

COMP170

Discrete Mathematical Tools for Computer Science

More Counting

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Discrete Math for Computer Science

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Section 1.2, pp. 9-19

1.2 Counting Lists, Permutations, and Subsets

- Using the Sum and Product Principles
- Lists and Functions
- The Bijection Principle
- k -Element Permutations of a Set
- k -Element Subsets of a Set
Binomial Coefficients

Some Simple Examples

- The entry code to our office is 4 characters long where each character is a number between $\{0 \dots 9\}$.
e.g., 1012, 2561.
How many door codes are there?
- A computer system's password must be 6 characters long, where each character is in $\{a \dots z, A \dots Z\}$.
e.g., *abcdEF*, *DHrsAQ*, *Mickey*.
How many passwords are there?
- Hong Kong car plates are of the form $L_1 L_2 D_1 D_2 D_3 D_4$ where the L_i are letters in $\{A \dots Z\}$ and the D_i are digits in $\{0, \dots 9\}$.
e.g., *AB1234*, *EC1357*.
How many car plates are there?

The entry code to our office is 4 characters long
where each character is a number between $\{0 \dots 9\}$.

E.G., 1012, 2561.

How many door codes are there?

A door code is of the form $X_1X_2X_3X_4$ where,
for $i = 1, 2, 3, 4$, $X_i \in \{0, 1, \dots, 9\}$
(\in means *is in* or *is a member of*)

There are

10 possibilities for X_1 10 possibilities for X_2

10 possibilities for X_3 10 possibilities for X_4

$\Rightarrow 10 \times 10 \times 10 \times 10 = 10^4 = 10,000$ possible door codes.

Note; This is really the *product principle*

What are the sets being used?

A computer system's password must be 6 characters long,
where each character is in $\{a \dots z, A \dots Z\}$.

E.G., *abcdEF*, *DHrsAQ*, *Mickey*.

How many passwords are there?

A password is of the form $D_1 D_2 D_3 D_4 D_5 D_6$ where
for $i = 1, 2, 3, 4, 5, 6$, $D_i \in \{a \dots z, A \dots Z\}$.

Each D_i has 52 possible choices so,
the total number of passwords is

$$52 \times 52 \times 52 \times 52 \times 52 \times 52 = 52^6$$

Hong Kong car plates are of the form $L_1L_2D_1D_2D_3D_4$
where the L_i are letters in $\{A \dots Z\}$
and the D_i are digits in $\{0, \dots, 9\}$.

e.g., $AB1234$, $EC1357$.

How many car plates are there?

Each L_i has 26 possibilities and
each D_i has 10 possibilities
so the total number of car plates is

$$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4 = 6,760,000$$

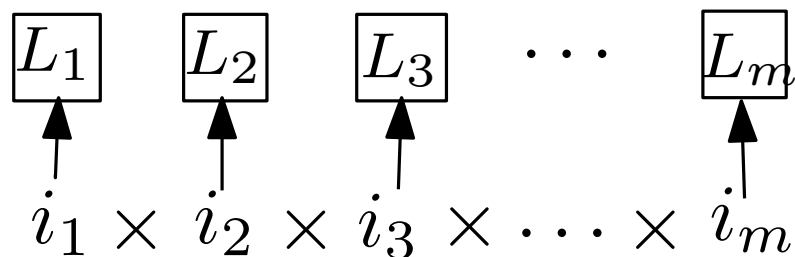
We have just seen examples of

Product Principle, Version 2

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S , and
2. for each $j > 1$
and each choice of the first $j - 1$ elements of a list in S ,
there are i_j choices of elements in position j
of those lists,

\Rightarrow there are $i_1 i_2 \cdots i_m = \prod_{k=1}^m i_k$ lists in S .



Combining Sum and Product Principles

A password for a certain computer system is supposed to be between 4 and 8 characters long and composed of lowercase and/or uppercase letters.

How many passwords are possible?

This is a counting problem!

Should we use sum or product principle?

Set of *all* passwords is *disjoint union* of passwords with 4, 5, 6, 7, or 8 letters.

So *sum principle* might help us.

Let P_i be set of i -letter passwords
and P set of *all* possible passwords.

Clearly

$$P = P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8.$$

P_i are mutually disjoint, thus apply *sum principle*

$$|P| = \sum_{i=4}^8 |P_i|.$$

By *product principle*, $|P_i|$ is 52^i .

\Rightarrow total number of passwords is

$$52^4 + 52^5 + 52^6 + 52^7 + 52^8.$$

Lists and Functions

We've already seen *lists*.

Informally, a list of k elements from T is

$$L = L_1 L_2 \dots L_k, \text{ where each } L_i \in T.$$

Rewriting a list in a different order yields a different list
(compare to sets, where changing the order doesn't change the set)

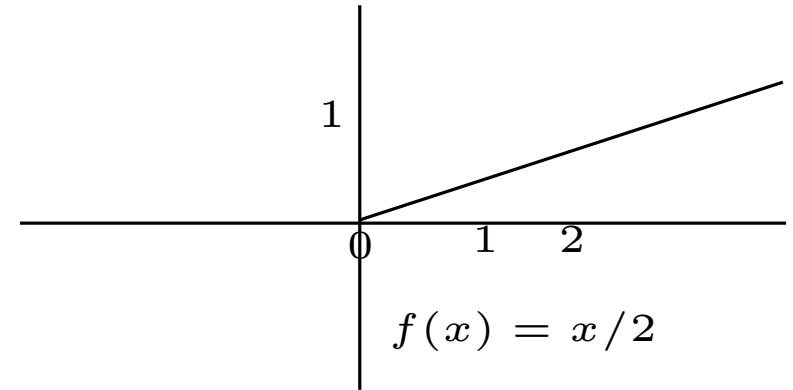
Example: *MORE* and *ROME* are
two different 4-letter lists from $T = \{A, B, \dots, Z\}$

$\{M, O, R, E\}$ and $\{R, O, M, E\}$ are
the same subset of T .

Lists and Functions (cont)

We now introduce **functions**.

A function f from set S to set T is a *relationship* that associates exactly one element of T with each element of S



In continuous math, e.g., basic calculus and algebra, you often have formulas, e.g., $f(x) = x/2$ that **describe** the function. In discrete math, where we often deal with finite sets, we can frequently state the full function explicitly: e.g.,

$$f(1) = \text{Sam}, \quad f(2) = \text{Mary}, \quad f(3) = \text{Sarah}$$

Lists and Functions (cont)

We often write a function from S to T as

$$f : S \rightarrow T$$

S is the domain of f ; T is the range of f .

Lists and Functions (cont)

A k -element List, $L = L_1 L_2 \dots L_k$ from set T can now be defined as a function

$$f : \{1, 2, \dots, k\} \rightarrow T \text{ where } f(i) = L_i$$

Example: 4-element lists from $\{A, \dots, Z\}$

MORE is the function

$$f(1) = M, f(2) = O, f(3) = R, f(4) = E$$

ROME is the function

$$f(1) = R, f(2) = O, f(3) = M, f(4) = E$$

Exercises

- Write down all the functions from the 2-element set $\{1, 2\}$ to the 2-element set $\{a, b\}$.
- How many functions are there from a 2-element set to a 3-element set?
- How many functions are there from a 3-element set to a 2-element set?

Write down all the functions from the two-element set $\{1, 2\}$ to the two-element set $\{a, b\}$.

Use f_1, f_2, \dots to denote the functions. To describe $f_i : \{1, 2\} \rightarrow \{a, b\}$, we must specify $f_i(1)$ and $f_i(2)$.

$$f_1(1) = a \quad f_1(2) = b$$

$$f_2(1) = b \quad f_2(2) = a$$

$$f_3(1) = a \quad f_3(2) = a$$

$$f_4(1) = b \quad f_4(2) = b$$

How do we know we've found *all* of them?

Set of *all* functions from $\{1, 2\}$ to $\{a, b\}$ is just the set of 2-element lists from $\{a, b\}$ which we already saw, by the product principle, has size $2 \times 2 = 4$.

How many functions are there from a two-element set to a three-element set?

Set of all functions is union of three sets,
one for each choice of $f_i(1)$.

Each of these sets has three elements,
one for each choice of $f_i(2)$.

Thus, by product principle,
we have $3 \cdot 3 = 9$ functions.

Note: This is the same as number of 2-element lists from a 3-element set

How many functions are there from a three-element set to a two-element set?

This is same as number of 3-element lists from a 2-item set, which is $2 \times 2 \times 2 = 8$.

A function f is **one-to-one**, or an **injection**, if
 $(x \neq y) \Rightarrow (f(x) \neq f(y))$.

$$f_1(1) = a \quad f_1(2) = b \quad \checkmark$$

$$f_2(1) = b \quad f_2(2) = a \quad \checkmark$$

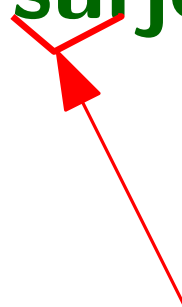
$$f_3(1) = a \quad f_3(2) = a \quad \times$$

$$f_4(1) = b \quad f_4(2) = b \quad \times$$

A function $f : X \rightarrow Y$ is **onto**, or a **surjection**, if
every element y in range Y is
 $f(x)$ for some x in domain X .

f_1 and f_2 are surjections

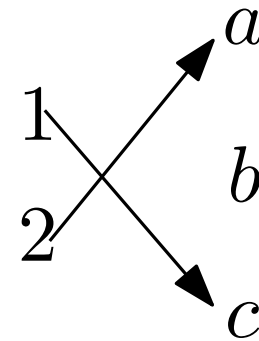
f_3 and f_4 are not surjections



In French
sur = "on"

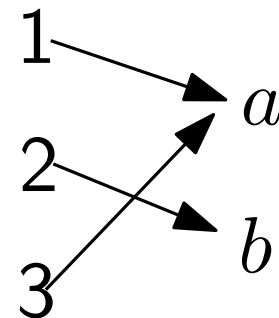
Using 2- or 3-element sets as domains & ranges, find an example of a *one-to-one* function that is not *onto*.

Function $f : \{1, 2\} \rightarrow \{a, b, c\}$
given by $f(1) = c, f(2) = a$.



Using 2- or 3-element sets as domains & ranges, find an example of a *onto* function that is not *one-to-one*.

Function $f : \{1, 2, 3\} \rightarrow \{a, b\}$
given by $f(1) = a, f(2) = b, f(3) = a$.

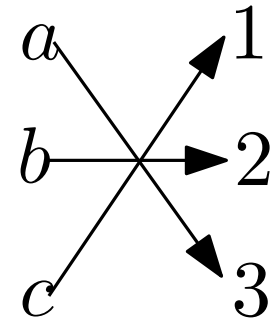


Bijections & Permutations

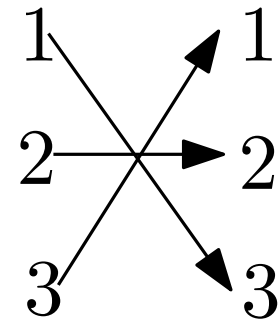
A function that is both one-to-one and onto is called a **bijection**, or a **one-to-one correspondence**.

A bijection from a set onto itself is called a **permutation**

$f : \{a, b, c\} \rightarrow \{1, 2, 3\}$ defined by
 $f(a) = 3, f(b) = 2, f(c) = 1$
is a bijection.



$f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ defined by
 $f(1) = 3, f(2) = 2, f(3) = 1$
is a permutation.



Bijections & Permutations

A function that is both one-to-one and onto is called a **bijection**, or a **one-to-one correspondence**.

A bijection from a set onto itself is called a **permutation**

In a bijection

exactly one arrow

leaves each item on the left

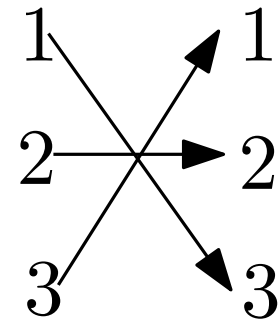
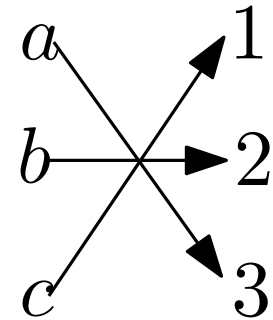
and

exactly one arrow

arrives at each item on the right

so the left and right sides

must have the same size



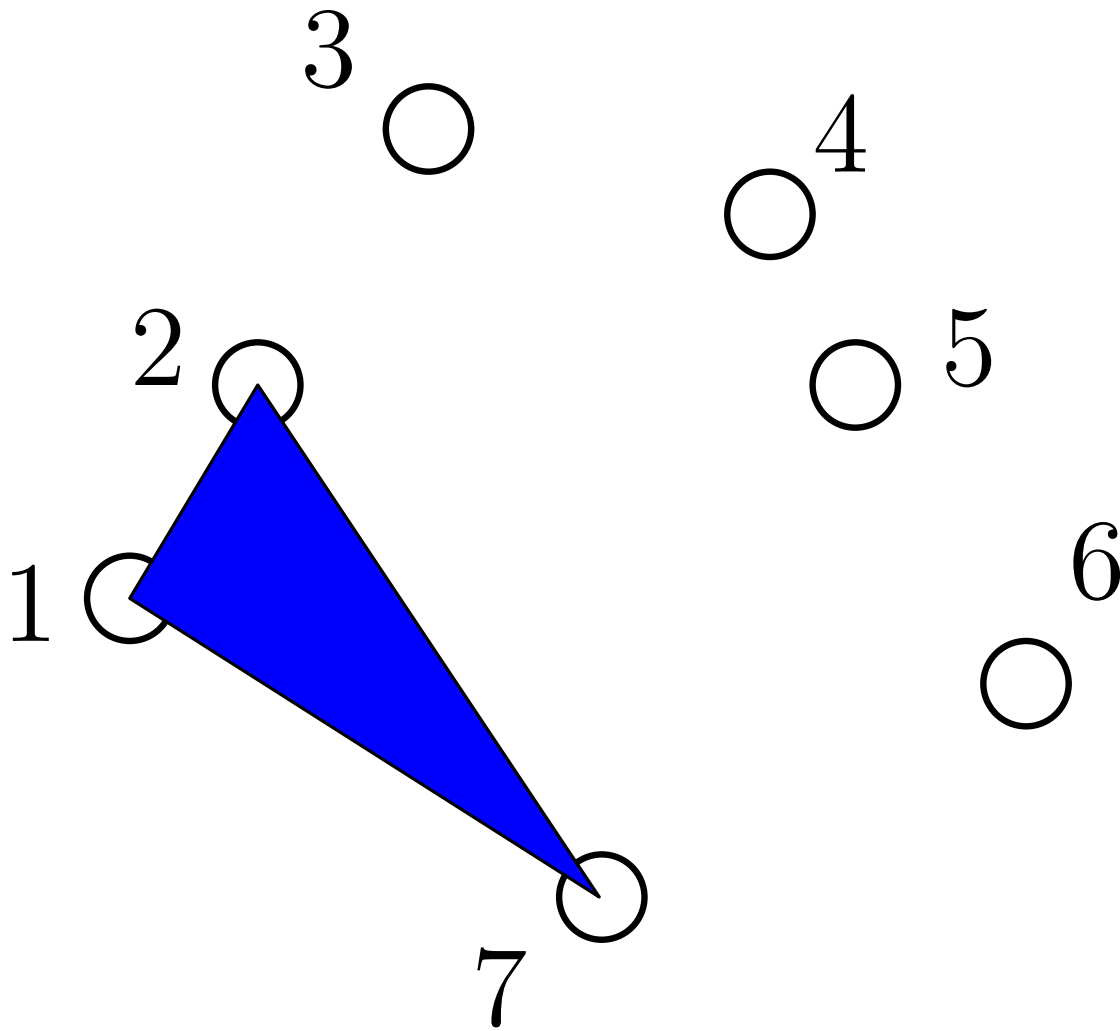
The Bijection Principle

The following loop is a part of program to determine the number of triangles formed by n points in the plane.

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

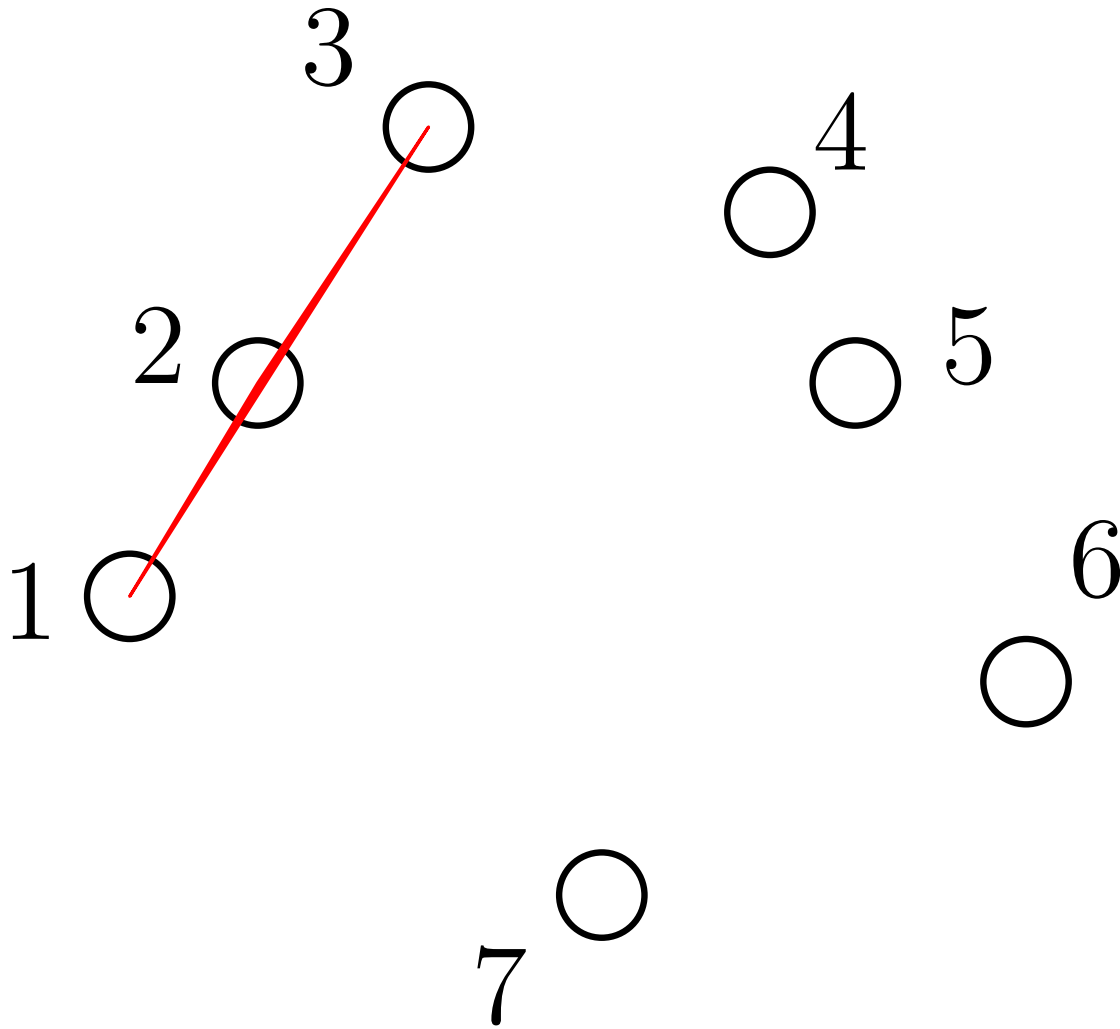
Among all iterations of line 5 in the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



1 – 2 – 7: yes

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)

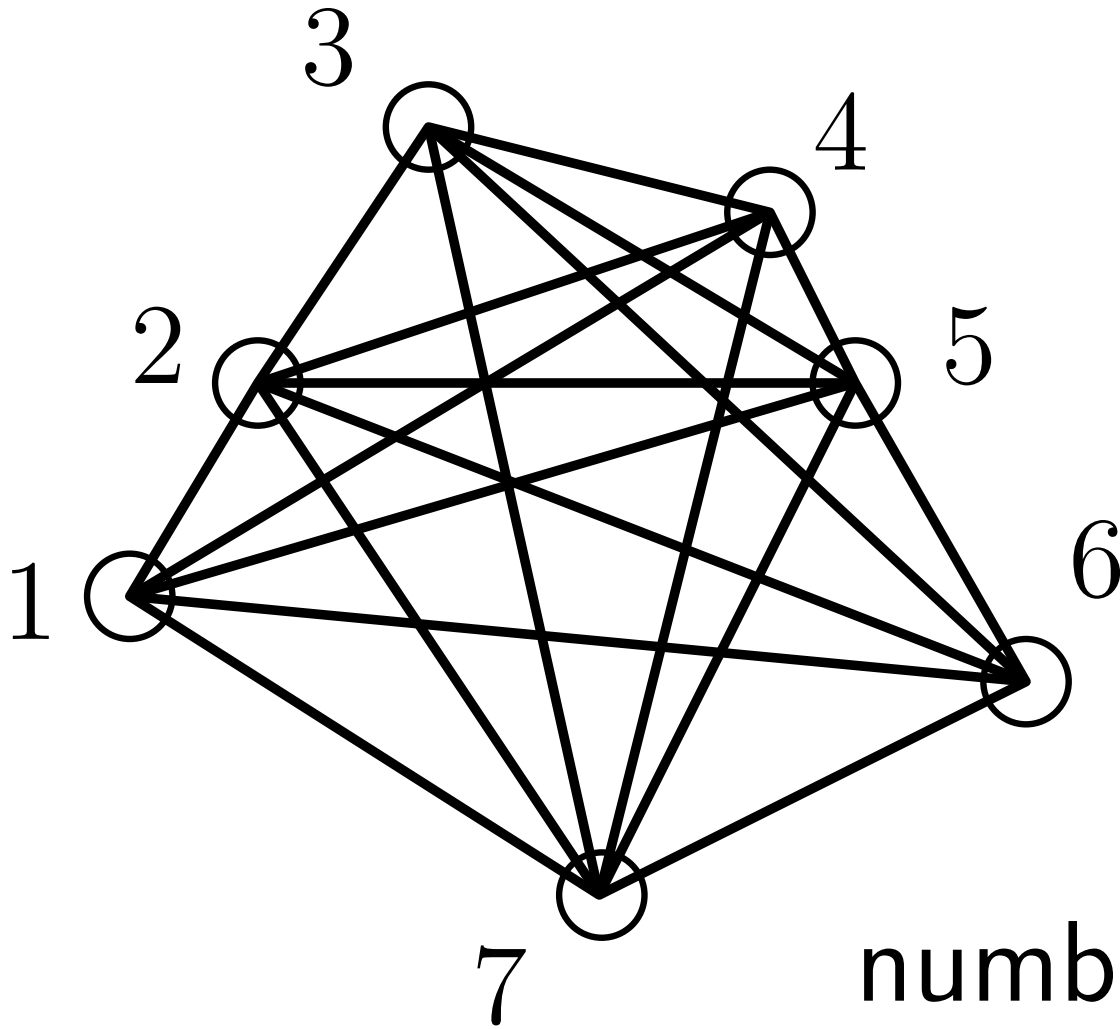


1 — 2 — 7: yes

1 — 2 — 5: yes

1 — 2 — 3: no

Counting Triangles: 3 points form a triangle if and only if they are not collinear (on a straight line)



1 — 2 — 7: yes

1 — 2 — 5: yes

1 — 2 — 3: no

1 — 5 — 6: yes

3 — 4 — 7: yes

4 — 5 — 6: no

number of triangles: 33

```
(1) trianglecount = 0
(2)   for i = 1 to n
(3)     for j = i+1 to n
(4)       for k = j+1 to n
(5)         if points i, j, k are not collinear
(6)           trianglecount = trianglecount + 1
```

A loop embedded in a loop embedded in another loop.

Second loop begins with $j = i + 1$ and j increases up to n .

Third loop begins with $k = j + 1$ and k increases up to n .

Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$.

Thus, compute number of increasing triples!

- Want to compute the number of
increasing triples (i, j, k) with $1 \leq i < j < k \leq n$.
 - Claim: Number of increasing triples is **exactly** the same
as number of 3-element subsets from $\{1, 2, \dots, n\}$
 - Why? Let X = set of increasing triples and
 Y = set of 3-element subsets from $\{1, 2, \dots, n\}$
 - Define $f : X \rightarrow Y$ by $f((i, j, k)) = \{i, j, k\}$
Claim: f is a **bijection** (why) so $|X| = |Y|$
- f is a bijection because
- f is **one-to-one**
if $(i, j, k) \neq (i', j', k') \Rightarrow f((i, j, k)) \neq f((i', j', k'))$
- f is **onto**
if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
where $i < j < k$ so $f((i, j, k)) = \gamma$.

- We've already seen something very similar.
The number of
increasing pairs (i, j) with $1 \leq i < j \leq n$
is the same as the number of
2-sets from $\{1, 2, \dots, n\}$
- Define $f : X \rightarrow Y$ by $f((i, j)) = \{i, j\}$
Claim: f is a **bijection** so $|X| = |Y|$
- We actually already saw that $|X| = |Y| = \binom{n}{2}$

(Bijection Principle)

Two sets have the same size if and only if there is a one-to-one function from one set onto the other.

- A standard first step in counting the size of a set is to use a bijection to show that it has the same size as a 2nd set, and then count the 2nd set instead.
- In practice, in real problems we often only *implicitly* use the bijection and don't *explicitly* describe it
- Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \dots, n\}$
- We will now see how to count the # of k -element permutations of $\{1, 2, \dots, n\}$. From this we will derive how to count # of k -element sets.

k -Element Permutations of a Set

- In how many ways can we choose an ordered triple of distinct elements from $\{1, 2, \dots, n\}$?
- More generally, in how many ways can we choose a list of k distinct elements from $\{1, 2, \dots, n\}$?
- A list of k *distinct* elements chosen from a set N is called a **k -element permutation of N** .
- Note that the case of $k = n$ is special;
An n -element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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- In how many ways can we choose an ordered triple of distinct elements from $\{1, 2, \dots, n\}$?
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- A list of k *distinct* elements chosen from a set N is called a **k -element permutation of N** .
 - To start, how many 3-element permutations of $\{1, 2, \dots, n\}$ are there?

How many three-element permutations of $\{1, 2, \dots, n\}$ are there?

- n choices for first number
 - For each way of choosing first number there are $n - 1$ choices for the second.
 - For each way of choosing first two numbers, there are $n - 2$ choices for the third number.

By version 2 of product principle, there are $n(n - 1)(n - 2)$ ways to choose the permutation.

By version 2 of product principle, there are $n(n-1)(n-2)$ ways to choose the permutation

Ex: When $n = 4$, there are $4 \times 3 \times 2 = 24$
3 -element permutations of $\{1, 2, 3, 4\}$

$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}$.

Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a **lexicographic ordering** and is used quite often.

- We just saw that there are $n(n-1)(n-2)$ ways to choose a 3-element permutation from $\{1, 2, \dots, n\}$
- How does this help us solve our original problem (from triangle program) of counting # of 3-element subsets?
- Note that every 3-element subset $\{i, j, k\}$ can be made into exactly 6 3-element perms.
- $\Rightarrow (\# \text{ 3-element perms}) = 6 \times (\# \text{ 3-element subsets})$
- Let $\binom{n}{3}$ denote # of 3-element subsets

$(1, 2, 3)$	$(1, 3, 2)$	$(2, 1, 3)$
$(2, 3, 1)$	$(3, 1, 2)$	$(3, 2, 1)$

$$\Rightarrow \frac{n(n-1)(n-2)}{6} = \binom{n}{3}$$

General formula to compute the number of k -element permutations of the set $\{1, 2, \dots, n\}$

- n choices for first element.
- Independent of first choice, we have $n - 1$ choices for second number
- Independent of first and 2nd choices, we have $n - 2$ choices for third number
- Given the first $i - 1$ elements, we always have exactly $n - (i - 1) = n - i + 1$ choices for the i th element.
- \dots

Thus, by version 2 of product principle, there are $n(n - 1) \cdots (n - k + 1)$ ways to choose a k -element permutation.

We just saw that there are

$$n(n-1) \cdots (n-k+1) = \prod_{i=0}^{k-1} (n-i)$$

ways to choose a k -element permutation.

In the special case of $k = n$ (a permutation) this reduces to

$$n(n-1) \cdots 3 \cdot 2 \cdot 1 = n!$$

So there are $n!$ different permutations of a set of size n .

Handy notation suggested by Donald E. Knuth, is $n^{\underline{k}}$, the k th falling factorial power of n , defined as

$$n^{\underline{k}} = n(n-1) \cdots (n-k+1) = \prod_{i=0}^{k-1} (n-i).$$

Theorem 1.1

The #of k -element permutations of an n -element set is

$$n^{\underline{k}} = \frac{n!}{(n-k)!}.$$

Handy notation suggested by Donald E. Knuth, is $n^{\underline{k}}$, the k **th falling factorial power of** n , defined as

$$n^{\underline{k}} = n(n-1) \cdots (n-k+1) = \prod_{i=0}^{k-1} (n-i).$$

Donald Ervin Knuth (born 1938) is *Professor Emeritus of the Art of Computer Programming* at Stanford University. He is most famous for his 3+-volume set, *The Art of Computer Programming* but he's done many other things, including designing the system, TeX, with which these notes were typeset.



- Have seen that # of k -element *permutations* is n^k .
- What does this say about # of k -element *subsets*?
- Let $\binom{n}{k}$ denote the # of k -element subsets
 The set of all k -element permutations of $\{1, 2, \dots, n\}$
 can be *partitioned* into $\binom{n}{k}$ disjoint blocks,
 Each block comprises all k -element permutations
 of the same k -element subset.
- However, the number of k -element permutations of a
 k -element set is $k!$ (Theorem 1.1 with $n = k$).
- Thus, by sum principle,

$$n^k = (\# \text{ of blocks}) \times (\# \text{ in each block}) = \binom{n}{k} k!.$$

Theorem 1.2

For integers n and k with $0 \leq k \leq n$, the number of k -element subsets of an n -element set is

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

Proof:

We just proved this except for case $k = 0$.

The only subset of an n -element set of size zero is the empty set, so we have exactly one such subset and should have $\binom{n}{0} = 1$

This will work if we define $0! = 1$.

Note: Both cases $k = 0$ and $k = n$ use fact that $0! = 1$.

Binomial Coefficients

The number of ways of choosing a
 k -element subset from a set of size n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This term is called a **binomial coefficient**.
Be aware that there are other, alternative, notations for the same thing, occasionally used, e.g.,
 $C(n, k)$ or nCk