# Supplementary Exercises on Logic and Proof

*Note: These exercises are meant to help you revise the material that you have learnt in class when preparing for the exam. Please note that when solutions are given below for the problems they are usually sketch solutions, and do not provide derivations of the answers. For assignments and exams you are expected to provide full derivations. See the posted solutions to assignments for examples of this.*

1. Construct a truth table for the statement $\neg((p \vee q) \wedge \neg p) \wedge \neg p$.

   SOLUTION:

| $p$ | $q$ | $\neg((p \vee q) \wedge \neg p) \wedge \neg p$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

2. Construct a truth table for the statement $(p \wedge (r \wedge (\neg p \vee q))) \Rightarrow ((p \wedge r) \wedge (\neg p \vee q))$.

   SOLUTION:

| $p$ | $q$ | $r$ | $(p \wedge (r \wedge (\neg p \vee q))) \Rightarrow ((p \wedge r) \wedge (\neg p \vee q))$ |
|---|---|---|---|
| T | T | T | T |
| T | T | F | T |
| T | F | T | T |
| T | F | F | T |
| F | T | T | T |
| F | T | F | T |
| F | F | T | T |
| F | F | F | T |

3. Determine if the following statement is always true:

$$(p \vee (\neg p \wedge q)) \Rightarrow (p \vee q)$$

   (A statement with all truth values in the truth table being true is called a *tautology*.)
   SOLUTION: Yes

4. Determine if the following statement is a tautology:

$$(((p \vee q) \wedge \neg p) \wedge q) \Rightarrow (q \wedge \neg q)$$

   SOLUTION: No

5. Are the following statements logically equivalent:

$$(\neg p \wedge (\neg p \wedge q)) \vee (p \wedge (p \wedge \neg q))$$

$$(\neg p \wedge q) \vee (p \wedge \neg q)$$

Solution: Yes

6. Are the following statements logically equivalent:

$$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

$$r$$

Solution: Yes

7. Are the following statements logically equivalent:

$$p \Rightarrow (q \Rightarrow r)$$

$$(p \wedge q) \Rightarrow r$$

Solution: Yes

8. Are the following statements logically equivalent:

$$(p \Rightarrow q) \Rightarrow r$$

$$(p \vee r) \wedge (q \Rightarrow r)$$

Solution: Yes

9. Are the following statements logically equivalent:

$$(p \vee q) \Rightarrow r$$

$$(p \Rightarrow r) \wedge (q \Rightarrow r)$$

Solution: Yes

10. Are the following statements logically equivalent:

$$p \Rightarrow (q \vee r)$$

$$(p \Rightarrow q) \vee (p \Rightarrow r)$$

Solution: Yes

11. What is the truth value of the following statement?

$$(2 + 2 = 7) \Leftrightarrow (5 + 3 = 4)$$

Solution: True

12. Let the universe be the set of all positive integers. Are the following quantified statements true?

    (a) $\exists x\ (\forall y\ (x^2 < y + 1))$
    (b) $\forall x\ (\exists y\ (x^2 + y^2 < 12))$
    (c) $\forall x\ (\forall y\ (x^2 + y^2 > 0))$

    SOLUTION: (a) Yes (set $x = 1$); (b) No (set $x = 4$);
    (c) Yes ($x^2 + y^2 \geq 1 + 1 = 2$)

13. Prove that $p \Rightarrow q$ and $q$ do not imply $p$.

    SOLUTION:

    | $p$ | $q$ | $p \Rightarrow q$ |
    |---|---|---|
    | T | T | T |
    | T | F | F |
    | F | T | T |
    | F | F | T |

    Consider the truth table above. When $p \Rightarrow q$ and $q$ are both true, $p$ may be true or false. Hence the result.

14. Prove that $p \Rightarrow \neg q$ and $q$ imply $\neg p$.
    SOLUTION: $p \Rightarrow \neg q$ is equivalent to its contrapositive $q \Rightarrow \neg p$. Applying modus ponens on $q \Rightarrow \neg p$ and $q$, we get $\neg p$.

15. Prove that $p \Rightarrow \neg q$, $r \Rightarrow q$ and $r$ imply $\neg p$.
    SOLUTION: $p \Rightarrow \neg q$ is equivalent to its contrapositive $q \Rightarrow \neg p$. From $r \Rightarrow q$ and $q \Rightarrow \neg p$, we have $r \Rightarrow \neg p$. Applying modus ponens on $r \Rightarrow \neg p$ and $r$, we get $\neg p$.

16. Construct a contrapositive proof that for all integers $a$, $b$ and $c$, if $a$ does not divide $bc$ then $a$ does not divide $b$.

17. Prove by induction on $n \geq 0$ that

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

18. Prove by induction on $n \geq 0$ that

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$$

19. Prove by induction on $n \geq 0$ that for every $a \neq 1$,

$$\sum_{i=1}^{n} ia^i = \frac{na^{n+2} - (n+1)a^{n+1} + a}{(a-1)^2}.$$

20. Prove by induction on $k \geq 1$ that

$$\sum_{i=1}^{n} i^k \leq \frac{n^k(n+1)}{2}.$$

21. Prove by induction on $n \geq 0$ that $n^3 + 2n$ is divisible by 3.

22. Prove by induction that the sum of the cubes of three successive natural numbers is divisible by 9.

23. Prove by induction on $n \geq 0$ that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$