COMP170
Discrete Mathematical Tools for Computer Science

Intro to Graphs

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 6.1, pp. 309-320

Graphs

- Basic Definitions
- The Degree of a Vertex
- Connectivity
- Cycles
- Trees

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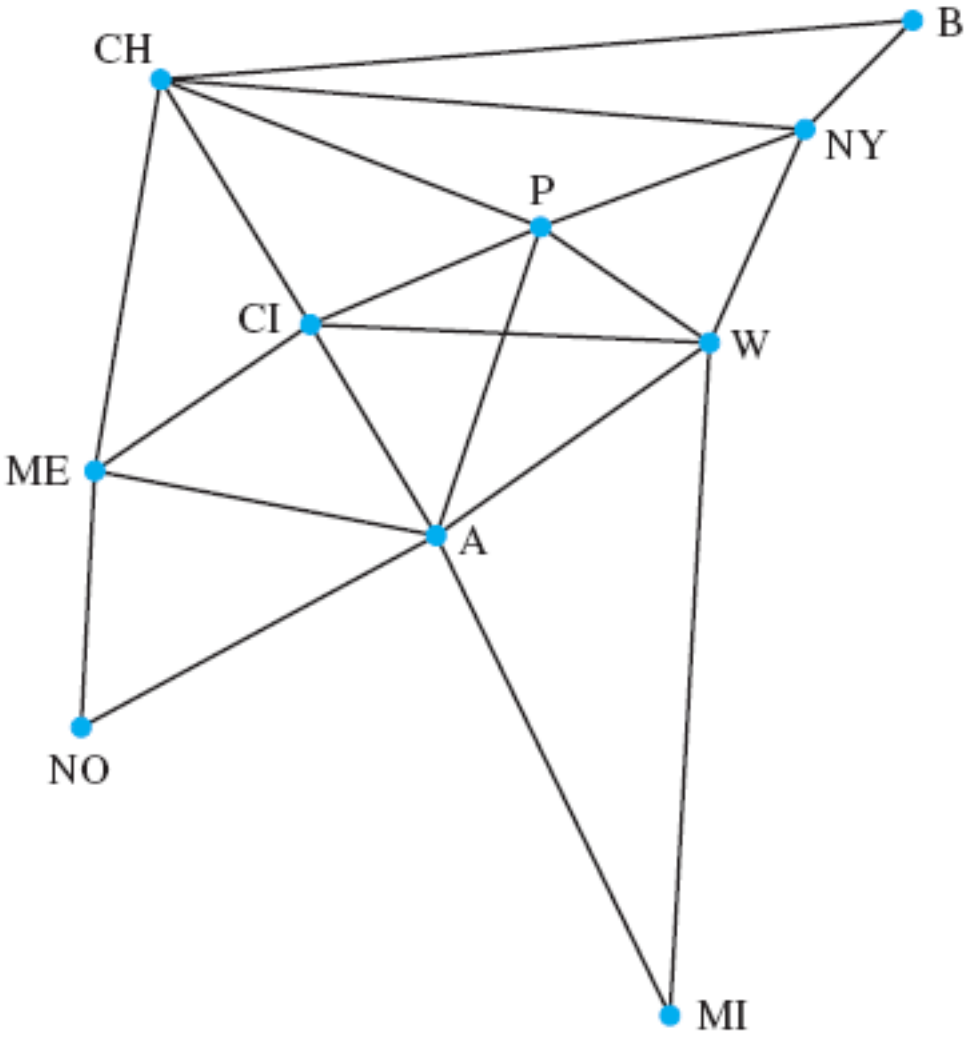
Example

Map of some cities in eastern US.

with communication lines existing
between certain pairs of these cities.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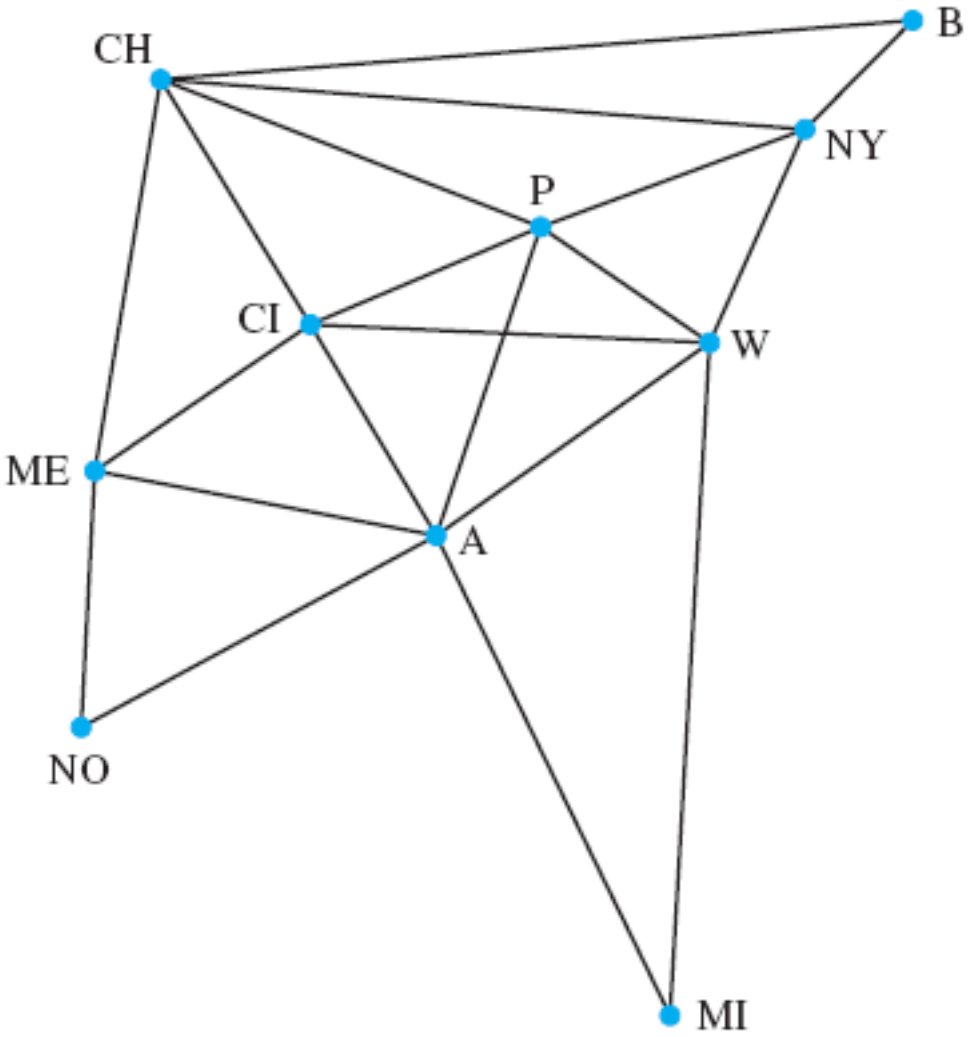

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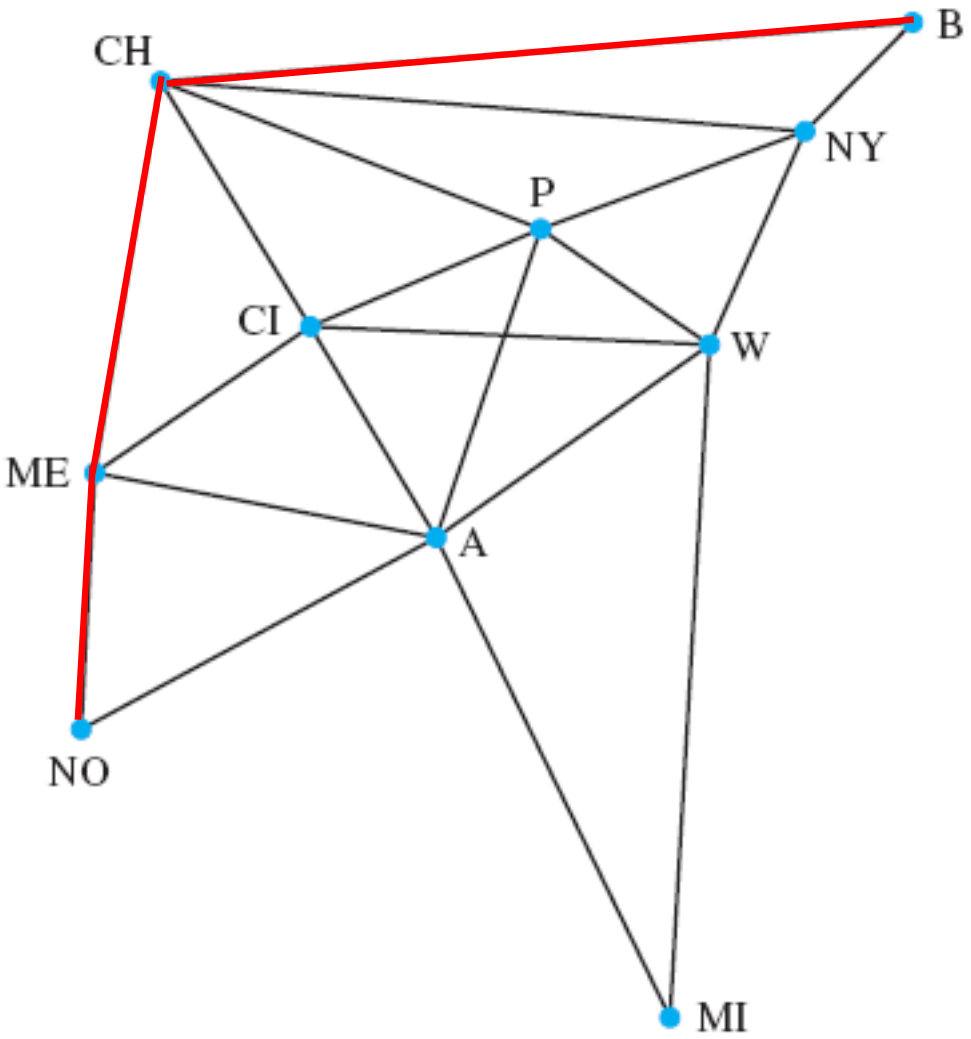

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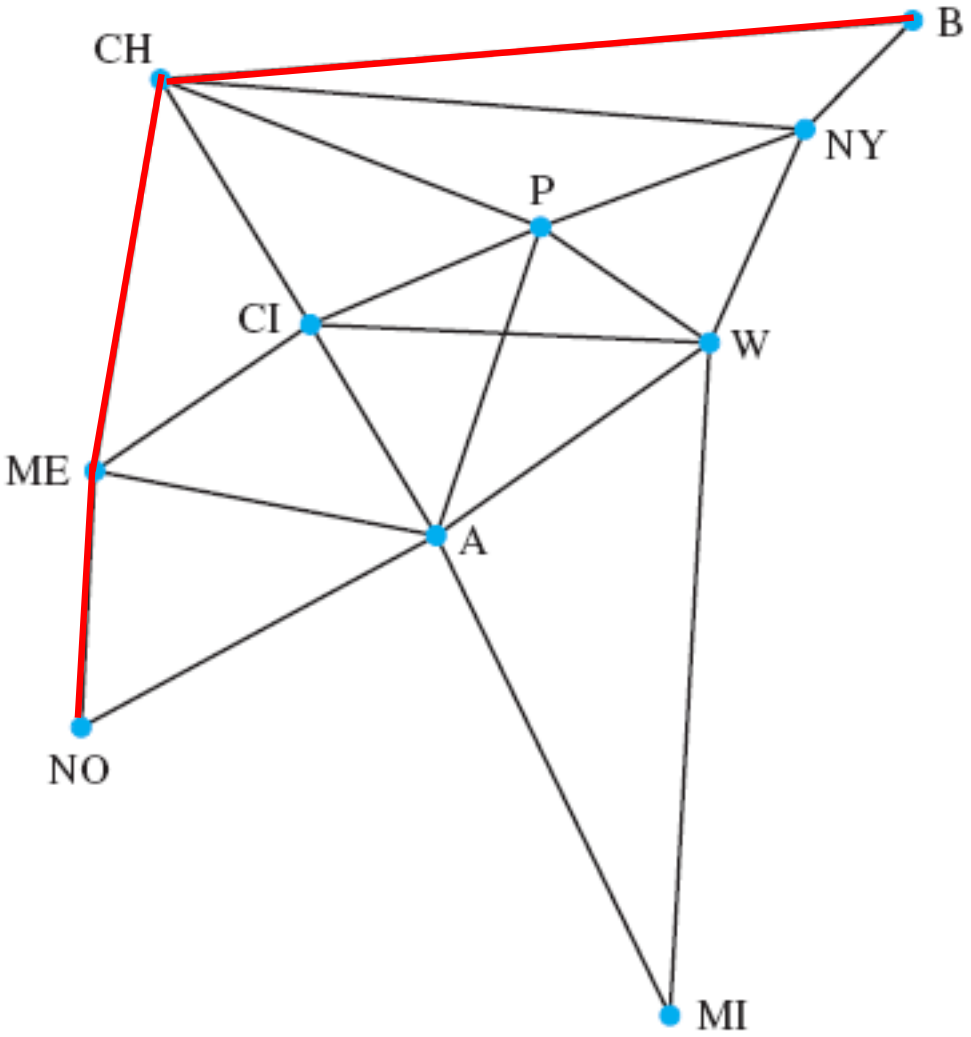

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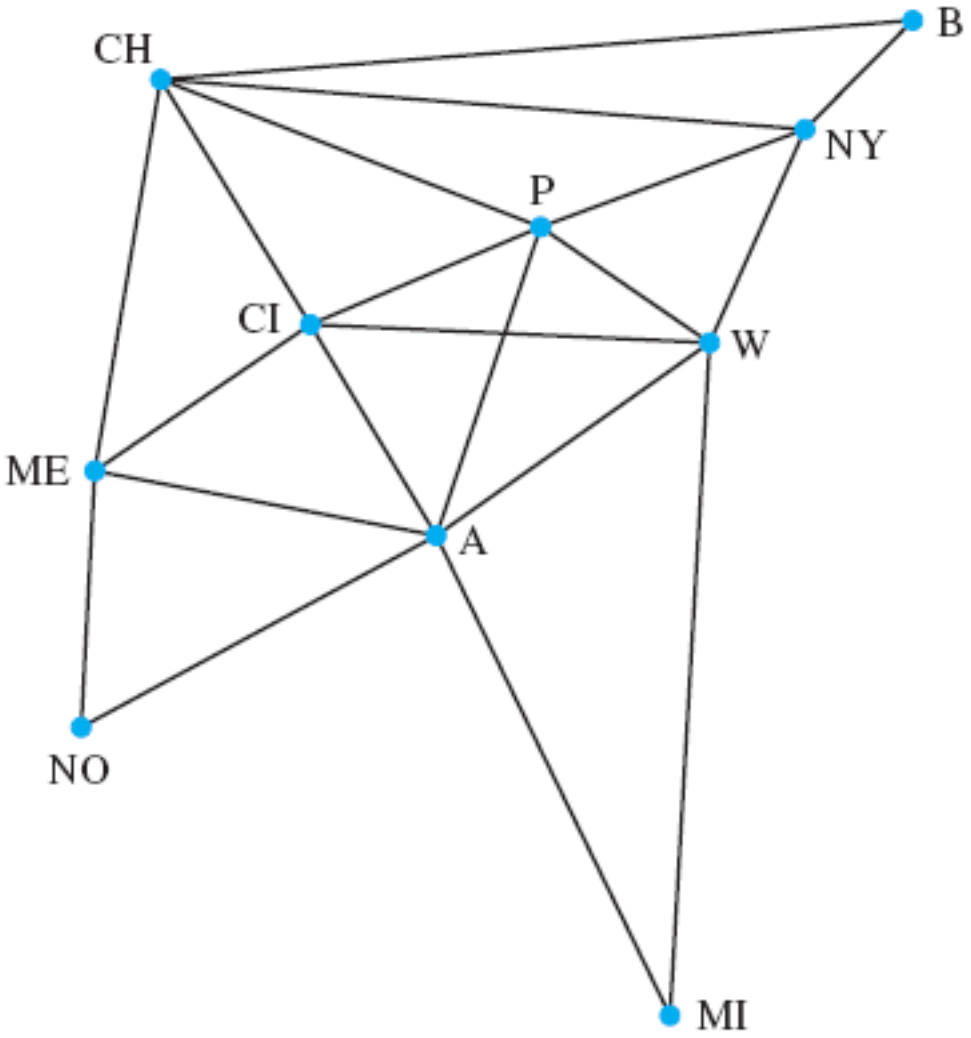

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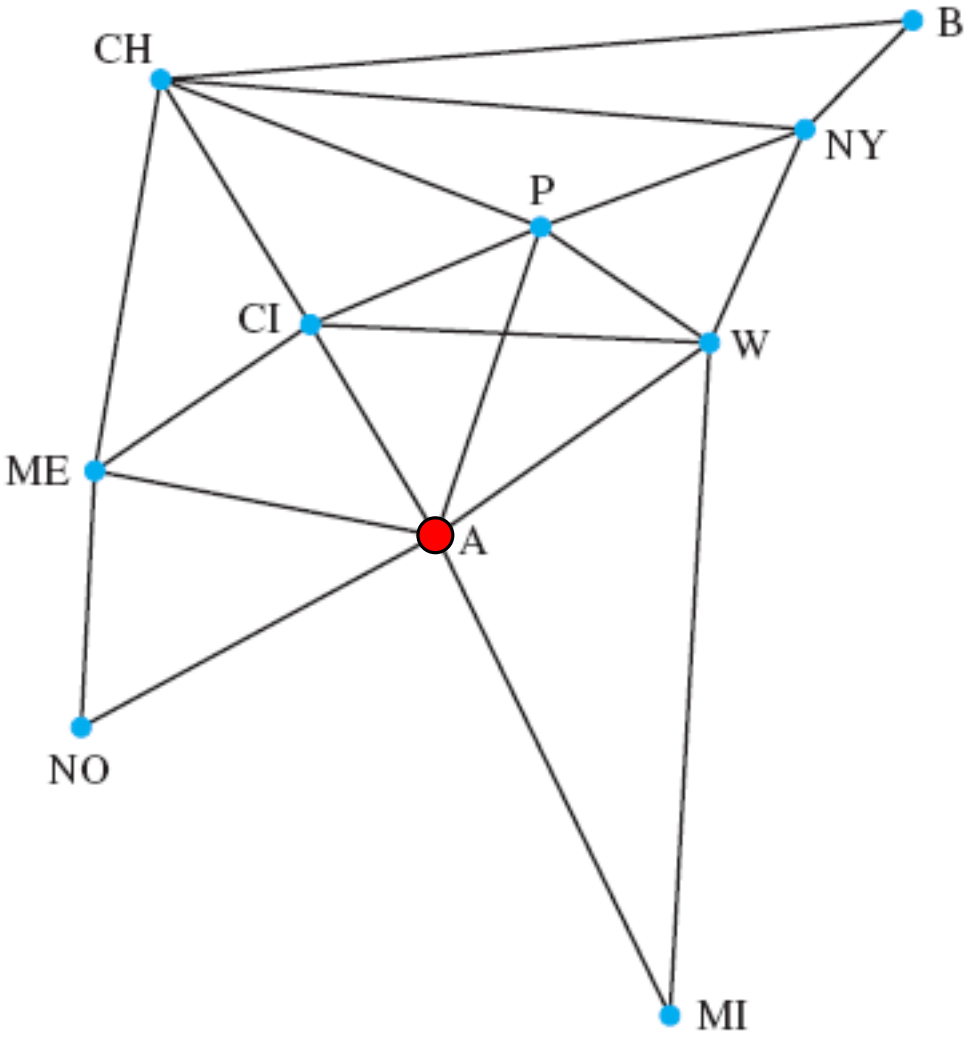

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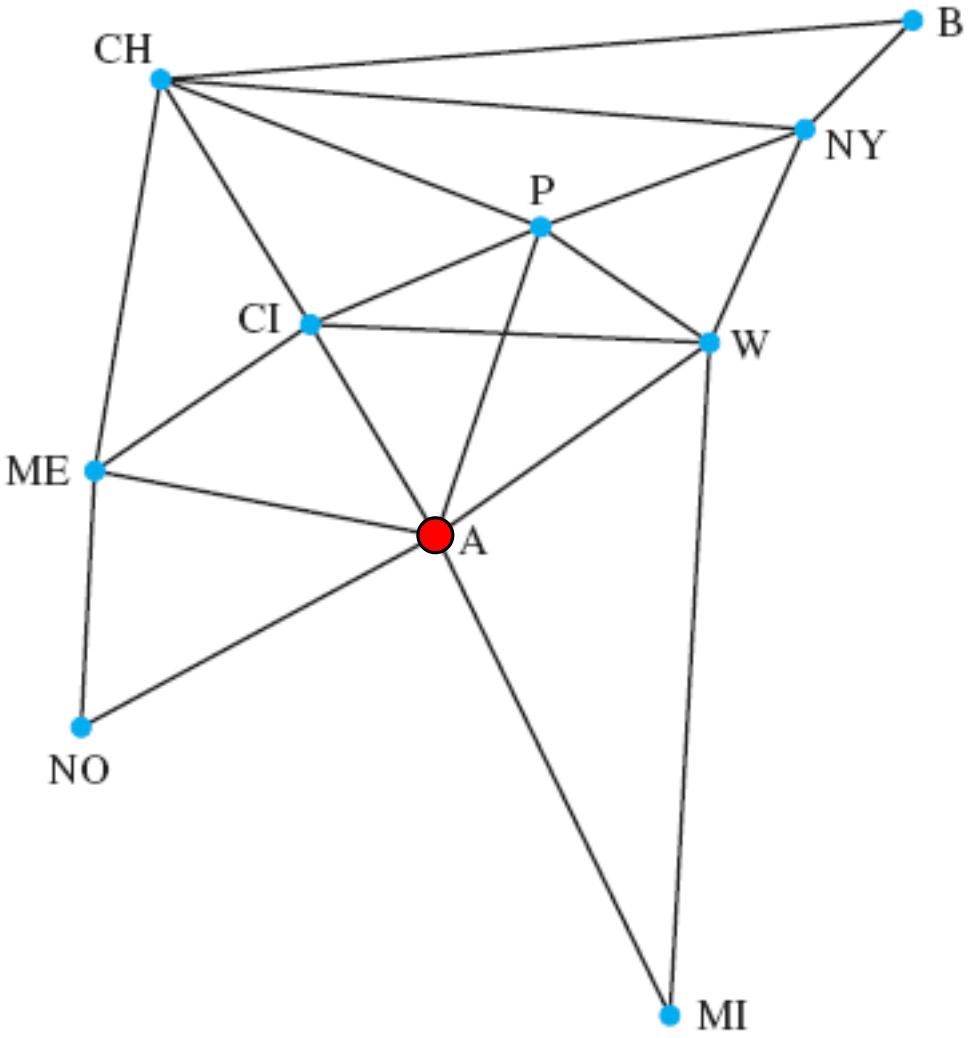

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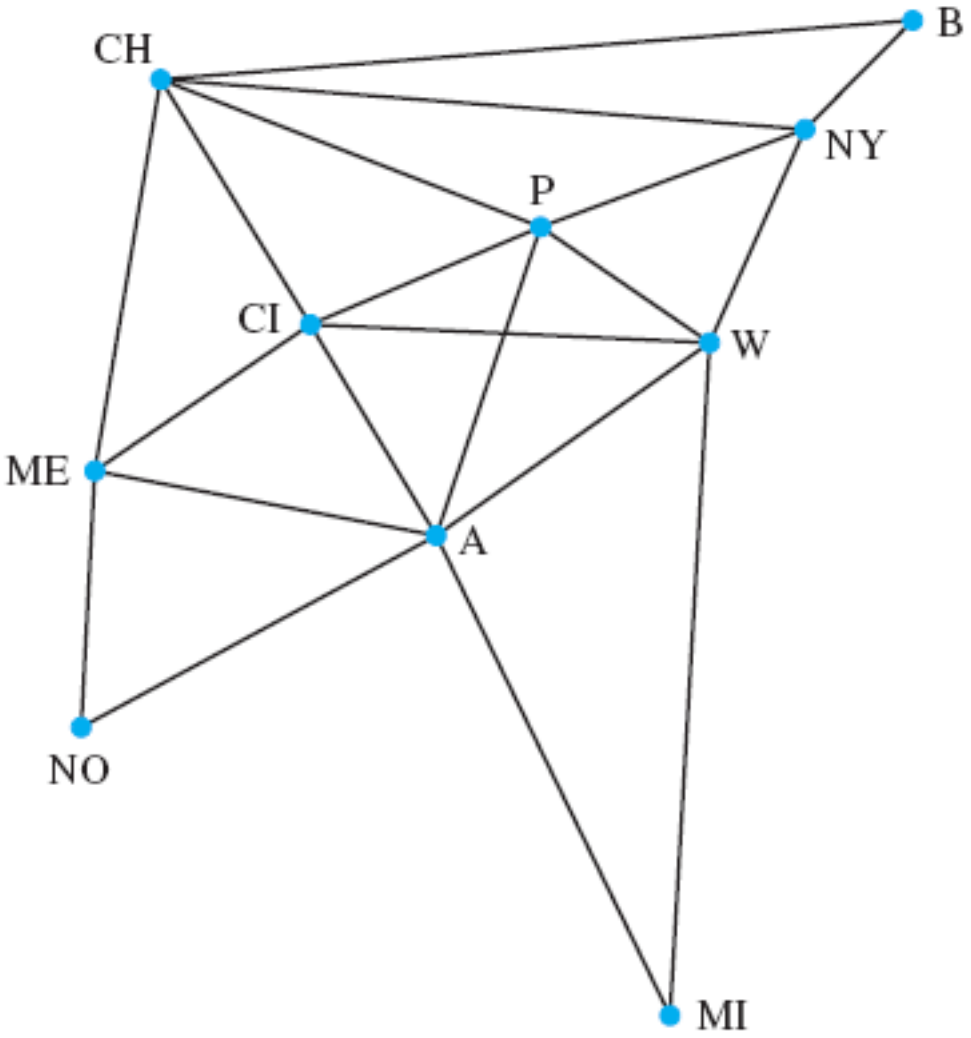

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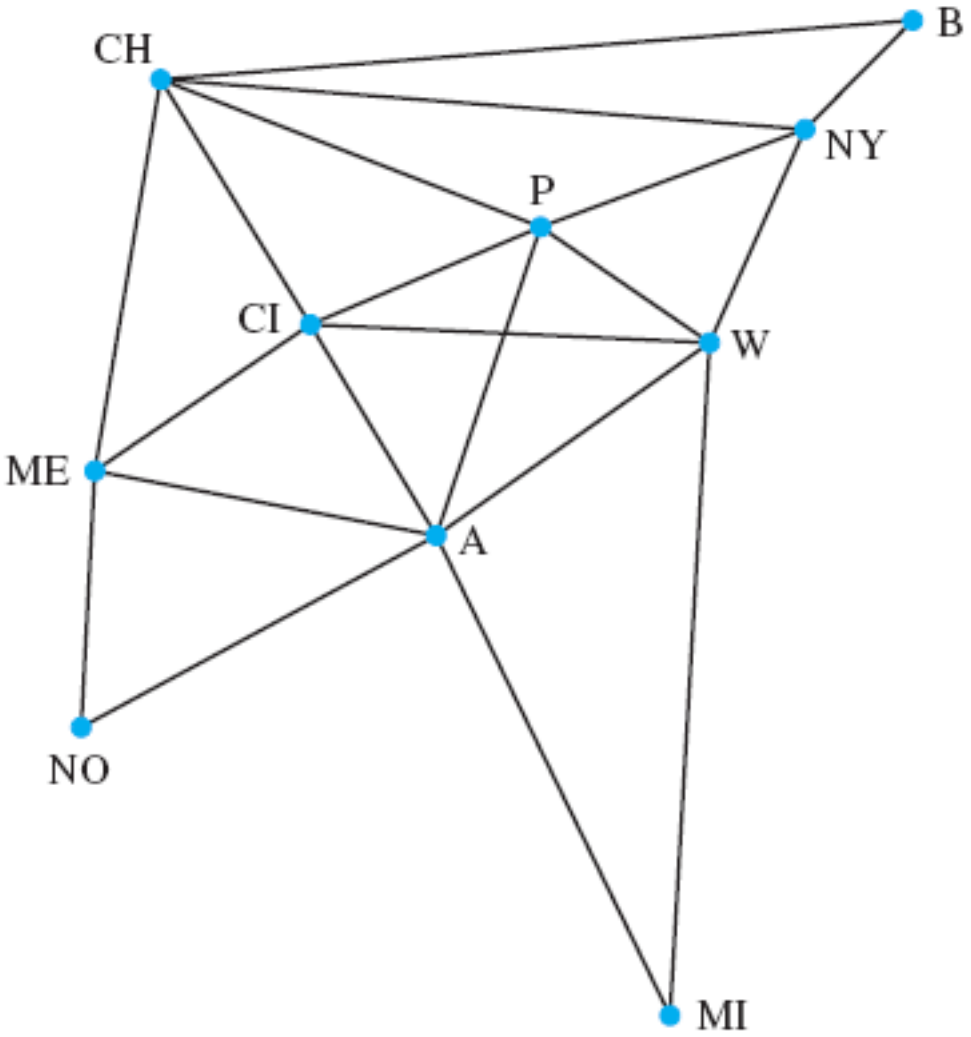

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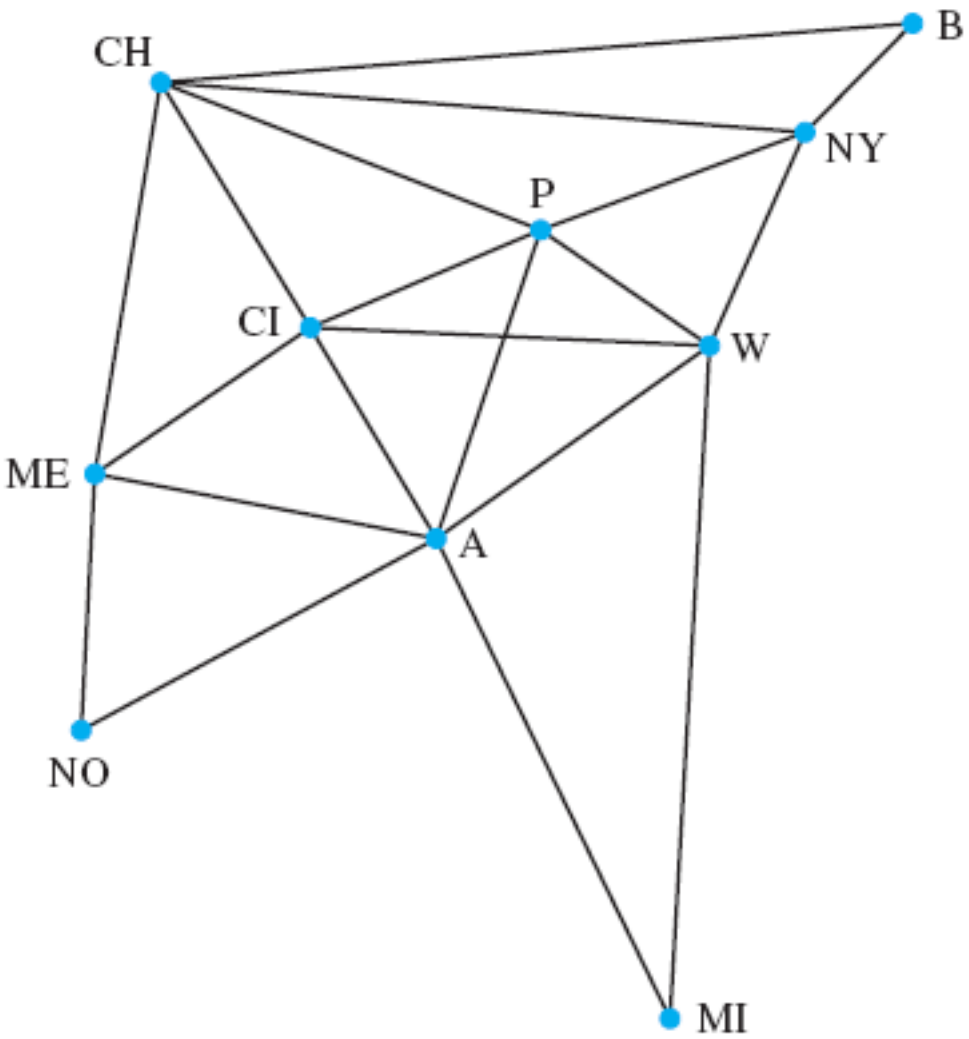

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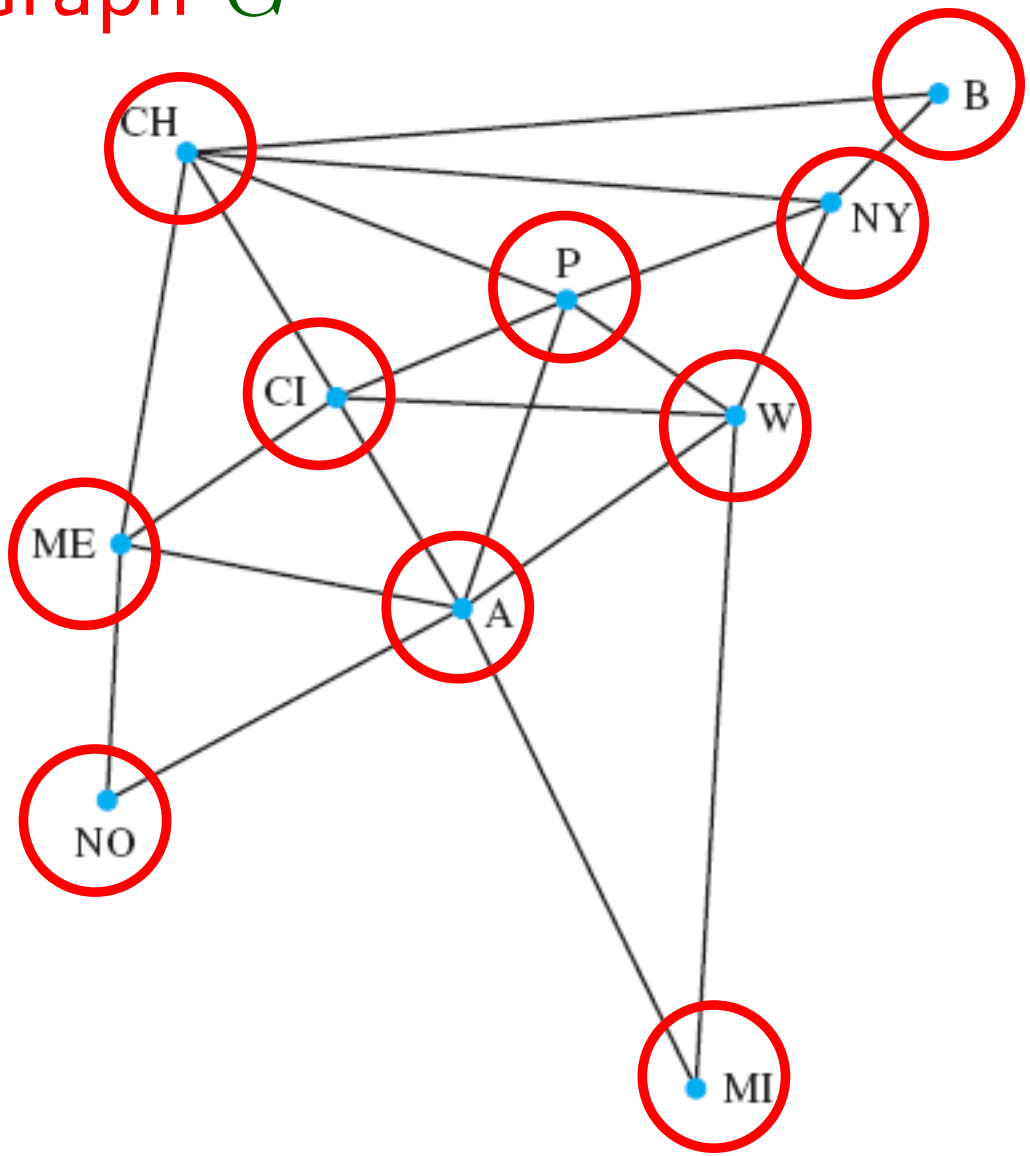

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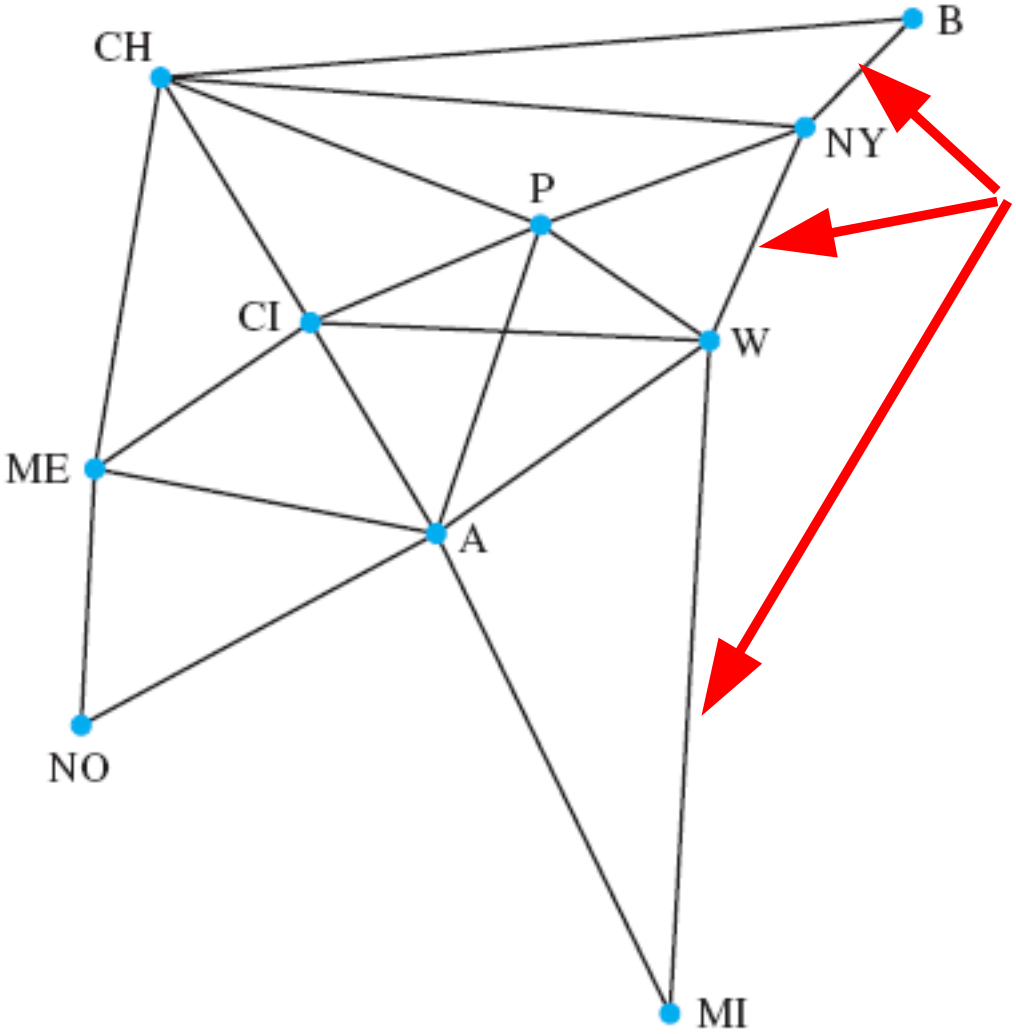

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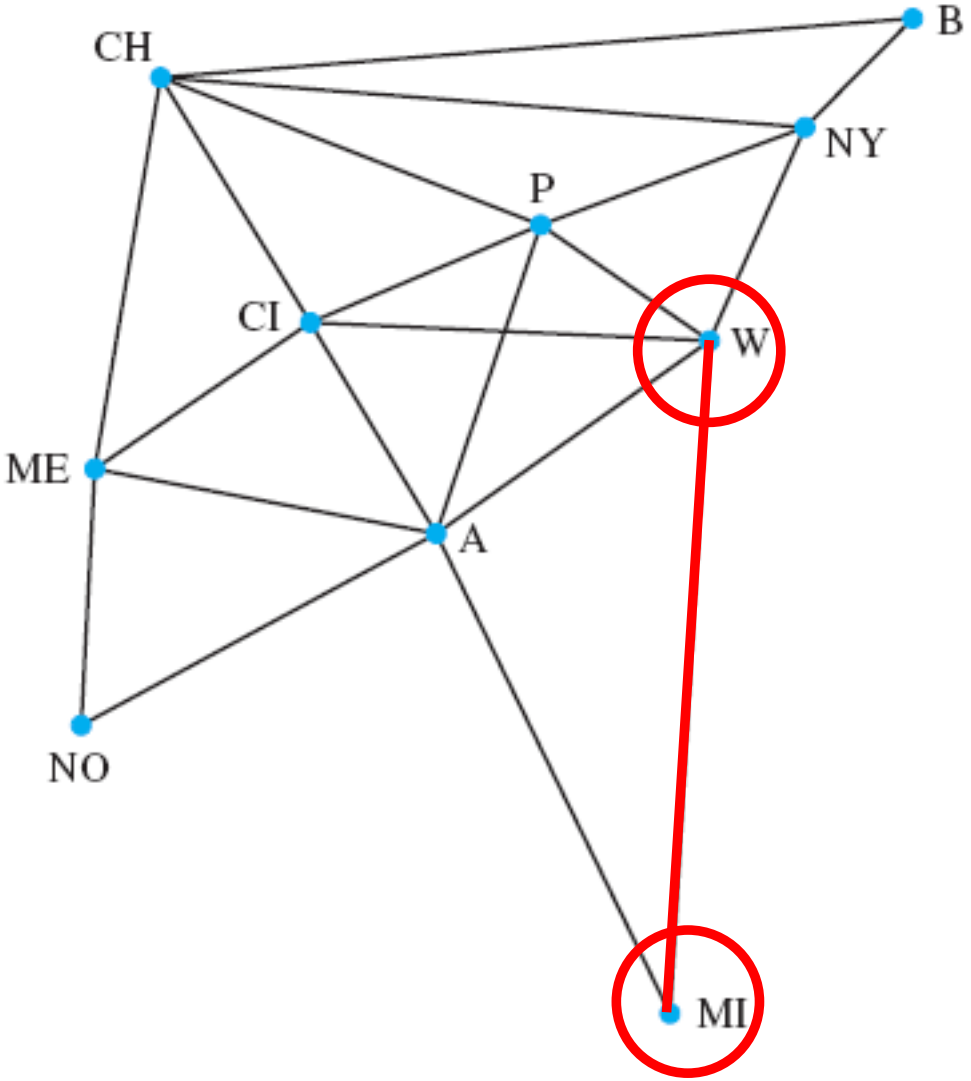

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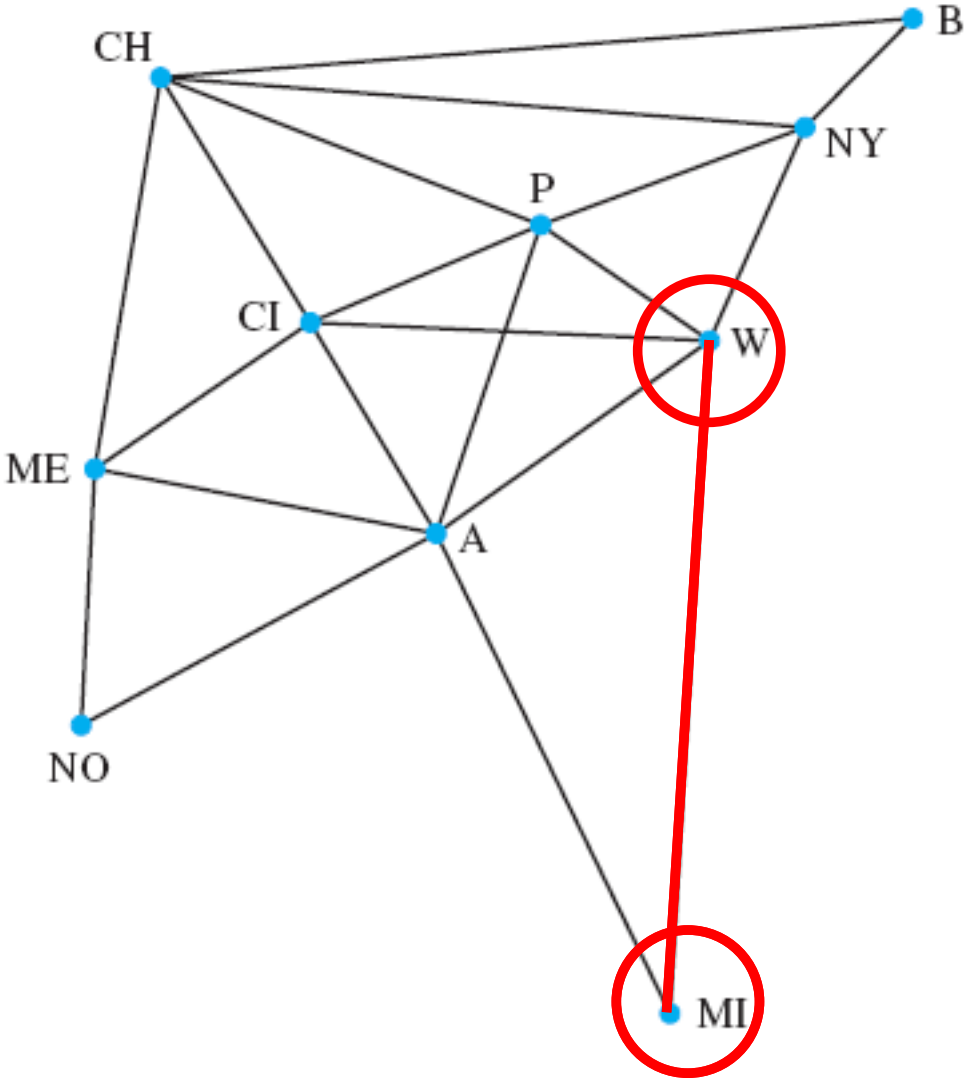

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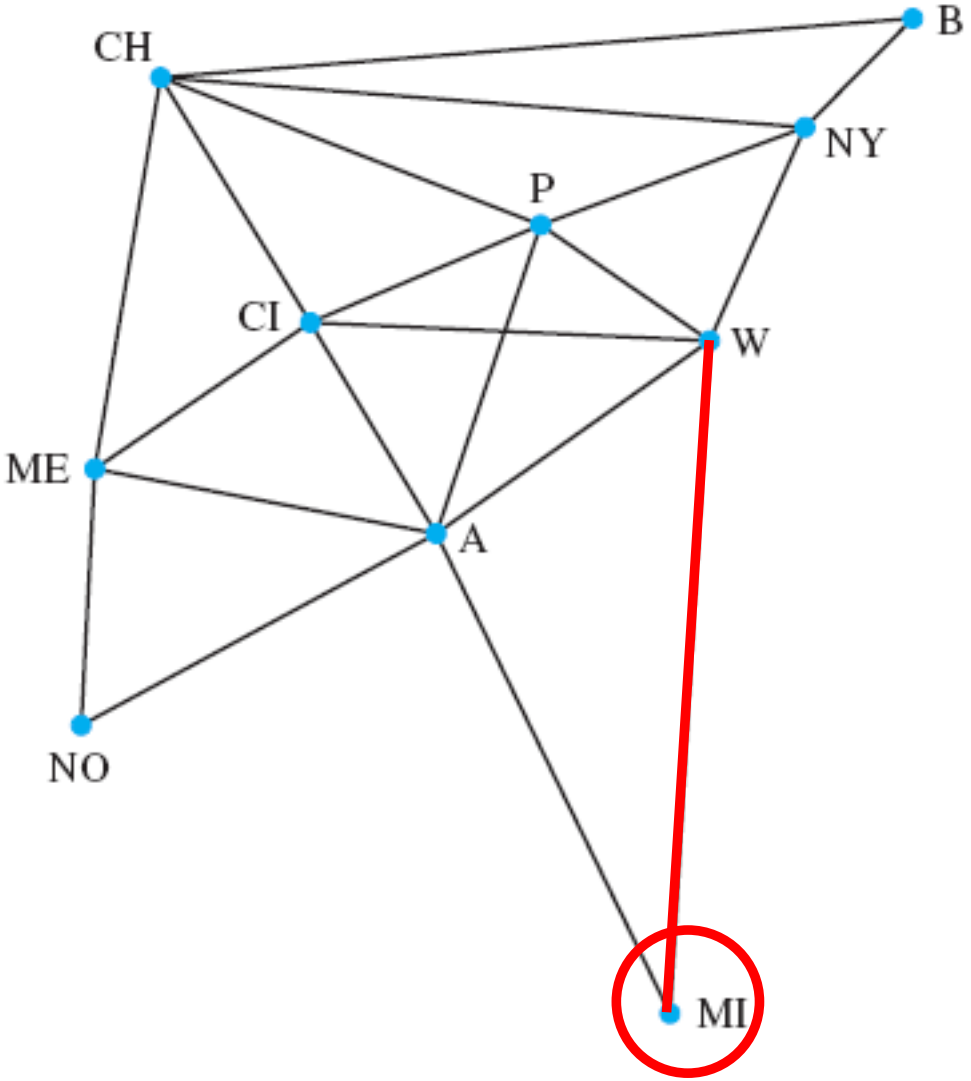

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When a vertex is an endpoint of an edge, we say that the edge and the vertex are **incident** to each other.

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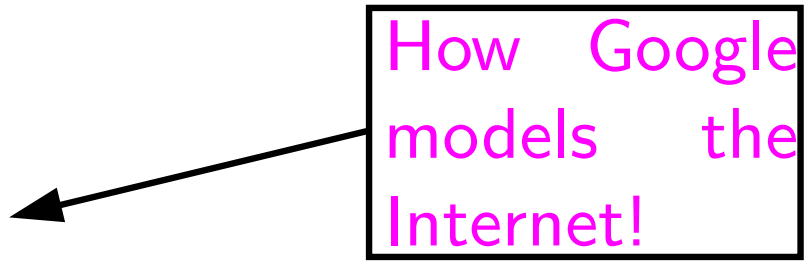

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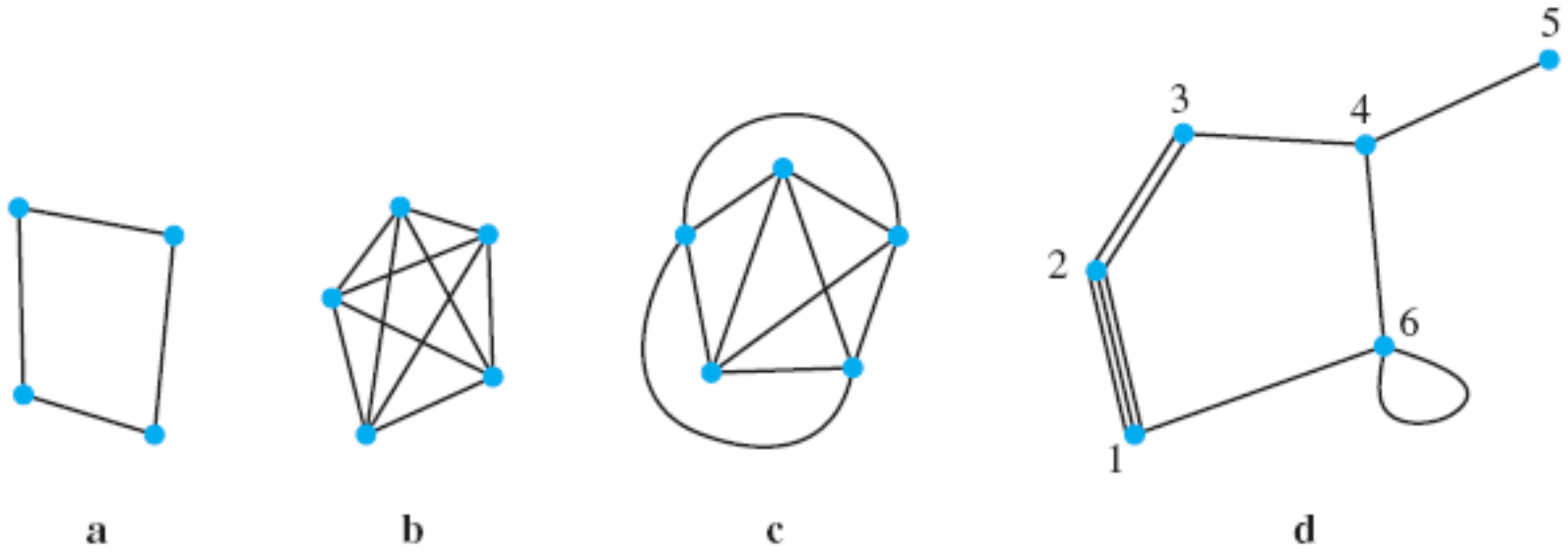

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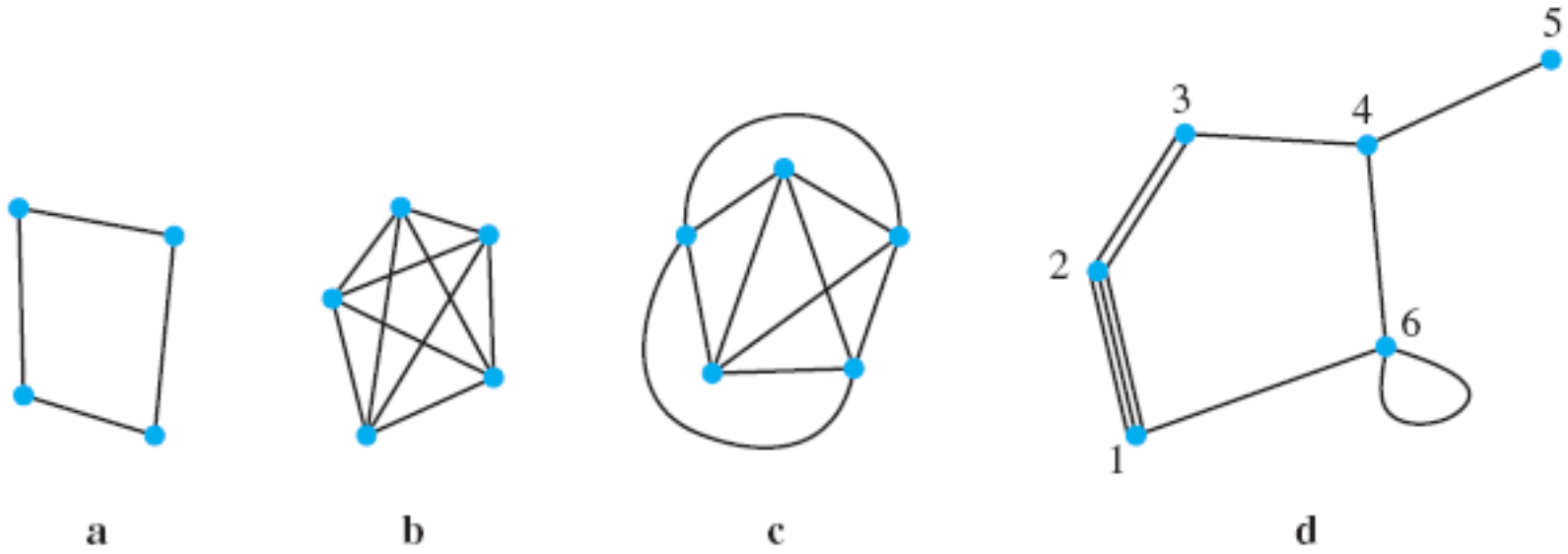

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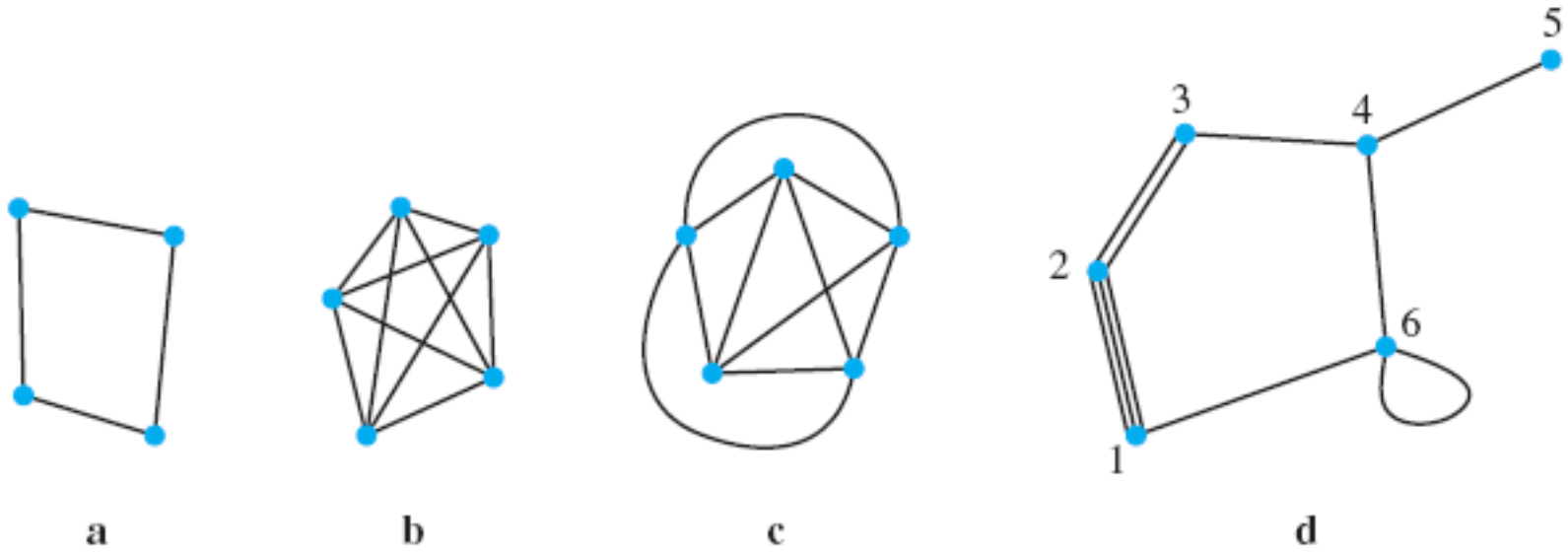

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- **Complete Graph** K_n (b, c): graph with n vertices that has an edge between each pair of vertices.

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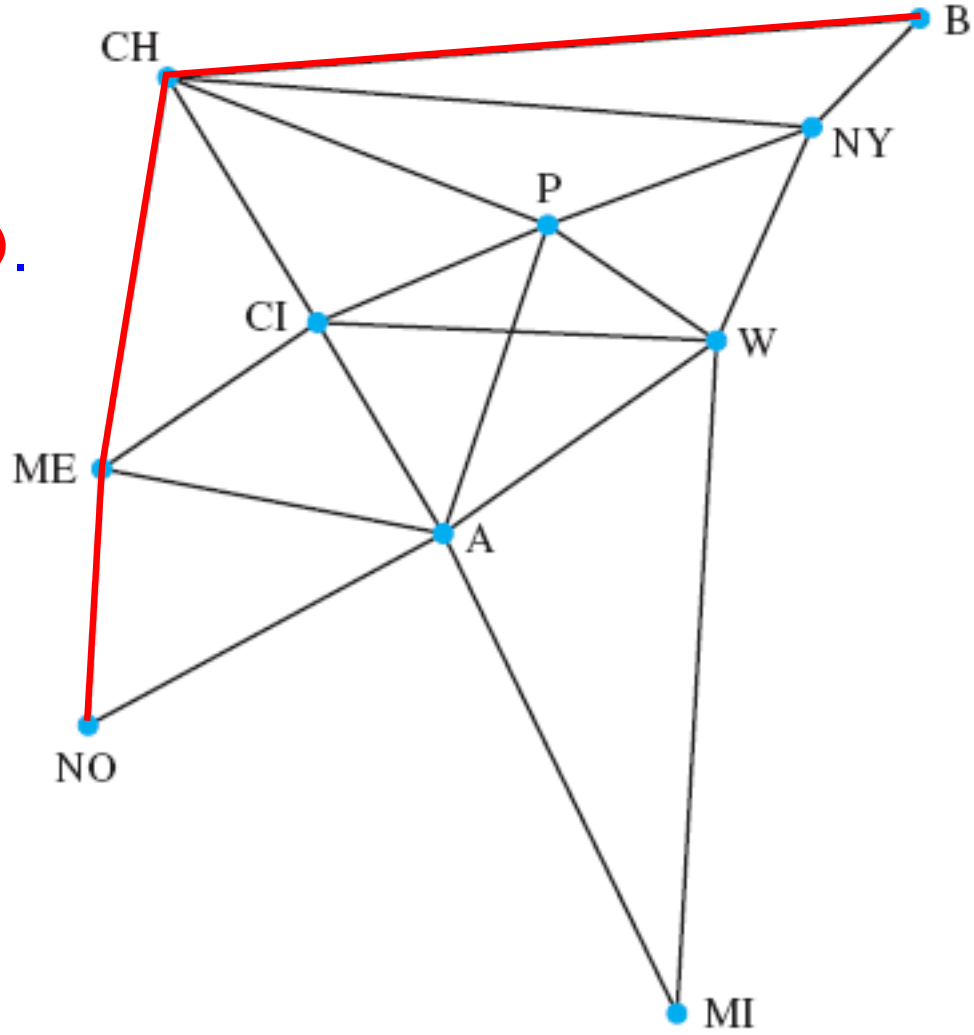
Length of a path $=$ # of edges on path

Example

Path from Boston to New Orleans is
B{B,CH}CH{CH,ME}ME{ME,NO}NO.



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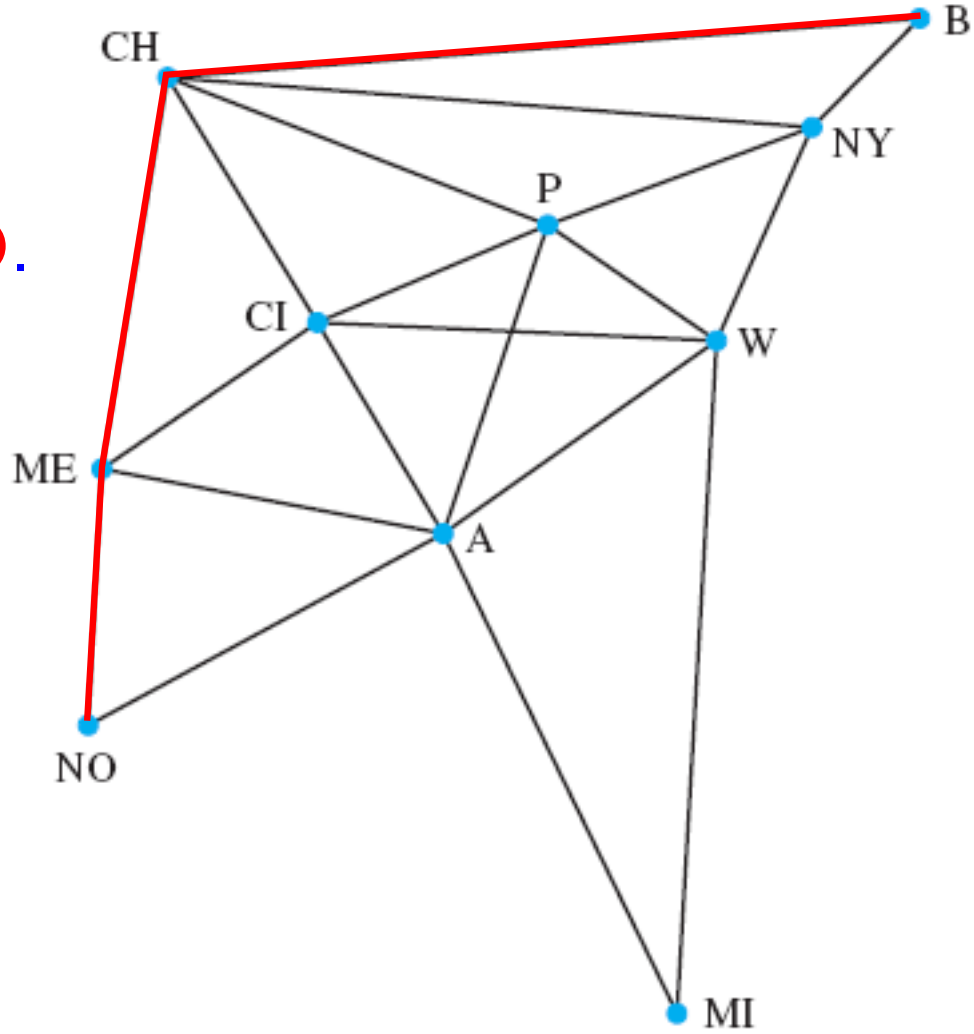

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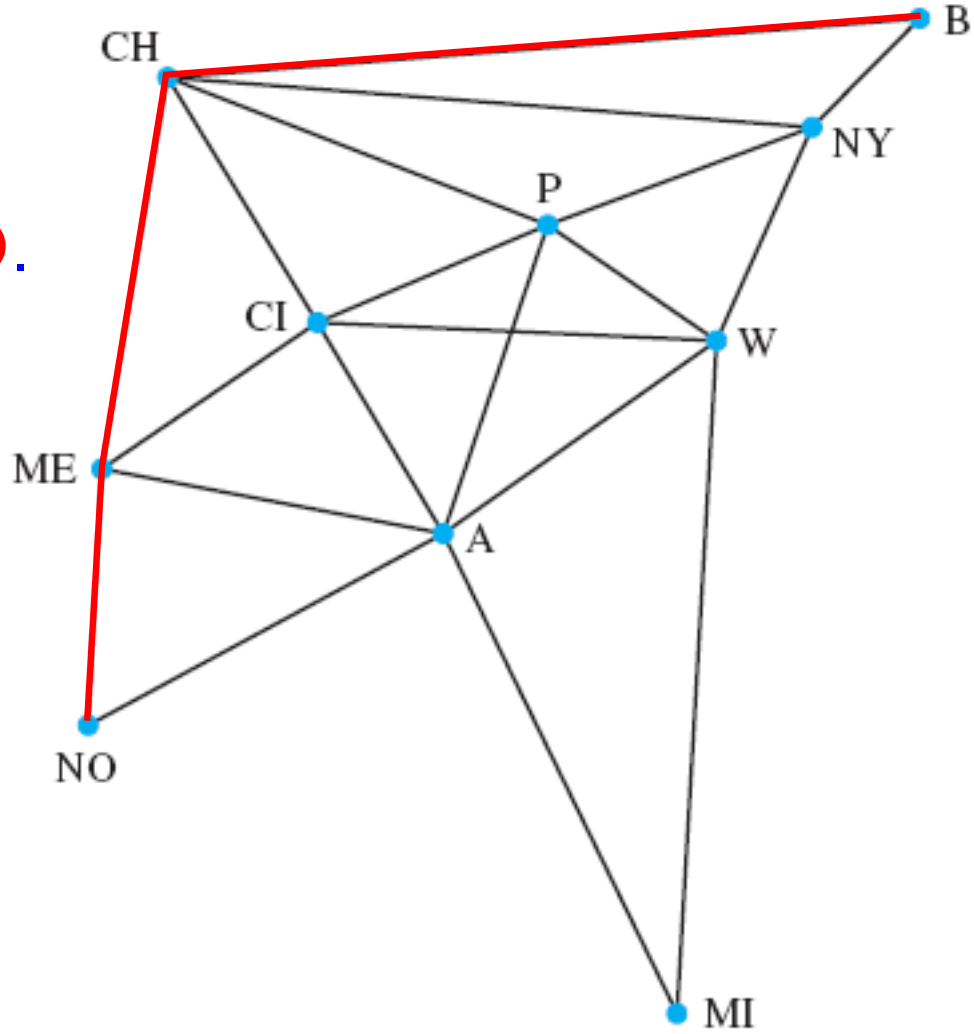
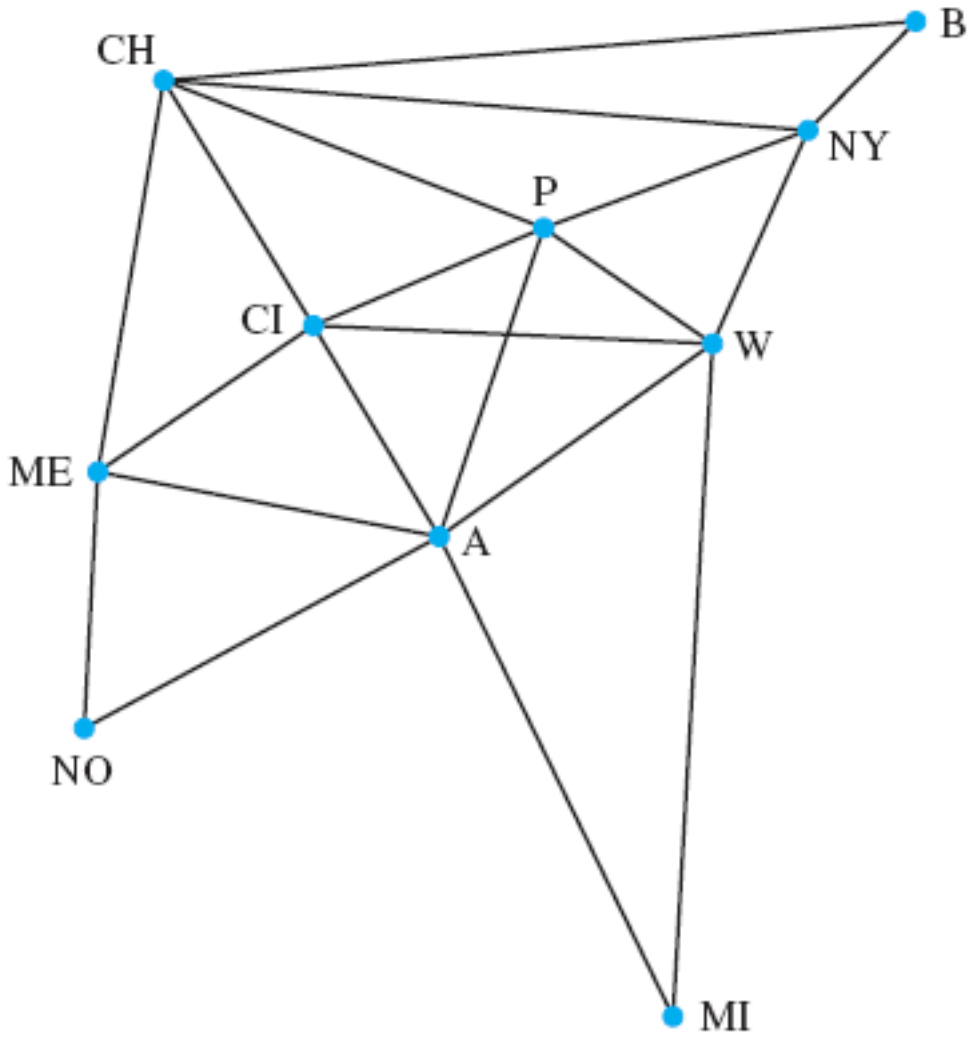


This path has length 3.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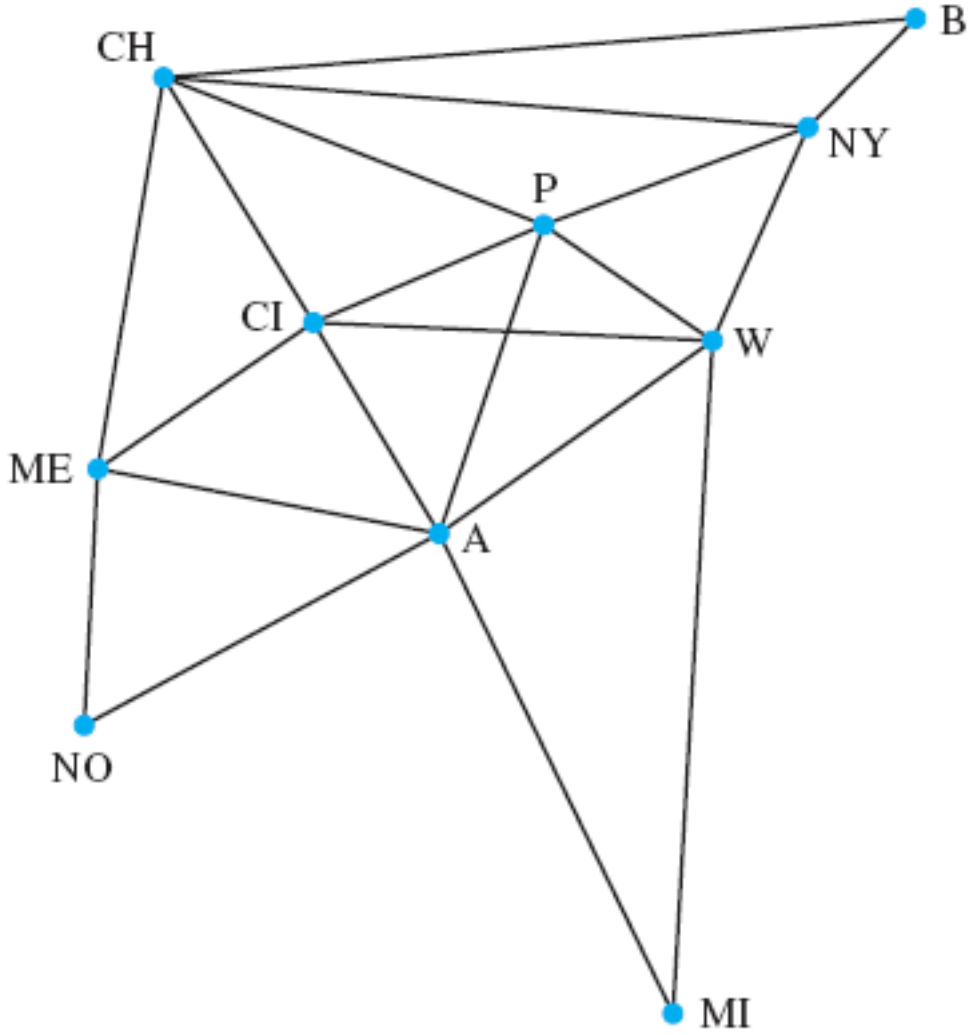
The distance between two vertices is the length of the shortest path between them.



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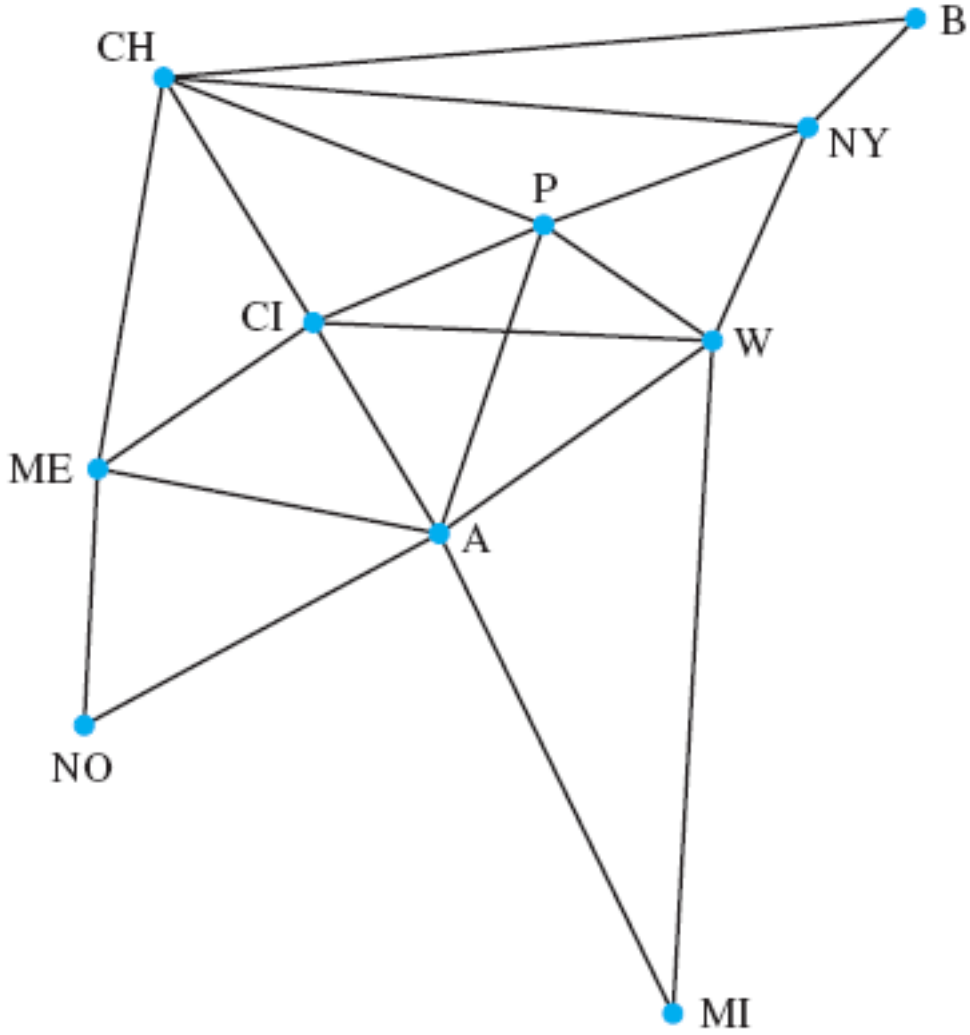

Examples:

$$\begin{aligned} \text{dist(CI, W)} &= 1 \\ \text{dist(CI, B)} &= 2 \\ \text{dist(CI, NO)} &= 2 \end{aligned}$$

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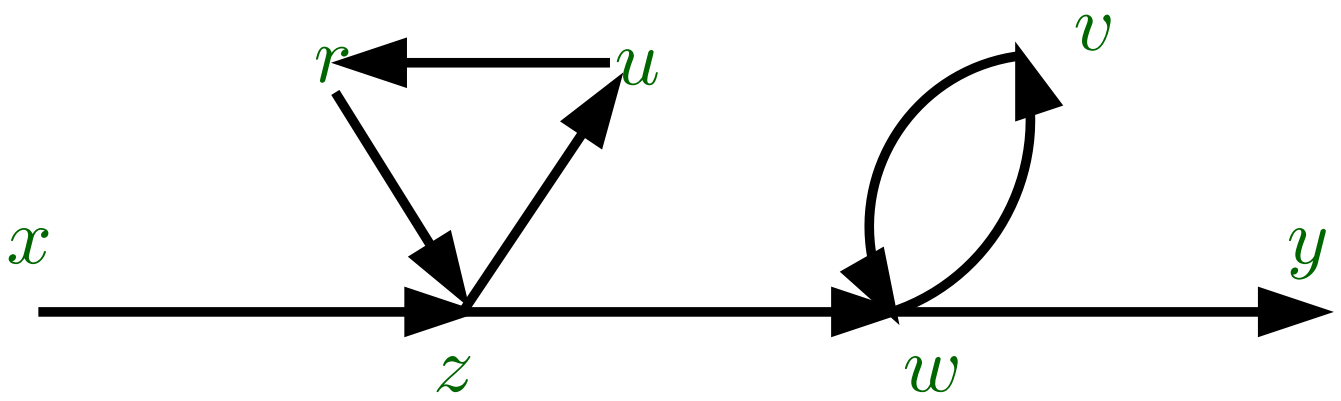

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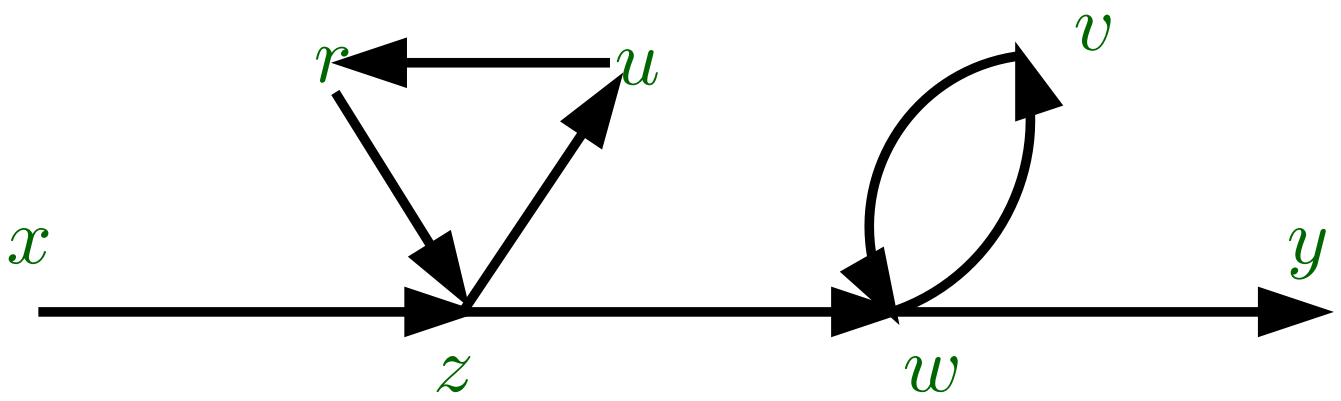

Walk from x to y

$x, z, u, r, z, w, v, w, y.$

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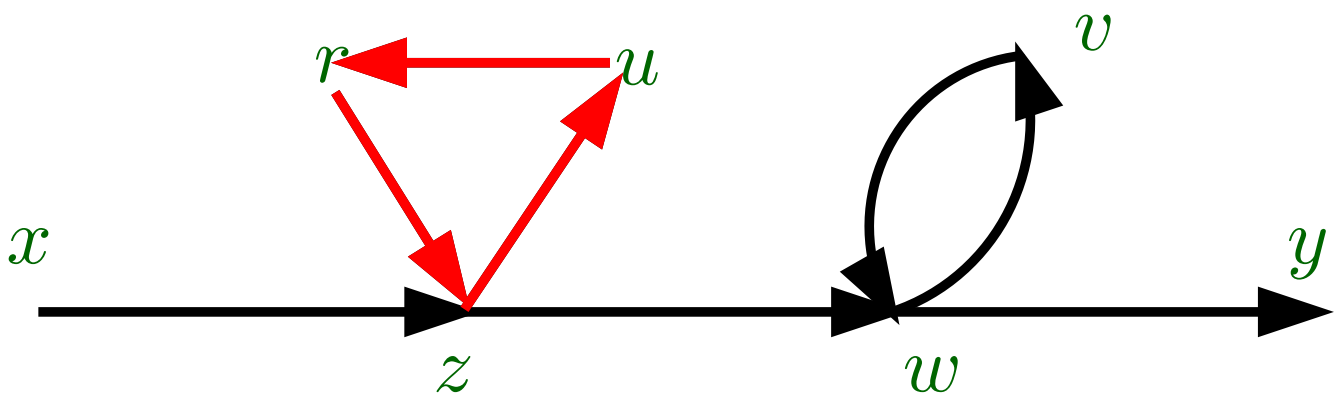

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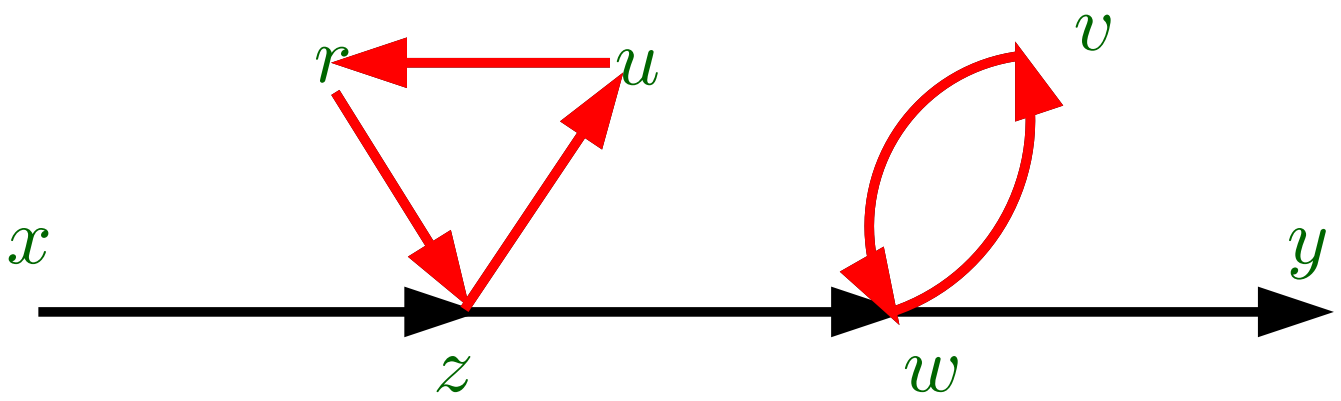

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→ Path from x to y
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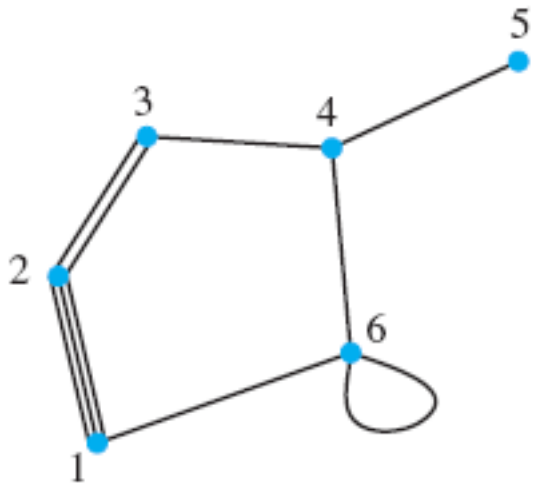

Example: Vertex 2 has degree 5, vertex 6 has degree 4 and vertex 4 has degree 3.

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Inductive Hypothesis:

Suppose that $m > 0$ and that the theorem is true whenever a graph has fewer than m edges.

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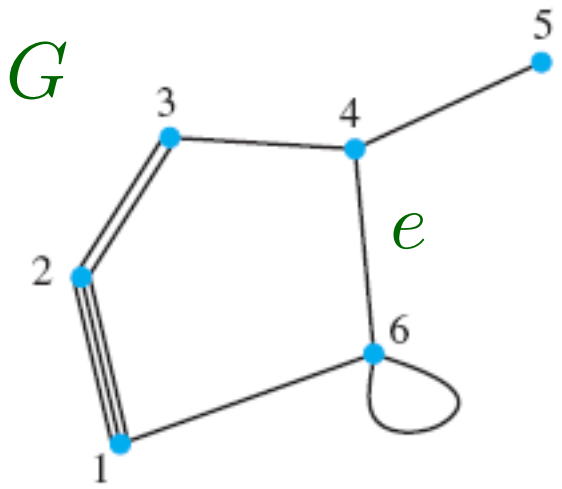

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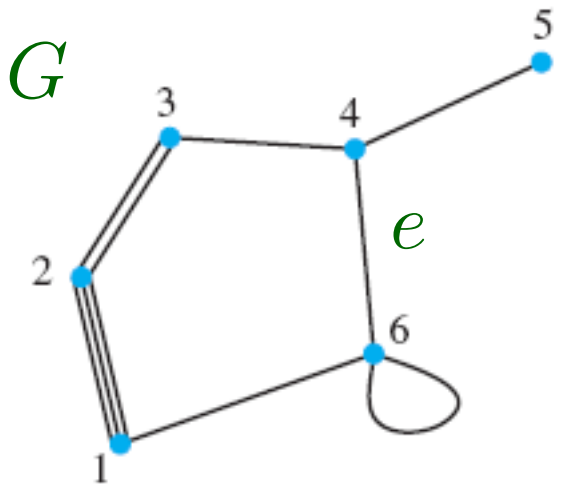

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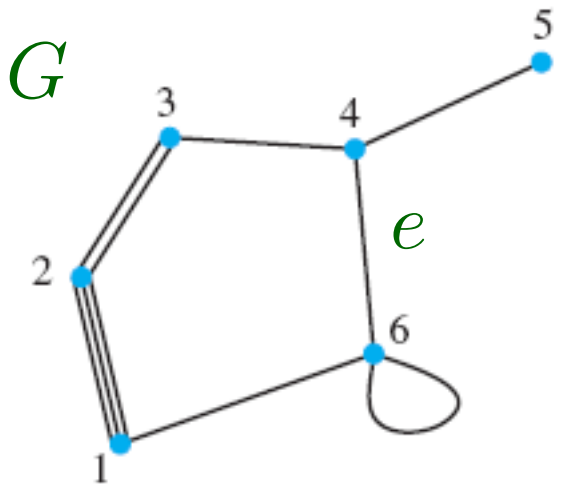

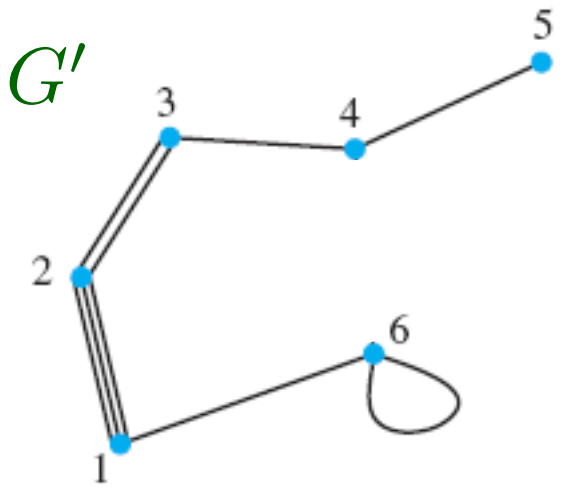

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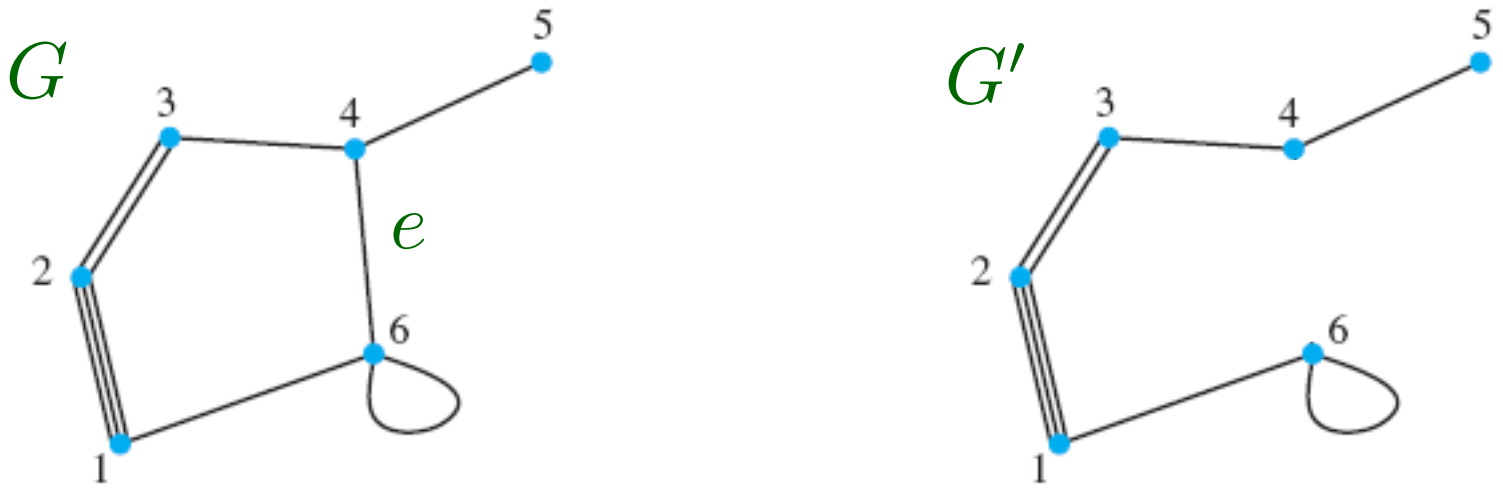

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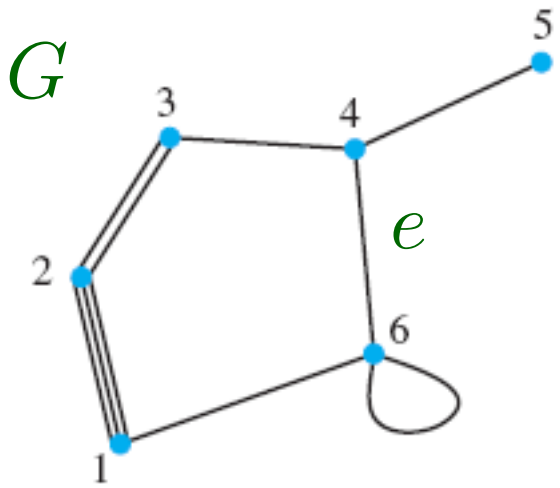

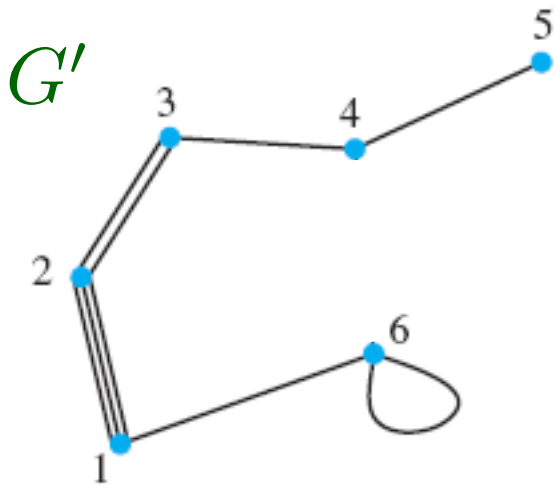

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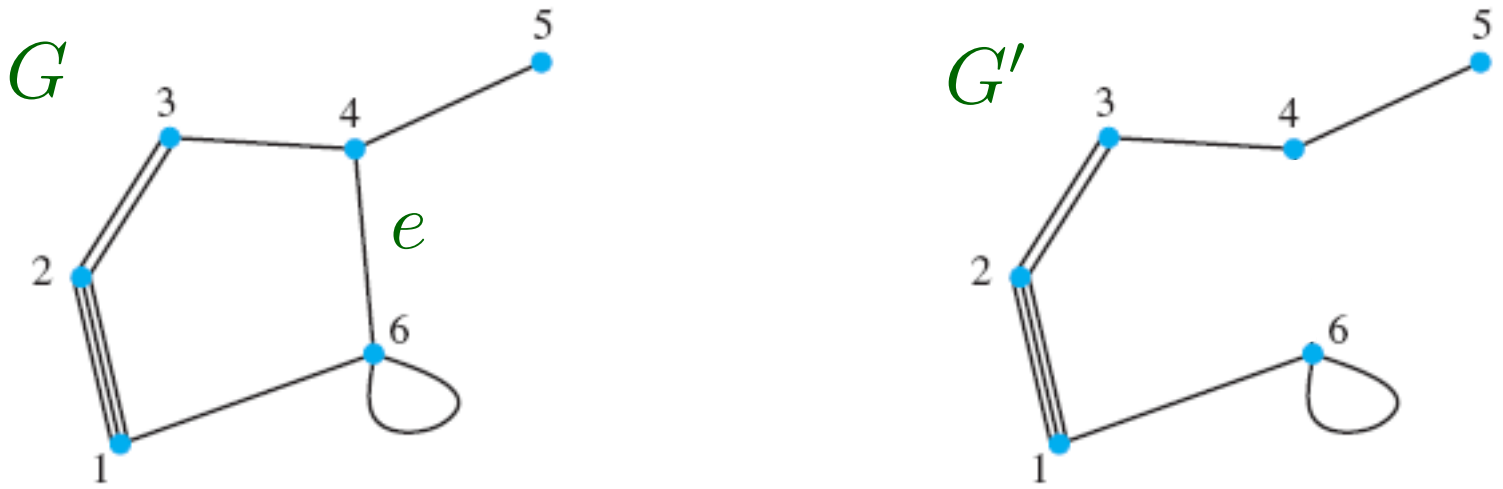

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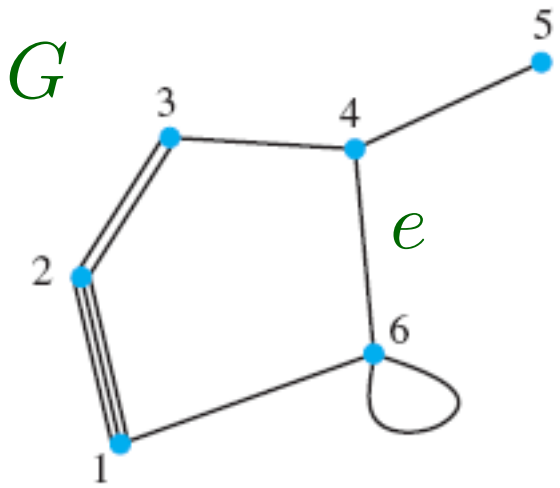

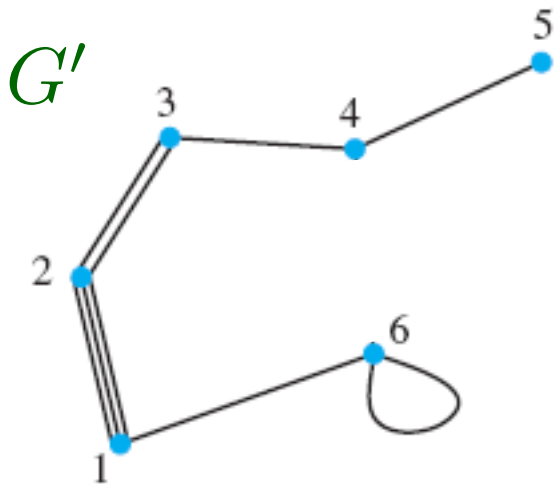

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Therefore, by principle of mathematical induction, theorem is true for a graph with any finite number of edges.

Graphs

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Connectivity

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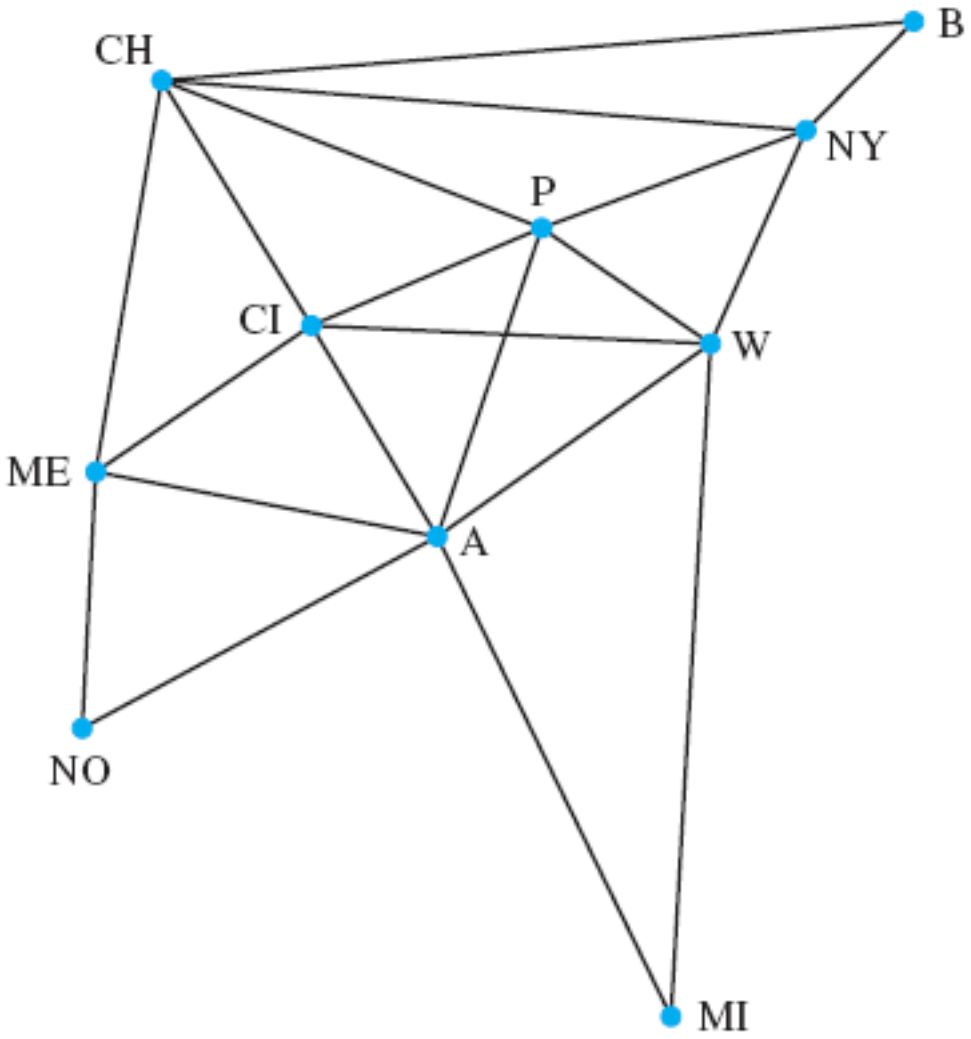

Company decides to lease only minimum number of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is **minimum** number of lines it needs to lease?

Choosing 10 edges?

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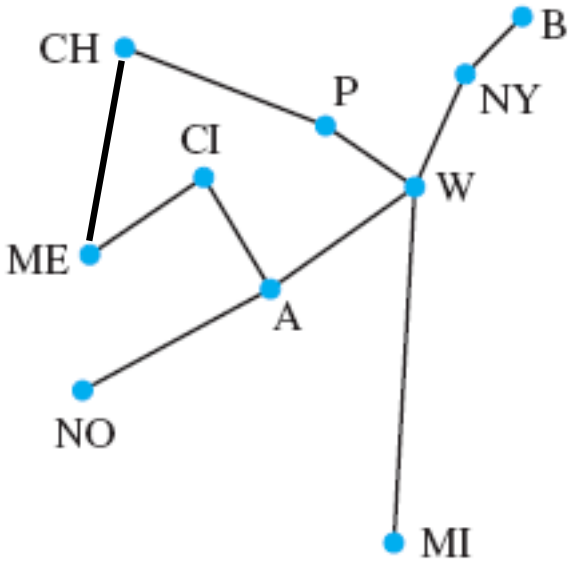

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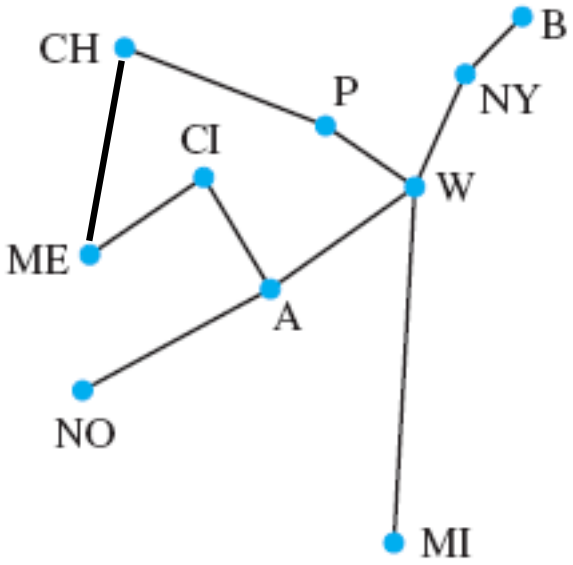

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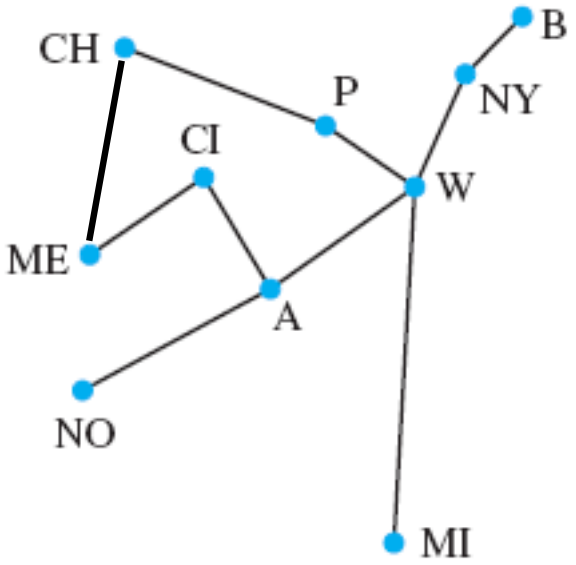

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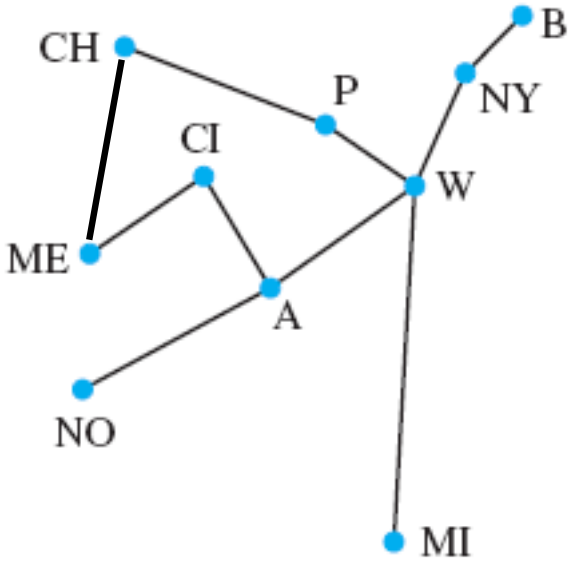

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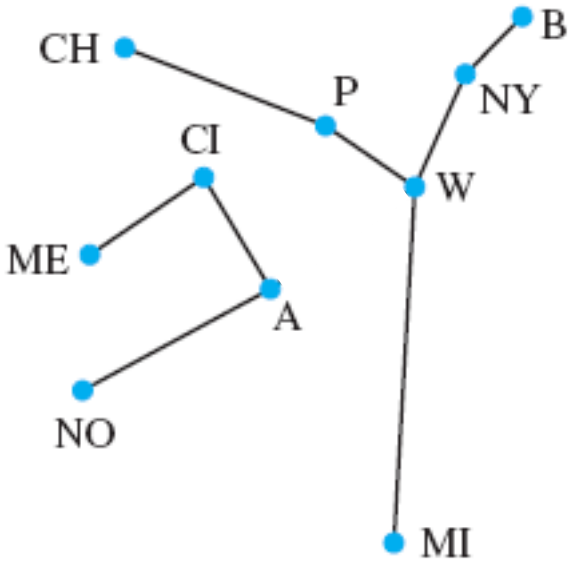

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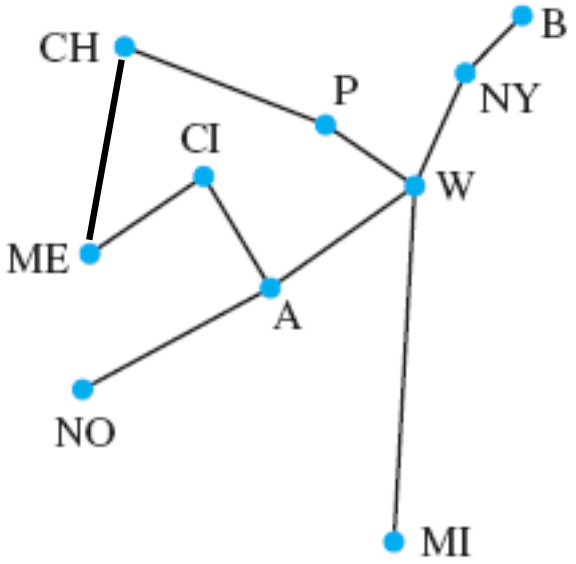

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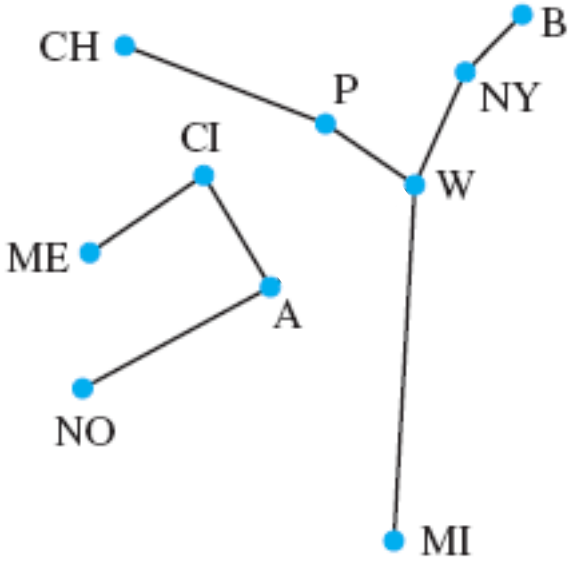

Not enough.

There is no path from, e.g., NO to B.

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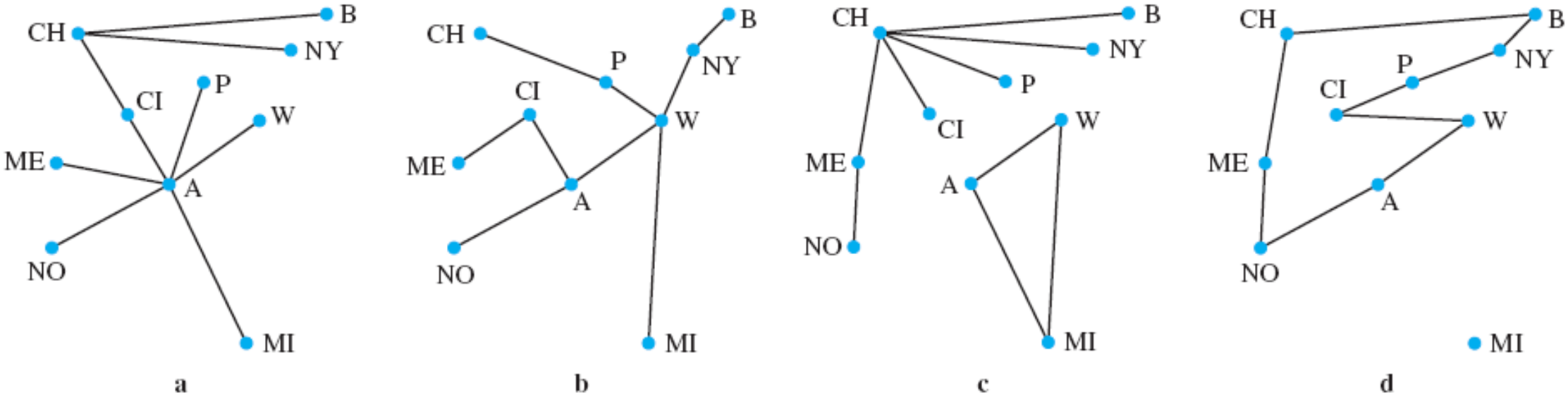

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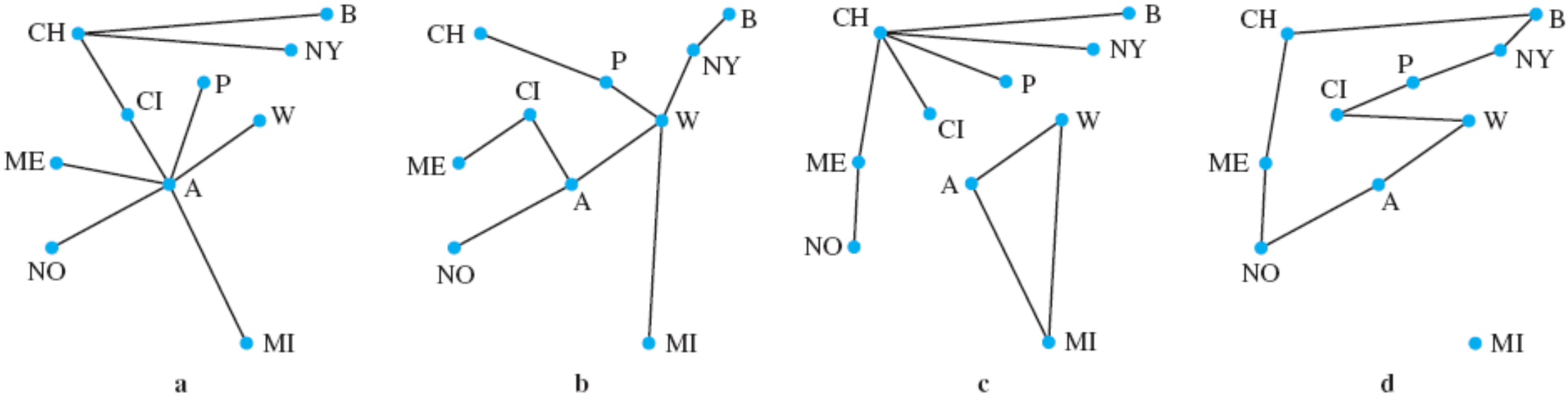

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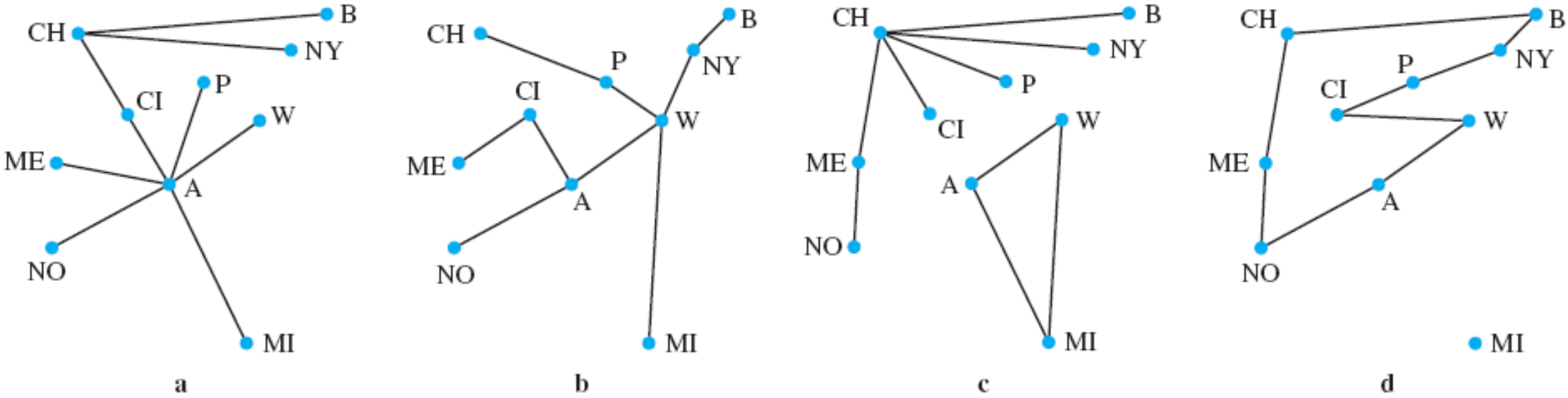

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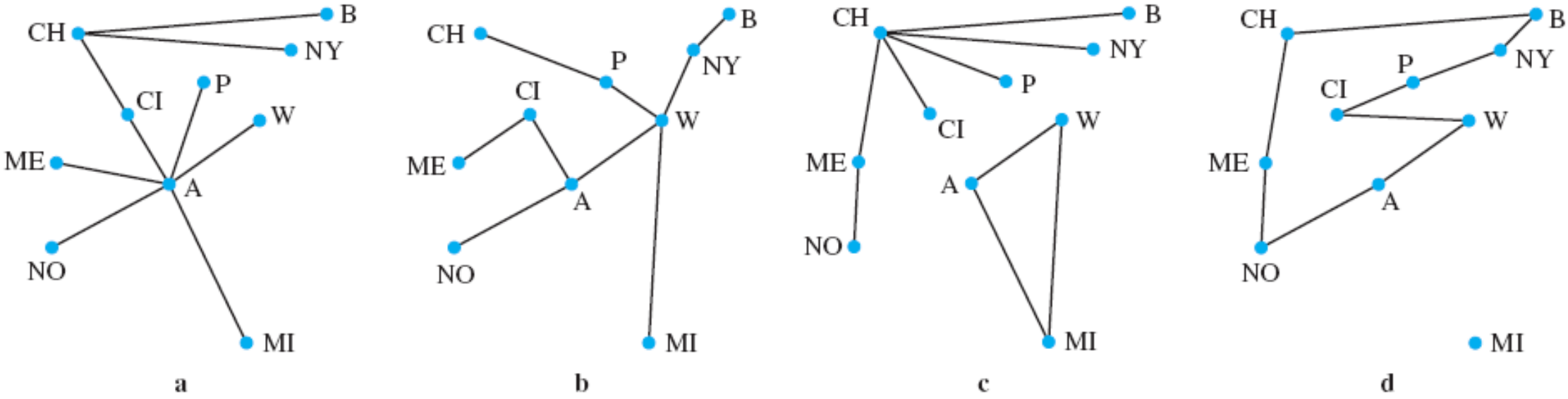

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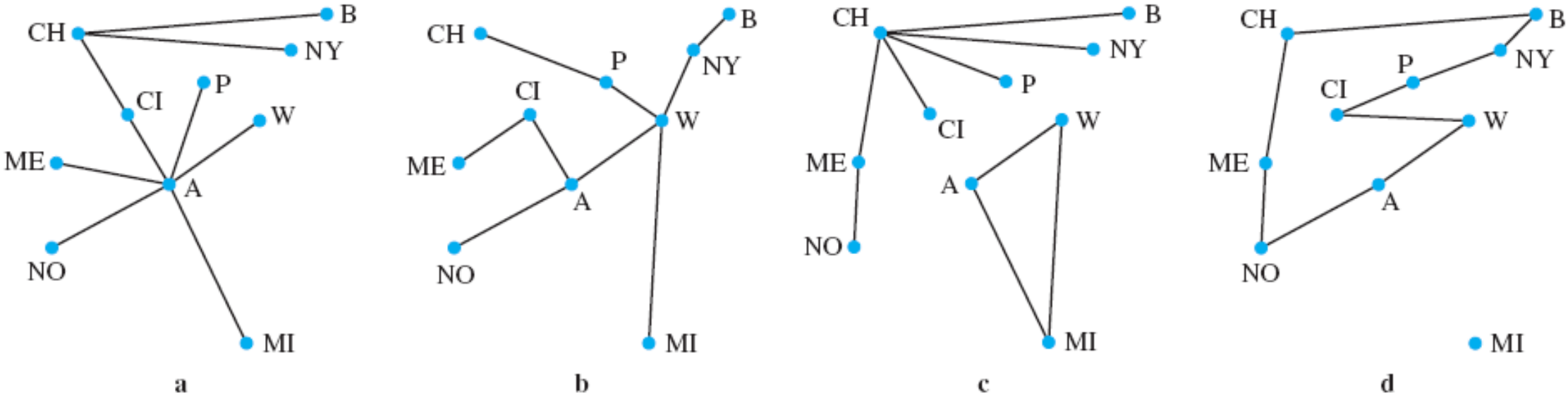

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In (d), we say that M(iami) is an **isolated vertex**.

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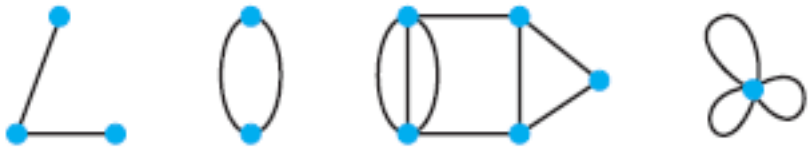
More Examples:

G_1 3 cc's



G_2 4 cc's

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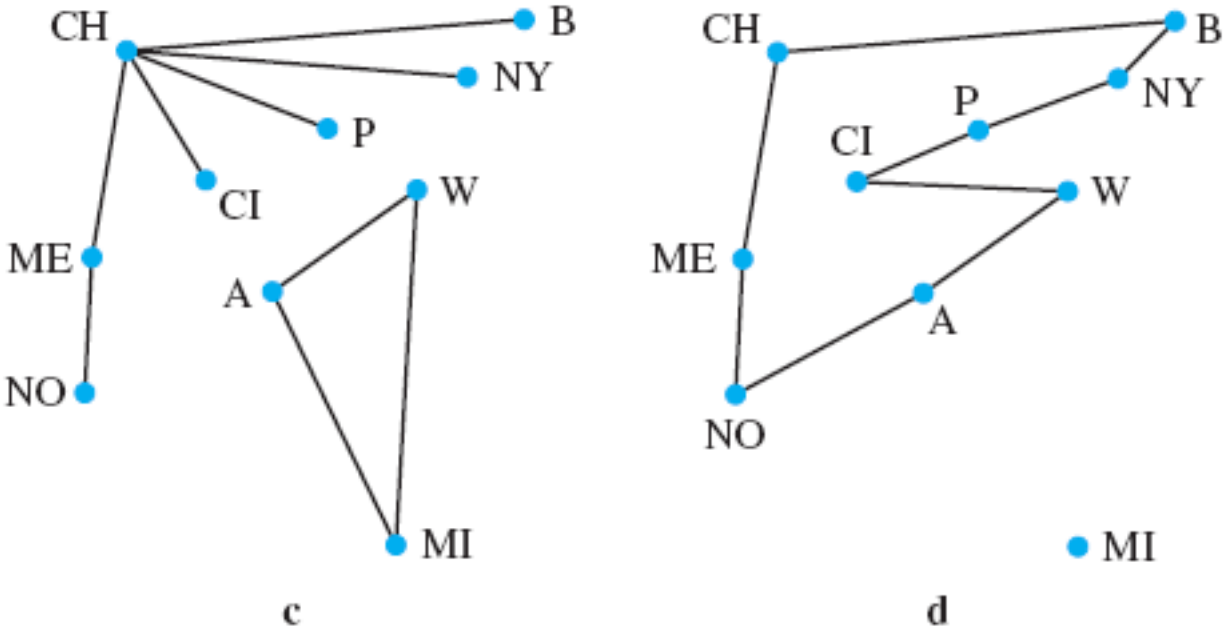

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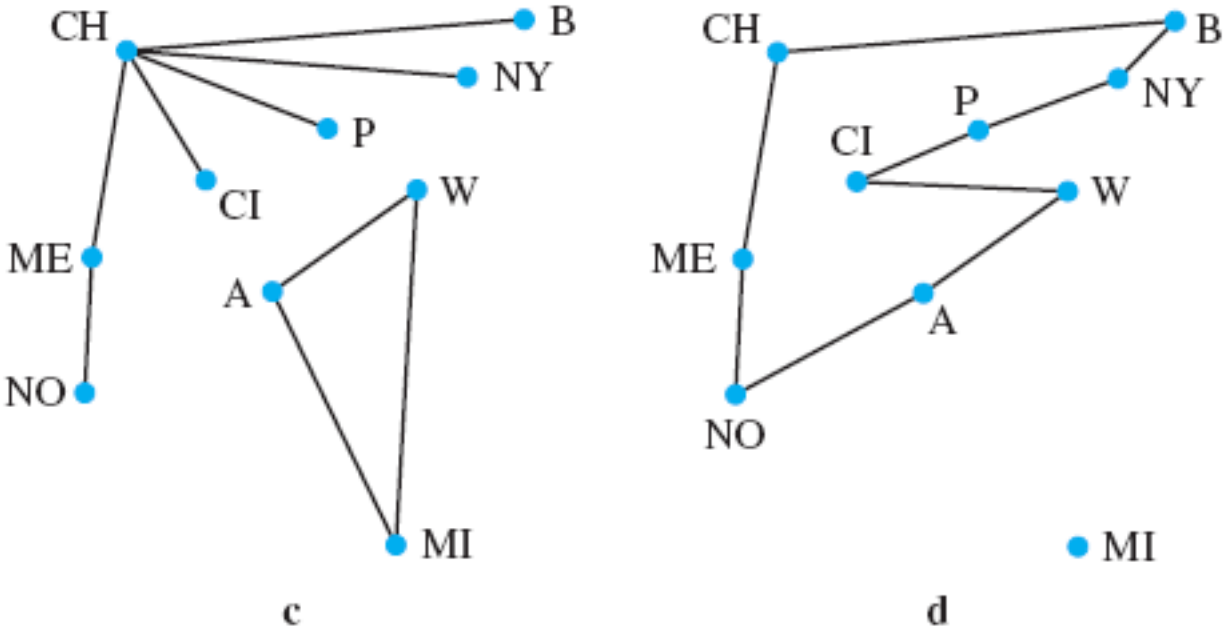

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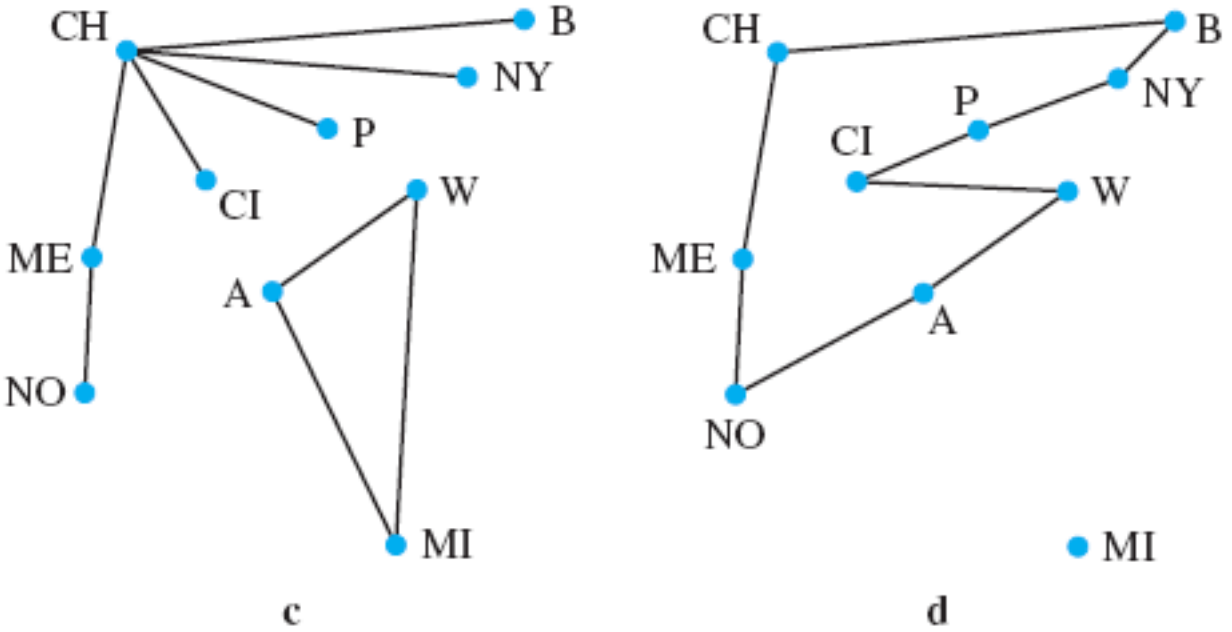

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Example: The closed walks in (c) and (d) are, respectively, cycles A, W, M, A and NO, ME, CH, B, NY, P, CI, W, A, NO.

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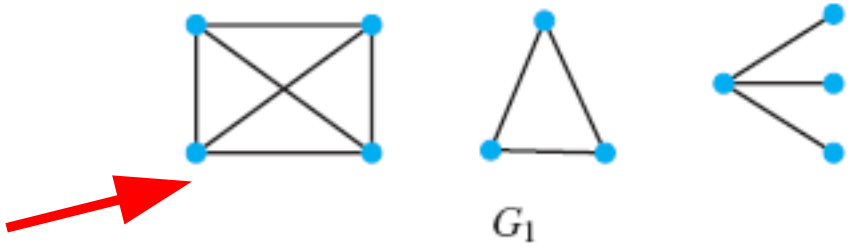

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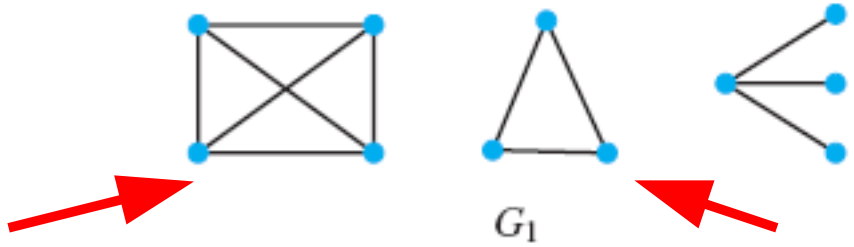

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G_2

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Examples:
Graph G_2 has an induced P_3 and an induced C_2 as subgraphs.

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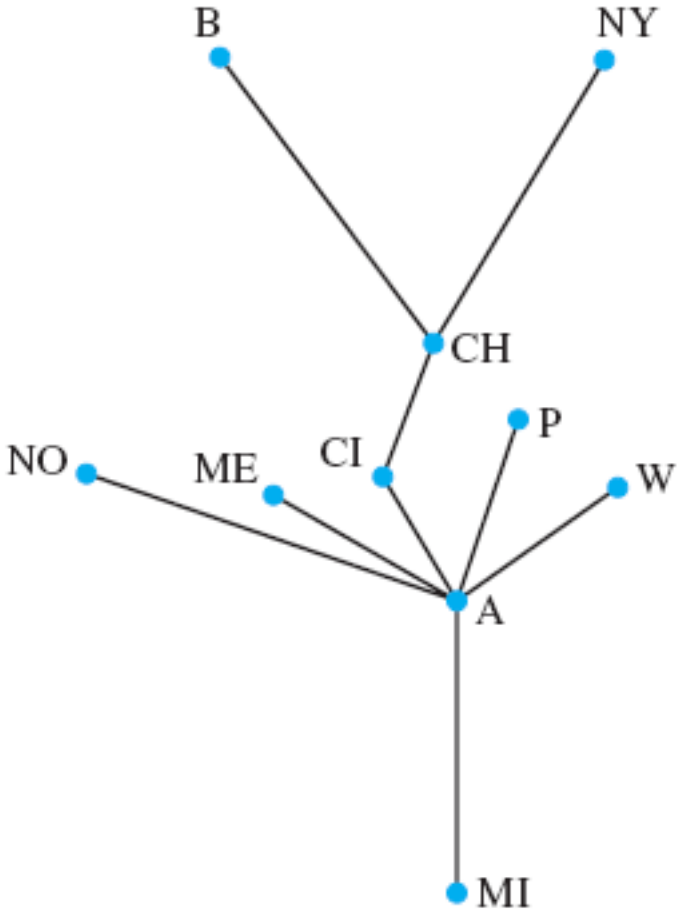
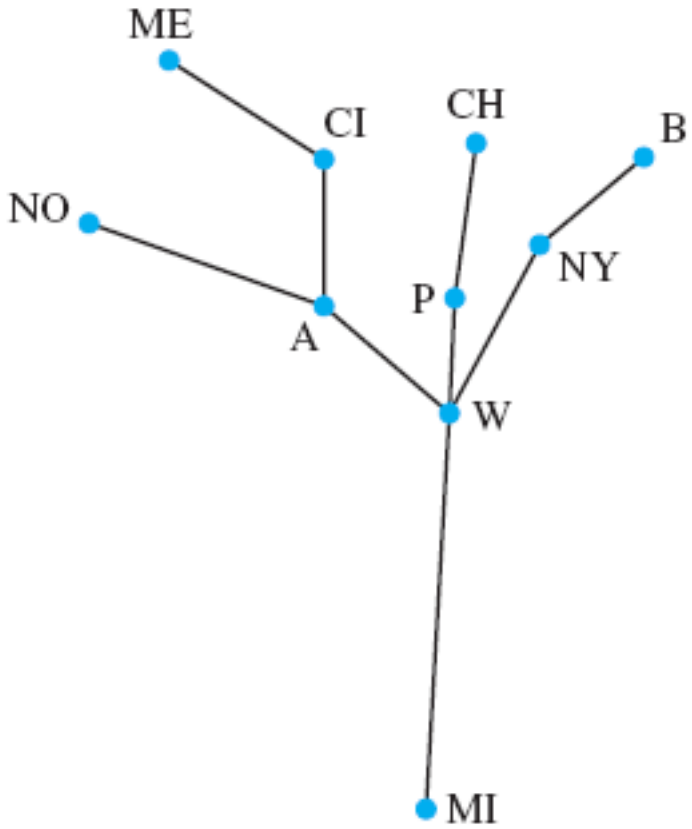


a



b

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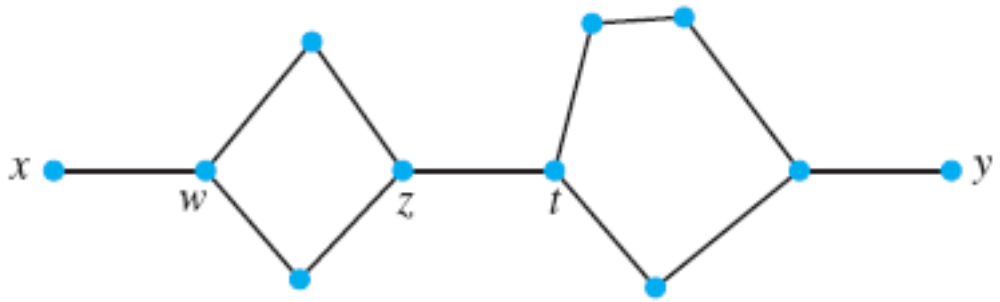

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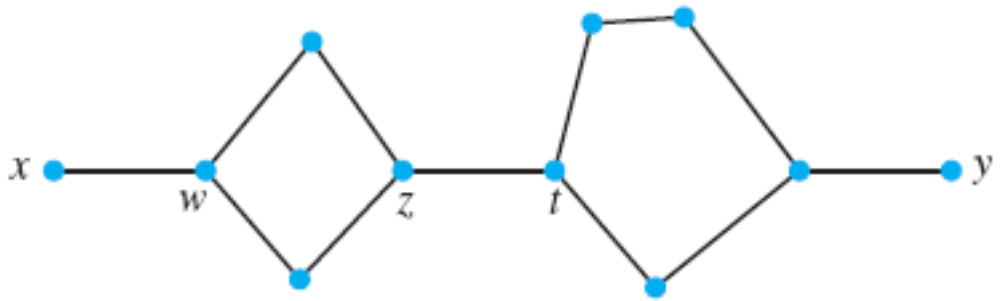

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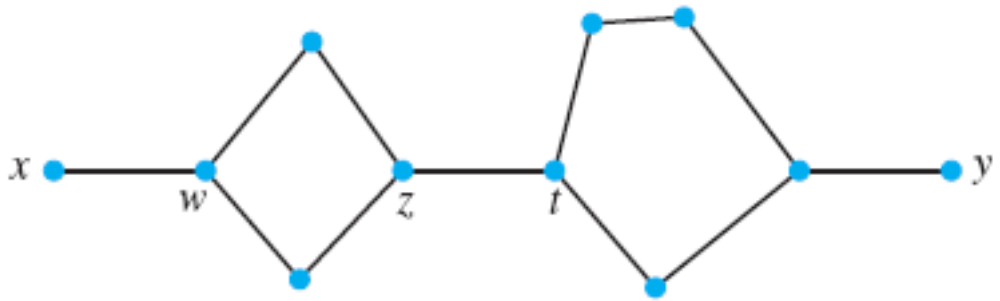

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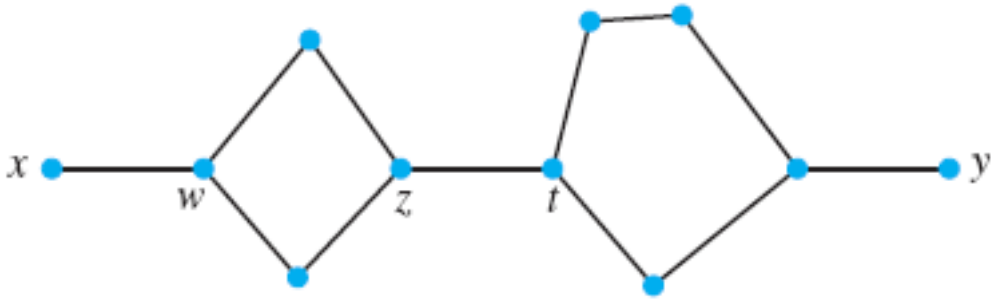
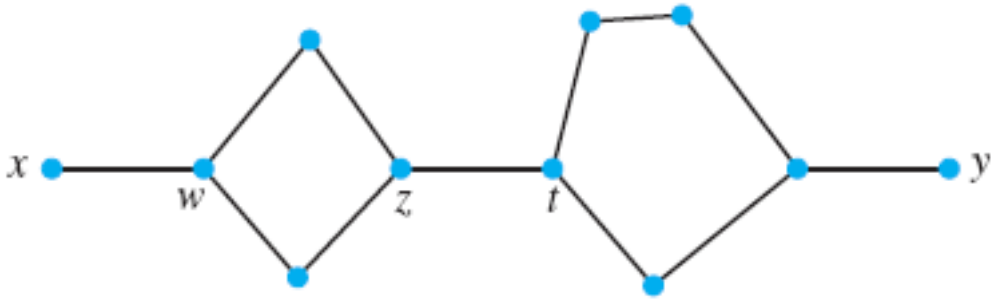
Let w be the last vertex after (or including) x that the paths share before they contain their first different edge.


x
w
z
t
y



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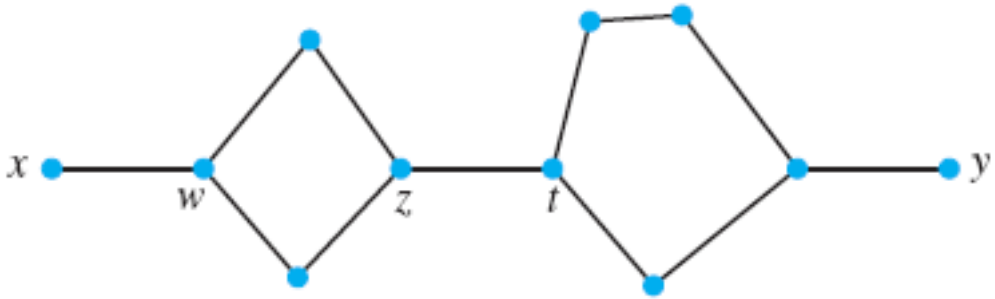

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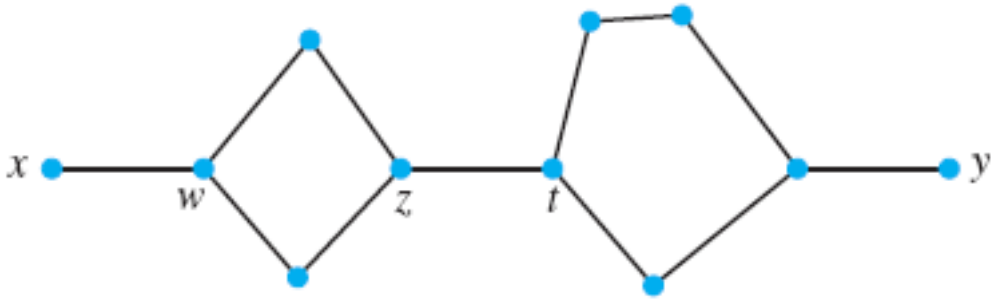

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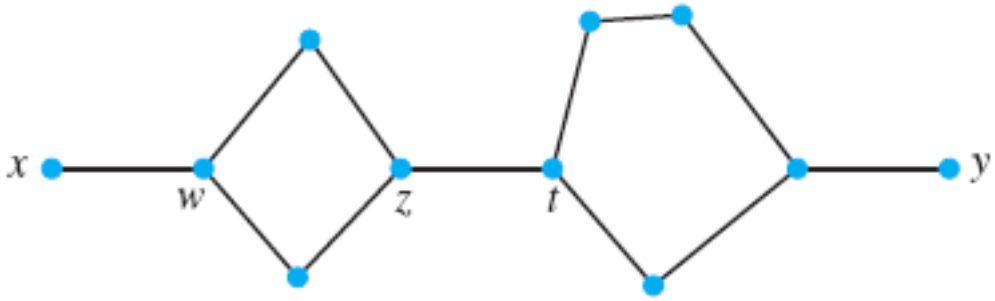

We now examine the two paths starting at w; they must join together again at y, though they might have joined at some earlier vertex.

Let z be the first vertex the paths have in common after w.

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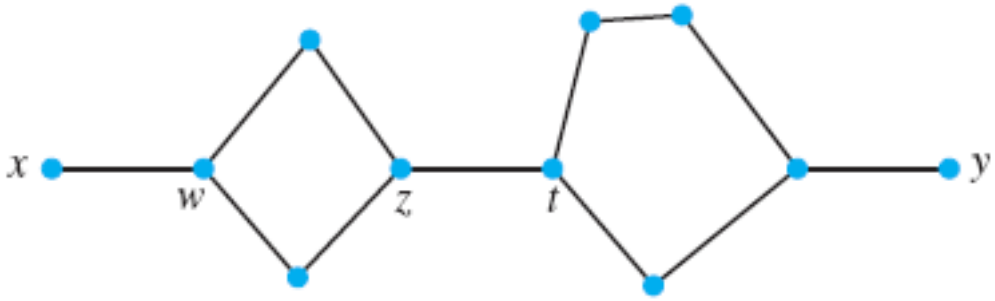
Travelling one of these paths from w to z and then travelling the other from z to w gives us cycle, and so the graph is not a tree.



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We have shown that if a graph has two distinct paths from x to y, then it is not a tree.

By contrapositive inference, then, if a graph is a tree, it does not have two distinct paths between two vertices x and y.

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Thus, there is exactly one path.

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Thus, it is not connected and is therefore not a tree.

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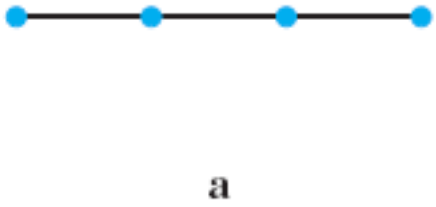
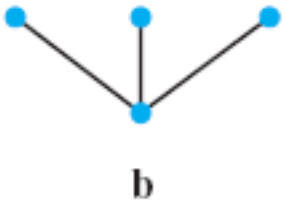
Because neither has any cycles (why?), both are trees.

Two trees on four vertices



a



b

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a

b

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Therefore, for all $n \geq 1$, a tree with n vertices has $n-1$ edges.

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$\Rightarrow$ by Theorem 6.5, G is not a tree.

$\Rightarrow$ by contrapositive inference,
if T is a tree, then T must have at least one vertex of degree 1.