Breadth-First Search

Version of September 23, 2016

Representations of Graphs: Adjacency List

- V: set of vertices, E: set of edges. (We will sometimes also simultaneously use V to denote the number of vertices, and E to denote the number of edges.)
- Adjacency list representation: $O(V + E)$ storage
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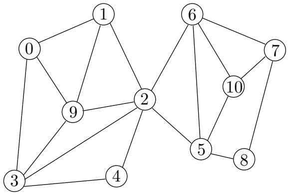
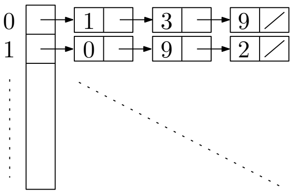
- Can retrieve all the neighbors of u in $O(\text{degree}(u))$ time.





- Adjacency matrix representation: $O(V^2)$ storage
  $A = [a_{ij}]$, $a_{ij} = 1$ if $(v_i, v_j) \in E$;
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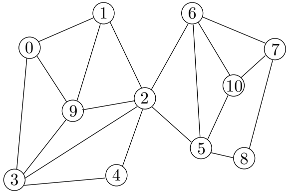
  For undirected graph, adjacency matrix is always symmetric.
- Can check if u and v are connected in $O(1)$ time.



	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	0	0	0	0	1	0
1	1	0	1	0	0	0	0	0	0	1	0
2	0	1	0	1	1	1	1	0	0	1	0
3	1	0	1	0	1	0	0	0	0	1	0
4	0	0	1	1	0	0	0	0	0	0	0
5	0	0	1	0	0	0	1	0	1	0	1
6	0	0	1	0	0	1	0	1	0	0	1
7	0	0	0	0	0	0	1	0	1	0	1
8	0	0	0	0	0	1	0	1	0	0	0
9	1	1	1	1	0	0	0	0	0	0	0
10	0	0	0	0	0	1	1	1	0	0	0

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3. $d[u]$: the distance from the source to vertex u

BFS Algorithm

BFS(G)

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// Initialize
foreach u in V do
    color[u] = WHITE; // undiscovered
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end
time=
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        BFSVisit(u);
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Q $= \emptyset$; Enqueue(Q,s);
while $Q \neq \emptyset$ **do**

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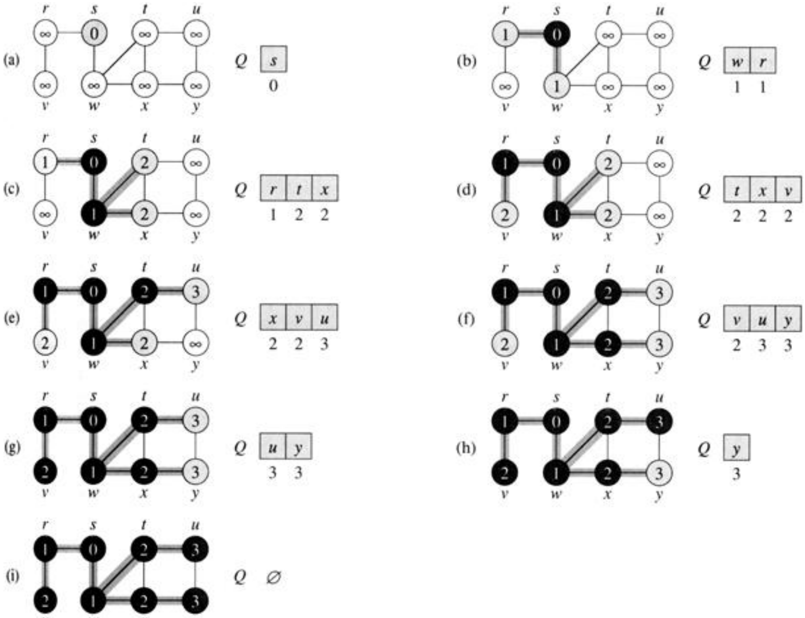
Question

Which graph representation shall we use?

BFS Example



(a) Q s 0
(b) Q w r 1 1
(c) Q r t x 1 2 2
(d) Q t x v 2 2 2
(e) Q x v u 2 2 3
(f) Q v u y 2 3 3
(g) Q u y 3 3
(h) Q y 3
(i) Q ∅
r s t u
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- Use $pred[v]$ to define a graph $F = (V, E_f)$ as follows:

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- In each tree, $d[v]$ gives the shortest distance to the initial vertex of the tree.
- Following $pred[v]$ gives *a* shortest path to the initial vertex of the tree.

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1. Shortest paths in a graph
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2. Finding connected components