COMP170
Discrete Mathematical Tools for Computer Science

Random Variables

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 5.4, pp. 249-262

Random Variables

- What Are Random Variables?
- Binomial Probabilities
- Expected Values
- Expected Values of Sums and Numerical Multiples
- Indicator Random Variables
- The Number of Trials until a First Success

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Rolling two dice

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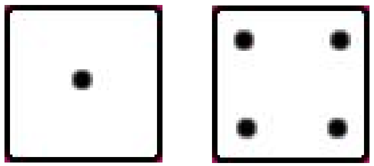
$X\Big($  $\Big) = 5$

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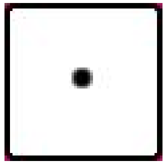
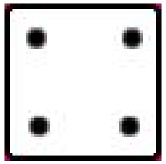
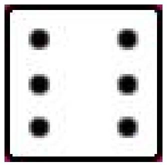
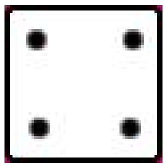
X ( ) $= 5$

X ( ) $= 10$

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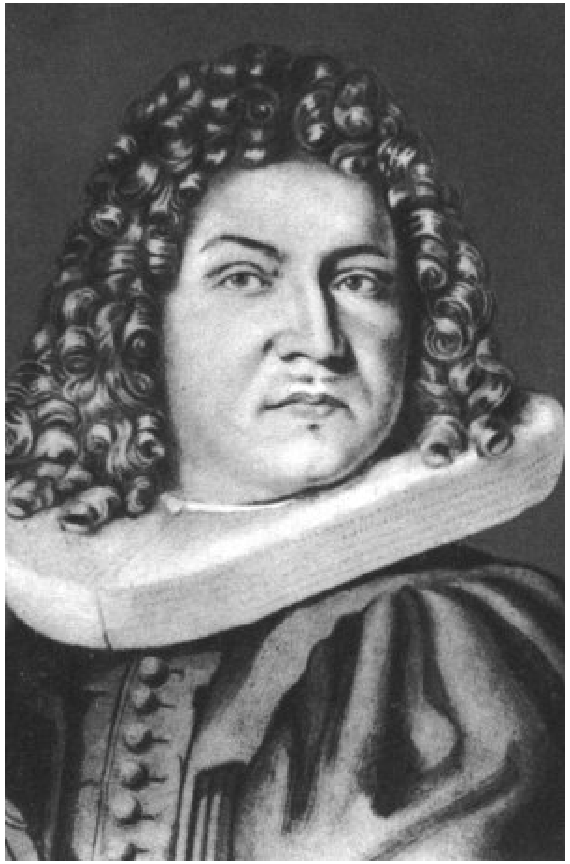
It is called a Bernoulli trial or Benoulli Random Variable with success probability p

Jakob Bernoulli

b. 1654, d. 1705



Swiss Mathematician and Scientist. Famous for his work on probability theory (where *Bernoulli trials* come from) and calculus.

He often collaborated with his brother Johann Bernoulli, another famous mathematician

For more information, please see
http://en.wikipedia.org/wiki/James_Bernoulli

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Note that this is the sum of Bernoulli Random Variables

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Theorem 5.8
The probability of having exactly k successes in a sequence of n independent trials with two outcomes and probability p of success on each trial is given by

$$P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

The Binomial Random Variable X (with parameters n, p) takes on integer values with probability distribution:

$$P(X=k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{Otherwise} \end{cases}$$

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Reality Check: This *is* a probability distribution since

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \Big(p + [1-p]\Big)^n = 1^n = 1$$

Example:

A student takes a ten-question objective test.

Suppose that a student who knows 80% of the course material has probability $.8$ of success on any question, independent of how (s)he did on any other problem.

What is the probability that (s)he earns a grade of 80 or better (out of 100)?

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$$\binom{10}{8}(.8)^8(.2)^2 + \binom{10}{9}(.8)^9(.2)^1 + \binom{10}{10}(.8)^{10}(.2)^0 \approx .678.$$

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$$\frac{0+1+1+1+2+2+2+3}{8} = 1.5.$$

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Therefore, it is reasonable to play this game
as long as the cost is at most $1.50.

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Another Example

(a) Throwing a fair die: Let X be the number of spots shown. Since each outcome is equally likely

$$E(X) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

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i	2	3	4	5	6	7	8	9	10	11	12
$Pr(Y = i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

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$$E(Y) = \sum_{i=2}^{12} i Pr(Y=i) = 7$$

Returning to the biased coin tossing

outcomes 's'	TTT,	TTH,	THT,	HTT,	THH,	HTH,	HHT,	HHH
$X(s)$	3	2	2	2	1	1	1	0
$P(s)$	$\frac{8}{27}$	$\frac{4}{27}$	$\frac{4}{27}$	$\frac{4}{27}$	$\frac{2}{27}$	$\frac{2}{27}$	$\frac{2}{27}$	$\frac{1}{27}$

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Notice that if, instead of using the formula $\sum_{i=1}^{k} x_i P(X = x_i)$ on the previous page, we instead summed up $X(s)$ over all outcomes s, weighted by $P(s)$, we get the same answer!

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$$= 3 \cdot 1 \cdot \frac{8}{27} + 2 \cdot 3 \cdot \frac{4}{27} + 1 \cdot 3 \cdot \frac{2}{27} + 0 \cdot 1 \cdot \frac{1}{27} = 2$$

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Lemma 5.9

If a random variable X is defined on a (finite) sample space S, then its expected value is given by

$$E(X) = \sum_{s : s \in S} X(s)P(s).$$

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Take items in sample space, group them together into events F_i, and rework sum into definition of expectation:

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This gives us $x_i P(X = x_i)$, which leads us to the definition of the expected value of X.

Random Variables

- What Are Random Variables?
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- Expected Values of Sums and Numerical Multiples
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We already saw that 7 is the correct answer.

We now see that this formula will *always* be true.

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Flip a fair coin and observe whether it comes up H or T.
Define the two random variables X, Y by

$$X = \begin{cases} 1 & \text{if H} \\ 0 & \text{if T} \end{cases} \qquad Y = \begin{cases} 1 & \text{if T} \\ 0 & \text{if H} \end{cases}$$

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$E(X+Y) = E(X)+E(Y)$ is *always true.* $E(X \cdot Y) = E(X) \cdot E(Y)$ *is* sometimes true and sometimes false *(more later).*

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They can tremendously simplify calculations of expected values

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We could evaluate this but, there is an easier way.

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By Theorem 5.10, expected number of successes in n trials is

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Because of linearity of expectation,
there is no need for events to be independent.

Example

Recall the problem of the ten-question exam in which the student has probability $.9$ of getting each question correct. We used the random variables

$$X_i = \begin{cases} 1 & \text{if question } i \text{ answered correctly} \\ 0 & \text{if question } i \text{ answered incorrectly} \end{cases}.$$

The fact that $X = X_1 + X_2 + \cdots X_9 + X_{10}$ and linearity of expectation, let us easily calculate

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_{10}) = 10 \cdot (.9) = 9.$$

These X_i are indicator random variables!

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e.g., when $n = 2$: either both students or neither student get own backpacks returned so $X_1 = X_2$.

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$$\Rightarrow \qquad E(X_i) = \frac{(n-1)!}{n!} = 1/n$$

We just showed that $E(X_i) = \frac{1}{n}$.

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This means that
$E($number of students who get their own backpack back $) = 1$

Note that this is independent of n.

Random Variables

- What Are Random Variables?
- Binomial Probabilities
- Expected Values
- Expected Values of Sums and Numerical Multiples
- Indicator Random Variables
- The Number of Trials until a First Success

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How many times should we expect to have to roll two dice until we see a sum of 7? Why?

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Does that mean we should expect to have to roll the dice six times before we see 7?

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The natural probability weight we would assign to F^iS would be $(1-p)^i p$.

Does this make sense?

$$P(S) = p, \quad P(FS) = (1-p)p, \ \ldots, \ P(F^iS) = (1-p)^i p, \ldots .$$

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$$\sum_{i=0}^{\infty} (1-p)^i p = p \sum_{i=0}^{\infty} (1-p)^i = p \frac{1}{1-(1-p)} = \frac{p}{p} = 1.$$

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Probability distribution $P(F^iS) = (1-p)^i p$ is called a **geometric distribution** because of the geometric series we used in proving that probabilities sum to 1.

Theorem 5.13

Suppose we have a sequence of trials in which each trial has two outcomes, success and failure, and in which the probability of success at each step is some $p > 0$. Then the expected number of trials until the first success is $1/p$.

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Probability that first success is on Trial i is $(1-p)^{i-1}p$, because for this to happen, there must be $i - 1$ failures followed by 1 success.

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if $|x| < 1$,

$$\sum_{j=0}^{\infty} j x^j = \frac{x}{(1-x)^2}$$

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$$\frac{1}{\frac{1}{2}} = 2.$$

Example

When throwing two fair dice, the probability of seeing a 7 is $\frac{1}{6}$. So, applying Theorem 5.13, we see that expected number of times we need to roll two dice until we get a 7 is

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When throwing two fair dice, the probability of seeing a 6 is $\frac{5}{36}$. So, applying Theorem 5.13, we see that expected number of times we need to roll two dice until we get a 6 is

$$\frac{1}{\frac{5}{36}} = \frac{36}{5} = 7.2$$