Probability: Part II

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Contents

Probability Function or Probability Measure

Definition 1

Let S be a sample space, A probability function p from the set of all events in S to the set of real numbers satisfies the following three axioms: For all events A and B in S,

Axiom 1: $0 \leq p(A) \leq 1$;

Axiom 2: $p(\emptyset) = 0$ and $p(S) = 1$; and

Axiom 3: if A and B are disjoint (i.e., $A \cap B = \emptyset$), then the probability of the union of A and B is

$$p(A \cup B) = p(A) + p(B).$$

A Property of Probability Functions

Theorem 2

Let (S, p) be a probability space. Then

$$\sum_{s \in S} p(s) = 1.$$

Proof.

It follows from Axioms 2 and 3. $\square$

Probability Function or Probability Measure

Example 3

Let S be any finite sample space with uniform probability distribution. For any event $E \subseteq S$, define $p(E) = |E|/|S|$. Then this function p satisfies the three axioms above, and is the probability function defined in the previous lecture.

Proof.

- By definition, $0 \leq p(E) = |E|/|S| \leq 1$, as $0 \leq |E| \leq |S|$.
- Secondly, $p(\emptyset) = |\emptyset|/|S| = 0/|S| = 0$, and $p(S) = |S|/|S| = 1$.
- If A and B are disjoint events, then $A \cap B = \emptyset$. It then follows that $|A \cup B| = |A| + |B|$ and

$$\frac{|A \cup B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|},$$

which means that $p(A \cup B) = p(A) + p(B)$.

$\square$

Probability of the Complement of an Event

Proposition 4

Let E be an event in a sample space S. Then the probability of the complement E^c of the event E is

$$p(E^c) = 1 - p(E).$$

Proof.

Note that $E \cap E^c = \emptyset$ and $S = E \cup E^c$. By Axioms 2 and 3,

$$1 = p(E) + p(E^c).$$

$\square$

Probability of a Union of Events

Proposition 5

Let E_1 and E_2 be two events in a sample space S. Then the probability of the union $E_1 \cup E_2$ of the events E_1 and E_2 is

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Probability of a Union of Events

Proof.

1. We claim that $A \cup B$ is the disjoint union of the three sets: $A \setminus (A \cap B)$, $B \setminus (A \cap B)$, and $A \cap B$. (It is clear from the Wenn diagram.)
2. We claim that for any events U and V in the sample space S, if $U \subseteq V$, then $p(V \setminus U) = p(V) - p(U)$.
   - Obviously, $V \setminus U$ and U are disjoint. Since $U \subseteq V$, $V \setminus U$ and U form a partition of V. By Axiom 3, $p(V) = p((V \setminus U) \cup U) = p(V \setminus U) + p(U)$.

It then follows from Conclusion 1 and Axiom 3 that

$$\begin{aligned} p(E_1 \cup E_2) &= p([E_1 \setminus (E_1 \cap E_2)] \cup [E_2 \setminus (E_1 \cap E_2)] \cup [E_1 \cap E_2)]) \\ &= p(E_1 \setminus (E_1 \cap E_2)) + p(E_2 \setminus (E_1 \cap E_2)) + p(E_1 \cap E_2) \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2). \end{aligned}$$

□

Expected Value of the Outcome

Definition 6

Suppose the possible outcomes of an experiment, or random process, are real numbers $a_1, a_2, a_3, \ldots, a_n$, which occur with probabilities $p_1, p_2, p_3, \ldots, p_n$. The expected value of the process is

$$\sum_{i=1}^{n} p_i a_i.$$

Expected Value of the Outcome: Example

Example 7

The experiment is to roll a balanced dice. The sample space is $\{1, 2, 3, 4, 5, 6\}$ with uniform probability distribution. Hence the expected value of the experiment is

$$\sum_{i=1}^{6} p_i a_i = \sum_{i=1}^{6} \frac{1}{6} i = \frac{7}{2}.$$

Conditional Probability

Definition 8

Let A and B be events in a sample space S. If $p(A) \neq 0$, then the conditional probability of B given A, denoted $p(B|A)$, is

$$p(B|A) = \frac{p(B \cap A)}{p(A)}. \tag{1}$$

Problem 9

Suppose a couple has two children, each of whom is equally likely to be a boy or a girl. Now suppose you are given the information that one is a boy. What is the probability that the other child is a boy?

Solution to Problem 9

Problem

Suppose a couple has two children, each of whom is equally likely to be a boy or a girl. Now suppose you are given the information that one is a boy. What is the probability that the other child is a boy?

Solution: The **four equally likely** combinations of gender for the children are:

$$BB, \;\; BG, \;\; GB, \;\; GG.$$

Let the 1st and 2nd letter denote the gender of the elder and younger child. Let A denote the event that at least one child is a boy and B the event that both are boys. Then

$$p(A) = \frac{3}{4}, \;\; p(A \cap B) = \frac{1}{4}.$$

Hence

$$p(B|A) = \frac{p(B \cap A)}{p(A)} = 1/3.$$

Independent Events

Definition 10

Two events A and B are <u>independent</u> if and only if $p(A \cap B) = p(A)p(B)$, or equivalently, $p(A|B) = p(A)$.

Remark

Disjoint events are usually NOT probabilistically independent.

Independent Events

Problem 11

Suppose a fair coin is tossed twice. Let A be the even that a head is obtained on the first toss and B be the event that a head is obtained on the second toss. Suppose that the coin is tossed randomly two times.

1. *Are the two events independent?*
2. *How do you prove your conclusion?*

Solution to Problem 11

Answer

Suppose a fair coin is tossed twice. Let A be the even that a head is obtained on the first toss and B be the event that a head is obtained on the second toss. Then the two events are **independent**.

Proof.

Since the coin is fair, the then the four outcomes HH, HT, TH, and TT are equally likely. By definition,

$$A = \{HH, HT\}, \ B = \{TH, HH\}, \ A \cap B = \{HH\}.$$

Hence, $p(A) = p(B) = 2/4 = 1/2$ and $p(A \cap B) = 1/4$. Then

$$p(A|B) = \frac{p(A \cap B)}{p(A)} = \frac{1}{2} = p(A), \quad p(B|A) = \frac{p(B \cap A)}{p(B)} = \frac{1}{2} = p(B).$$

□

The Birthday Problem

Problem 12

Assuming that all years have 365 days and all birthdays occur with equal probability, how large must n be so that in any randomly chosen group of n people, the probability that two or more have the same birthday is at least 1/2?

Solution to the Birthday Problem

Solution: Let A be the event that at least two people in the group have the same birthday. Then A^c is the event that no two people in the group have the same birthday. Since $p(A) = 1 - p(A^c)$, we now compute $p(A^c)$.

The first person can have any birthday. The second person's birthday has to be different. There are 364 different days to choose from, so the probability that two people have different birthdays is 364/365. That leaves 363 birthdays out of 365 open for the third person.

So the probability that the n people in the group have different birthdays is

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-n+1}{365} = \frac{365!}{(365-n)! * 365^n}.$$

Hence

$$p(A) = 1 - p(A^c) = 1 - \frac{365!}{(365-n)! * 365^n}.$$

The minimum n for $p(A) \geq 0.5$ is $n = 23$.

The Bernoulli Trial

Definition 13

Each performance of an experiment with two possible outcomes is called a Bernoulli trial. The two possible outcomes are usually referred to as **success** anf **failure**.

Example 14

A coin is biased so that the probability of a head is $2/3$. What is the probability that exactly 4 heads come up when the coin is flipped 7 times , assuming that the flips are independent?

Remark

A more general solution to this problem is given in the next slide.

The Bernoulli Trial

Proposition 15

The probability of exactly k success in n independent Bernoulli trials, with probability p of success and probability $q = 1 - p$ of failure is, $\binom{n}{k} p^k q^{n-k}$.

Proof.

Let $(t_1, t_2, \ldots, t_n) \in \{S, F\}^n$ denote the outcome when n Bernoulli trials are carried out, where S and F denote success and failure respectively. Since the n trials are **independent**, the probability of each outcome of n trials consisting of k successes and $n - k$ failures in any order is $p^k q^{n-k}$. Note that there are $\binom{n}{k}$ n-tuples of S's and F's that contain k S's. The probability of k successes is $\binom{n}{k} p^k q^{n-k}$. $\square$

The Binomial Distribution

Definition 16

The probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$, denoted by $b(k; n, p)$, is given by

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}.$$

When considered as a function of k, it is called the binomial distribution.

Online Problem

Problem 17

You are given 12 coins, of which at most one is counterfeit. Design a procedure of using a balance at most three times, so that you are able to find out if there is a counterfeit coin, and the specific counterfeit coin and whether it is heavier or lighter in the case there is a counterfeit coin.

Problem 18

Do you have a solution if the number of coins in the problem above is 13?