All-Pairs Shortest Paths

Version of November 5, 2014

Outline

- A third example of dynamic programming
- Will see **two** different dynamic programming formulations for same problem.

Outline

- The all-pairs shortest path problem.
- A first dynamic programming solution.
- The Floyd-Warshall algorithm
- Extracting shortest paths.

The All-Pairs Shortest Paths Problem

Input: weighted digraph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$

Find: lengths of the shortest paths (i.e., distance) between all pairs of vertices in G.

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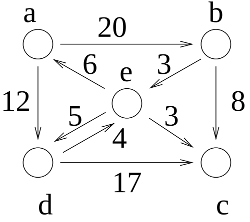
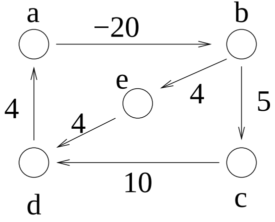


without negative cost cycle



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Solution 1: Using Dijkstra's Algorithm

- Where there are no negative cost edges.
  - Apply Dijkstra's algorithm n times, once with each vertex (as the source) of the shortest path tree.
  - Recall that Dijkstra algorithm runs in $\Theta(e \log n)$
    - $n = |V|$ and $e = |E|$.
  - This gives a $\Theta(ne \log n)$ time algorithm
  - If the digraph is dense, this is a $\Theta(n^3 \log n)$ algorithm.

- When negative-weight edges are present:
  - Run the Bellman-Ford algorithm from each vertex.
  - $O(n^2 e)$, which is $O(n^4)$ for dense graphs.
  - We don't learn Bellman-Ford in this class.

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- Adjacency matrix: graph is represented by an $n \times n$ matrix containing edge weights

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Output Format: an $n \times n$ matrix $D = [d_{ij}]$ in which d_{ij} is the length of the shortest path from vertex i to j.

Step 1: Space of Subproblems

For $m = 1, 2, 3 \ldots,$

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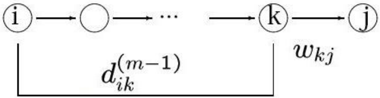
A path without cycles can have length at most $n-1$ (since a longer path must contain some vertex twice, that is, contain a cycle). $\square$

Step 2: Building $D^{(m)}$ from $D^{(m-1)}$.

Consider a shortest path from i to j that contains at most m edges.



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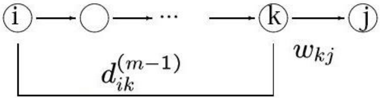

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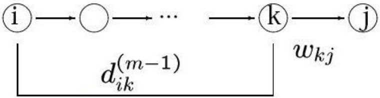

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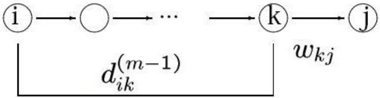

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$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}$$

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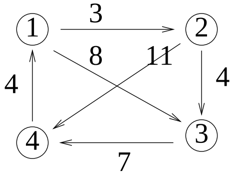

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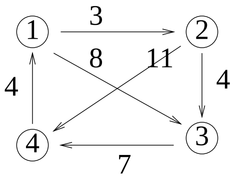

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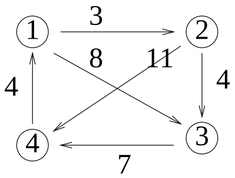

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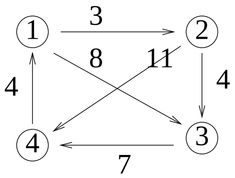

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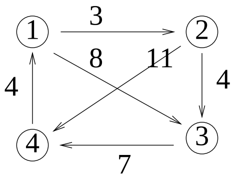

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$D^{(3)}$ gives the distances between any pair of vertices.

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}$$

```
for m = 1 to n − 1 do
    for i = 1 to n do
        for j = 1 to n do
            min = ∞;
            for k = 1 to n do
                new = d_ik^(m−1) + w_kj;
                if new < min then
                    min = new
                end
            end
            d_ij^(m) = min;
        end
    end
end
```

Notes

Running time $O(n^4)$, much worse than the solution using Dijkstra's algorithm.

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Question

Can we improve this?

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We can therefore calculate all of

$$D^{(2)}, D^{(4)}, D^{(8)}, \ldots, D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n-1)}$$

in $O(n^3 \log n)$ time, improving our running time.

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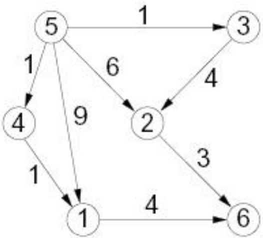

$d_{5,6}^{(0)}$ = INF (no path)

$d_{5,6}^{(1)}$ = 13 (5,1,6)

$d_{5,6}^{(2)}$ = 9 (5,2,6)

$d_{5,6}^{(3)}$ = 8 (5,3,2,6)

$d_{5,6}^{(4)}$ = 6 (5,4,1,6)

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- Original Problem: $D = D^{(n)}$, i.e. $d_{ij}^{(n)}$ is the shortest distance from i to j

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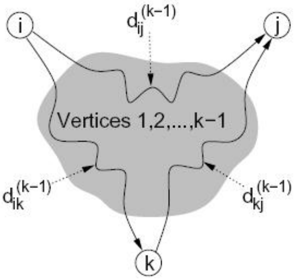
(Impossible for k to appear in path twice. Why?) So:

$$d_{ij}^{(k)} = \min\left\{ d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}$$



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Proof.

- Consider a shortest path from i to j with intermediate vertices from the set $\{1, 2, \ldots, k\}$. Either it contains vertex k or it does not.
- If it does not contain vertex k, then its length must be $d_{ij}^{(k-1)}$.
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- Each subpath can only contain intermediate vertices in $\{1, ..., k-1\}$, and must be as short as possible. Hence they have lengths $d_{ik}^{(k-1)}$ and $d_{kj}^{(k-1)}$.
- Hence the shortest path from i to j has length $\min\left\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\}$.

□

Step 3: Bottom-up Computation

- Initialization: $D^{(0)} = [w_{ij}]$, the weight matrix.
- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, ..., n$.

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                d^(k)[i,j] = d^(k-1)[i,k] + d^(k-1)[k,j];
                pred[i,j] = k;
            else
```

The Floyd-Warshall Algorithm: Version 1

Floyd-Warshall(w, n): $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$

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for i = 1 to n do
    for j = 1 to n do
        d^(0)[i,j] = w[i,j]; pred[i,j] = nil; // initialize
    end
end
// dynamic programming
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            if (d^(k-1)[i,k] + d^(k-1)[k,j] < d^(k-1)[i,j]) then
                d^(k)[i,j] = d^(k-1)[i,k] + d^(k-1)[k,j];
                pred[i,j] = k;
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            end
            d^(k)[i,j] = d^(k-1)[i,j];
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end
return d^(n)[1..n, 1..n]
```

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Comments on the Floyd-Warshall Algorithm

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- It is possible to reduce this to $\Theta(n^2)$ space by keeping only one matrix instead of n.
- Algorithm is on next page. Convince yourself that it works.

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Outline

- The all-pairs shortest path problem.

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- The Floyd-Warshall algorithm
- Extracting shortest paths.

Extracting the Shortest Paths

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- To find the shortest path from i to j, we consult $pred[i, j]$.
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Path(i, j)

```
if pred[i, j] = nil then
    // single edge
    output (i, j);
else
    // compute the two parts of the path
    Path(i, pred[i, j]);
    Path( pred[i, j], j);
end
```

Example of Extracting the Shortest Paths

Find the shortest path from vertex 2 to vertex 3.

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25..6..3	Path$(5,6)$	$pred[5,6] = nil$	*Output*

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256..3	Path(6, 3)	$pred[6, 3] = 4$	

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256..3	Path(6, 3)	$pred[6,3] = 4$	
256..4..3	Path(6, 4)	$pred[6,4] = nil$	*Output*

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