# Voronoi and Delaunay Diagrams

Siu-Wing Cheng

Room 3514
Phone: 2358–6973
scheng@cse.ust.hk
http://www.cse.ust.hk/faculty/scheng

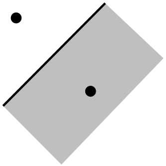
Department of Computer Science and Engineering
The Hong Kong University of Science and Technology
Hong Kong

# Voronoi Diagram

Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.

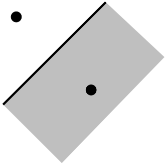
# Voronoi Diagram

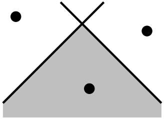
Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.



# Voronoi Diagram

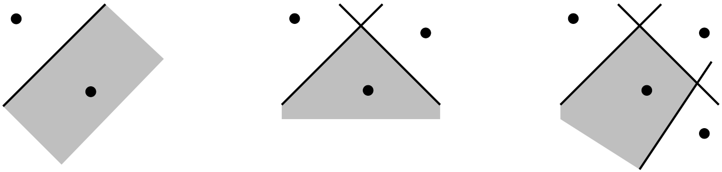
Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.





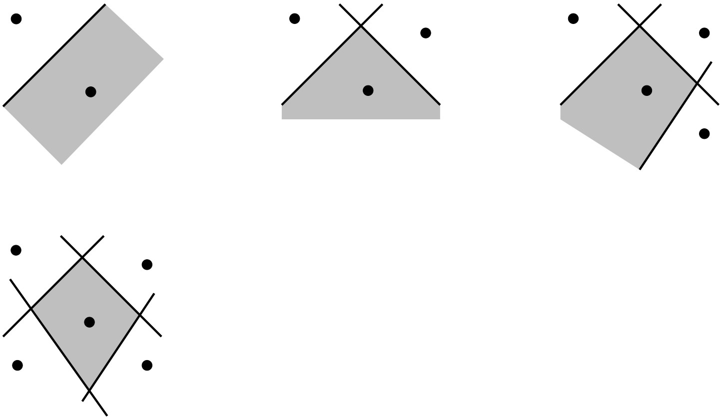
# Voronoi Diagram

Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.

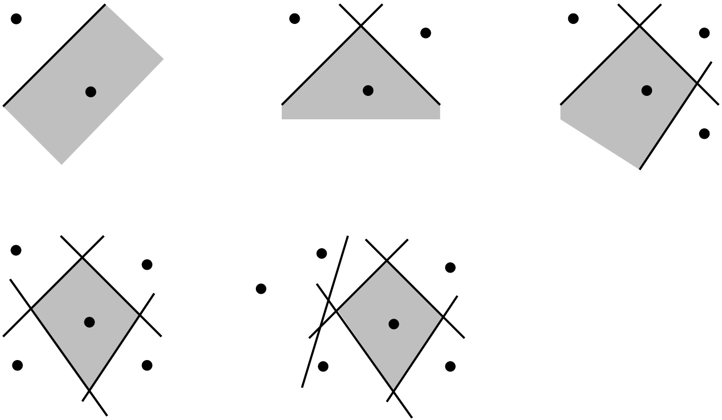


# Voronoi Diagram

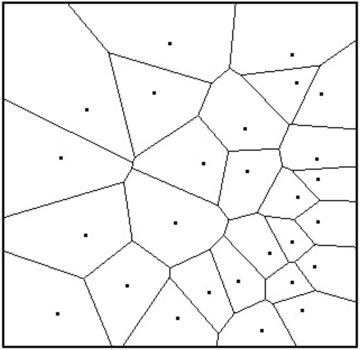
Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.



Given $n$ point sites, partition $\mathbb{R}^2$ into $n$ Voronoi cells so that each cell consists of points closest to one site.

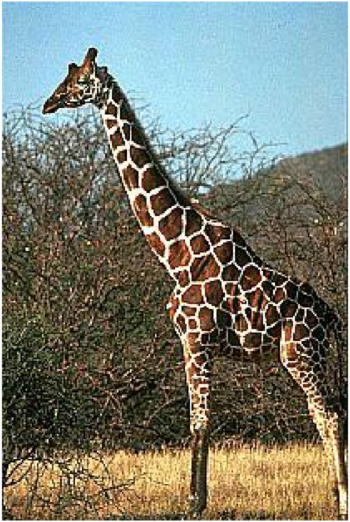


# Examples



www.amath.washington.edu/~dnlennon/voronoi/

# Examples

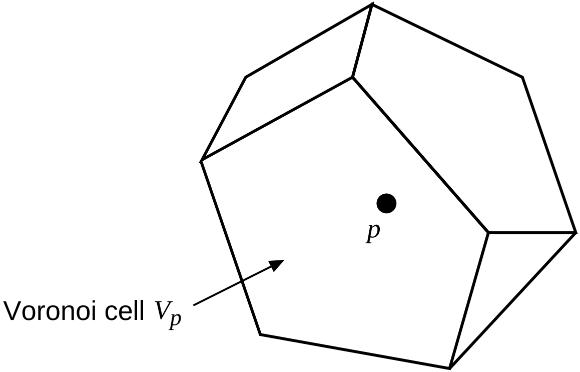


www.amath.washington.edu/~dnlennon/voronoi/

# Examples

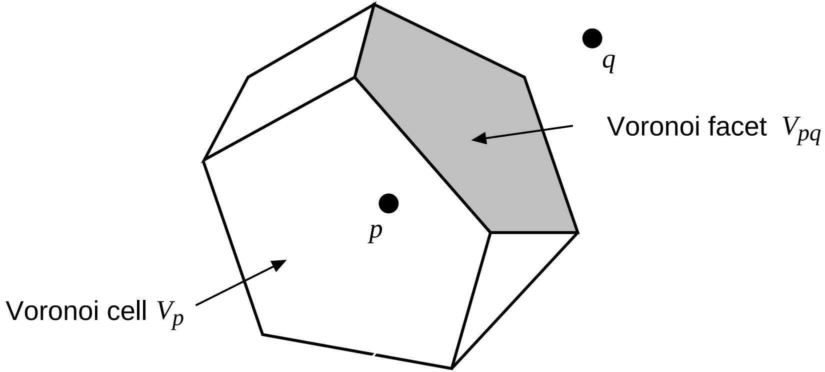
www.amath.washington.edu/~dnlennon/voronoi/

In $\mathbb{R}^3$, the Voronoi cell $V_p$ of a site $p$ is a convex polyhedron.



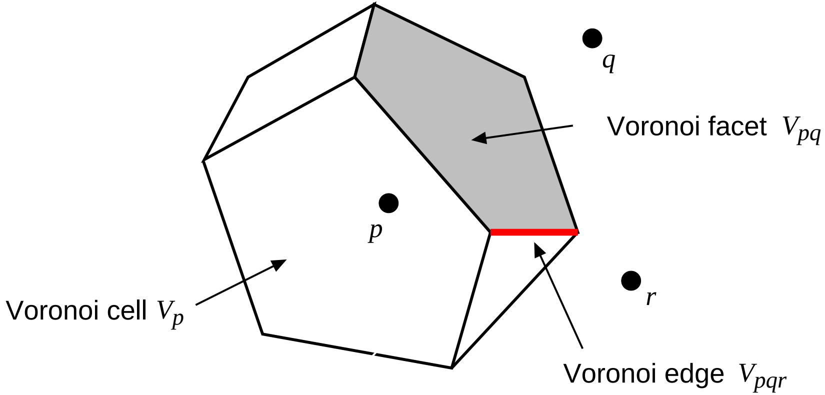
# Voronoi Diagram

In $\mathbb{R}^3$, the Voronoi cell $V_p$ of a site $p$ is a convex polyhedron.



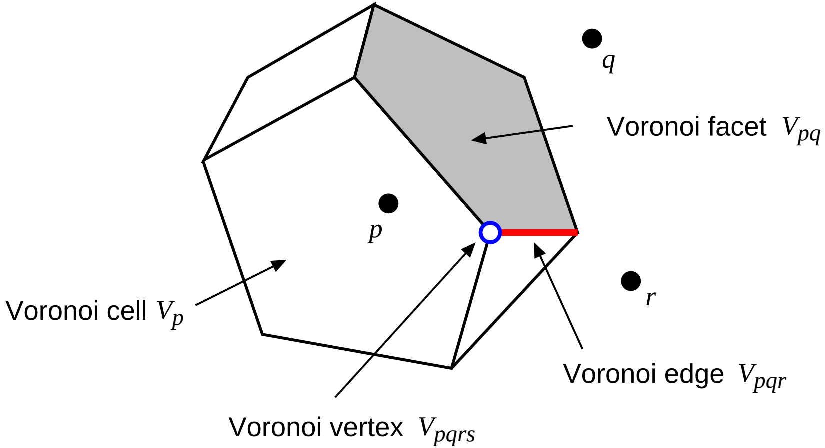
# Voronoi Diagram

In $\mathbb{R}^3$, the Voronoi cell $V_p$ of a site $p$ is a convex polyhedron.

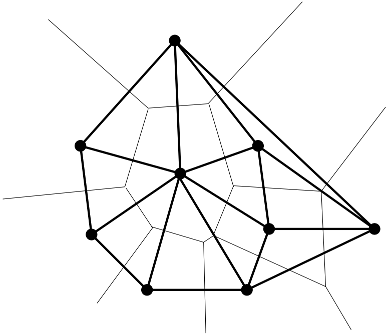


# Voronoi Diagram

In $\mathbb{R}^3$, the Voronoi cell $V_p$ of a site $p$ is a convex polyhedron.



# Delaunay Triangulation



# Delaunay Triangulation in $\mathbb{R}^2$

Dual of the Voronoi diagram:

| | |
|---|---|
| Voronoi edge $V_{pq}$ | Delaunay edge $pq$ |
| Voronoi vertex $V_{pqr}$ | Delaunay triangle $pqr$ |

Equivalent definition: given $k$ vertices, $2 \leq k \leq 3$, they form a Delaunay simplex iff they have an empty circumcircle.

# Delaunay Triangulation in $\mathbb{R}^2$

Dual of the Voronoi diagram:

| | |
|---|---|
| Voronoi edge $V_{pq}$ | Delaunay edge $pq$ |
| Voronoi vertex $V_{pqr}$ | Delaunay triangle $pqr$ |

Equivalent definition: given $k$ vertices, $2 \leq k \leq 3$, they form a Delaunay simplex iff they have an empty circumcircle.

# Delaunay Triangulation in $\mathbb{R}^3$

Dual of the Voronoi diagram:

| | |
|---|---|
| Voronoi facet $V_{pq}$ | Delaunay edge $pq$ |
| Voronoi edge $V_{pqr}$ | Delaunay triangle $pqr$ |
| Voronoi vertex $V_{pqrs}$ | Delaunay tetrahedron $pqrs$ |

Equivalent definition: given $k$ vertices, $2 \leq k \leq 4$, they form a Delaunay simplex iff they have an empty circumsphere.

Always defined, always a valid 3D triangulation, efficient implementation (e.g., CGAL http://www.cgal.org).

# Delaunay Triangulation in $\mathbb{R}^3$

Dual of the Voronoi diagram:

| | |
|---|---|
| Voronoi facet $V_{pq}$ | Delaunay edge $pq$ |
| Voronoi edge $V_{pqr}$ | Delaunay triangle $pqr$ |
| Voronoi vertex $V_{pqrs}$ | Delaunay tetrahedron $pqrs$ |

Equivalent definition: given $k$ vertices, $2 \leq k \leq 4$, they form a Delaunay simplex iff they have an empty circumsphere.

Always defined, always a valid 3D triangulation, efficient implementation (e.g., CGAL http://www.cgal.org).

# Delaunay Triangulation in $\mathbb{R}^3$

Dual of the Voronoi diagram:

| | |
|---|---|
| Voronoi facet $V_{pq}$ | Delaunay edge $pq$ |
| Voronoi edge $V_{pqr}$ | Delaunay triangle $pqr$ |
| Voronoi vertex $V_{pqrs}$ | Delaunay tetrahedron $pqrs$ |

Equivalent definition: given $k$ vertices, $2 \leq k \leq 4$, they form a Delaunay simplex iff they have an empty circumsphere.

Always defined, always a valid 3D triangulation, efficient implementation (e.g., CGAL http://www.cgal.org).

# Restricted Delaunay Triangulation in $\mathbb{R}^2$

- Given a curve $S$ and points on $S$, a Delaunay edge $e$ is restricted Delaunay if $V_e$ intersects $S$.
- The restricted Delaunay triangulation $T$ is the set of vertices and restricted Delaunay edges.

# Restricted Delaunay Triangulation in $\mathbb{R}^3$

- Given a surface $S$ and points on $S$, a Delaunay edge or triangle $\sigma$ is restricted Delaunay if $V_\sigma$ intersects $S$.
- The restricted Delaunay triangulation $T$ is the set of vertices, restricted Delaunay edges, and restricted Delaunay triangles.

# Topological Ball Property

Topological ball property:

- For any triangle $pqr \in T$, $V_{pqr} \cap S$ is a single point.
- For any edge $pq \in T$, $V_{pq} \cap S$ is a single arc.
- For any vertex $p \in T$, $V_p \cap S$ is a topological disk.

**Theorem (Edelsbrunner & Shah)**

*Given the topological ball property, $T$ has the same topology as $S$.*