

COMP170

Discrete Mathematical Tools for Computer Science

More on *“time until first success”*

Version 2: Last updated, Nov 30, 2005

Example 1

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Then, by linearity of expectation,

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_6) = 6 \cdot 6 = 36.$$

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	2	2	5	2	3	3	2	3	6	5	2	6	5	1	2	1	5	6	1	3	4
i	1	2	3						4					5							6
N_i	2	5	3						6					1							4
X_i	1	2	2						4					5							7

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$i = 1$: $X_1 = 1$ so $E(X_1) = 1$.

$i = 2$: Once N_1 is chosen, X_2 is the number of times we need to throw the die until we see something that is *not* N_1 . Since being "*not* N_1 " occurs with probability $\frac{5}{6}$, X_2 is geometric with $p = \frac{5}{6}$ so $E(X_2) = \frac{1}{p} = \frac{6}{5}$.

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$i = 3$: Similarly, once N_1, N_2 are chosen, X_3 is the number of times we need to throw the die until we see something that is *not* N_1, N_2 . Since being “*not* N_1, N_2 ” occurs with probability $\frac{4}{6}$, X_3 is geometric with $p = \frac{4}{6}$ so $E(X_3) = \frac{1}{p} = \frac{6}{4}$.

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General i In the general case, once N_1, N_2, \dots, N_{i-1} are chosen, X_i is the number of times we need to throw the die until we see something that is *not* N_1, N_2, \dots, N_{i-1} . Since being “*not* N_1, N_2, \dots, N_{i-1} ” occurs with probability $\frac{6-(i-1)}{6}$, X_i is geometric with $p = \frac{6-(i-1)}{6}$ so $E(X_i) = \frac{1}{p} = \frac{6}{6-(i-1)}$.

$$\Rightarrow E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6)$$

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$$\begin{aligned} \Rightarrow E(X) &= E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6) \\ &= \frac{6}{6} + \frac{6}{5} + \dots + \frac{6}{1} = 6 \sum_{i=1}^6 \frac{1}{i} = 6 \cdot \frac{49}{20} = \frac{147}{10}. \end{aligned}$$

Compare this to previous problem in which we needed $6 \cdot 6$ flips on average, to see the numbers in order.

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$$X_1 = 1.$$

For $i > 1$: X_i = time needed to receive i th new coupon after having received $(i - 1)$ st new coupon.

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$$E(X) = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n \frac{n}{n - (i - 1)} = n \sum_{i=1}^n \frac{1}{i}$$

We just showed that

$$E(X) = n \sum_{i=1}^n \frac{1}{i}$$

$H_n = \sum_{i=1}^n 1/i$ has a special name. It is called the n^{th} **harmonic number**.

It is also known that $\forall n \ |H_n - \ln n| \leq 2$.

So H_n grows like $\ln n$ and

$E(X)$ grows like $n \ln n$.