Implicit Function

Siu-Wing Cheng

Room 3514
Phone: 2358–6973
scheng@cse.ust.hk
http://www.cse.ust.hk/faculty/scheng

Department of Computer Science and Engineering
The Hong Kong University of Science and Technology
Hong Kong

Implicit Function in $\mathbb{R}^2$

In general, a function $f(x_1, x_2) = 0$ describes a curve in $\mathbb{R}^2$.
Examples:

- $x_1^2 + x_2^2 = 1$.
- $x_1^2 - x_2 = 0$.

Implicit Function in $\mathbb{R}^2$

In general, a function $f(x_1, x_2) = 0$ describes a curve in $\mathbb{R}^2$.
Examples:

- $x_1^2 + x_2^2 = 1$.
- $x_1^2 - x_2 = 0$.

What are the sets of points $f(x_1, x_2) > 0$ and $f(x_1, x_2) < 0$?

Implicit Function in $\mathbb{R}^3$

In general, a function $f(x_1, x_2, x_3) = 0$ describes a surface in $\mathbb{R}^3$.
Examples:

- $x_1^2 + x_2^2 + x_3^2 = 1$.
- $x_1^2 + x_2^2 - x_3 = 0$.

Implicit Function in $\mathbb{R}^3$

In general, a function $f(x_1, x_2, x_3) = 0$ describes a surface in $\mathbb{R}^3$.
Examples:

- $x_1^2 + x_2^2 + x_3^2 = 1.$
- $x_1^2 + x_2^2 - x_3 = 0.$

What are the sets of points $f(x_1, x_2) > 0$ and $f(x_1, x_2) < 0$?

Approximation Surface in $\mathbb{R}^3$

Given a set P of point samples, our goal is to construct an implicit function $f(x_1, x_2, x_3) = 0$ so that f approximates the unknown surface.

Approximation Surface in $\mathbb{R}^3$

Given a set P of point samples, our goal is to construct an implicit function $f(x_1, x_2, x_3) = 0$ so that f approximates the unknown surface.

Once we know f, we can then use f to compute the restricted Delaunay triangulation. The triangulation gives us a triangulated surface that approximates the unknown surface.

Approximation Surface in $\mathbb{R}^3$

Desirable properties of f:

Approximation Surface in $\mathbb{R}^3$

Desirable properties of f:

- The surface $f(x_1, x_2, x_3) = 0$ should pass close to the given point samples.

Approximation Surface in $\mathbb{R}^3$

Desirable properties of f:

- The surface $f(x_1, x_2, x_3) = 0$ should pass close to the given point samples.
- The surface normals of $f(x_1, x_2, x_3) = 0$ in the vicinity of the given point samples should approximate the surface normals of the unknown surface at these point samples.

Approximation Surface in $\mathbb{R}^3$

Desirable properties of f:

- The surface $f(x_1, x_2, x_3) = 0$ should pass close to the given point samples.
- The surface normals of $f(x_1, x_2, x_3) = 0$ in the vicinity of the given point samples should approximate the surface normals of the unknown surface at these point samples.
- The function $f(x_1, x_2, x_3)$ can be evaluated quickly.

Vectors

A vector $\mathbf{n}$ in $\mathbb{R}^3$ has a direction and a magnitude. It is represented by three coordinates (n_1, n_2, n_3) just like a point.

The representation (n_1, n_2, n_3) means that $\mathbf{n}$ can be viewed as the line segment directed from the origin to the point (n_1, n_2, n_3). However, we are actually free to translate $\mathbf{n}$. So $\mathbf{n}$ is not fixed in any particular position.

The magnitude of $\mathbf{n}$ is equal to $\sqrt{n_1^2 + n_2^2 + n_3^2}$. This can be viewed as the length of the vector.

Vectors

Given two points $x, y \in \mathbb{R}^3$, we can use $x - y$ to denote the vector directed from y to x. Its representation is

$$(\, x_1 - y_1,\, x_2 - y_2,\, x_3 - y_3 \,).$$

Surface Normal

Given a point p on a surface,

Surface Normal

Given a point p on a surface,

- A surface normal at p is a vector that is orthogonal to the surface at p.

Surface Normal

Given a point p on a surface,

- A surface normal at p is a vector that is orthogonal to the surface at p.
- A unit normal at p is a normal with unit length.

Surface Normal

Given a point p on a surface,

- A surface normal at p is a vector that is orthogonal to the surface at p.
- A unit normal at p is a normal with unit length.
- A unit outward normal at p is a unit normal that points outside the surface.

A Motivating Example

Suppose that the unknown surface is a plane. Let p be a given point sample on the plane.

Suppose that we known the normal $\mathbf{n}_p$ to the plane at the point p. Given a point $x \in \mathbb{R}^3$, how would you test whether x lies the plane or not?

A Motivating Example

Suppose that the unknown surface is a plane. Let p be a given point sample on the plane.

Suppose that we known the normal $\mathbf{n}_p$ to the plane at the point p. Given a point $x \in \mathbb{R}^3$, how would you test whether x lies the plane or not?

The point x lies on the plane iff the inner product $\langle \mathbf{n}_p, (x - p) \rangle = 0$.

A Motivating Example

Suppose that the unknown surface is a plane. Let p be a given point sample on the plane.

Suppose that we known the normal $\mathbf{n}_p$ to the plane at the point p. Given a point $x \in \mathbb{R}^3$, how would you test whether x lies the plane or not?

The point x lies on the plane iff the inner product $\langle \mathbf{n}_p, (x-p) \rangle = 0$.

The inner product $\langle \mathbf{n}_p, (x-p) \rangle$ is equal to

$$|\mathbf{n}_p| \cdot |x-p| \cdot \cos \angle \mathbf{n}_p, (x-p).$$

It can also be evaluated as

$$n_{p,1}(x_1-p_1) + n_{p,2}(x_2-p_2) + n_{p,3}(x_3-p_3).$$

A Motivating Example

Since any surface is quite flat in a small neighborhood around p, given a point x very close to p, we can use

$$\langle \mathbf{n}_p, (x - p) \rangle = 0$$

to tell if x is on the surface or not.

A Motivating Example

Since any surface is quite flat in a small neighborhood around p, given a point x very close to p, we can use

$$\langle \mathbf{n}_p, (x - p) \rangle = 0$$

to tell if x is on the surface or not.

If a point x is far from p, we should find another given point sample q very close to x, compute $\langle \mathbf{n}_q, (x - q) \rangle$, and make the decision.

Local Support and Weight Function

Define a weight function $w : P \times \mathbb{R}^3 \to \mathbb{R}$:

$$w(p, x) = \frac{e^{-(d(p,x)/r(x))^2}}{\sum_{q \in P} e^{-(d(q,x)/r(x))^2}}$$

where $r(x)$ is the distance from x to the fifth nearest sample point in P.

Local Support and Weight Function

Define a weight function $w : P \times \mathbb{R}^3 \to \mathbb{R}$:

$$w(p,x) = \frac{e^{-(d(p,x)/r(x))^2}}{\sum_{q \in P} e^{-(d(q,x)/r(x))^2}}$$

where $r(x)$ is the distance from x to the fifth nearest sample point in P.

Intuitively speaking,

- $w(p,x)$ is a value between 0 and 1.
- $w(p,x)$ increases as x moves towards p.
- $w(p,x)$ drops rapidly as x moves away from p.

An Implicit Function

$$f(x) = \sum_{p \in P} \langle \mathbf{n}_p, (x - p) \rangle \cdot w(p, x)$$

An Implicit Function

$$f(x) = \sum_{p \in P} \langle \mathbf{n}_p, (x - p) \rangle \cdot w(p, x)$$

Remaining issue: How to obtain $\mathbf{n}_p$?

An Implicit Function

$$f(x) = \sum_{p \in P} \langle \mathbf{n}_p, (x - p) \rangle \cdot w(p, x)$$

Remaining issue: How to obtain $\mathbf{n}_p$?

Can be done using Principal Component Analysis.
Details skipped.