COMP 170 Discrete Mathematical Tools for CS
2007 Fall Semester
Suplementary Problems for Written Assignment # 6

Note: These are extra problems (with solutions) on the same topic as Written Assignment 6. They are only meant to provide you with extra opportunity to revise the material and should not be submitted to be marked. Some of these problems might be discussed in the tutorials.

Problem 1: Use contradiction to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all integers $n \geq 1$.

SOLUTION: Let us denote the statement by $p(n)$.
Assume $p(n)$ is false for some integer $n \geq 1$, and choose k to be the smallest n that makes it false.
Since $p(n)$ gives $2 = 2$ when $n = 1$, so $k > 1$.
From the assumption that k is the smallest n that makes $p(n)$ false, $p(k-1)$ must be true and so

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k-1)k = \frac{(k-1)k(k+1)}{3}.$$

Adding $k(k+1)$ to both sides of $p(k-1)$ gives

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + (k-1)k + k(k+1) &= \frac{(k-1)k(k+1)}{3} + k(k+1) \\ &= \frac{(k-1)k(k+1) + 3k(k+1)}{3} \\ &= \frac{k(k+1)((k-1)+3)}{3} \\ &= \frac{k(k+1)(k+2)}{3}. \end{aligned}$$

Thus $p(k)$ is true, contradicting the assumption that it is not. Thus, $p(n)$ must be true for all integers $n \geq 1$.

Problem 2: Use induction to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all integers $n \geq 1$.

SOLUTION: Denote the statement to prove by $p(n)$.

Base case: For $n = 1$, we have $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$. So $p(1)$ is true.

Inductive hypothesis: Suppose $p(n-1)$ is true for some $n \geq 2$, i.e.

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n = \frac{(n-1)n(n+1)}{3}.$$

Inductive step: Adding $n(n+1)$ to both sides of $p(n-1)$, we have

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n + n(n+1) &= \frac{(n-1)n(n+1)}{3} + n(n+1) \\ &= \frac{(n-1)n(n+1) + 3n(n+1)}{3} \\ &= \frac{n(n+1)((n-1)+3)}{3} \\ &= \frac{n(n+1)(n+2)}{3}, \end{aligned}$$

which shows that $p(n)$ is true, and so $p(n-1) \Rightarrow p(n)$.

Inductive conclusion: By the principle of mathematical induction, we can conclude that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

is true for all integers $n \geq 1$.

Problem 3: Let $0 \leq j \leq n$, prove that

$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}.$$

In addition to an inductive proof, there is a nice combinatorial proof of this formula. It is well worth trying to figure out both proofs.

SOLUTION: INDUCTIVE PROOF:

1. **Base case:** For $n = 0$, the equation says $\binom{0}{0} = \binom{1}{1}$, which is true.
2. **Inductive hypothesis:** Suppose for $n = k - 1 \geq 0$, the equation is true.
3. **Inductive step:** From the inductive hypothesis (second equality) and the Pascal relationship (third equality), we have

$$\sum_{i=j}^{k} \binom{i}{j} = \sum_{i=j}^{k-1} \binom{i}{j} + \binom{k}{j} = \binom{k}{j+1} + \binom{k}{j} = \binom{k+1}{j+1}.$$

   Thus the equation is true for $n = k$.

4. **Inductive conclusion:** From the principle of mathematical induction, the equation is true for all integers $n \geq 0$.

NONINDUCTIVE PROOF:

When we choose $j+1$ numbers from $\{1, 2, \ldots, n+1\}$, the largest number we choose is at least $j+1$. Suppose k is the largest number we choose. The other j numbers are chosen from among 1 through $k-1$. Each i between j and n can be this $k-1$, and for each such i, there are $\binom{i}{j}$ ways to choose the j additional numbers. By the sum principle, we can compute the total number of choices by adding up the number of choices for each possible largest number.

Problem 4: Prove by induction that the number of subsets of an n-element set is 2^n for all $n \geq 0$.

SOLUTION: Let us denote the statement to prove by $p(n)$.

Base case: For $n = 0$, the set has no elements, so it is the empty set. The only subset of the empty set is the empty set. Since $2^0 = 1$, $p(0)$ is true.

Inductive hypothesis: Suppose $p(n-1)$ is true for some $n \geq 1$, i.e., the number of subsets of an $(n-1)$-element set is 2^{n-1}.

Inductive step: For any set S of size $n \geq 1$, we can choose a single element x and remove it from S. The subsets of S consist of the subsets containing x and those that do not contain x. The number of subsets not containing x is the number of subsets of $S - \{x\}$, which, by the inductive hypothesis, is 2^{n-1}. The number of subsets containing x must be the same, because by removing x from each we get a subset not containing x. Thus, the total number of subsets is $2^{n-1} + 2^{n-1} = 2^n$.

Inductive conclusion: By the principle of mathematical induction, the number of subsets of an n-element set is 2^n for all $n \geq 0$.