COMP170
Discrete Mathematical Tools for Computer Science

Inverses and GCDs

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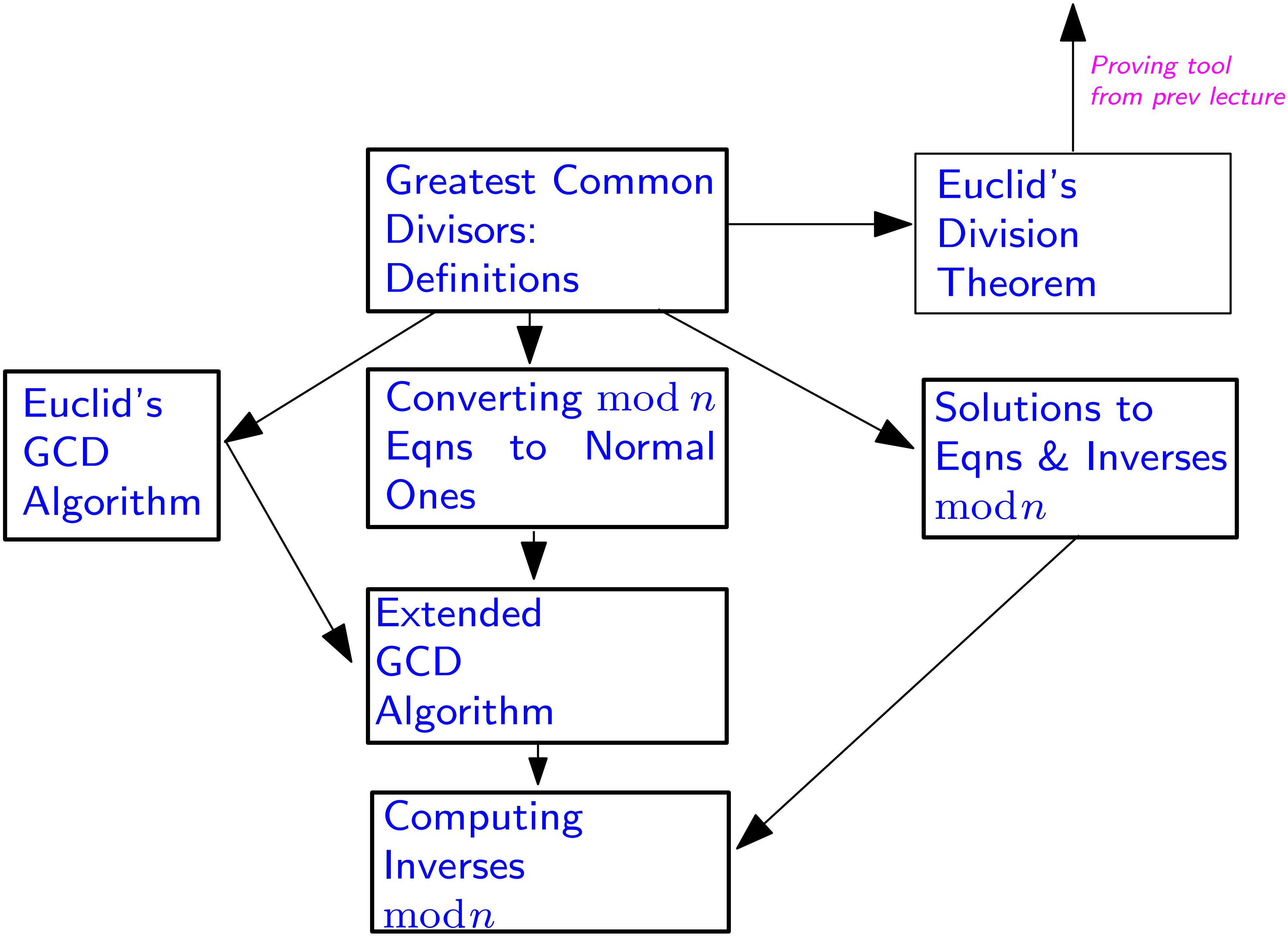
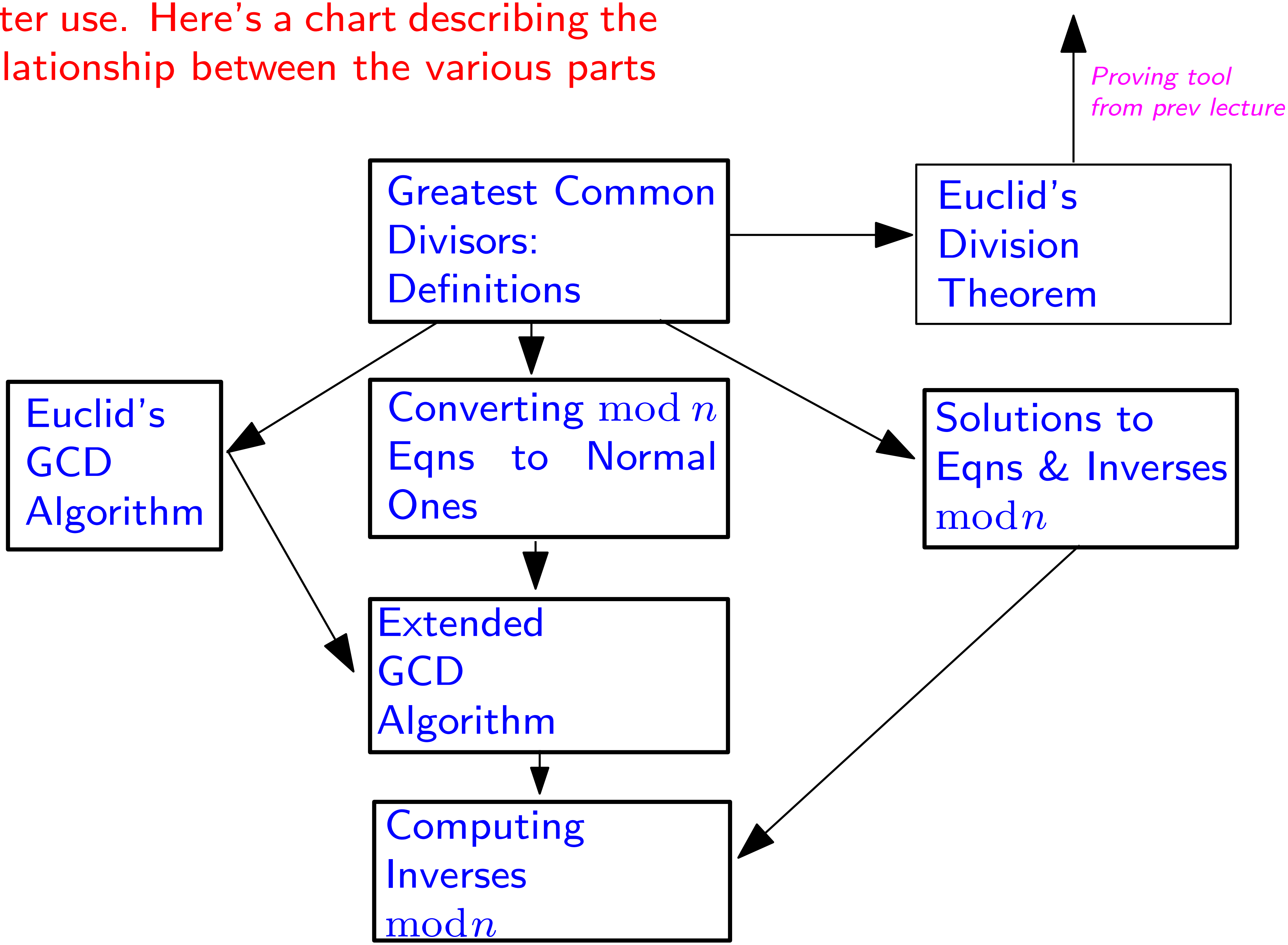
Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 2.2, pp. 56-69

2.2 Inverses and Greatest Common Divisors

- Greatest Common Divisors
- Euclid's Division Theorem
- Euclid's GCD Algorithm
- Solutions to Equations and Inverses mod n
- Converting Modular Equations to Normal Equations
- Extended GCD Agorithm
- Computing Inverses


Proving tool
from prev lecture
Greatest Common Divisors: Definitions
Euclid's Division Theorem
Euclid's GCD Algorithm
Converting mod n Eqns to Normal Ones
Solutions to Eqns & Inverses mod n
Extended GCD Algorithm
Computing Inverses mod n

This lecture develops lots of tools for later use. Here's a chart describing the relationship between the various parts



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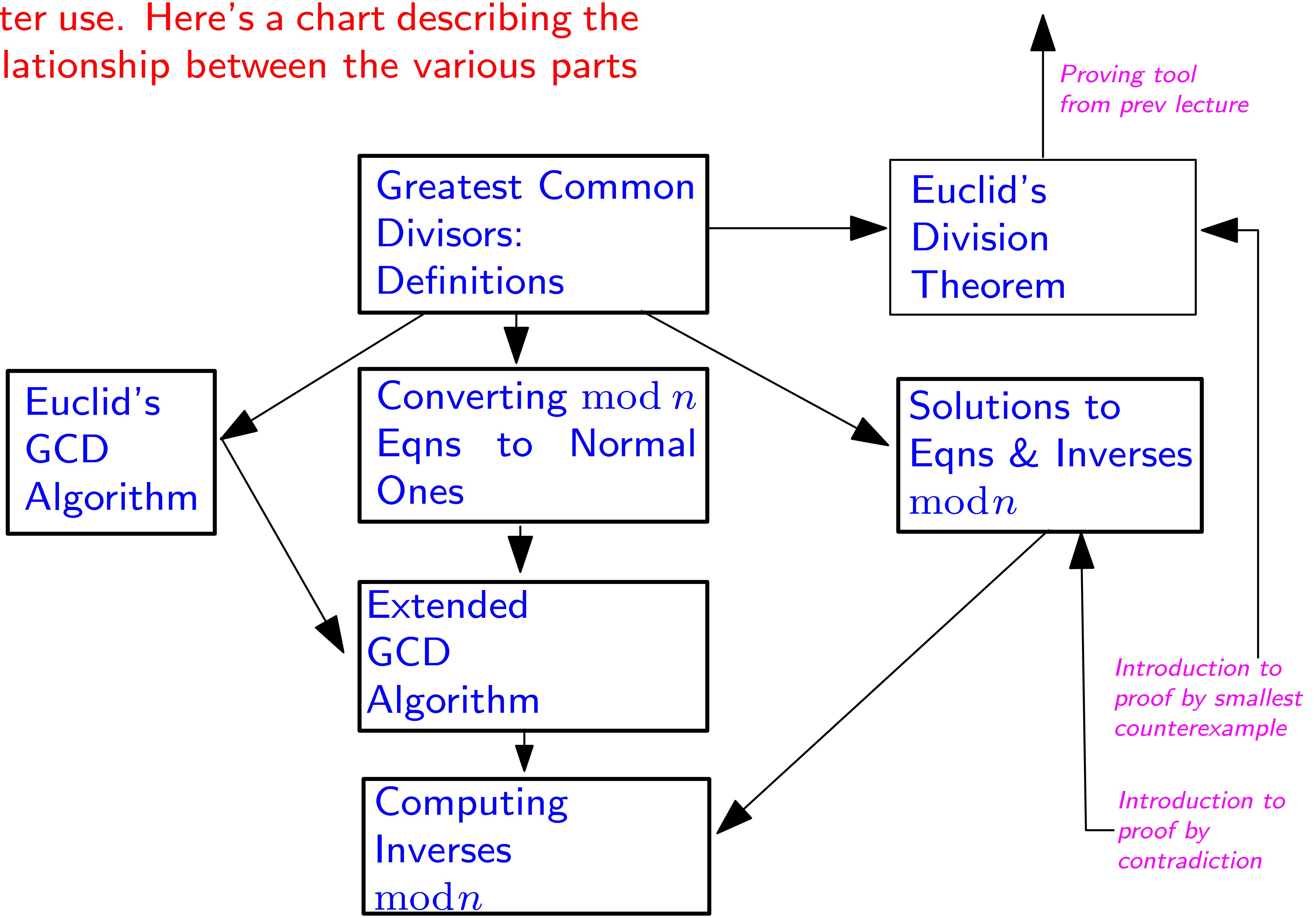

Definition:

- Positive integer m is a divisor of integer n
  if $n = mq$ for some integer q

- if m is a divisor of n we write $m|n$.
  (say) "m divides n"

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Examples:

- $1|30, \quad 5|30, \quad 5|35, \quad 5 \not| 31$

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  p is a common divisor of m and n

- $gcd(m, n)$ denotes the greatest common divisor of m and n.
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Examples:

- $\{1,2,3,6\}$ are *all* of the common divisors of 24 and 30.
- $gcd(24,30) = 6$

Definition:

- Positive integer $p > 1$ is prime if its only divisors are 1 and itself . If p is not prime, it is composite.
- m and n are relatively prime if they have no common divisor other than 1, i.e., $gcd(m, n) = 1$.

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Examples:

- $2, 3, 5, 7, 11$ are prime.
  $33 = 3 \cdot 11$ is composite
- $gcd(77, 34) = 1$, so 77 and 34 are relatively prime
  $gcd(77, 33) = 11$, so 77 and 33 are *not* relatively prime

The main goal of this lecture is to prove the Theorem and Corollary below and also to show how to calculate the corresponding x and y and multiplicative inverses.

In order to get to that point we will have to develop a lot of auxillary machinery. We will see in the next lecture that this auxillary machinery will be useful for implementing RSA public-key cryptography.

Theorem 2.15: Two positive integers j, k are relatively prime, i.e., $gcd(j,k) = 1$, if and only if there are integers x and y such that $jx + ky = 1$.

Corollary 2.16: For any positive integer n, an element $a \in Z_n$ has a multiplicative inverse if and only if $gcd(a,n) = 1$.

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Recall that in the last section we learnt about Euclid's division theorem and proved facts based upon it. In this subsection, we prove the correctness of Euclid's division theorem

Euclid's Division Theorem

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Note 2: This is **restricted** *because we assume that* m *is nonnegative. Book problem shows how to extend this to negative* m *as well.*

Theorem 2.12 (Euclid's Division Theorem, Restricted Version): Let n be a positive integer. Then for every nonnegative integer m, there exist unique integers q, r such that $m = nq + r$ and $0 \leq r < n$.

Proof:

(i) First, show that, for each m, there is at least one pair of integers q, r satisfying

(*) $m = qn + r$ with $0 \leq r < n$

(ii) Then show that this pair q, r is *unique*

Assume, (proof by contradiction), that there is a nonnegative integer m for which no such q and r exist.

(*) $m = qn + r$ with $0 \leq r < n$

(i) Assume (proof by contradiction) that there is a nonnegative integer m for which no q, r satisfying (*) exists

Choose the **smallest** m for which q, r satisfying (*) does not exist.

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If $m < n$, $\Rightarrow m = 0 \cdot n + m$ so

(*) is satisfied with $q = 0$, $r = m$

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$$0 = n(q - q^*) + r - r^* \quad \Rightarrow \quad n(q - q^*) = r^* - r.$$

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Therefore, $q = q^*$ and $r = r^*$,
proving that q and r satisfying (*) are unique.

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This method is closely related to a proof method called *proof by induction* *(to be seen later)*

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$1 \quad 2 \quad 3 \quad 4 \quad 5$

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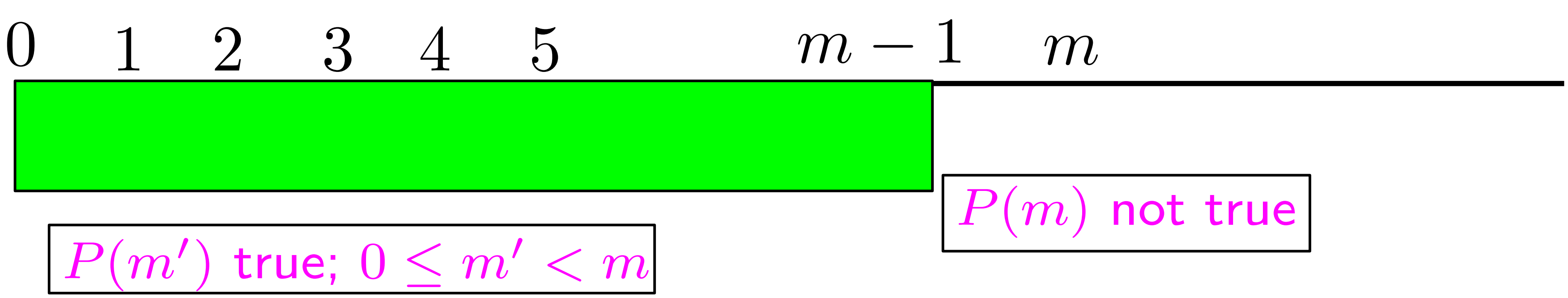

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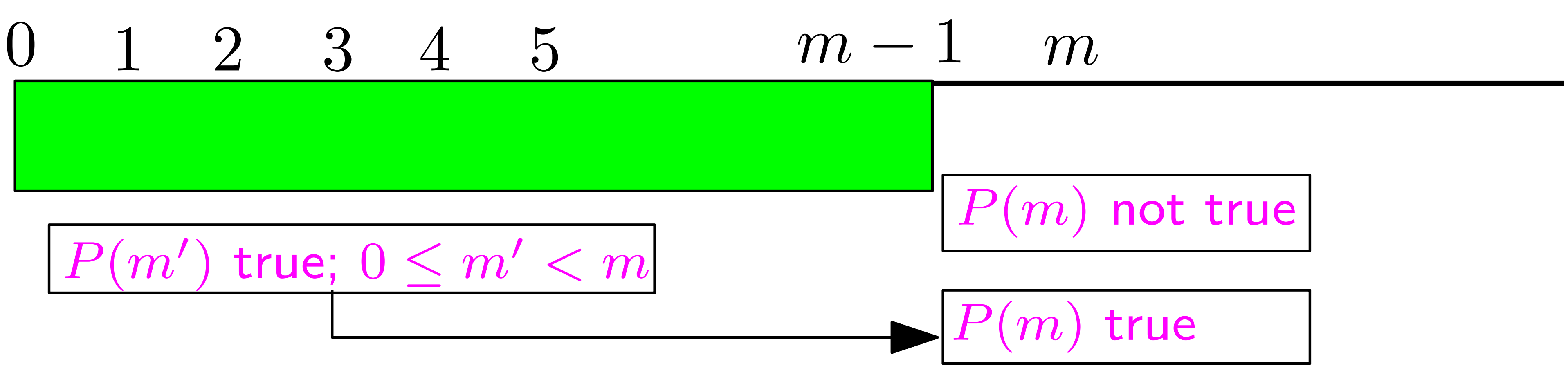

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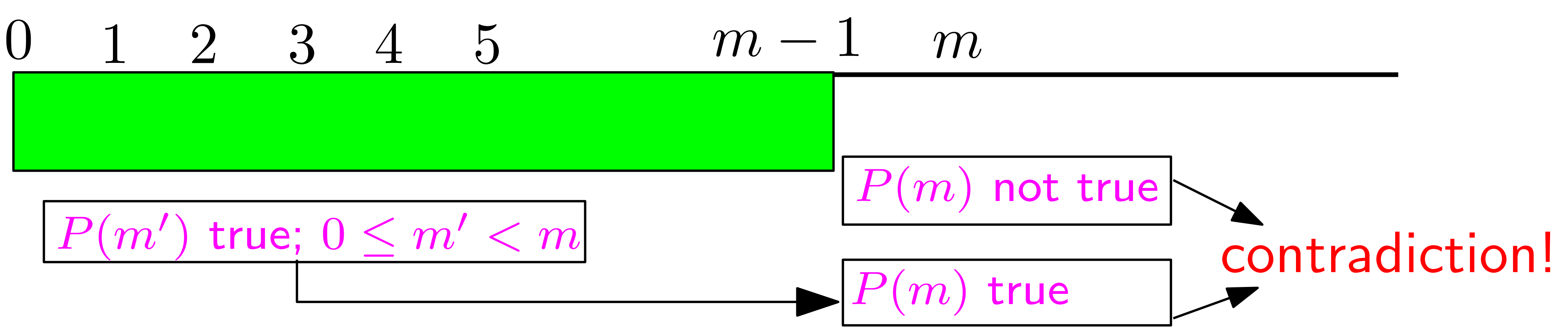

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Proof:

(i) $r = 0$:

Then $gcd(r,j) = j$ since *every number* divides 0.

But $k = jq$ so $gcd(k,j) = j = gcd(j,r)$ and we are done.

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$\Rightarrow k = i_1 d$ and $j = i_2 d$ for some nonnegative i_1, i_2.

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$$\Rightarrow \quad d = gcd(j, k) \text{ iff } d = gcd(r, j)$$

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Note that r is nonegative, and every time line 4 is executed, $r < j$, so the value of r **decreases**. Therefore, in a finite number of steps, process reaches $j = 0$ and **terminates**

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Example: Find $gcd(102, 70)$

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k	$=$	$j(q)$	$+$	r

k	j	r	q
102	70		

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k	$=$	$j(q)$	$+$	r
102	$=$	$70(1)$	$+$	32

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102	70	32	1

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$$k = j(q) + r$$

$$102 = 70(1) + 32$$

k	j	r	q
102	70	32	1
70	32		

1) $GCD(k, j)$ where $0 \le j < k$

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3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Answer is $GCD(j, r)$

Example: Find $gcd(102, 70)$

$$
\begin{array}{ccccccc}
k & = & j(q) & + & r \\
102 & = & 70(1) & + & 32 \\
70 & = & 32(2) & + & 6
\end{array}
$$

k	j	r	q
102	70	32	1
70	32		

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\end{array}
$$

k	j	r	q
102	70	32	1
70	32	6	2

1) $GCD(k, j)$ where $0 \le j < k$

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32	6		

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k	j	r	q
102	70	32	1
70	32	6	2
32	6	2	5
6	2	0	3
2	0		

$gcd(102, 70) = 2$

1) $GCD(k, j)$ where $0 \leq j < k$

2) If $j = 0$ answer is k

3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Answer is $GCD(j, r)$

Example: Find $gcd(252, 189)$

1) $GCD(k, j)$ where $0 \leq j < k$

2) If $j = 0$ answer is k

3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Answer is $GCD(j, r)$

Example: Find $gcd(252, 189)$

$k \quad = \quad j(q) \quad + \quad r$

k	j	r	q
252	189		

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Example: Find $gcd(252, 189)$

$$k = j(q) + r$$

$$252 = 189(1) + 63$$

k	j	r	q
252	189		

1) $GCD(k,j)$ where $0 \le j < k$

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Example: Find $gcd(252, 189)$

$$k \quad = \quad j(q) \quad + \quad r$$

$$252 \quad = \quad 189(1) \quad + \quad 63$$

k	j	r	q
252	189	63	1

1) $GCD(k,j)$ where $0 \leq j < k$

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\begin{array}{ccccccc}
k & = & j(q) & + & r \\
252 & = & 189(1) & + & 63
\end{array}
$$

k	j	r	q
252	189	63	1
189	63		

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3) Else

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Example: Find $gcd(252, 189)$

$$\begin{array}{ccccc} k & = & j(q) & + & r \\ 252 & = & 189(1) & + & 63 \\ 189 & = & 63(3) & + & 0 \end{array}$$

k	j	r	q
252	189	63	1
189	63		

1) $GCD(k,j)$ where $0 \leq j < k$

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Example: Find $gcd(252,189)$

$$\begin{array}{ccccc} k & = & j(q) & + & r \\ 252 & = & 189(1) & + & 63 \\ 189 & = & 63(3) & + & 0 \end{array}$$

k	j	r	q
252	189	63	1
189	63	0	3

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3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Answer is $GCD(j, r)$

Example: Find $gcd(252, 189)$

k	$=$	$j(q)$	$+$	r
252	$=$	$189(1)$	$+$	63
189	$=$	$63(3)$	$+$	0

k	j	r	q
252	189	63	1
189	63	0	3
63	0		

1) $GCD(k, j)$ where $0 \le j < k$

2) If $j = 0$ answer is k

3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Answer is $GCD(j, r)$

Example: Find $gcd(252, 189)$

k	$=$	$j(q)$	$+$	r
252	$=$	$189(1)$	$+$	63
189	$=$	$63(3)$	$+$	0

k	j	r	q
252	189	63	1
189	63	0	3
63	0		

$$gcd(252, 189) = 63$$

Euclid of Alexandria

ca. 325BC – 265BC

If he existed, most probably a Greek Mathematician who taught at Alexandria (Egypt)

Most famous for his *Elements*, considered to be one of history's most successful textbooks.

The *Elements* contains 13 books. Book 7 is on number theory and contains the GCD algorithm

See *http://en.wikipedia.org/wiki/Euclid and*
http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Euclid.html

2.2 Inverses and Greatest Common Divisors

- Greatest Common Divisors
- Euclid's Division Theorem
- Euclid's GCD Algorithm
- Solutions to Equations and Inverses mod n
- Converting Modular Equations to Normal Equations
- Extended GCD Agorithm
- Computing Inverses

Solutions to Equations and Inverses $\mathrm{mod}\, n$

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- Given a, to decide whether $a \cdot_n x = b$ has a *unique solution* in Z_n, it helps to know whether a has a **multiplicative inverse** in Z_n.

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- A **multiplicative inverse** is a' such that $a' \cdot_n a = 1$.

Solutions to Equations and Inverses $\bmod n$

- Given a, to decide whether $a \cdot_n x = b$ has a *unique solution* in Z_n, it helps to know whether a has a **multiplicative inverse** in Z_n.

- A **multiplicative inverse** is a' such that $a' \cdot_n a = 1$.

- Example: in Z_9
  $2 \cdot_9 5 = 1$ so the inverse of 2 is 5
  3 does not have an inverse because
  $3 \cdot_9 x = 1$ does not have a solution.
  This can be verified by checking the 9 possible values for x.

Lemma 2.5: If a has multiplicative inverse $a' \in Z_n$, then for any $b \in Z_n$, the equation $a \cdot_n x = b$ has the solution $x = a' \cdot_n b$, and this solution is unique.

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i) $a' \cdot_n (a \cdot_n x) = a' \cdot_n b$ Multiply both sides by a'

ii) $(a' \cdot_n a) \cdot_n x = a' \cdot_n b$ By the associative law

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Since this is valid for *any* x that satisfies (*), we conclude that *only* $x = a' \cdot_n b$ could satisfy (*).

To see that $x = a' \cdot_n b$ satisfies (*) just multiply to find that

$$\boxed{a \cdot_n x = a \cdot_n (a' \cdot_n b) = b}$$

Theorem 2.7: If element $a \in Z_n$ has a multiplicative inverse, then the inverse is unique

Theorem 2.7: If element $a \in Z_n$ has a multiplicative inverse, then the inverse is unique

Proof:

Let a have some inverse $a' \in Z_n$.
Now apply the previous lemma with $b = 1$. It says that

If $a \cdot_n x = 1 \quad \Rightarrow \quad x = a' \cdot_n 1 = a'$.

This can be read as saying that,
"if a' is an inverse of a in Z_n
and x is also an inverse of a in Z_n
then $x = a'$",
so the inverse is unique.

For each $n = 5, 6, 7, 8,$ and 9, determine which nonzero elements $a \in Z_n$ have mutiplicative inverses and, if they do, what the inverses are.

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Z_5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

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Z_5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\longrightarrow$

a	1	2	3	4
a'	1	3	2	4

For each $n = 5, 6, 7, 8,$ and 9, determine which nonzero elements $a \in Z_n$ have mutiplicative inverses and, if they do, what the inverses are.

Z_5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\longrightarrow$

a	1	2	3	4
a'	1	3	2	4

Z_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

For each $n = 5, 6, 7, 8,$ and 9, determine which nonzero elements $a \in Z_n$ have mutiplicative inverses and, if they do, what the inverses are.

Z_5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\longrightarrow$

a	1	2	3	4
a'	1	3	2	4

Z_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

$\longrightarrow$

a	1	2	3	4	5
a'	1	X	X	X	5

X denotes no inverse

Z_5:

a	1	2	3	4
a'	1	3	2	4

Z_6:

a	1	2	3	4	5
a'	1	X	X	X	5

Z_7:

a	1	2	3	4	5	6
a'	1	4	5	2	3	6

Z_8:

a	1	2	3	4	5	6	7
a'	1	X	3	X	5	X	7

Z_9:

a	1	2	3	4	5	6	7	8
a'	1	5	X	7	2	X	4	8

- We've just seen how to find inverses (or the lack of them) by scanning through the entire multiplication table. Is there a more efficient way?

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- We will now see a way of proving that an inverse does not exist,

- We will then develop an efficient way of calculating inverses when they do exist.

Corollary 2.6: Suppose there is a $b \in Z_n$ such that $a \cdot_n x = b$ does not have a solution. Then a does not have a multiplicative inverse in Z_n.

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Proof (by contradiction!):

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Proof (by contradiction!):

i) Assume (*) $a \cdot_n x = b$ does not have a solution.

Corollary 2.6: Suppose there is a $b \in Z_n$ such that $a \cdot_n x = b$ does not have a solution. Then a does not have a multiplicative inverse in Z_n.

Proof (by contradiction!):

i) Assume (*) $a \cdot_n x = b$ does not have a solution.

ii) Suppose further that

(**) a does have a multiplicative inverse $a' \in Z_n$.

Corollary 2.6: Suppose there is a $b \in Z_n$ such that $a \cdot_n x = b$ does not have a solution. Then a does not have a multiplicative inverse in Z_n.

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iii) Then by Lemma 2.5,

$x = a' \cdot_n b$ is a solution to $a \cdot_n x = b$.

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iii) Then by Lemma 2.5,

$x = a' \cdot_n b$ is a solution to $a \cdot_n x = b$.

iv) This contradicts the hypothesis (*) that

$a \cdot_n x = b$ does not have a solution.

One of the assumptions —

(*) $a \cdot_n x = b$ does not have a solution.

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Assuming both (*) and (**) led to a contradiction. It must therefore be the case that, if (*) is true, then (**) can not be true.

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Assuming both (*) and (**) led to a contradiction. It must therefore be the case that, if (*) is true, then (**) can not be true.

Thus, if $a \cdot_n x = b$ does not have a solution, then a does not have a multiplicative inverse $a' \in Z_n$.

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(*) $a \cdot_n x = b$ does not have a solution.

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The only other assumption we made was

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Assuming both (*) and (**) led to a contradiction.
It must therefore be the case that, if (*) is true, then (**) can not be true.

Thus, if $a \cdot_n x = b$ does not have a solution, then a does not have a multiplicative inverse $a' \in Z_n$.

$\boxed{\text{\textit{A classical example of} \textbf{proof by contradiction}.}}$

Principle 2.1 (Proof by Contradiction):
If, by assuming a statement we want to prove is false,
we are led to a contradiction,
then the statement we are trying to prove
must be true.

Corollary 2.6: Suppose there is a $b \in Z_n$ such that $a \cdot_n x = b$ does not have a solution. Then a does not have a multiplicative inverse in Z_n.

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Now consider Z_6. The equation $2 \cdot_6 x = 3$ can not have a solution because $2x$ will always be even so $2x \bmod 6$ will always be even.

Corollary 2.6: Suppose there is a $b \in Z_n$ such that $a \cdot_n x = b$ does not have a solution. Then a does not have a multiplicative inverse in Z_n.

Now consider Z_6. The equation $2 \cdot_6 x = 3$ can not have a solution because $2x$ will always be even so $2x \bmod 6$ will always be even.

The corollary therefore tells us that 2 does not have a multiplicative inverse in Z_6. We originally discovered this by checking all of the possibilities, but now we don't have to.

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The corollary therefore tells us that 2 does not have a multiplicative inverse in Z_6. We originally discovered this by checking all of the possibilities, but now we don't have to.

Z_6:

a	1	2	3	4	5
a'	1	X	X	X	5

Z_5:

a	1	2	3	4
a'	1	3	2	4

Z_6:

a	1	2	3	4	5
a'	1	X	X	X	5

Z_7:

a	1	2	3	4	5	6
a'	1	4	5	2	3	6

Z_8:

a	1	2	3	4	5	6	7
a'	1	X	3	X	5	X	7

Z_9:

a	1	2	3	4	5	6	7	8
a'	1	5	X	7	2	X	4	8

Z_5:

a	1	2	3	4
a'	1	3	2	4

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a	1	2	3	4	5
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Note that $5, 7$ are prime and all of the elements in Z_5, Z_7 have inverses.

For the non-prime $n \in 6, 8, 9$ the elements in Z_n that have inverses are exactly those elements that are relatively prime to n.

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For the non-prime $n \in 6, 8, 9$ the elements in Z_n that have inverses are exactly those elements that are relatively prime to n.

Nice pattern!
Is this aways true?
Yes!

2.2 Inverses and Greatest Common Divisors

- Greatest Common Divisors
- Euclid's Division Theorem
- Euclid's GCD Algorithm
- Solutions to Equations and Inverses mod n
- Converting Modular Equations to Normal Equations
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Converting Modular Equations to Normal Equations

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So, $\boxed{\text{if } a \cdot_n x = 1}$ $\Rightarrow$ we can write $ax + (-q)n = 1$ in form (*).

$\boxed{\text{If (*) for some } y}$ then $ax = (-y)n + 1$ so
by definition of mod, $ax \bmod n = 1 \Rightarrow a \cdot_n x = 1$.

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This can be restated as

Theorem 2.9: A number a has a multiplicative inverse in Z_n if and only if there are integers x, y such that $ax + ny = 1$.

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Multiple appl of Lemma 2.3

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$$\begin{aligned} 1 &= ax + ny \\ &= skx + qky \\ &= k(sx + qy). \end{aligned}$$

- But then k is a divisor of 1.
  Since *only* divisors of 1 are $1, -1 \Rightarrow k = 1$ or -1.

We just saw that, if $ax + ny = 1$ for integers x, y then the only common divisors of a, n are $1, -1$.

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Lemma 2.11: Given a and n, if there exist integers x and y such that $ax + ny = 1$, then $\gcd(a, n) = 1$ — that is, a and n are relatively prime.

The Story So Far ....

- **Theorem 2.9**: a has a multiplicative inverse in Z_n if and only if there are integers x, y such that $ax + ny = 1$.
- **Corollary 2.10**: If $a \in Z_n$ and x, y are integers s.t. $ax + ny = 1$, then the solution to $a \cdot_n \overline{x} = 1$ is $\overline{x} = x \bmod n$.
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We will be able to find the x, y using the
Extended GCD Algorithm.
As a side effect, it will also prove that, if $gcd(a, n) = 1$,
there aways exists x, y s.t. $ax + ny = 1$.
Combining with Lemma 2.11 this will show that
$gcd(a, n) = 1$ iff there exists x, y s.t. $ax + ny = 1$

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and x', y' s.t. $gcd(r,j) = rx' + jy'$.

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so $\boxed{gcd(k,j)} = gcd(r,j) = jx + ky$
where $\boxed{y = x'}$ and $\boxed{x = y' - qx'}$.

1) $GCD(k,j)$ where $0 \leq j < k$

Returns $gcd(k,j)$ and

x, y s.t. $jx + ky = gcd(k,j)$

2) If $k = jq$, return $gcd(k,j) = j$, $x = 1$, $y = 0$

3) Else

4) Write $k = jq + r$ where $r = k \bmod j$

5) Run $GCD(r,j)$ to find $gcd(r,j)$

and x', y' s.t. $gcd(r,j) = rx' + jy'$

6) Return $gcd(r,j)$, $x = y' - qx'$ and $y = x'$

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Can implement this in two different ways

(i) Recursively (if you know about recursion already) or

(ii) Iteratively. First run the standard GCD algorithm "top-down", calculating all of the k, j, r, q.

Then run the extended part "bottom-up", calculating the values of the x, y.

We will now see an example of the iterative version.
We start at $i = 0$ with our original j, k and increase i each time we descend. This means that, given $j[i], k[i]$, we calculate

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$q[i], r[i]$ such that $k[i] = j[i]q[i] + r[i]$ where
$$r[i] = k[i] \bmod j[i]$$

and also $x[i], y[i]$ such that
$$j[i]x[i] + k[i]y[i] = gcd(k[i], j[i])$$

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Note that, in this notation
$y[i-1] = x[i]$ and $x[i-1] = y[i] - q[i-1]x[i]$

Recall that (**) $y[i-1] = x[i]$ and (*) $x[i-1] = y[i] - q[i-1]x[i]$

and we want $j[i]x[i] + k[i]y[i] = gcd(k[i], j[i])$

Example: $k = 24, j = 14$

Recall that (**) $y[i-1] = x[i]$ and (*) $x[i-1] = y[i] - q[i-1]x[i]$

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Example: $k = 24, j = 14$

i	$k[i]$	$=$	$j[i]q[i]$	$+$	$r[i]$	$k[i]$	$j[i]$	$r[i]$	$q[i]$
0	24	$=$	14(1)	$+$	10	24	14	10	1
1	14	$=$	10(1)	$+$	4	14	10	4	1
2	10	$=$	4(2)	$+$	2	10	4	2	2
3	4	$=$	2(2)	$+$	0	4	2	0	2

1) First run the regular GCD algorithm: get $gcd(24, 14) = 2$

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1	14	$=$	10(1)	$+$	4	14	10	4	1			
2	10	$=$	4(2)	$+$	2	10	4	2	2		1	-2
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0	24	=	14(1)	+	10	24	14	10	1	3	−5
1	14	=	10(1)	+	4	14	10	4	1	−2	3
2	10	=	4(2)	+	2	10	4	2	2	1	−2
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3) Continue bottom-up, calculating the $x[i], y[i]$ from (*) and (**)

4) We are done! Note that $24(3) + 14(-5) = 2 = gcd(24, 14)$.

Euclid's extended GCD algorithm then gives

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Theorem 2.14: Given two integers j, k, Euclid's extended GCD algorithm computes $gcd(j, k)$ and two integers x, y such that $gcd(j, k) = jx + ky$.

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Proof: *"if"* comes from Lemma 2.11
"only if" comes from Theorem 2.14

Recall

Lemma 2.8 The equation $a \cdot_n x = 1$ has a solution in Z_n iff there exist integers x, y such that $ax + ny = 1$.

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Corollary 2.17: For any prime p, every nonzero $a \in Z_p$ has a mutiplicative inverse.

Z_5:

a	1	2	3	4
a'	1	3	2	4

Z_6:

a	1	2	3	4	5
a'	1	X	X	X	5

Z_7:

a	1	2	3	4	5	6
a'	1	4	5	2	3	6

Z_8:

a	1	2	3	4	5	6	7
a'	1	X	3	X	5	X	7

Z_9:

a	1	2	3	4	5	6	7	8
a'	1	5	X	7	2	X	4	8

We noted that $5, 7$ are prime and all of the elements in Z_5, Z_7 have inverses.

For the non-prime $n \in 6, 8, 9$ the elements in Z_9 that have inverses are exactly those elements that are relatively prime to n.

Nice pattern!!
We now know that it's aways true

2.2 Inverses and Greatest Common Divisors

- Greatest Common Divisors
- Euclid's Division Theorem
- Euclid's GCD Algorithm
- Solutions to Equations and Inverses mod n
- Converting Modular Equations to Normal Equations
- Extended GCD Agorithm
- Computing Inverses

Computing Inverses

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Corollary 2.18: If an element $a \in Z_n$ has an inverse, we can compute it by running Euclid's extended GCD algorithm to determine integers x, y so that $ax + ny = 1$. The inverse of $a \in Z_n$ is $x \bmod n$.

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Example: Given $a = 27$, $n = 58$ we can use the Extended GCD algorithm to find that

$$27(-15) + 58(7) = 1.$$

Thus the multiplicative inverse of 27 in Z_{58} is

$$-15 \bmod 58 = 43.$$

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Reality check: $27 \cdot 43 = 1161 = 20 \cdot 58 + 1$

We now know how to *efficiently* find inverses $\bmod n$.

We are almost ready to learn the RSA public-key algorithm.