Sequences and Induction

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Sequences

Definition 1

A sequence is a function s whose domain is either $\{i \in \mathbb{Z} : m \leq i \leq n\}$, where $m \leq n$, or $\{i \in \mathbb{Z} : i \geq m\}$, and where the range of this function could be any set (also called the **alphabet**).

Definition 2

Finite sequences are usually defined by

$$s_m s_{m+1} s_{m+2} \dots s_n \text{ or } s_m, s_{m+1}, s_{m+2}, \dots, s_n$$

where m and n are integers with $m \leq n$.

Sequences

Definition 3

Infinite sequences are usually defined by

$$s_m s_{m+1} s_{m+2} \dots \text{ or } s_m, s_{m+1}, s_{m+2}, \dots,$$

where m is an integer.

In this course, we focus on the cases that $m = 0$ and $m = 1$, and denote such a sequence by $(s_i)_{i=m}^{\infty}$ or (s_i).

Example 4

The following is an infinite sequence:

$$s_i = i \text{ for all } i \in \mathbb{N}.$$

Sequences

Definition 5

An infinite sequence $(s_i)_{i=0}^{\infty}$ is called periodic with period n if

$$s_{n+i} = s_i \text{ for all } \geq 0.$$

Such least n is called the least period of the sequence.
An infinite sequence is called eventually periodic or ultimately periodic if the sequence becomes periodic after deleting a finite number of the initial terms.

Example 6

The following alternating sequence is periodic with least period 2:

$$s_i = (-1)^i \text{ for all } i \geq 0.$$

The Summation Notation

Definition 7

If m and n are positive integers with $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read **the summation from** m **to** n **of a-sub-**k, is defined by

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n,$$

where k is called the index, m the lower limit, and n the upper limit of of the summation.

The Product Notation

Definition 8

If m and n are positive integers with $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read **the product from** m **to** n **of a-sub-**k, is defined by

$$\prod_{k=m}^{n} a_k = a_m \times a_{m+1} \times \ldots \times a_n.$$

Properties of Summations and Products

The proof of the following properties is straightforward and omitted.

Theorem 9

If $(a_i)_{i=m}^{\infty}$ and $(b_i)_{i=m}^{\infty}$ are sequences of real numbers and c is any real number, then following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$.
2. $c\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} ca_k$.
3. $(\prod_{k=m}^{n} a_k)(\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} a_k b_k$.

The sum of a sequence in arithmetic progression

$S := \sum_{k=m}^{n} k$

Solution: We have

$$\begin{aligned} 2S &= [m+(m+1)+\cdots(n-1)+n]+[m+(m+1)+\cdots(n-1)+n] \\ &= [m+(m+1)+\cdots(n-1)+n]+[n+(n-1)+\cdots(m+1)+m] \\ &= (m+n)+(m+1+n-1)+\cdots+(n-1+m+1)+(n+m) \\ &= (m+n)(n-m+1). \end{aligned}$$

Consequently,

$$S = \frac{(m+n)(n-m+1)}{2}.$$

The sum of a geometric sequence

$S := \sum_{k=0}^{n} r^k$, where $r \neq 1$

Solution: We have

$$S = 1 + r + r^2 + \cdots + r^{n-1} + r^n$$

and

$$rS = r + r^2 + r^3 + \cdots + r^n + r^{n+1}.$$

It then follows that

$$(r-1)S = r^{n+1} - 1.$$

Hence

$$S = \frac{r^{n+1} - 1}{r - 1}.$$

What is Mathematical Induction?

In general, mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer.

Principle of Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. Then the statement for all integers $n \geq a$, $P(n)$ is true.

Remark

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the **principle** of mathematical induction rather than as a theorem.

Method of Proof by Mathematical Induction

Consider a statement of the form, “For all integers $n \geq a$, a property $P(n)$ is true.” To prove such a statement, perform the following two steps:

1. (Basis step) Show that $P(a)$ is true.
2. (Inductive step) **Suppose** that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$. Then **show** $p(k+1)$ is true.

Proof by Induction: Sum of the Arithmetic Sequence

Example 10

Let $S(n) = \sum_{i=1}^{n} i$ for all $n \in \mathbb{N}$. Prove that $S(n) = n(n+1)/2$.

Proof.

Basis step: By definition, $S(1) = 1 = 1(1+1)/2$. Hence $S(n) = n(n+1)/2$ holds for $n = 1$.

Inductive step: Suppose that $S(k) = k(k+1)/2$ for any $k \in \mathbb{N}$. We have

$$S(k+1) = S(k) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

Hence $S(n) = n(n+1)/2$ holds for $n = k+1$. This completes proof. $\square$

Proof by Induction: Sum of the Geometric Sequence

Example 11

Let $S(n) = \sum_{i=0}^{n} r^i$ for all integer $n \geq 0$, where r is any real number with $r \neq 1$. Prove that $S(n) = (r^{n+1} - 1)/(r - 1)$.

Proof.

Basis step: By definition, $S(0) = 1 = (r^{0+1} - 1)/(r - 1)$. Hence $S(n) = (r^{n+1} - 1)/(r - 1)$ holds for $n = 0$.
Inductive step: Suppose that $S(k) = (r^{k+1} - 1)/(r - 1)$ for any $k \in \mathbb{N}$. We have

$$S(k+1) = S(k) + r^{k+1} = \frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}.$$

Hence $S(n) = (r^{n+1} - 1)/(r - 1)$ holds for $n = k + 1$. This completes proof. $\square$

Strong Mathematical Induction

What is strong mathematical induction?

- Strong mathematical induction is similar to ordinary mathematical induction in that it is a technique for establishing the truth of a sequence of statements about integers.
- Also, a proof by strong mathematical induction consists of a basis step and an inductive step.
- However, the basis step may contain proofs for several initial values, and in the inductive step the truth of the predicate $P(n)$ is assumed not just for one value of n but for all values through k, and then the truth of $P(k+1)$ is proved.

Principle of Strong Mathematical Induction

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), p(a+1), p(a+2), \ldots, p(b)$ are all true (**basis step**).
2. For all integers $k \geq b$, if $P(i)$ is true for all i from a through k, then $P(k+1)$ is true. Then the statement for all integers $n \geq a$, $P(n)$ is true.

Remarks

- Any statement that can be proved with ordinary mathematical induction can be proved with strong mathematical induction.
- Any statement that can be proved with strong mathematical induction can be proved with ordinary mathematical induction.

Applying Strong Mathematical Induction

Theorem 12

Prove that any integer greater than 1 is divisible by a prime number.

Proof.

Let the property $P(n)$ be the sentence "n is divisible by a prime number".

- $P(2)$ is true, as 2 divides 2 and 2 is a prime.
- For any integer $k \geq 2$, suppose $P(i)$ is true for all integers i from 2 through k. We now prove that $P(k+1)$ is also true as follows:
    1. $k+1$ **is a prime:** In this case $k+1$ is divisible by a prime number, namely itself.
    2. $k+1$ **is not a prime:** In this case $k+1 = ab$, where $1 < a < k+1$ and $1 < b < k+1$.
       Thus, in particular, $2 \leq a \leq k$, and so by inductive hypothesis, a is divisible by a prime number p. Hence $k+1$ is divisible by p.

This completes the proof. $\square$

Applying Strong Mathematical Induction

Example 13

Define a sequence $(s_i)_{i=0}^{\infty}$ by

$$s_0 = 0, \; s_1 = 4, \; s_k = 6s_{k-1} - 5s_{k-2} \text{ for all integers } k \geq 2.$$

Prove that $s_n = 5^n - 1$.

Proof.

Let the property $P(n)$ be the sentence "$s_n = 5^n - 1$".

- $P(0)$ and $P(1)$ are clearly true.
- For any integer $k \geq 1$, suppose $P(i)$ is true for all integers i from 0 through k. We now prove that $P(k+1)$ is also true. We have

  $$s_{k+1} = 6s_k - 5s_{k-1} = 6(5^k - 1) - 5(5^{k-1} - 1) = 5^{k+1} - 1.$$

  Hence, $P(k+1)$ is also true.

□

An Exercise

The problem

Observe that

$$
\begin{aligned}
1 &= 1, \\
1-4 &= -(1+2), \\
1-4+9 &= 1+2+3, \\
1-4+9-16 &= -(1+2+3+4), \\
1-4+9-16+25 &= 1+2+3+4+5.
\end{aligned}
$$

Guess a general formula and prove it by mathematical induction.