Union Find

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Disjoint Set Union-Find

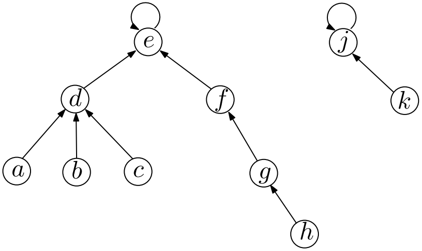
A disjoint set Union-Find data structure supports three operations on collections of disjoint sets over some universe U. For any $x, y \in U$:

1. Create-Set(x)
    - Create a set containing a single item x.
2. Find-Set(x)
    - Find the set that contains x
3. Union(x, y)
    - Merge the set containing x, and another set containing y to a single set.
    - After this operation, we have Find-Set(x) $=$ Find-Set(y).

Outline

- The Disjoint Set Union-Find data structure
    - The basic implementation
    - An improvement

Up-Tree Implementation



- Every item is in a tree. (Do not confuse these with the subtrees formed by Kruskal's algorithm.)
- The root of the tree is the representative item of all items in that tree
  - i.e., the root of the tree represents the whole items.
  - use the root's ID as the unique ID of the set.
- In this up-tree implementation, every node (except the root) has a pointer pointing to its parent.
  - The root element has a pointer pointing to itself.

Create-Set(x) and Find-Set(x)

Create-Set(x): easy

```
x.parent=x;
```

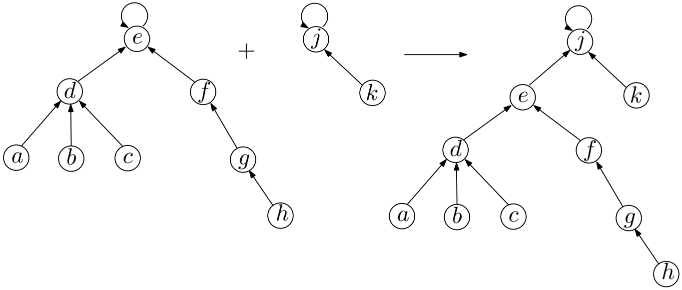
Find-Set(x): also easy

- simply trace the parent point until we hit the root, then return the root element.

```
while x ≠ x.parent do
    x = x.parent;
end
return x
```

Naive solution:

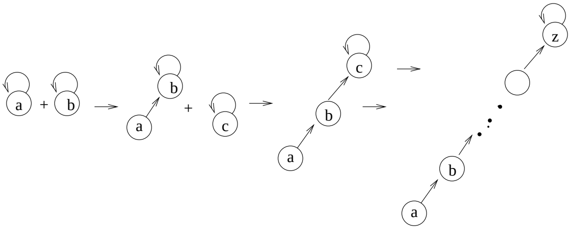
- put the parent pointer of the representation of x pointing to the representation of y.



Question

Is this a good idea?

Problem



May become a linked-list at the end! Hence it is not efficient.

Question

Can we do better?

Simple trick (Union by height):

- when we union two trees together, we always make the root of the taller tree the parent of shorter tree.

Up-Tree Implementation : Union by Height

- The root of every tree also holds the height of the tree.
- In case two trees have the same height, we choose the root of the first tree point to the root of the second. And the tree height is increased by 1.

Union(x, y)

```
a=Find-Set(x);
b=Find-Set(y);
if a.height ≤ b.height then
    if a.height == b.height then
        b.height++;
    end
    a.parent=b;
else
    b.parent=a;
end
```

Lemma

For the root x of any tree, let $size(x)$ denote the number of nodes and $h(x)$ be the height of the tree. Then $size(x) \geq 2^{h(x)}$.

Proof.

(By induction)

1. At beginning, $h(x) = 0$, and $size(x) = 1$. We have $1 \geq 2^0 = 1$.
2. Suppose the assumption is true for any x and y before Union(x, y). Let the size and height of the resulting tree be $size(x')$, and $h(x')$.
   - $h(x) < h(y)$, we have
     $$size(x') = size(x) + size(y) \geq 2^{h(x)} + 2^{h(y)} \geq 2^{h(y)} = 2^{h(x')}.$$
   - $h(x) = h(y)$, we have
     $$size(x') = size(x) + size(y) \geq 2^{h(x)} + 2^{h(y)} = 2^{h(y)+1} = 2^{h(x')}.$$
   - $h(x) > h(y)$, is similar to the first case

$\square$

Lemma

For n items, the running time of

- *Create-Set is $O(1)$,*
- *Find-Set is $O(\log n)$, and*
- *Union is $O(\log n)$*

respectively.

Proof.

- Obviously, Create-Set(x) is $O(1)$, and the running time of Union(x, y) depends on Find-Set(x).
- Since the running time of Find-Set(x) depends on the height of the tree. From previous lemma, for any tree, we have

$$\begin{aligned} n \geq 2^h &\Rightarrow h \leq \log n \\ &\Rightarrow h = O(\log n) \end{aligned}$$

Hence we have Find-Set(x) $= O(\log n)$. □

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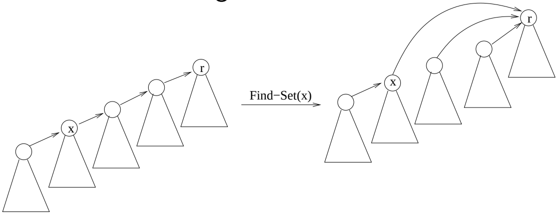
Up-Tree Implementation: Path Compression

- We can make the running time even faster if we add another trick.
- In Find-Set(x), we trace the path from x to the root.
- Let r be the root of the tree, and the path from x to r is $x a_1 a_2 \ldots a_k r$.
- As a by-product, we also make all the parent pointers of $x, a_1, a_2, \ldots a_k$ pointing to r directly.
  - Shortens the time of some future calls to Find-Set.
  - Does not increase height.



- This idea is called path compression.

Path Compression...

Question

Does path compression improves the running time of union-find?

$\lg^{(i)} n$: defined recursively for nonnegative integers i as

$$\lg^{(i)} n = \begin{cases} n & \text{if } i = 0 \\ \lg(\lg^{(i-1)} n) & \text{if } i > 0 \text{ and } \lg^{(i-1)} n > 0, \\ \text{undefined} & \text{if } i > 0 \text{ and } \lg^{(i-1)} n \leq 0, \text{ or } \lg^{(i-1)} n \text{ is undefined.} \end{cases}$$

The iterated logarithm is defined as

$$\lg^* n = \min \{i \geq 0 : \lg^{(i)} n \leq 1\}$$

- a very slow growing function.
- e.g.,
  $\lg^* 2 = 1, \lg^* 4 = 2, \lg^* 16 = 3, \lg^* 65536 = 4, \lg^* 2^{65536} = 5.$

Path Compression...

The following theorem is stated without proof.

Theorem

A sequence of m Create-Set, Find-Set and Union operations, n of which are Create-Set operations, can be performed on a disjointed-set forest with union by height and path compression in worst-case time $O(m \lg^* n)$.

Question

What is the running time of Kruskal's algorithm if we employ this implementation of disjoint set Union-Find?