COMP170
Discrete Mathematical Tools for Computer Science
Lecture 11

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.2, pp. 143-153

Recursion, Recurrences and Induction

- Recursion
- Recurrences
- Iterating a Recurrence
- Geometric Series
- First-Order Linear Recurrences

Recursion

- Recursive computer programs or algorithms often lead to inductive analyses
- A classic example of this is the Towers of Hanoi problem

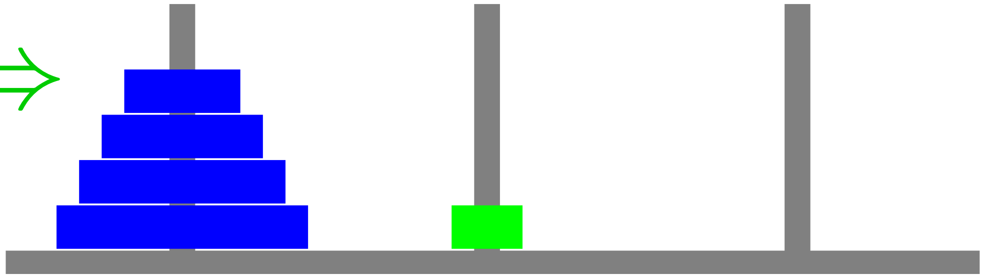
Towers of Hanoi

- 3 pegs; n disks of different sizes.
- A legal move takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- Problem: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

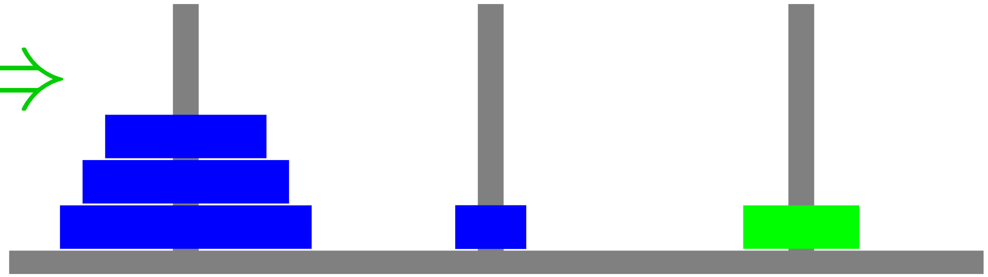
legal move

$\Rightarrow$



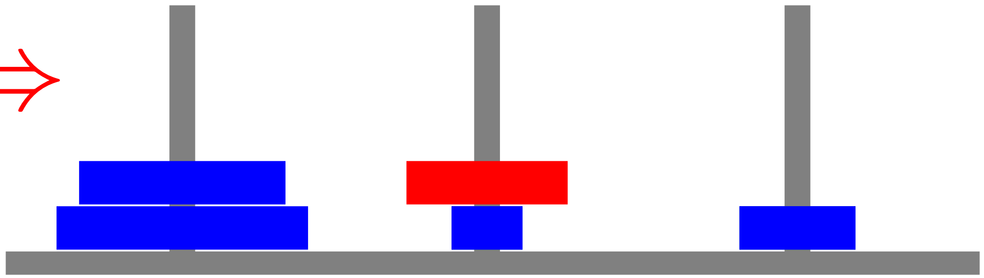
legal move

$\Rightarrow$



not legal

$\Rightarrow$



legal move

$\Rightarrow$

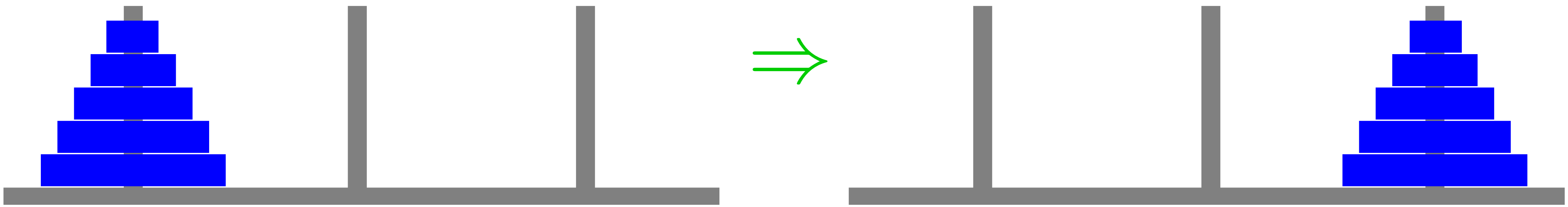
Towers of Hanoi

Problem

Start with n disks on leftmost peg

using only legal moves

move all disks to rightmost peg.



Given $i, j \in \{1,2,3\}$ let $\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$

i.e., $\overline{\{1,2\}} = 3$, $\overline{\{1,3\}} = 2$, $\overline{\{2,3\}} = 1$.

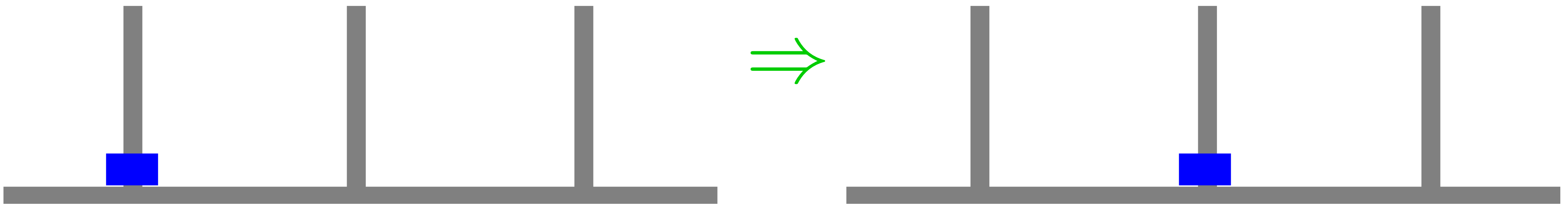
Towers of Hanoi

General Solution

Recursion Base:

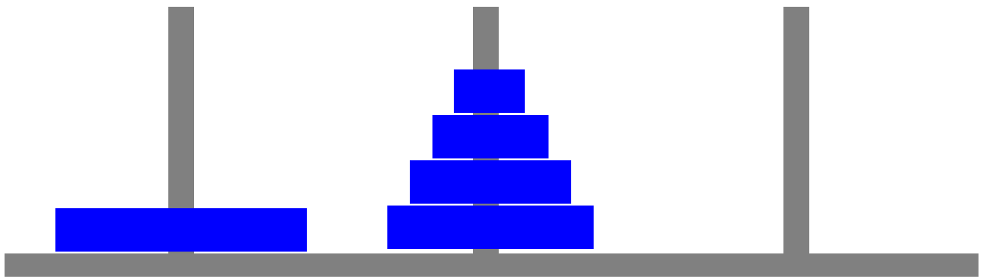
If $n = 1$ moving one disk from i to j is easy.

Just move it.

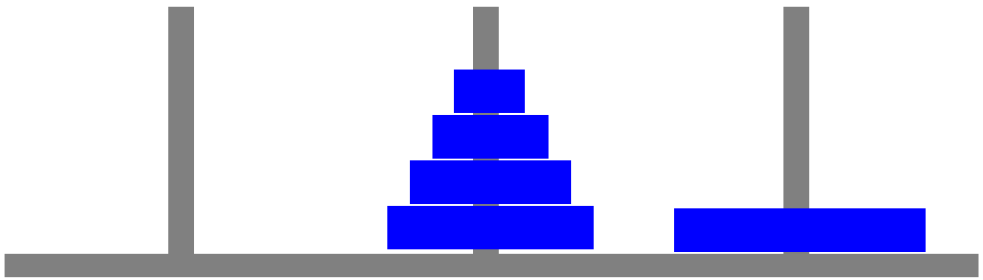


Towers of Hanoi

To move $n > 1$ disks from i to j

1) 

move top $n-1$ disks from i to $\overline{\{i,j\}}$

2) 

move largest disk from i to j

3) 

move top $n-1$ disks from $\overline{\{i,j\}}$ to j.

- To prove Correctness of solution we are implicitly using induction

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

iii) move top $n-1$ disks from $\overline{\{i,j\}}$ to j.

- $p(n)$ is statement that algorithm is correct for n

- $p(1)$ is statement that algorithm works for $n=1$ disks, which is obviously true

- $p(n-1) \Rightarrow p(n)$ is "recursion" statement that if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Running Time

$M(n)$ is number of disk moves needed for n disks

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

iii) move top $n-1$ disks from $\overline{\{i,j\}}$ to j.

- $M(1) = 1$
- If $n > 1$, then $M(n) = 2M(n-1) + 1$

- We saw that $M(1) = 1$ and that
- $M(n) = 2M(n-1) + 1$ for $n > 1$.

- Iterating the recurrence gives
  $$M(1) = 1, \;\; M(2) = 3, \;\; M(3) = 7,$$
  $$M(4) = 15, \;\; M(5) = 31, \ldots$$

- We guess that $M(n) = 2^n - 1$.
  - We'll prove this by induction
  - Later, we'll see how to solve without guessing

Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2M(n-1) + 1 & \text{otherwise.} \end{cases}$$

we show that $M(n) = 2^n - 1$

Proof: (by induction),

The base case $n = 1$ is true, since $2^1 - 1 = 1$.
For the inductive step, assume that
$M(n-1) = 2^{n-1} - 1$ for $n > 1$. Then

$$\begin{aligned} M(n) &= 2M(n-1) + 1 && \textit{def} \\ &= 2(2^{n-1} - 1) + 1 && \textit{ind hyp} \\ &= 2^n - 1. && \textit{ind conc} \end{aligned}$$

Note that we used induction twice

The first time was to derive Correcteness of Algorithm and the recurrence

$$M(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2M(n-1) + 1 & \text{otherwise.} \end{cases}$$

The second time was to derive the closed form solution

$$M(n) = 2^n - 1$$

of the recurrence

Francois Edouard Anatole Lucas

b. 1842, d. 1891

French mathematician.
Best known for his results in number theory.

He is also famous for being a creator of mathematical puzzles, among the most well-known of which is the Tower of Hanoi puzzle (1883).

Recursion, Recurrences and Induction

- Recursion
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A **recurrence equation** or **recurrence** for a function defined on the set of integers greater than or equal to some number b is one that tells us how to compute the nth value from some or all the first $(n-1)$ values.

To completely specify a function on the basis of a recurrence, we have to give the **initial condition(s)** (also called the *base case(s)*) for the recurrence.

$$M(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2M(n-1)+1 & \text{otherwise.} \end{cases}$$

Towers of Hanoi

$$F(n) = \begin{cases} 1 & \text{if } n = 0, 1, \\ F(n-1)+F(n-2) & \text{otherwise.} \end{cases}$$

Fibonacci Numbers

Example 2:

Let $S(n)$ be the number of subsets of a set of size n. What is the formula for $S(n)$?

The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

It is not difficult to see that
$S(1) = 2$, $S(2) = 4$, $S(3) = 8$,

We "guess" that $S(n) = 2^n$ but, in order to prove formula, we'll need to think recursively.

Consider the eight subsets of $\{1, 2, 3\}$:

$$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

$$\begin{array}{cccc} \emptyset & \{1\} & \{2\} & \{1,2\} \\ \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array}$$

First four subsets do not contain 3, but second four do.

First four subsets are exactly the subsets of $\{1,2\}$, while second four are the subsets of $\{1,2\}$ with 3 added into each one.

So, we get a subset of $\{1,2,3\}$ either by taking a subset of $\{1,2\}$ or by adjoining 3 to a subset of $\{1,2\}$.

This suggests that the recurrence for the number of subsets of an n-element set ($\{1,2,...,n\}$) is

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Proof (inductive) of correctness of this recurrence:

The subsets of $\{1, 2, \ldots, n\}$ can be partitioned according to whether or not they contain element n.

Each subset S containing n can be constructed in a unique fashion by adding n to the subset $S - \{n\}$ not containing n.

Each subset S not containing n can be constructed by removing n from the unique set $S \cup \{n\}$ containing n.

So, the number of subsets containing n is exactly the same as the the number of subsets not containing n.

The number of subsets not containing n is just the number of subsets of the $(n-1)$-element set $\{1, 2, \ldots, n-1\}$ which is $S(n-1)$.

Thus, if $n > 0$, then $S(n) = 2S(n-1)$.

We already observed that $\emptyset$ has only one subset (itself), so $S(0) = 1$ and we have proved the correctness of the recurrence.

If

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then, $S(n) = 2^n$ for all $n \geq 0$.

Proof: by induction

i) if $n = 0$ then $S(0) = 2^0 = 1$.

ii) If the statement is true for $n - 1$ then $S(n-1) = 2^{n-1}$ so

$$S(n) = 2S(n-1) = 2 \cdot 2^{n-1} = 2^n$$

and we are done!

Example 3:

When paying off a loan with initial amount A and monthly payment M at an interest rate of p percent, the total amount $T(n)$ of the loan still due after n months is computed by adding $p/12$ percent to the amount due after $n-1$ months and then subtracting the monthly payment M.

Convert this description into a recurrence for the amount owed after n months.

Answer

$$T(n) = (1 + \frac{0.01p}{12}) \cdot T(n-1) - M, \text{ with } T(0) = A.$$

We will now see a general tool for deriving closed form solution to these type of recurrence relations

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Iterating a Recurrence

Let $T(n) = rT(n-1) + a$,
where r and a are constants.

Find a recurrence that expresses
$T(n)$ in terms of $T(n-2)$
$T(n)$ in terms of $T(n-3)$
$T(n)$ in terms of $T(n-4)$
$\vdots$

Can you generalize this to find a closed form solution to
$T(n) = rT(n-1) + a$?

Note that $T(n) = rT(n-1) + a$, implies that,

$\forall i < n$, $T(n-i) = rT(n-(i-1)) + a$.

Then

$$\begin{aligned}
T(n) &= rT(n-1) + a \\
&= r(rT(n-2) + a) + a \\
&= r^2T(n-2) + ra + a \\
&= r^2(rT(n-3) + a) + ra + a \\
&= r^3T(n-3) + r^2a + ra + a \\
&= r^3(rT(n-4) + a) + r^2a + ra + a \\
&= r^4T(n-4) + r^3a + r^2a + ra + a.
\end{aligned}$$

From this, we can "guess" that

$$T(n) = r^nT(0) + a\sum_{i=0}^{n-1} r^i = \boxed{r^n b + a\sum_{i=0}^{n-1} r^i.}$$

The method we used to guess the solution is called **iterating the recurrence**, because we repeatedly (iteratively) use the recurrence.

Another aproach is to iterate from the "bottom-up" instead of "top-down".

$$
\begin{aligned}
T(0) &= b \\
T(1) &= rT(0) + a = rb + a \\
T(2) &= rT(1) + a = r(rb + a) + a = r^2 b + ra + a \\
T(3) &= rT(2) + a = r^3 b + r^2 a + ra + a
\end{aligned}
$$

This could lead to the same guess

$$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

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Geometric Series

$\sum_{i=0}^{n-1} r^i$ is a **finite geometric series with common ratio** r.

$\sum_{i=0}^{n-1} ar^i$ is a **finite geometric series with common ratio** r **and initial value** a .

It is known that, for all $r \neq 1$,

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}.$$

Note: We will see another proof of this soon.

Theorem 4.1

If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

for all nonnegative integers n.

Proof by induction:

$$T(0) = r^0 b + a\frac{1-r^0}{1-r} = b.$$

So, the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1-r^{n-1}}{1-r}.$$

Then we have
$$
\begin{aligned}
T(n) &= rT(n-1) + a \\
&= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\
&= r^n b + \frac{ar - ar^n}{1-r} + a \\
&= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\
&= r^n b + a\frac{1-r^n}{1-r}.
\end{aligned}
$$

Therefore, by the principle of mathematical induction, our formula holds for all integers $n \geq 0$.

Theorem 4.1

If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

for all nonnegative integers n.

Example:

$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

Plugging $r = 3$, $a = 2$, $b = 5$ into the formula, gives

$$T(n) = 3^n \cdot 5 + 2\frac{1-3^n}{1-3} = 3^n \cdot 6 - 1$$

Corollary 4.2: The formula for the sum of a geometric series with $r \neq 1$ is

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

Proof:

Let $T(0) = 0$, and $T(n) = \sum_{i=1}^{n-1} r^i$ for $n > 0$.

Then $T(n) = rT(n-1) + 1$.
Applying Theorem 4.1 with $b = 0$ and $a = 1$ gives

$$T(n) = \frac{1 - r^n}{1 - r}$$

Lemma 4.3: Let $r \neq 1$ be a positive value independent of n. Let $t(n)$ be the largest term in the geometric series

$$\sum_{i=0}^{n-1} r^i$$

Then the value of the geometric series is $O(t(n))$.

Proof: There are two cases.

i) $r < 1$: in which case, $t(n) = r^0 = 1$.

ii) $r > 1$: in which case $t(n) = r^{n-1}$

(i) is easy because in this case

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r} < \frac{1}{1 - r} \quad \text{which is } O(1) = O(t(n))$$

In case (ii), $r > 1$, $t(n) = r^{n-1}$ and

$$\sum_{i=0}^{n-1} r^n = \frac{1 - r^n}{1 - r} = \frac{r^n - 1}{r - 1} \leq \frac{r^n}{r - 1} = r^{n-1} \frac{r}{r - 1}$$

Thus, $\sum_{i=0}^{n-1} r^i = O\left(r^{n-1}\right) = O(t(n))$.

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First-Order Linear Recurrences

A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a **first-order linear recurrence**.

- First Order because it is only dependent upon going back one step, i.e., $T(n-1)$.

  If it was dependent upon $T(n-2)$, it would be a second-order recurrence, e.g., $T(n) = T(n-1) + 2T(n-2)$.

- Linear because $T(n-1)$ only appears to the first power.

  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

$$T(n) = f(n)T(n-1) + g(n)$$

When $f(n)$ is a constant, say r, the general solution is almost as easy to write as in Theorem 4.1. Iterating the recurrence gives

$$\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2T(n-2) + rg(n-1) + g(n) \\
&= r^3T(n-3) + r^2g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^nT(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}$$

Theorem 4.5 For any positive constants a and r, and any function g defined on the nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0, \\ a & \text{if } n = 0, \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i). \qquad (*)$$

Proof by induction:

Because the sum (*) has no terms when $n = 0$, the formula gives $T(0) = 0$ and, so, is valid when $n = 0$.

We now assume that n is positive and $T(n-1) = r^{n-1}a + \sum_{i=1}^{n-1} r^{(n-1)-i} g(i)$.

Using the definition of the recurrence and the inductive hypothesis, we get that

$$
\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r\left(r^{n-1}a + \sum_{i=1}^{n-1} r^{(n-1)-i} g(i)\right) + g(n) \\
&= r^n a + \sum_{i=1}^{n-1} r^{(n-1)+1-i} g(i) + g(n) \\
&= r^n a + \sum_{i=1}^{n-1} r^{n-i} g(i) + g(n) \\
&= r^n a + \sum_{i=1}^{n} r^{n-i} g(i).
\end{aligned}
$$

Therefore, by the principle of mathematical induction, the solution to the recurrence is given by (*) for all nonnegative integers n.

Example: Solve $T(n) = 4T(n-1) + 2^n$ with $T(0) = 6$.

Using Theorem 4.5

$$
\begin{aligned}
T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + 4^n \cdot \frac{1}{2} \cdot \sum_{i=0}^{n-1} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - (\frac{1}{2})^n) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n.
\end{aligned}
$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

Using Theorem 4.5

$$T(n) = 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i$$

$$= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i}$$

We now need the following well known theorem
(can be proven by induction or see book for another proof)

Theorem 4.6
For any real number $x \neq 1$,

$$\sum_{i=1}^{n} i x^i = \frac{n x^{n+2} - (n+1) x^{n+1} + x}{(1-x)^2}.$$

Example: Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$.

Using Theorem 4.5

$$
\begin{aligned}
T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\tfrac{3}{2}(n+1)3^{-(n+1)} - \tfrac{3}{4}3^{-(n+1)} + \tfrac{3}{4} \right) \\
&= \tfrac{43}{4}3^n - \tfrac{n+1}{2} - \tfrac{1}{4}
\end{aligned}
$$