

# COMP170

## Discrete Mathematical Tools for Computer Science

### Binomial Coefficients

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*Discrete Math for Computer Science*

*K. Bogart, C. Stein and R.L. Drysdale*

*Section 1.3, pp. 19-26*

# 1.3 Binomial Coefficients

- Pascal's Triangle
- A Proof using the Sum Principle
- The Binomial Theorem
- Labeling and Trinomial Coefficients

# Some properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  is the number of  $k$ -element subsets of an  $n$ -element set.
- $\binom{n}{0} = 1$  only one set of size 0.
- $\binom{n}{n} = 1$  only one set of size  $n$ .
- $\binom{n}{k} = \binom{n}{n-k}$  Obvious from equation. Can you think of a simple bijection that explains this?

# Some properties of Binomial Coefficients (cont)

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

Use Sum Principle

Let  $P$  = set of all subsets of  $\{1, 2, \dots, n\}$

$S_i$  = set of all  $i$  subsets of  $\{1, 2, \dots, n\}$

$$\Rightarrow |P| = \sum_{i=0}^n |S_i| = \sum_{i=0}^n \binom{n}{i}$$

Let  $L = L_1 L_2 \dots L_n$  be a list of size  $n$  from  $\{0, 1\}$

If  $\mathcal{L}$  = set of all such lists  $\Rightarrow |\mathcal{L}| = 2^n$

There is a *bijection* (next page) between  $\mathcal{L}$  and  $P$  so

$|P| = 2^n$  and we are done.

Let  $P =$  set of all subsets of  $\{1, 2, \dots, n\}$

Let  $L = L_1 L_2 \dots L_n$  be a list of size  $n$  from  $\{0, 1\}$   
and  $\mathcal{L} =$  set of all such lists

Define the following function  $f : \mathcal{L} \rightarrow P$

If  $L \in \mathcal{L}$  then  $f(L)$  is the set  $S \subseteq \{1, 2, \dots, n\}$  defined by

$$i \in S \Leftrightarrow L_i = 1$$

$f$  is a *bijection* between  $\mathcal{L}$  and  $P$  (why?) so  $|\mathcal{L}| = |P|$

Ex:  $n = 5$

$$f(10101) = \{1, 3, 5\}, \quad f(11101) = \{1, 2, 3, 5\}, \quad f(00000) = \emptyset.$$

*Note:  $L$  is sometimes called the incidence vector or membership vector associated with  $L$*

**Example:**  $n = 4$ ,  $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{cccccc} & \{1\} & \{1, 2\} & \{1, 3\} & \{1, 2, 3\} & \{1, 2, 3, 4\} \\ \{\} & \{2\} & \{1, 4\} & \{2, 3\} & \{1, 2, 4\} & \\ & \{3\} & \{2, 4\} & \{3, 4\} & \{1, 3, 4\} & \\ & \{4\} & & & \{2, 3, 4\} & \end{array} \right\}$$

**Example:**  $n = 4$ ,  $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{c|cc|cc|c} \{\} & \{1\} & \{1, 2\} & \{1, 3\} & \{1, 2, 3\} & \{1, 2, 3, 4\} \\ & \{2\} & \{1, 4\} & \{2, 3\} & \{1, 2, 4\} & \\ & \{3\} & \{2, 4\} & \{3, 4\} & \{1, 3, 4\} & \\ & \{4\} & & & \{2, 3, 4\} & \end{array} \right\}$$

$$P = \{S_0, S_1, S_2, S_3, S_4\}$$

**Example:**  $n = 4$ ,  $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{cccccc} & \{1\} & \{1, 2\} & \{1, 3\} & \{1, 2, 3\} & \{1, 2, 3, 4\} \\ \{\} & \{2\} & \{1, 4\} & \{2, 3\} & \{1, 2, 4\} & \\ & \{3\} & \{2, 4\} & \{3, 4\} & \{1, 3, 4\} & \\ & \{4\} & & & \{2, 3, 4\} & \end{array} \right\}$$

$$|S_0| = \binom{4}{0}, |S_1| = \binom{4}{1}, |S_2| = \binom{4}{2}, |S_3| = \binom{4}{3}, |S_4| = \binom{4}{4}$$



**Example:**  $n = 4$ ,  $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{l} \{\} \\ \{1\} \\ \{2\} \\ \{3\} \\ \{4\} \\ \{1, 2\} \\ \{1, 3\} \\ \{1, 2, 3\} \\ \{1, 2, 3, 4\} \\ \{1, 4\} \\ \{2, 3\} \\ \{1, 2, 4\} \\ \{2, 4\} \\ \{3, 4\} \\ \{1, 3, 4\} \\ \{2, 3, 4\} \end{array} \right\}$$

$$\begin{aligned} |P| &= |S_0| + |S_1| + |S_2| + |S_3| + |S_4| \\ &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\ &= 2^4 = 16 \end{aligned}$$

# Binomial Coefficients

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Each row begins with a 1 because  $\binom{n}{0} = 1$

Each row ends with a 1 because  $\binom{n}{n} = 1$ .

Each row increases at first and then decreases.  
*(will see why in homework)*

Second half of each row is the reverse of the first half.

Sum of items on  $n^{\text{th}}$  row is  $2^n$

# Pascal's Triangle

Take the table

and shift each row slightly  
so that middle element is  
in middle

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

				1				
			1		1			
		1		2		1		
	1	3		3		1		
	1	4		6		4		1
1	5	10		10		5		1
1	6	15	20	15	6		1	

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## Pascal relationship

Each (non-1) **entry** in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's relationship says that, for  $0 < k < n$ ,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

A purely *algebraic* proof (manipulating formulas) is possible.

In discrete mathematics, though, we prefer to derive intuitive explanations. In this case, that would involve interpreting Pascal's relationship as a statement describing *relationships among sets*.

# A Proof Using the Sum Principle

From Theorem 1.2 and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

we know  $\binom{n}{k}$  is the number of  $k$ -element subsets of an  $n$ -element set.

Therefore, each term (left and right) in

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

represents the number of subsets of a particular size chosen from an appropriately sized set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Number of  $k$ -subsets of an  $n$ -element set.

Number of  $(k-1)$ -subsets of an  $(n-1)$ -element set.

Number of  $k$ -subsets of an  $(n-1)$ -element set.

Try to use sum principle to explain relationship among these three terms.

Example:  $n = 5, k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider  $S = \{A, B, C, D, E\}$ .

Set  $S_1$  of 2-subsets of  $S$  can be partitioned into 2 disjoint parts.

$S_2$  the 2-subsets that contain  $E$  and

$S_3$ , the set of 2-subsets that do not contain  $E$ .

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

$S_2$  is equivalent to choosing 1 item out of  $\{A, B, C, D\}$ :  $|S_2| = \binom{4}{1}$

$S_3$  chooses 2 items out of  $\{A, B, C, D\}$ :  $|S_3| = \binom{4}{2}$

Sum Principle:  $\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}$



## Theorem 1.3

If  $n$  and  $k$  are integers satisfying  $0 < k < n$ , then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Proof:** Apply sum principle.

Partition set of  $k$ -element subsets of an  $n$ -element set into *disjoint union* of two other disjoint sets.

Suppose  $S = \{x_1, x_2, \dots, x_n\}$ .

Let  $S_1$  be set of all  $k$ -element subsets.  $|S_1| = \binom{n}{k}$

To apply sum principle, partition  $S_1$  into  $S_2$  and  $S_3$ .

Let  $S_2$  be set of  $k$ -element subsets that contain  $x_n$ .

Let  $S_3$  be set of  $k$ -element subsets that don't contain  $x_n$ .

$|S_3| = \binom{n-1}{k}$  since this is just how to choose a  $k$ -element subset from a  $(n-1)$  size set

$|S_2| = \binom{n-1}{k-1}$  since this is just how to choose a  $(k-1)$ -element subset from a  $(n-1)$  size set

$$\Rightarrow \binom{n}{k} = |S_1| = |S_2| + |S_3| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

# Blaise Pascal

- Born 1623; Died 1662
- French Mathematician
- A Founder of Probability Theory
- Inventor of one of the first (the 2nd?) mechanical calculating machines
- Pascal Programming Language named for him



# The Binomial Theorem

$$(x + y) = \binom{1}{0}x + \binom{1}{1}y$$

$$(x + y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}x^1y^1 + \binom{2}{2}y^2$$

$$\begin{aligned}(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\end{aligned}$$

# The Binomial Theorem

Number of  $k$ -element subsets of an  $n$ -element set is called a **binomial coefficient** because of its role in the algebraic expansion of a binomial  $(x + y)^n$ .

## Theorem 1.4 (Binomial Theorem)

For any integer  $n \geq 0$ ,

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

or, in summation notation,

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

Example:  $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

First,

$$\begin{aligned}
 & (x+y)(x+y)(x+y) \\
 &= [x(x+y) + y(x+y)](x+y) \\
 &= (xx + yx + xy + yy)(x+y) \\
 &= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y \\
 &= xxx + xyx + yxx + yxy + xx y + xy y + yxy + yyy.
 \end{aligned}$$

Each *monomial* term in the final result is of form  $x^{3-i}y^i$  and is the product of – one blue, one red, and one green.

For each color we can choose either an  $x$  or a  $y$ .

Coefficient of  $x^{3-i}y^i$  is

$$\begin{aligned}
 & \# \text{ of ways of choosing } i \text{ } y\text{'s from three colors} \\
 &= \binom{3}{i}
 \end{aligned}$$

Example:  $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

First,

$$\begin{aligned}
 & (x+y)(x+y)(x+y) \\
 &= [x(x+y) + y(x+y)](x+y) \\
 &= (xx + yx + xy + yy)(x+y) \\
 &= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y \\
 &= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.
 \end{aligned}$$

Alternatively, can think of the monomial as *lists* where each item of the list is either  $x$  or  $y$ .

Coefficient of  $x^{3-i}y^i$  is

$$\begin{aligned}
 & \# \text{ of lists containing } i \text{ } y\text{'s (and } (3-i) \text{ } x\text{'s)} \\
 &= \binom{3}{i}
 \end{aligned}$$

# Proof of the Binomial Theorem

Use same explanation as in example just seen:

When multiplying three factors of  $x + y$ , we get a sum of eight products (monomials, lists)

Each factor  $x + y$  doubles the number of monomials

Thus, product of  $n$  binomials,  $(x + y)^n$ ,  
is sum of  $2^n$  monomials

Each monomial is a length- $n$  list of  $x$ 's and  $y$ 's.

In each list, the  $i$ th entry  
comes from the  $i$ th binomial factor.



A list that becomes  $x^{n-k}y^k$  (after applying commutative law) will have a  $y$  in  $k$  places and an  $x$  in the remaining  $(n - k)$  places.

Number of lists that have a  $y$  in  $k$  places is thus the number of ways to select  $k$  binomial factors to contribute a  $y$  to our list.

Number of ways to select  $k$  binomial factors from  $n$  binomial factors is simply  $\binom{n}{k}$ .

Therefore, the coefficient of  $x^{n-k}y^k$ , is  $\binom{n}{k}$ .

What is  $(x + 1)^4$ ?

By applying the binomial theorem

$$(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

What is  $(2 + y)^4$ ?

$$(2 + y)^4 = 16 + 32y + 24y^2 + 8y^3 + y^4.$$

What is  $(x + y)^4$ ?

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

At the beginning of this lesson we gave a combinatorial proof that

$$\sum_{i=0}^n \binom{n}{i} = 2^n.$$

Here's another, simple, algebraic, proof.

We just saw that

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$$

Now set  $x = y = 1$ . This gives

$$2^n = (1 + 1)^n = \sum_{i=0}^n \binom{n}{i}$$

# Labelling and Trinomial Coefficients

- Suppose we have  $k$  labels of one kind, e.g., red and  $n - k$  labels of another, e.g., green. In how many different ways can we apply these labels to  $n$  objects?
- Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., green, and  $k_3 = n - k_1 - k_2$  labels of a third kind, e.g., orange, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to  $n$  objects
- What is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x + y + z)^n$

- Suppose we have  $k$  labels of one kind, e.g., red and  $n - k$  labels of another, e.g., green. In how many different ways can we apply these labels to  $n$  objects?

There are  $\binom{n}{k}$  ways to choose the items with red labels.  
The other  $n - k$  items will then get the green labels  
So this is just  $\binom{n}{k}$

- Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., green, and  $k_3 = n - k_1 - k_2$  labels of a third kind, e.g., orange, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to  $n$  objects

There are  $\binom{n}{k_1}$  ways to choose the red items

There are then  $\binom{n-k_1}{k_2}$  ways to choose the green items from the remaining  $n - k_1$ .

The remaining  $k_3$  items get labelled orange

Using the *product principle* the total number of labellings is

$$\begin{aligned} \binom{n}{k_1} \binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!} \end{aligned}$$

When  $k_1 + k_2 + k_3 = n$ , we call

$$\frac{n!}{k_1!k_2!k_3!}$$

a *trinomial coefficient* and denote it as

$$\binom{n}{k_1 \ k_2 \ k_3}$$

Note that this slightly modifies the notation for *binomial coefficients*. If we really wanted the notation to be consistent (which we don't) we could write the binomial coefficient  $\binom{n}{k}$  as

$$\binom{n}{k \ (n-k)}$$

We really just saw that the Trinomial Coefficient

$$\binom{n}{k_1 \ k_2 \ k_3}$$

is the number of ways to partition a set of size  $n$  into three subsets (where order of the subsets does not count) of sizes  $k_1$ ,  $k_2$  and  $k_3$ .



- What is the coefficient of  $x^{k_1} y^{k_2} z^{k_3}$  in  $(x + y + z)^n$

Example:

$$\begin{aligned} & (x + y + z)(x + y + z)(x + y + z)(x + y + z) \\ &= xxxx + xxxy + xxxz + xxyx + \cdots + zzzz. \end{aligned}$$

After opening the parentheses and multiplying,  
there will be, in total,  $3^4 = 81$  different monomial terms (lists)

Each term, (after rewriting using commutativity),  
is in the form  $x^{k_1} y^{k_2} z^{k_3}$  where  $k_1 + k_2 + k_3 = 4$

The coefficient of  $x^{k_1} y^{k_2} z^{k_3}$  is exactly the number of ways of  
writing a list of size 4 with  $k_1$   $x$ 's,  $k_2$   $y$ 's, and  $k_3$   $z$ 's such that  
 $k_1 + k_2 + k_3 = 4$ , which is

$$\binom{4}{k_1 \ k_2 \ k_3}$$

- What is the coefficient of  $x^{k_1} y^{k_2} z^{k_3}$  in  $(x + y + z)^n$

After opening the parentheses and multiplying,  
there will be, in total,  $3^n$  different monomial terms (lists) .

Each term, (after rewriting using commutativity),  
is in the form  $x^{k_1} y^{k_2} z^{k_3}$  where  $k_1 + k_2 + k_3 = n$

The coefficient of  $x^{k_1} y^{k_2} z^{k_3}$  is exactly the number of ways of  
writing a list of size  $n$  with  $k_1$   $x$ 's,  $k_2$   $y$ 's, and  $k_3$   $z$ 's such that  
 $k_1 + k_2 + k_3 = n$ , which is

$$\binom{n}{k_1 \ k_2 \ k_3}$$