Propositional Logic

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Contents

Propositions

Definition 1

A proposition is a **declarative statement** that is either true (T) or false (F), but not both.

Example 2

Each of the following statements is a proposition.

1. $1+1=2$. (T)
2. $2+2=3$. (F)

Propositions

Remark

A statement cannot be true or false unless it is declarative.
Each of the following is not a proposition.

1. No parking.
2. Who has an iMac?

Remark

Declarations about semantic tokens of non-constant value are NOT propositions.

- For example: $x+2=5$.

This is because this statement does not have the value T or F.

Propositions

Remark

A declarative statement is a proposition even if no one knows if it is true.

1. For example: There are infinitely many twin prime numbers.
2. $(3,5)$, $(5,7)$, $(11,13)$, $\cdots$
3. This is a unsettled conjecture (called, twin-prime conjecture).

Remark

Often, a proposition is condition-based.

- For example: If you would pay me ten million dollars, you will become the President of HKUST.

Truth Tables

Definition 3

The boolean domain is the set $\{T, F\}$. Either of its elements is called a boolean value.

Definition 4

An n-tuple $(p_1, \ldots, p_n)$ of boolean values is called a boolean n-tuple.

Example 5

- (T, T, F, T, F) is a boolean 5-tuple.

Truth Tables

Definition 6

An *n-operand truth table* is a table that assigns a boolean value to the set of all boolean *n*-tuples.

Example 7

Table 1 : A 2-operand truth table.

Boolean 2-tuples	Boolean value
(T, T)	T
(T, F)	T
(F, T)	T
(F, F)	F

Propositional Operators or Logical Operators

Definition 8

A propositional operator is a **rule** defined by a truth table.
An operator is monadic if it has only one argument. It is dyadic if it has two arguments.

Example 9

- The truth table in Table 1 defines a dyadic operator, called **"disjunction"**, read **"or"**, and denoted by "$\vee$".
- The following truth table defines a monadic operator, called **"negation"**, read **"not"**, and denoted by "$\sim$".

Table 2 : A 1-operand truth table.

Boolean 1-tuples	Boolean value
T	F
F	T

The Negation Operator "Not"

Recall of definition: the negation $\sim$

p	$\sim p$ ("not p")
T	F
F	T

Example 10

- p: It is sunny.
- $\sim p$: It is NOT sunny.

The Disjunction Operator "Or"

Recall of definition: the disjunction $\vee$

p	q	$p \vee q$ ("p or q")
T	T	T
T	F	T
F	T	T
F	F	F

Example 11

Let c, a and b be real numbers.

- p: $c < a$.
- q: $c = a$.
- $p \vee q$: $c \leq a$.

The Conjunction Operator “And”

Definition 12

Table 3 : The conjunction operator “and”, $\wedge$

p	q	$p \wedge q$ (“p and q”)
T	T	T
T	F	F
F	T	F
F	F	F

Example 13

Let c, a and b be real numbers.

- p: $c \geq a$.
- q: $c \leq b$.
- $p \wedge q$: $a \leq c \leq b$.

The Exclusive-or Operator "$\oplus$"

Definition 14

It is denoted by "$p \oplus q$", and defined to be $(p \vee q) \wedge (\sim (p \wedge q))$. It means that "*p* **or** *q* **but not both**".

Table 4 : The exclusive-or operator, $\oplus$

p	q	$p \oplus q$ ("p or q but not both")
T	T	F
T	F	T
F	T	T
F	F	F

The Conditional Operator “implies”

Definition 15

The conditional operator is denoted by $p \to q$, read implies, and defined by the following truth table:

Table 5 : The conditional operator $\to$

p	q	$p \to q$ (“if p then q”)
T	T	T
T	F	F
F	T	T
F	F	T

Example 16

If $0 = 1$, then $1 = 2$. Is this a true statement?

The Conditional Operator "implies"

Remarks

- In the form $p \to q$, p is called the antecedent ot hypothesis, and q is called the consequent or conclusion.

Example 17

If the Yankees win the World Series, then they give Lou Gehrig a $1,000 bonus.

The Biconditional Operator “if and only if”

Definition 18

The biconditional operator is denoted by $\leftrightarrow$, read if and only if, and defined by the following truth table:

Table 6 : The biconditional operator $\leftrightarrow$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 19

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Remark

The phrases *necessary condition* and *sufficient condition*, as used in formal English, correspond exactly to their definitions in logic.

Propositional Variables

Definition 20

A propositional variable is a variable such as p, q, r (possibly subscripted, e.g. p_j) over the boolean domain.

Atomic Propositional Forms

Definition 21

An atomic propositional form is either a boolean constant or a propositional variable.

Example 22

Boolean constants: T and F.

Atomic propositional forms: p, q, r, etc.

Compound Propositional Forms

Definition 23

A compound propositional form is derived from atomic propositional forms by application of propositional operators.

Example 24

Some compound propositional forms on two variables:

- $p \vee q$, $p \wedge q$, $p \oplus q$, $p \rightarrow q$, $p \leftrightarrow q$
- $\sim p$, $(p \vee \sim q) \rightarrow q$

Evaluating Compound Propositional Forms

Remark

Any compound propositional form can be evaluated by a truth table.

Problem 25

Evaluating the compound propositional form $(p \vee \sim q) \rightarrow q$ *by a truth table.*

Order of Operations for Logical Operators

$\sim$: Evaluate negations first.

$\vee$ and $\wedge$: Evaluate $\vee$ and $\wedge$ second. When both are present, parenthesis may be needed.

$\rightarrow$ and $\leftrightarrow$: Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parenthesis may be needed.

Evaluating Compound Propositional Forms

Example 26

Evaluate the compound propositional form $(p \vee \sim q) \rightarrow q$ in the following order.

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \rightarrow q$
T	T			
T	F			
F	T			
F	F			

Evaluating Compound Propositional Forms

Example 26

Evaluate the compound propositional form $(p \vee \sim q) \to q$ in the following order.

		Step 1		
p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \to q$
T	T	F		
T	F	T		
F	T	F		
F	F	T		

Evaluating Compound Propositional Forms

Example 26

Evaluate the compound propositional form $(p \vee \sim q) \rightarrow q$ in the following order.

		Step 1	Step 2	
p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \rightarrow q$
T	T	F	T	
T	F	T	T	
F	T	F	F	
F	F	T	T	

Evaluating Compound Propositional Forms

Example 26

Evaluate the compound propositional form $(p \vee \sim q) \to q$ in the following order.

		Step 1	Step 2	Step 3
p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \to q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F