Minimum Spanning Trees and Prim's Algorithm

Version of September 23, 2016

Outline

- Spanning trees and minimum spanning trees (MST).

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- Prim's algorithm for the MST problem.
  - The idea
  - The algorithm
  - Analysis

Spanning Trees

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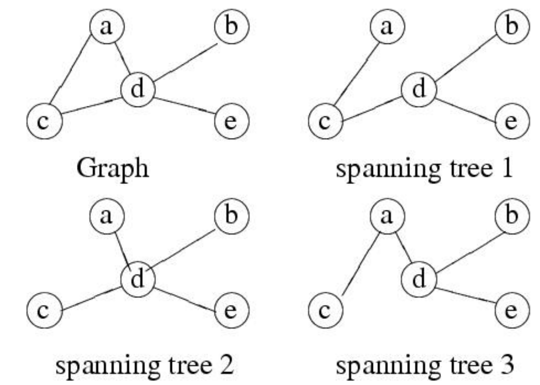

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Theorem

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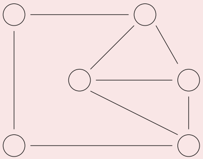
Given a connected graph G, how can you find a spanning tree of G?



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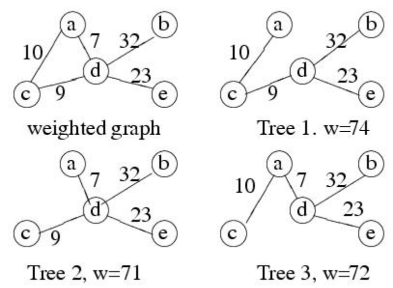

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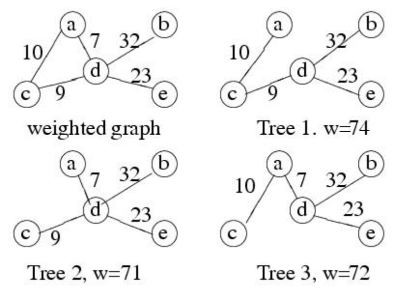

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Weight of a graph: The sum of the weights of all edges

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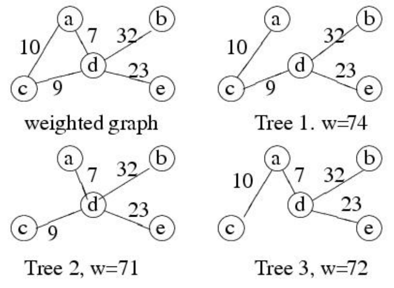

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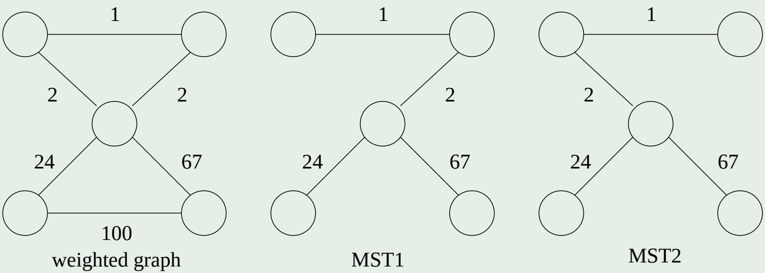

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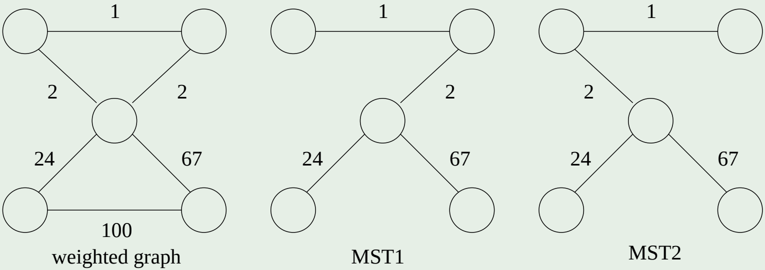

Note: if the weights of all the edges are distinct, MST is provably unique (proof will follow from later results).

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Hard part is ensuring (3)!

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Generic-MST(G, w)

```
begin
    A= EMPTY;
    while A does not form a spanning tree do
        find an edge (u, v) that is safe for A;
        add (u, v) to A;
    end
    return A
end
```

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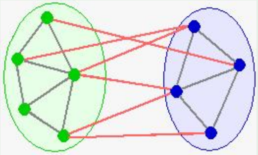

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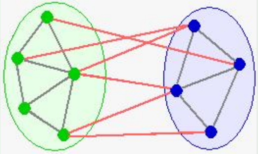

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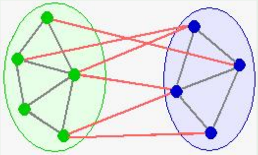

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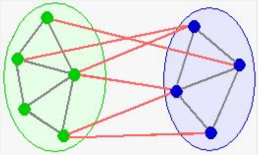

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An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

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Lemma

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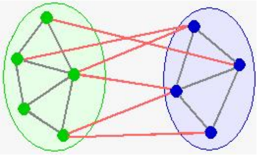

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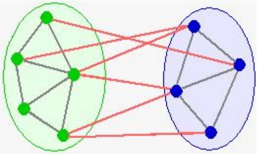

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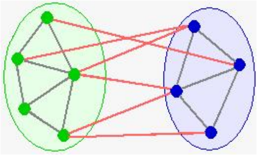

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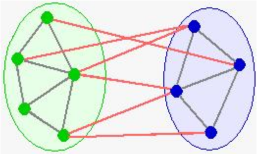

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That light edge is a safe edge.

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  - Hence (u, v) is safe for A.

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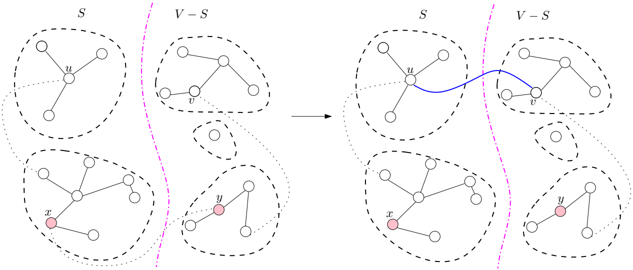

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- But $A \cup \{(u, v)\} \subseteq T^{'}$, so (u, v), is safe for A.
- The Lemma is proved.

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- Different ways of chosing cuts correspond to different algorithms.
- The two major ones are Prim's algorithm and Kruskal's algorithm,

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We will show that these steps can be implemented in total $O(E \cdot \log V)$.

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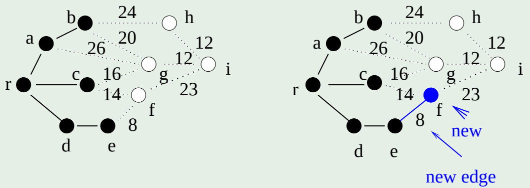

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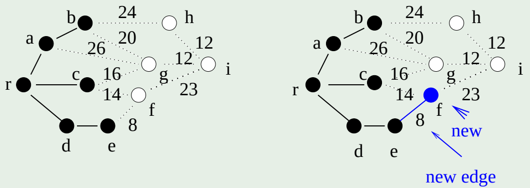

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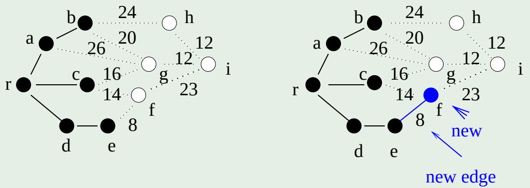

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More Details

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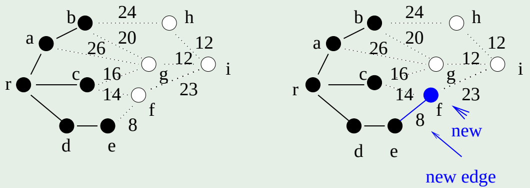

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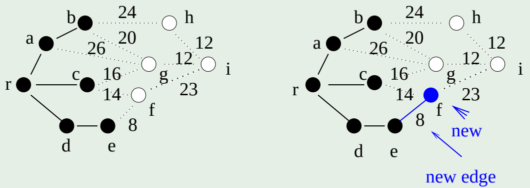

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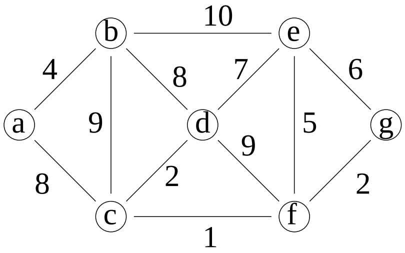
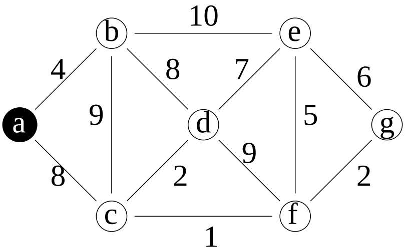
Worked Example



Connected graph



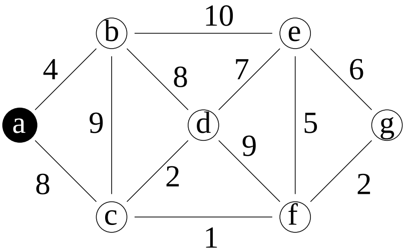
Step 0

S={a}

V \ S = {b,c,d,e,f,g}

lightest edge = {a,b}

Prim's Example – Continued



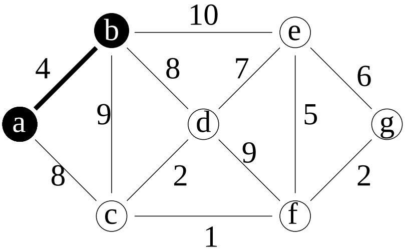
Step 1.1 before

S={a}

V \ S = {b,c,d,e,f,g}

A={}

lightest edge = {a,b}



Step 1.1 after

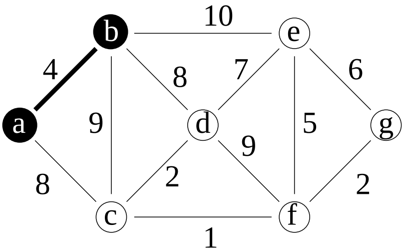
S={a,b}

V \ S = {c,d,e,f,g}

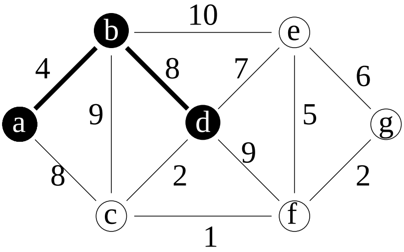
A={{a,b}}

lightest edge = {b,d}, {a,c}

Prim's Example – Continued

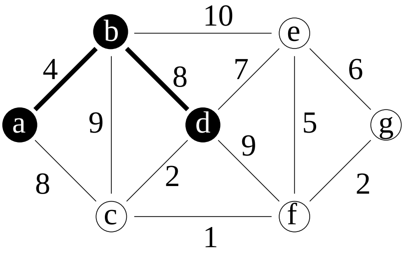


Step 1.2 before
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}

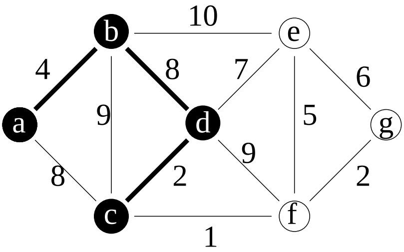


Step 1.2 after
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}

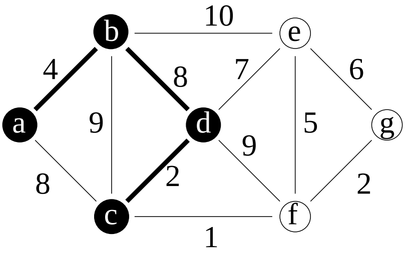
Prim's Example – Continued



Step 1.3 before
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}



Step 1.3 after
S={a,b,c,d}
V \ S = {e,f,g}
A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}



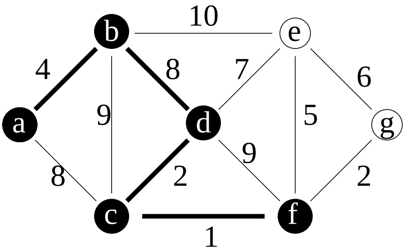
Step 1.4 before

S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}



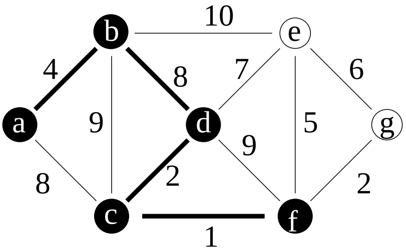
Step 1.4 after

S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}



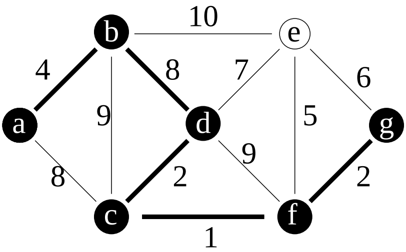
Step 1.5 before

S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}



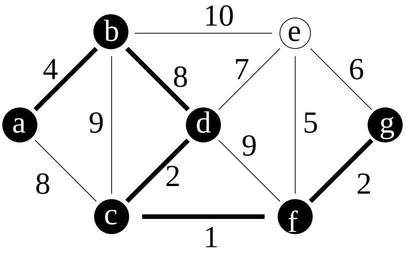
Step 1.5 after

S={a,b,c,d,f,g}

V \ S = {e}

A={{a,b},{b,d},{c,d},{c,f},
{f,g}}

lightest edge = {f,e}



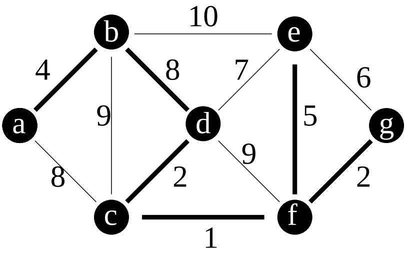
Step 1.6 before

S={a,b,c,d,f,g}

V \ S = {e}

A={{a,b},{b,d},{c,d},{c,f},
{f,g}}

lightest edge = {f,e}



Step 1.6 after

S={a,b,c,d,e,f,g}

V \ S = {}

A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}

MST completed

- Spanning trees and minimum spanning trees (MST).

Outline

- Spanning trees and minimum spanning trees (MST).
- Strategy for solving the MST problem.

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- Spanning trees and minimum spanning trees (MST).
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- Prim's algorithm for the MST problem.
  - The idea
  - The algorithm
  - Analysis

Recall Idea of Prim's Algorithm

Step 0: Choose any element r and set $S = \{r\}$ and $A = \emptyset$. (Take r as the root of our spanning tree.)

Step 1: Find a lightest edge such that one endpoint is in S and the other is in $V \setminus S$.
Add this edge to A and its (other) endpoint to S.

Step 2: If $V \setminus S = \emptyset$, then stop and output the minimum spanning tree (S, A); Otherwise go to Step 1.

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Questions

1. Why does this produce a minimum spanning tree?
2. How does the algorithm find the lightest edge and update A efficiently?
3. How does the algorithm update S efficiently?

Prim's Algorithm

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How does the algorithm update S efficiently?

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Answer: Color the vertices.

- Initially all are white.

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How does the algorithm find a lightest edge and update A efficiently?

Answer:

1. Use a priority queue to find the lightest edge.

Prim's Algorithm

Question

How does the algorithm update S efficiently?

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- Initially all are white.
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Question

How does the algorithm find a lightest edge and update A efficiently?

Answer:

1. Use a priority queue to find the lightest edge.
2. Use pred[v] to update A.

Priority Queue is a data structure

- can be implemented as a heap

Supports the following operations:

Reviewing Priority Queues

Priority Queue is a data structure

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Insert(u, key): Insert u with the key value key in Q.

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Decrease-Key(u, $new\text{-}key$): Decrease u's key value to $new\text{-}key$.

Remark: We already saw how to implement Insert and Extract-Min (and Delete) in $O(\log |Q|)$ time.
Same ideas can also be used to implement Decrease-Key in $O(\log |Q|)$ time.
Alternatively, can implement Decrease-Key using Delete followed by Insert.

Using a Priority Queue to Find the Lightest Edge

Each item of the queue is a pair $(u, key[u])$, where

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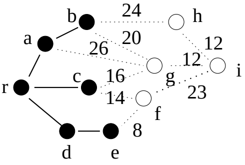


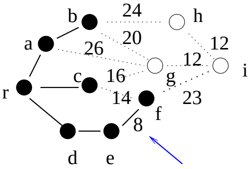
key[f] = 8, pred[f] = e

key[i] = infinity, pred[i] = nil

key[g] = 16, pred[g] = c

key[h] = 24, pred[h] = b

→ f has the minimum key



new edge

key[i] = 23, pred[i] = f

After adding the new edge and vertex f, update the key[v] and pred[v] for each vertex v adjacent to f

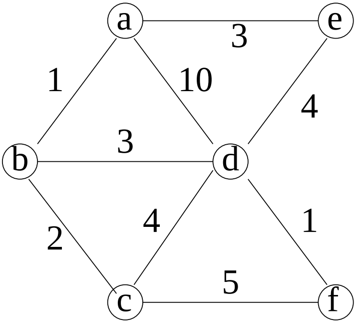
```
begin
    foreach u ∈ V do
        color[u] = WHITE; key[u] = +∞; // initialize
    end
    key[r] = 0; pred[r] = NIL; // start at root
    Q = new PriQueue(V); // put vertices in Q
    while Q is nonempty do
        u= Q.Extract-Min(); // lightest edge
        foreach v ∈ adj[u] do
            if (color[v] = WHITE)&&(w[u, v] < key[v]) then
                key[v] = w[u, v]; // new lightest edge
                Q.Decrease-Key(v, key[v]);
                pred[v] = u;
            end
        end
        color[u] = BLACK;
    end
end
```

Description of Prim's Algorithm...

When the algorithm terminates, $Q = \emptyset$ and the MST is

$$T = \{\{v, pred[v]\} : v \in V \setminus \{r\}\}.$$

- The pred pointers define the MST as an inverted tree rooted at r.



u	a	b	c	d	e	f
key[u]						
pred[u]						

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```
begin
    foreach u ∈ V do
        key[u] = +∞; color[u] = WHITE; // O(V)
    end
    key[r] = 0; pred[r] = NIL;
    Q = new PriQueue(V); // O(V)
    while Q is nonempty do
        u= Q.Extract-Min(); // Do this for each vertex
        foreach v ∈ adj[u] do
            // Do the following for each edge twice
            if (color[v] = WHITE)&&(w[u, v] < key[v]) then
                key[v] = w[u, v]; pred[v] = u;
                Q.Decrease-Key(v, key[v]); // This is bottleneck
            end
        end
        color[u] = BLACK;
    end
end
```

The data structure PriQueue (heap) supports the following two operations:

- ($O(|V|)$ for creating new Priority Queue
- $O(\log V)$ for Extract-Min on a PriQueue of size at most V.
  Total cost: $O(V \log V)$
- $O(\log V)$ time for Decrease-Key on a PriQueue of size at most V.
  Total cost: $O(E \log V)$.

Total cost is then $O((V + E) \log V) = O(E \log V)$

Going Further

A more advanced Priority Queue implementation called *Fibonnaci Heaps* allow

- $O(1)$ for inserting each item
- $O(\log |V|)$ for Extract-Min
- **O(1) (amortized) for each Decrease-Key**

Since algorithm performs $|V|$ Inserts, $|V|$ Extract-Mins and at most E Decrease-Keys this leads to a $O(|E| + |V| \log |V|)$ algorithm, improving upon the $O(E \log V)$ more naive implementation.