# Illustration of the Proof of Lemma 5.9

*Version 2: Last updated, Dec 1, 2005*

The Expectation of Random Variable $X$ is defined as

$$E(X) = \sum_{i=1}^{k} x_i P(X = x_i).$$

In class, we proved the equivalence of the following alternative method of calculating $E(X)$

**Lemma 5.9**

If a random variable $X$ is defined on a (finite) sample space $S$, then its expected value is given by

$$E(X) = \sum_{s: s \in S} X(s) P(s).$$

In these sides, we illustrate the proof of Lemma 5.9 with an example.

Our example will be the space of 3 coin flips.

$X$ will be the number of heads in the flip

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Possible values for $X$ are $x_1, = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$.

We group the outcomes by their $X$ values

$$\begin{array}{lllll}
F_1 & = & (X = x_1) & = & \{HHH\} \\
F_2 & = & (X = x_2) & = & \{THH, HTH, HHT\} \\
F_3 & = & (X = x_3) & = & \{HTT, THT, TTH\} \\
F_4 & = & (X = x_4) & = & \{TTT\}
\end{array}$$

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\end{array}
$$

On the next page, the Right Hand Side lists the formulas that appeared in the proof of Lemma 5.9. The Left Hand Side illustrates, on our example, what each of those formulas means.

$$\sum_{s \in S} X(s)P(s)$$

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$$= \underbrace{X(HHH)P(HHH)}_{F_1}$$

$$+ \underbrace{X(HHT)P(HHT) + X(HTH)P(HTH) + X(THH)P(THH)}_{F_2}$$

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$$+ \underbrace{X(TTT)P(TTT)}_{F_4}$$

$$= 0 \cdot \underbrace{P(HHH)}_{F_1} \quad +1 \underbrace{\left(P(HHT) + P(HTH) + P(THH)\right)}_{F_2}$$

$$+2 \cdot \underbrace{\left(P(HTT) + P(TTH) + P(THT)\right)}_{F_3} +3 \cdot \underbrace{P(TTT)}_{F_4}$$

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