COMP170
Discrete Mathematical Tools for Computer Science

Lecture 10 *Version 2: Last updated, Oct 25, 2005*

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.1, pp. 127-142

4.1 Mathematical Induction

- Smallest Counterexamples
- The Principle of Mathematical Induction
- Strong Induction
- Induction in General

- We start by reviewing proof by smallest counterexample to try and understand what it is really doing

- This will lead us to transform the indirect proof technique of proof by counterexample to direct proof technique. This direct proof technique will be induction

- We conclude by distinguishing between the weak principle of mathematical induction and the strong principle of mathematical induction

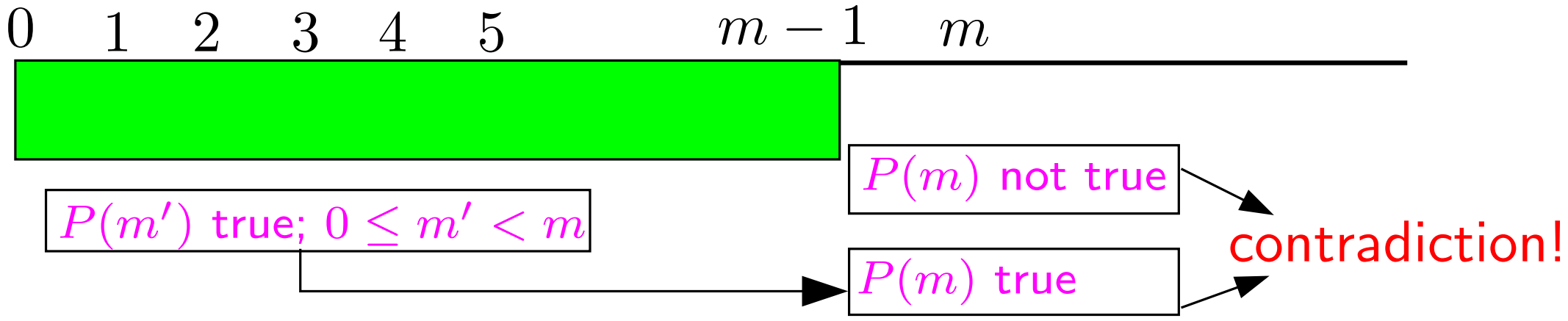
    - Note that the strong principle can actually be derived from the weak principle. The difference between them has less to do with the power of the techniques, than with proof format

Proof by smallest counterexample that
statement $P(n)$ is true for all $n = 0, 1, 2 \ldots$ works by

(i) Assuming that a non-zero counterexample exists, i.e.,
There is some $n > 0$ for which $P(n)$ is not true

(ii) Letting $m \geq 0$ be *smallest* value for which $P(m)$ is not true

(iii) Then use fact that $P(m')$ is true for all $0 \leq m' < m$
to show that $P(m)$ is true,
contradicting original choice of m.
$\Rightarrow P(n)$ true for **all** $n = 0, 1, 2, \ldots$



Example 1:

Use proof by s.c. to show that, $\forall n \in N$, (non-negative ints)

$$(*) \quad 0 + 1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}.$$

- Suppose (*) is not always true
- Then there must be a smallest $n \in N$ such that (*) does not hold for n
- Then, for any positive integer $i \leq n$,
$$1 + 2 + \ldots + i = \frac{i(i+1)}{2}.$$
- Because $0 = 0 \cdot 1/2$, (*) holds when $n = 0$.
- Therefore, the smallest counterexample n is larger than 0.

- So, (i) smallest counterexample n is greater than 0, and
  (ii) (*) holds for $n-1$

- Substituting $n-1$ for i gives
$$1+2+\ldots+n-1=\frac{(n-1)n}{2}.$$

- Adding n to both sides gives
$$1+2+\ldots+n-1+n=\frac{(n-1)n}{2}+n$$
$$=\frac{n(n+1)}{2}.$$

- Thus, n is not a counterexample after all.

- Therefore, there is no counterexample for (*).

- Hence, (*) holds for all positive integers n.

What implication did we have to prove?

The crucial step was proving that

$$p(n-1) \Rightarrow p(n)$$

where $p(n)$ is the statement $1+2+\ldots+n = \frac{n(n+1)}{2}$.

Example 2:

Use proof by s.c. to show that, $\forall n \in N$,

$$2^{n+1} \geq n^2 + 2.$$

Let $p(n)$ be the statement $\quad 2^{n+1} \geq n^2 + 2.$

We start by assuming that the statement

$$\forall n \in N \; p(n)$$

is false.

When a for all statement is false
there must be some n for which it is false.

Let n be the smallest nonnegative integer
for which $2^{n+1} \not\geq n^2 + 2$.

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for which $2^{n+1} \not\geq n^2 + 2$.

This means that, for all $i \in N$ with $i < n$,
$2^{i+1} \geq i^2 + 2$

Since $2^{0+1} \geq 0^2 + 2$ we know that $n > 0$
so $n - 1$ is a nonnegative integer less than n.

Thus, setting $i = n - 1$ gives

$$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

$$\text{or } 2^n \geq n^2 - 2n + 1 + 2$$

$$= n^2 - 2n + 3. \qquad (*)$$

(*) will let us draw a contradiction to $2^{n+1} \not\geq n^2 + 2$.

We are given $2^n \geq n^2 - 2n + 3.$ (*)

To derive contradiction to $2^{n+1} \not\geq n^2 + 2$,
we want to convert left side of (*) to 2^{n+1}.

Multiply both sides by 2, giving
$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot (n^2 - 2n + 3) = 2n^2 - 4n + 6.$

What next?

To get a contradiction, we want to convert the right side into
$n^2 + 2$ plus an additional nonnegative term.

Thus, we write
$$\begin{aligned} 2^{n+1} &\geq 2n^2 - 4n + 6 \\ &= (n^2 + 2) + (n^2 - 4n + 4) \\ &= n^2 + 2 + (n-2)^2 \\ &\geq n^2 + 2. \qquad (**) \end{aligned}$$

contradiction!

Let $p(n)$ be the statement $2^{n+1} \geq n^2 + 2$.

We just showed that

(a) $p(0)$ is True and (b) If $n > 0$ then $p(n-1) \Rightarrow p(n)$

(*)
- Suppose there is some n for which $p(n)$ is False
- Let n be the smallest counterexample
  i.e., the smallest value for which $p(n)$ is False.
- Then, from (a),$n > 0$, so $p(n-1)$ is True.
- Therefore, from (b), using direct inference, $p(n)$ is True.
- This contradicts (*).
- Thus, $p(n)$ is True for all $n \in N$.

What did we really do?

Let $p(n)$ be the statement $2^{n+1} \geq n^2 + 2$. We showed that

(a) $p(0)$ is True and (b) If $n > 0$ then $p(n-1) \Rightarrow p(n)$

We then used Proof by smallest counterexample
to derive that $p(n)$ is True for all $n \in N$.

This is an indirect proof.
Is it possible to prove this fact directly?

Since $p(n-1) \Rightarrow p(n)$, we see that
$p(0)$ implies $p(1)$, $p(1)$ implies $p(2)$, $p(2)$ implies $p(3)$, ...

This should permit us to directly derive $p(n)$ for every n!

4.1 Mathematical Induction

- Smallest Counterexamples
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The Principle of Mathematical Induction

The **well-ordering** principle permits us to assume that every set of nonnegative integers has a smallest element, allowing us use the technique smallest counterexample.

This is actually equivalent to the
principle of **mathematical induction**

Principle 4.1 (The Weak Principle of Mathematical Induction)
(a) If the statement $p(b)$ is True, and
(b) the statement $p(n-1) \Rightarrow p(n)$ is True for all $n > b$,
then $p(n)$ is True for all integers $n \geq b$.

Proof by induction that $\forall n \geq 0,\ 2^{n+1} \geq n^2 + 2$:

Let $p(n)$ be statement that $2^{n+1} \geq n^2 + 2$

(i) Note that for $n = 0$, $2^{0+1} = 2 \geq 2 = 0^2 + 2$. $p(0)$

(ii) Suppose that $n > 0$ and that (*) $2^n \geq (n-1)^2 + 2$.

Then $2^{n+1} = 2 \cdot 2^n \geq 2(n-1)^2 + 4$

$$= (n^2 + 2) + (n^2 - 4n + 4)$$

$$= n^2 + 2 + (n-2)^2$$

$$\geq n^2 + 2.$$

We have therefore just proven that, for $n > 0$, $p(n-1) \Rightarrow p(n)$

so, by mathematical induction, $\forall n > 0,\ 2^{n+1} \geq n^2 + 2$

Proof by induction that $\forall n \geq 2,\ 2^{n+1} > n^2 + 3$:

(i) Note that for $n = 2$, $2^{2+1} = 8 > 7 = 2^2 + 3$.

(ii) Suppose that $n > 2$ and that (*) $2^n > (n-1)^2 + 3$.

Multiply both sides of (*) by 2, giving

$$\begin{aligned} (**)\quad 2^{n+1} &> 2(n^2 - 2n + 1) + 6. \\ &= n^2 + 3 + n^2 - 4n + 4 + 1 \\ &= n^2 + 3 + (n-2)^2 + 1 \qquad > n^2 + 3 \end{aligned}$$

Therefore, $\Big(2^n > (n-1)^2 + 3\Big) \Rightarrow \Big(2^{n+1} > n^2 + 3\Big)$.

Then, by the principle of mathematical induction,

$\forall n \geq 2,\ \ 2^{n+1} > n^2 + 3$

(i) Note that for $n = 2$, $2^{2+1} = 8 > 7 = 2^2 + 3$.

is the **Base Case**

It consisted of proving that $p(b)$ is True,
where in this case $p(n)$ is $2^{n+1} > n^2 + 3$, and $b = 2$

(ii) Suppose that $n > 2$ and that (*) $2^n > (n-1)^2 + 3$.

is the **Inductive Hypothesis**.

This is the assumption that $p(n-1)$ is True.

The implication $p(n-1) \Rightarrow p(n)$

was then proven in the **Inductive Step**.

The final sentence of the proof is called the **Inductive Conclusion**.

Example: Use mathematical induction to show that

$$\forall k \in Z^+,\ 1+3+5+\ldots+(2k-1) = k^2$$

i) Base Case: The formula holds when $k = 1$.

ii) Inductive Hypothesis:
Assume that formula holds when $k = n - 1$, i.e.,

$$(*)\quad 1 + 3 + \ldots + (2n-3) = (n-1)^2$$

Adding $2n - 1$ to both sides of (*) gives

$$1 + 3 + \ldots + (2n-3) + (2n-1) = n^2 - 2n + 1 + 2n - 1 = n^2$$

Let $p(n)$ be that $1 + 3 + 5 + \ldots + (2k-1) = k^2$.
We have just proven the inductive step that $p(n-1) \Rightarrow p(n)$

So by the principle of mathematical induction,
we see the formula holds for all $k \in Z^+$.

Example 2: For what values of n is $2^n > n^2$?

Note that $2 = 2^1 > 1^2 = 1$;
the inequality then fails for $n = 2, 3, 4$.
However, for $n = 5$, $32 > 25$.

Assume inductively that for $n > 5$, we have

$$2^{n-1} > (n-1)^2$$

Multiplying both sides by 2 gives

$$\begin{aligned} 2^n &> 2(n^2 - 2n + 1) \\ &= n^2 + n^2 - 4n + 2 \\ &> n^2 + n^2 - n \cdot n \\ &= n^2. \end{aligned}$$

since $n > 5$ implies $-4n > -n \cdot n$

Thus, by the principle of mathematical induction,
$2^n > n^2$ for all $n \geq 5$.

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Strong Induction

Recall that when we used Proof by smallest counterexample in Euclid's Division Theorem we actually

> chose a smallest counterexample m to the EDT property $p(n)$, and observed that $m - n$ was a non-negative integer less than m. Therefore $p(m - n)$ had to be true. This in turn implied that $p(m)$ was true, yielding a contradiction.

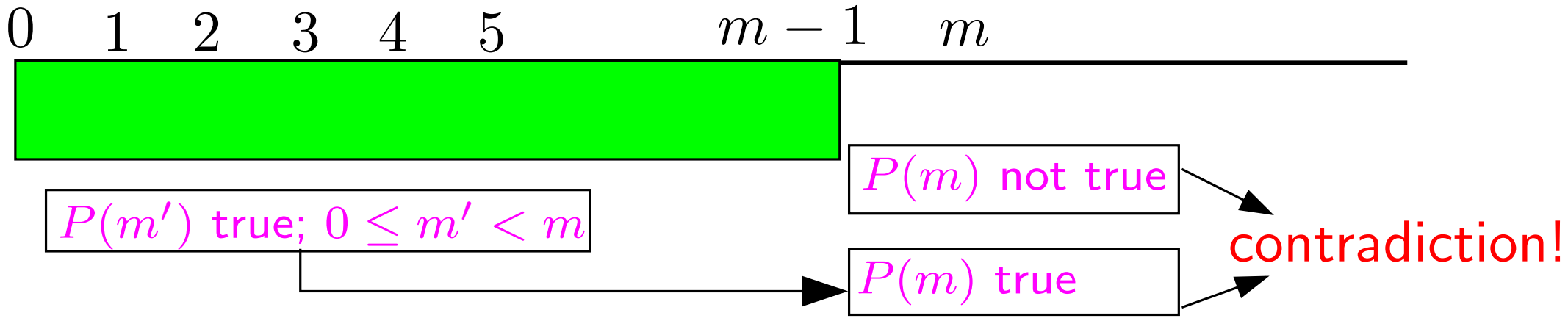
Note that the contradiction came from $p(m - n)$ and not from $p(m - 1)$. We strongly used the fact that $p(i)$ was True for all $i < m$ and not just for $i = m - 1$.

Proof by smallest counterexample that
statement $P(n)$ is true for all $n = 0, 1, 2 \ldots$ works by

(i) Assuming that a non-zero counterexample exists, i.e.,
There is some $n > 0$ for which $P(n)$ is not true

(ii) Letting $m \geq 0$ be *smallest* value for which $P(m)$ is not true

(iii) Then use fact that $P(m')$ is true for all $0 \leq m' < m$
to show that $P(m)$ is true,
contradicting original choice of m.
$\Rightarrow P(n)$ true for **all** $n = 0, 1, 2, \ldots$



The essence of our method for proving Euclid's division theorem was:

1. We have a statement $q(k)$ that we want to prove for all k larger than some integer.

2. We suppose this is not true; so, there is a smallest k for which $q(k)$ is false.

3. This means we may assume $q(k')$ is true for *all* $k' \leq k$.

4. We then use this assumption to derive a proof of $q(k)$, thus generating our contradiction.

We can turn this into a direct proof as follows

- First suppose that we have a proof of $q(0)$.
- Next suppose that we have a proof that, $\forall k > 0$,

$$q(0) \wedge q(1) \wedge q(2) \wedge \ldots \wedge q(k-1) \Rightarrow q(k)$$

- Then, $q(0)$ implies $q(1)$;
  $q(0) \wedge q(1)$ implies $q(2)$;
  $q(0) \wedge q(1) \wedge q(2)$ implies $q(3)$; $\ldots$
- Iterating gives us a proof of $q(n)$ for every n
- This is another form of the
  principle of mathematical induction.

Principle 4.2

(The Strong Principle of Mathematical Induction)

(a) If the statement $p(b)$ is True and

(b) for all $n > b$, the statement

$$p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$$

is True

then $p(n)$ is True for all integers $n \geq b$.

Example: Prove that every positive integer is a power of a prime number or the product of powers of prime numbers.

- Base Case: 1 is a power of a prime number, e.g., $1 = 2^0$.
- Inductive hypothesis: Suppose that we know that every number less than n is a power of a prime number or a product of powers of prime numbers.
- Then, if n is not a prime number, it is a product of two smaller numbers, each of which is, by the induction hypothesis, a power of a prime number or a product of powers of prime numbers.
- n is therefore a power of a prime number or the product of powers of prime numbers
- Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime number or a product of powers of prime numbers.

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We have just seen

The Weak Principle of Mathematical Induction

and

The Strong Principle of Mathematical Induction

Note that, in practice, we do not usually explicitly distinguish between the weak and strong forms.

In reality, they are equivalent to each other in that the weak form is a special case of the strong form and the strong form can be derived from the weak form.

Induction – A summary

A typical proof by mathematical induction, showing that a statement $p(n)$ is true for all integers $n \geq b$ consists of three steps:

1. We show that $p(b)$ is True.
This is called **establishing a base case**.

2.We then, $\forall n > b$, show either

(*) $$p(n-1) \Rightarrow p(n)$$

or

(**) $$p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$$

We need to make the inductive hypothesis of
either $p(n-1)$ or $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1)$.

We then use either (*) or (**) to derive $p(n)$

3. We conclude on the basis of the principle of mathematical induction that $p(n)$ is true for all integers $n \geq b$

The second step, proving (*) or (**), is the real core of an inductive proof.

This is usually where hard work, creativity and insights are most needed