# The Price of Routing Unsplittable Flow

Baruch Awerbuch Yossi Azar Amir Epstein

presented by Yajun Wang (yalding@cs.ust.hk)

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# Problem Formulation

- Graph $G = (V, E)$ and $k$ source-destination pairs $\{s_i, t_i\}$
- $\mathcal{Q}_i$ denotes the set of (simple) $s_i - t_i$ paths, and
- Latency function $f_e \,:\, \mathcal{R}^+ \rightarrow \mathcal{R}^+$
- Bandwidth request $(s_j, t_j, w_j)$ $w_j \in \mathcal{R}^+$
- A flow is a function vector $(l_j)$.

$$l_j \,:\, \mathcal{Q}_j \rightarrow \mathcal{R}^+$$

- A flow is feasible if :

$$\textstyle\sum_{Q \in \mathcal{Q}_j} l_j(Q) = w_j$$

# Flow and Strategy

- Splittable Flow

$$l_j(Q) \in [0, w_j]$$

- Unsplittable Flow

$$l_j(Q) \in \{0, w_j\}$$

Pure Strategies:

User $j$ selectes a single path $Q \in \mathcal{Q}_j$.

Mixed Strategies:

User $j$ selectes a probability distribution $\{p_{Q,j}\}$ over $\mathcal{Q}_j$.

# Latency for Users

- Pure Strategies:

  Let $\mathcal{S}$ be the system of strategies.

  Let $Q_j$ be the choice of user $j$, and $\mathcal{Q} = \cup_j Q_j$.

  Define $J(e) = \{j \mid e \in \mathcal{Q}\}$ and $l_e = \sum_{j \in J(e)} w_j$.

Latency (per unit) of user $j$ for select path $Q$ (instead of $Q_j$):

$$c_{Q,j} = \sum_{(e \in Q) \wedge (e \in Q_j)} f_e(l_e) + \sum_{(e \in Q) \wedge (e \notin Q_j)} f_e(l_e + w_j)$$

## Latency for Users

- Mixed Strategies:

  Let $\mathcal{S}$ be the system of strategies with $\{p_j\}$

  Let $\{X_{Q,j}\}$ be the set of indicator random variables: whether request $j$ is assigned to $Q$.

  $X_{e,j} = \sum_{Q|e \in Q} X_{Q,j} \qquad l_e = \sum_{j=1}^{n} X_{e,j} w_j$

Expected latency (per unit) of user $j$ for select path $Q$ in $\mathcal{S}$

$$\begin{aligned} c_{Q,j} &= E[\sum_{e \in Q} f_e(l_e) \,|\, X_{Q,j} = 1] \\ &= E[\sum_{e \in Q} f_e(\sum_{i=1, i \neq j}^{n} X_{e,i} w_i + w_j)] \\ &= \sum_{e \in Q} E[f_e(l_e + (1 - X_{e,j}) w_j)] \end{aligned}$$

# Nash Equilibrium

A system $\mathcal{S}$ is at Nash equilibrium if and only if for every $j \in \{1, 2, \ldots, n\}$ and $Q, Q' \in \mathcal{Q}_j$, with $p_{Q,j} > 0 (Q = Q_j)$

$$c_{Q,j} \leq c_{Q',j}$$

Social cost (expected) for system $\mathcal{S}$ is:

$$C(\mathcal{S}) = E[\sum_{e \in E} f_e(l_e) l_e]$$

Coordination Ration (Price of Anarchy) is:

$$R = \max_{\mathcal{S}} \frac{C(\mathcal{S})}{C(\mathcal{S}^*)}$$

$\mathcal{S}$ takes over all Nash equilibrium(N.E), and $\mathcal{S}^*$ is the Social Optimal(S.O) solution.

# Nash Equilibrium for Linear Latency Functions

**Theorem** For linear latency functions and pure strategies, the worse-case coordination ratio $R$ is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$

Proof: Let $Q_j$ be the path assigned for request $j$ in N.E. Let $Q_j^*$ be the path assigned for request $j$ in S.O.

$$\begin{aligned}\sum_{e \in Q_j} a_e l_e + b_e &\leq \sum_{(e \in Q_j^*) \wedge (e \in Q_j)} a_e l_e + b_e + \sum_{(e \in Q_j^*) \wedge (e \notin Q_j)} a_e (l_e + w_j) + b_e \\ &\leq \sum_{e \in Q_j^*} a_e (l_e + w_j) + b_e\end{aligned}$$

$$\sum_j \sum_{e \in Q_j} (a_e l_e + b_e) w_j \leq \sum_j \sum_{e \in Q_j^*} (a_e l_e + b_e) w_j + a_e w_j^2$$

$$\sum_{e \in E} \sum_{j \in J(e)} (a_e l_e + b_e) w_j \leq \sum_{e \in E} \sum_{j \in J^*(e)} (a_e l_e + b_e) w_j + a_e w_j^2$$

# Nash Equilibrium for Linear Latency Functions

Proof (cont'):

$$\sum_{e\in E}\sum_{j\in J(e)}(a_e l_e + b_e)w_j \leq \sum_{e\in E}\sum_{j\in J^*(e)}(a_e l_e + b_e)w_j + a_e w_j^2$$

$$\sum_{j\in J(e)} w_j = l_e, \quad \sum_{j\in J^*(e)} w_j = l_e^*, \quad \sum_{j\in J^*(e)} w_j^d \leq (l_e^*)^d$$

$$\begin{aligned}\sum_{e\in E}(a_e l_e + b_e)l_e &\leq \sum_{e\in E}(a_e l_e + b_e)l_e^* + a_e {l_e^*}^2 \\ &= \sum_{e\in E} a_e l_e l_e^* + \sum_{e\in E}(a_e l_e^* + b_e)l_e^*\end{aligned}$$

# Nash Equilibrium for Linear Latency Functions

## Proof (cont'):

$$\sum_{e \in E} (a_e l_e + b_e) l_e \leq \sum_{e \in E} a_e l_e l_e^* + \sum_{e \in E} (a_e l_e^* + b_e) l_e^*$$

$$\sum_{e \in E} a_e l_e l_e^* \quad \leq \quad \sqrt{\sum_{e \in E} a_e l_e^2 \sum_{e \in E} a_e {l_e^*}^2} \quad \text{Cauchy-Schwartz Inequality}$$

$$\leq \quad \sqrt{\sum_{e \in E} (a_e l_e + b_e) l_e \sum_{e \in E} (a_e l_e^* + b_e) l_e^*}$$

$$x = \sqrt{\frac{C(\mathcal{S})}{C(\mathcal{S}^*)}} \qquad x^2 \leq x + 1, \;\; x^2 \leq \frac{3+\sqrt{5}}{2}$$

# Nash Equilibrium for Linear Latency Functions

Unweighted Demand: $w_j = 1$

**Theorem** For linear latency functions, unweighted demand and pure strategies, the worse-case coordination ratio $R$ is at most $2.5$

Proof:

$$\sum_{e \in E} \sum_{j \in J(e)} (a_e l_e + b_e) w_j \leq \sum_{e \in E} \sum_{j \in J^*(e)} (a_e l_e + b_e) w_j + a_e w_j^2$$

$$\sum_{e \in E} (a_e l_e + b_e) l_e \leq \sum_{e \in E} a_e l_e l_e^* + a_e l_e^* + b_e l_e^*$$

# Nash Equilibrium for Linear Latency Functions

Proof:

$$\sum_{e\in E}(a_e l_e + b_e) l_e \le \sum_{e\in E} a_e l_e l_e^* + a_e l_e^* + b_e l_e^*$$

$$\begin{aligned}
(a_e l_e + b_e) l_e &\le a_e l_e^2 + \frac{3}{2} b_e l_e = \frac{3}{2}(a_e l_e^2 + b_e l_e) - \frac{1}{2} a_e l_e^2 \\
&\le \frac{3}{2}(a l_e l_e^* + a l_e^* + b l_e^*) - \frac{1}{2} a l_e^2 \\
&= \frac{1}{2} a(3 l_e l_e^* + 3 l_e^* - l_e^2) + \frac{3}{2} b_e l_e^* \\
&\le \frac{5}{2} a_e {l_e^*}^2 + \frac{3}{2} b_e l_e^* \qquad 3ij + 3j - i^2 \le 5j^2 \\
&\le \frac{5}{2}(a_e l_e^* + b_e) l_e^*
\end{aligned}$$

# Nash Equilibrium for Linear Latency Functions

**Theorem** For linear latency functions and mixed strategies, the worse-case coordination ratio $R$ is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$

Proof:

$$
\begin{aligned}
c_{Q,j} &= E[\sum_{e \in Q} f_e(l_e) \,|\, X_{Q,j} = 1] \\
&= E[\sum_{e \in Q} f_e(\sum_{i=1, i \neq j}^{n} X_{e,i} w_i + w_j)] \\
&= \sum_{e \in Q} E[f_e(l_e + (1 - X_{e,j}) w_j)]
\end{aligned}
$$

The change from $X_{Q,j}$ to $X_{e,j}$ does not affect the proofs. In particular, the proof of **Lemma 3.4** is still correct, if we replace $p_{Q,j} - p_{Q,j}^2$ by $(1 - p_{e,j}) p_{Q,j}$.

# Nash Equilibrium for Linear Latency Functions

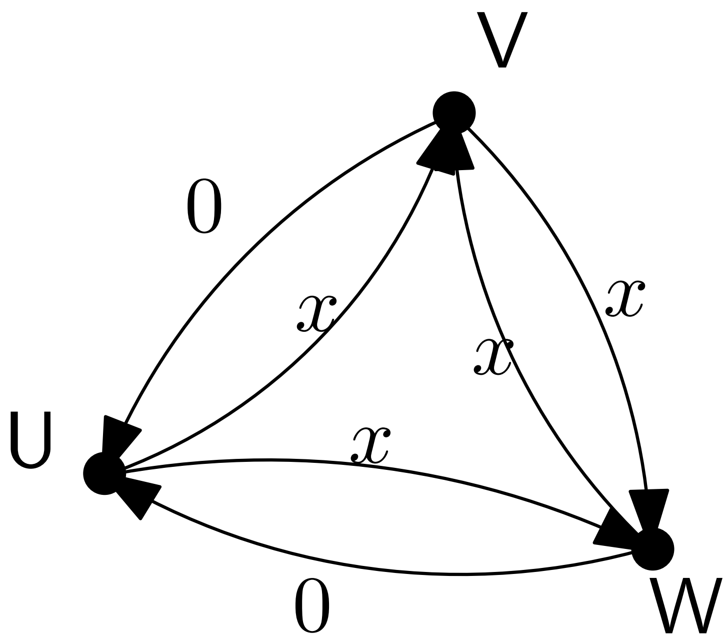
Remarks:

If we allow splittable flows, the price of anarchy is bounded by $\frac{4}{3}$ [Roughgarden, SODA 05]

> Though I am doubt on this result, as the Proposition 1 there is counter intuitive to me.

Unweighted demand will not achieve better ratio in mixed strategies. Because we lose the properties for integers.

# Lower Bounds for Linear Latency Functions




Demands: $\phi = \frac{1+\sqrt{5}}{2}, 1$

- User 1: $(U, V, \phi)$
- User 2: $(U, W, \phi)$
- User 3: $(V, W, 1)$
- User 4: $(W, V, 1)$

Optimal: $2\phi^2 + 2$

- User 1: $UV$
- User 2: $UW$
- User 3: $VW$
- User 4: $WV$

N.E $2\phi^2 + 2(\phi + 1)^2$

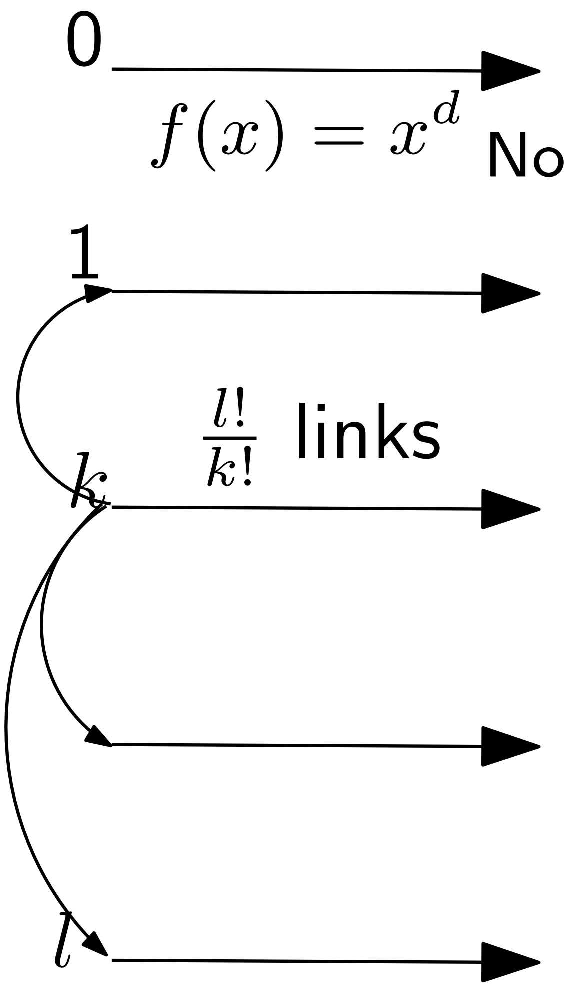
- User 1: $UWV$
- User 2: $UVW$
- User 3: $VUW$
- User 4: $WUV$

# Nash Equilibrium for Polynomial Latency Functions

**Theorem** For polynomial latency functions of degree $d$ and pure and mixed strategies, the worse-case coordination ratio $R$ is $O(2^d d^{d+1})$

**Theorem** For polynomial latency functions of degree $d$ and pure strategies, the worse-case coordination ratio $R$ is $\Omega(d^{d/2})$

# Lower Bounds for Polynomial Latency Functions




Optimal:

Group $k$ assigns jobs to links of group $k-1$.

Nash Equilibrium:

Group $k$ assigns jobs to links of group $k$.

$$OPT = \sum_{k=0}^{l-1} \frac{l!}{k!} 1^d = l! \sum_{k=0}^{l-1} \frac{1}{k!} \approx l! \cdot e$$

$$NE = \sum_{k=1}^{l} \frac{l!}{k!} k^d \geq \frac{l!}{(d/2)^d} \cdot (d/2)^d = l! \cdot \Omega(d^{d/2})$$

## Remaining:

Lower bounds for mixed strategies.

Gap in the bounds of polynomial latency functions: $O(2^d d^{d+1})$ and $\Omega(d^{d/2})$.