Huffman Coding

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Outline

- Coding and Decoding
- The optimal source coding problem
- Huffman coding: A greedy algorithm
- Correctness

Encoding

Example

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  - a code is a set of codewords.
  - e.g., $\{000, 001, 010, 011, 100, 101\}$ and $\{0, 101, 100, 111, 1101, 1100\}$

  are codes over the binary alphabet $\Sigma = \{0, 1\}$.

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- In a fixed-length code each codeword has the same length.
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- Goal is to save space!

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$\{a = 0,\ b = 110,\ c = 01,\ d = 111\}$ is *not* a prefix code.
$\{a = 0,\ b = 110,\ c = 10,\ d = 111\}$ is a prefix code.

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We are therefore interested in finding *good* (best compression) prefix-free codes.

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The Optimal Source Coding Problem

Huffman Coding Problem

Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that minimizes the number of bits

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needed to encode a message of $\sum_{i=1}^{n} f(a_i)$ characters, where

- c_i is the codeword for encoding a_i, and
- $L(c_i)$ is the length of the codeword c_i.

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Remark: We will see later that this is the optimum (lowest cost) prefix code.

Correspondence between Binary Trees and Prefix Codes

1-1 correspondence between leaves and characters.

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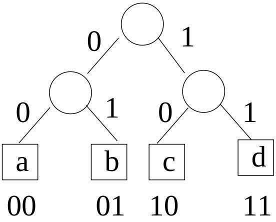

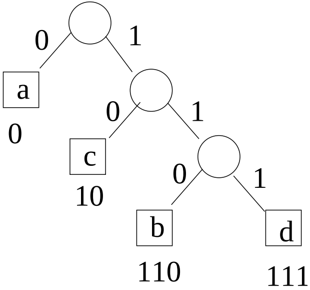

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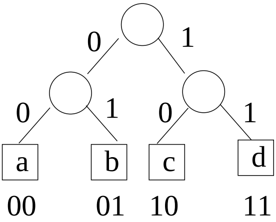

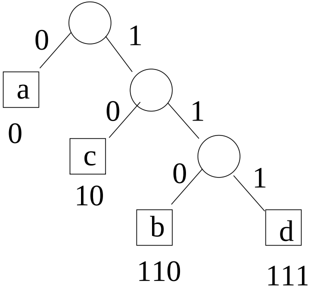

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- The binary string on a path from the root to a leaf is the codeword associated with the character at the leaf.

- d_i, the depth of leaf a_i, is equal to $L(c_i)$, the depth of the codeword associated with that leaf.

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Definition (Minimum-Weight External Pathlength Problem)

Given weights $f(a_1), \ldots, f(a_n)$, find a tree T with n leaves labeled $a_1, \ldots, a_n$ that has minimum weighted external path length.

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The Huffman encoding problem is equivalent to the minimum-weight external pathlength problem.

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- It encodes the optimum (minimum-cost) prefix code for the given frequency distribution.

Example of Huffman Coding

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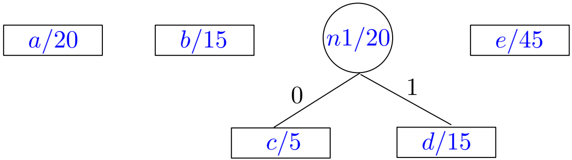

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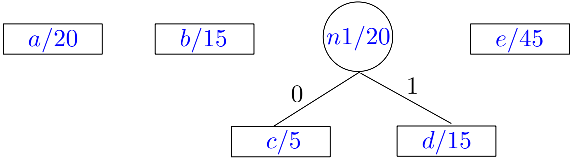

Now have $S = \{a/20, b/15, \mathbf{n1/20}, e/45\}$.

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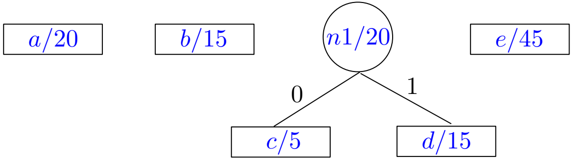

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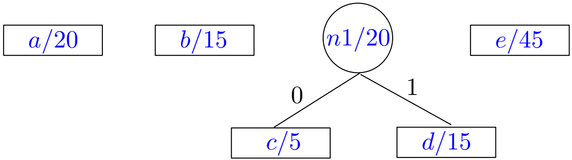

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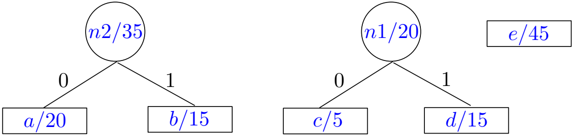

Now have $S = \{n2/35, n1/20, e/45\}$.

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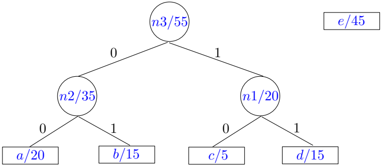

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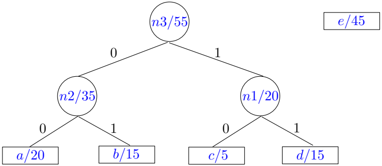

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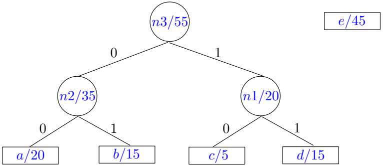

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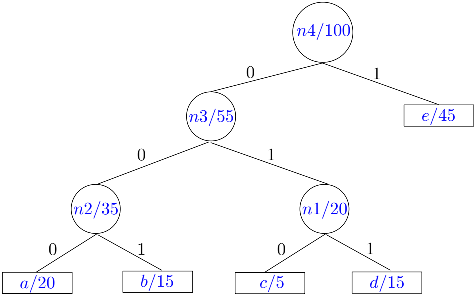

The Huffman code is:
$a = 000$, $b = 001$,
$c = 010$, $d = 011$,
$e = 1$.

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Running time is $O(n \log n)$, as each priority queue operation takes time $O(\log n)$.

Outline

- Coding and Decoding
- The optimal source coding problem
- Huffman coding: A greedy algorithm
- Correctness

Lemma 1

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An optimal prefix code tree must be "full", i.e., every internal node has exactly two children.

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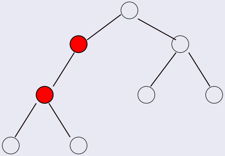

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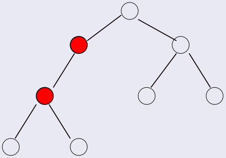

then we could simply get rid of this node and replace it with its unique child. This would decrease the total cost of the encoding. □

Lemma (2)

Let T be prefix code tree and T' the tree obtained by swapping two leaves x and b in T. If,

$$f(x) \leq f(b), \quad \textit{and} \quad d(x) \leq d(b)$$

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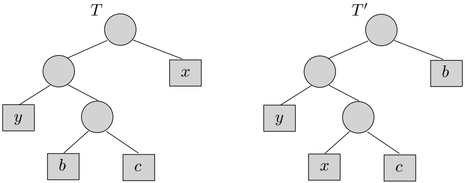
i.e., swapping a lower-frequency character downward in T does not increase T's cost.



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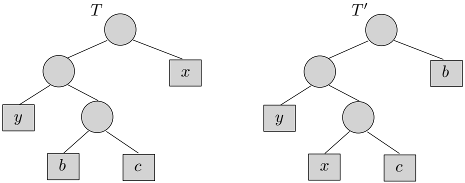

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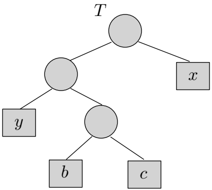

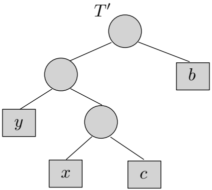

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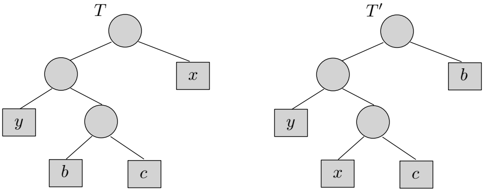

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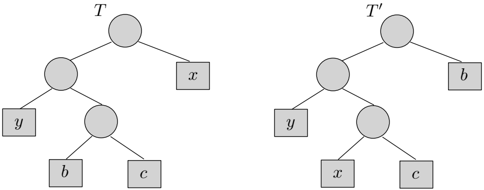

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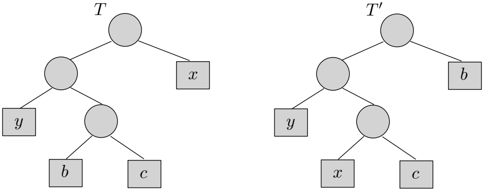

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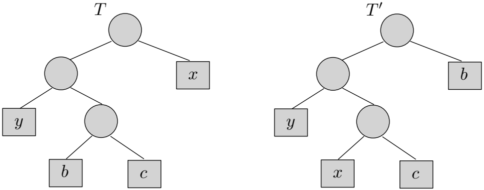

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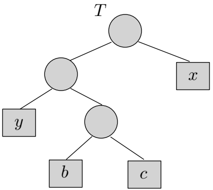

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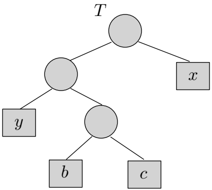

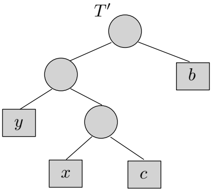

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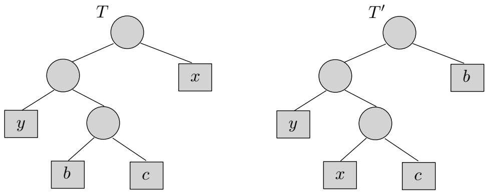

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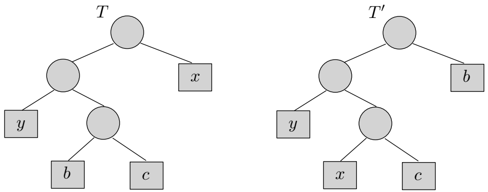

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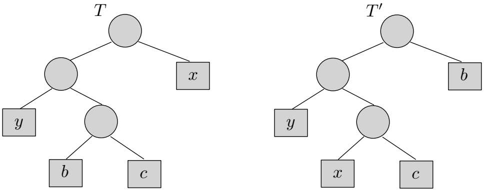

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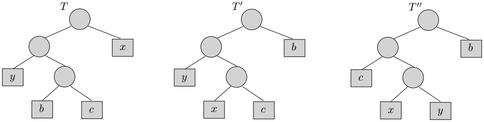

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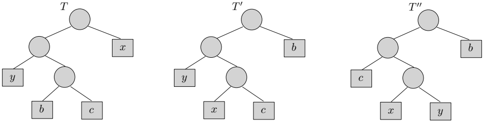

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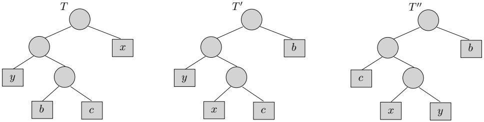

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- (If necessary) swap x with b and swap y with c.
- Proof follows from Lemma 2.

Lemma 4

Lemma (4)

- *Let T be a prefix code tree and x, y two sibling leaves.*
- *Let T' be obtained from T by removing x and y, naming the parent z, and setting $f(z) = f(x) + f(y)$*
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- **Induction hypothesis**: Huffman's algorithm produces optimal tree in the case of $n - 1$ characters.

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Proof: (By induction on n, the number of characters).

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    - By the induction hypothesis, H' is optimal for S'.
    - By Lemma 4, $\mathbf{B(H) = B(H') + f(x) + f(y)}$.

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  - Therefore, H must be optimal!