Solving Recurrence Relations

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Contents

Objectives of this Lecture

Recursions and linear recursions were introduced in the previous lecture. The objectives of this lecture are the following.

- Recall the definitions of linear recurrence relations.
- Introduce general techniques for solving linear recurrence relations.
- Solving a number of important types of linear recurrence relations.
- Solving nonlinear recurrence relations.

These techniques will be fundamental in the design and analysis of computer algorithms.

Linear Recurrence Relations

Definition 1

A linear recurrence relation with constant coefficients for a sequence $(s_i)_{i=0}^{\infty}$ is a formula that relates each term s_i to its predecessors $s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}$ in the form

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} + d \text{ for all } i \geq \ell, \tag{1}$$

where ℓ is some fixed integer and d is a constant.

Example 2

Let $(s_i)_{i=0}^{\infty}$ be defined by $s_i = i$ for all integers $i \geq 0$. Then $s_i = s_{i-1} + 1$ is a linear recurrence relation for the sequence with the initial condition that $s_0 = 0$.

Linear Homogeneous Recurrence Relations

Definition 3

A **linear homogeneous recurrence relation of degree ℓ with constant coefficients** (in sort, LHRRCC) for a sequence $(s_i)_{i=0}^{\infty}$ is a formula that relates each term s_i to its predecessors $s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}$ in the form

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell, \tag{2}$$

where ℓ is some fixed integer, and c_i's are real constants with $c_\ell \neq 0$.
The equation

$$x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0 \tag{3}$$

is called the characteristic equation of the linear recursion of (2), and its roots are referred to as the characteristic roots.
The polynomial $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell$ is called the **characteristic polynomial** of the sequence.

Solving Linear Homogeneous Recurrence Relations

Question 1

Given a sequence $(s_i)_{i=i_0}^{\infty}$ defined by a linear homogeneous recurrence relation with constant coefficients, how do you solve the LHRRCC so that you are able to find a mathematical formula for each term of the sequence?

Example 4

Let $(s_i)_{i=0}^{\infty}$ be defined by the following linear homogeneous recurrence relation of degree 2:

$$s_{i+1} = 2s_i - s_{i-1} \text{ for all } i \geq 1$$

with initial conditions $s_0 = 1$ and $s_1 = 3$. Find a mathematical formula in terms of i for each s_i.

When the Characteristic Roots Have Multiplicity 1

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell}$ for all $i \geq \ell$.
Characteristic equation: $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0$.

Theorem 5

If the characteristic equation has distinct roots $r_1, r_2, \ldots, r_\ell$, then a sequence $(s_i)_{i=0}^{\infty}$ satisfies the linear recurrence relation if and only if

$$s_i = \alpha_1 r_1^i + \alpha_2 r_2^i + \ldots + \alpha_\ell r_\ell^i \text{ for integers } i \geq 0, \tag{4}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are constants.

Remarks

A proof of the necessity will be presented in a tutorial. The proof of the sufficiency will be left as an assignment problem.

When the Characteristic Roots Have Multiplicity 1

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell}$ for all $i \geq \ell$.
Characteristic equation: $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0$.

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out all the distinct roots $r_1, r_2, \ldots, r_\ell$.
2. Use the initial conditions $s_0, s_1, \ldots, s_{\ell-1}$ and the roots r_i to solve the following set of equations,

$$s_i = \alpha_1 r_1^i + \alpha_2 r_2^i + \ldots + \alpha_\ell r_\ell^i, \;\; i = 0, 1, 2, \ldots, \ell - 1.$$

This will determine $\alpha_1, \alpha_2, \ldots, \alpha_\ell$.

Solving the First-order Linear Homogeneous Recurrence Relations

Recurrence: $s_i = c_1 s_{i-1}$ for all $i \geq 1$.
Characteristic equation: $x - c_1 = 0$.

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out the unique root $r_1 = c_1$.
2. Use the initial condition s_0 and the root r_1 to solve the following equation

$$s_0 = \alpha_1 .$$

   This will determine $\alpha_1 = s_0$.

Hence, $s_i = s_0 c_1^i$ for all integers $i \geq 0$. This is the geometric sequence.

Solving the Second-order Linear Homogeneous Recurrence Relations

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2}$ for all $i \geq 2$.
Characteristic equation: $x^2 - c_1 x - c_2 = 0$.

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out the two distinct roots r_1, r_2.
2. Use the initial conditions s_0, s_1 and the roots r_1, r_2 to solve the following set of equations,

$$s_0 = \alpha_1 + \alpha_2, \; s_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

This yields α_1 and α_2.

By Theorem 5, we have

$$s_i = \frac{s_1 - s_0 r_2}{r_1 - r_2} r_1^i + \frac{s_0 r_1 - s_1}{r_1 - r_2} r_1^i$$

The Fibonacci Sequence

Problem 6

The sequence $(F_i)_{i=0}^{\infty}$ *is defined by the linear homogeneous recursion*

$$F_i = F_{i-1} + F_{i-2} \text{ for all } i \geq 2,$$

with initial condition $F_0 = 0$ *and* $F_1 = 1$. *Solve this linear recurrence relation.*

Solution 7

The characteristic equation $x^2 - x - 1 = 0$ *has the following distinct roots*

$$r_1 = \frac{1+\sqrt{5}}{2}, \; r_2 = \frac{1-\sqrt{5}}{2}.$$

Hence,

$$F_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$$

An Exercise

Problem 8

Solve the following linear recurrence relation

$$s_i = 6s_{i-1} - 11s_{i-2} + 6s_{i-3} \text{ for all } i \geq 3$$

with initial conditions $s_0 = 2$, $s_1 = 5$ *and* $s_2 = 15$.

When the Characteristic Roots Have Multiplicity > 1

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell}$ for all $i \geq \ell$.
Characteristic equation: $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0$.

Theorem 9

If the characteristic equation has distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that all m_i's are positive and $\sum_{i=1}^{t} m_i = \ell$, then a sequence $(s_i)_{i=0}^{\infty}$ satisfies the linear recurrence relation if and only if

$$
\begin{aligned}
s_i \quad = \quad & (\alpha_{1,0} + \alpha_{1,1} i + \ldots + \alpha_{1,m_1-1} i^{m_1-1}) r_1^i + \\
& (\alpha_{2,0} + \alpha_{2,1} i + \ldots + \alpha_{2,m_2-1} i^{m_2-1}) r_2^i + \ldots + \\
& (\alpha_{t,0} + \alpha_{t,1} i + \ldots + \alpha_{t,m_t-1} i^{m_t-1}) r_t^i \text{ for all } i \geq 0, \qquad (5)
\end{aligned}
$$

where all $\alpha_{i,j}$'s are constants.

When the Characteristic Roots Have Multiplicity > 1

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell}$ for all $i \geq \ell$.
Characteristic equation: $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0$.

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out all the distinct roots $r_1, r_2, \ldots, r_t$ and their multiplicities.
2. Use the initial conditions $s_0, s_1, \ldots, s_{\ell-1}$ and the roots r_i's and their multiplicities m_i to solve the following set of equations,

$$\begin{aligned} s_i &= (\alpha_{1,0} + \alpha_{1,1} i + \ldots + \alpha_{1,m_1-1} i^{m_1-1}) r_1^i + \\ &= (\alpha_{2,0} + \alpha_{2,1} i + \ldots + \alpha_{2,m_2-1} i^{m_2-1}) r_2^i + \ldots + \\ &= (\alpha_{t,0} + \alpha_{t,1} i + \ldots + \alpha_{t,m_t-1} i^{m_t-1}) r_t^i, \quad i = 0, 1, \ldots, \ell - 1. \end{aligned}$$

This will determine $\alpha_{i,j}$'s.

Remark: We will not present a proof for this theorem.

When the Characteristic Roots Have Multiplicity > 1

Recurrence: $s_i = 6s_{i-1} - 9s_{i-2}$ for all $i \geq 2$ with $s_0 = 1$ and $s_1 = 6$.
Characteristic equation: $x^2 - 6x + 9 = 0$.

Solution 10

Note that $x^2 - 6x + 9 = 0$ has the only root $x = 3$ with multiplicity 2. By Theorem 9,

$$s_i = \alpha_1 3^i + \alpha_2 i 3^i.$$

Using the initial conditions, we obtain that $\alpha_1 = \alpha_2 = 1$. Hence,

$$s_i = (i+1)3^i.$$

Rational Functions

Definition 11

A **rational function** is the quotient of two "polynomials" of finite degree over the set of real numbers.

Example 12

$$\frac{x+x^2}{1-3x+3x^2-x^3}.$$

Sequences Defined by Rational Functions

Theorem 13 (Power series expansion of a rational function)

Every rational function $f(x)/g(x)$ can be expressed as

$$\frac{f(x)}{g(x)} = \sum_{i=0}^{\infty} s_i x^i$$

where $\gcd(f(x), g(x)) = 1$, $\deg(f) < \deg(g)$ *and* $g(0) \neq 0$.

Proof.

Let

$$f(x) = g(x) \sum_{i=0}^{\infty} s_i x^i.$$

Solving the polynomial equation above yields s_i one by one. □

Remark: $(s_i)_{i=0}^{\infty}$ is the **sequence** defined by the rational function $f(x)/g(x)$.

Sequences Defined by Rational Functions

Example 14

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

The sequence is $(1)_{i=0}^{\infty}$.

Example 15

$$\frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k.$$

The sequence is $(2^i)_{i=0}^{\infty}$.

Generating Functions of Sequences

Definition 16

The **generating function** of an infinite sequence $(s_i)_{i=0}^{\infty}$ is defined by

$$S(x) = \sum_{i=0}^{\infty} s_i x^i.$$

Example 17

The generating function of the constant sequence $(1)_{i=0}^{\infty}$ is defined by

$$S(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

Generating Functions of Sequences

Definition

The **generating function** of an infinite sequence $(s_i)_{i=0}^{\infty}$ is defined by

$$S(x) = \sum_{i=0}^{\infty} s_i x^i.$$

Questions

- Can the generating function of a sequence always be expressed as a rational function?
- If the answer is Yes, please give a proof.
- If the answer is No, please give a counter-example and derive conditions under which the generating function of a sequence can be expressed as a rational function.

Generating Functions and Linear Recursions

Example 18

Let $(s_i)_{i=0}^{\infty}$ be a sequence defined by $s_i = 5s_{i-1} - 6s_{i-2}, \;\; i \geq 2$, with initial condition $s_0 = 1$ and $s_1 = -2$. Employing this linear recurrence relation,

$$\begin{array}{rcccccccccccc} S(x) & = & s_0 & + & s_1 x & + & s_2 x^2 & + & s_3 x^3 & + & s_4 x^4 & + & \cdots \\ -5xS(x) & = & & - & 5s_0 x & - & 5s_1 x^2 & - & 5s_2 x^3 & - & 5s_3 x^4 & - & \cdots \\ 6x^2 S(x) & = & & & & + & 6s_0 x^2 & + & 6s_1 x^3 & + & 6s_2 x^4 & + & \cdots \end{array}$$

Hence, $(1 - 5x + 6x^2)S(x) = s_0 + (s_1 - 5s_0)x = 1 - 7x$. The generating function is given by $S(x) = \frac{1-7x}{1-5x+6x^2}$.

Question

Do you see any relation between the denominator in the generating function above and the linear recurrence formula of the sequence?

Reciprocals of Polynomials

Definition 19

Let $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial. Its **reciprocal polynomial**, denoted by $a^*(x)$, is defined by

$$a^*(x) = a_n + x_{n-1} x + a_{n-2} x^2 + \cdots + a_0 x^n.$$

Example 20

The reciprocal of $a(x) = 1 + 3x + 2x^5$ is $a^*(x) = 2 + 3x^4 + x^5$.

From Linear Recursions to Generating Functions

Theorem 21

Let $(s_i)_{i=0}^{\infty}$ *be a sequence satisfying the following linear recurrence relation*

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell,$$

where $c_\ell \neq 0$. *Then its generating function is given by* $S(x) = P(x)/Q(x)$, *where*

$$Q(x) = 1 - c_1 x - c_2 x^2 - \cdots - c_\ell x^\ell,$$

which is the reciprocal of the characteristic polynomial of the sequence, and $P(x)$ *is some polynomial of degree less than* ℓ.

Proof.

Define $P(x) = S(x)Q(x)$. It is straightforward to determine $P(x)$ and prove that its degree is at most $\ell - 1$. □

From Generating Functions to Linear Recursions

Theorem 22

Let

$$Q(x) = 1 - c_1 x - c_2 x^2 - \cdots - c_\ell x^\ell,$$

where $c_\ell \neq 0$. Let $P(x)$ be a polynomial of degree less than ℓ. If $(s_i)_{i=0}^{\infty}$ is a sequence with generating function $S(x) = P(x)/Q(x)$, then the sequence must satisfy the following linear recurrence relation

$$s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell.$$

Proof.

The proof is straightforward and left as an exercise. □

Solving Nonlinear Recurrence Relations

Comments

- Most recurrence relations are not linear, and may be very hard to solve.
- However, some of them are solvable. In this case, there is no general approach to solving nonlinear recursions.

Example 23

Solve the recurrence relation $s_i = s_{i-1} + i$ for all $i \geq 1$ with the initial condition $s_0 = 0$.

Example 24

Solve the recurrence relation $s_i = s_{i-1} + i^2$ for all $i \geq 1$ with the initial condition $s_0 = 0$.