Quick Review of Linearity of Expectation

COMP 3711H - HKUST
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Linearity of Expectation is one of the simplest and most useful tools used in the analysis of randomized algorithms.

In its easiest form it just says that,
if X, Y are *any* two random variables (not necessarily independent) then

$$E(X + Y) = E(X) + E(Y).$$

The iterated version is that
if $X_1, X_2, \ldots, X_n$ are *any* random variables, then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i).$$

Example: Let Z be the value seen when rolling two dice. $Z = X_1 + X_2$ where X_i is the value seen when rolling single die $i = 1, 2$. It's easy to calculate that

$$E(X_i) = \sum_{j=1}^{6} j \Pr(X_i = j) = \sum_{j=1}^{6} j \frac{1}{6} = \frac{7}{2}.$$

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Then

$$E(Z) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7.$$

When flipping n coins, what is the expected number of heads?

$Z = \sum_{i=1} X_i$, where
$X_i = 1$ if coin i is a head and 0 if it is a tail.
Set $\Pr(X_i = 1) = p_i$ and $\Pr(X_i = 0) = 1 - p_i$.
Then X_i is a *Bernoulli Random Variable* with probability p_i.

Note that $E(X_i) = 1 \cdot \Pr(X_i = 1) = p_i$ so

$$E(Z) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \Pr(X_i = 1) = \sum_{i=1}^{n} p_i.$$

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$p_i = \frac{1}{i} \quad \Rightarrow E(Z) = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{n} = H_n \sim \ln n$

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Let $x_1, x_2, \ldots, x_n$ be the list of coin tosses, i.e., $x_i \in \{\text{H(ead)}, \text{T(ail)}\}$.

$Z = \sum_{i=3}^{n} X_i$ where $X_i = 1$ iff $x_{i-2}x_{i-1}x_i = HHH$

$\Pr(X_i = 1) = \frac{1}{8}$, so

$$E(Z) = \frac{n-2}{8}.$$

Suppose an algorithm's input is a permutation of n numbers.
Let $x_1, x_2, \ldots, x_n$ be the input in its given order.

x_i is a *left to right maxima* if it's bigger than $x_1, x_2, \ldots, x_{i-1}$.

For example, the red items in these two permutations are the l.t.r. maxima:
5 4 7 8 1 6 3 2   1 3 5 7 2 4 6 8

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One way of generating a random permutation is to first randomly choose the first i items equally likely among all posible $\binom{n}{i}$ subsets. Then choose a random permutation among the i possible permutations to order them as $x_1, \ldots x_i$. Then randomly order the remaining items as $x_{i+1}, \ldots x_n$.

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Probability x_i is l.t.r. maxima is prob it's largest in first i items which is now $\frac{1}{i}$. So X_i is Bernouli Random Variable with $p_i = 1/i$ and $E(Z_i) = H_n$.

Suppose we flip n coins.
ith coin is Heads with probability $p_i = 1/i$.

If ith coin is Heads you get i dollars; Tails, you get nothing.
What is expected amount you receive?

Let Z be total amount. $Z = \sum_{i=1}^{n} X_i$ where $X_i = i$ if ith coin is heads and is otherwise 0. Then

$$E(X_i) = ip_i = 1$$

So, $$E(Z) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 = n$$

Make a minor change from the previous page.
Suppose we flip n coins. ith coin is Heads with probability $p_i = 1/i$.

If ith coin is Heads you run another random process Y_i to tell you how much money you receive. All you know is that $E(Y_i) = i$.
If ith coin is Tails, you get nothing. What is expected amount you receive?

Let Z be total amount. $Z = \sum_{i=1}^{n} X_i Y_i$ where
$X_i = 1$ if ith coin is Heads and is otherwise 0, so $E(X_i) = p_i$. $E(Y_i) = i$.
Then, *because X_i and Y_i are independent*

$$E(X_i Y_i) = E(X_i)E(Y_i) = \frac{1}{i} i = 1$$

So,

$$E(Z) = E\left(\sum_{i=1}^{n} X_i Y_i\right) = \sum_{i=1}^{n} E(X_i Y_i) = \sum_{i=1}^{n} 1 = n$$