# COMP170
# Discrete Mathematical Tools for Computer Science

## Intro to Probability

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*Discrete Math for Computer Science*
*K. Bogart, C. Stein and R.L. Drysdale*
*Section 5.1, pp. 213-221*

# Introduction to Probability

- Why Study Probability?

- Probability Spaces and Distributions
- Complementary Probabilities
- Probability and Hashing
- The Uniform Probability Distribution

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In Computer Science we often deal with random events. Some involve randomness imposed from the outside, e.g., networking, when requests from computers on the network enter the network at “random” time.

Some involve randomness that we introduce, e.g., hashing, which is a techinique often used to compactly store information in a computer for later quick retrieval.

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In Computer Science we often deal with random events. Some involve randomness imposed from the outside, e.g., networking, when requests from computers on the network enter the network at "random" time.

Some involve randomness that we introduce, e.g., hashing, which is a techinique often used to compactly store information in a computer for later quick retrieval.

Studying the performance of computer systems in the presence of these types of randomness, requires understanding randomness, which is the study of probability.

# Hashing

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The records are stored in a table. Each table location, called a bucket or slot, holds a list of records. We are also given a hash function $h(x)$. A record with key $key$ is stored in the bucket with index $h(key)$.

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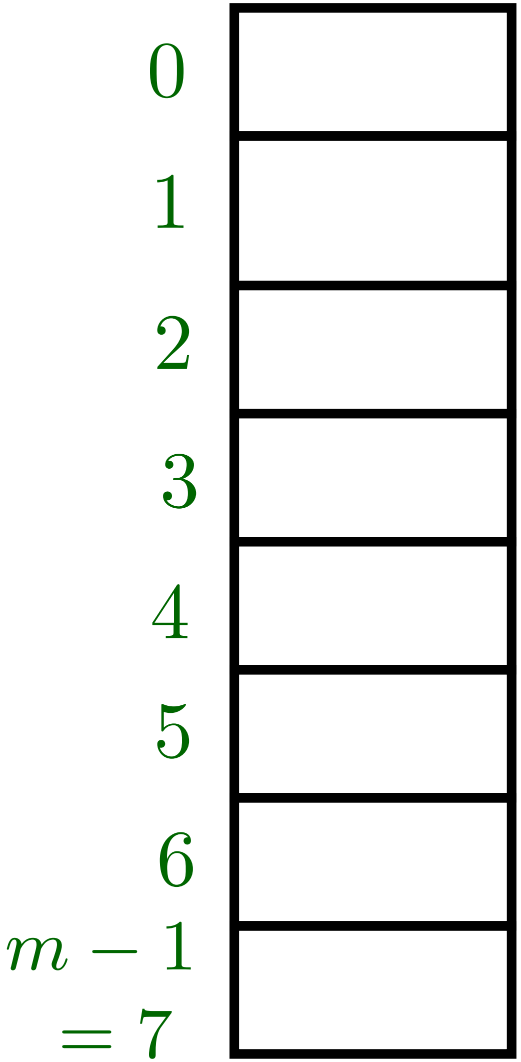
Hash Table $T$



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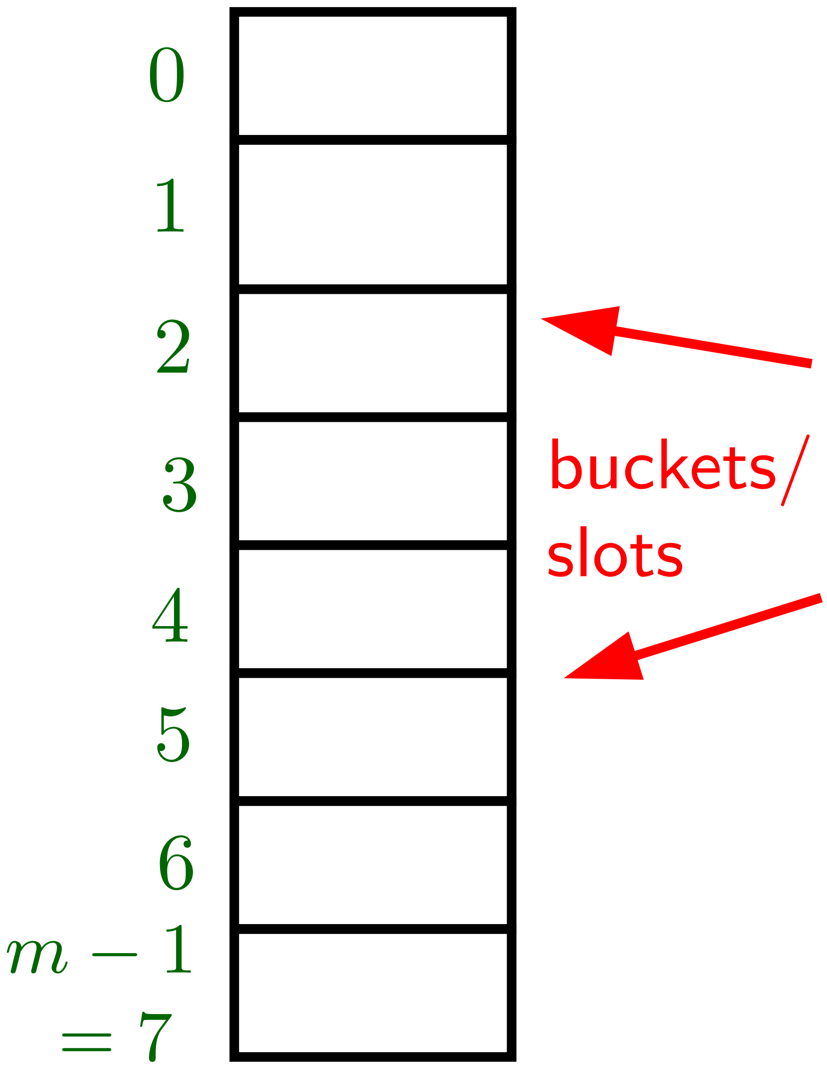

$0$

$1$

$2$

$3$

$4$

$5$

$6$

$m - 1$
$= 7$

buckets/
slots

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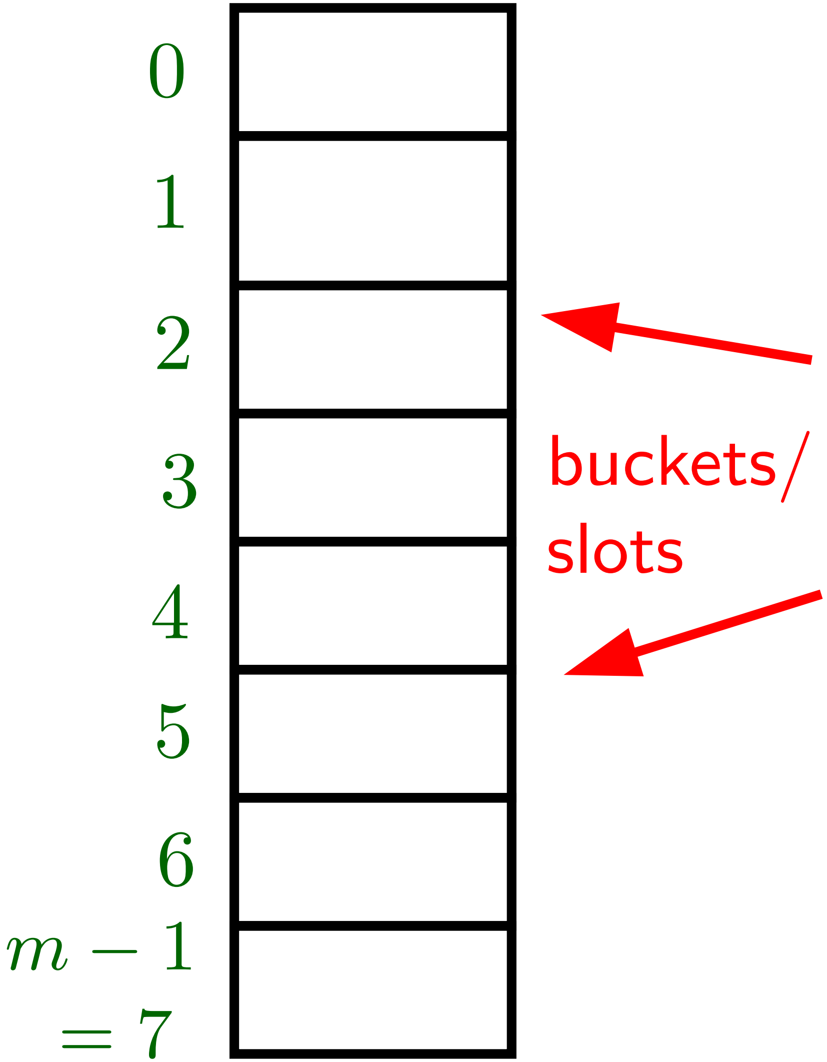

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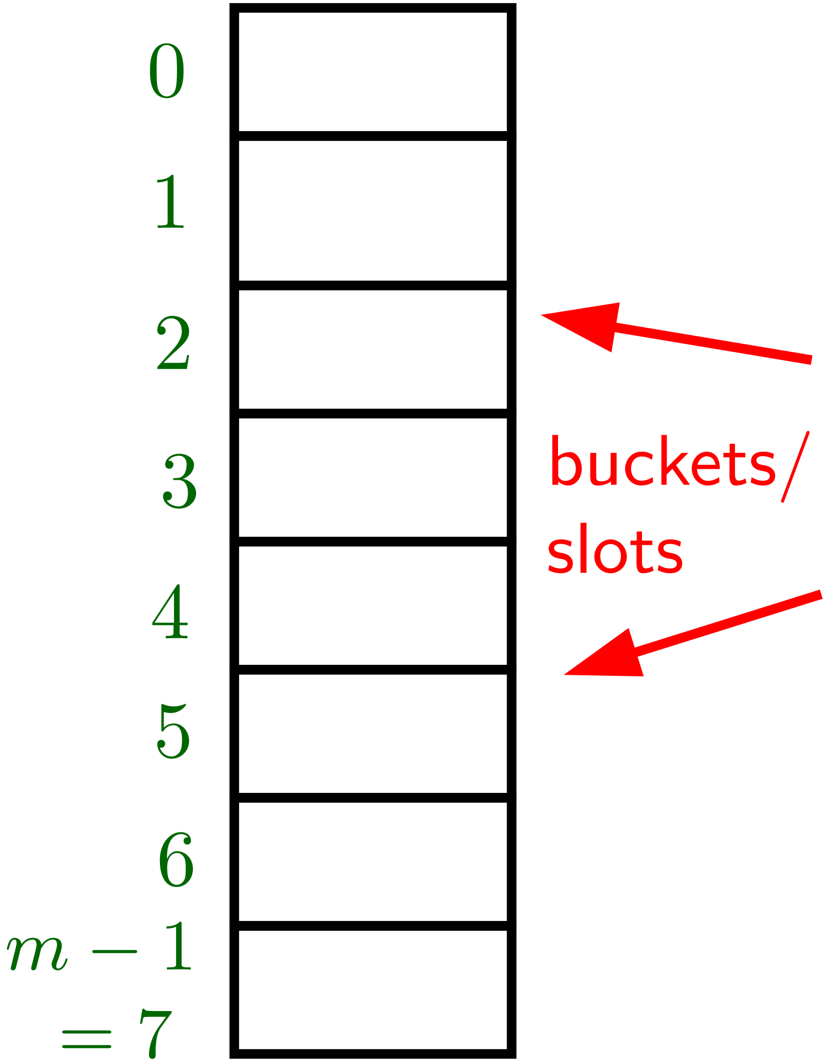

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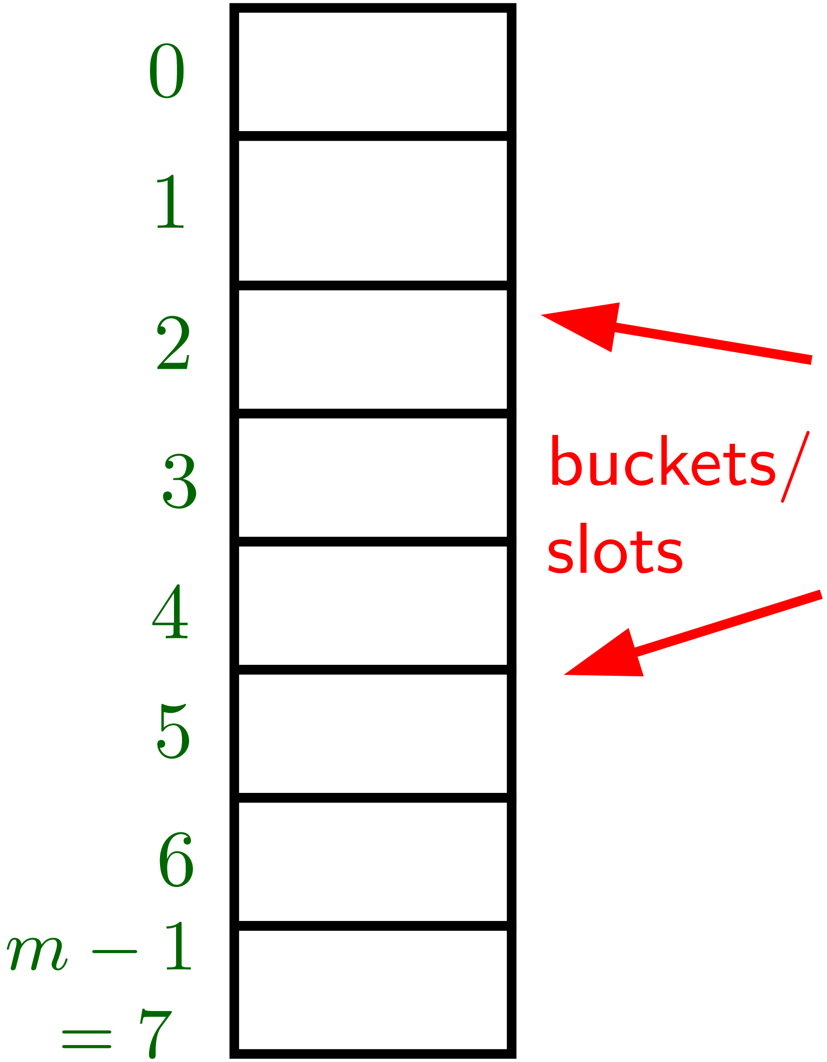

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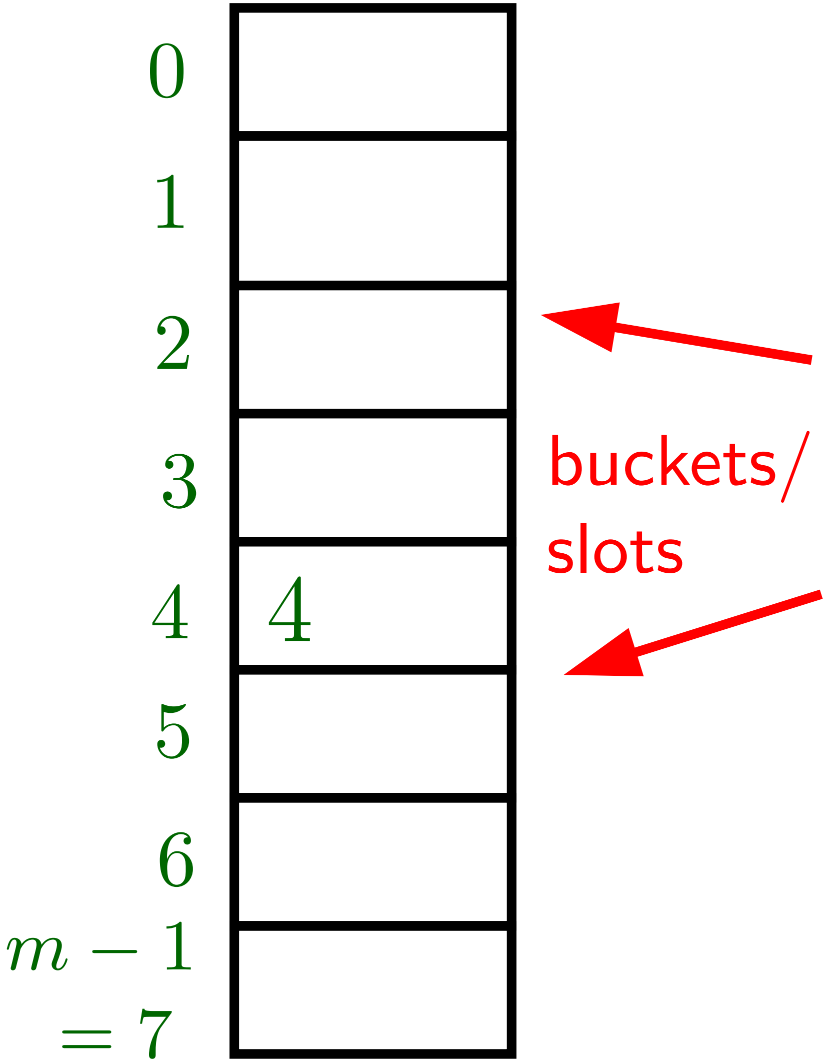

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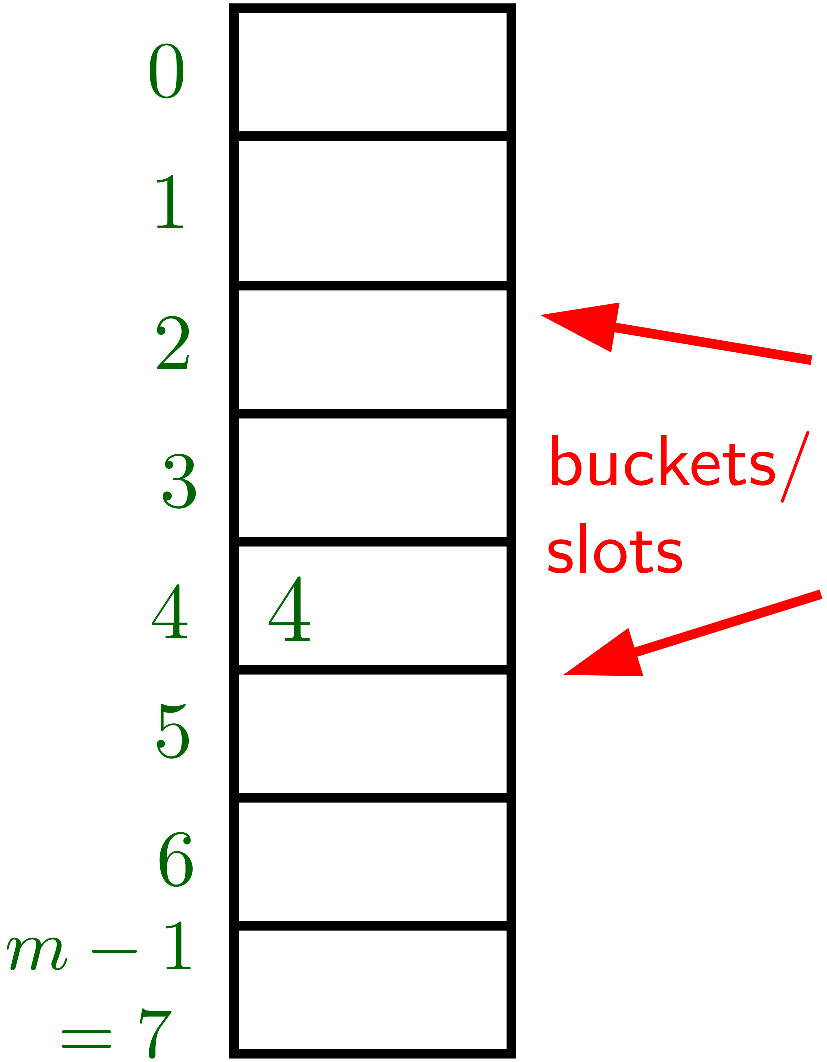

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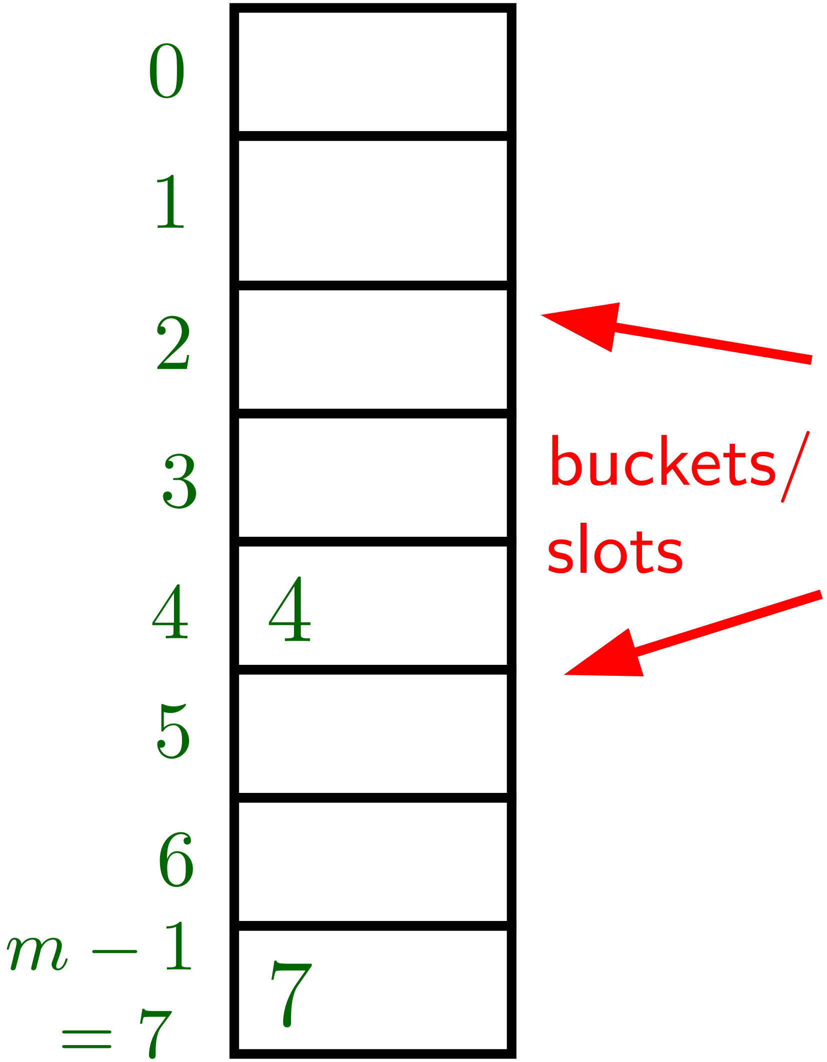

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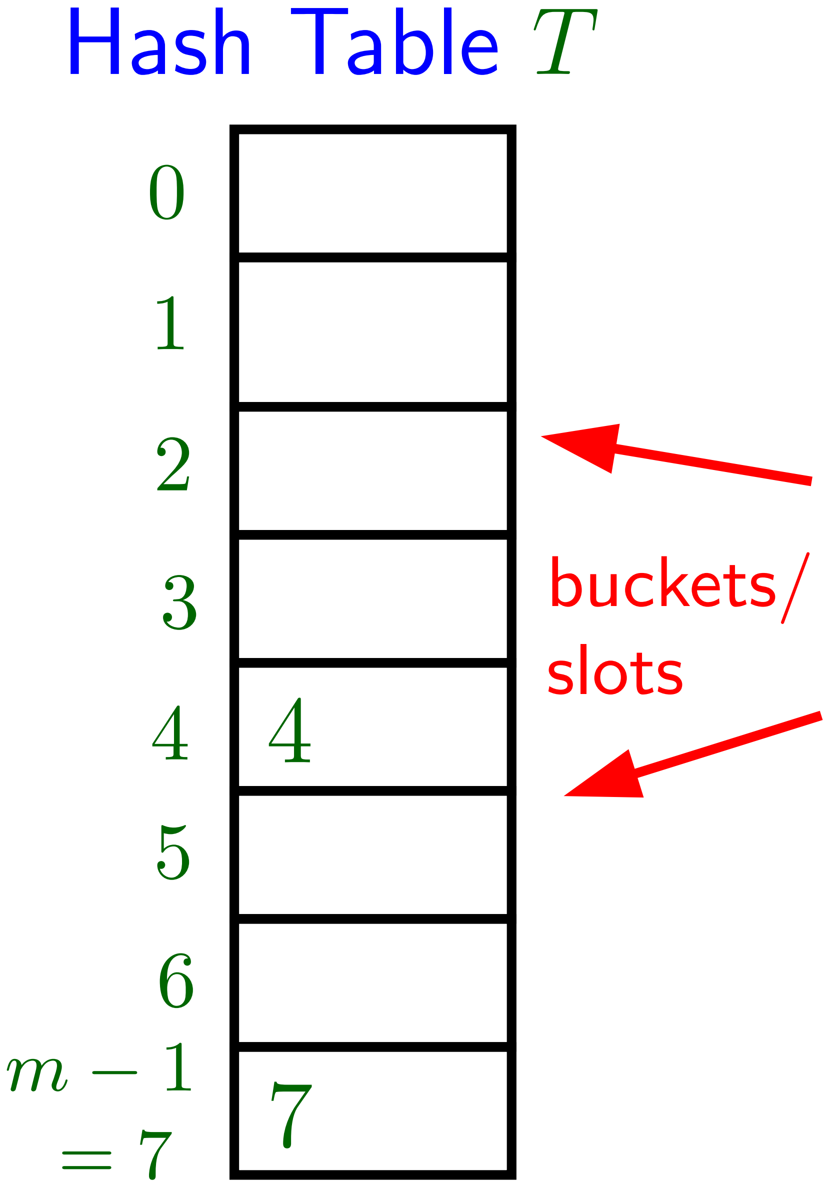

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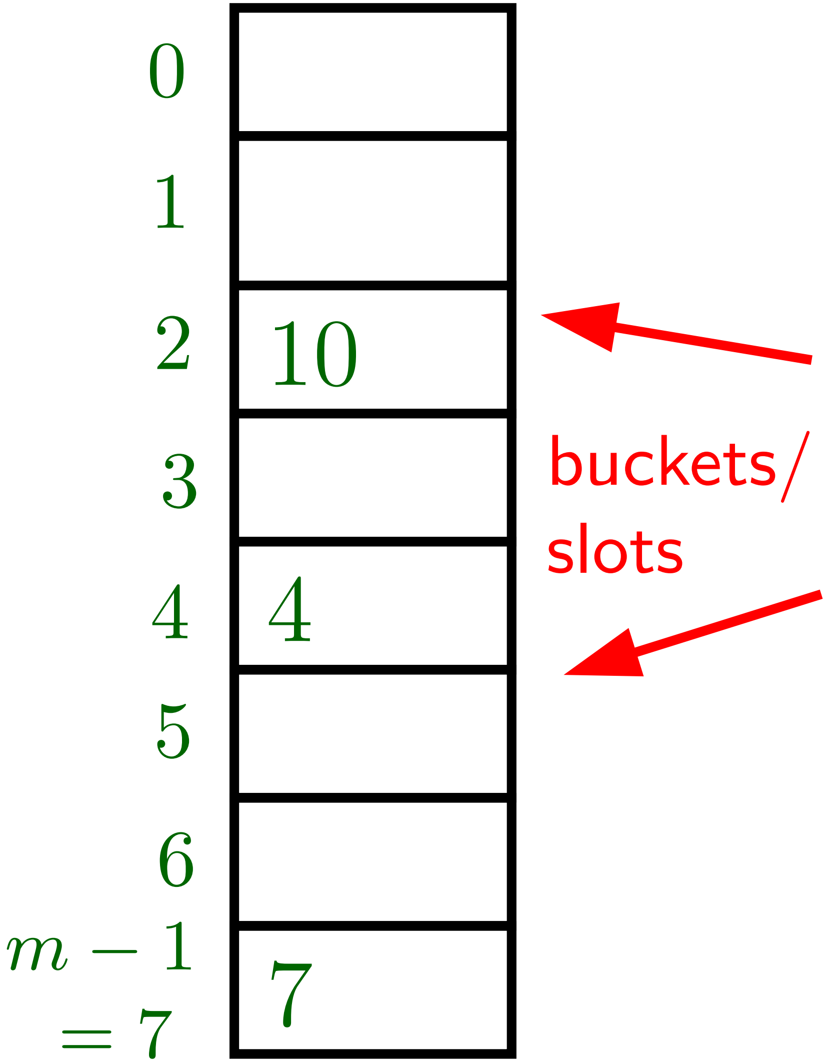

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | 10 |
| 3 | |
| 4 | 4 |
| 5 | |
| 6 | |
| $m-1$<br>$=7$ | 7 |

buckets/
slots

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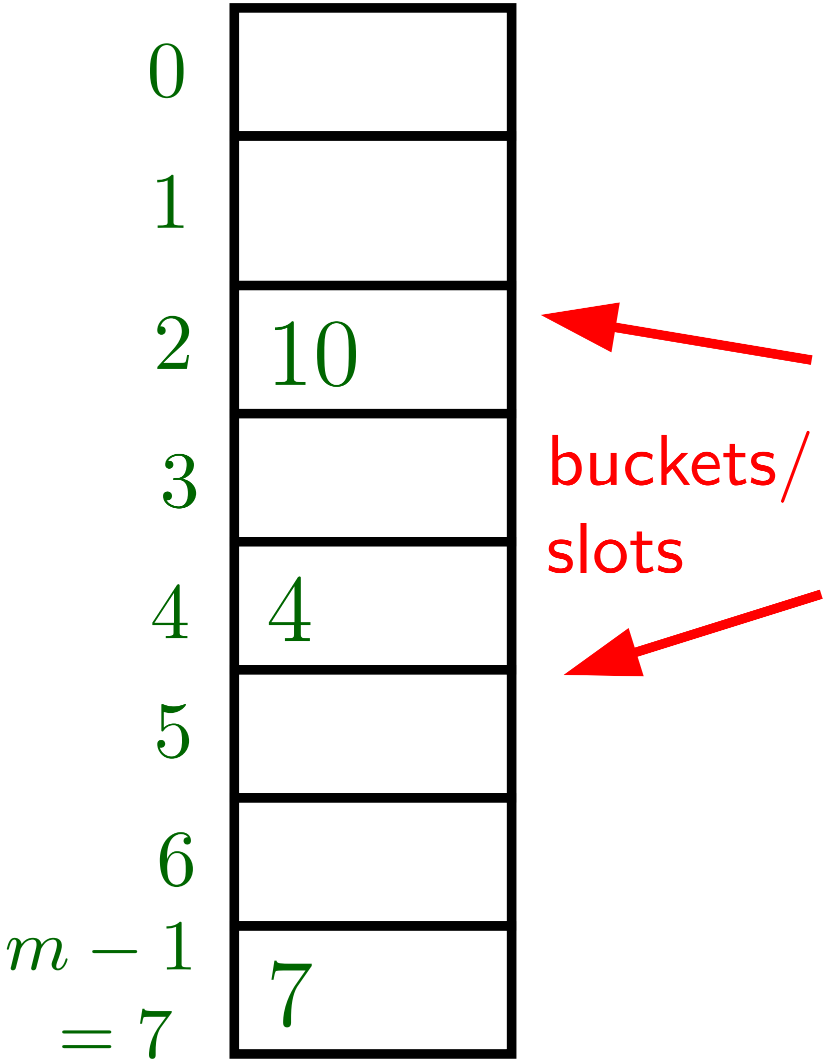

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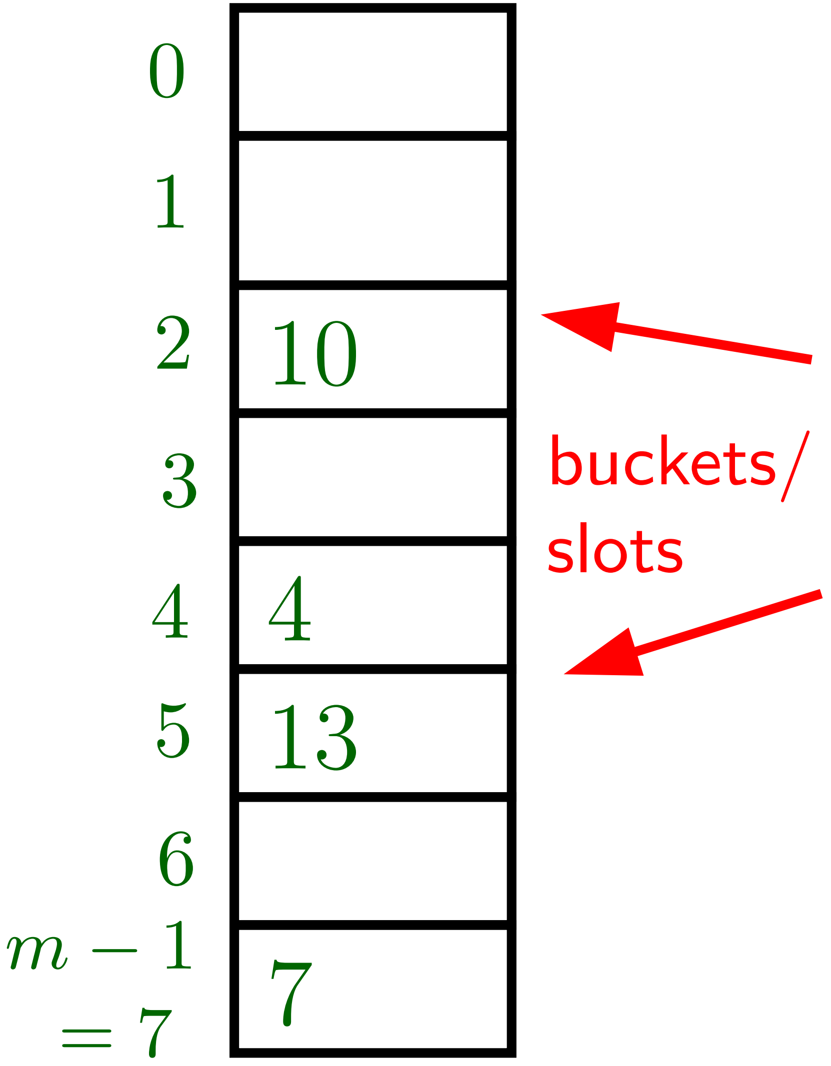

| | |
|---|---|
| $0$ | |
| $1$ | |
| $2$ | $10$ |
| $3$ | |
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## Our Hash Function:

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$4 \quad 7 \quad 10 \quad 13 \quad 15$

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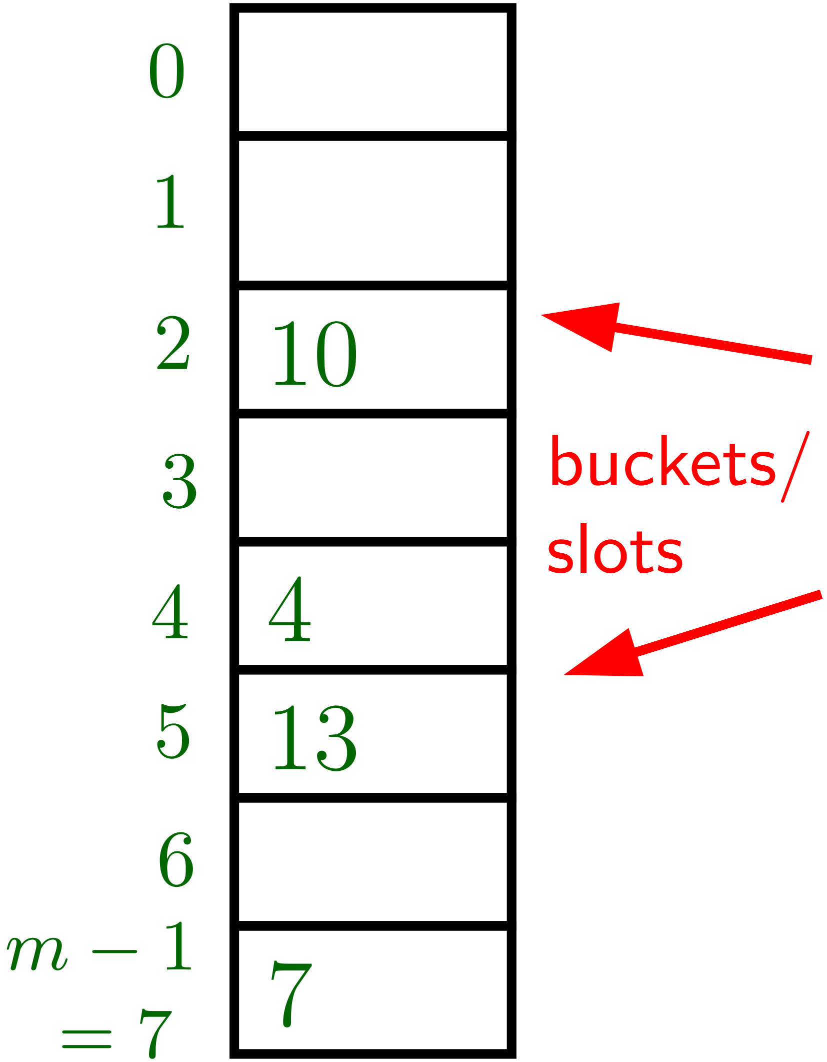

| | |
|---|---|
| $0$ | |
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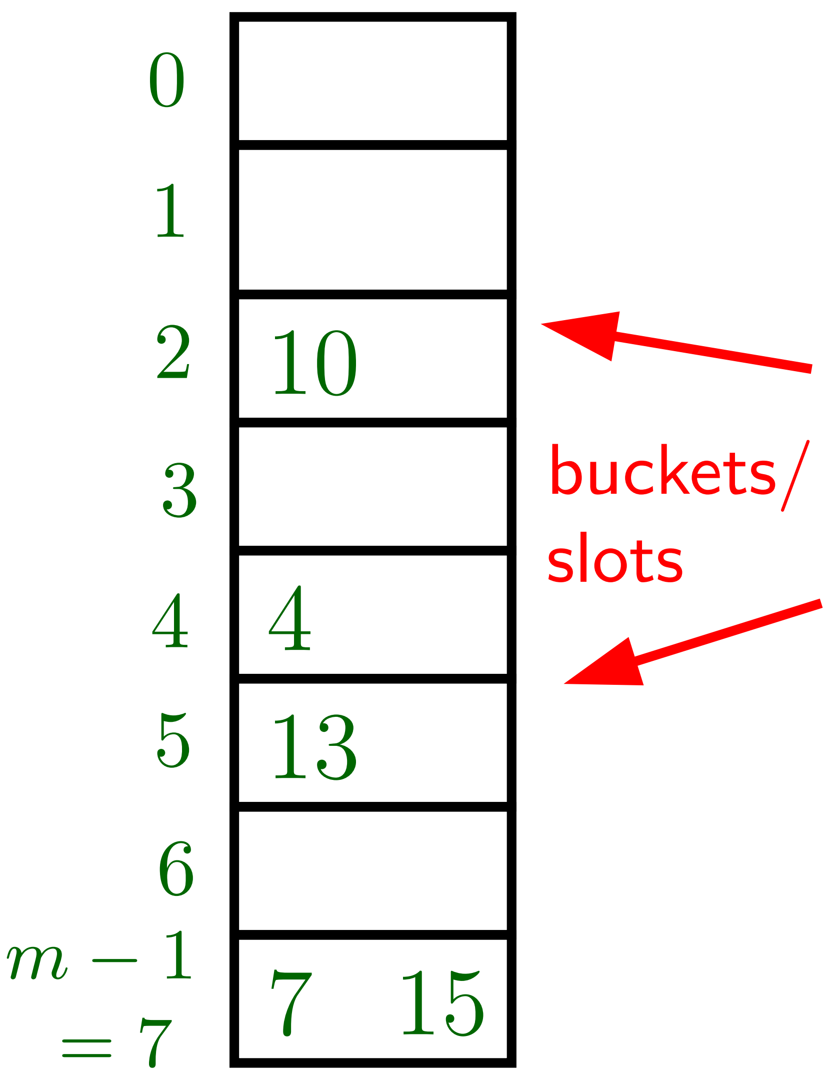

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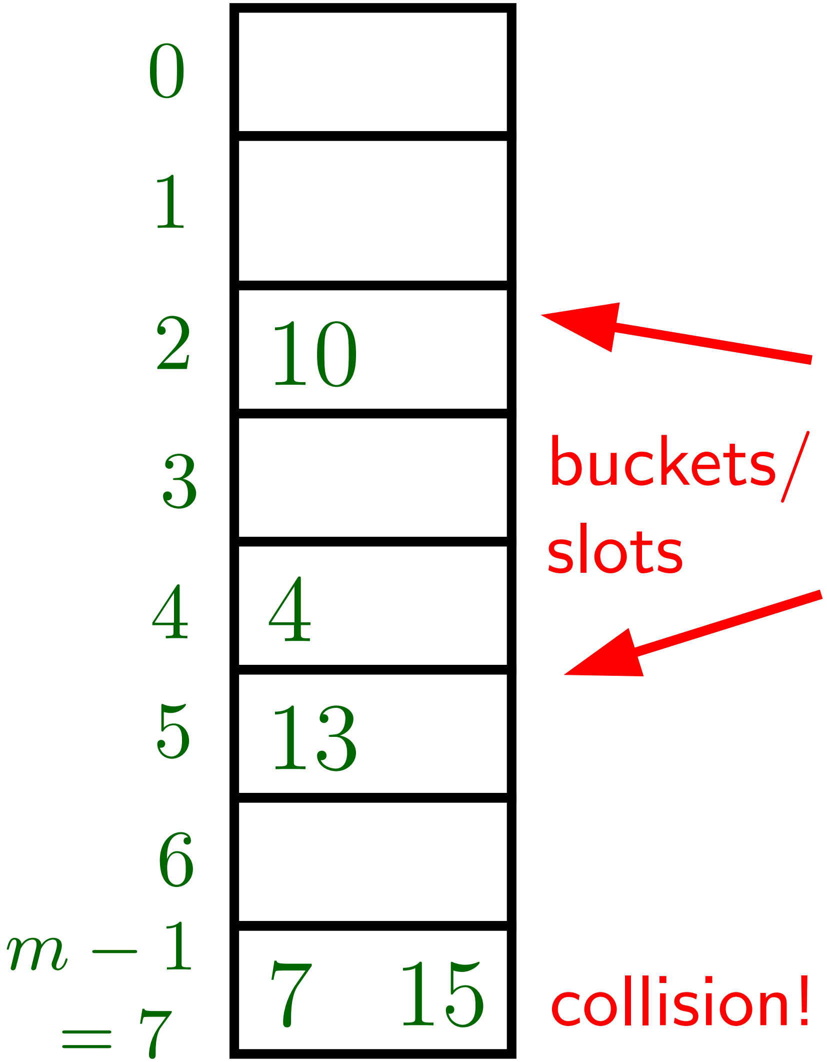

0

1

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buckets/
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collision!

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When searching for a record you might have to look at *every* record in the appropriate bucket, so

Good hash function spreads keys evenly among buckets.

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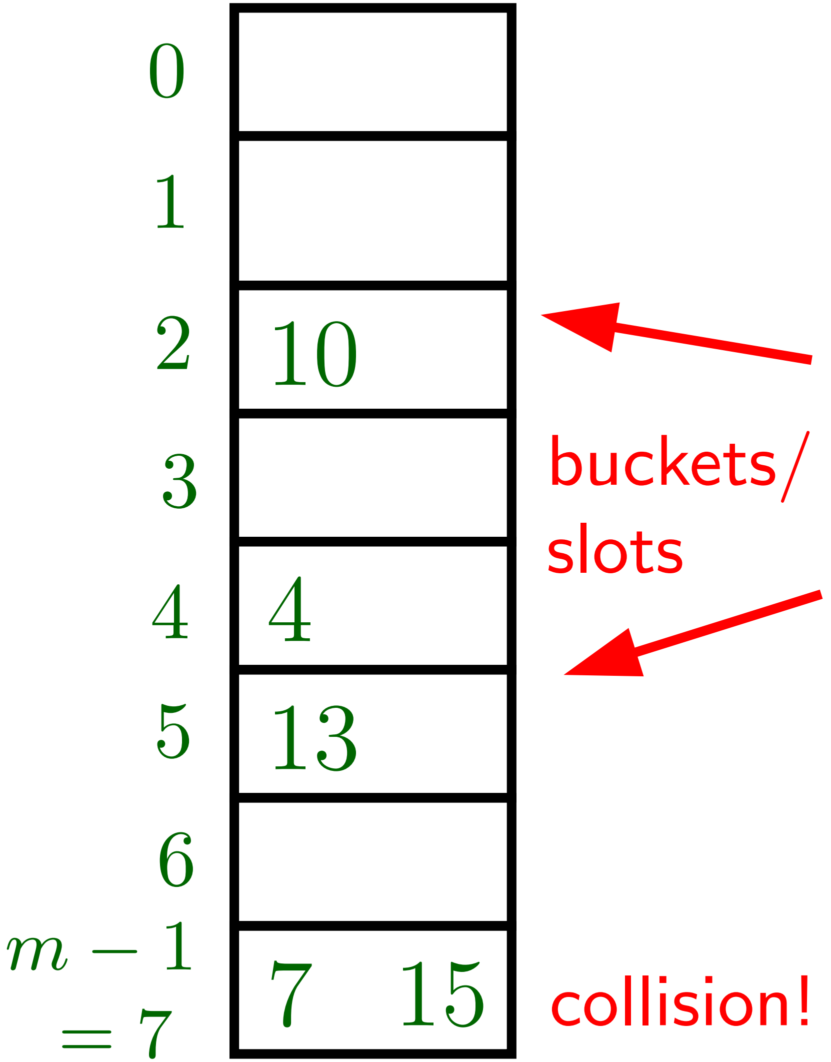

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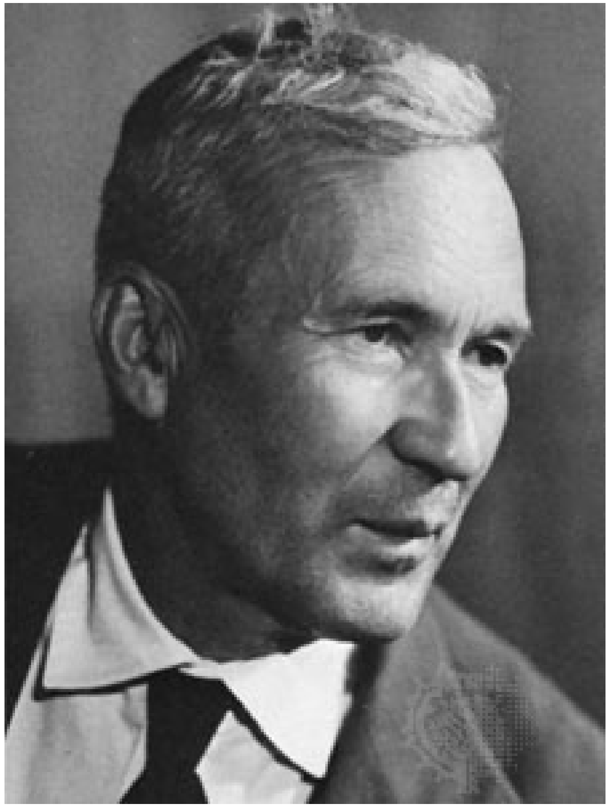
- The Weight of an element in the sample space Gives a Probability Distribution (Measure)

# Andrei Nikolaevich Kolmogorov

Russian Mathematician

*b. 1903. d. 1987*



The birth of probabiity theory is often dated to 1654, when Pascal and Fermat, trying to solve a gambling problem, developed the fundamentals.

It wasn't until the work of Kolmogorav in 1933, though, that we had a "definition of probability that is precise enough for use in mathematics, yet comprehensive enough to be applicable to a wide range of phenomena".

*Ref. A Short History of Probability by Tom Apostol*
*http://www.cc.gatech.edu/classes/cs6751_97_winter/Topics/stat-meas/probHist.html*

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*Note: TTT corresponds to all answers being true, etc..*

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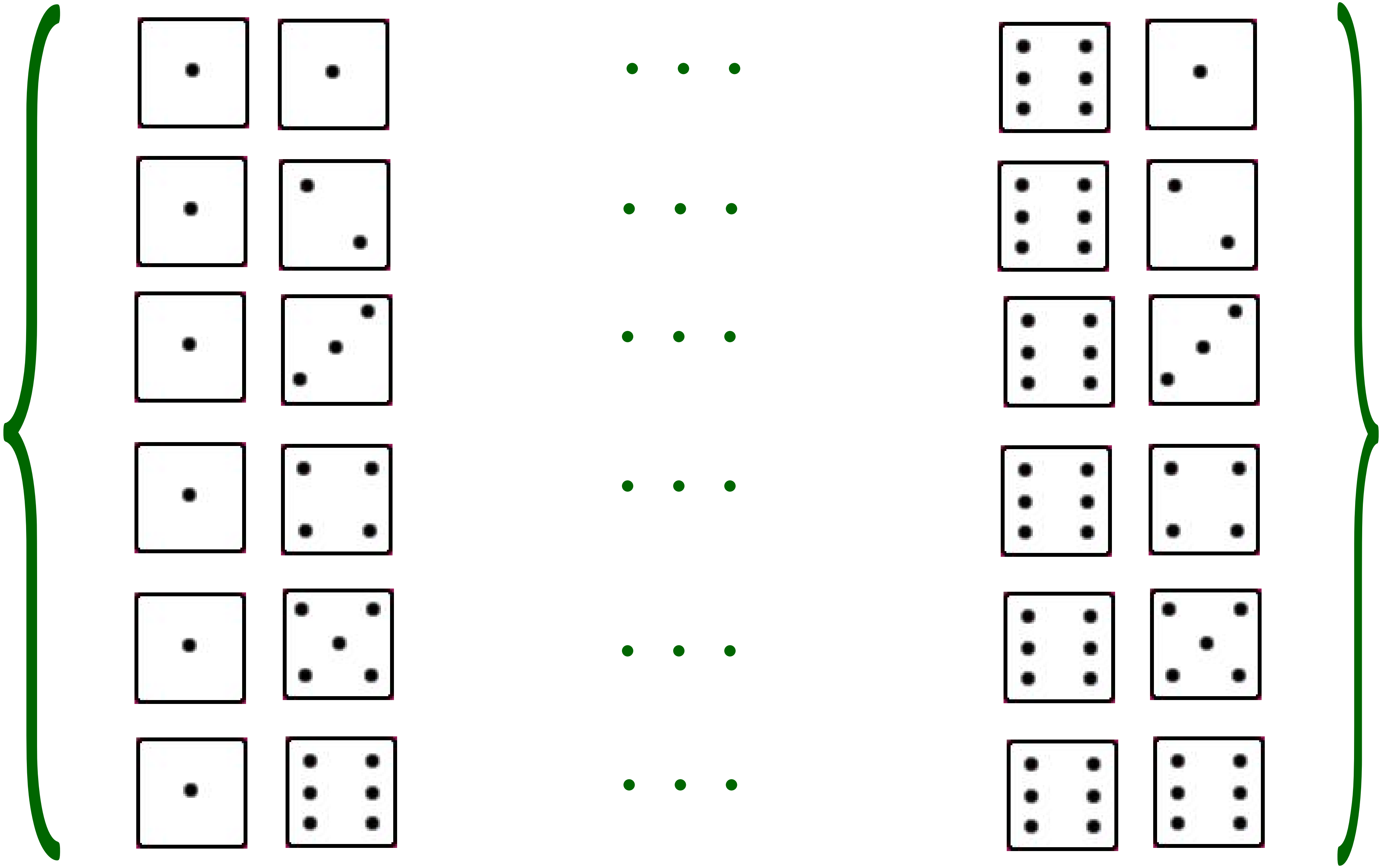

Example 3:

Throw a two-sided (Heads, H, and Tails, T) coin, until you see a Head. This sample space contains an *infinite* number of outcomes.

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{`H, TH, TTH, TTTH, TTTTH, TTTTTH, ...`}

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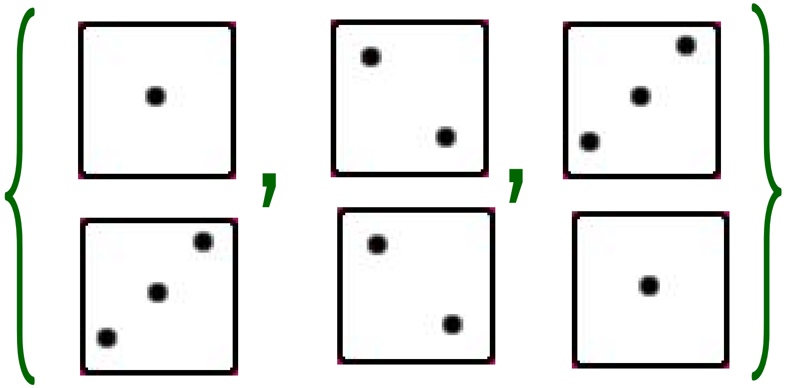

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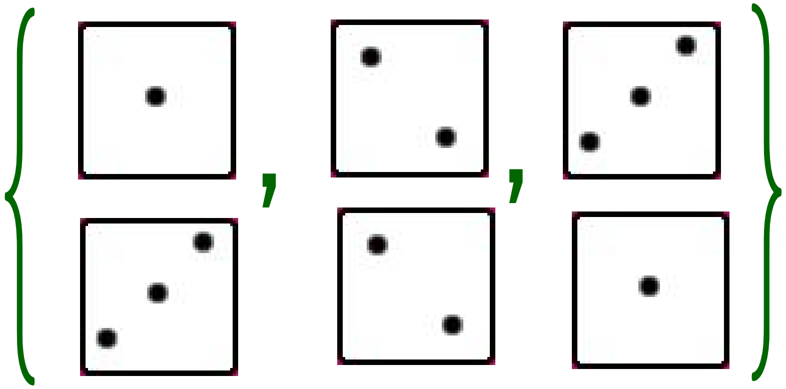

3) Event of a Head occurring in the first 3 flips: {H, TH, TTH}.

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$$P(E) = \sum_{x:x \in E} P(x)$$

*read: "The probability of event* $E$ *is the sum, over all* $x$ *such that* $x$ *is in* $E$*, of* $P(x)$*."*

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*A function $P$ satisfying these rules is called a* **probability distribution** *or a* **probability measure**.

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$$\{\texttt{TTT},\ \texttt{TTF},\ \texttt{TFT},\ \texttt{FTT},\ \texttt{TFF},\ \texttt{FTF},\ \texttt{FFT},\ \texttt{FFF}\}.$$

Suppose that each sequence of T and F is equally likely. We then want to assign each outcome the same *(uniform)* probability weight. Since the sum of the weights must add up to $1$, we assign each of the $8$ outcomes a weight of $\frac{1}{8}$.

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The event "the first answer is true"
is $\{$TTT, TTF, TFT,TFF$\}$
so its probability is

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$$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

# Example 1b:

The professor's three-question quiz with sample space

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"there is exactly one true answer"
is $\{\text{TFF, FTF, FFT}\}$
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$$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

# Example 1c:

Now suppose that the professor has a bias towards true answers and a T is more likely than a F. Let's assume that a T has a probability of $2/3$ and a F has a probability of $1/3$. Then, we will see later, we will get the following different weights on the *same* sample space as before

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Now suppose that the professor has a bias towards true answers and a T is more likely than a F. Let's assume that a T has a probability of $2/3$ and a F has a probability of $1/3$. Then, we will see later, we will get the following different weights on the *same* sample space as before

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| | | | | | | | |
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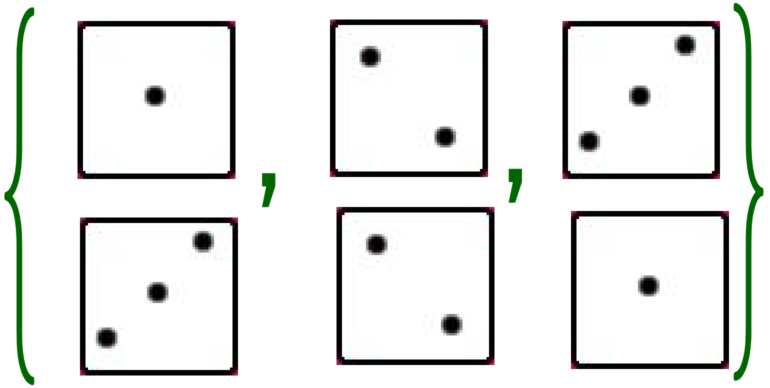

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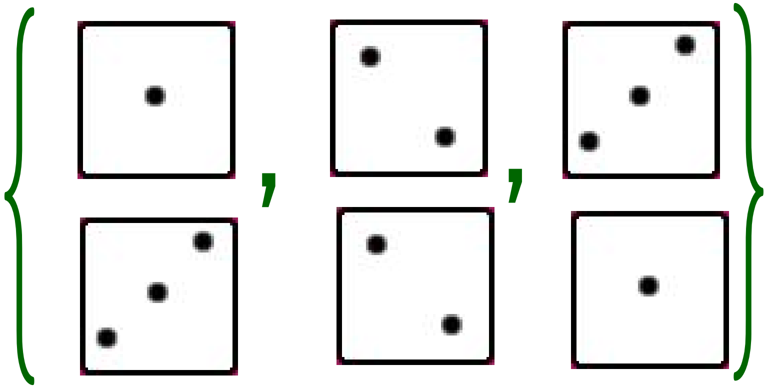

There are three outcomes in this event, so its probability is

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}$$

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When throwing a coin until the first H is seen,
the sample space is

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Note that this is a legal probability distribution, since

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

# Introduction to Probability

- Why Study Probability?

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Note that $S = A \cup B$ (or $B = S - A$) and that $P(A) + P(B) = 1$.

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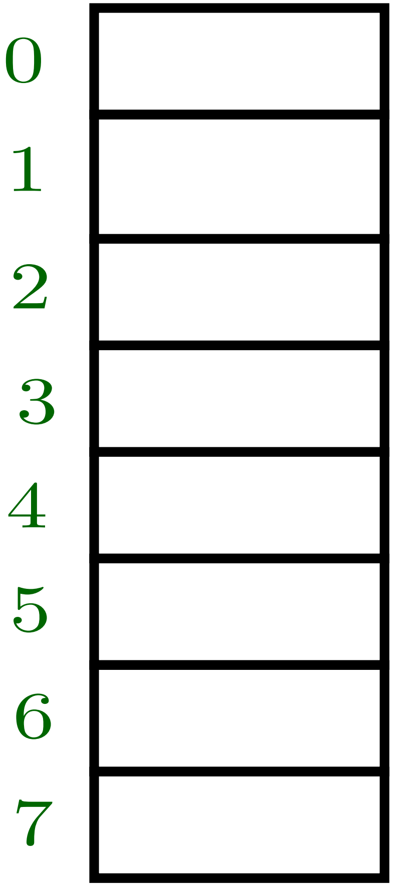
Example:



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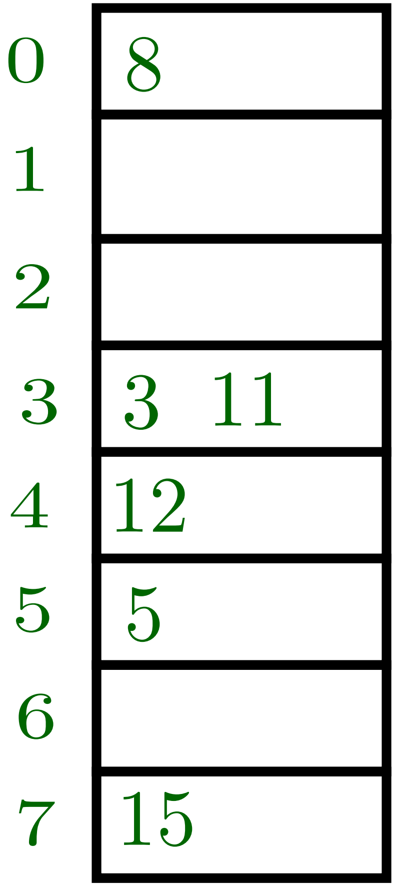
Example: $3, 12, 15, 8, 11, 5$

is outcome $(3, 4, 7, 0, 3, 5)$



| | |
|---|---|
| 0 | 8 |
| 1 | |
| 2 | |
| 3 | 3 11 |
| 4 | 12 |
| 5 | 5 |
| 6 | |
| 7 | 15 |

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Weight function: Assuming that hash function is "random" then every $n$-tuple is equally likely. So, every $n$-tuple should have (the same) weight $1/m^n$.

Hash a list of $3$ keys into hash table with $20$ locations.

What is the probability of event A in which all three keys hash to different locations, i.e., no collisions?

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$$\Rightarrow \boxed{P(A) = \frac{20 \cdot 19 \cdot 18}{20^3} = .855}$$

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$$P(B) = 1 - P(A) = 1 - .855 = .145$$

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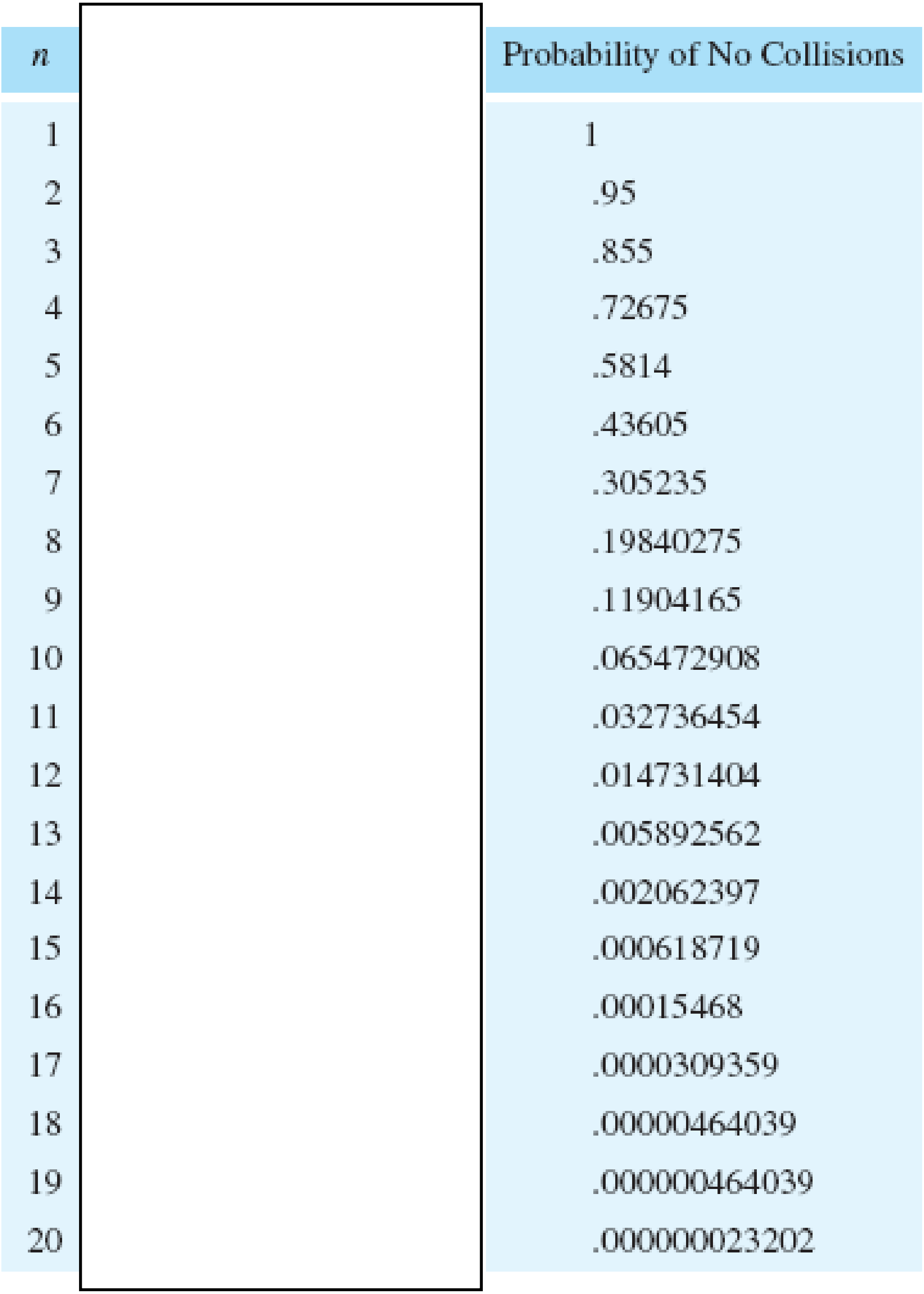
Since events A and B are complementary

$$P(B) = 1 - P(A) = 1 - \frac{20^{\underline{\mathsf{n}}}}{20^n}$$



| *n* | Probability of No Collisions |
|---|---|
| 1 | 1 |
| 2 | .95 |
| 3 | .855 |
| 4 | .72675 |
| 5 | .5814 |
| 6 | .43605 |
| 7 | .305235 |
| 8 | .19840275 |
| 9 | .11904165 |
| 10 | .065472908 |
| 11 | .032736454 |
| 12 | .014731404 |
| 13 | .005892562 |
| 14 | .002062397 |
| 15 | .000618719 |
| 16 | .00015468 |
| 17 | .0000309359 |
| 18 | .00000464039 |
| 19 | .000000464039 |
| 20 | .000000023202 |

Probabilities that all elements of set hash to different entries of hash table of size $20$ is

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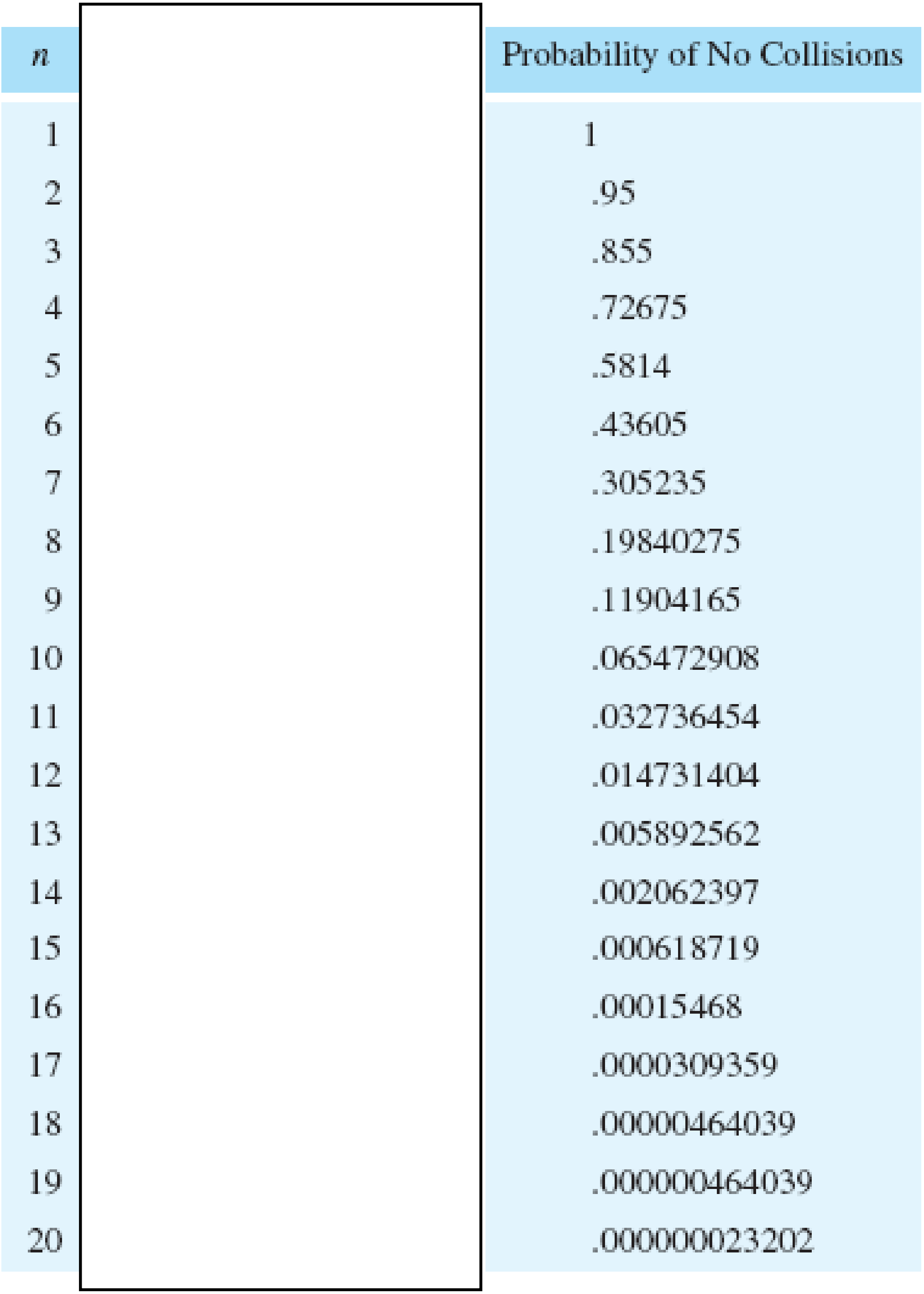

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$$p_{n+1} = p_n \frac{20-n}{20} < p_n,$$

$p_n$ decreases as $n$ increases

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*Note: We have implicitlly used this theorem many times already*

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$$\begin{aligned} 1 &= P(S) \\ &= P(x_1 \cup x_2 \cup \cdots \cup x_{|S|}) \\ &= P(x_1) + P(x_2) + \cdots + P(x_{|S|}) \\ &= p|S|. \end{aligned}$$

## Proof:

Let $S = \{x_1, x_2, \ldots, x_{|S|}\}$.

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$$\Rightarrow p = \frac{1}{|S|}$$

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Combining these equations gives

$$P(E) = |E|p = |E|(1/|S|) = |E|/|S|.$$

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Use Theorem 5.2

$\Rightarrow$ probability is $4/8 = 1/2$ by Theorem 5.2.

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Theorem 5.2 doesn't apply to this distribution. For example, let E be the event that the outcome is not positive.

Then $E = \{0\}$ but

$$P(E) = \frac{1}{8} \neq \frac{1}{4} = \frac{|E|}{|S|}$$

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which is exactly the non-uniform distribution we just saw.