

COMP170

Discrete Mathematical Tools for Computer Science

Quantifiers

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Discrete Math for Computer Science

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Section 3.2, pp. 104-114

3.2 Variables and Quantifiers

- Variables and Universes
- Quantifiers
- Standard Notation for Quantification
- Statements about Variables
- Proving Quantified Statements True or False
- Negation of Quantified Statements
- Implicit Quantification

Variables and Universes

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In statements such as $m^2 > m$, variable m is *not constrained*.
Unconstrained variables are called *free variables*.

Each possible value of a free variable gives a new statement.
The Truth or Falsehood of this new statement, is determined
by substituting in the new value for the variable.

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- For the universe of real numbers, the statement is True for every value of m except for $0 \leq m \leq 1$

Again consider the statement: $(*) \quad m^2 > m$

- For which values of m is $(*)$ **True** and for which values is it **False**?
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The answer depends upon which **universe** we assume
 - For the universe of **non-negative integers**, the statement is **True** for every value of m except $m = 0, 1$.
 - For the universe of **real numbers**, the statement is **True** for every value of m except for $0 \leq m \leq 1$

Two main points:

- Clearly state the universe
- A statement about a variable can be **True** for some values of a variable and **False** for others.

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- While $m^2 > m$ is True for values such as $m = -3$ or $m = 9$ it is False for $m = 0$ or $m = 1$.
- Thus, it is not True that $m^2 > m$ for every integer m , so (*) is False

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- A phrase like for every integer m that converts a symbolic statement about potentially any member of our universe into a statement about the universe is called a **quantifier**
- A quantifier that asserts a statement about a variable is true for every value of the variable in its universe, is called a **universal quantifier**.

Examples of universal quantifiers

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The statement

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 - Example for (***) : set $m = 2$

- What would you have to do to show that a statement about one variable with an existential quantifier is False?

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Examples: Use Z for universe of all integers

- For all integers n , $n^2 \geq n$ becomes $\forall n \in Z (n^2 \geq n)$
- There exists an integer n such that $n^2 \not\geq n$ becomes
 $\exists n \in Z (n^2 \not\geq n)$

Using the given notation let's rewrite the part of Euclid's division theorem (Theorem 2.12) that states

For every positive integer n and every nonnegative integer m , there are integers q and r , with $0 \leq r < n$, such that $m = qn + r$

Let \mathbb{Z}^+ be the positive integers and \mathbb{N} the nonnegative integers.

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Let Z^+ be the positive integers and N the nonnegative integers.

$$\forall n \in Z^+ \quad (\forall m \in N \quad (\exists q \in N \quad (\exists r \in N \\ ((r < n) \wedge (m = qn + r)) \quad))))$$

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We now rewrite Euclid's division theorem.

Let $p(m, n, q, r)$ denote $m = nq + r$ with $0 \leq r < n$.

Leave out references to universes to clearly see the order in which the quantifiers occur.

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$$\forall n (\forall m (\exists q (\exists r p(m, n, q, r))))$$

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$$\text{a') } \forall x \in R ((x > 0) \Rightarrow (x > 1))$$

$$\text{b') } \exists x \in R ((x > 0) \wedge (x > 1))$$

Theorem 3.2:

Let U_1, U_2 be two universes with $U_1 \subseteq U_2$.

Suppose that $q(x)$ is a statement such that

$$(*) \quad U_1 = \{x \in U_2 \mid q(x) \text{ is true}\}.$$

Then, if $p(x)$ is a statement about U_2 , it may also be interpreted as a statement about U_1 , and

a. $\forall x \in U_1 (p(x))$ is equiv. to $\forall x \in U_2 (q(x) \Rightarrow p(x))$,
and

b. $\exists x \in U_1 (p(x))$ is equiv. to $\exists x \in U_2 (q(x) \wedge p(x))$.

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Proof:

By $(*)$, $q(x)$ must be **True** for all $x \in U_1$ and
False for all $x \in U_2$ but $x \notin U_1$.

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For each of the following, state **T** or **F** and explain why.

a) $\forall x \in R^+ (x > 1)$

b) $\exists x \in R^+ (x > 1)$

c) $\forall x \in R (\exists y \in R (y > x))$

d) $\forall x \in R (\forall y \in R (y > x))$

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d) $\forall x \in R (\forall y \in R (y > x))$ **F**. Let $x = 1, y = 0$

e) $\exists x \in R ((x \geq 0) \wedge \forall y \in R^+ (y > x))$
T. Let $x = 0$.

Principle 3.2

(The Meaning of Quantified Statements)

- The statement $\exists x \in U (p(x))$ is **True** if there exists at least one value of $x \in U$ for which the statement $p(x)$ is **True**.
- The statement $\exists x \in U (p(x))$ is **False** if there is **no** $x \in U$ for which $p(x)$ is **True**.
- The statement $\forall x \in U (p(x))$ is **True** if $p(x)$ is **True** for every value of $x \in U$.
- The statement $\forall x \in U (p(x))$ is **False** if $p(x)$ is **False** for at least one value of $x \in U$.

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i.e., **there exists** some integer n s.t. $n^2 \leq 0$,

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Thus, the negation of our for all (\forall) statement is a
there exists (\exists) statement.

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Proof:

$p(x)$	$\neg p(x)$	$\forall x \in U(p(x))$	$\neg \forall x \in U(p(x))$	$\exists x \in U(\neg p(x))$
always true	always false	true	false	false
not always true	not always false	false	true	true

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Example: Let $p(x)$ be the statement $x^2 > 0$. Then

$$\neg \forall n \in Z (n^2 > 0) \quad \text{is equivalent to} \quad \exists n \in Z (n^2 \leq 0)$$

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Negating both statements gives

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Now, setting $q(x) = \neg p(x)$ gives

$\forall x \in U(\neg p(x))$ and $\neg \exists x \in U(p(x))$ are equivalent,

and proves the corollary.

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Let $p(x)$ be $2x$ is odd.

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Then $\neg p(x)$ is $2x$ is even.

The corollary then says that

$$\neg \exists x \in Z (2x \text{ is odd})$$

is equivalent to

$$\forall x \in Z (2x \text{ is even})$$

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Our original statement really means that

For every two even integers, m, n , $m + n$ is even

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In symbols

$$\forall m \in Z (\forall n \in Z ((p(m) \wedge p(n)) \Rightarrow p(m + n)))$$