COMP170 Discrete Mathematical Tools for Computer Science

Lecture 18 *Version 3: Last updated, Dec 1, 2005*

Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 5.7, pp. 294-303

Probability Distributions and Variance

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- Distributions of Random Variables

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- Variance

Distributions of Random Variables

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Expected value

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Flip a coin 100 times, expected number of $\mathtt{H}$ is 50.

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Flip a coin 100 times, expected number of `H` is 50.

To what extent do we expect to see 50 heads?

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Flip a coin 100 times, expected number of `H` is 50.

To what extent do we expect to see 50 heads?

Is it surprising to see $55, 60$, or 65 heads instead?

Distributions of Random Variables

Expected value

Example:
Flip a coin 100 times, expected number of `H` is 50.

To what extent do we expect to see 50 heads?

Is it surprising to see $55, 60$, or 65 heads instead?

General Question: how much do we expect a random variable to **deviate** from its expected value.

The **distribution function** D of a random variable X with finitely many values is the function on the values of X defined by $D(x) = P(X = x)$.

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Visualize the distribution function using a diagram called a **histogram**.

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Visualize the distribution function using a diagram called a **histogram**.

> Graphs that show, for each integer value x of X, a rectangle of width 1 centered at x, whose height (and thus area) is proportional to the probability $P(X = x)$.

Examples:

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10 coin flips

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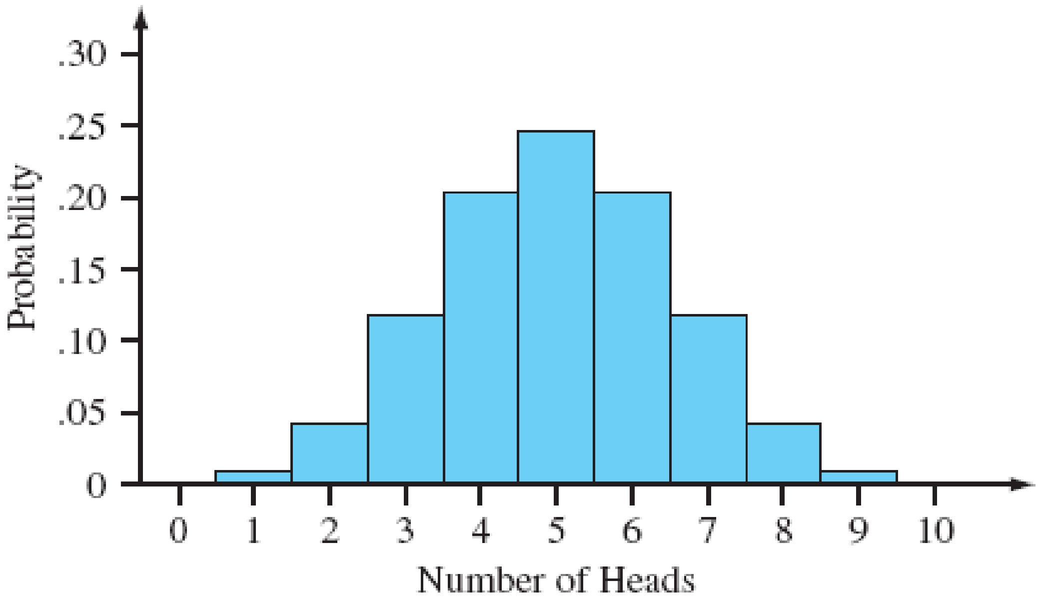

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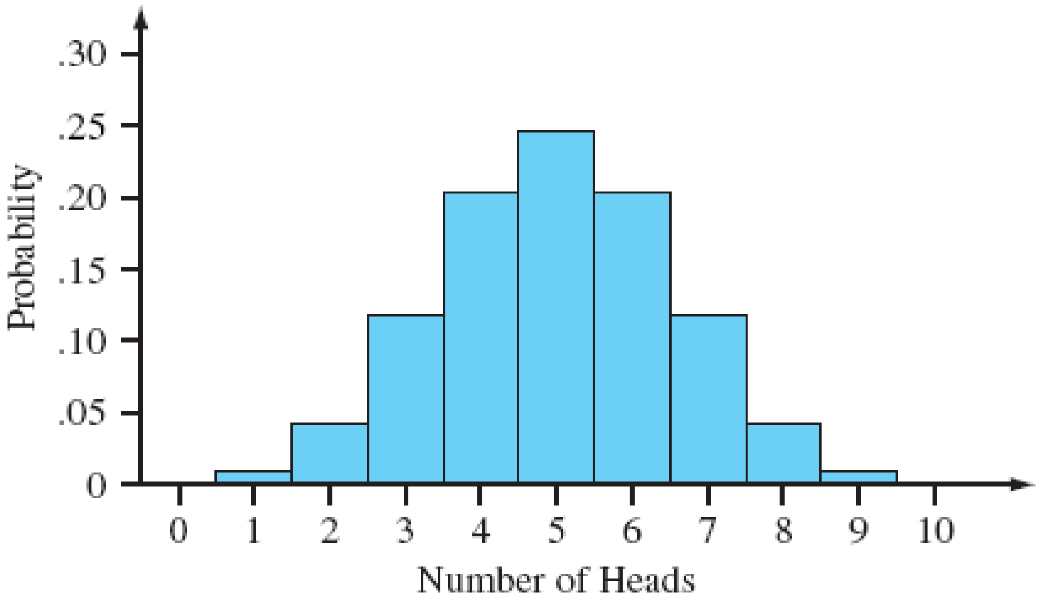

Ten-question test with probability .8 of getting a correct answer.

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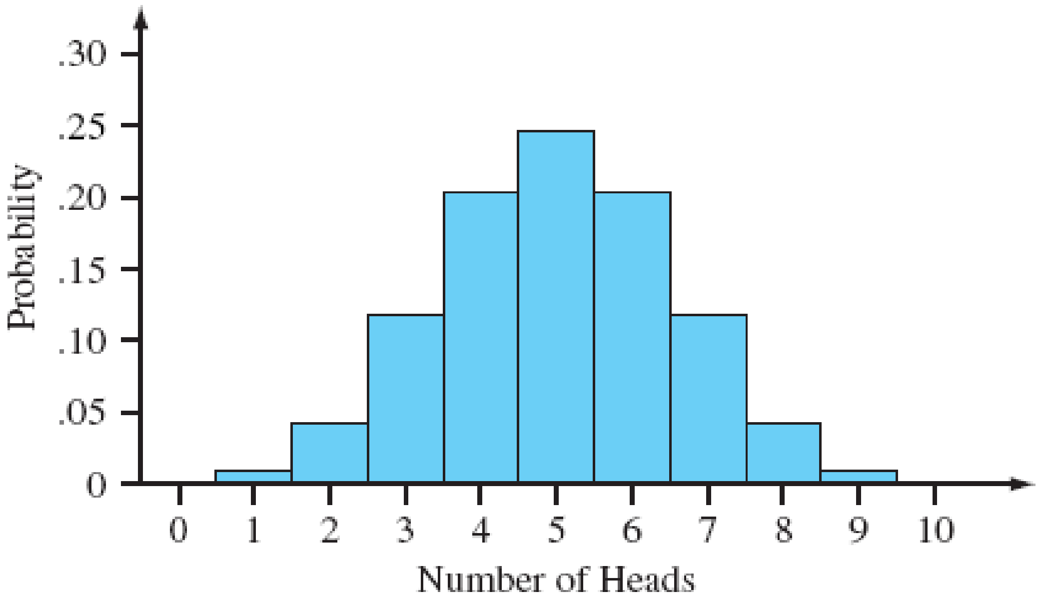

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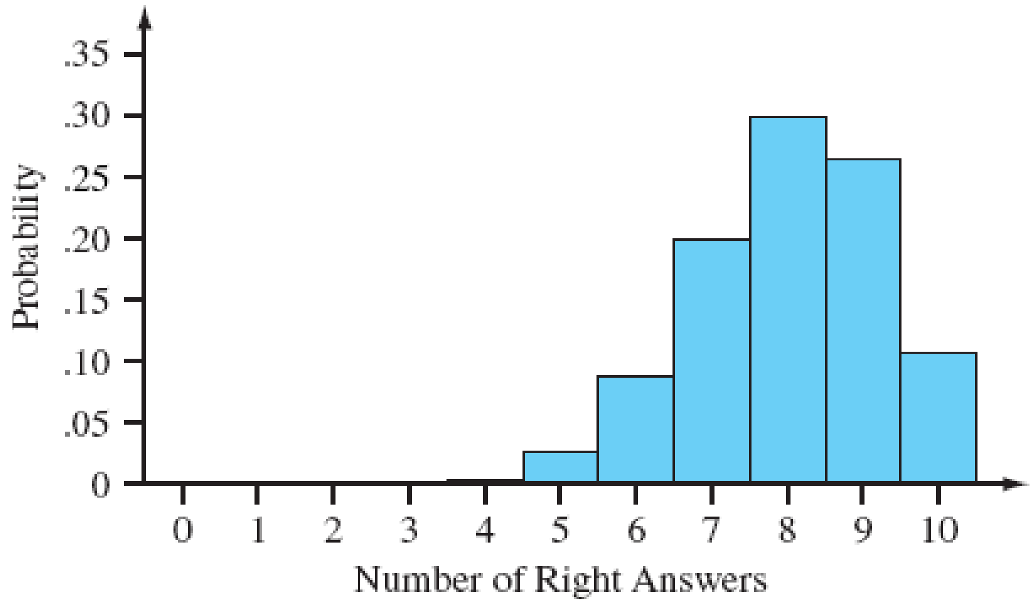

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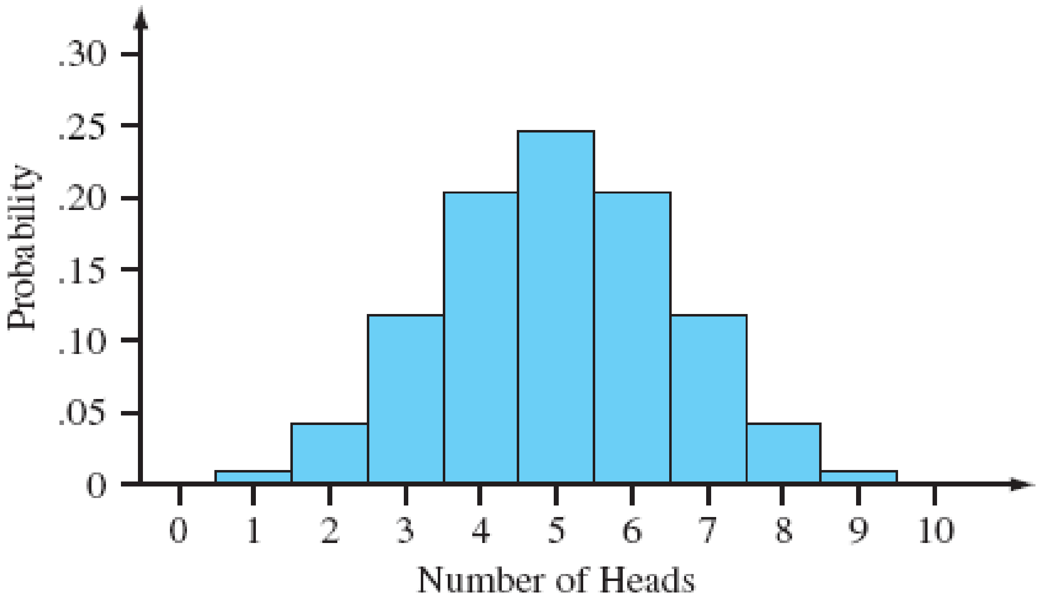

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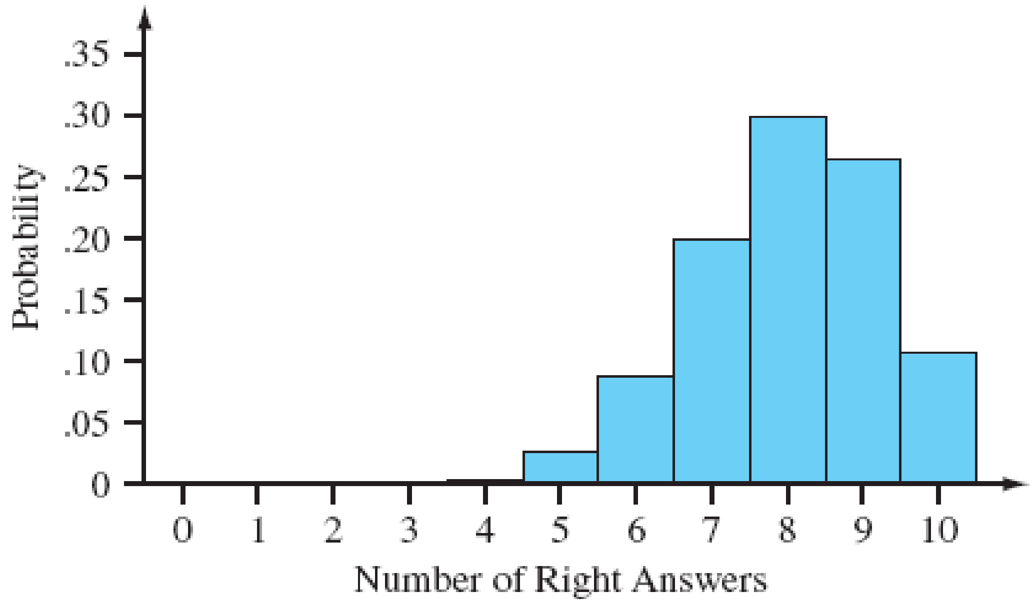

Area of Rectangles with bases ranging from $x = a$ to $x = b$ is probability that X is between a and b.

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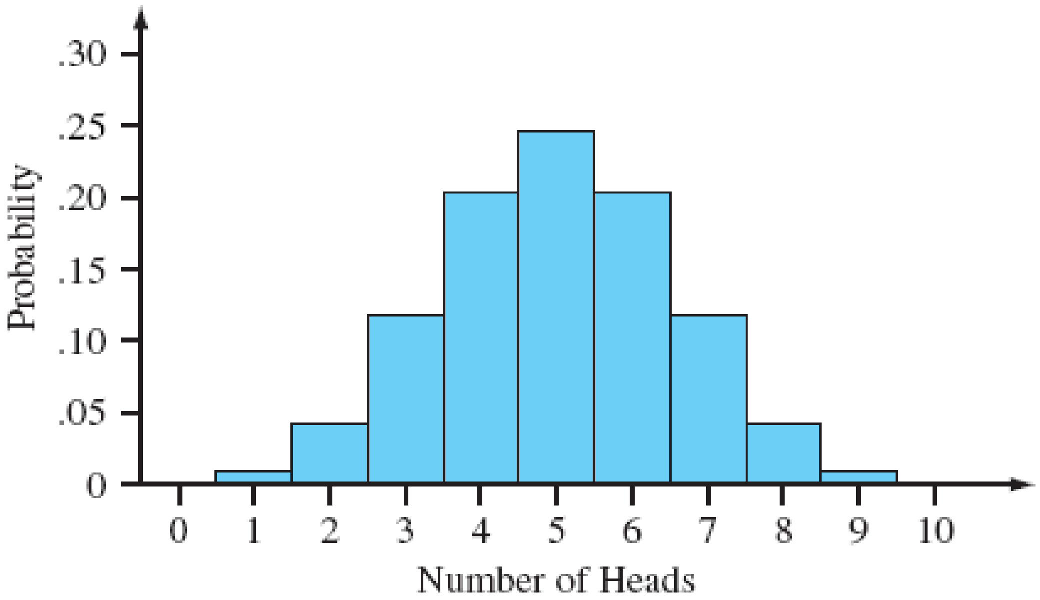

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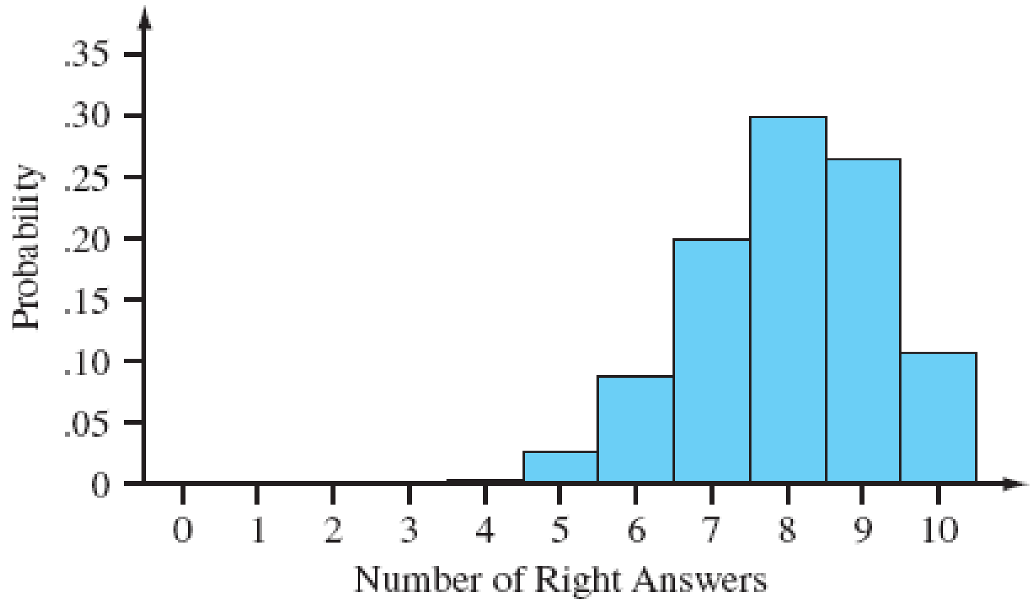

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Cumulative distribution function D:

$$D(a, b) = P(a \leq X \leq b).$$

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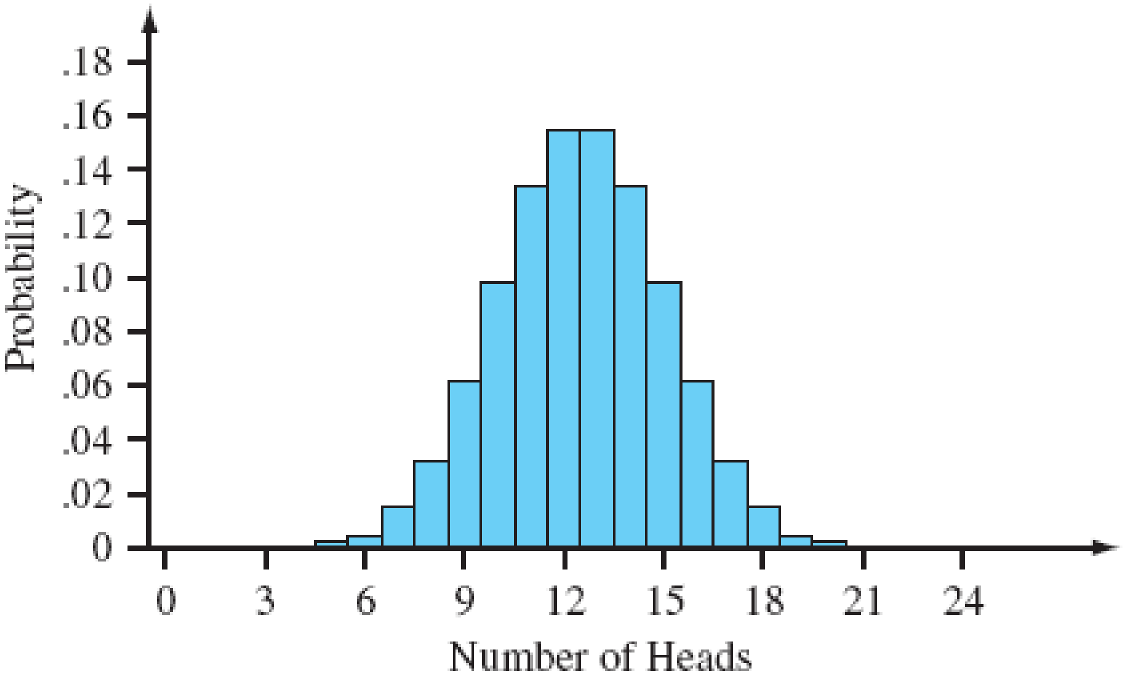
25 trials:



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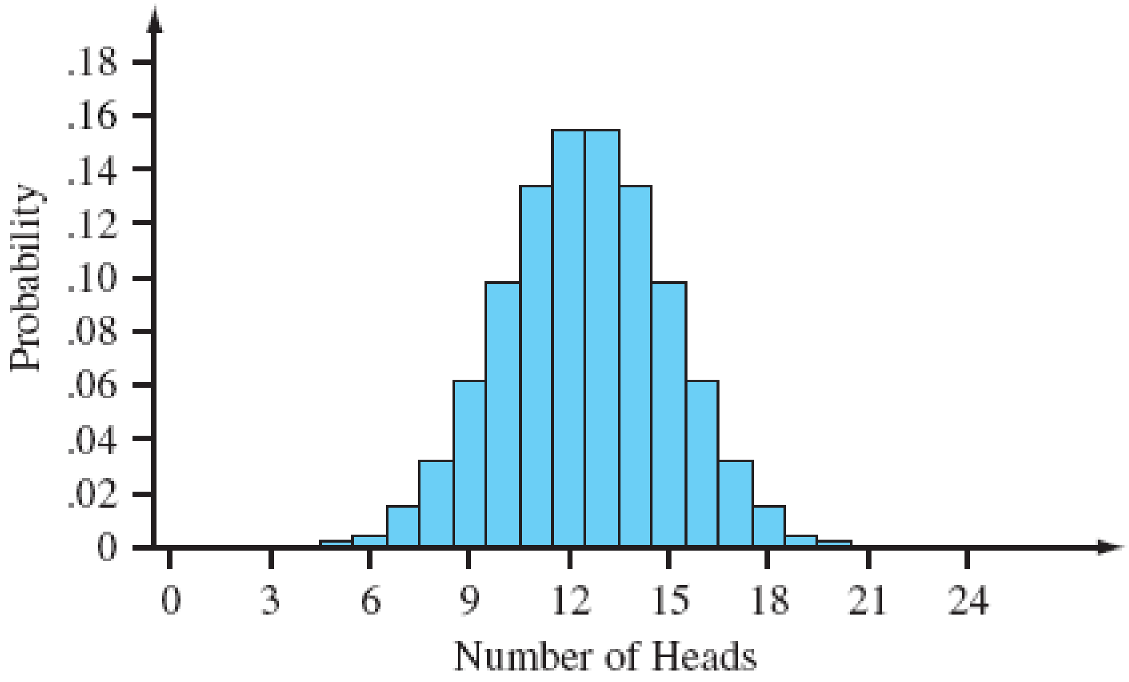

Expected number of heads is 12.5.

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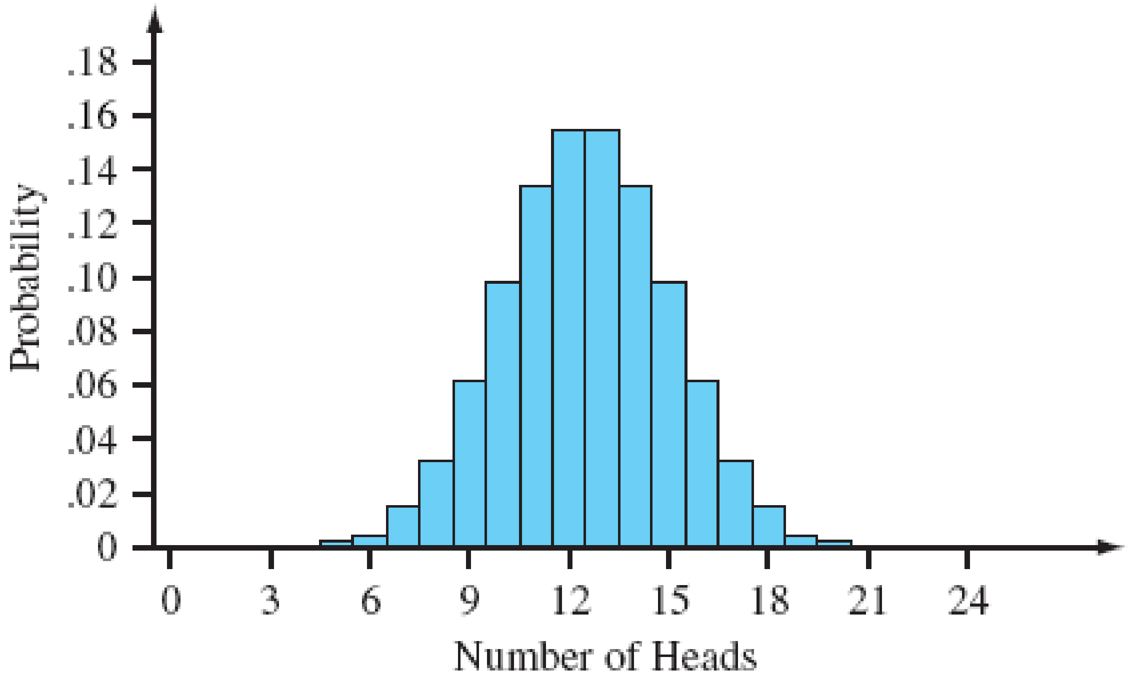

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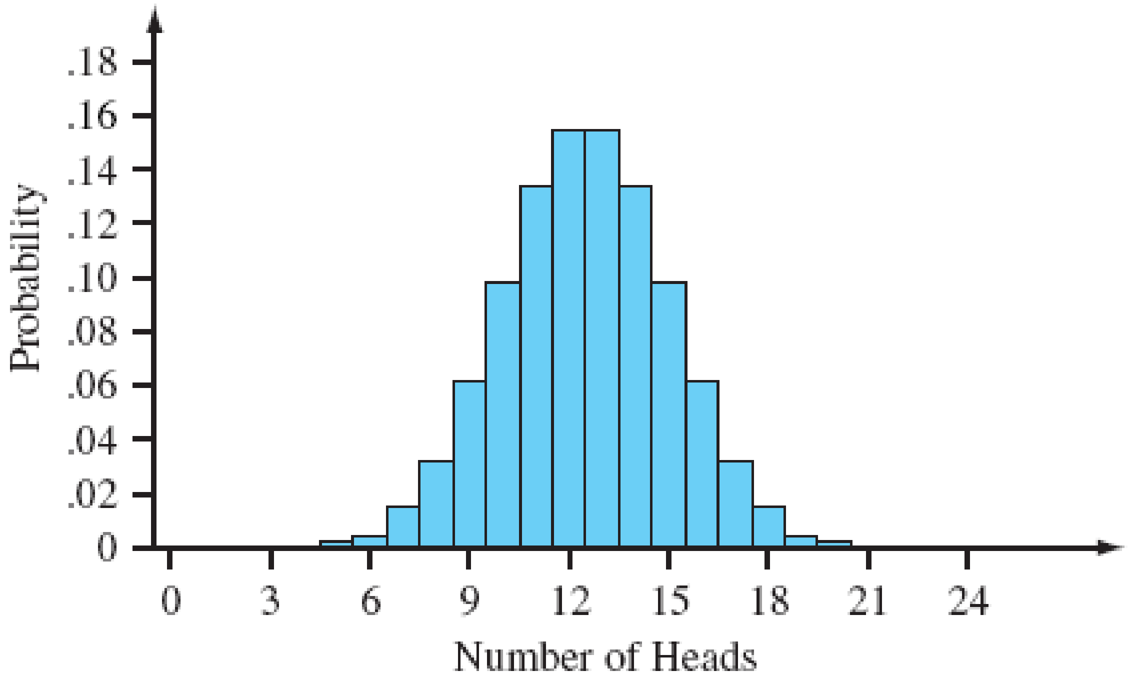

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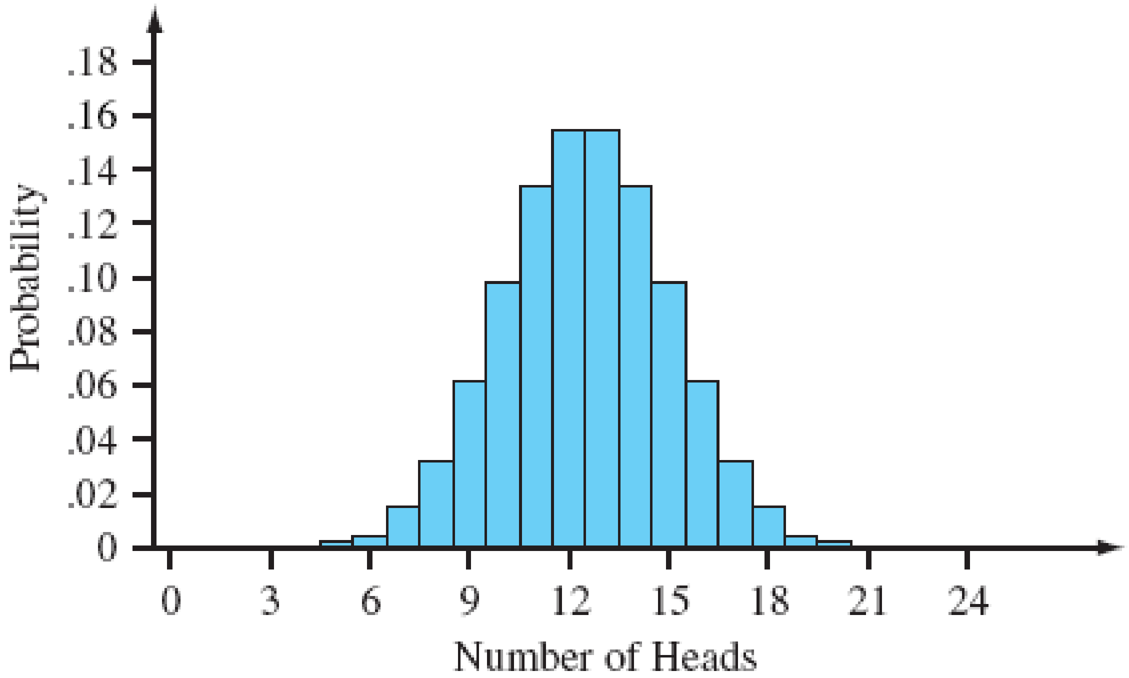

Expected number of heads is 12.5.

Histogram says: vast majority of results between 9 and 16 heads. Virtually all results lie between 5 and 20.

Thus, results are not spread as broadly (relatively speaking) as they were with just 10 flips.

Test score histogram with 25 questions

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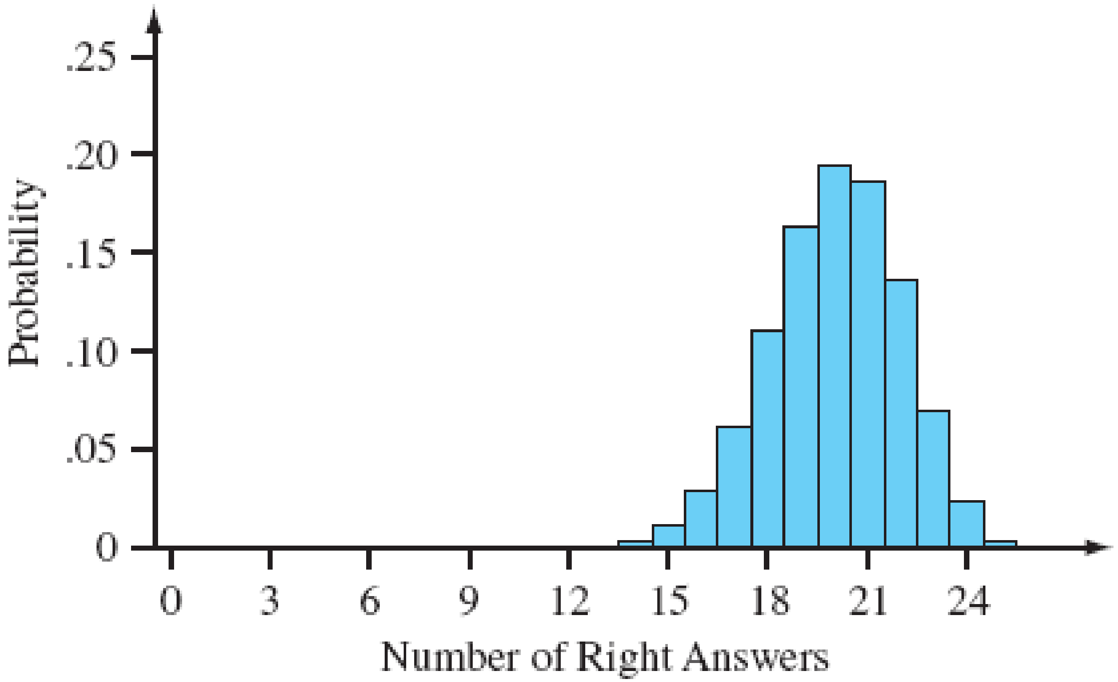

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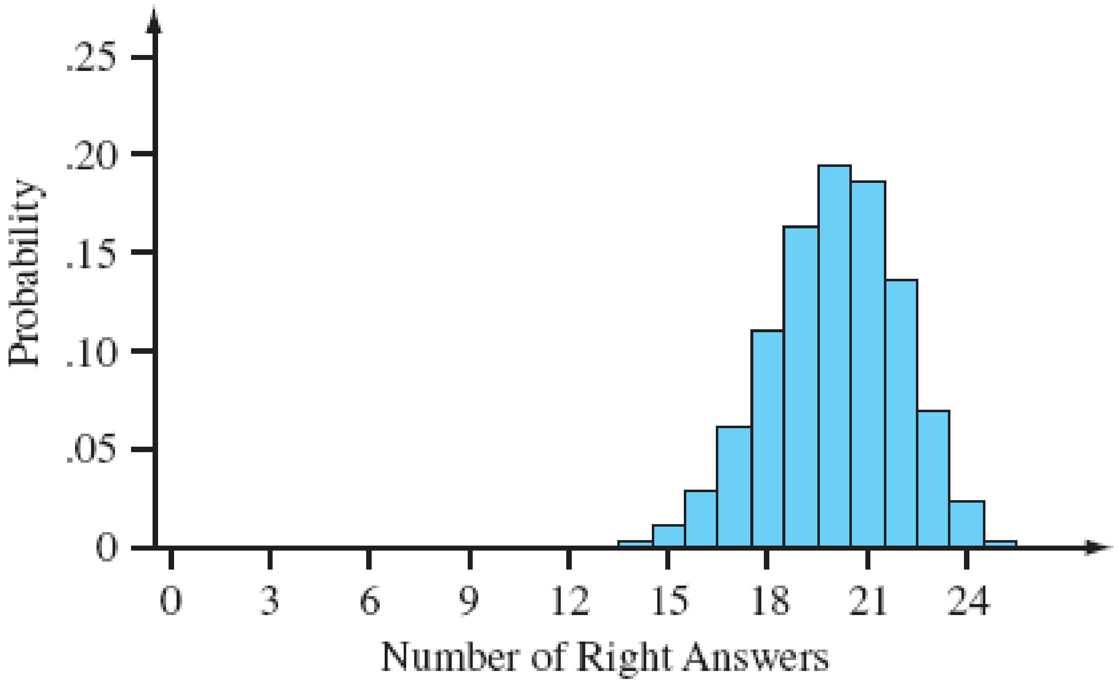

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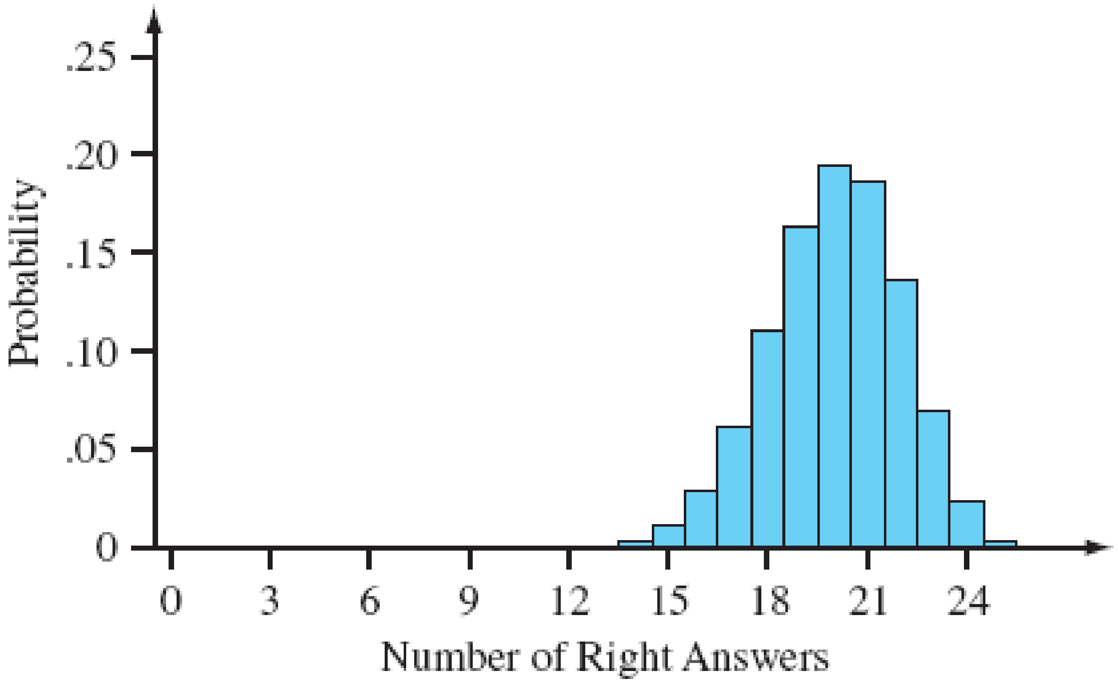
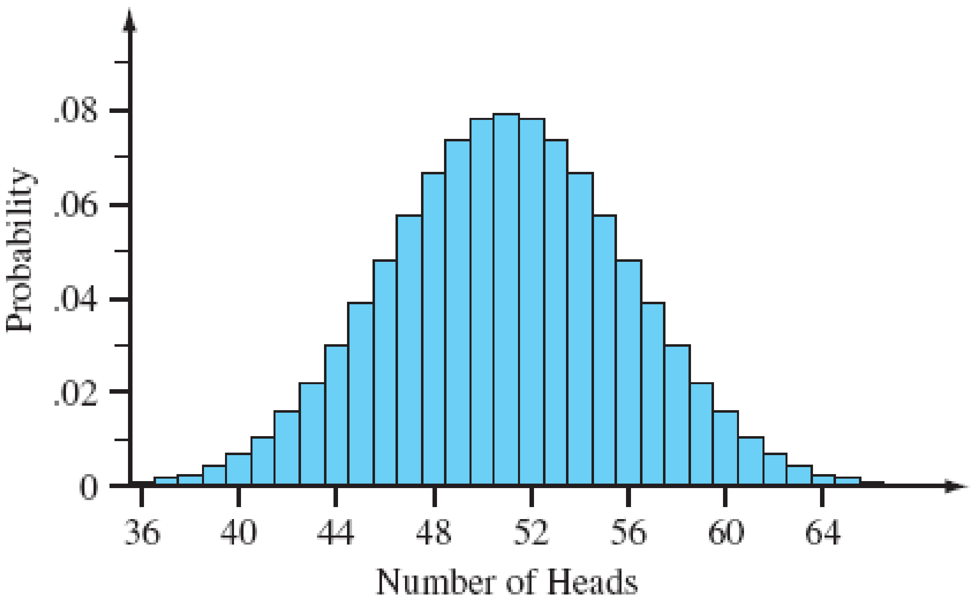


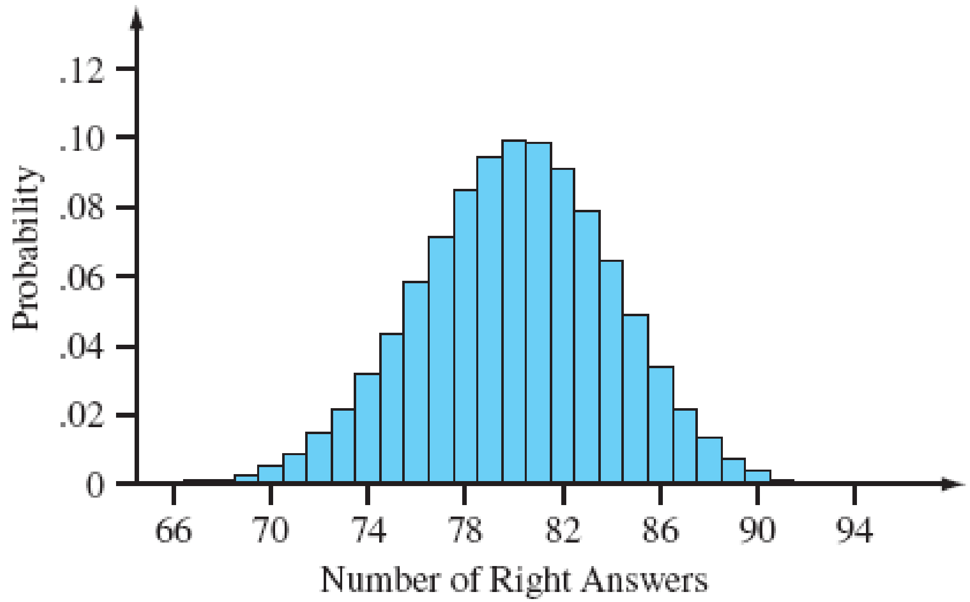
Compared to coin flipping, even more tightly packed around its expected value.

Essentially, all scores lie between 14 and 25.

100 flips of a coin



100-question test



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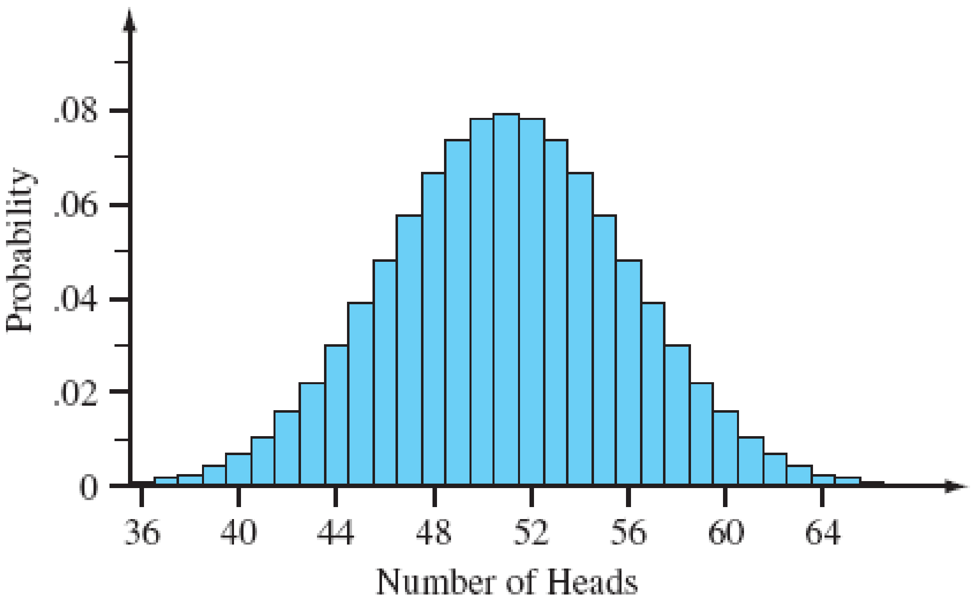

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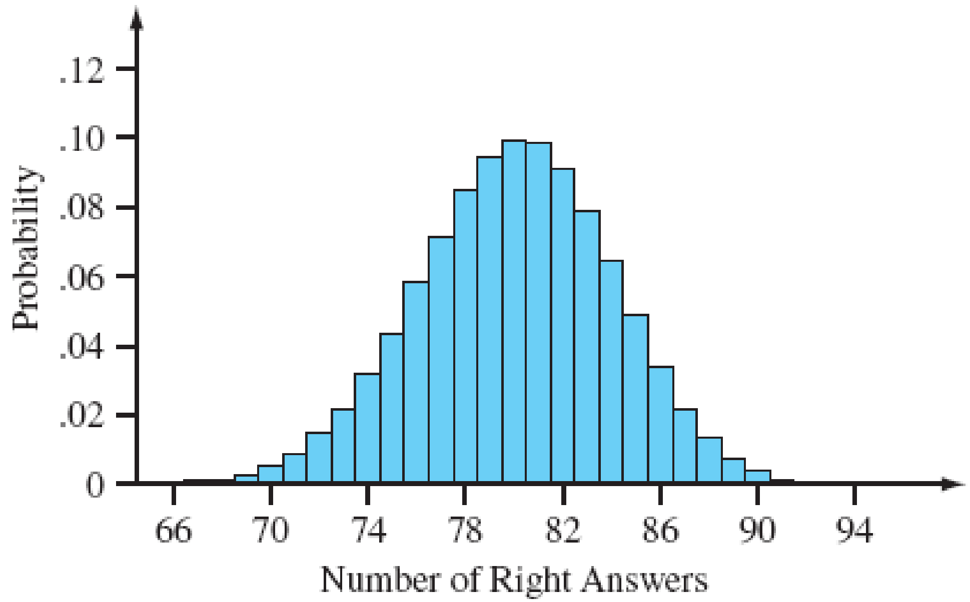
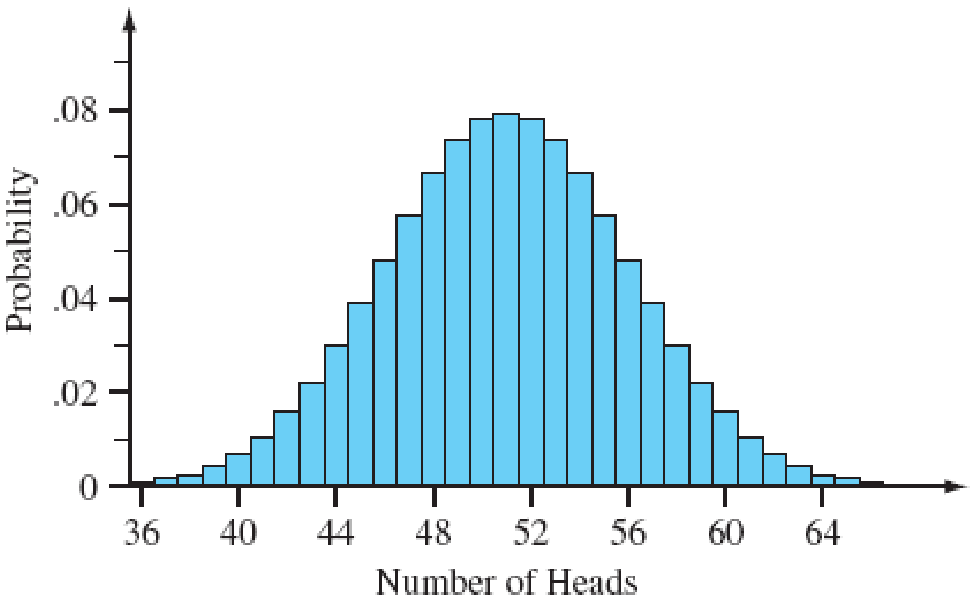
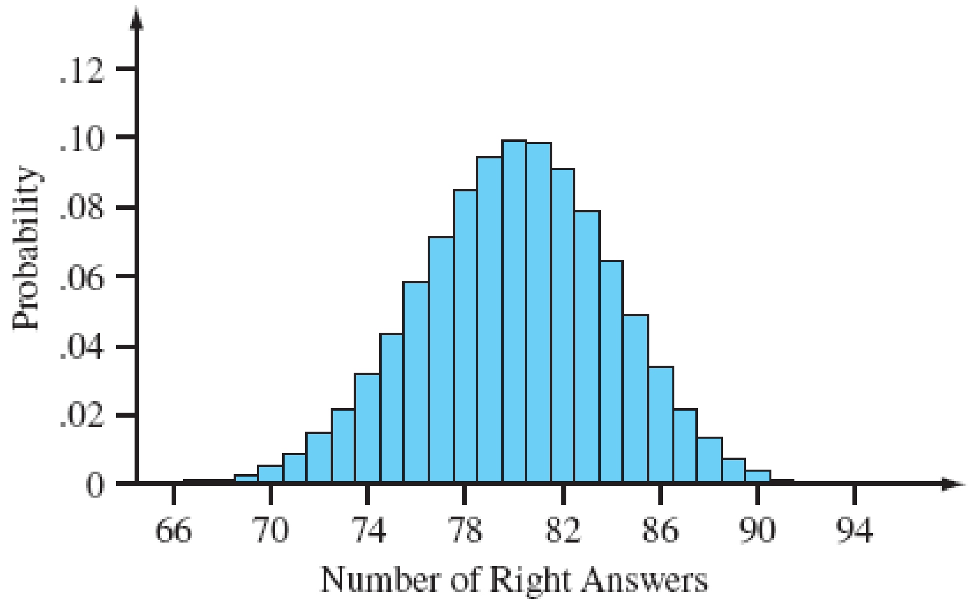


Two histograms have almost same shape.

100 flips of a coin 100-question test





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Number of heads has virtually no chance of deviating by more than 15 from its expected value, and test score has almost no chance of deviating by more than 11.

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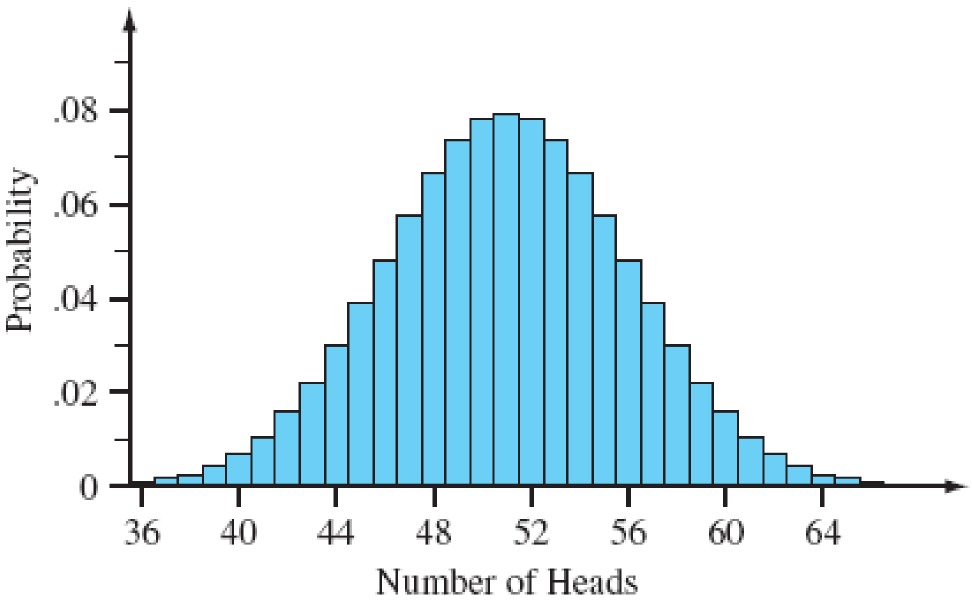

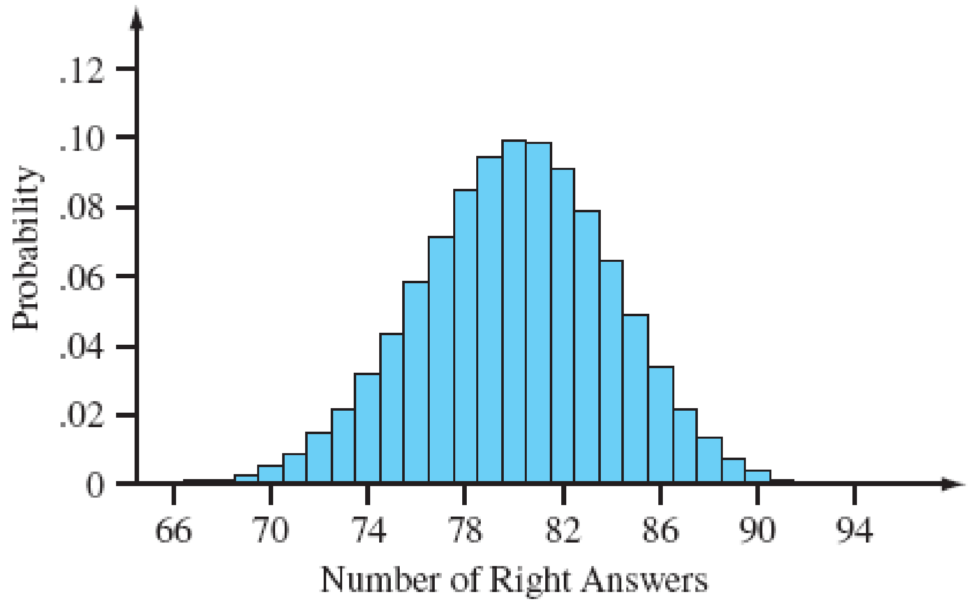

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Number of heads has virtually no chance of deviating by more than 15 from its expected value, and test score has almost no chance of deviating by more than 11.

Thus, spread has only doubled, even though number of trials has quadrupled.

We need about 30 values to see the most relevant probabilities for 100 trials, whereas we need 15 values to see the most relevant probabilities for 25 independent trials.

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How many values to see essentially all results in 400 trials.

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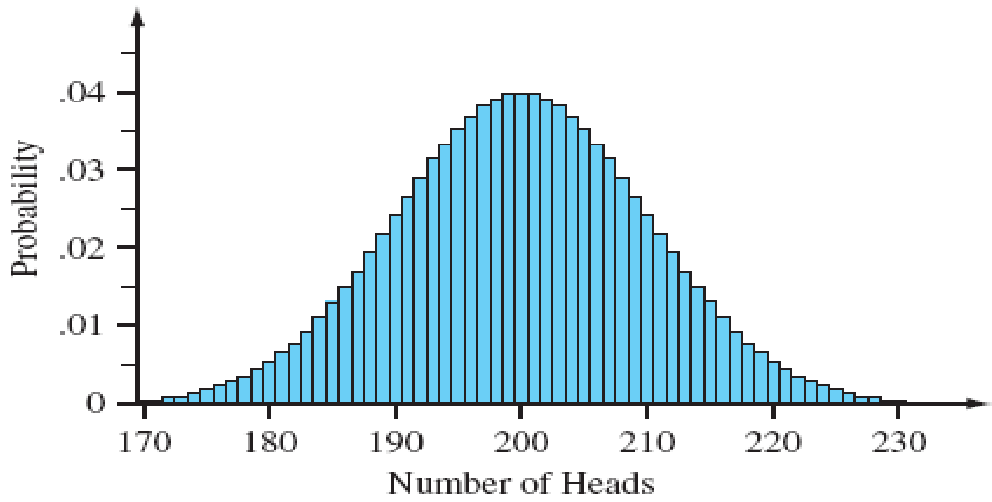
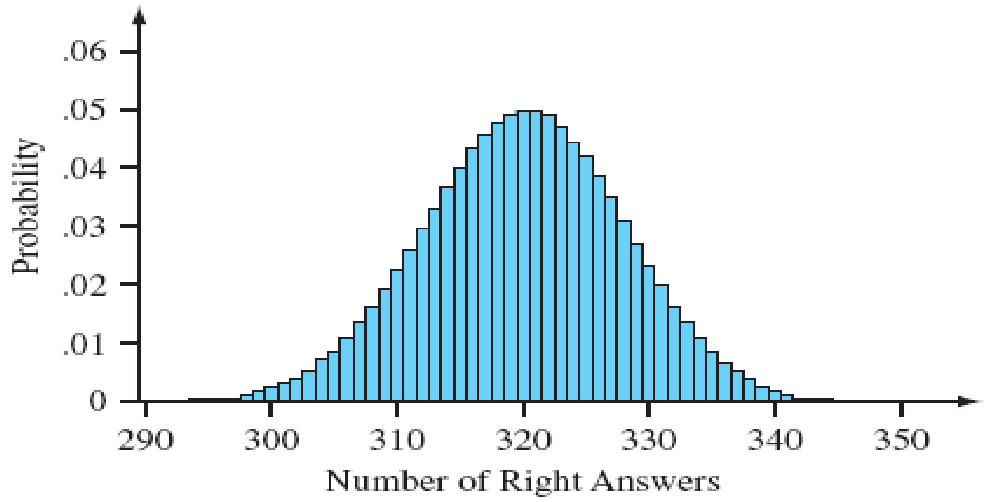
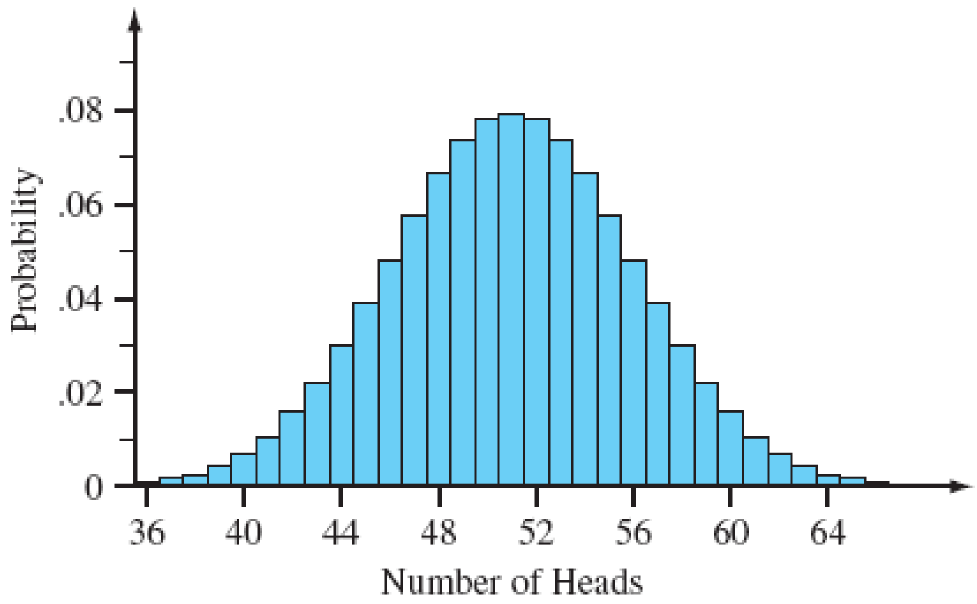
How many values to see essentially all results in 400 trials.

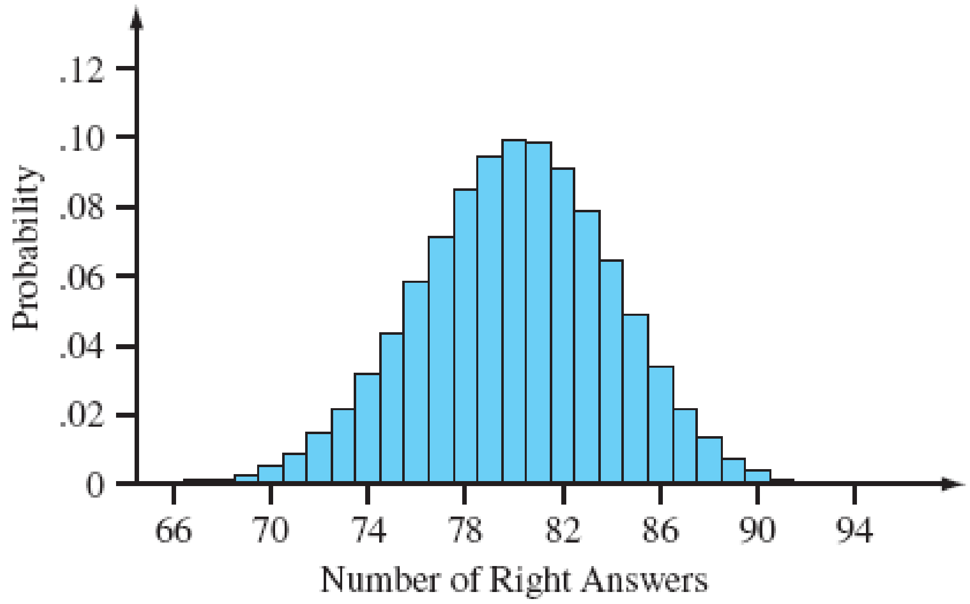
Only about 60 values.



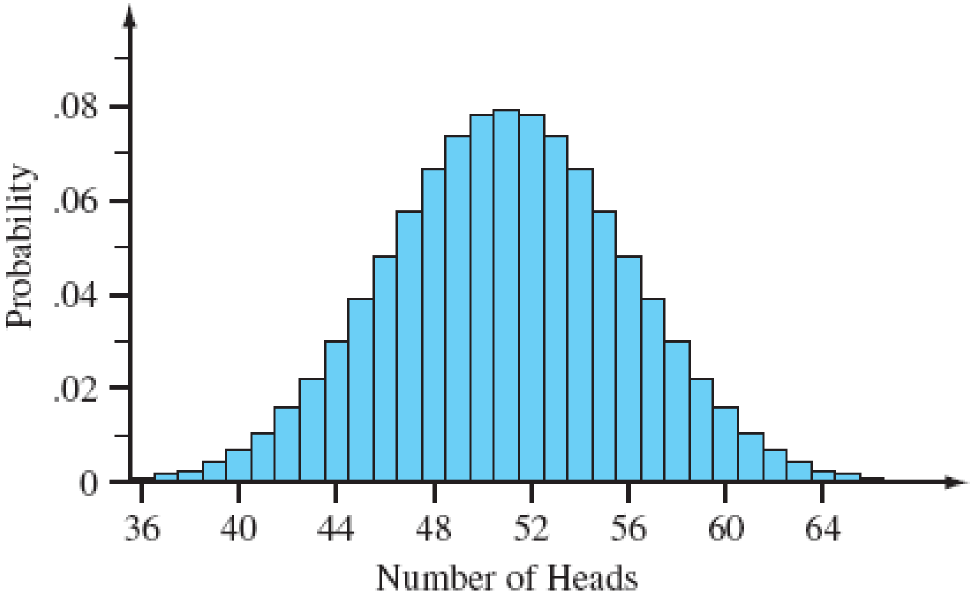


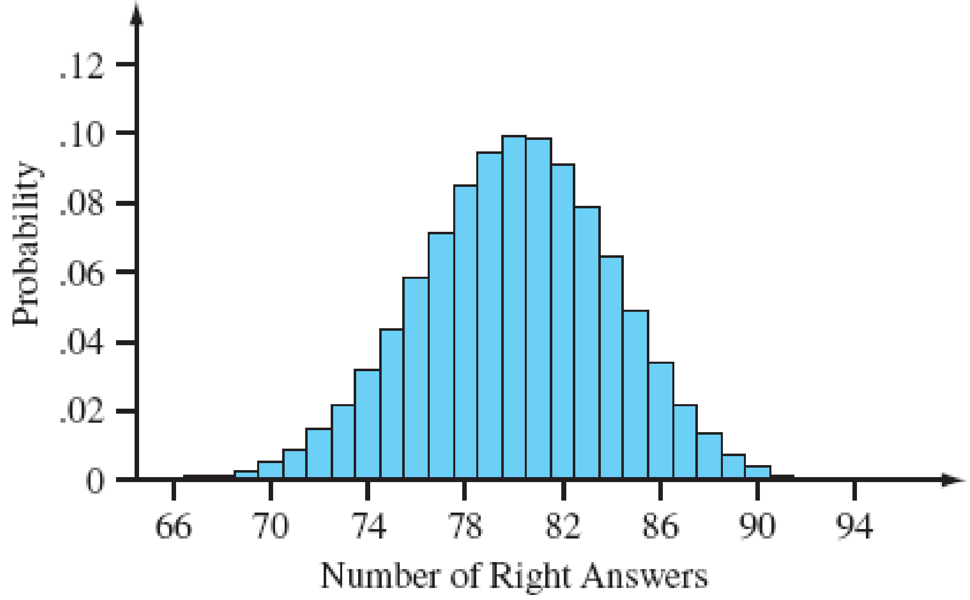


.08
.06
.04
.02
0
Probability
36
40
44
48
52
56
60
64
Number of Heads

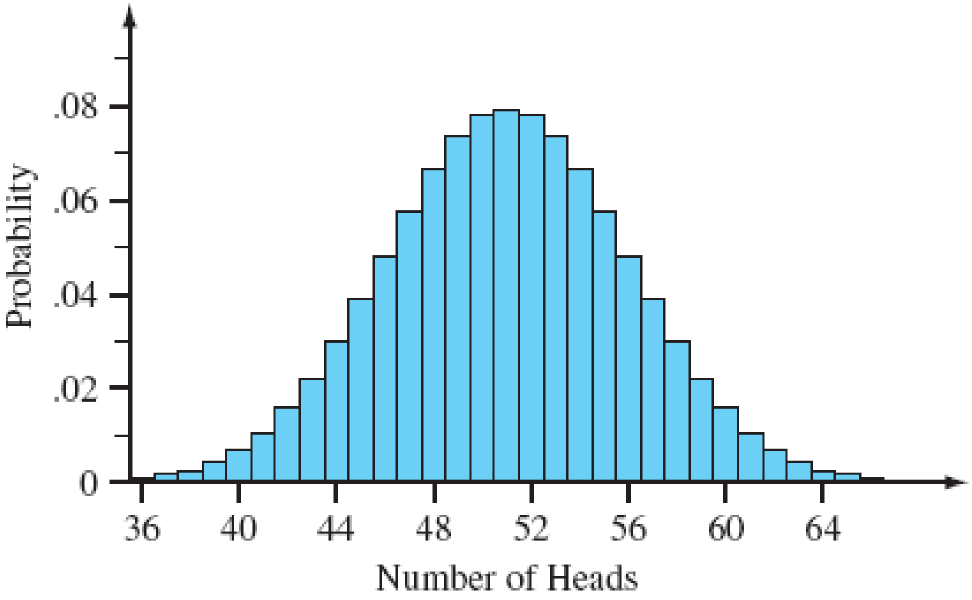


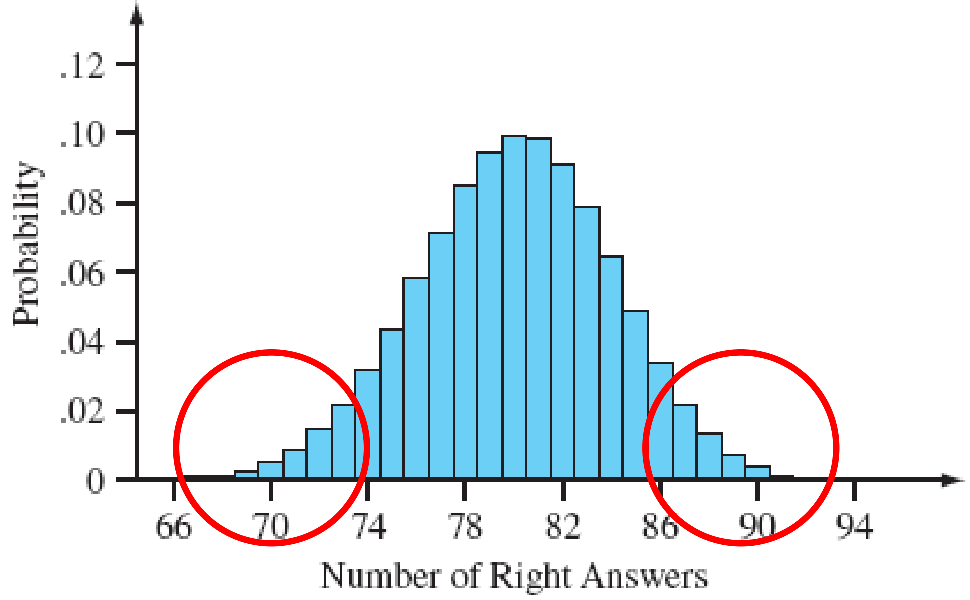
.12
.10
.08
.06
.04
.02
0
Probability
66
70
74
78
82
86
90
94
Number of Right Answers





In both cases, curve formed by tops of rectangles seems quite similar to bell-shaped curve, called **normal curve**.





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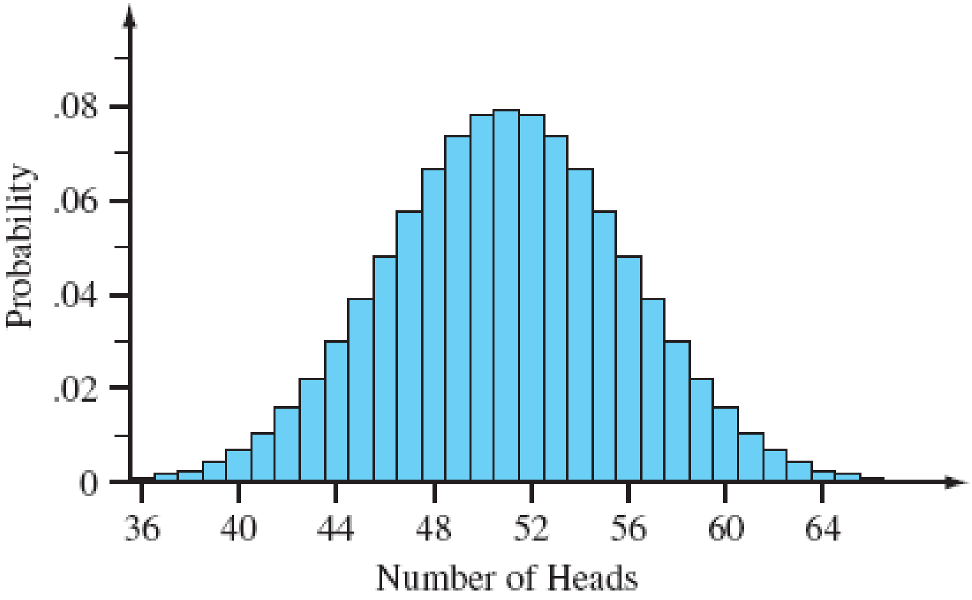

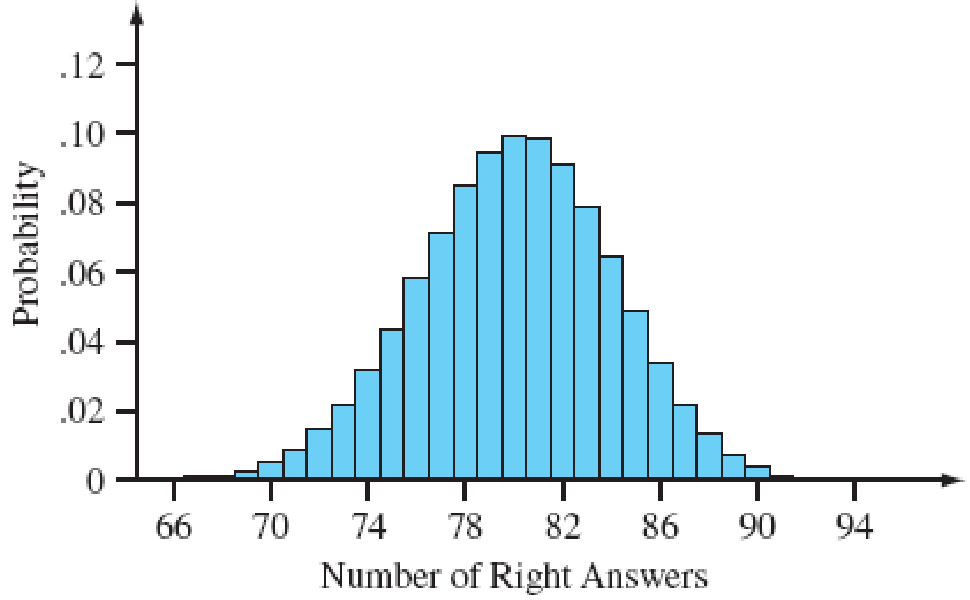

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We want an algebraic way to measure the difference between a random variable and its expected value.

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Proof:

When we think of $E(X)$ as a random variable,
it has a constant value traditionally denoted by μ.
By Lemma 5.26, we have that $E(E(x)) = E(\mu) = \mu = E(x)$.

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Our next attempt will be to look at $E(Y^2)$.

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$$\begin{aligned} V(x) &= E\Big((X - 2)^2\Big) \\ &= (0-2)^2 \cdot \frac{1}{16} + (1-2)^2 \cdot \frac{1}{4} + (2-2)^2 \cdot \frac{3}{8}. \\ &\qquad + (3-2)^2 \cdot \frac{1}{4} + (4-2)^2 \cdot \frac{1}{16} = 1 \end{aligned}$$

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- Calculating variances from scratch is very time consuming. Many R.V.s, X, such as the binomial distribution, can actually be built as the sum of simpler R.Vs, i.e., $X = \sum_{i=1}^{n} X_i$.

- Is there any way of constructing $V(X)$ from the $V(X_i)$?

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We already saw that the variance for 4 coin flips is 1, which is 4 times the variance for one coin flip.

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Let's compute variance for number of right answers when solving for 1 and 5 questions.
Is there a relationship between these two variances?

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1 question:

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Let's compute variance for number of right answers
when solving for 1 and 5 questions.
Is there a relationship between these two variances?

1 question:

$$V(X) = .2(0-.8)^2 + .8(1-.8)^2 = .16.$$

5 questions:

$$V(X) = 4^2 \cdot (.2)^5 + 3^2 \cdot 5 \cdot (.2)^4 \cdot (.8) + 2^2 \cdot 10 \cdot (.2)^3 \cdot (.8)^2$$

$$+1^2 \cdot 10 \cdot (.2)^2 \cdot (.8)^3 + 0^2 \cdot 5 \cdot (.2)^1 \cdot (.8)^4 + 1^2 \cdot (.8)^5 = .8.$$

Example 2

Getting correct answer to a question has probability $.8$.
Let's compute variance for number of right answers when solving for 1 and 5 questions.
Is there a relationship between these two variances?

1 question:

$$V(X) = .2(0 - .8)^2 + .8(1 - .8)^2 = .16.$$

5 questions:

$$V(X) = 4^2 \cdot (.2)^5 + 3^2 \cdot 5 \cdot (.2)^4 \cdot (.8) + 2^2 \cdot 10 \cdot (.2)^3 \cdot (.8)^2$$

$$+1^2 \cdot 10 \cdot (.2)^2 \cdot (.8)^3 + 0^2 \cdot 5 \cdot (.2)^1 \cdot (.8)^4 + 1^2 \cdot (.8)^5 = .8.$$

Result is five times variance for one question.


1
ONE DOLLAR
5
FIVE DOLLARS

Withdraw one coin.

What is expected amount of money we withdraw?

What is the variance?

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$$V(X) = .5(1-3)^2 + .5(5-3)^2 = 4.$$

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Now let $X = X_1 + X_2$ be the *total* amount withdrawn.
Since both coins are withdrawn, $X = 6$ so
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$$\Rightarrow \boxed{V(X_1 + X_2) \neq V(X_1) + V(X_2)}$$

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Before continuing, we need to introduce concept of Independent Random Variables

(as opposed to the Independent events that we have already seen).

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Example: Roll two dice. X is the amount rolled on the first die, Y the amount on the second. They are independent because, for every $1 \leq i, j \leq 6$,

$$P\big((X = i) \wedge (Y = j)\big) \;=\; \frac{1}{36} \;=\; P(X = i) \cdot P(Y = j)$$

To show that variance of a sum of independent R.V.s is sum of their variances, we first show that expected value of product of two independent R.V.s is product of their expected values, i.e., To show that variance of a sum of independent R.V.s is sum of their variances, we first show that expected value of product of two independent R.V.s is product of their expected values, i.e.,

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Lemma 5.28

If X and Y are independent random variables on sample space S with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

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Flip two fair coins and observe whether they come up H or T.
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Result of coin 1

$$X = \begin{cases} 1 & \text{if H} \\ 0 & \text{if T} \end{cases}$$

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Note that $XZ = 0$ (why)? and $E(Z) = \frac{1}{2}$

$$\Rightarrow \; E(XZ) = 0 \neq \frac{1}{4} = E(X)E(Z)$$

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and the proof is complete.

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Variance for 400 flips is 100.

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By Theorem 5.29,

$$V(X) = V(X_1) + V(X_2) + \cdots V(X_n) = np(1-p)$$

Material in these slides, from this point on

will not be on the exam

Returning to our previous histograms (illustrating coin flip and answer distributions) we see that when number of trials grew by a factor of 4, spread observed in histograms grew by factor of two.

Theorem on previous page tells us that when number of trials grows by 4, Variance grows by 4 as well.

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This quantity, is callled the standard deviation of RV X and is usually denoted by $\sigma(X) = \sqrt{V(X)}$, or sometimes just by σ.

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For 25 flips, Variance $= 25 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{25}{4} \;\Rightarrow\; \sigma = \frac{5}{2}$.

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Note: In both 100-flip case and 400-flip case, "spread" observed in histogram was ± 3 standard deviations from expected value.

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For 25 flips, Variance $= 25 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{25}{4} \;\Rightarrow\; \sigma = \frac{5}{2}$.

So, ± 3 standard deviations from expected value is a range of 15 points, which is, again, what we observed.

Central Limit Theorem

Central Limit Theorem

s.d. is standard deviation

Assume a relatively large number of independent trials with two outcomes.

Percentage of results within 1 s.d. of mean is about 68%;
percentage within 2 s.d.s of mean is about 95.5%;
percentage within 3 s.d.s of mean is about 99.7%.

Central Limit Theorem tells us

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If $a = -1.5$ and $b = 2$, then theorem tells us an approximate probability that sum is between 1.5 standard deviations less than its expected value and 2 standard deviations more than its expected value.

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The distribution given by

$$P(a \le X \le b) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx$$

is called the **normal distribution**.

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So, we want an n such that $2\sqrt{n}/2 = .01(.5n)$.

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Squaring both sides gives $n = 25 \cdot 10^{-6}n^2$,

which gives $n = 10^6/25 = 40000$.

Therefore, need to flip a coin $40,000$ times to be 95% sure that number of heads will be within 1% of expected value of $20,000$.