Approximating the bandwidth via volume respecting embeddings

Uriel Feige

Presenter
Qin Zhang
March 6, 2007

Outline

- Problems and Results
- Intuition and General Idea
- The Framework of the Method
- Algorithm and Explanation
- Major Techniques and Key Proofs
- Conclusion and Comments

Problems and Results

- Given an undirected graph $G = (V, E), |V| = n, |E| = m$ The *Minimum Bandwidth Problem* is to find a one-to-one mapping $f : V(G) \stackrel{1-1}{\longrightarrow} [n]$ to minimize

$$bw(f) = max_{(i,j) \in E} |f(i) - f(j)|$$

- This paper presents a randomized algorithm that runs in nearly linear time and outputs a linear arrangement whose bandwidth is within a $O(polylog(n))$ factor of optimal.

Outline

- Problems and Results
- Intuition and General Idea
- The Framework of the Method
- Algorithm and Explanation
- Major Techniques and Key Proofs
- Conclusion and Comments

Intuition and General Idea

- The idea: randomly map the points in V to real line $\mathbb{R}$, denote the mapping ψ. Such an one to one mapping ψ defines a natural ordering on the points according to their positions in the real line, with the following two properties:

Intuition and General Idea

- The idea: randomly map the points in V to real line $\mathbb{R}$, denote the mapping ψ. Such an one to one mapping ψ defines a natural ordering on the points according to their positions in the real line, with the following two properties:

  - Let $S \subseteq V$ and $|S| = k$. We say S is bad if $\psi(S) \subseteq [tl, (t+1)l)$ for some integer t. We want to make sure that the chance of *"S is bad"* is small.

    Suppose the real line is already divided into intervals of length l

  - The image $\psi(v_i), \psi(v_j)$ of two endpoints of any edge $e = (v_i, v_j) \in E$ are not far apart.

The key techniques

- (η, k)-well-separated mapping/volume respecting embedding

A contracting mapping $\phi : V \rightarrow \mathbb{R}^L$ is called (η, k)-well-separated if the following condition holds.

For each set $S \subseteq V, s.t. |S| = k$, there exists an ordering $\{s_0, s_1, \ldots, s_{k-1}\}$ of S such that, for all i, let $L_i = span\{\phi(s_0), \phi(s_1), \ldots, \phi(s_{i-1})\}$, and

$$dist(\phi(s_i), L_i) \geq \frac{q_i}{\eta}$$

where $q_i = d_G(s_i, \{s_0, s_1, \ldots, s_{i-1}\})$ is the minimum length between s_i and some point in $\{s_0, s_1, \ldots, s_{i-1}\}$.

The key techniques

- (η, k)-well-separated mapping/volume respecting embedding

  A contracting mapping $\phi : V \rightarrow \mathbb{R}^L$ is called (η, k)-well-separated if the following condition holds.

  For each set $S \subseteq V, s.t. |S| = k$, there exists an ordering $\{s_0, s_1, \ldots, s_{k-1}\}$ of S such that, for all i, let $L_i = span\{\phi(s_0), \phi(s_1), \ldots, \phi(s_{i-1})\}$, and

  $$dist(\phi(s_i), L_i) \geq \frac{q_i}{\eta}$$

  where $q_i = d_G(s_i, \{s_0, s_1, \ldots, s_{i-1}\})$ is the minimum length between s_i and some point in $\{s_0, s_1, \ldots, s_{i-1}\}$.

- And we let $\psi = \varphi \circ \phi$, where φ is a random projection $\mathbb{R}^L \rightarrow \mathbb{R}$

The utility of this technique

- The simplex (convex-hull of $\phi(s_i)$) obtained by the well-separated mapping must be "fat" (i.e. has a large volume), and there is a big chance to obtain a well separated points configuration after the line projection.

The utility of this technique

- The simplex (convex-hull of $\phi(s_i)$) obtained by the well-separated mapping must be "fat" (i.e. has a large volume), and there is a big chance to obtain a well separated points configuration after the line projection.

- Since the mapping is contracting, the length of the vectors corresponding to all edges after the mapping would be less than 1.

The utility of this technique

- The simplex (convex-hull of $\phi(s_i)$) obtained by the well-separated mapping must be "fat" (i.e. has a large volume), and there is a big chance to obtain a well separated points configuration after the line projection.

- Since the mapping is contracting, the length of the vectors corresponding to all edges after the mapping would be less than 1.

- The rest of the life:
  - How to find such a well-separated mapping ϕ with a small η?
  - How to use such a well-separated mapping to prove the first property?

Outline

- Problems and Results
- Intuition and General Idea
- The Framework of the Method
- Algorithm and Explanation
- Major Techniques and Key Proofs
- Conclusion and Comments

Two definitions

- The Local Density D of graph G is defined as

$$D = D(G) = \max_{v,r} \frac{|B(v,r)| - 1}{2r}$$

where $B(v,r)$ is the ball centered at v containing all the vertices that are within distance r from v, include v itself.

Observation: D is a lower bound of the minimum bandwidth

Two definitions

- The Local Density D of graph G is defined as

$$D = D(G) = \max_{v,r} \frac{|B(v,r)| - 1}{2r}$$

where $B(v,r)$ is the ball centered at v containing all the vertices that are within distance r from v, include v itself.

Observation: D is a lower bound of the minimum bandwidth

- An auxiliary concept — the Tree Volume. Let $S \subseteq V$. Consider the complete graph G_S on S, and let the length of an edge d_e for $e = (u,v) \in S \times S$. We define

$$Tvol = \prod_{e \in T} d_e$$

where T is the minimum spanning tree on G_S.

The formalization of the two properties

- Claim 1: S-bad set is rare with large $|S|$.

$$Pr[S \text{ is } l\text{-bad}] \leq \frac{O(\eta l)^{|S|-1}}{Tvol(S)}$$

The formalization of the two properties

- Claim 1: S-bad set is rare with large $|S|$.

$$Pr[S \text{ is } l\text{-bad}] \leq \frac{O(\eta l)^{|S|-1}}{Tvol(S)}$$

Claim 1^+: an extension of Claim 1.

$$\sum_{S \subseteq V : |S| = k} \frac{1}{Tvol(S)} \leq n \cdot O(D \log n)^{k-1}$$

The formalization of the two properties

- Claim 1: S-bad set is rare with large $|S|$.

$$Pr[S \text{ is } l\text{-bad}] \leq \frac{O(\eta l)^{|S|-1}}{Tvol(S)}$$

Claim 1^+: an extension of Claim 1.

$$\sum_{S \subseteq V : |S| = k} \frac{1}{Tvol(S)} \leq n \cdot O(D \log n)^{k-1}$$

- Claim 2: End points of any edge are not far apart

$$Pr[|\overline{\psi(v_i)\psi(v_j)}| \geq l : e = (v_i, v_j) \in E] \leq \frac{1}{2m}$$

Towards getting a good upperbound

- Claim 1 and 1^+ give us

$$E[\#(\text{bad sets})] \leq n \cdot O(\eta l D \log n)^{k-1}$$

Using Markov's inequality, one can conclude that

$$Pr[\#(\text{bad sets}) \leq c \cdot n \cdot O(\eta l D \log n)^{k-1}] \geq \frac{1}{c}$$

for some constant c.

Towards getting a good upperbound

- Claim 1 and 1^+ give us

$$E[\#(\text{bad sets})] \leq n \cdot O(\eta l D \log n)^{k-1}$$

  Using Markov's inequality, one can conclude that

$$Pr[\#(\text{bad sets}) \leq c \cdot n \cdot O(\eta l D \log n)^{k-1}] \geq \frac{1}{c}$$

  for some constant c.

- On the other hand, we can conclude from Claim 2 that with probability more than $\frac{1}{2}$, all the edges' length are no more than l.

Towards getting a good upperbound

- Notice that one edge spans at most 2 intervals.

$$\#\text{bad sets} \geq \binom{\frac{B}{2}}{k} \geq \left(\frac{B}{2k}\right)^k$$

Towards getting a good upperbound

- Notice that one edge spans at most 2 intervals.

$$\#\text{bad sets} \geq \binom{\frac{B}{2}}{k} \geq \left(\frac{B}{2k}\right)^k$$

- Double Counting:

$$\left(\frac{B}{2k}\right)^k \leq c \cdot n \cdot O(\eta l D \log n)^{k-1} \Rightarrow B \leq (3n)^{1/k} \cdot O(k \eta l D \log n)$$

Set $k = \log n$, we have

$$B \leq O(D \eta l \log^2 n)$$

Towards getting a good upperbound

- Notice that one edge spans at most 2 intervals.

$$\#\text{bad sets} \geq \binom{\frac{B}{2}}{k} \geq \left(\frac{B}{2k}\right)^k$$

- Double Counting:

$$\left(\frac{B}{2k}\right)^k \leq c \cdot n \cdot O(\eta l D \log n)^{k-1} \Rightarrow B \leq (3n)^{1/k} \cdot O(k \eta l D \log n)$$

Set $k = \log n$, we have

$$B \leq O(D \eta l \log^2 n)$$

- Remark: we can further set $l = \Theta(\sqrt{\log n})$ and $\eta = \Theta(\log^{3/2} n)$ (could be improved to $\eta = \Theta(\log n \sqrt{\log \log n})$), and then obtain an $O(polylog(n))$ approximation ratio.

Sketch proof of Claim 2

We pick a random vector $\vec{r} = \{r_1, r_2, \ldots, r_L\}$ in $\mathbb{R}^L$ and each $r_i \sim N(0,1)$. Then for a vector $\vec{v} = \overline{\phi(v_i)\phi(v_j)}$ of at most unit length (recall that ϕ is a contracting mapping), we have

$$\begin{aligned} Pr[|\langle \vec{r}, \vec{v} \rangle| > 2(\sqrt{\log n})] &= Pr[\|\vec{v}\|_2 \times |X| > 2(\sqrt{\log n})] \\ &\leq Pr[|X| > 2(\sqrt{\log n})] \end{aligned}$$

where X is a $N(0,1)$ random variable. Using the fact that $Pr[|X| > t] \leq \frac{2}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$, we have

$$\begin{aligned} Pr[|\langle \vec{r}, \vec{v} \rangle| > 2(\sqrt{\log n})] &\leq \frac{2}{2\sqrt{\log n}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(2\sqrt{\log n})^2} \\ &< \frac{1}{n^2} < \frac{1}{2m} \end{aligned}$$

Trust Claim 1+

From the definition of *local density* we know that at most n/k vertices at distance at most $n/(k \cdot 2D)$ from v.

Hence, when selecting k vertices randomly to form the set S, the expected distance from v to the closest other vertex in S would be $\Omega(n/(kD))$. And then $Tvol(S)$ would be $\Omega((n/kD)^{k-1})$. Finally we have

$$\sum_{S \subseteq V : |S| = k} \frac{1}{Tvol(S)} \leq \binom{n}{k}((n/kD)^{k-1}) < \frac{n}{k}(eD)^{k-1}$$

Sketch proof of Claim 1

We fix a set $S = \{s_0, s_1, \ldots, s_{k-1}\}$.

Recall that $\phi(s_i)$ is the image after the embedding and $L_i = span\{\phi(s_0), \phi(s_1), \ldots, \phi(s_{i-1})\}$. W.l.o.g, we assume that $s_0 = \vec{0}$, and $L_i = span\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_{i-1}}\}$ is the subspace spanned by the first $i-1$ basis vector. Let $\phi(s_i) = (s_{i1}, s_{i2}, \ldots, s_{ii}, 0, \ldots, 0)$. Here comes the key observation

$$(\eta, k)\text{-well-separatedness} \Rightarrow s_{ii} \geq \frac{q_i}{\eta}$$

Sketch proof of Claim 1, Cont.

Next we define

$$\psi(x) = \langle \phi(x), \vec{r} \rangle$$

where $\vec{r} = (r_1, \ldots, r_k)$ and $r_i \sim N(0,1)$ Then we have the following expression, let $I = [0, l)$.

$$\begin{aligned} Pr[\psi(S) \subseteq I] &= Pr[\psi(s_0) \in I] \times Pr[\psi(s_1) \in I | \psi(s_0) \in I] \times \\ &\quad \ldots \times Pr[\psi(s_{k-1}) \in I | \wedge_{j=1}^{k-2} \psi(s_j) \in I] \end{aligned}$$

Consider the expression

$$Pr[\psi(s_i) \in I | \wedge_{j=1}^{i-1} \psi(s_j) \in I \wedge (r_1 = \widehat{r_1} \wedge \ldots \wedge r_{i-1} = \widehat{r_{i-1}})]$$

Sketch proof of Claim 1, Cont.

Let $Z = \sum_{j<i} s_{ij}\widehat{r_j}$. If $\psi(s_i)$ fall into the interval I, it must be the case that $s_{ii}r_i \in [-Z, l-Z)$ Since the value r_i is independent of all the conditions, we have $s_{ii}r_i \sim N(0, s_{ii}^2)$ with $s_{ii} \geq q_i/\eta$, and hence[a]

$$Pr[\psi(s_i) \in I | \wedge_{j=1}^{i-1} \psi(s_j) \in I \wedge (r_1 = \widehat{r_1} \wedge \ldots \wedge r_{i-1} = \widehat{r_{i-1}})] \leq \frac{\eta l}{\sqrt{2\pi} q_i}$$

Since the inequality holds for every possible value of $r_j (j < i)$, we have

$$Pr[\psi(s_i) \in I | \wedge_{j=1}^{i-1} \psi(s_j) \in I] \leq \frac{\eta l}{\sqrt{2\pi} q_i}$$

[a] According to the fact that provided $X \sim N(0, \hat{\sigma}^2)$ with $\hat{\sigma} \geq \sigma$ and I be an interval of length l in the real line, we have $Pr[X \in I] \leq \frac{l}{\sqrt{2\pi}\sigma}$

A proof leak and solution

Notice that here we choose a random vector $\vec{r}$ in space $\mathbb{R}^k$. But we want to get a mapping from $\mathbb{R}^L$ to the real line. And intuitively, a vector would be shrunk by the random projection from $\mathbb{R}^L$ to $\mathbb{R}^k$. Fortunately, the following proposition comes to rescue.

Proposition: Let r be a random unit vector in $\mathbb{R}^L$ chosen with spherical symmetry, and let $\mathbb{R}^k$ be a subspace of $\mathbb{R}^L$. Let l denote the length of the projection of r on R^k. Then:

- Small projection: for every $0 < \epsilon < 1$, $Pr_r[l < \epsilon\sqrt{k}/\sqrt{L}] \leq (\beta\epsilon)^k$, for some universal $\beta > 0$.
- Large projection: for $c > 1$ and $k = 1$, $Pr_r[l > \sqrt{c/L}] \leq e^{-c/4}$. When L is large, the exponent tends to $-c/2$.

Outline

- Problems and Results
- Intuition and General Idea
- The Framework of the Method
- Algorithm and Explanation
- Major Techniques and Key Proofs
- Conclusion and Comments

The main algorithm

Algorithm Bandwidth(G)

1. Embed G to R^L using an (η, k)-well-separated mapping, where $L = c \log n \log D$, c is some constant and D is the diameter of the original graph.

2. Project all the vertices in the embedding on a random line, obtain for each vertex a point $h(v)$ on the line.

3. Sort $h(v)$, and output the sorted list of vertices as the linear arrangement.

To find a well separated mapping

Algorithm Well_Separated_Map(G)

1. Let $L = c \log n \log D$
2. For $t = 1$ to $\log D$
   (a) Let $\Delta = 2^t$
   (b) For $j = 1$ to $ck \log n$
   $f_{tj} = \mathbf{Generate\_Coordinate}(G, \Delta)$
3. Let $f = \oplus_{tj} f_{tj}$
4. Let $\phi = f/\sqrt{4L}$ for all t, j. $/*$ *to preserve "contracting"* $*/$

Gererate the coordinates of the embedding

Algorithm Generate_Coordinate(G, Δ)

1. Let $G' = G$, $S_{tj} = \emptyset$.
2. While $G' \neq \emptyset$ do
   (a) Pick up an arbitrary vertices $v \in G'$, and build a BFS tree rooted at v. Let $l(u)$ denotes the distance (level) from a vertex $u \in G'$ to v.
   (b) Let $r = \Delta/(4 \log n)$.
   (c) Define layers $Lay_k = \{u \mid l(u) \in [(k-1)r, kr)\}$ for all possible k, such that every layer contains r levels.
   (d) Pick one of the layers randomly, with the k^{th} layer chosen with probability $p_k = 2^{-k}$, and pick a level l uniformly at random within that layer.
   (e) Add all the vertices at distance l to S_{tj} and delete them as well as the whole component C bounded by those vertices from G'.
3. Choose a parameter $\gamma_{tj}(C) \in [1, 2]$ independently, uniformly at random for all the components resulting from the above decomposition. Assign the value $f_{tj}(v)$ for a vertex $v \in C$ with $f_{tj}(v) = \gamma_{tj} \cdot d_G(S_{tj}, v)$.

A few explanations

- The meaning of the two subscripts t, j in f_{tj}.
  - t: As shown in the algorithm Well_Separated_Map(G), the value of t denotes the value $\Delta \in \{1, 2, 4, \ldots, D\}$. Obviously, each q_i must fall into a particular $[\Delta, 2\Delta]$ where $\Delta = 2^t$. Therefore if we can prove that f_{tj} will be large (say, at least q_i/η for some small η) in many tj^{th} coordinates (t is fixed and j varies), we are done.
  - j: the $ck \log n$ copies of js is used to guarantee that with high possibility, at least a constant fractional of tj^{th}s will be large.

Outline

- Problems and Results
- Intuition and General Idea
- The Framework of the Method
- Algorithm and Explanation
- Major Techniques and Key Proofs
- Conclusion and Comments

Correctness of the well-separated mapping construction

- Claim 3: Algorithm Well_Separated_Map(G) is a randomized construction such that for an arbitrary point $x \in L_i$

$$Pr\left[\|\phi(s_i) - x\| > 3\frac{q_i}{\eta}\right] \geq 1 - n^{-3k}$$

where η is chosen to be $\log^{3/2} n$

The intuition of Claim 3

- If we choose an $2/\eta$-net for $B(\phi(s_i), n) \cap L_i$, where $B(\phi(s_i), n)$ is a ball centered at ϕs_i with radius n. The number of points of the $2/\eta$-net would be at most $(n\eta/2)^{i-1} = O(n^{2k})$ by a volume argument. And then we can claim that with high probability $(1 - n^{-k})$, $\phi(s_i)$ is at least $3q_i/\eta$ away from each point in the $2/\eta$-net.
- Thus for every point y in the space L_i, suppose x is the closest point to y in the $2/\eta$-net

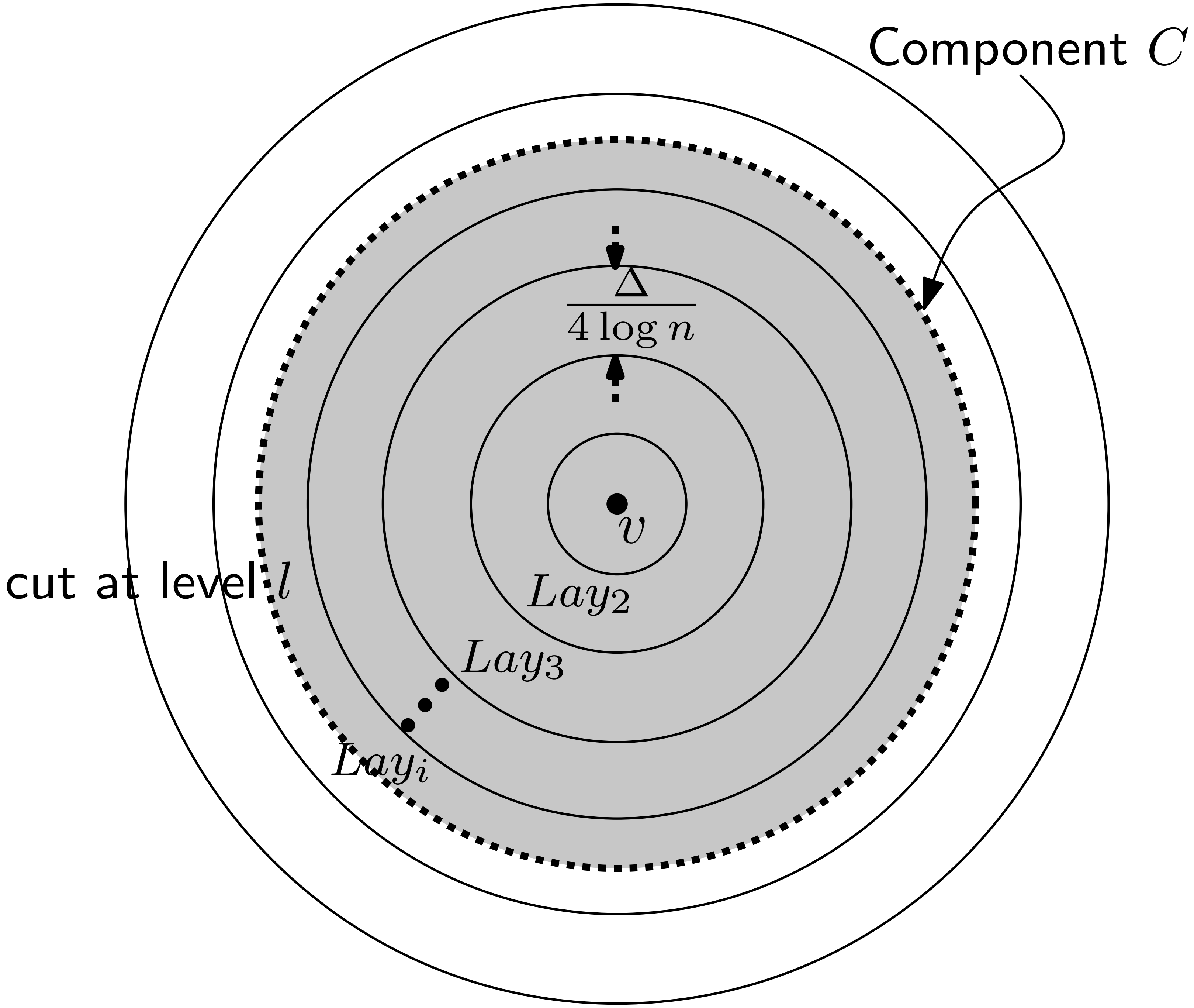
$$
\begin{aligned}
\|\phi(s_i) - y\|^2 &\geq \|\phi(s_i) - x\|^2 - \|x - y\|^2 \\
&\geq \left(\frac{3q_i}{\eta}\right)^2 - \left(\frac{2}{\eta}\right)^2 \geq \left(\frac{q_i}{\eta}\right)^2
\end{aligned}
$$

Gererate the coordinates of the embedding

Algorithm Generate_Coordinate(G, Δ)

1. Let $G' = G$, $S_{tj} = \emptyset$.
2. While $G' \neq \emptyset$ do
   (a) Pick up an arbitrary vertices $v \in G'$, and build a BFS tree rooted at v. Let $l(u)$ denotes the distance (level) from a vertex $u \in G'$ to v.
   (b) Let $r = \Delta/(4 \log n)$.
   (c) Define layers $Lay_k = \{u \mid l(u) \in [(k-1)r, kr)\}$ for all possible k, such that every layer contains r levels.
   (d) Pick one of the layers randomly, with the k^{th} layer chosen with probability $p_k = 2^{-k}$, and pick a level l uniformly at random within that layer.
   (e) Add all the vertices at distance l to S_{tj} and delete them as well as the whole component C bounded by those vertices from G'.
3. Choose a parameter $\gamma_{tj}(C) \in [1, 2]$ independently, uniformly at random for all the components resulting from the above decomposition. Assign the value $f_{tj}(v)$ for a vertex $v \in C$ with $f_{tj}(v) = \gamma_{tj} \cdot d_G(S_{tj}, v)$.

The illustration of decomposition



(a)

Proof idea of Claim 3

- Our task is trying to prove that $\phi(s_i)$ is far away from the subspace spanned by $\phi(s_0), \phi(s_1), \ldots, \phi(s_{i-1})$. Each point x in that subspace could be expressed as $x = \sum_{j<i} \lambda_j \phi(s_j)$ with $\sum_{j<i} \lambda_i = 1$
  - Prove that $f_{tj}(s_i)$ is far away from $f_{tj}(s_j)$ for every s_j $(j < i)$
  - Use the variational parameter γ_{tj} to prove that $f_{tj}(s_i)$ is far from $\lambda_j f_{tj}(s_j)$ for every s_j $(j < i)$ at a constant fractional of coordinates.

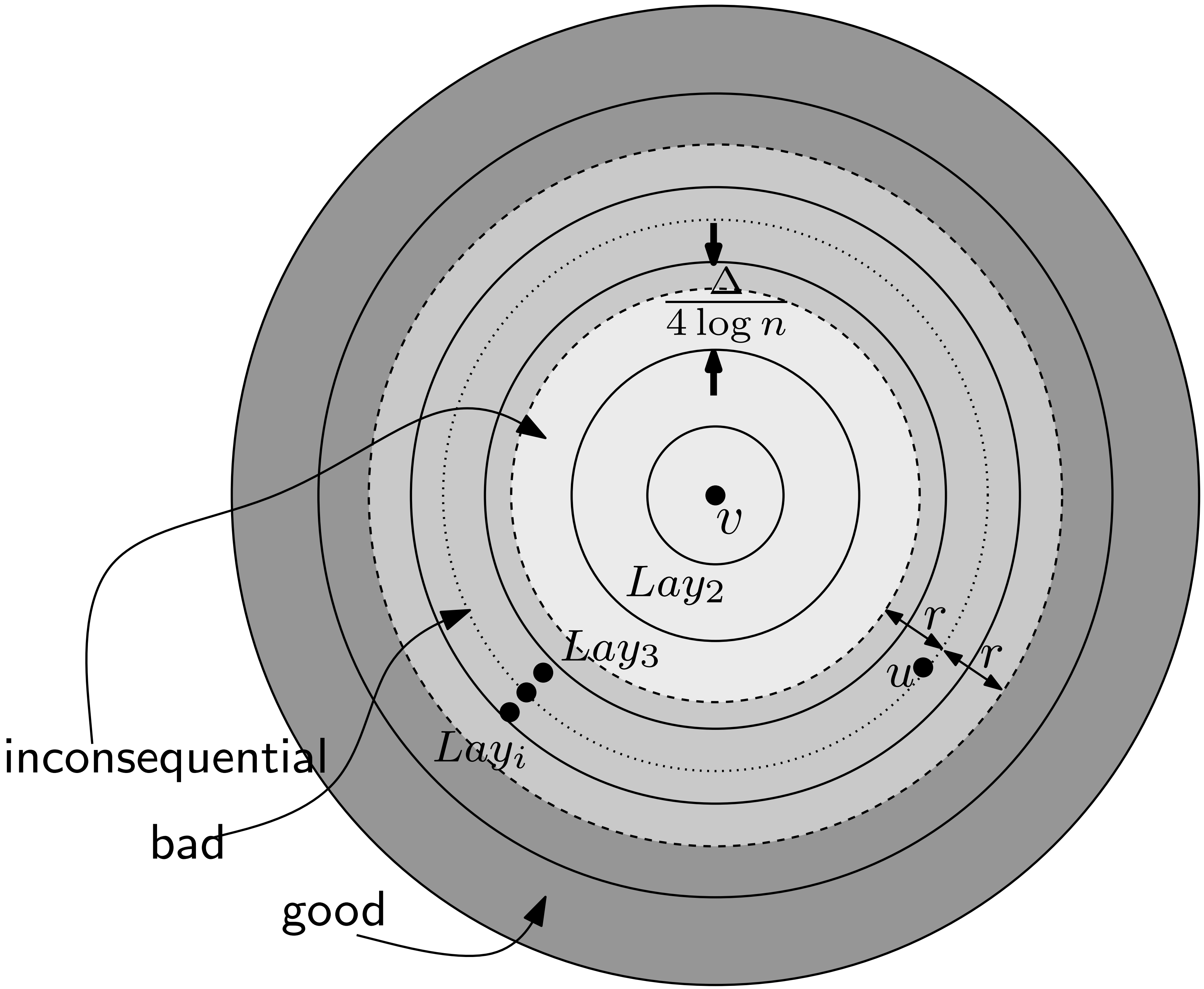
Proof sketch of Claim 3

- We say a particular coordinate f_{tj} created by the algorithm is eligible if all the components created during its creation have a diameter of at most Δ. For a given set S_{tj} defining a coordinate f_{tj}, we say a node u is δ-good if $d(S_{tj}, u) \geq \delta$.

  - **Observation 1:** A coordinate is eligible with probability $1 - \frac{1}{n}$

  - **Observation 2:** Each node $u \in G$ is $r = \Delta/(4 \log n)$-good with constant probability.

Each node $u \in G$ is r-good



(b)

Proof of the two observations

- Proof of the first observation:

  Notice a coordinate is not eligible only if a level greater than $\Delta/2$ is choose. Such a level lies in a layer greater than $(2\log n)$. But this layer is chosen with probability at most $2^{-2\log n} = 1/n^2$. Meanwhile, there are at most n components at that time.

Proof of the two observations, cont.

- Proof of the second observation:
  Suppose that at a particular time, $v \in G$ is chosen to be the root, u lies in the neighborhood of v and a cut happens:
  - A cut falls into a level in $[0, l(u) - r)$, it will not effect u
  - A cut falls into a level in $[l(u) - r, l(u) + r]$, we call it bad since $d(S_{tj}, u)$ is less than δ.
  - A cut falls into a level $> l(u) + r$, we call it good since u will be cut off this time and it will be at least δ far away from the boundary S_{tj} forever.

Proof of the two observations, cont.

- Proof of the second observation:
  Suppose that at a particular time, $v \in G$ is chosen to be the root, u lies in the neighborhood of v and a cut happens:
  - A cut falls into a level in $[0,\ l(u) - r)$, it will not effect u
  - A cut falls into a level in $[l(u) - r,\ l(u) + r]$, we call it bad since $d(S_{tj}, u)$ is less than δ.
  - A cut falls into a level $> l(u) + r$, we call it good since u will be cut off this time and it will be at least δ far away from the boundary S_{tj} forever.
- Note: u is δ-good if and only if a good cut happens before a bad one. And one can easily see that this is a constant, by our choices of p_ks.

Proof sketch of Claim 3, cont.

- For point s_i, we choose the coordinate f_{tj}s created when the procedure "Generate_Coordinate(G, Δ)" is called with $\Delta \in [q_i/2, q_i]$. With the above two observations, we notice that with some constant possibility β, a coordinate $f_{tj}(s_i)$ is both eligible and $\Delta/(4\log n)$-good.

Proof sketch of Claim 3, cont.

- For point s_i, we choose the coordinate f_{tj}s created when the procedure "Generate_Coordinate(G, Δ)" is called with $\Delta \in [q_i/2, q_i]$. With the above two observations, we notice that with some constant possibility β, a coordinate $f_{tj}(s_i)$ is both eligible and $\Delta/(4 \log n)$-good.
  - "The coordinate is eligible" means that s_i lies in a component C different from all the s_j $j < i$ during that decomposition procedure.
  - "The coordinate is also $\Delta/(4 \log n)$-good" means that $f_{tj}(s_i)$ is at least $\gamma_{tj}(C) \cdot \Delta/(4 \log n)$ away from all the $f_{tj}(s_j)$ $(j < i)$ at that coordinate.

Proof sketch of Claim 3, cont.

- For point s_i, we choose the coordinate f_{tj}s created when the procedure "Generate_Coordinate(G, Δ)" is called with $\Delta \in [q_i/2, q_i]$. With the above two observations, we notice that with some constant possibility β, a coordinate $f_{tj}(s_i)$ is both eligible and $\Delta/(4\log n)$-good.
  - "The coordinate is eligible" means that s_i lies in a component C different from all the s_j $j < i$ during that decomposition procedure.
  - "The coordinate is also $\Delta/(4\log n)$-good" means that $f_{tj}(s_i)$ is at least $\gamma_{tj}(C) \cdot \Delta/(4\log n)$ away from all the $f_{tj}(s_j)$ $(j < i)$ at that coordinate.
- The final power of randomization lies in $\gamma_{tj}(c)$

Proof sketch of Claim 3, cont.

By the fact that $\gamma_{tj}(c)$ varies in the interval $[1,2]$, we can conclude that $f_{tj}(s_i)$ is at least $\Delta/(12\log n)$ away from $\lambda_j f_{tj}(s_j)$ with possibility $\beta/3$.

Since there are $ck\log n$ copies of j, we conclude that there is a constant fraction (say, ρ) of coordinates that contribute to $\|\phi(s_i)-x\|$ with high probability $(1-n^{-3k})$.
Therefore, after dividing by $\sqrt{4L}=\sqrt{4ck\log n\log D}$, we conclude that

$$\|\phi(s_i)-x\| \geq O\left(\sqrt{\left(\frac{\Delta}{12\log n}\right)^2(\rho ck\log n)}\cdot\frac{1}{\sqrt{4L}}\right) = O\left(\frac{q_i}{\log^{3/2} n}\right)$$

The End

THANK YOU

Q and A