# COMP170
# Discrete Mathematical Tools for Computer Science
# Lecture 3

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*Discrete Math for Computer Science*
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*Section 1.3, pp. 19-26*

# 1.3 Binomial Coefficients

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- Pascal's Triangle

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- A Proof using the Sum Principle

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- Pascal's Triangle
- A Proof using the Sum Principle
- The Binomial Theorem
- Labeling and Trinomial Coefficients

# Some properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of $k$-element subsets of an $n$-element set.

- $\binom{n}{0} = 1$ only one set of size $0$.

- $\binom{n}{n} = 1$ only one set of size $n$.

- $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

# Some properties of Binomial Coefficients (cont)

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$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

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Use Sum Principle

Let $P =$ set of all subsets of $\{$1,2,. . . ,n$\}$

$S_i =$ set of all $i$ subsets of $\{$1,2,. . . ,n$\}$

# Some properties of Binomial Coefficients (cont)

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Use Sum Principle

Let $P$ = set of all subsets of $\{$1,2,. . . ,n$\}$

$S_i$ = set of all $i$ subsets of $\{$1,2,. . . ,n$\}$

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

# Some properties of Binomial Coefficients (cont)

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Use Sum Principle

Let $P$ = set of all subsets of {1,2,. . . ,n}

$S_i$ = set of all $i$ subsets of {1,2,. . . ,n}

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Let $L = L_1 L_2 \ldots L_n$ be a list of size $n$ from $\{0, 1\}$

If $\mathcal{L}$ = set of all such lists $\Rightarrow$ $|\mathcal{L}| = 2^n$

There is a *bijection* (next page) between $\mathcal{L}$ and $P$ so $|P| = 2^n$ and we are done.

Let $P =$ set of all subsets of $\{$1,2,...,n$\}$

Let $L = L_1 L_2 \ldots L_n$ be a list of size $n$ from $\{0, 1\}$

and $\mathcal{L} =$ set of all such lists

Let $P$ = set of all subsets of {1,2,. . . ,n}

Let $L = L_1L_2 \ldots L_n$ be a list of size $n$ from $\{0, 1\}$

and $\mathcal{L}$ = set of all such lists

Define the following function $f : \mathcal{L} \rightarrow P$

If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, , 2, \ldots, n\}$ defined by

$$i \in S \;\Leftrightarrow\; L_i = 1$$

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$f$ is a *bijection* between $\mathcal{L}$ and $P$ (why?) so $|\mathcal{L}| = |P|$

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Ex: $n = 5$

$f(10101) = \{1, 3, 5\},\ f(11101) = \{1, 2, 3, 5\},\ f(00000) = \emptyset.$

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Let $L = L_1 L_2 \ldots L_n$ be a list of size $n$ from $\{0, 1\}$

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If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, , 2, \ldots, n\}$ defined by

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Ex: $n = 5$

$f(10101) = \{1, 3, 5\},\; f(11101) = \{1, 2, 3, 5\},\; f(00000) = \emptyset.$

*Note: $L$ is sometimes called the* incidence vector *or* membership vector *associated with $L$*

**Example:** $n = 4$, $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{ccccc} & \{1\} & \{1,2\}\;\{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\}\;\{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\}\;\{3,4\} & \{1,3,4\} & \\ & \{4\} & & \{2,3,4\} & \end{array} \right\}$$

**Example:** $n = 4$, $S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{c:c:cc:c:c} \{\} & \{1\} & \{1,2\} & \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ & \{2\} & \{1,4\} & \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\} & \{3,4\} & \{1,3,4\} & \\ & \{4\} & & & \{2,3,4\} & \end{array} \right\}$$

$$P = \{S_0, S_1, S_2, S_3, S_4\}$$

**Example:** $n = 4,\ S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{ccccc} & \{1\} & \{1,2\}\ \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\}\ \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\}\ \{3,4\} & \{1,3,4\} & \\ & \{4\} & & \{2,3,4\} & \end{array} \right\}$$

$$|S_0| = \binom{4}{0},\ |S_1| = \binom{4}{1},\ |S_2| = \binom{4}{2},\ |S_3| = \binom{4}{3},\ |S_4| = \binom{4}{4}$$

**Example:** $n = 4,\ S = \{1, 2, 3, 4\}$

$$P = \left\{ \begin{array}{cccccc} & \{1\} & \{1,2\} & \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{\} & \{2\} & \{1,4\} & \{2,3\} & \{1,2,4\} & \\ & \{3\} & \{2,4\} & \{3,4\} & \{1,3,4\} & \\ & \{4\} & & & \{2,3,4\} & \end{array} \right\}$$

$$\begin{aligned} |P| &= |S_0| + |S_1| + |S_2| + |S_3| + |S_4| \\ &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\ &= 2^4 = 16 \end{aligned}$$

# Binomial Coefficients

# Binomial Coefficients

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

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|---|---|---|---|---|---|---|---|
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| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
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Each row begins with a $1$ because $\binom{n}{0} = 1$

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|---|---|---|---|---|---|---|---|
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| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

Each row begins with a $1$ because $\binom{n}{0} = 1$

Each row ends with a $1$ because $\binom{n}{n} = 1$.

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|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

Each row begins with a $1$ because $\binom{n}{0} = 1$

Each row ends with a $1$ because $\binom{n}{n} = 1$.

Each row increases at first and then decreases.
*(will see why in homework)*

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| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
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Second half of each row is the reverse of the first half.

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|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
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Each row begins with a $1$ because $\binom{n}{0} = 1$

Each row ends with a $1$ because $\binom{n}{n} = 1$.

Each row increases at first and then decreases.
*(will see why in homework)*

Second half of each row is the reverse of the first half.

Sum of items on $n^{\text{th}}$ row is $2^n$

# Pascal's Triangle

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Take the table

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

# Pascal's Triangle

Take the table

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 2 | 1 | | | | |
| 3 | 1 | 3 | 3 | 1 | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |

and shift each row slightly so that middle element is in middle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

What is the next row in the table?

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 2 | | 1 | | | | | |
| | | | | 1 | | 3 | | 3 | | 1 | | | | |
| | | | 1 | | 4 | | 6 | | 4 | | 1 | | | |
| | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | |
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | |
| 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 |

What is the next row in the table?

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 2 | | 1 | | | | | |
| | | | | 1 | | 3 | | 3 | | 1 | | | | |
| | | | 1 | | 4 | | 6 | | 4 | | 1 | | | |
| | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | |
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | |
| 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 |

**Pascal relationship**

Each (non-$1$) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

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| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 2 | | 1 | | | | | |
| | | | | 1 | | 3 | | 3 | | 1 | | | | |
| | | | 1 | | 4 | | 6 | | 4 | | 1 | | | |
| | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | |
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | |
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1

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1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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1

1 1

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1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 2 | | 1 | | | | | |
| | | | | 1 | | 3 | | 3 | | 1 | | | | |
| | | | 1 | | 4 | | 6 | | 4 | | 1 | | | |
| | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | | |
| | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 | |
| 1 | | 7 | | 21 | | 35 | | 35 | | 21 | | 7 | | 1 |

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | 1 | | | | | | |
| 1 | 2 | 1 | | | | | |
| 1 | 3 | 3 | 1 | | | | |
| 1 | 4 | 6 | 4 | 1 | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

**Pascal relationship**

Each (non-$1$) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's relationship says that, for $0 < k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

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A purely *algebraic* proof (manipulating formulas) is possible.

In discrete mathematics, though, we prefer to derive intuitive explanations. In this case, that would involve interpreting Pascal's relationship as a statement describing *relationships among sets*.

# A Proof Using the Sum Principle

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From Theorem 1.2 and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

we know $\binom{n}{k}$ is the number of $k$-element subsets of an $n$-element set.

# A Proof Using the Sum Principle

From Theorem 1.2 and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

we know $\binom{n}{k}$ is the number of $k$-element subsets of an $n$-element set.

Therefore, each term (left and right) in

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

represents the number of subsets of a particular size chosen from an appropriately sized set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\boxed{\binom{n}{k}} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Number of $k$-subsets of an $n$-element set.

$$\binom{n}{k} = \boxed{\binom{n-1}{k-1}} + \binom{n-1}{k}$$

Number of $k$-subsets of an $n$-element set.

Number of $(k-1)$-subsets of an $(n-1)$-element set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \boxed{\binom{n-1}{k}}$$

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Number of $k$-subsets of an $n$-element set.

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Try to use sum principle to explain relationship among these three terms.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Number of $k$-subsets of an $n$-element set.

Number of $(k-1)$-subsets of an $(n-1)$-element set.

Number of $k$-subsets of an $(n-1)$-element set.

Try to use sum principle to explain relationship among these three terms.

Example: $n = 5$, $k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

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Consider $S = \{A, B, C, D, E\}$.

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Consider $S = \{A, B, C, D, E\}$.

Set $S_1$ of $2$-subsets of $S$

$$\begin{aligned} S_1 \quad = \quad & \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ & \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}. \end{aligned}$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider $S = \{A, B, C, D, E\}$.

Set $S_1$ of $2$-subsets of $S$ can be partitioned into 2 disjoint parts.

$S_2$ the $2$ subsets that contain $E$ and

$S_3$, the set of $2$-subsets that do not contain $E$.

$$\begin{aligned} S_1 \quad = \quad & \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ & \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}. \end{aligned}$$

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$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

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Consider $S = \{A, B, C, D, E\}$.

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$S_2$ the $2$ subsets that contain $E$ and

$S_3$, the set of $2$-subsets that do not contain $E$.

$$\begin{aligned} S_1 \quad = \quad & \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ & \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}. \end{aligned}$$

$S_2$ is equivalent to choosing 1 item out of $\{A, B, C, D\}$: $|S_2| = \binom{4}{1}$

$S_3$ chooses 2 items out of $\{A, B, C, D\}$: $|S_3| = \binom{4}{2}$

Sum Principle: $\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}$

**Theorem 1.3**

If $n$ and $k$ are integers satisfying $0 < k < n$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

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**Proof:** Apply sum principle.

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**Proof:** Apply sum principle.

Partition set of $k$-element subsets of an $n$-element set into *disjoint union* of two other disjoint sets.

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Partition set of $k$-element subsets of an $n$-element set into *disjoint union* of two other disjoint sets.

Suppose $S = \{x_1, x_2, \ldots, x_n\}$.

## Theorem 1.3

If $n$ and $k$ are integers satisfying $0 < k < n$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Proof:** Apply sum principle.

Partition set of $k$-element subsets of an $n$-element set into *disjoint union* of two other disjoint sets.

Suppose $S = \{x_1, x_2, \ldots, x_n\}$.

Let $S_1$ be $\binom{n}{k}$-element set of all $k$-element subsets.

To apply sum principle, partition $S_1$ into $S_2$ and $S_3$.

Let $S_2$ be set of $k$-element subsets that contain $x_n$.

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$$\Rightarrow \binom{n}{k} = |S_1| = |S_2| + |S_3| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

# Blaise Pascal

- Born 1623; Died 1662
- French Mathematician
- A Founder of Probability Theory
- Inventor of one of the first (the 2nd?) mechanical calculating machines
- Pascal Programming Language named for him

# The Binomial Theorem

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or, in summation notation,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

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&= (xx + yx + xy + yy)(x+y)\\
&= (xx + yx + xy + yy)x + (xx + yx + xy + yy)y\\
&= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.
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Each *monomial* term in the final result is of form $x^{3-i}y^i$ and is the product of – one blue, one red, and one green.

For each color we can choose either an $x$ or a $y$.

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$\#$ of ways of choosing $i$ $y$'s from three colors

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Aternatively, can think of the monomial as *lists* where each item of the list is either $x$ or $y$.

Coefficient of $x^{3-i}y^i$ is
# of lists containing $i$ $y$'s (and $(3-i)$ $x$'s)
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Each monomial is a length-$n$ list of $x$'s and $y$'s.

In each list, the $i$th entry
comes from the $i$th binomial factor.

A list that becomes $x^{n-k}y^k$ *(after applying commutative law)* will have a $y$ in $k$ places and an $x$ in the remaining $(n-k)$ places.

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Number of ways to select $k$ binomial factors from $n$ binomial factors is simply $\binom{n}{k}$.

Therefore, the coefficient of $x^{n-k}y^k$, is $\binom{n}{k}$ .

What is $(x+1)^4$?

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By applying the binomial theorem

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Now set $x = y = 1$. This gives

$$\boxed{2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i}}$$

# Labelling and Trinomial Coefficients

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- Show that if we have $k_1$ labels of one kind, e.g., red, $k_2$ labels of a second kind, e.g., green, and $k_3 = n - k_1 - k_2$ labels of a third kind, e.g., orange, then there are $\frac{n!}{k_1!k_2!k_3!}$ ways to apply these labels to $n$ objects

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There are $\binom{n}{k}$ ways to choose the items with red labels.

The other $n-k$ items will then get the green labels

So this is just $\binom{n}{k}$

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There are $\binom{n}{k_1}$ ways to choose the red items

There are then $\binom{n-k_1}{k_2}$ ways to choose the green items from the remaining $n - k_1$.

The remaining $k_3$ items get labelled orange

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Using the *product principle* the total number of labellings is

$$\begin{aligned}
\binom{n}{k_1}\binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\
&= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}
\end{aligned}$$

When $k_1 + k_2 + k_3 = n$, we call

$$\frac{n!}{k_1!k_2!k_3!}$$

a *trinomial coefficient* and denote it as

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Note that this slightly modifies the notation for binomial coefficients. If we really wanted the notation to be consistent (which we don't) we coud write the binomial coefficient $\binom{n}{k}$ as

$$\binom{n}{k \quad (n-k)}$$

We really just saw that the Trinomial Coefficient

$$\binom{n}{k_1\ k_2\ k_3}$$

is the number of ways to partition a set of size $n$ into three subsets (where order of the subsets does not count) of sizes $k_1$, $k_2$ and $k_3$.

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After opening the parentheses and multiplying,
there will be, in total, $3^4 = 81$ different monomial terms (lists)

Each term, (after rewriting using commutativity),
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$$\binom{n}{k_1 \quad k_2 \quad k_3}$$