Example 1

$\boxed{\text{Claim}}$: $\left.\begin{array}{l}\text{If } T(1) > 0,\ T(n) \leq 2T(\frac{n}{2}) + cn \\ \qquad\qquad \text{for some constant } c\end{array}\right\} A$

then $T(n) = O(n \log n)$

Rewrite conclusion:

$\exists n_0, k$ s.t. $\forall n$, $\underbrace{\text{if } n > n_0,\ T(n) \leq kn\log n}_{p(n,\, n_0,\, k)}$

Need to prove:

$$A \Longrightarrow \exists n_0, k \ \forall n\ p(n, n_0, k)$$

plan: Assume A. Pick n_0, k

Show $\forall n\ p(n, n_0, k)$ $\quad (*)$

strategy:

* Try to prove (*) by induction

* Find constraints on n_0, k in the process

* pick n_0, k accordingly

$\boxed{\text{proof}}$

Base case: $n = n_0 + 1$

Need: $T(n_0+1) \le k\,(n_0+1)\log(n_0+1)$ (★)

- (★) cannot be true when $n_0 = 0$

So, $n_0 > 0$

pick $n_0 = 1$ (c1)

- Now (★) becomes

$$T(2) \le k\, 2 \log 2 = 2k$$

So, must make sure : $\boxed{k \ge T(2)/2}$ (C2

If C1 & C2 are true, $P(n, n_0, k)$

is true at base case.

Induction Hypothesis

$P(n, n_0, k)$ true for $n = 2^{i-1}$

$$T(2^{i-1}) \le k\, 2^{i-1} \log 2^{i-1}$$

Induction step : Consider $n = 2^i$

$$T(n) \le 2T\left(\frac{n}{2}\right) + cn$$

$$= 2\,T(2^{i-1}) + cn$$

$$\le 2k\, 2^{i-1} \log 2^{i-1} + cn$$

$= k2^i \log 2^i - k2^i \log 2 + cn$

$= kn \log n - kn + cn$

Need: $T(n) \le kn\log n$

So must have:

$$-kn + cn \le 0$$

or $\boxed{k \ge c \qquad (C_3)}$

Now, if we pick

$$\boxed{\begin{array}{l} n_0 = 1 \\ k = \max\left\{ \frac{T(2)}{2}, c \right\} \end{array}}$$

C_1, C_2, C_3 are true.

$\Rightarrow$ $P(n, n_0, k)$ true in base case

true in induction step

$\Rightarrow$ $P(n, n_0, k)$ true for all n.

proved.

Advanced Induction

* pick n_0, k, show $\forall n > n_0$

$P(n, n_0, k)$

Base case: $n = n_0 + 1$

what n_0, k should be for $P(n, n_0, k)$ to hold?

$n_0 = 1$

$k \geq \frac{T(2)}{2}$

Induction step

$P(2^{i-1} \cdot n_0, k)$

$\Rightarrow P(2^i \cdot n_0, k)$

$k \geq c$

what n_0, k should be for the implication to hold?

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pick $n_0 = 1$, $k = \max\{\frac{T(2)}{2}, c\}$

both step hold.

Example 3 : Induction step

$T(m) \le k m^2 \quad m = 2^j, \ 0 \le j < i$

$\Rightarrow T(n) \le k n^2, \quad n = 2^i$

No k can make the implication true.

Goal: $\quad T(n) = O(n^2) \quad$ ①

Try 1: $\exists\, n_0, k, \ \forall n > n_0 \quad T(n) \le k n^2$

② $\quad$ X

② ⇒ ①, but cannot prove ②

Try 2: $\exists\, n_0, k_1, k_2$

$\forall n > n_0, \ T(n) \le k_1 n^2 - k_2 n$

③

③ ⇒ ①

Prove ③

Base case:

$n_0 = 0$

$T(1) \leq k_1 - k_2$

Induction step

$T(2^{i-1}) \leq k_1 (2^{i-1})^2 - k_2 2^{i-1}$

$\Rightarrow T(n) \leq k_1 n^2 - k_2 n, \quad n = 2^i$

True if set $k_2 = c$

Set $k_1 = T(1) + c$

$k_2 = c$

Base case true

induction step true.

Proved.