Recapitulation from Previous Lectures

Recap: Name Analysis and Scopes

Name analysis follows parsing, works from *abstract syntax trees* (AST)

⇒ link every *name occurrence* to its logical *binder*

⇒ AST becomes a *graph* data structure, possibly cyclic (recursive definitions)

Static scoping is the standard scoping discipline

Associates occurrences with binders *statically* (at compile time), based on lexical structure

Makes program understanding and refactoring easy

Recap: Name Analysis and Scopes

Name analysis follows parsing, works from *abstract syntax trees* (AST)

⇒ link every *name occurrence* to its logical *binder*

⇒ AST becomes a *graph* data structure, possibly cyclic (recursive definitions)

Static scoping is the standard scoping discipline

Associates occurrences with binders *statically* (at compile time), based on lexical structure

Makes program understanding and refactoring easy

Next: ***type checking***, to validate the usage of language features

Type Systems

Why types are good

Prevent errors: many simple errors are caught by types

Ensure memory safety or other desired properties

Document the program (purpose of parameters)

Make it easier to change program

Make compilation more efficient: remove checks, specialize operations

An unsound (broken) type system

A type system that aims to ensure some property but, in fact, fails

Example: Suppose we have a system that aims to ensure that if parameter is of type Int, then it is only invoked with values of type Int.
But we find a (tricky) program that passes the type checker and ends up invoking the function by passing a *string reference* instead. This is unsoundness.

Sometimes unsoundness is an *intentional* compromise:

- type casts in C and Scala (through *asInstanceOf*)
- covariance for function arguments and arrays

Often *unintentional* (soundness bugs in type systems), due to subtle interactions between e.g. subtyping, generics, mutation, higher-order functions, recursion

Java and Scala's Type Systems are Unsound *

The Existential Crisis of Null Pointers

Nada Amin
EPFL, Switzerland
nada.amin@epfl.ch

Ross Tate
Cornell University, USA
ross@cs.cornell.edu

Abstract

We present short programs that demonstrate the unsoundness of Java and Scala's current type systems. In particular, these programs provide parametrically polymorphic functions that can turn any type into any type without (down)casting. Fortunately, parametric polymorphism was not integrated into the Java Virtual Machine (JVM), so these examples do not demonstrate any unsoundness of the JVM. Nonetheless, we discuss broader implications of these findings on the field of programming languages.

ture, we often develop a minimal calculus employing that feature and then verify key properties of that calculus. But these results provide no guarantees about how the feature in question will interact with the many other common features one might expect for a full language. The unsoundness we identify results from such an interaction of features. Thus, in addition to valuing the development and verification of minimal calculi, our community should explore more ways to improve our chances of identifying abnormal interactions of features within reasonable time but without unreasonable resources and distractions. Ideally our community could pro-

1. Introduction

In 2004, Java 5 introduced generics, i.e. parametric polymorphism, to the Java programming language. In that same year, Scala was publicly released, introducing path-dependent types as a primary language feature. Upon their release 12 years ago, both languages were unsound; the examples we will present were valid even in 2004. But despite the fact that Java has been formalized repeatedly [3, 4, 6, 9, 10, 18, 26, 38], this unsoundness has not been discovered until now. It was found in Scala in 2008 [40], but the bug was deferred and its broader significance was not realized until now.

–same paper, published in November 2016

Goal of today's lecture

Explain that "expression has a type" is an *inductively defined relation*

Goal of today's lecture

Explain that "expression has a type" is an *inductively defined relation*

Define precisely a small language:

- its abstract syntax (as certain math expressions)
- its operational semantics (interpreter written in math)
- its typing rules

Goal of today's lecture

Explain that "expression has a type" is an *inductively defined relation*

Define precisely a small language:

- its abstract syntax (as certain math expressions)
- its operational semantics (interpreter written in math)
- its typing rules

Show that our type system prevents certain kinds of errors

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$?

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$?

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$?

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r} \text{ (zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r} \text{ (increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r} \text{ (increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r} \text{ (decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$? Yes

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$? Yes, but not the *smallest*

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$? Yes, but not the *smallest*

What is the **smallest** r (wrt. $\subseteq$) for which rules hold? $\emptyset$?

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$? Yes, but not the *smallest*

What is the **smallest** r (wrt. $\subseteq$) for which rules hold? $\emptyset$? No.

Background: inductively defined relations and sets

Define relation $r \subseteq \mathbb{Z} \times \mathbb{Z}$ using these rules (x, y range over $\mathbb{Z}$):

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

For which of the following relations r are all the above rules true?

- $r = \{(x,y) \mid x = 0 \vee y = 0\}$? No (increase right)
- $r = \{(x,y) \mid x \leq 0 \wedge 0 \leq y\}$? No
- $r = \mathbb{Z} \times \mathbb{Z}$? Yes, but not the *smallest*

What is the **smallest** r (wrt. $\subseteq$) for which rules hold? $\emptyset$? No. $r = \{\ (x,y)\ \mid\ x \leq y\ \}$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$(0,0) \in r$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$$\frac{(0,0) \in r}{(0,1) \in r}$$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$$\cfrac{\cfrac{(0,0) \in r}{(0,1) \in r}}{(0,2) \in r}$$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$$\cfrac{\cfrac{\cfrac{(0,0) \in r}{(0,1) \in r}}{(0,2) \in r}}{(-1,1) \in r}$$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{(0,0) \in r}}{(0,1) \in r}}{(0,2) \in r}}{(-1,1) \in r}}{(-2,0) \in r}$$

Example derivation of $(-3,-1) \in r$

$$\frac{}{(0,0) \in r}\ \text{(zero)} \qquad \frac{(x,y) \in r}{(x,y+1) \in r}\ \text{(increase right)}$$

$$\frac{(x,y) \in r}{(x+1,y+1) \in r}\ \text{(increase both)} \qquad \frac{(x,y) \in r}{(x-1,y-1) \in r}\ \text{(decrease both)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{(0,0) \in r}}{(0,1) \in r}}{(0,2) \in r}}{(-1,1) \in r}}{(-2,0) \in r}}{(-3,-1) \in r}$$

Proof that our rules define $\{(x, y) \mid x \leq y\}$

Establish two directions:

- if there exists a derivation, then $x \leq y$
  Strategy: induction on derivation, go through each rule

Proof that our rules define $\{(x, y) \mid x \leq y\}$

Establish two directions:

- if there exists a derivation, then $x \leq y$
  Strategy: induction on derivation, go through each rule
- if $x \leq y$ then there exists a derivation
  Strategy (problem-specific): we can find an algorithm that given x, y finds derivation tree (what is the algorithm?)

Proof that our rules define $\{(x,y) \mid x \leq y\}$

Establish two directions:

- if there exists a derivation, then $x \leq y$
  Strategy: induction on derivation, go through each rule
- if $x \leq y$ then there exists a derivation
  Strategy (problem-specific): we can find an algorithm that given x, y finds derivation tree (what is the algorithm?)

  Example algorithm: start from $(0,0)$, then

  derive $(0, y-x)$ in $y-x$ steps of "increase right",

  then depending on whether $x < 0$ or $x > 0$,

  apply "increase both" or "decrease both" rule $|x|$ times.

Context-Free Grammars as Inductively Defined Relations

Inductive definitions work on multiple relations as well

Context-free grammars: mutually defined sets of strings (sets are relations)

Each non-terminal corresponds to a set of strings. Let $A = \{a, b\}$

context-free grammar rule	inductive rule ($S, N \subseteq A^*$)
$S ::= aN$	$\dfrac{w \in L(N)}{aw \in L(S)}$
$N ::= \varepsilon$	$\dfrac{}{\varepsilon \in L(N)}$
$N ::= aNNb$	$\dfrac{w_1 \in L(N), w_2 \in L(N)}{aw_1 w_2 b \in L(N)}$

Sets of first symbols for each non-terminal is also an inductively definable relation

Inductively defined relations

We can use inductive rules to define type systems, grammars, interpreters, ...

We define a relation r using **rules** of the form

$$\frac{t_1(\bar{x}) \in r, \ldots, t_n(\bar{x}) \in r}{t(\bar{x}) \in r}$$

where $t_i(\bar{x}) \in r$ are assumptions and $t(\bar{x}) \in r$ is the conclusion.

When $n = 0$ (no assumptions), the rule is called an axiom.

A derivation tree has nodes marked by tuples $t(\bar{a})$ for some specific values $\bar{a}$ of $\bar{x}$.

We define relation r as the set of all tuples for which there exists a derivation tree.
One can prove (in general case) that tuples for which there exists a derivation tree give us precisely the smallest relation that satisfies the rules!

Amyli language

Tiny language similar to one in the project.

Works only on integers and booleans.

Amyli language

Tiny language similar to one in the project.

Works only on integers and booleans.

(Initial) program is a pair (e_{top}, t_{top}) where

- e_{top} is the top-level environment mapping function names to function definitions
- t_{top} is the top-level term (expression) that starts execution

Amyli language

Tiny language similar to one in the project.

Works only on integers and booleans.

(Initial) program is a pair (e_{top}, t_{top}) where

- e_{top} is the top-level environment mapping function names to function definitions
- t_{top} is the top-level term (expression) that starts execution

Function definition for a given function name is a tuple of: parameter list $\bar{x}$, parameter types $\bar{\tau}$, expression representing function body t, and result type τ_0.

Amyli language

Tiny language similar to one in the project.

Works only on integers and booleans.

(Initial) program is a pair (e_{top}, t_{top}) where

- e_{top} is the top-level environment mapping function names to function definitions
- t_{top} is the top-level term (expression) that starts execution

Function definition for a given function name is a tuple of: parameter list $\bar{x}$, parameter types $\bar{\tau}$, expression representing function body t, and result type τ_0.

Expressions are formed by invoking primitive functions $(+, -, \leq, \&\&)$, invocations of defined functions, or **if** expressions.

No local **val** definitions nor **match**. e will remain fixed

Amyli: abstract syntax of terms

$$t := \mathit{true} \mid \mathit{false} \mid c \mid f(t_1, \ldots, t_n) \mid \textbf{if } (t) \ t_1 \textbf{ else } t_2$$

where

- $c \in \mathbb{Z}$ denotes integer constant
- f denotes either a user-defined function or a primitive operator

Program representation as a mathematical structure

$p_{fact} = (e, fact(2))$ where environment e is defined by:

$$
\begin{array}{lll}
e(fact) = (& n, & \textit{(parameters)} \\
 & Int, & \textit{(their types)} \\
 & \textbf{if } (n \leq 1)\ 1\ \textbf{else } n * fact(n-1), & \textit{(body)} \\
 & Int & \textit{(result type)} \\
 &) &
\end{array}
$$

Program representation as a mathematical structure

$p_{fact} = (e, fact(2))$ where environment e is defined by:

$$
\begin{array}{lll}
e(fact) = (& n, & \textit{(parameters)} \\
 & Int, & \textit{(their types)} \\
 & \textbf{if } (n \leq 1)\ 1\ \textbf{else } n * fact(n-1), & \textit{(body)} \\
 & Int & \textit{(result type)} \\
 &) &
\end{array}
$$

Note: "**if** $(n \leq 1)$ 1 **else** $n * fact(n-1)$" is shorthand notation for an AST, not a plain code string!

Program representation as a mathematical structure

$p_{fact} = (e, fact(2))$ where environment e is defined by:

$$
\begin{array}{lll}
e(fact) = (& n, & \textit{(parameters)} \\
 & Int, & \textit{(their types)} \\
 & \textbf{if } (n \leq 1)\ 1\ \textbf{else } n * fact(n-1), & \textit{(body)} \\
 & Int & \textit{(result type)} \\
 &) &
\end{array}
$$

Note: "**if** $(n \leq 1)$ 1 **else** $n * fact(n-1)$" is shorthand notation for an *AST*, not a plain code string!

Represented program:

```
def fact(n: Int): Int = if n < 1 then 1 else n * fact(n-1);
fact(2)
```

Operational semantics of Amyli: **if** expression

Given a program with environment e, we specify the result of executing the program as an inductively defined binary (infix) relation "$\rightsquigarrow$" on expressions

If expression becomes a constant c after some number of steps of $\rightsquigarrow$ (possibly zero), we have computed the result: $t \overset{*}{\rightsquigarrow} c$

Rules for **if**:

$$\frac{b \rightsquigarrow b'}{(\textbf{if}\ (b)\ t_1\ \textbf{else}\ t_2) \rightsquigarrow (\textbf{if}\ (b')\ t_1\ \textbf{else}\ t_2)}$$

$$\frac{}{(\textbf{if}\ (\mathit{true})\ t_1\ \textbf{else}\ t_2) \rightsquigarrow t_1}$$

$$\frac{}{(\textbf{if}\ (\mathit{false})\ t_1\ \textbf{else}\ t_2) \rightsquigarrow t_2}$$

b, b', t_1, t_2 range over terms

Operational semantics of Amyli: primitives

Logical operators:

$$\frac{b_1 \rightsquigarrow b_1'}{(b_1 \;\&\&\; b_2) \rightsquigarrow (b_1' \;\&\&\; b_2)}$$

$$\frac{}{(\mathit{true} \;\&\&\; b_2) \rightsquigarrow b_2}$$

$$\frac{}{(\mathit{false} \;\&\&\; b_2) \rightsquigarrow \mathit{false}}$$

Arithmetic:

$$\frac{k_1 \rightsquigarrow k_1'}{(k_1 + k_2) \rightsquigarrow (k_1' + k_2)}$$

$$\frac{k_2 \rightsquigarrow k_2'}{(c + k_2) \rightsquigarrow (c + k_2')} \quad c \in \mathbb{Z}$$

$$\frac{}{(c_1 + c_2) \rightsquigarrow c} \text{ if } \mathit{value}(c) = \mathit{value}(c_1) + \mathit{value}(c_2)$$

Operational semantics: user function f

If $c_1, \ldots, c_{i-1}$ are constants, then (as expected in call-by-value)

$$\frac{t_i \rightsquigarrow t_i'}{f(c_1, \ldots, c_{i-1}, t_i, \ldots) \rightsquigarrow f(c_1, \ldots, c_{i-1}, t_i', \ldots)}$$

Let the environment e define f by $e(f) = ((x_1, \ldots, x_n), \bar{\tau}, t_f, \tau_0)$

- $(x_1, \ldots, x_n)$ is the list of formal parameters of f
- t_f is the body of the function f

Operational semantics: user function f

If $c_1, \ldots, c_{i-1}$ are constants, then (as expected in call-by-value)

$$\frac{t_i \rightsquigarrow t_i'}{f(c_1, \ldots, c_{i-1}, t_i, \ldots) \rightsquigarrow f(c_1, \ldots, c_{i-1}, t_i', \ldots)}$$

Let the environment e define f by $e(f) = ((x_1, \ldots, x_n), \bar{\tau}, t_f, \tau_0)$

- $(x_1, \ldots, x_n)$ is the list of formal parameters of f
- t_f is the body of the function f

Then we have a rule:

$$\frac{}{f(c_1, \ldots, c_n) \rightsquigarrow t_f[x_1 := c_1, \ldots, x_n := c_n]}$$

Notation: $t[x_1 := t_1, \ldots, x_n := t_n]$ denotes the *substitution* in t of each variable x_i by t_i

Operational semantics: user function f

If $c_1, \ldots, c_{i-1}$ are constants, then (as expected in call-by-value)

$$\frac{t_i \rightsquigarrow t_i'}{f(c_1, \ldots, c_{i-1}, t_i, \ldots) \rightsquigarrow f(c_1, \ldots, c_{i-1}, t_i', \ldots)}$$

Let the environment e define f by $e(f) = ((x_1, \ldots, x_n), \bar{\tau}, t_f, \tau_0)$

- $(x_1, \ldots, x_n)$ is the list of formal parameters of f
- t_f is the body of the function f

Then we have a rule:

$$\overline{f(c_1, \ldots, c_n) \rightsquigarrow t_f[x_1 := c_1, \ldots, x_n := c_n]}$$

Notation: $t[x_1 := t_1, \ldots, x_n := t_n]$ denotes the *substitution* in t of each variable x_i by t_i

Example: if $t = x + 1$, then $t[x := c] = c + 1$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$fact(2) \rightsquigarrow$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$fact(2) \rightsquigarrow$$
$$\textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
&fact(2) \rightsquigarrow \\
&\textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&\textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
& fact(2) \rightsquigarrow \\
& \textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& \textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& 2 * fact(2-1) \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
&fact(2) \rightsquigarrow \\
&\textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&\textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&2 * fact(2-1) \rightsquigarrow \\
&2 * fact(1) \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
&fact(2) \rightsquigarrow \\
&\textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&\textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&2 * fact(2-1) \rightsquigarrow \\
&2 * fact(1) \rightsquigarrow \\
&2 * (\textbf{if}\ (1 \leq 1)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
& fact(2) \rightsquigarrow \\
& \textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& \textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& 2 * fact(2-1) \rightsquigarrow \\
& 2 * fact(1) \rightsquigarrow \\
& 2 * (\textbf{if}\ (1 \leq 1)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow \\
& 2 * (\textbf{if}\ (true)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
&fact(2) \rightsquigarrow \\
&\textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&\textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
&2 * fact(2-1) \rightsquigarrow \\
&2 * fact(1) \rightsquigarrow \\
&2 * (\textbf{if}\ (1 \leq 1)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow \\
&2 * (\textbf{if}\ (true)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow \\
&2 * 1 \rightsquigarrow
\end{aligned}
$$

Execution of factorial example program

$p_{fact} = (e, fact(2))$

where $e(fact) = (n, Int,\ \textbf{if}\ (n \leq 1)\ 1\ \textbf{else}\ n * fact(n-1), Int)$

$$
\begin{aligned}
& fact(2) \rightsquigarrow \\
& \textbf{if}\ (2 \leq 1)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& \textbf{if}\ (false)\ 1\ \textbf{else}\ 2 * fact(2-1) \rightsquigarrow \\
& 2 * fact(2-1) \rightsquigarrow \\
& 2 * fact(1) \rightsquigarrow \\
& 2 * (\textbf{if}\ (1 \leq 1)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow \\
& 2 * (\textbf{if}\ (true)\ 1\ \textbf{else}\ 1 * fact(1-1)) \rightsquigarrow \\
& 2 * 1 \rightsquigarrow \\
& 2
\end{aligned}
$$

Conclusion: $fact(2) \overset{*}{\rightsquigarrow} 2$

Getting stuck

If term t makes no sense, introduce no rule to evaluate it,

so there is no t' such that $t \rightsquigarrow t'$

Example: consider this top-level expression: **if** (5) 3 **else** 7

Getting stuck

If term t makes no sense, introduce no rule to evaluate it,

so there is no t' such that $t \rightsquigarrow t'$

Example: consider this top-level expression: **if** (5) 3 **else** 7

Term 5 cannot be evaluated further and is a constant,

but there are no rules for when condition of **if** is a number constant;

there are only rules for boolean constants.

Getting stuck

If term t makes no sense, introduce no rule to evaluate it,

so there is no t' such that $t \rightsquigarrow t'$

Example: consider this top-level expression: **if** (5) 3 **else** 7

Term 5 cannot be evaluated further and is a constant,

but there are no rules for when condition of **if** is a number constant;

there are only rules for boolean constants.

Such terms that are not constants *and* do not reduce further

are called **stuck**, because no further steps are possible

Stuck terms indicate errors.

Getting stuck

If term t makes no sense, introduce no rule to evaluate it,

so there is no t' such that $t \rightsquigarrow t'$

Example: consider this top-level expression: **if** (5) 3 **else** 7

Term 5 cannot be evaluated further and is a constant,

but there are no rules for when condition of **if** is a number constant;

there are only rules for boolean constants.

Such terms that are not constants *and* do not reduce further

are called **stuck**, because no further steps are possible

Stuck terms indicate errors.

Type checking is a way to prevent stuck terms **statically**,

i.e., without trying to evaluate the program to see if it gets stuck

Typing Rules: Program

After ***operational semantics***, we now define ***typing rules*** (also inductively)

Typing Rules: Program

After ***operational semantics***, we now define ***typing rules*** (also inductively)

Typing context: given initial program (e, t) define

$$\Gamma_0 \;=\; \left\{\; (f, \tau_1 \times \cdots \times \tau_n \rightarrow \tau_0) \;\mid\; (f, xs, (\tau_1, \ldots, \tau_n), t_f, \tau_0) \in e \;\right\}$$

Typing Rules: Program

After ***operational semantics***, we now define ***typing rules*** (also inductively)

Typing context: given initial program (e, t) define

$$\Gamma_0 \;=\; \left\{\; (f, \tau_1 \times \cdots \times \tau_n \rightarrow \tau_0) \;\mid\; (f, xs, (\tau_1, \ldots, \tau_n), t_f, \tau_0) \in e \;\right\}$$

We say a program type checks iff:

(1) the *top-level expression* type checks:

$$\Gamma_0 \vdash t : \tau$$

Typing Rules: Program

After ***operational semantics***, we now define ***typing rules*** (also inductively)

Typing context: given initial program (e, t) define

$$\Gamma_0 \;=\; \left\{\; (f, \tau_1 \times \cdots \times \tau_n \rightarrow \tau_0) \;\mid\; (f, xs, (\tau_1, \ldots, \tau_n), t_f, \tau_0) \in e \;\right\}$$

We say a program type checks iff:

(1) the *top-level expression* type checks:

$$\Gamma_0 \vdash t : \tau$$

(2) each *function body* type checks:

$$\Gamma_0 \oplus \{(x_1, \tau_1), \ldots, (x_n, \tau_n)\} \vdash t_f : \tau_0$$

for each $(f, (x_1, \ldots, x_n), (\tau_1, \ldots, \tau_n), t_f, \tau_0) \in e$

Typing Rules

$$\frac{\Gamma \vdash b : \mathit{Bool}, \quad \Gamma \vdash t_1 : \tau, \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\textbf{if } (b) \ t_1 \textbf{ else } t_2) : \tau}$$

Typing Rules

$$\frac{\Gamma \vdash b : \mathit{Bool}, \quad \Gamma \vdash t_1 : \tau, \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\textbf{if } (b) \ t_1 \textbf{ else } t_2) : \tau}$$

$$\frac{(x, \tau) \in \Gamma}{\Gamma \vdash x : \tau}$$

Typing Rules

$$\frac{\Gamma \vdash b : \mathit{Bool}, \quad \Gamma \vdash t_1 : \tau, \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\textbf{if } (b) \ t_1 \ \textbf{else } t_2) : \tau}$$

$$\frac{(x, \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \text{e.g., } x = f \text{ and } \tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$$

Typing Rules

$$\frac{\Gamma \vdash b : \mathit{Bool}, \quad \Gamma \vdash t_1 : \tau, \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\textbf{if } (b) \ t_1 \ \textbf{else } t_2) : \tau}$$

$$\frac{(x, \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \text{e.g., } x = f \text{ and } \tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$$

$$\frac{\Gamma \vdash f : \tau_1 \times \cdots \times \tau_n \to \tau_0, \quad \Gamma \vdash t_1 : \tau_1, \ \ldots, \ \Gamma \vdash t_n : \tau_n}{\Gamma \vdash f(t_1, \ldots, t_n) : \tau_0}$$

Typing Rules

$$\frac{\Gamma \vdash b : \mathit{Bool}, \quad \Gamma \vdash t_1 : \tau, \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash (\textbf{if } (b) \ t_1 \ \textbf{else } t_2) : \tau}$$

$$\frac{(x, \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \text{e.g., } x = f \text{ and } \tau = \tau_1 \times \cdots \times \tau_n \to \tau_0$$

$$\frac{\Gamma \vdash f : \tau_1 \times \cdots \times \tau_n \to \tau_0, \quad \Gamma \vdash t_1 : \tau_1, \ \ldots, \ \Gamma \vdash t_n : \tau_n}{\Gamma \vdash f(t_1, \ldots, t_n) : \tau_0}$$

We treat primitives like applications of functions e.g.

$+ : \mathit{Int} \times \mathit{Int} \to \mathit{Int}$

$\leq : \mathit{Int} \times \mathit{Int} \to \mathit{Bool}$

$\&\& : \mathit{Bool} \times \mathit{Bool} \to \mathit{Bool}$

Soundness through progress and preservation

Soundness theorem: *if program type checks, its evaluation does not get stuck*.

Proof uses the following two lemmas (common approach):

- **Progress:** if a program type checks, it is not stuck.

  If $\Gamma \vdash t : \tau$, then *either*

  t is a constant (execution is done)

  or there exists a t' such that $t \rightsquigarrow t'$

- **Preservation:** if a program type checks and makes one "$\rightsquigarrow$" step,

  then the result again type checks.

  In our simple system: it type checks *and has the same type*:

  if $\Gamma \vdash t : \tau$ and $t \rightsquigarrow t'$, then $\Gamma \vdash t' : \tau$.

Proof of progress and preservation — case of if

We prove both progress and preservation by induction on term t such that $\Gamma \vdash t : \tau$. The operational semantics defines the non-error cases of an interpreter, which enables case analysis. Consider **if**. By type checking rules, **if** can only type check if its condition b type checks and has type Bool. By inductive hypothesis and progress *either b is constant or it can be reduced to a* b'. If it is constant, one of these rules apply (so we get progress):

$$\frac{}{(\textbf{if}\ (\mathit{true})\ t_1\ \textbf{else}\ t_2) \rightsquigarrow t_1}$$

$$\frac{}{(\textbf{if}\ (\mathit{false})\ t_1\ \textbf{else}\ t_2) \rightsquigarrow t_2}$$

and the result, by typing rule for **if**, has type τ (preservation). If b' is not constant, the assumption of the rule

$$\frac{b \rightsquigarrow b'}{(\textbf{if}\ (b)\ t_1\ \textbf{else}\ t_2) \rightsquigarrow (\textbf{if}\ (b')\ t_1\ \textbf{else}\ t_2)}$$

applies, so t also makes progress. By preservation IH, b' also has type Bool, so the entire expression can be typed as τ re-using the type derivations for t_1 and t_2.

Progress and preservation — user defined functions

Following the cases of operational semantics, either all arguments of a function have been evaluated to a constant, or some are not yet constant.

If they are not all constants, the case is as for the condition of **if**, and we establish progress and preservation analogously.

Otherwise rule

$$\frac{}{f(c_1, \ldots, c_n) \rightsquigarrow t_f[x_1 := c_1, \ldots, x_n := c_n]}$$

applies, so progress is ensured. For preservation, we need to show

$$\Gamma \vdash t_f[x_1 := c_1, \ldots, x_n := c_n] : \tau \qquad (*)$$

where $e(f) = ((x_1, \ldots, x_n), (\tau_1, \ldots, \tau_n), t_f, \tau_0)$ and t_f is the body of f. According to typing rules $\tau = \tau_0$ and $\Gamma \vdash c_i : \tau_i$.

Progress and preservation — substitution and types

Function f definition type checks, so $\Gamma' \vdash t_f : \tau_0$ where $\Gamma' = \Gamma \oplus \{ (x_1, \tau_1), \ldots, (x_n, \tau_n) \}$.

Consider the type derivation tree for t_f and replace each use of $\Gamma' \vdash x_i : \tau_i$ with $\Gamma \vdash c_i : \tau_i$. The result is a type derivation for $(*)$:

$$\Gamma \vdash t_f[x_1 := c_1, \ldots, x_n := c_n] : \tau \qquad (*)$$

Therefore, the preservation holds in this case as well.

Progress and preservation — substitution and types

Function f definition type checks, so $\Gamma' \vdash t_f : \tau_0$ where $\Gamma' = \Gamma \oplus \{ (x_1, \tau_1), \ldots, (x_n, \tau_n) \}$.

Consider the type derivation tree for t_f and replace each use of $\Gamma' \vdash x_i : \tau_i$ with $\Gamma \vdash c_i : \tau_i$. The result is a type derivation for $(*)$:

$$\Gamma \vdash t_f[x_1 := c_1, \ldots, x_n := c_n] : \tau \qquad (*)$$

Therefore, the preservation holds in this case as well.

Exercise: prove the above step, i.e., that replacing variables with constants of the same type transforms a term that has type derivation with type τ into a term that again has a derivation with type τ. Is there a more general statement?