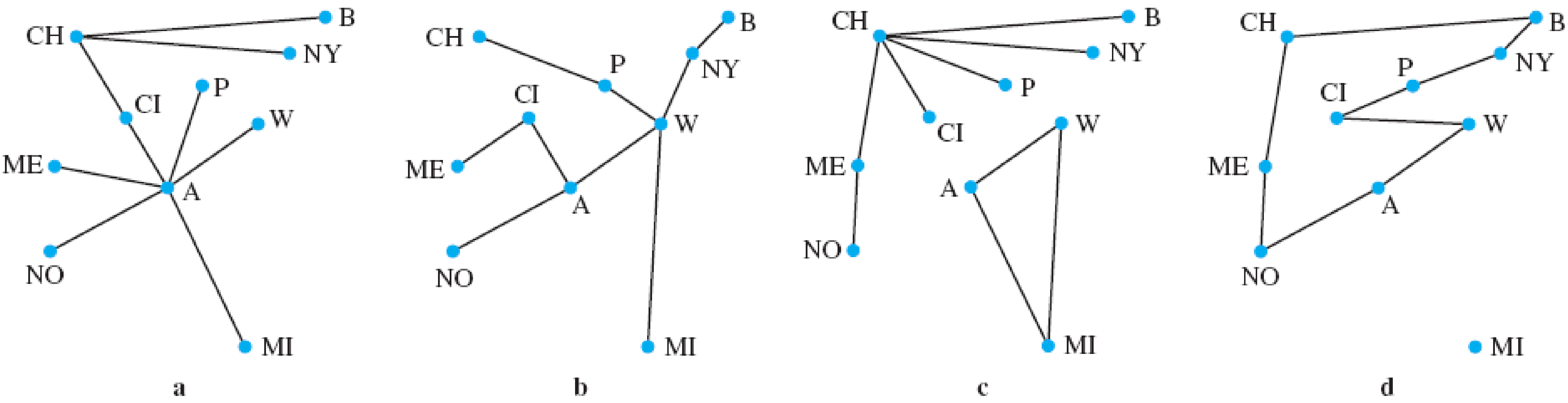

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In (d), we say that M(iami) is an **isolated vertex**.

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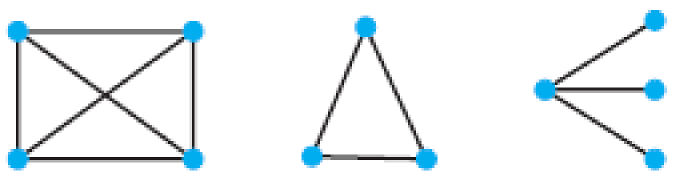
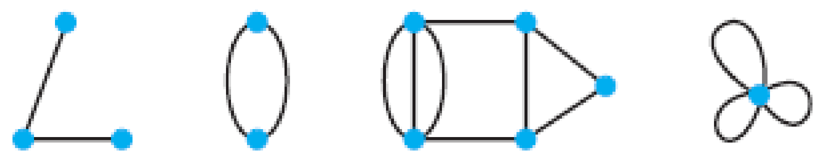
More Examples:



G_1 3 cc's



G_2 4 cc's

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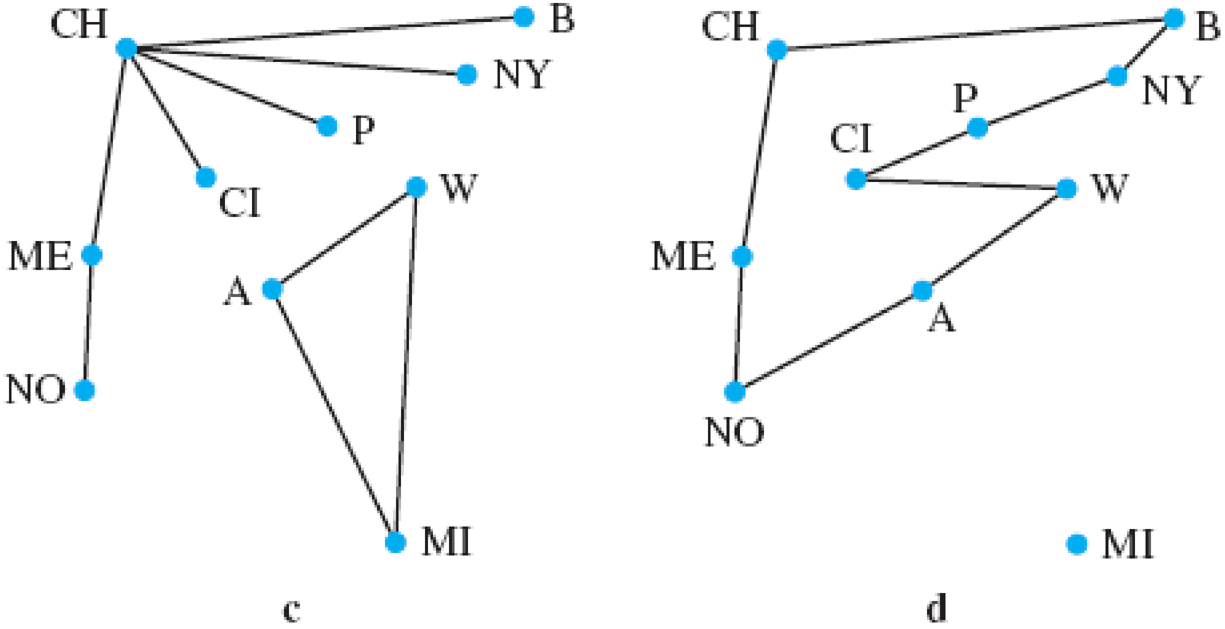

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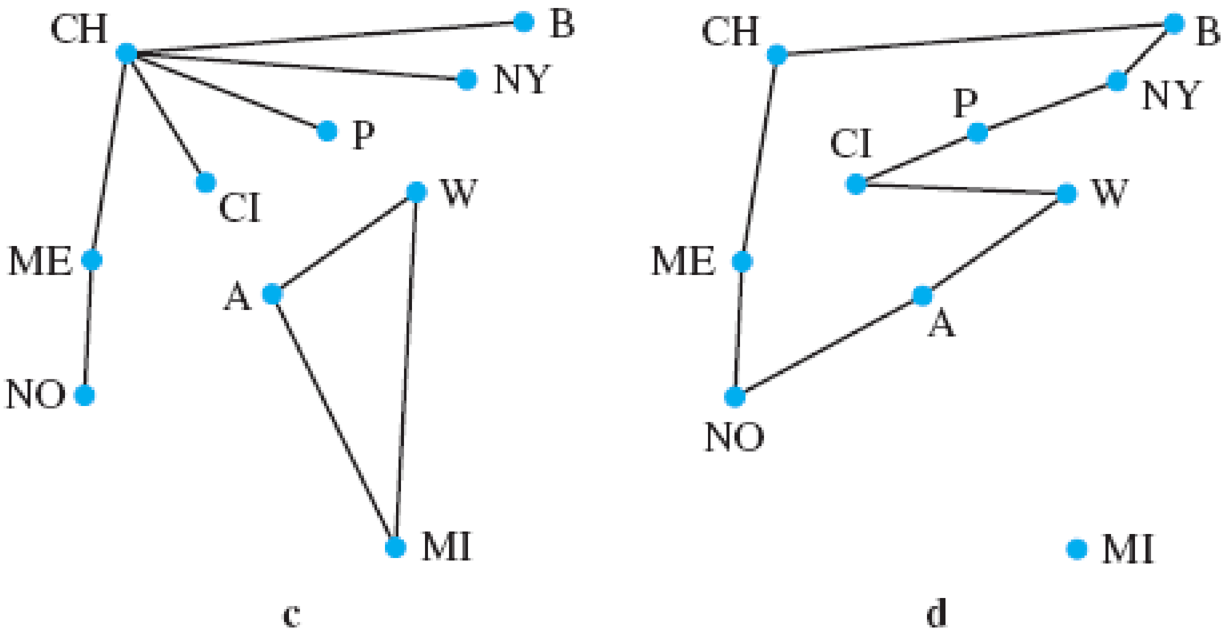
Cycles



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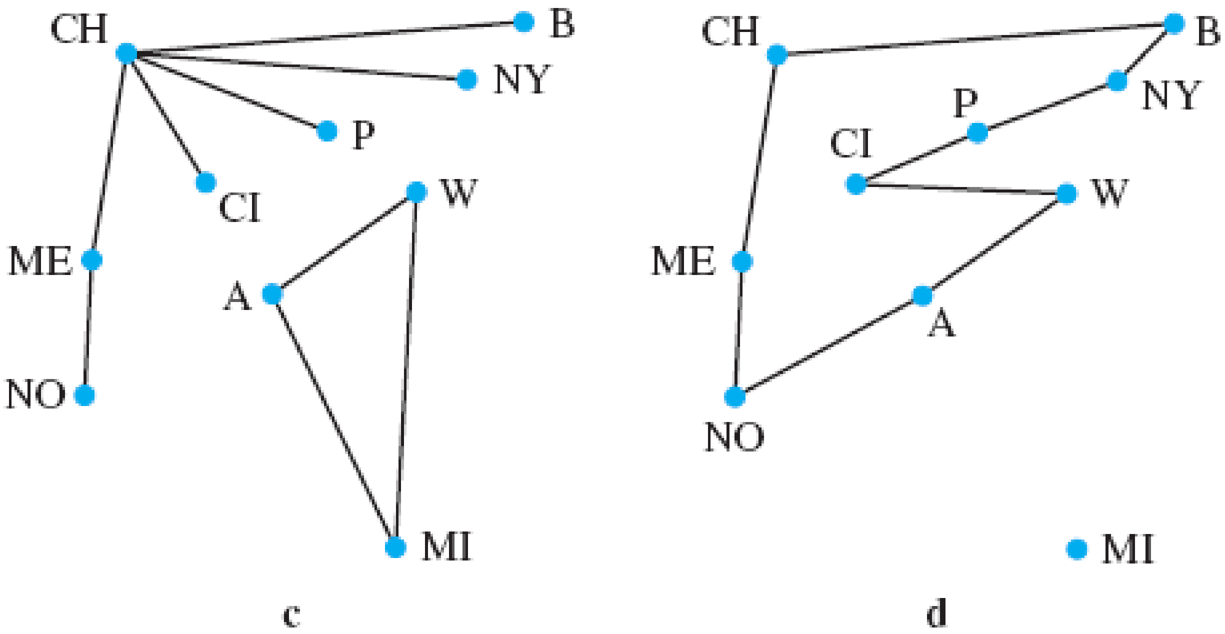

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Example: The closed walks in (c) and (d) are, respectively, cycles A, W, M, A and NO, ME, CH, B, NY, P, CI, W, A, NO.

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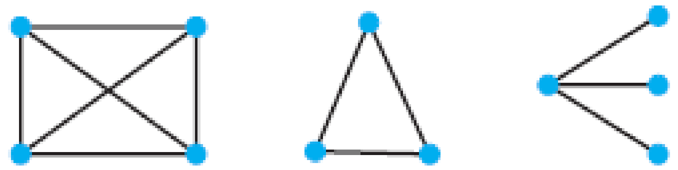

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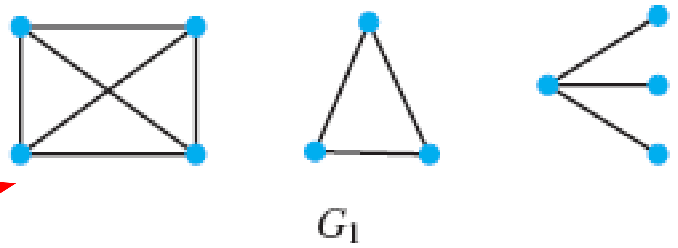

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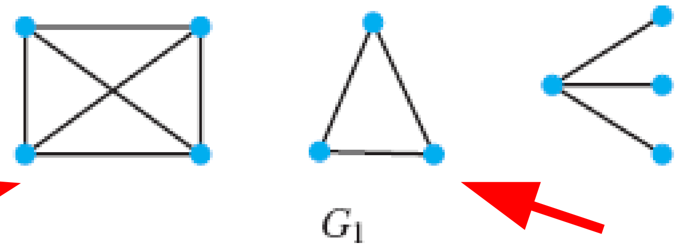
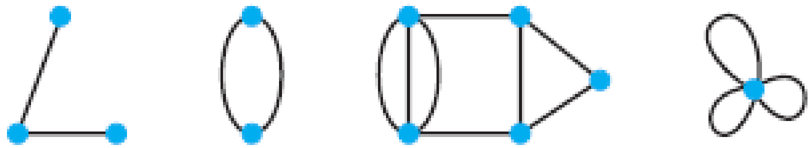
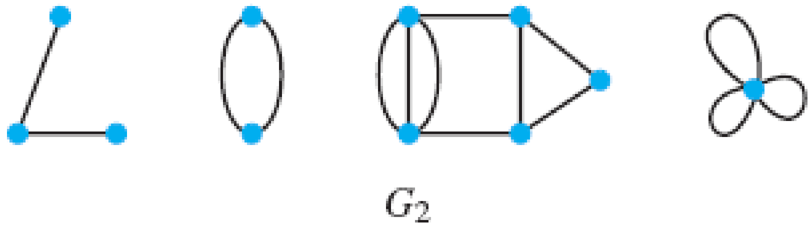


induced K_4

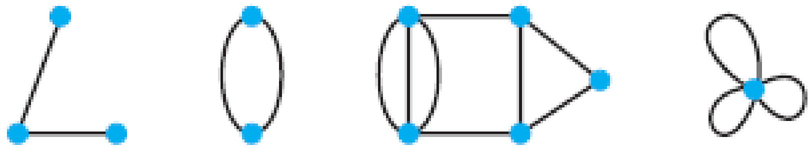
induced K_3



G_2



A graph is a **cycle on n vertices**, or an n-cycle, denoted by C_n, if its vertex set is the vertex set of a cycle and its edge set is the edge set of that cycle.



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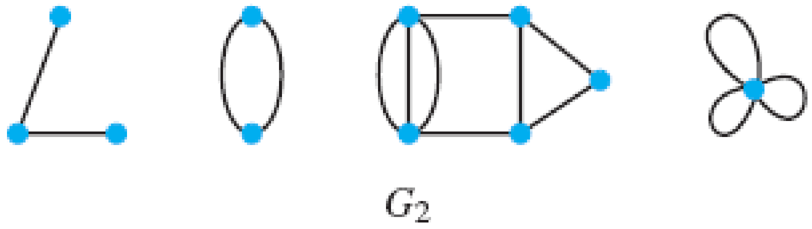

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Examples:
Graph G_2 has an induced P_3 and an induced C_2 as subgraphs.

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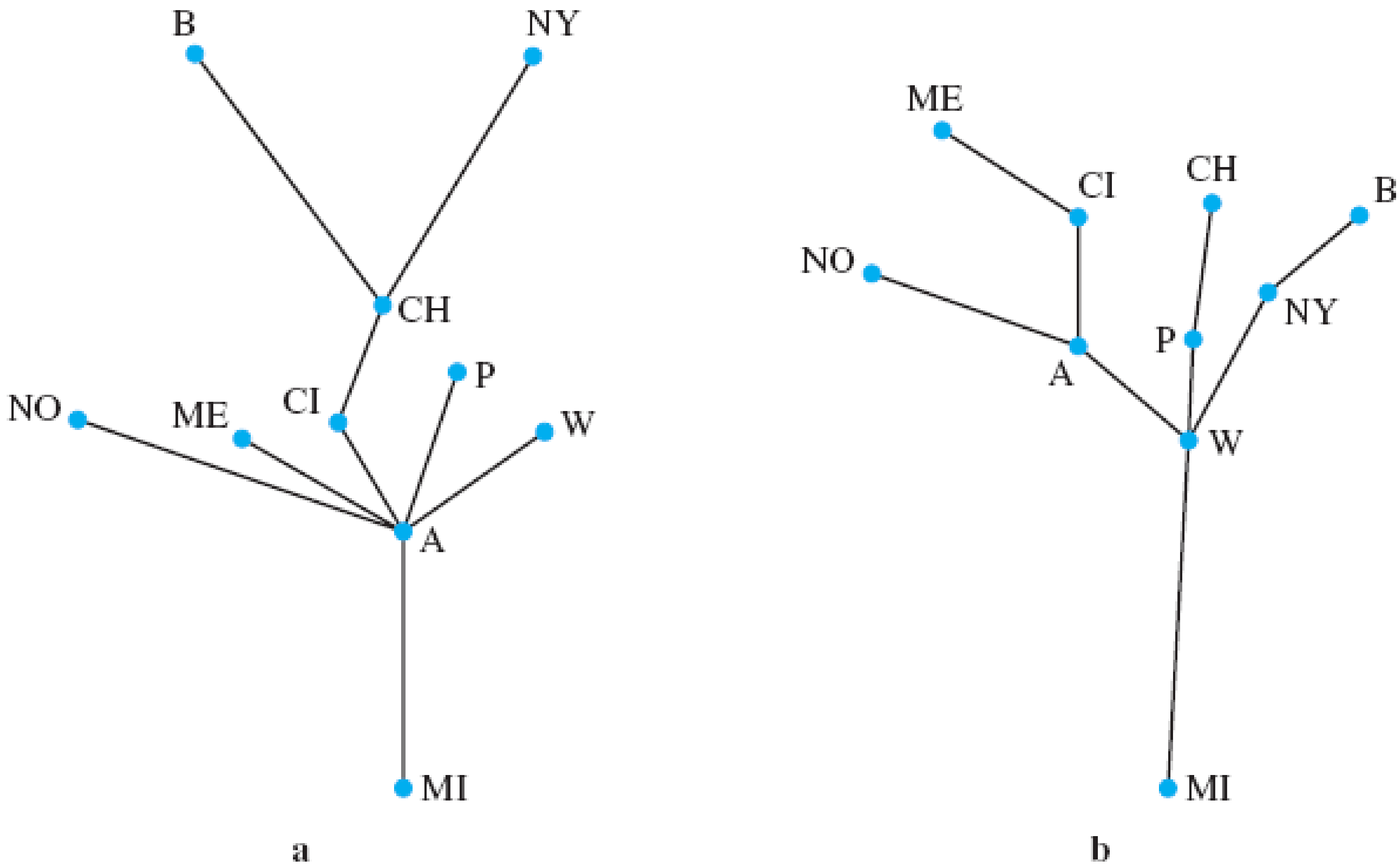

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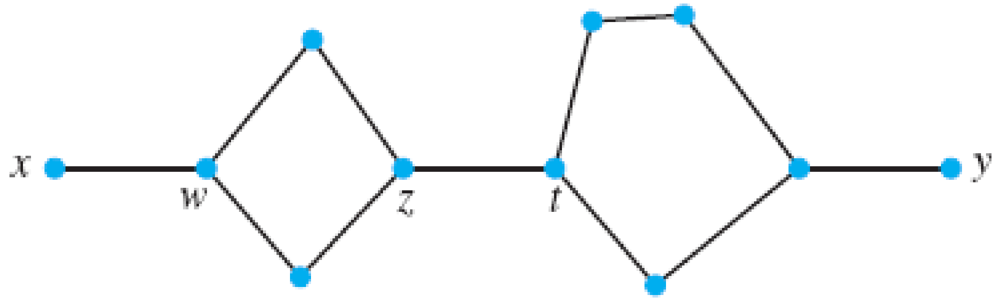

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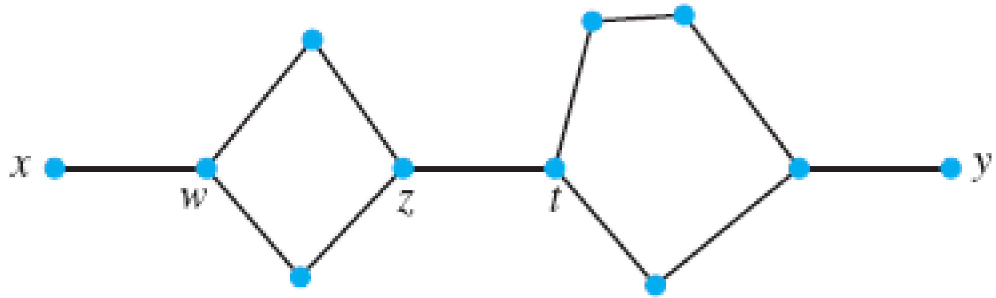

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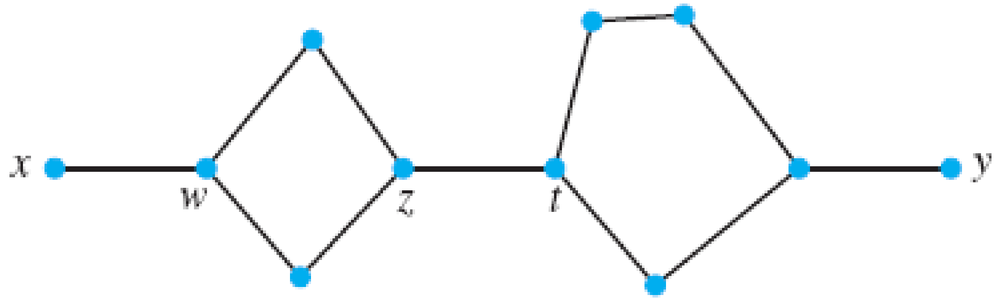

The paths begin with the same vertex x
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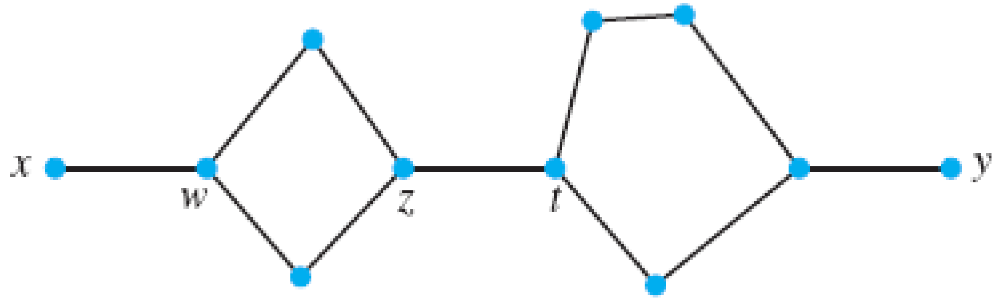
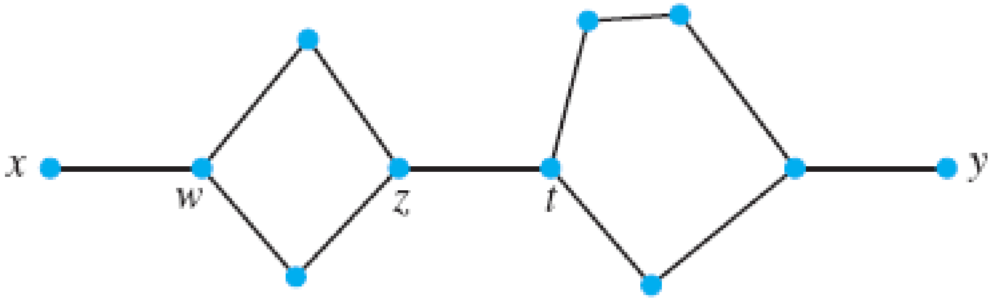
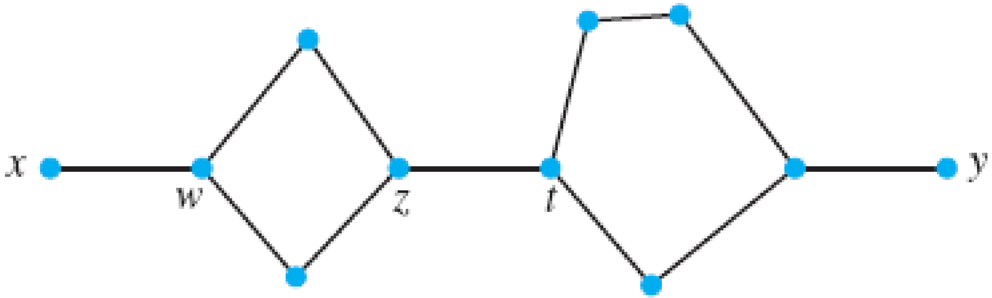
Let w be the last vertex after (or including) x that the paths share before they contain their first different edge.


x
w
z
t
y



We now examine the two paths starting at w; they must join together again at y, though they might have joined at some earlier vertex.



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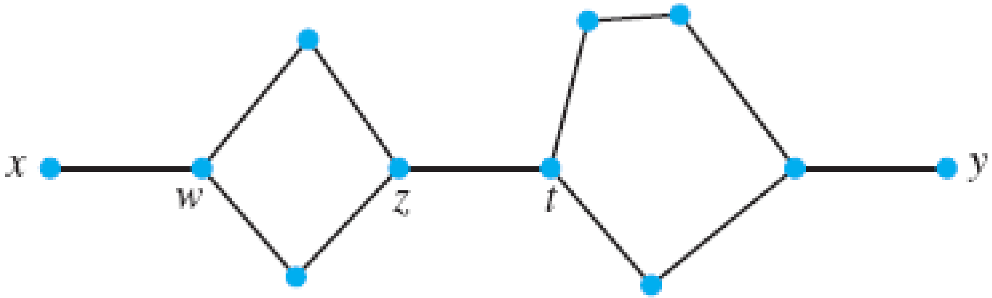

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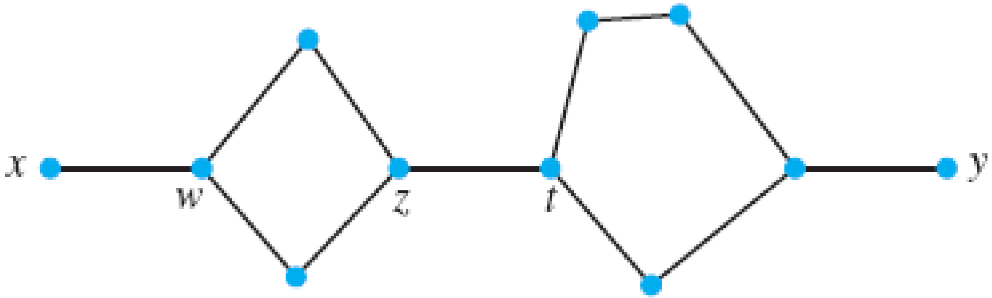

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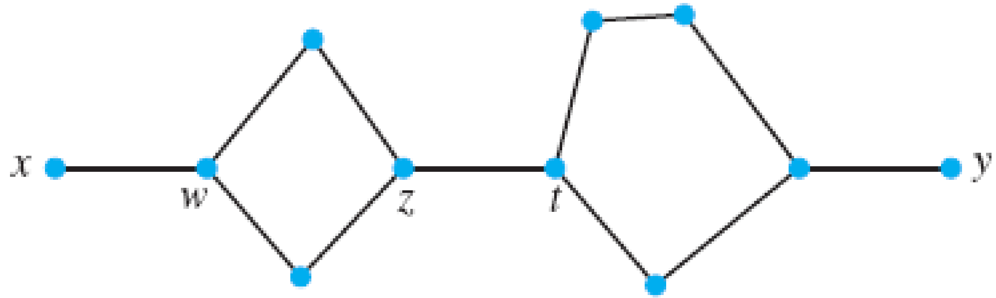

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We have shown that if a graph has two distinct paths from x to y, then it is not a tree.

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Thus, there is exactly one path.

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Thus, it is not connected and is therefore not a tree.

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But adding the edge back in can only reduce the number of connected components by one.

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Removing one edge from the edge set of a tree gives a graph with two connected components, each of which is a tree.

Proof:

Suppose that e is an edge from x to y in a tree.

We have seen that the graph that we get by deleting e from the edge set of the tree is not connected.

So the graph has at least two connected components.

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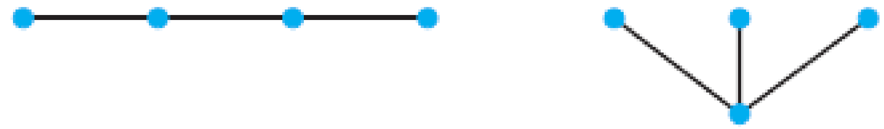
Because neither has any cycles (why?), both are trees.

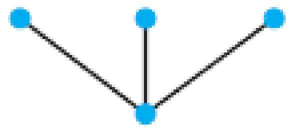
Two trees on four vertices



a **b**

Two trees on four vertices

a



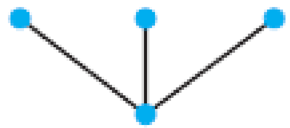
b

Theorem 6.5

For all integers $n \geq 1$, a tree with n vertices has $n - 1$ edges.

Two trees on four vertices

a



b

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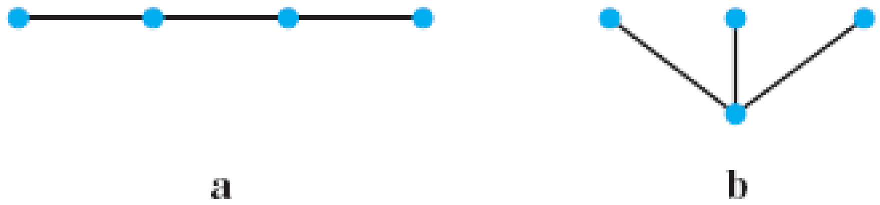
Two trees on four vertices



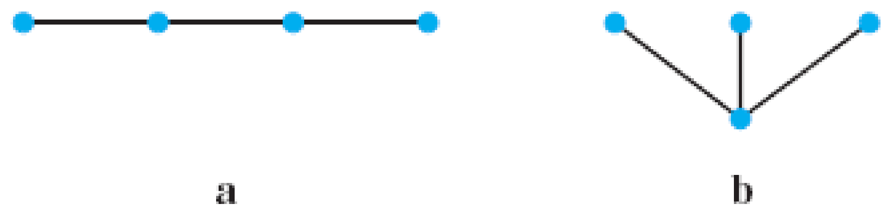
Theorem 6.5

For all integers $n \geq 1$, a tree with n vertices has $n - 1$ edges.

Proof:

If a tree has 1 vertex, it can have no edges,
since any edge would have to connect that vertex to itself giving a cycle.

Two trees on four vertices



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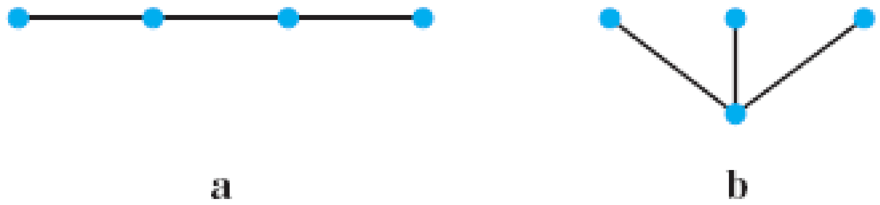

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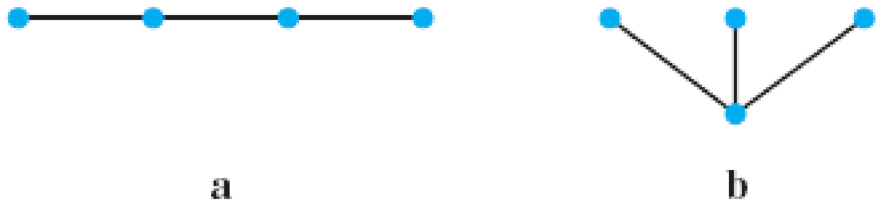

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We can use the deletion of an edge $+$ Lemma 6.4 to complete an inductive proof that a tree with n vertices has $n-1$ edges.

Therefore, for all $n \geq 1$, a tree with n vertices has $n-1$ edges.

Corollary 6.6

A finite tree with more than one vertex has at least one vertex of degree 1.

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$\Rightarrow$ by Theorem 6.5, G is not a tree.

$\Rightarrow$ by contrapositive inference,
if T is a tree, then T must have at least one vertex of degree 1.