

COMP170

Discrete Mathematical Tools for Computer Science

Lecture 2

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Discrete Math for Computer Science

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Section 1.2, pp. 9-19

1.2 Counting Lists, Permutations, and Subsets

- Using the Sum and Product Principles
- Lists and Functions
- The Bijection Principle
- k -Element Permutations of a Set
- k -Element Subsets of a Set
Binomial Coefficients

Some Simple Examples

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Note; This is really the *product principle*

What are the sets being used?

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A password is of the form $D_1 D_2 D_3 D_4 D_5 D_6$ where
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Each D_i has 52 possible choices so,
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Each L_i has 26 possibilities and
each D_i has 10 possibilities
so the total number of car plates is

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4 = 6,760,000$$

We have just seen examples of

Product Principle, Version 2

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S , and
2. for each $j > 1$
and each choice of the first $j - 1$ elements of a list in S ,
there are i_j choices of elements in position j
of those lists,

\Rightarrow there are $i_1 i_2 \cdots i_m = \prod_{k=1}^m i_k$ lists in S .

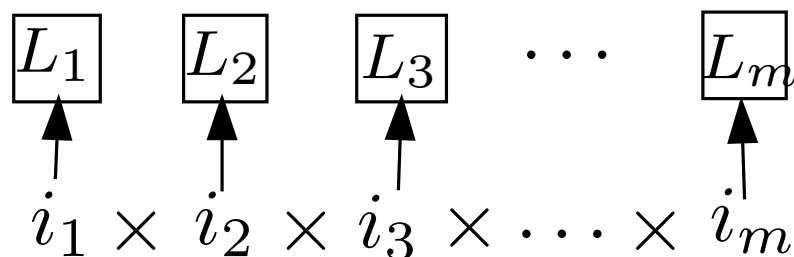
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So *sum principle* might help us.

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\Rightarrow total number of passwords is

$$52^4 + 52^5 + 52^6 + 52^7 + 52^8.$$

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Example: *MORE* and *ROME* are
two different 4-letter lists from $T = \{A, B, \dots, Z\}$

$\{M, O, R, E\}$ and $\{R, O, M, E\}$ are
the same subset of T .

Lists and Functions (cont)

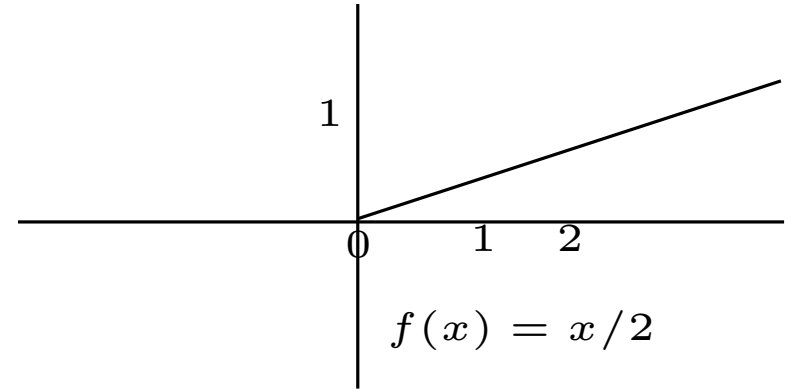
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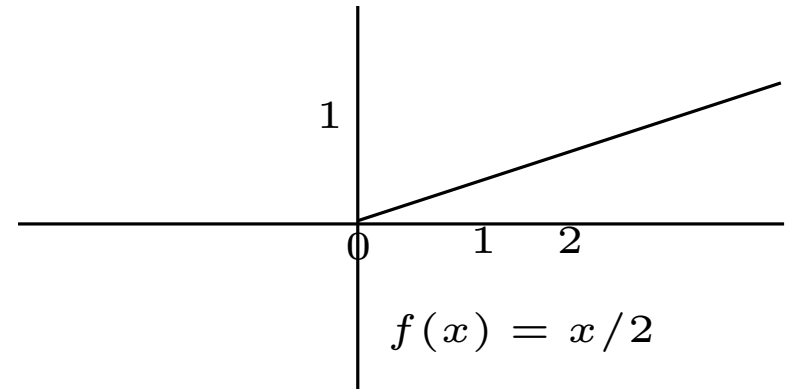
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In continuous math, e.g., basic calculus and algebra, you often have formulas, e.g., $f(x) = x/2$ that **describe** the function. In discrete math, where we often deal with finite sets, we can frequently state the full function explicitly: e.g.,

$$f(1) = \text{Sam}, f(2) = \text{Mary}, f(3) = \text{Sarah}$$

Lists and Functions (cont)

We often write a function from S to T as

$$f : S \rightarrow T$$

S is the domain of f ; T is the range of f .

Lists and Functions (cont)

A k element List, $L = L_1 L_2 \dots L_k$ from set T can now be defined as a function

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Example: Four element lists from $\{A, \dots, Z\}$

MORE is the function

$$f(1) = M, f(2) = O, f(3) = R, f(4) = E$$

ROME is the function

$$f(1) = R, f(2) = O, f(3) = M, f(4) = E$$

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Use f_1, f_2, \dots to denote the functions. To describe $f_i : \{1, 2\} \rightarrow \{a, b\}$, we must specify $f_i(1)$ and $f_i(2)$.

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Set of *all* functions from $\{1, 2\}$ to $\{a, b\}$ is just the set of 2-element lists from $\{a, b\}$ which we already saw, by the product principle, has size $2 \times 2 = 4$.

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In French
sur = “on”

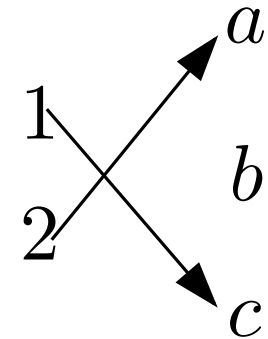
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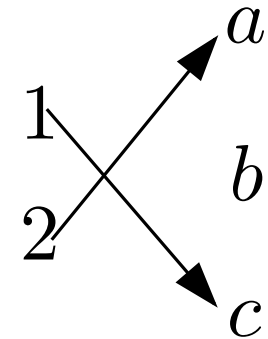
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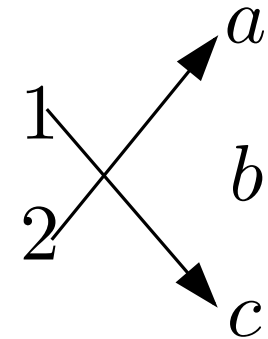
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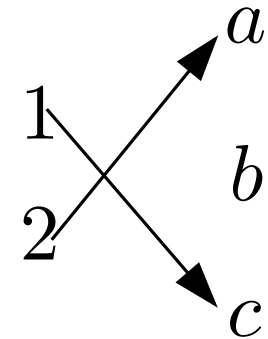


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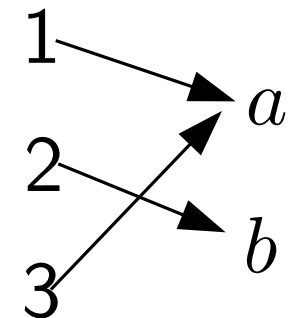
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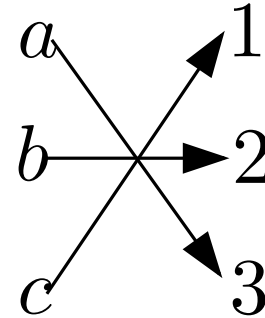
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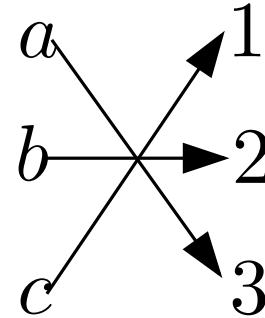


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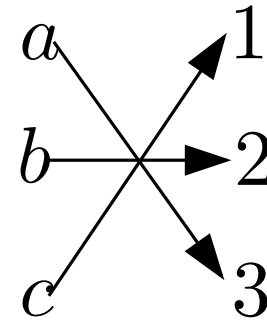
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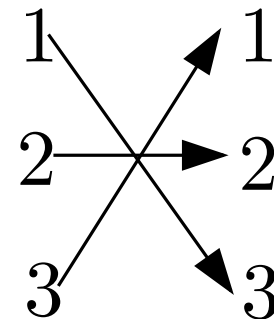
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In a bijection

exactly one arrow

leaves each item on the left

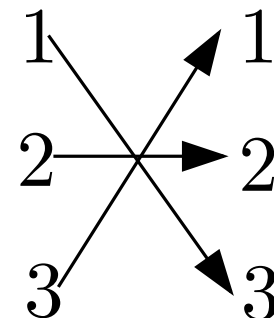
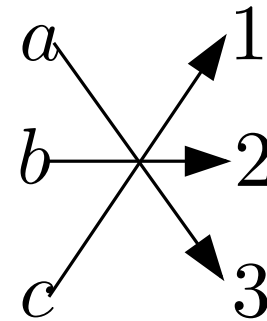
and

exactly one arrow

arrives at each item on the right

so the left and right sides

must have the same size



The Bijection Principle

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The following loop is a part of program to determine the number of triangles formed by n points in the plane.

The Bijection Principle

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The Bijection Principle

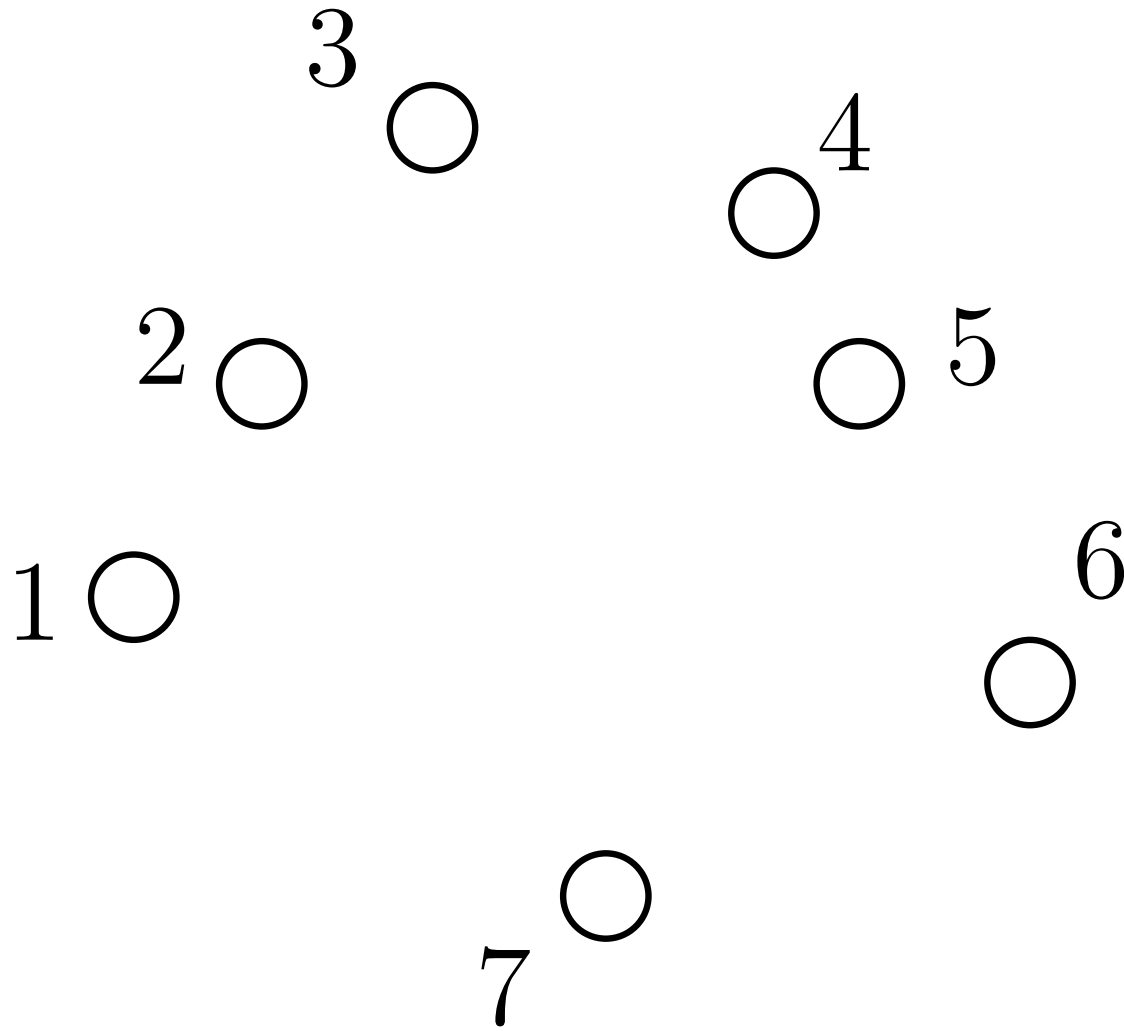
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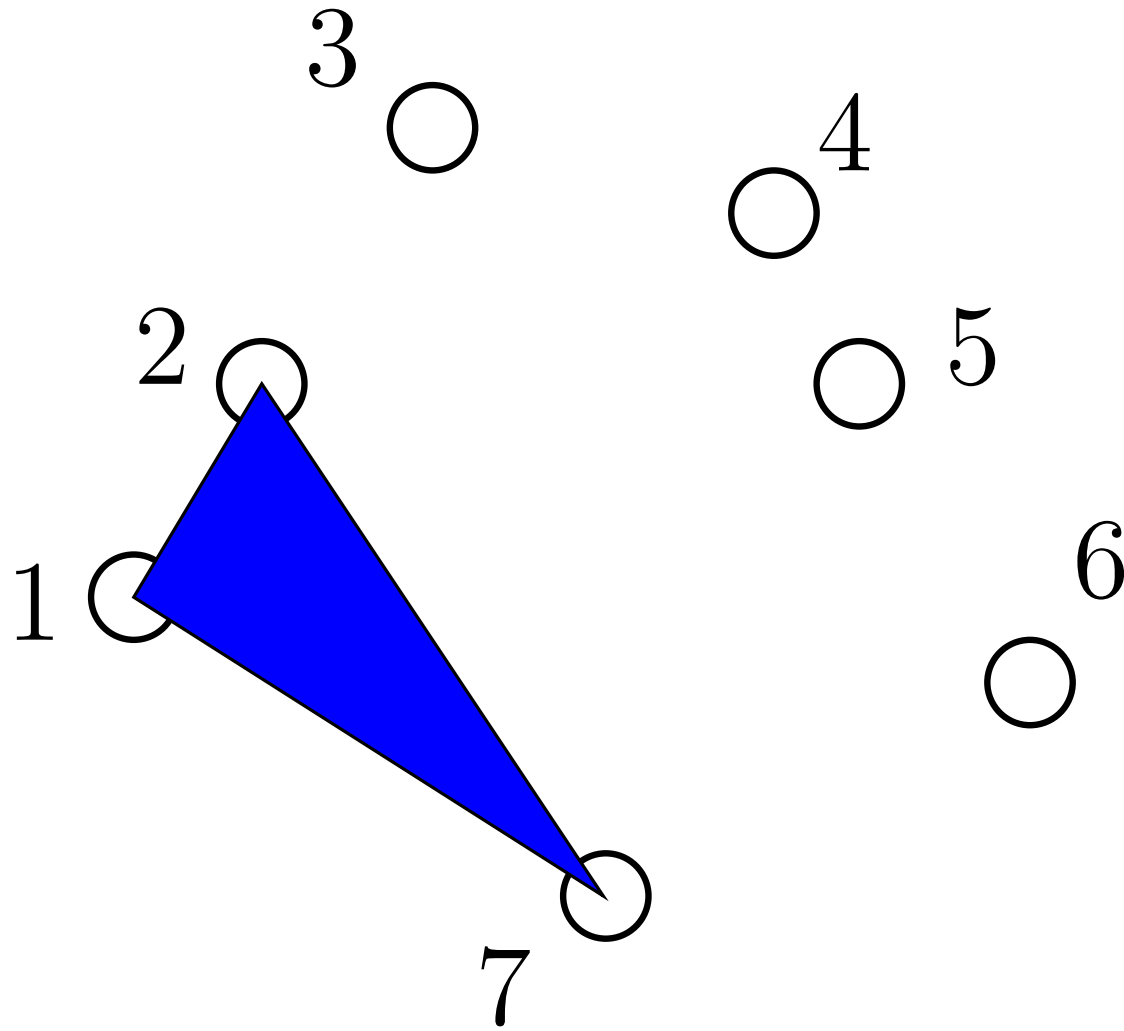
Among all iterations of line 5 in the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)

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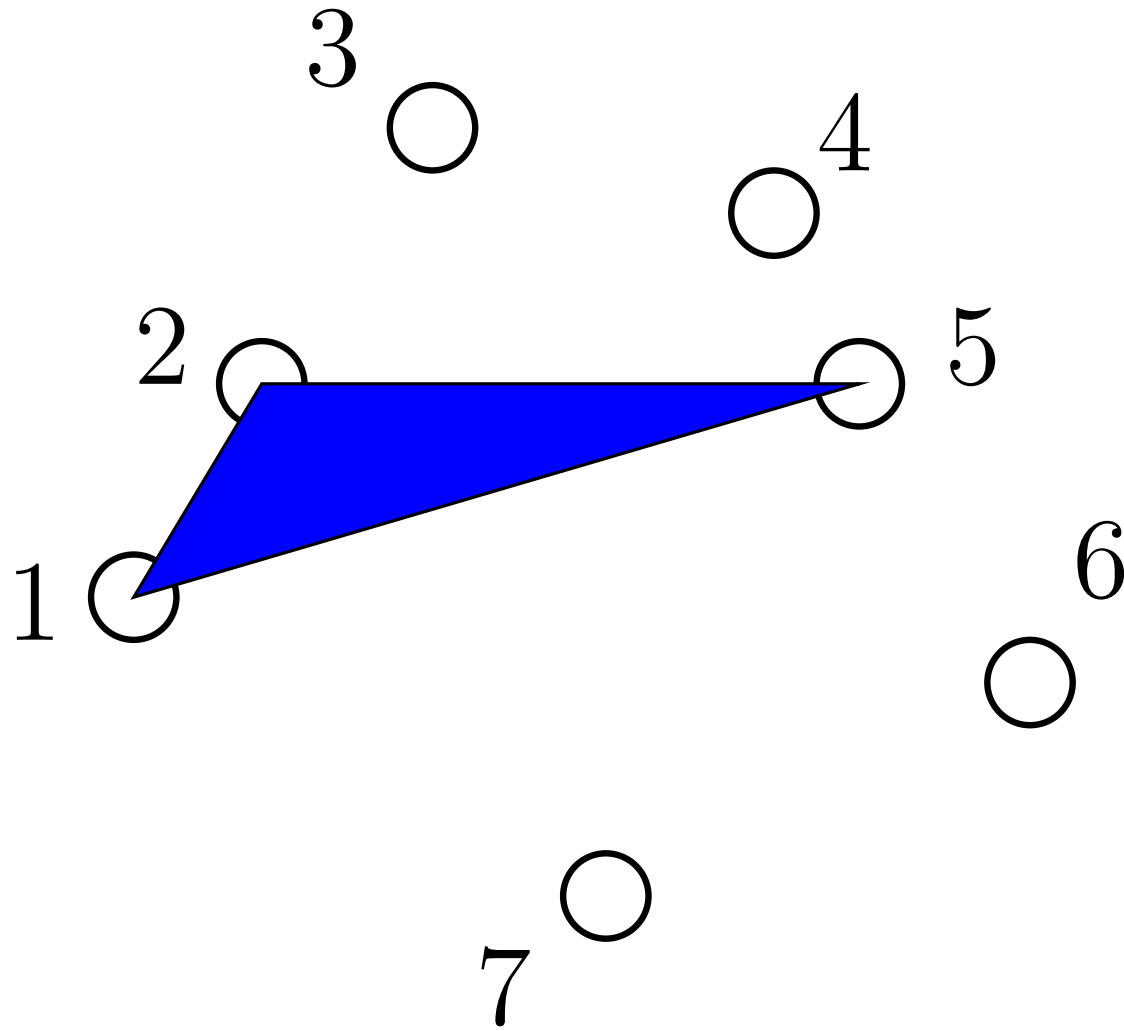


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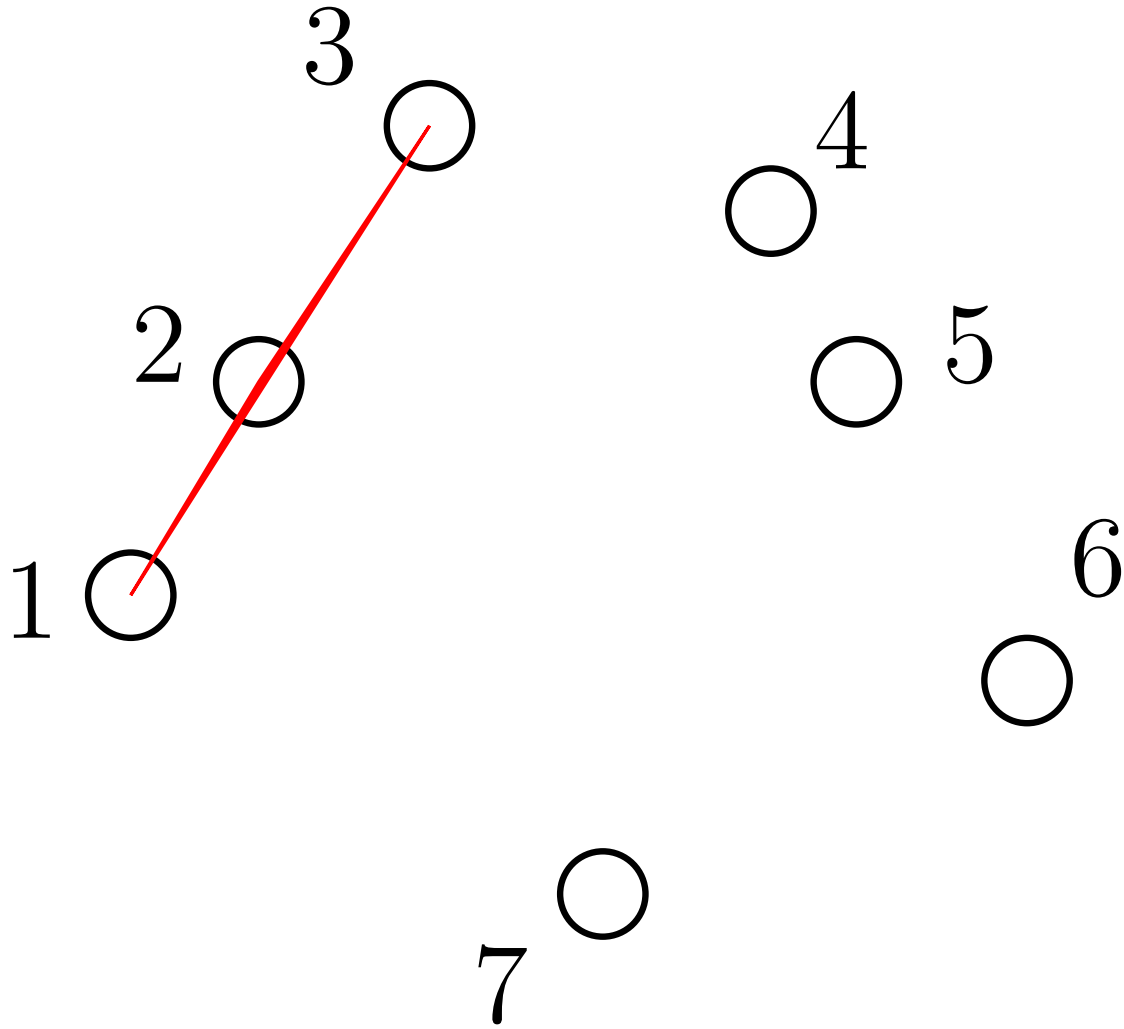
1 — 2 — 7: yes

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1 — 2 — 7: yes
1 — 2 — 5: yes

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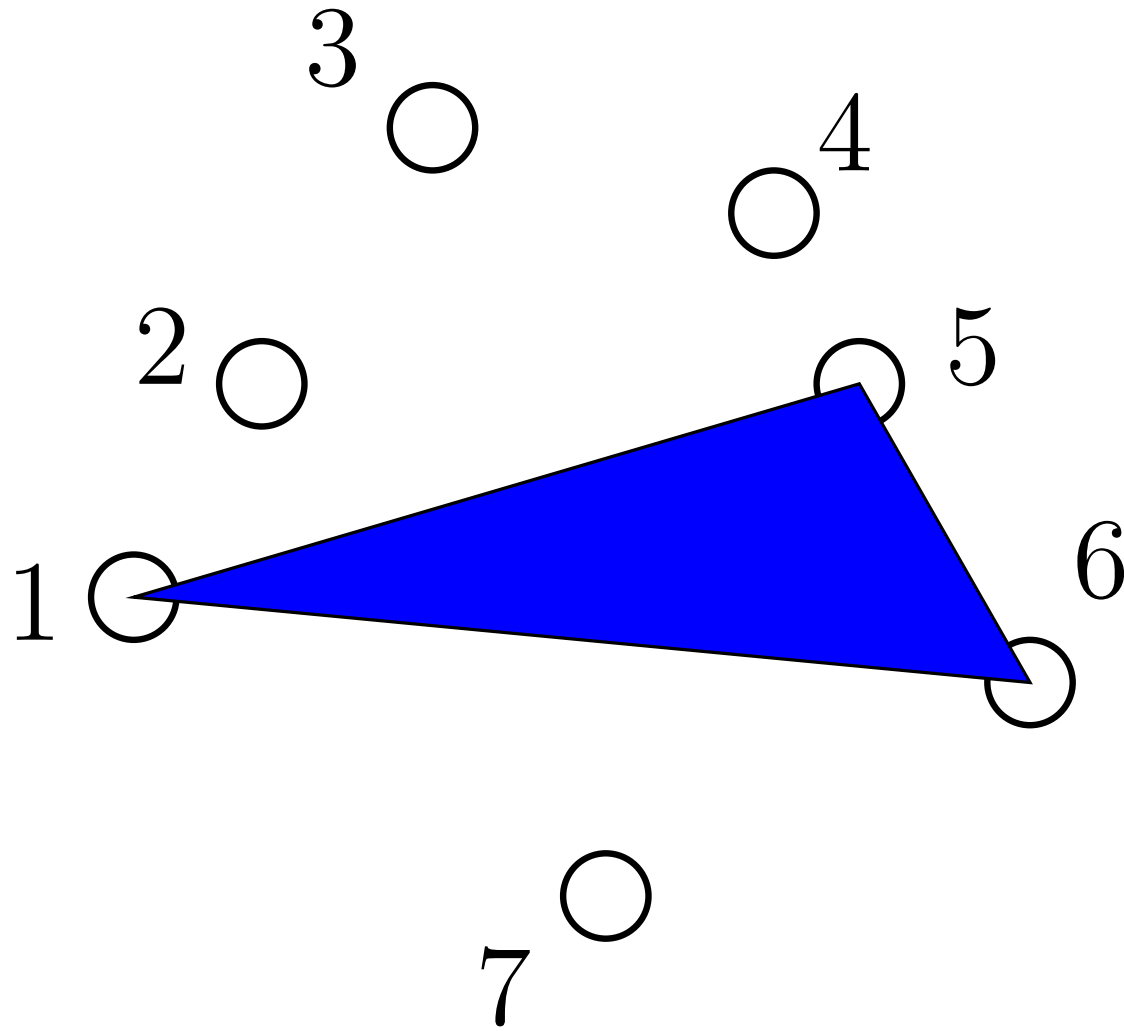


1 — 2 — 7: yes

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Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



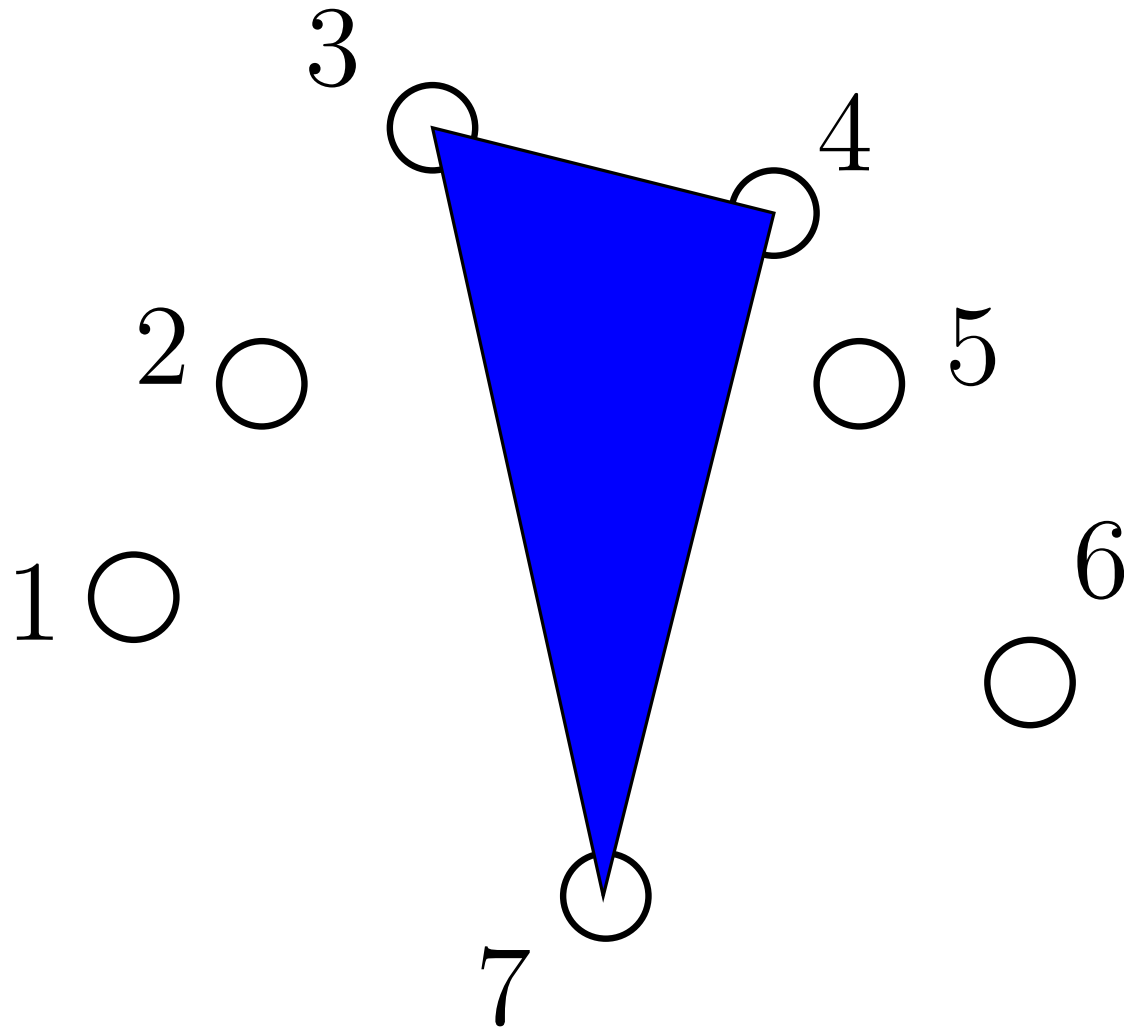
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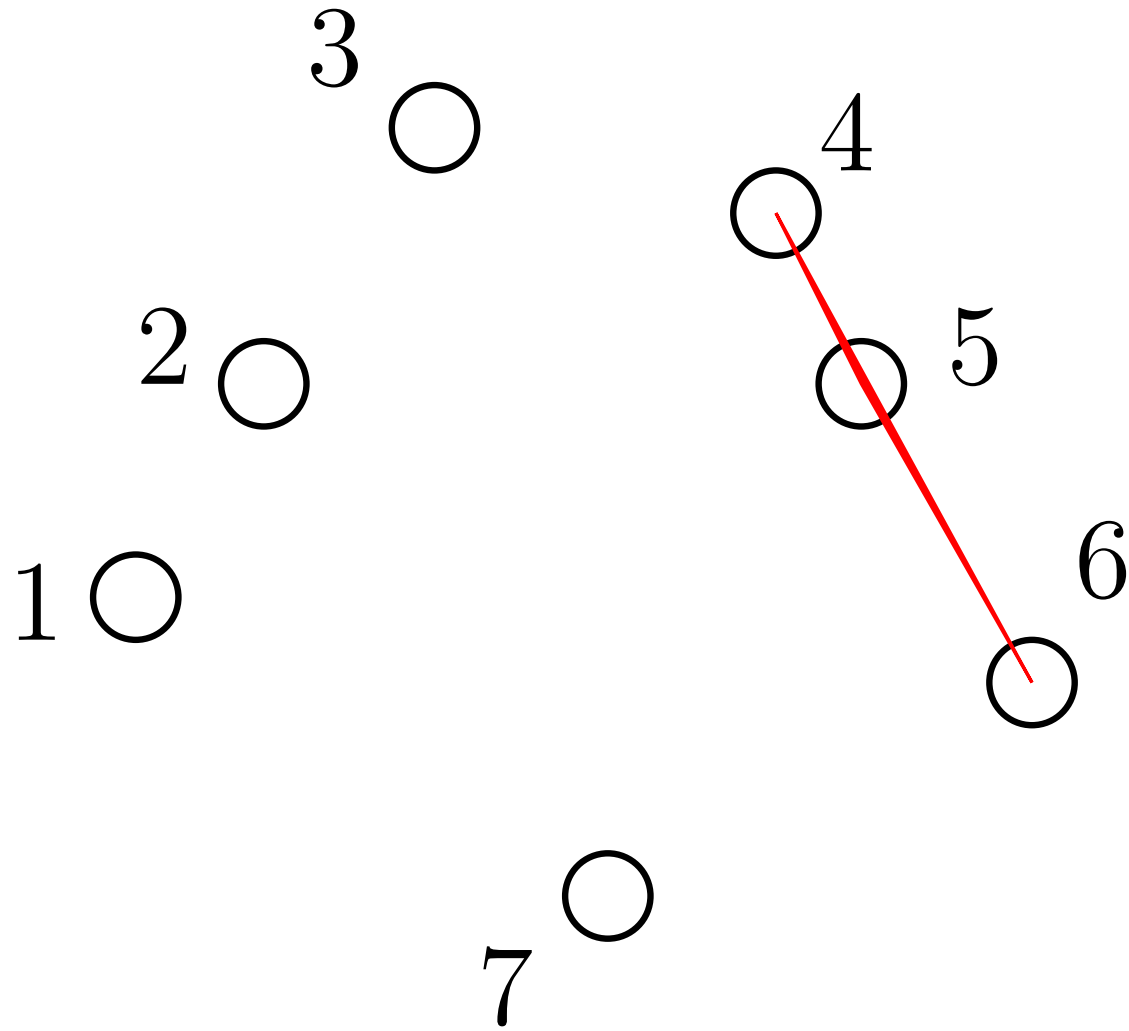
1 — 5 — 6: yes

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



1 — 2 — 7: yes
1 — 2 — 5: yes
1 — 2 — 3: no
1 — 5 — 6: yes
3 — 4 — 7: yes

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



1 — 2 — 7: yes

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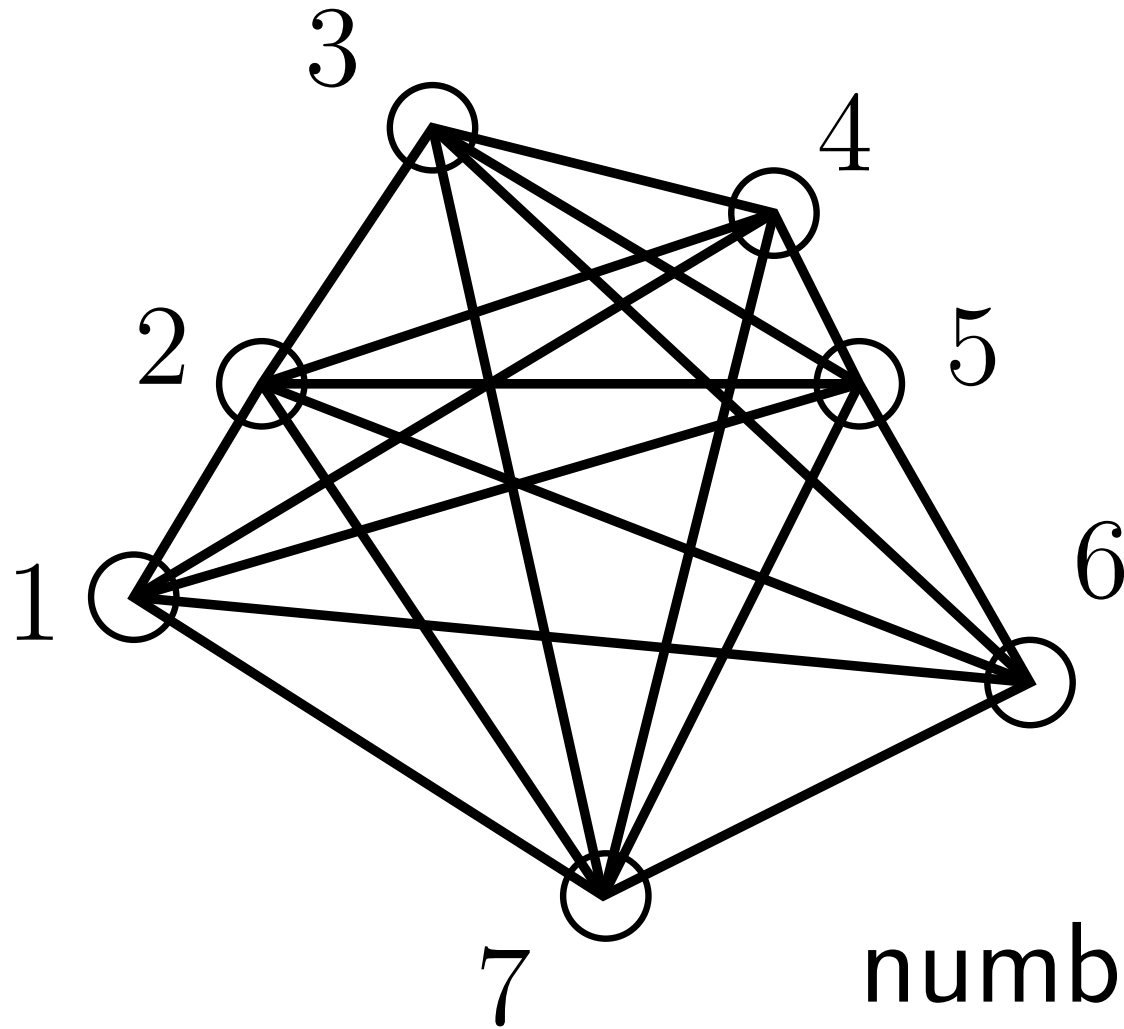
1 — 2 — 3: no

1 — 5 — 6: yes

3 — 4 — 7: yes

4 — 5 — 6: no

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



1 — 2 — 7: yes

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3 — 4 — 7: yes

4 — 5 — 6: no

number of triangles: 33

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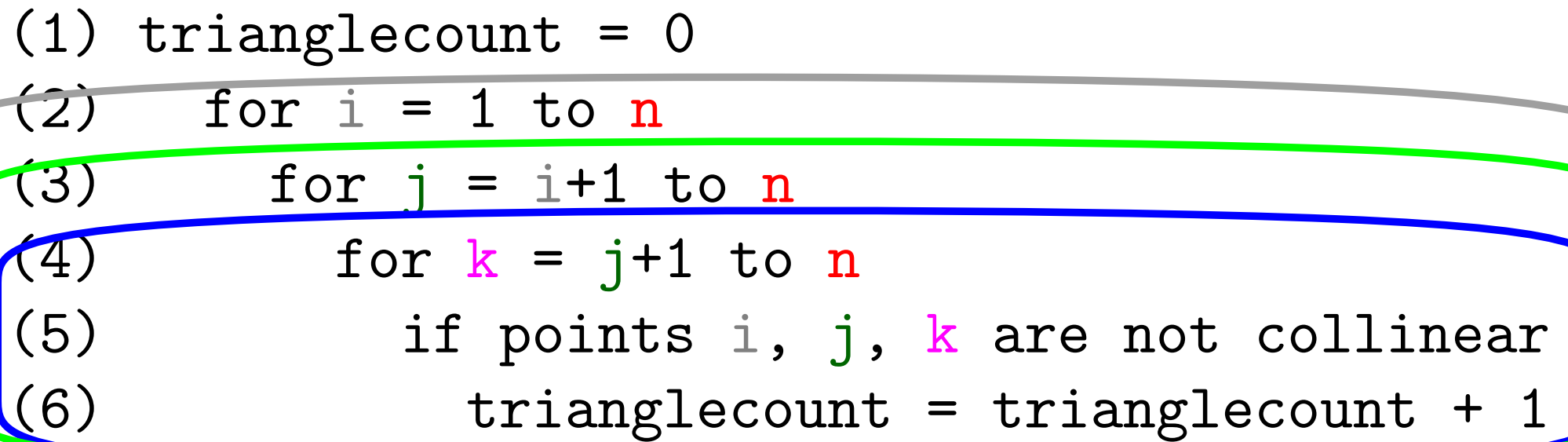
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A loop

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A loop embedded in a loop

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The diagram illustrates three nested loops. The outermost loop, 'for i = 1 to n', is highlighted with a red 'n' and enclosed in a grey rounded rectangle. The middle loop, 'for j = i+1 to n', is highlighted with a red 'n' and enclosed in a green rounded rectangle. The innermost loop, 'for k = j+1 to n', is highlighted with a red 'n' and enclosed in a blue rounded rectangle. The conditional statement 'if points i, j, k are not collinear' and the increment 'trianglecount = trianglecount + 1' are also enclosed within the blue rounded rectangle. The variables 'i', 'j', and 'k' in the conditional statement are highlighted in red, green, and blue respectively, matching their loop's color.

A loop embedded in a loop embedded in another loop.

(1) trianglecount = 0

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Second loop begins with $j = i + 1$ and j increases up to n .

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.


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Thus, compute number of increasing triples!

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- f is a bijection because
- f is **one-to-one**
if $(i, j, k) \neq (i', j', k') \Rightarrow f((i, j, k)) \neq f((i', j', k'))$
- f is **onto**
if X is a 3-element subset then it can be written as $X = \{i, j, k\}$
where $i < j < k$ so $f((i, j, k)) = X$.

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The number of
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- We actually already saw that $|X| = |Y| = \binom{n}{2}$

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- Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \dots, n\}$
- We will now see how to count the # of k -element permutations of $\{1, 2, \dots, n\}$. From this we will derive how to count # of k -element sets.

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- Note that the case of $k = n$ is special;
An n -element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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 - To start, how many 3-element permutations of $\{1, 2, \dots, n\}$ are there?

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3 -element permutations of $\{1, 2, 3, 4\}$

$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}$.

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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a **lexicographic ordering** and is used quite often.

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$(2, 3, 1)$	$(3, 1, 2)$	$(3, 2, 1)$

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$$\Rightarrow \frac{n(n-1)(n-2)}{6} = \binom{n}{3}$$

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Thus, by version 2 of product principle, there are $n(n - 1) \cdots (n - k + 1)$ ways to choose a k -element permutation.

We just saw that there are

$$n(n-1) \cdots (n-k+1) = \prod_{i=0}^{k-1} (n-i)$$

ways to choose a k -element permutation.

In the special case of $k = n$ (a permutation) this reduces to

$$n(n-1) \cdots 3 \cdot 2 \cdot 1 = n!$$

So there are $n!$ different permutations of a set of size n .

Handy notation suggested by Donald E. Knuth, is $n^{\underline{k}}$, the k **th falling factorial power of** n , defined as

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Donald Ervin Knuth (born 1938) is *Professor Emeritus of the Art of Computer Programming* at Stanford University. He is most famous for his 3+-volume set, *The Art of Computer Programming* but he's done many other things, including designing the system, TeX, with which these notes were typeset.



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- Thus, by sum principle,

$$n^k = (\# \text{ of blocks}) \times (\# \text{ in each block}) = \binom{n}{k} k!.$$

Theorem 1.2

For integers n and k with $0 \leq k \leq n$, the number of k -element subsets of an n -element set is

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

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This will work if we define $0! = 1$.

Note: Both cases $k = 0$ and $k = n$ use fact that $0! = 1$.

Binomial Coefficients

The number of ways of choosing a
 k -element subset from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This term is called a **binomial coefficient**.

Be aware that there are other, alternative, notations for the same thing, occasionally used, e.g.,

$C(n, k)$ or nCk