# Illustration of the Proof of Lemma 5.28

*Version 2: Last updated, Dec 1, 2005*

In class, we proved that the expectation of the product of two independent random variables, is the product of their expectations. Formally

In class, we proved that the expectation of the product of two independent random variables, is the product of their expectations. Formally

**Lemma 5.28**

If $X$ and $Y$ are independent random variables on sample space $S$ with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

In class, we proved that the expectation of the product of two independent random variables, is the product of their expectations. Formally

**Lemma 5.28**
If $X$ and $Y$ are independent random variables on sample space $S$ with values $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_m$, respectively, then $E(XY) = E(X)E(Y)$.

In these sides, we illustrate the proof of Lemma 5.28 with an example.

Suppose that we have two independent random variables, $X, Y$ that each can take on the values $1, 2, 4$, but with different probability weights.

Suppose that we have two independent random variables, $X, Y$ that each can take on the values $1, 2, 4$, but with different probability weights.

$$P(X = 1) = 1/3 \qquad P(Y = 1) = 1/2$$
$$P(X = 2) = 1/3 \qquad P(Y = 2) = 1/4$$
$$P(X = 4) = 1/3 \qquad P(Y = 4) = 1/4$$

Suppose that we have two independent random variables, $X, Y$ that each can take on the values $1, 2, 4$, but with different probability weights.

$$\begin{array}{lll} P(X=1)=1/3 & P(Y=1)=1/2 & \\ P(X=2)=1/3 & P(Y=2)=1/4 & \Rightarrow \\ P(X=4)=1/3 & P(Y=4)=1/4 & \end{array} \quad \begin{array}{l} E(X)=7/3 \\ E(Y)=2 \\ E(X)(EY)=14/3 \end{array}$$

Suppose that we have two independent random variables, $X, Y$ that each can take on the values $1, 2, 4$, but with different probability weights.

$$\begin{aligned} P(X=1) &= 1/3 & P(Y=1) &= 1/2 & & E(X) = 7/3 \\ P(X=2) &= 1/3 & P(Y=2) &= 1/4 & \Rightarrow\ & E(Y) = 2 \\ P(X=4) &= 1/3 & P(Y=4) &= 1/4 & & E(X)(EY) = 14/3 \end{aligned}$$

---

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$.

Suppose that we have two independent random variables, $X, Y$ that each can take on the values $1, 2, 4$, but with different probability weights.

$$\begin{array}{lll} P(X=1)=1/3 & P(Y=1)=1/2 & \\ P(X=2)=1/3 & P(Y=2)=1/4 & \Rightarrow \\ P(X=4)=1/3 & P(Y=4)=1/4 & \end{array} \quad \begin{array}{l} E(X)=7/3 \\ E(Y)=2 \\ E(X)(EY)=14/3 \end{array}$$

---

$Z = XY$ can only take on the values $1, 2, 4, 8, 16.$

$$P(Z=1) = P(X=1 \ \wedge Y=1) = \frac{1}{6}$$

$$P(Z=2) = P(X=1 \ \wedge \ Y=2) + P(X=2 \ \wedge \ Y=1) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$$

$$\begin{aligned} P(Z=4) = {} & P(X=1 \ \wedge \ Y=4) + P(X=4 \ \wedge \ Y=1) \\ & + P(X=2 \ \wedge \ Y=2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} \end{aligned}$$

$$P(Z=8) = P(X=2 \ \wedge \ Y=4) + P(X=4 \ \wedge \ Y=2) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$P(Z=16) = P(X=4 \ \wedge Y=4) = \frac{1}{12}$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16.$ So

$$
\begin{aligned}
E(XY) &= E(Z) \\
&= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\
&\qquad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\
&= \frac{14}{3} \\
&= E(X) \cdot E(Y)
\end{aligned}
$$

$Z = XY$ can only take on the values $1, 2, 4, 8, 16$. So

$$
\begin{aligned}
E(XY) &= E(Z) \\
&= 1 \cdot P(Z = 1) + 2 \cdot P(Z = 2) + 4 \cdot P(Z = 4) \\
&\qquad + 8 \cdot P(Z = 8) + 16 \cdot P(z = 16) \\
&= \frac{14}{3} \\
&= E(X) \cdot E(Y)
\end{aligned}
$$

*On the next page, we mimic the proof of Lemma 5.28, using these $X, Y$. Reading the proof with this example in mind, might make the proof more understandable.*

$E(X)E(Y)$

$$E(X)E(Y) = \sum_{x \in \{1,2,4\}} xP(X = x) \sum_{y \in \{1,2,4\}} yP(Y = y)$$

$$E(X)E(Y) = \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y)$$

$$= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)$$

$$
\begin{aligned}
E(X)E(Y) &= \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y) \\
&= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y) \\
&= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P(X=x)P(Y=y)
\end{aligned}
$$

$$E(X)E(Y) = \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y)$$

$$= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)$$

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P(X=x)P(Y=y)$$

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P\big((X=x)\wedge(Y=y)\big)$$

$$
\begin{aligned}
E(X)E(Y) &= \sum_{x \in \{1,2,4\}} xP(X = x) \sum_{y \in \{1,2,4\}} yP(Y = y) \\
&= \sum_{x \in \{1,2,4\}} \sum_{y \in \{1,2,4\}} xyP(X = x)P(Y = y) \\
&= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x, y \in \{1,2,4\} \\ xy = z}} P(X = x)P(Y = y) \\
&= \sum_{z \in \{1,2,4,8,16\}} z \sum_{\substack{x, y \in \{1,2,4\} \\ xy = z}} P\big( (X = x) \wedge (Y = y) \big)
\end{aligned}
$$

Ind of $X, Y$

$$
\begin{aligned}
E(X)E(Y) &= \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y) \\
&= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y) \\
&= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\} \\ xy=z}} P(X=x)P(Y=y) \\
& \qquad\qquad \Big\updownarrow \text{ Ind of } X,Y \\
&= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\} \\ xy=z}} P\big((X=x)\wedge(Y=y)\big) \\
&= \sum_{z\in\{1,2,4,8,16\}} zP(Z=z) = E(Z) = E(XY)
\end{aligned}
$$

$$\boxed{E(X)E(Y)} = \sum_{x\in\{1,2,4\}} xP(X=x) \sum_{y\in\{1,2,4\}} yP(Y=y)$$

$$= \sum_{x\in\{1,2,4\}} \sum_{y\in\{1,2,4\}} xyP(X=x)P(Y=y)$$

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P(X=x)P(Y=y)$$

Ind of $X, Y$

$$= \sum_{z\in\{1,2,4,8,16\}} z \sum_{\substack{x,y\in\{1,2,4\}\\ xy=z}} P\big((X=x)\wedge(Y=y)\big)$$

$$= \sum_{z\in\{1,2,4,8,16\}} zP(Z=z) = E(Z) \boxed{= E(XY)}$$