Disjoint sets with union

- a fixed set U is partitioned into disjoint subsets

- maintain these subsets under operations

  Create-Set(x)

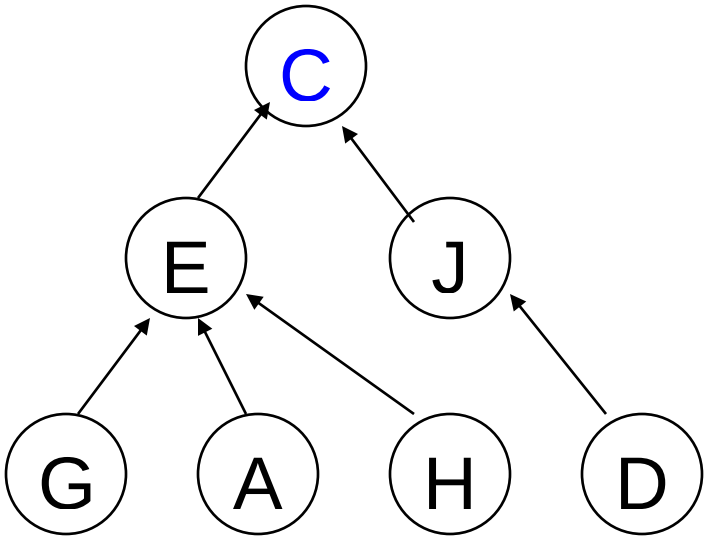
  Union(S, T)

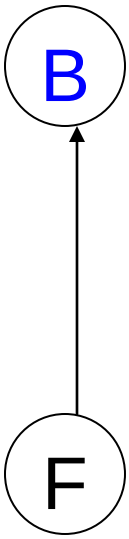
  Find-set(x)

S,T sets. x an element

N.B. No Insert, Delete,DeleteMin, FindMin

Up-trees





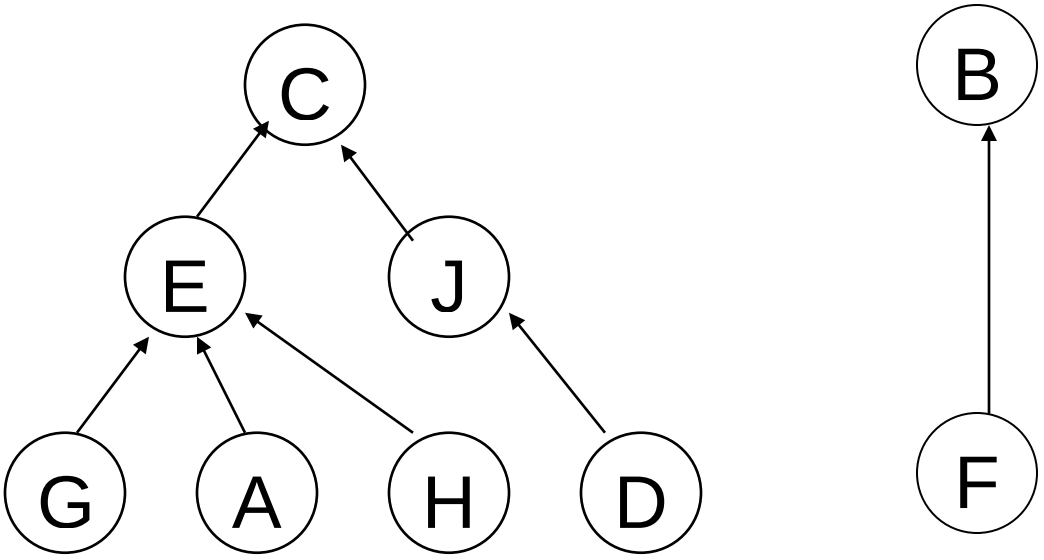
{ A, C, D, E, G, H, J }

{ B, F }

Use C to
denote this set

Use B to
denote this set

1. *Root points to itself.*
2. *Name of set is root name*
3. *No limit on number of children*



Find-set(x)

z ← x

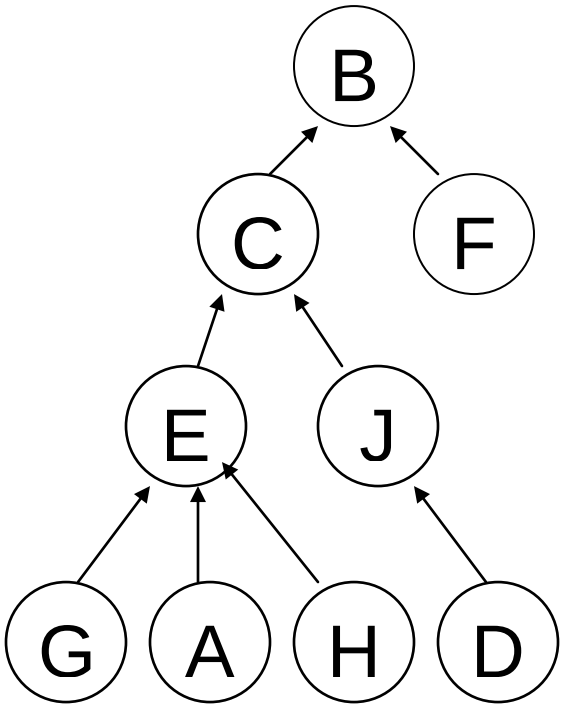
loop if z = parent [z]

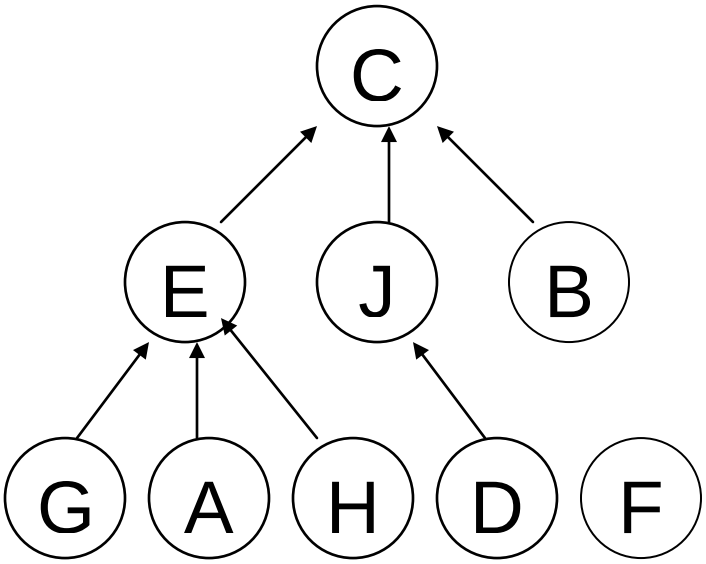
then return z

else z ← parent [z]

Find walks up tree
until reaching root.

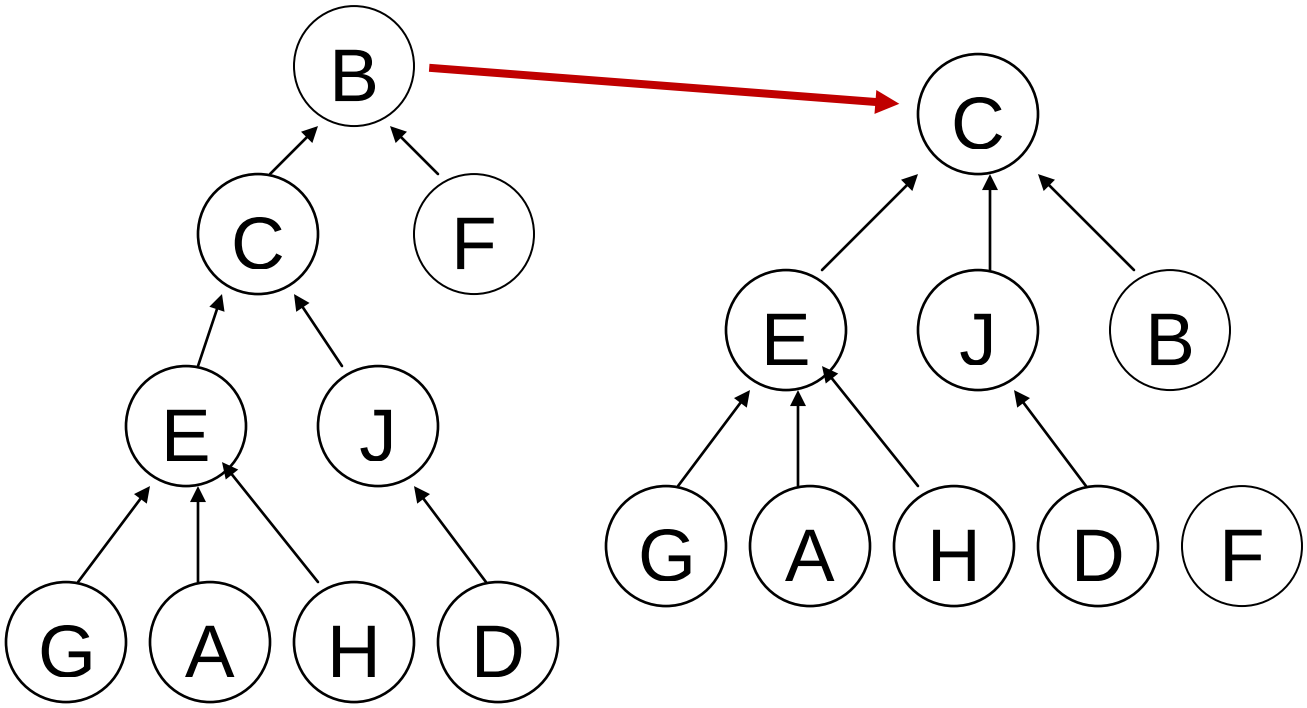
Union (C, B) *two possibilities*





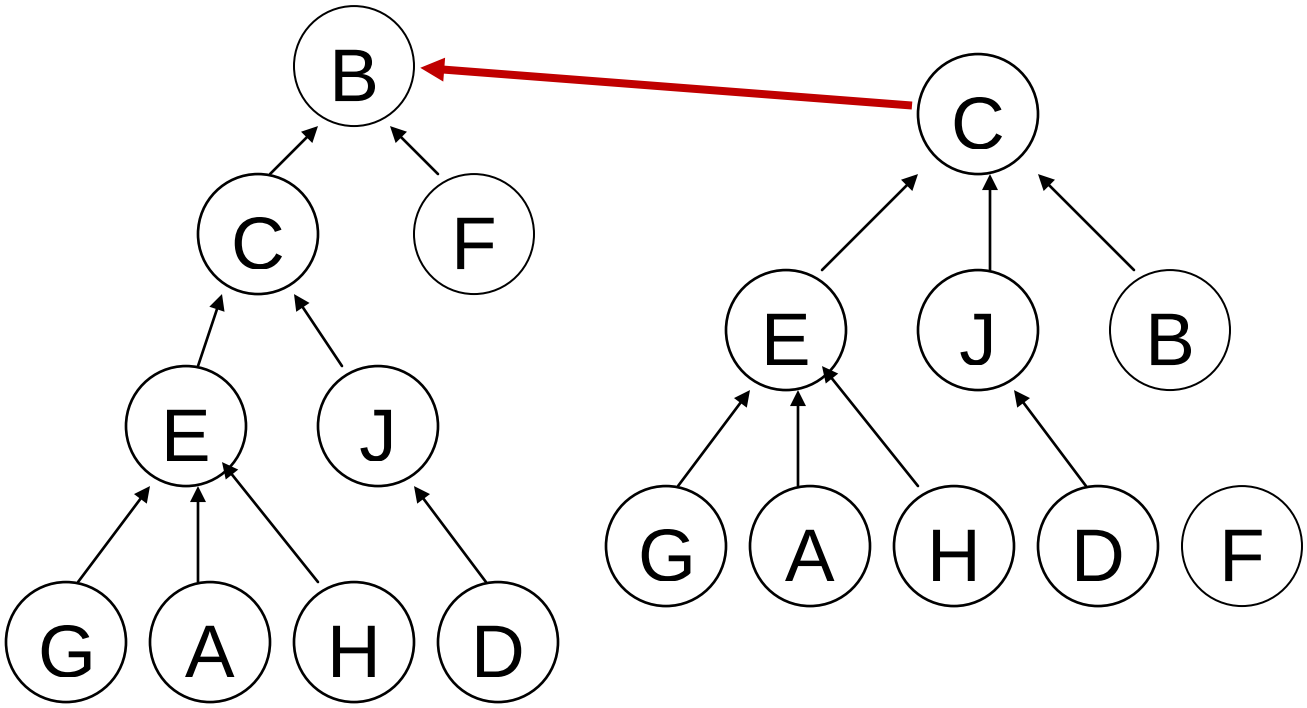
Either
B points to C
or
C points to B

Union (C, B) *two possibilities*



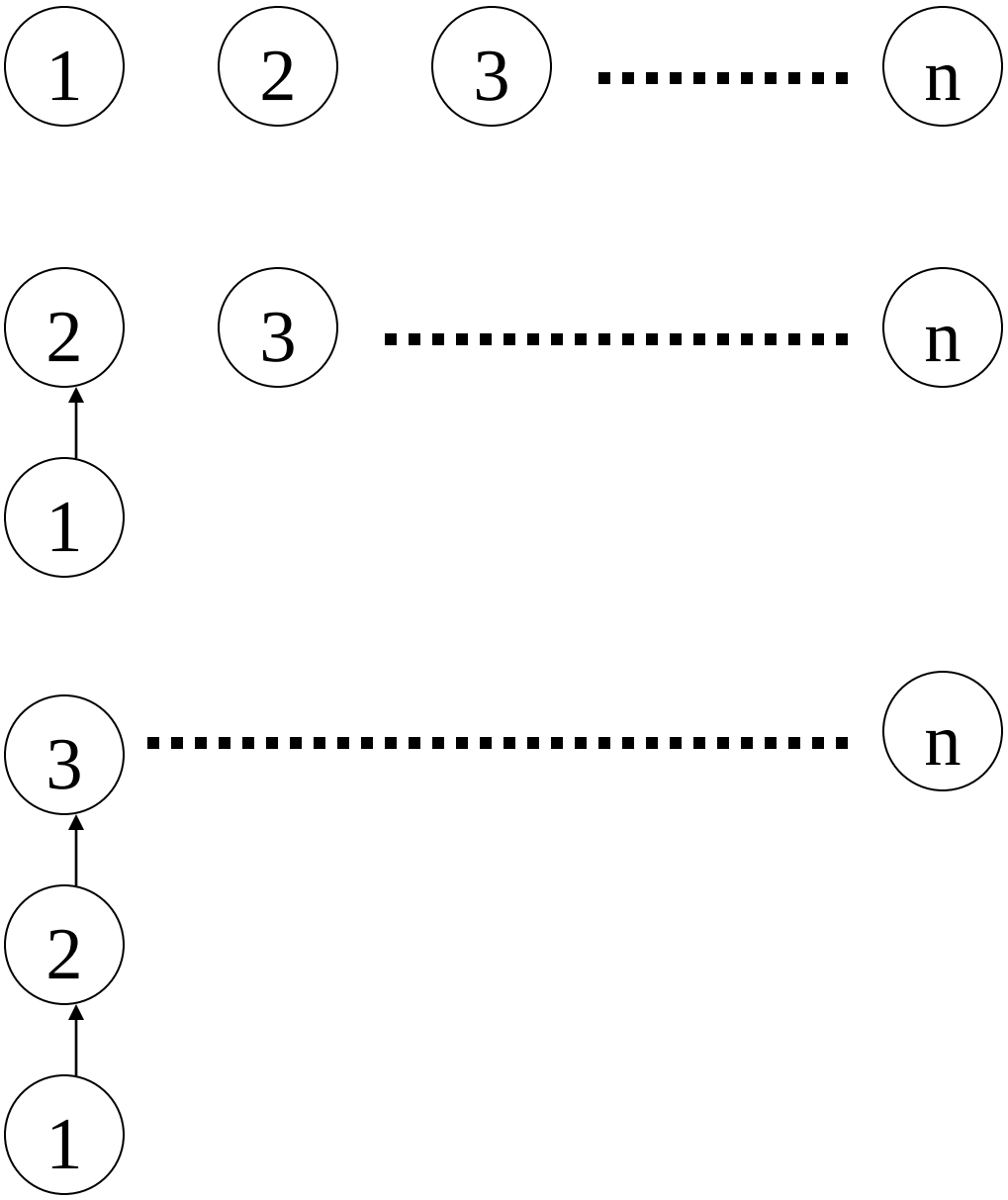
Either
B points to C
or
C points to B

Union (C, B) *two possibilities*



Either
B points to C
or
C points to B

Efficiency concern:



Possible to become a long single linked list.

Union by height

- the root of every tree holds the height of the tree.

- merge the shorter tree into the taller.

(make root of taller tree the parent of root of shorter tree)

(in case of ties, make root of first tree point to root of second)

CREATE-SET (x)

parent [x] ← x

height [x] ← 0

UNION (x, y)

if height [x] > height [y]

then parent [y] ← x

else parent [x] ← y

If height [x] = height [y]

Then height [y]++

LEMMA 1. For any root x,
$size(x) \geq 2^{height(x)}$

size(x): # descendants of x, including x

PROOF (by induction)
BASE CASE: At beginning, all heights are 0 and each tree has size 1.

INDUCTIVE STEP: Assume true just before a union(x, y).
DEF: size'(x) and height'(x) after union

CASE 1. height(x) < height(y)

Then size'(y) = size(x) + size(y)

$$\geq 2^{height(x)} + 2^{height(y)}$$

$$\geq 2^{height(y)}$$

$$= 2^{height'(y)}$$

CASE 2. height(x) = height(y)

Then size'(y) = size(x) + size(y)

$$\geq 2^{height(x)} + 2^{height(y)}$$

$$= 2^{height(y)+1}$$

$$= 2^{height'(y)}$$

CASE 3. height(x) >height(y)

same as Case 1

COROLLARY

Every node has height <= $\lfloor \lg n \rfloor$.

PROOF

Let h' > lg n.

There are at most $n/2^{h'} < 1$ nodes of height h'.

$\Rightarrow$ There are zero nodes with height > lg n.

THM.
Create-Set(x) uses O(1) time
Union(x,y) uses O(1) time when
x,y are roots of respective trees
Find-Set(x) uses O(log n) time

PROOF
Create-Set and Union are obviously O(1) time.

Find operation is O(h) where h is the max height of any tree.

By corollary, $h = \lfloor \lg n \rfloor$.
$\Rightarrow$ Find-Set(x) uses O(log n) time

Note:
Union(x,y) used by Kruskal's algorithm is actually the combination of three commands:

A = Find-Set(x)
B = Find-Set(y)
Union(A,B)

And therefore requires O(log n) time.

Note:
It is possible to improve the Union-Find data-structure so that it works even faster but that is beyond scope of this course.

Recall that running time of Kruskal's algorithm is dominated by the

$$O(|E| \log |E|)$$

sorting first stage, so improving Union-Find won't speed up Kruskal's algorithm.