

COMP170

Discrete Mathematical Tools for Computer Science

Variance of RVs

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*Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 5.7, pp. 294-303*

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Probability Distributions and Variance

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- Distributions of Random Variables

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- Variance

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General Question: how much do we expect a random variable to **deviate** from its expected value.

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Graphs that show, for each integer value x of X , a rectangle of width 1 centered at x , whose height (and thus area) is proportional to the probability $P(X = x)$.

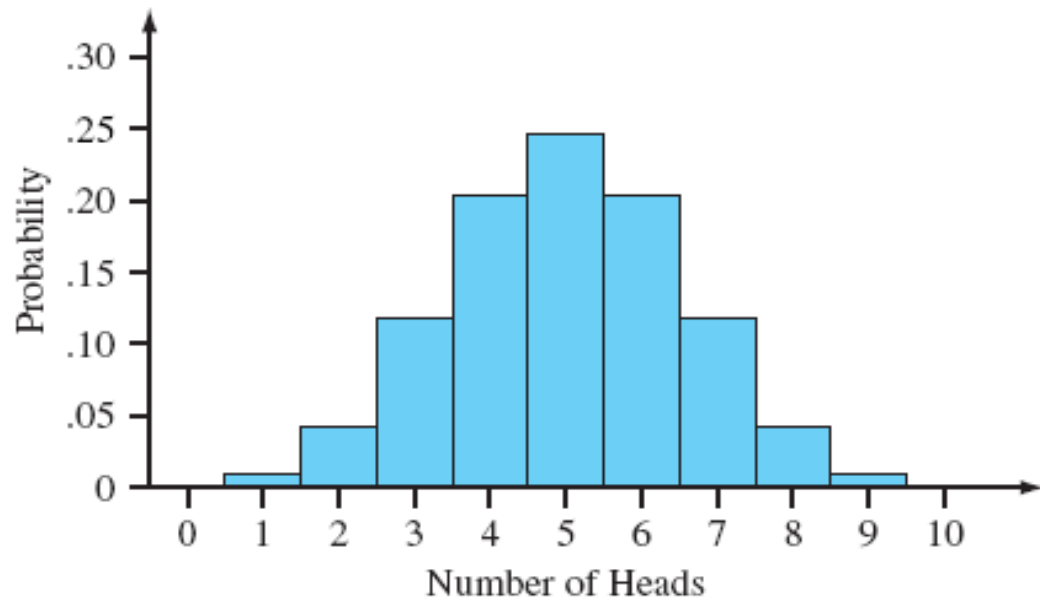
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10 coin flips

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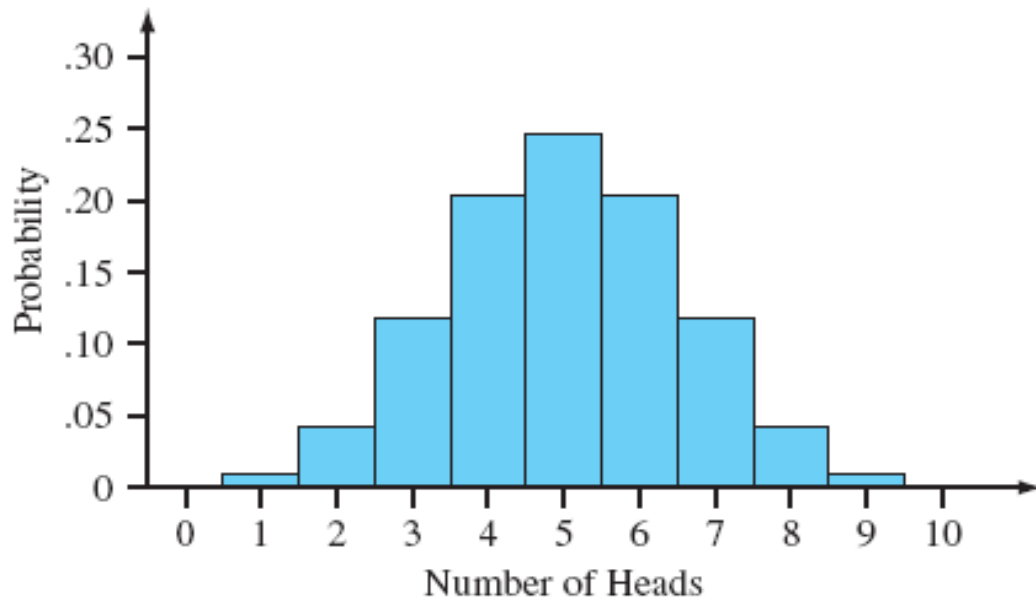
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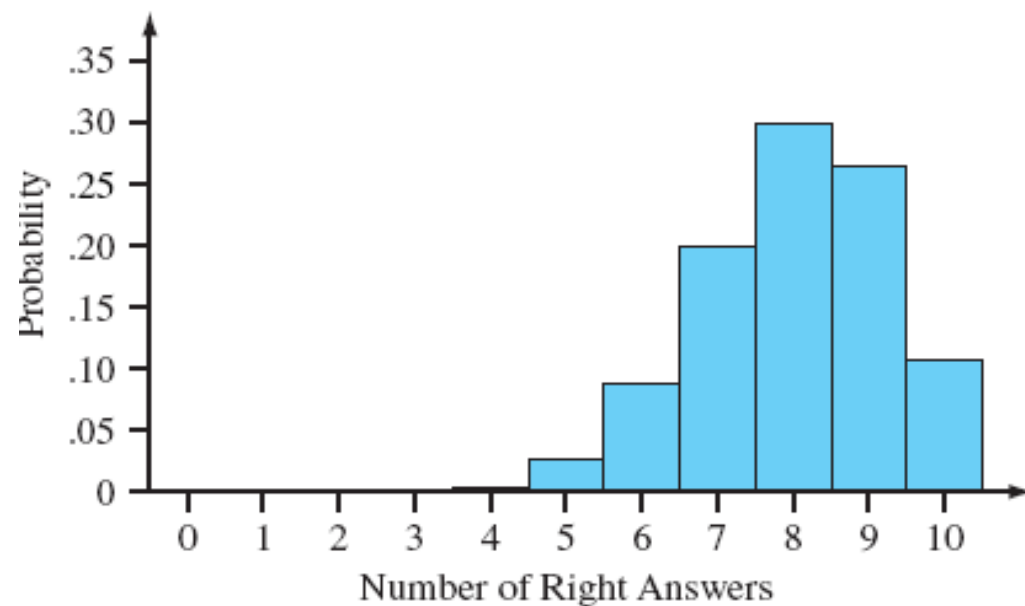
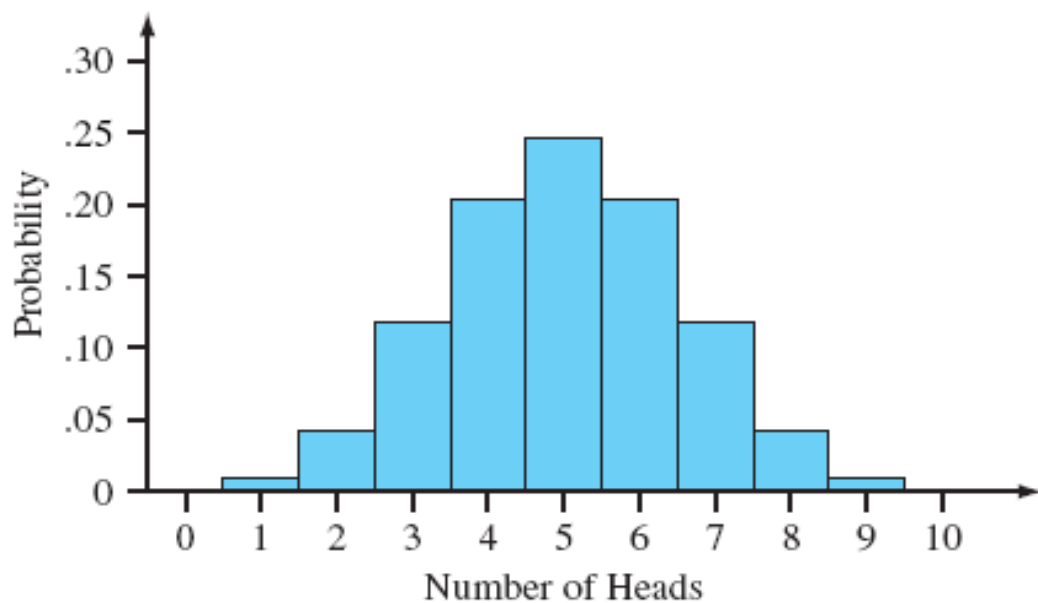
Ten-question test with probability
.8 of getting a correct answer.



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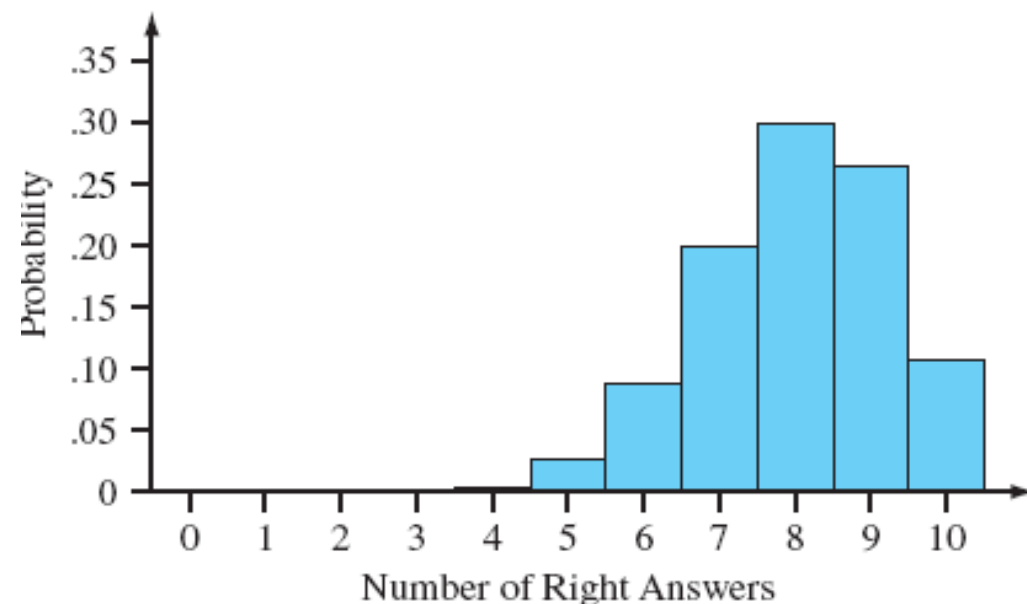
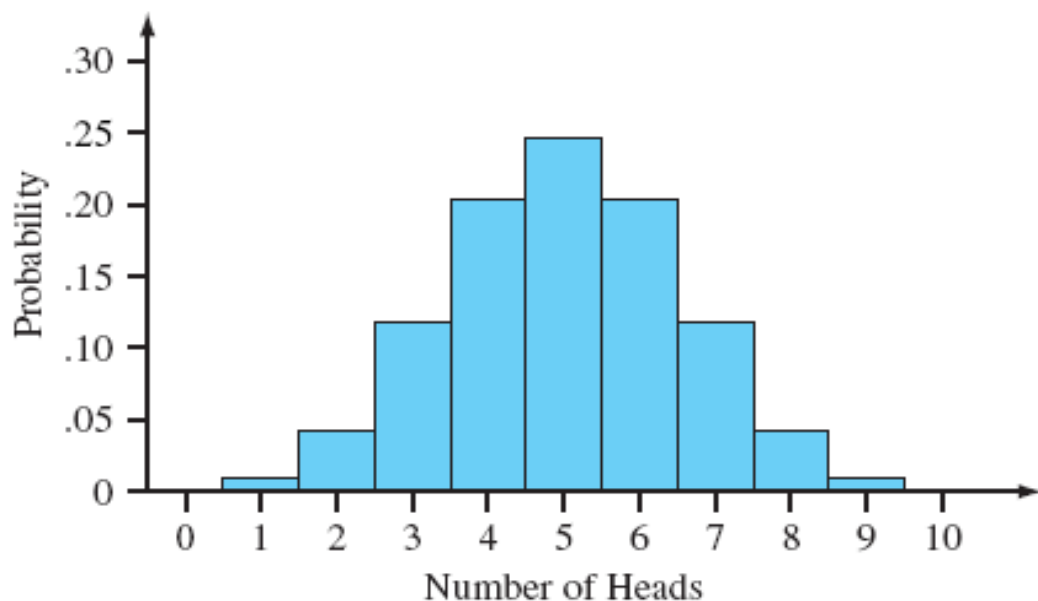
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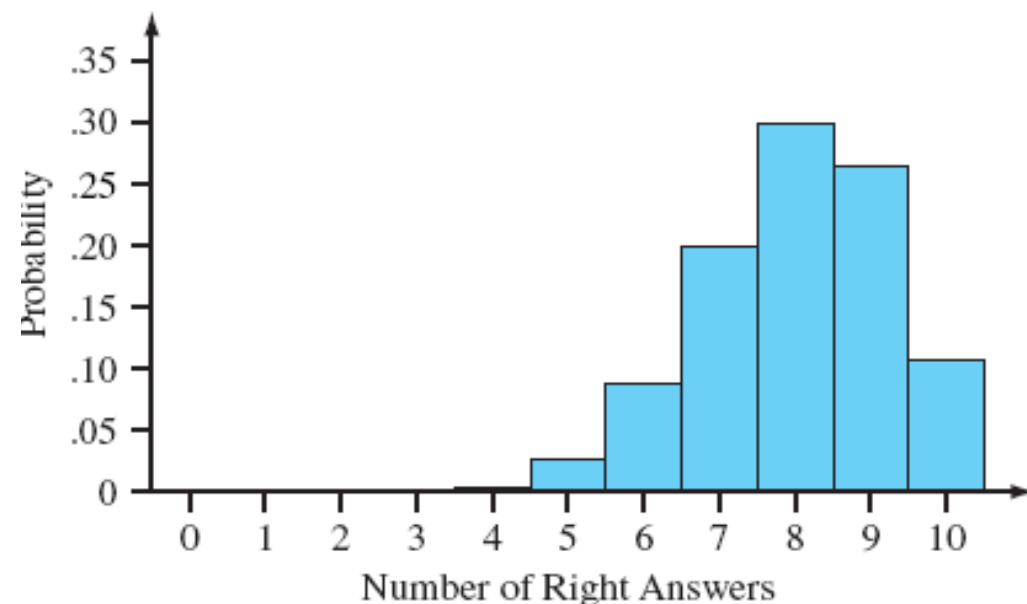
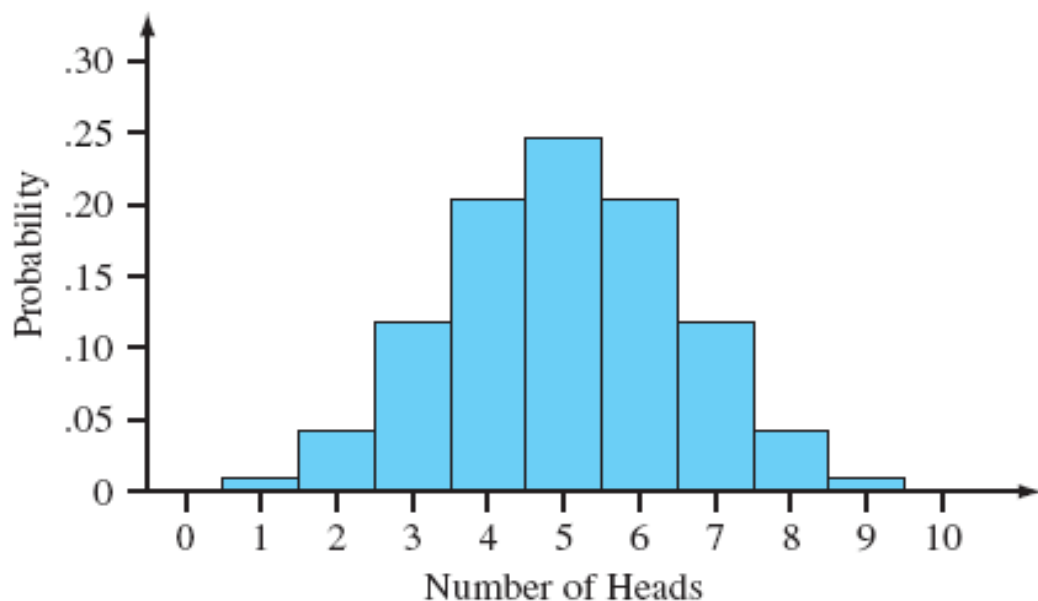


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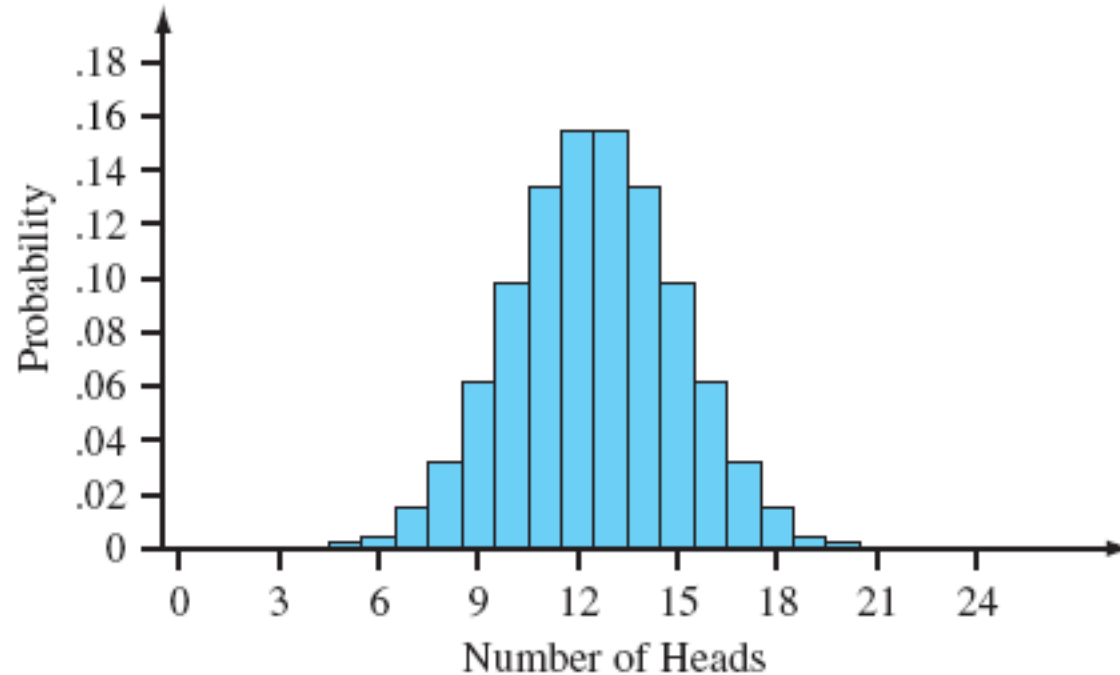
Cumulative distribution function D :

$$D(a, b) = P(a \leq X \leq b).$$

With more coin flips or more questions, will the results spread out?

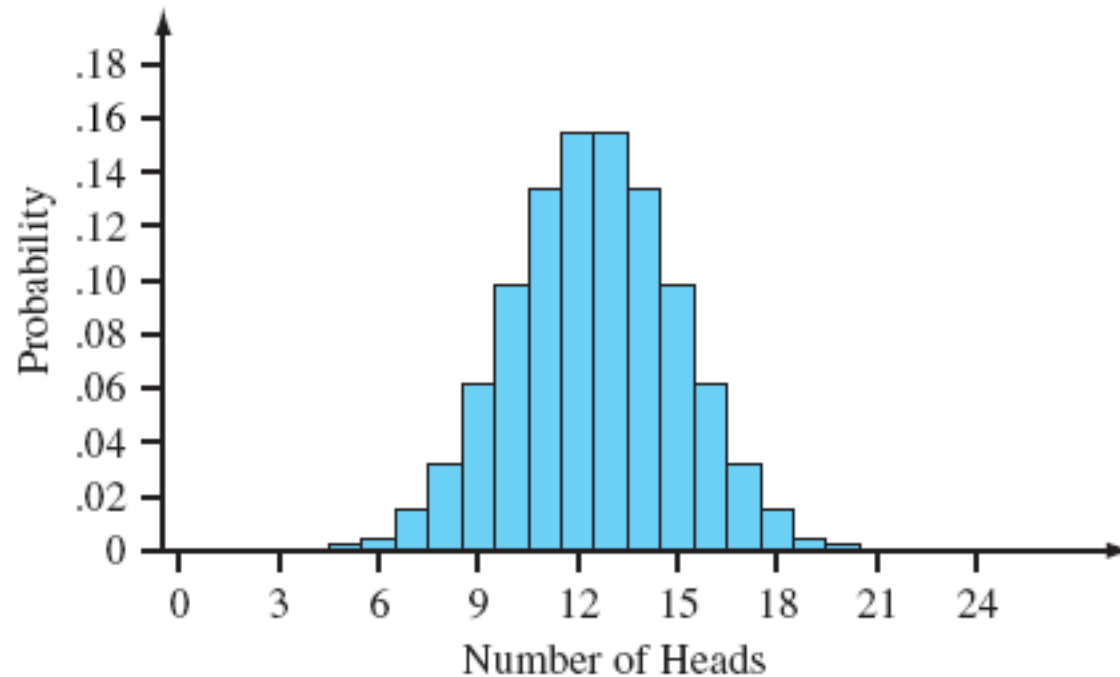
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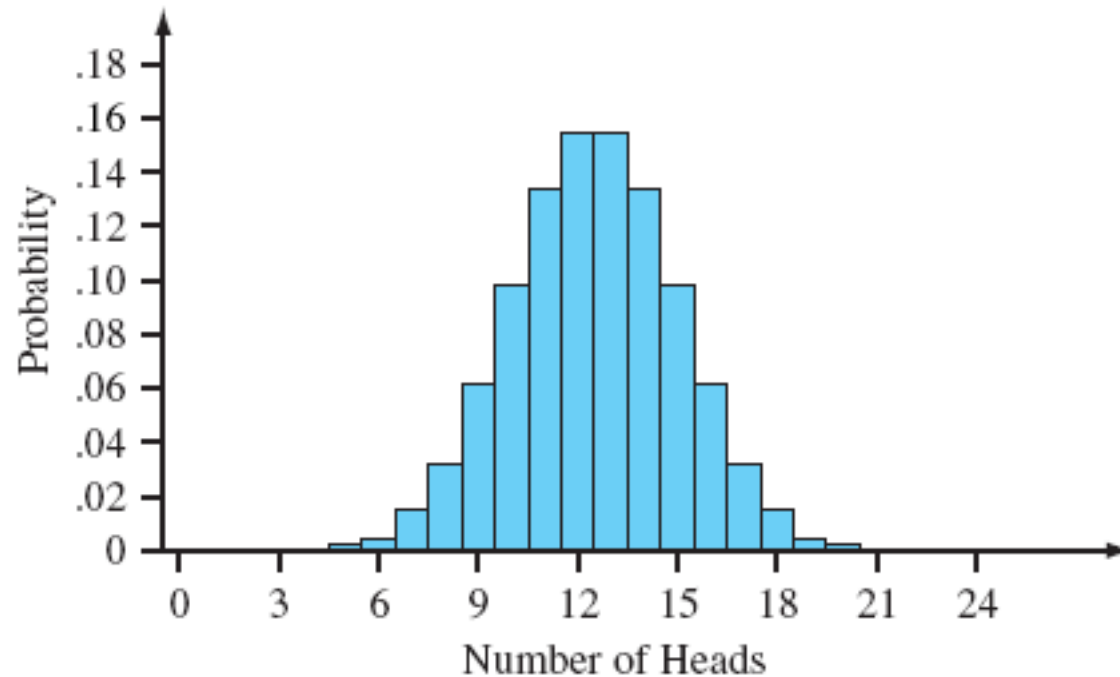
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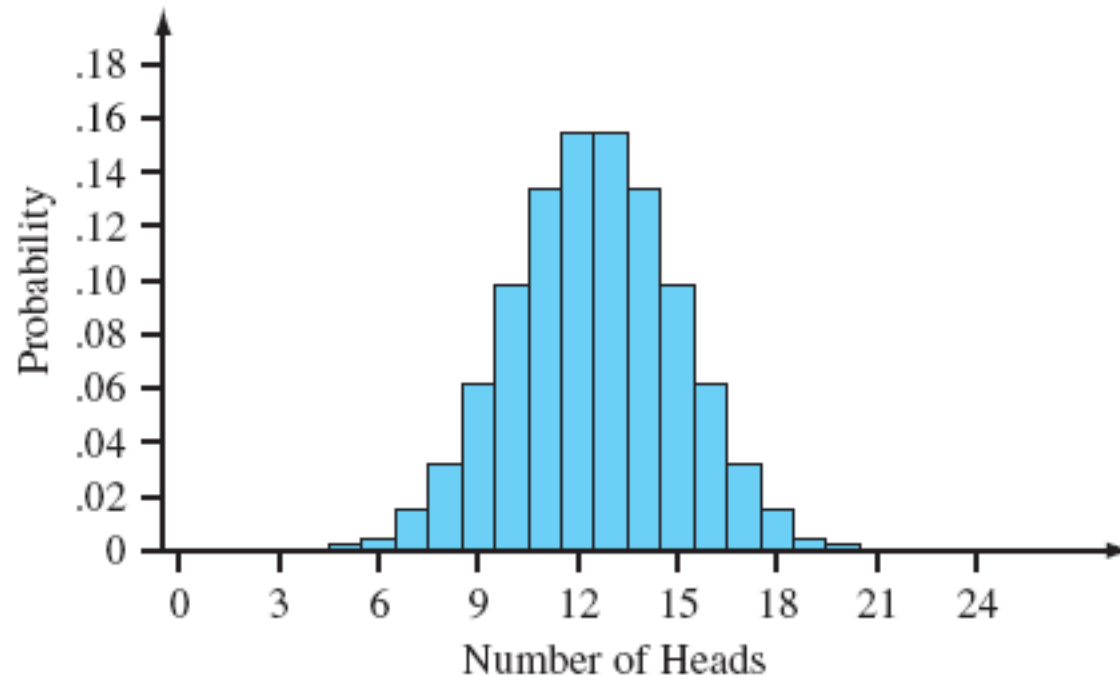


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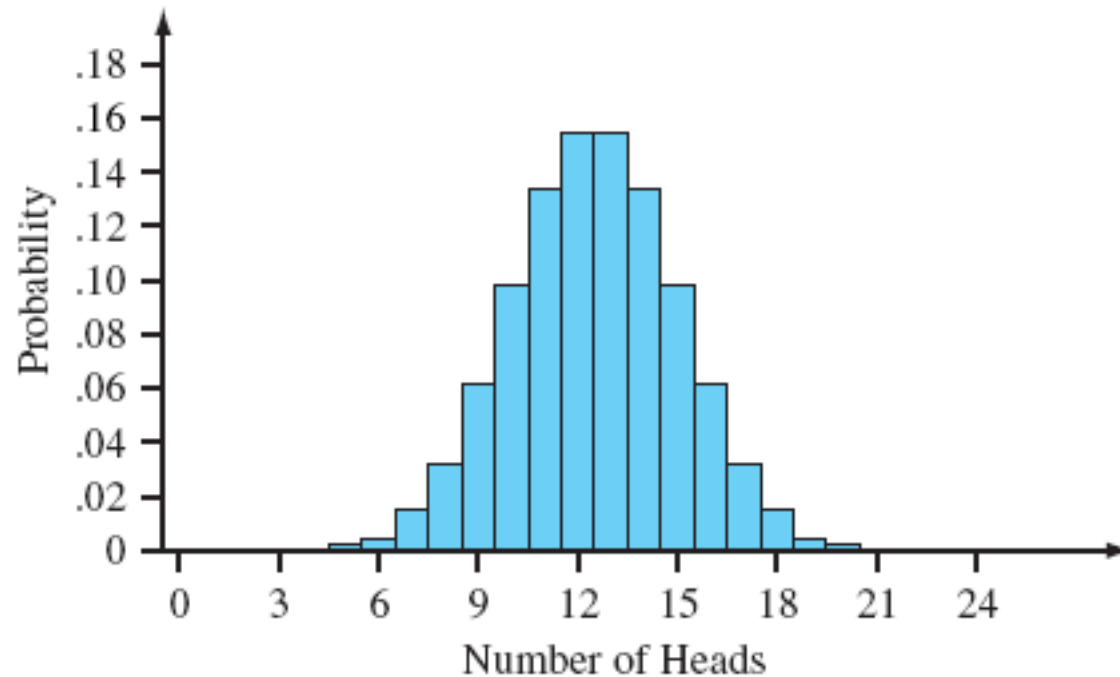


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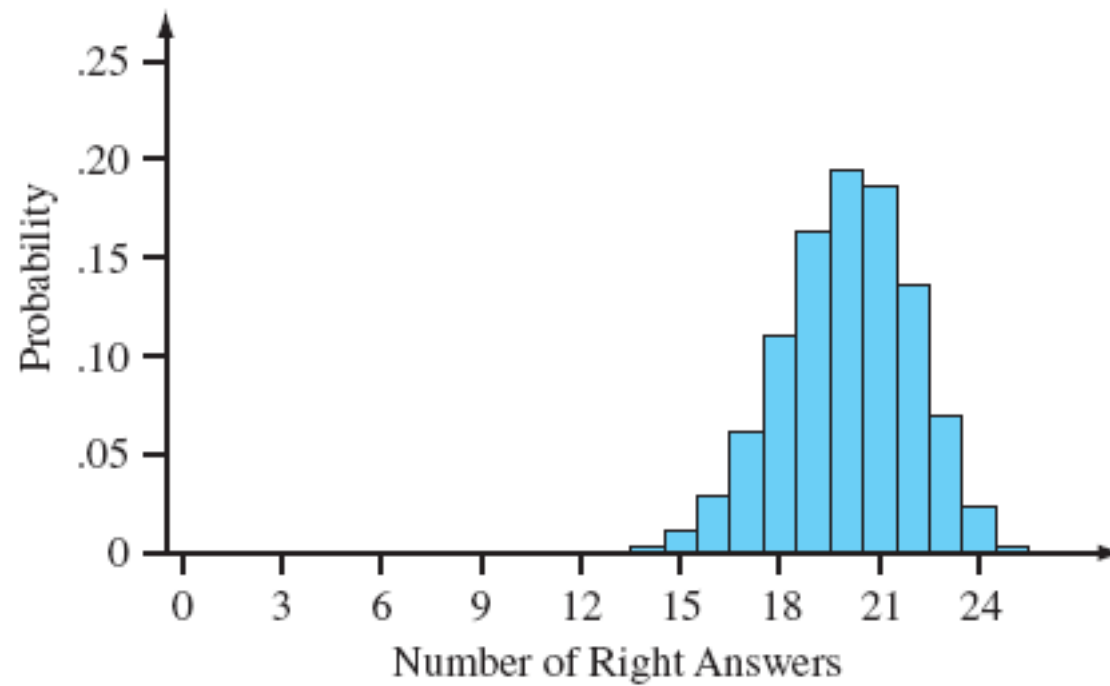
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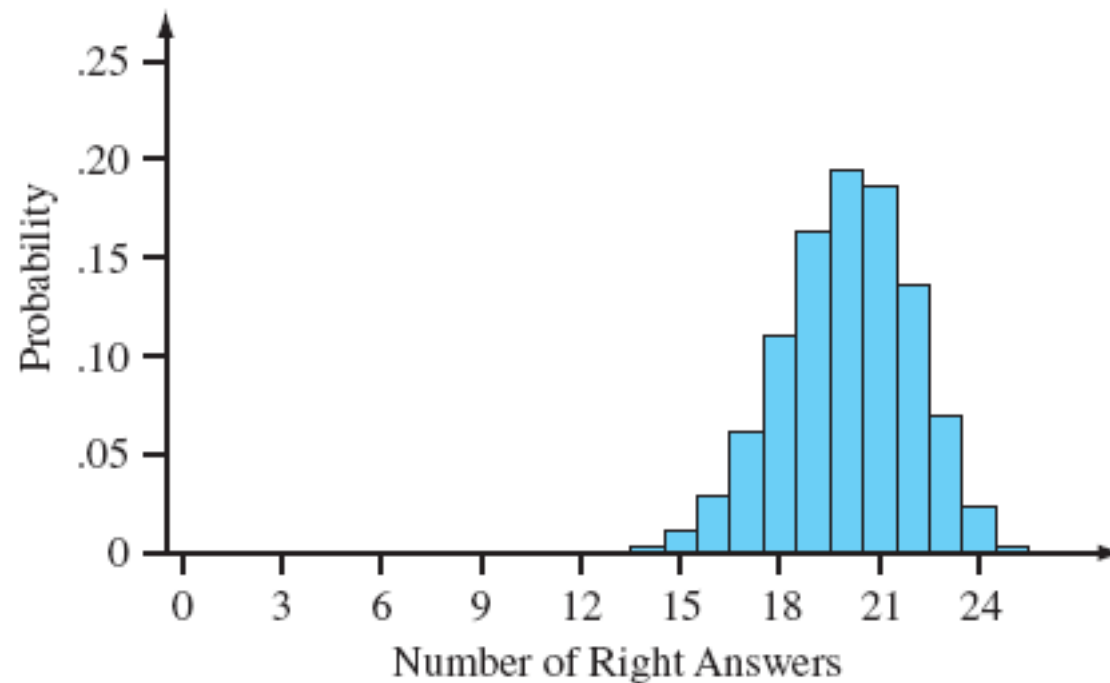
Thus, results are not spread as broadly (relatively speaking) as they were with just 10 flips.

Test score histogram with 25 questions

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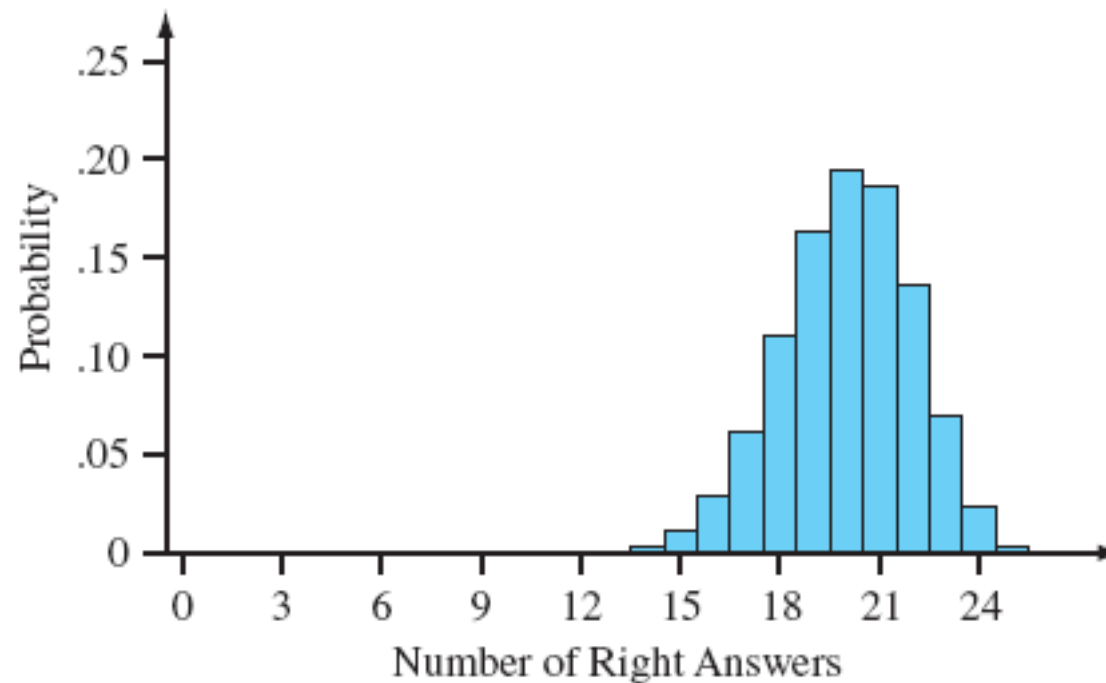


Test score histogram with 25 questions



Compared to coin flipping, even more tightly packed around its expected value.

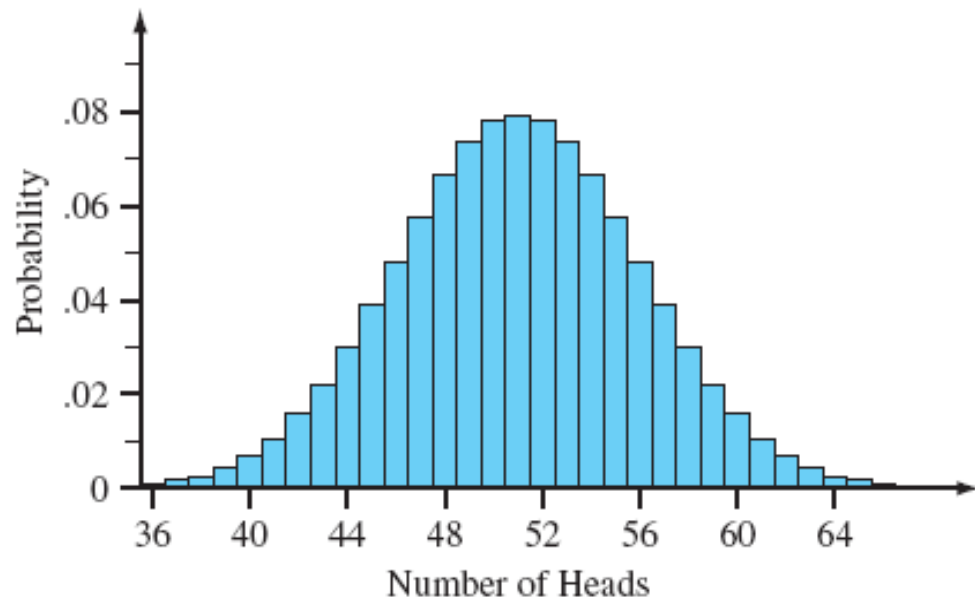
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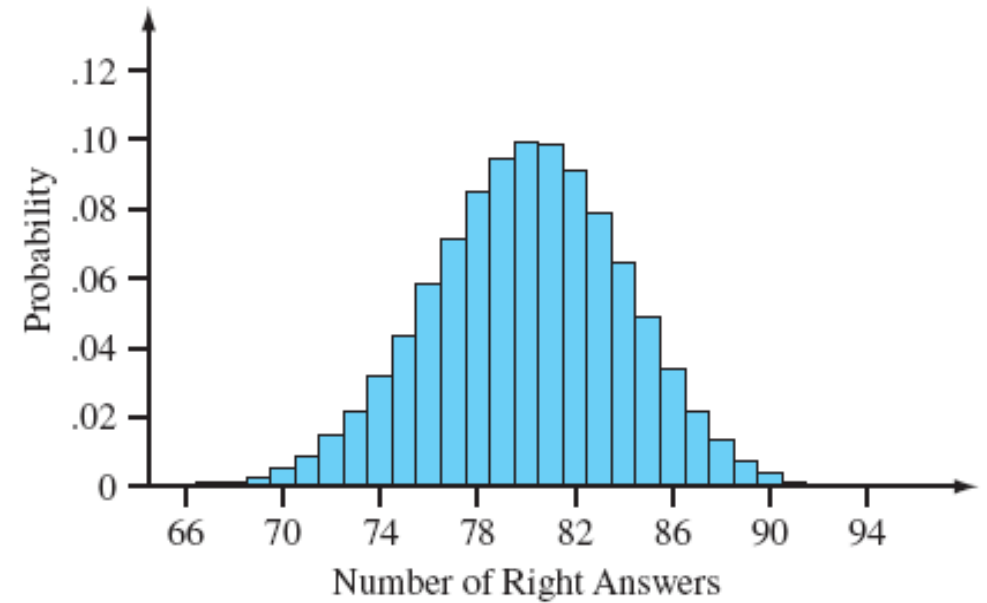
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Essentially, all scores lie between 14 and 25.

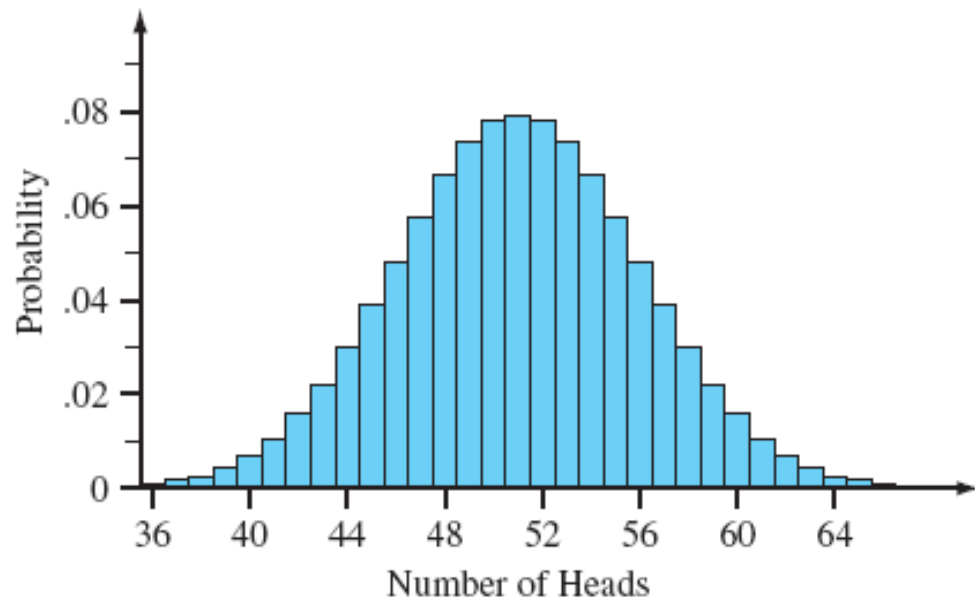
100 flips of a coin



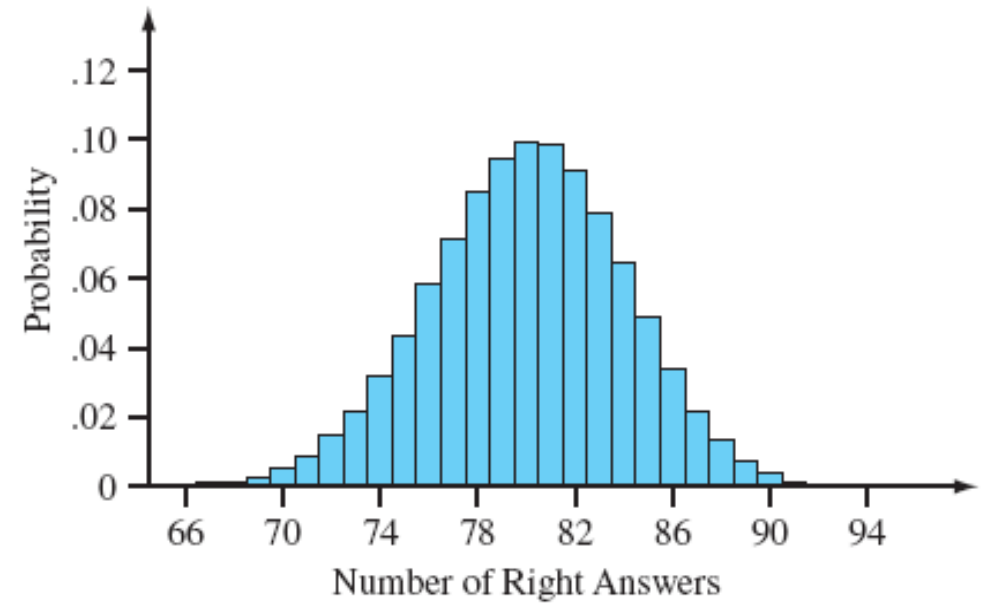
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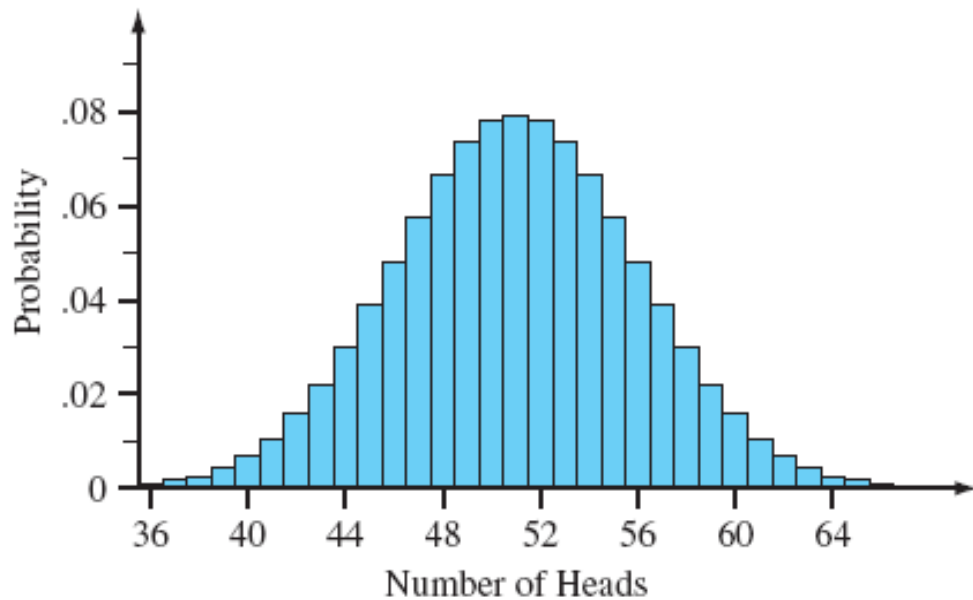


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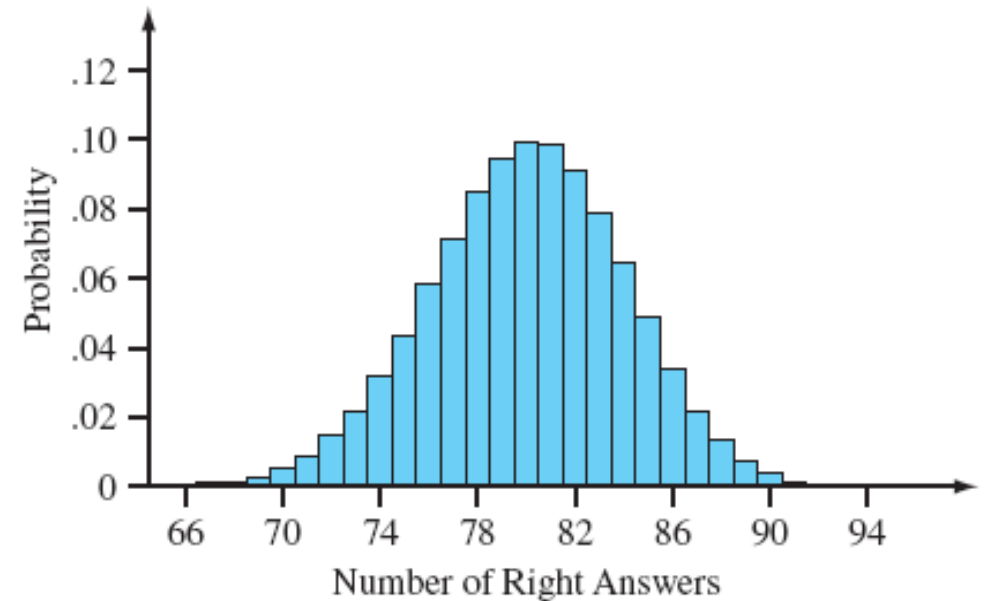


Two histograms have almost same shape.

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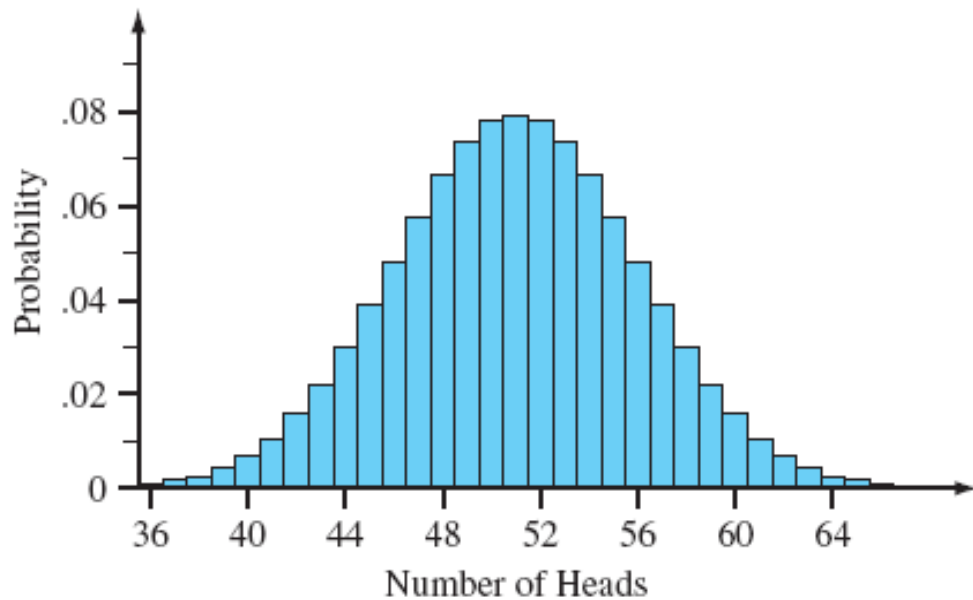
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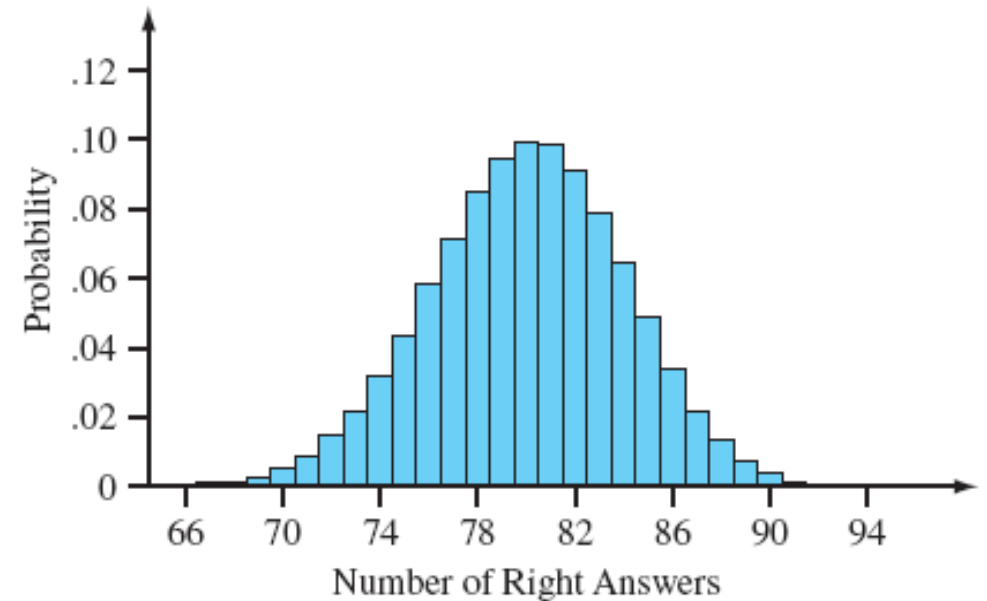
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Number of heads has virtually no chance of deviating by more than 15 from its expected value, and test score has almost no chance of deviating by more than 11.

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Number of heads has virtually no chance of deviating by more than 15 from its expected value, and test score has almost no chance of deviating by more than 11.

Thus, spread has only doubled, even though number of trials has quadrupled.

We need about 30 values to see the most relevant probabilities for 100 trials, whereas we need 15 values to see the most relevant probabilities for 25 independent trials.

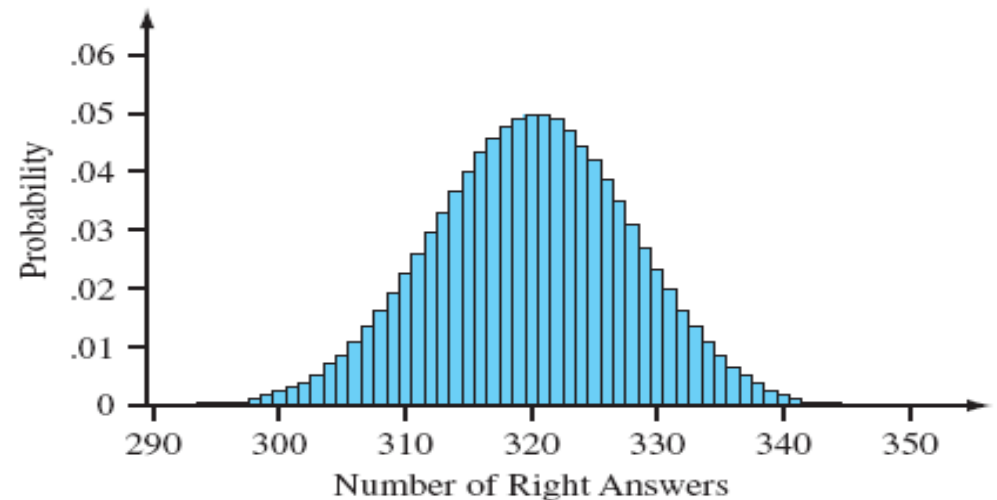
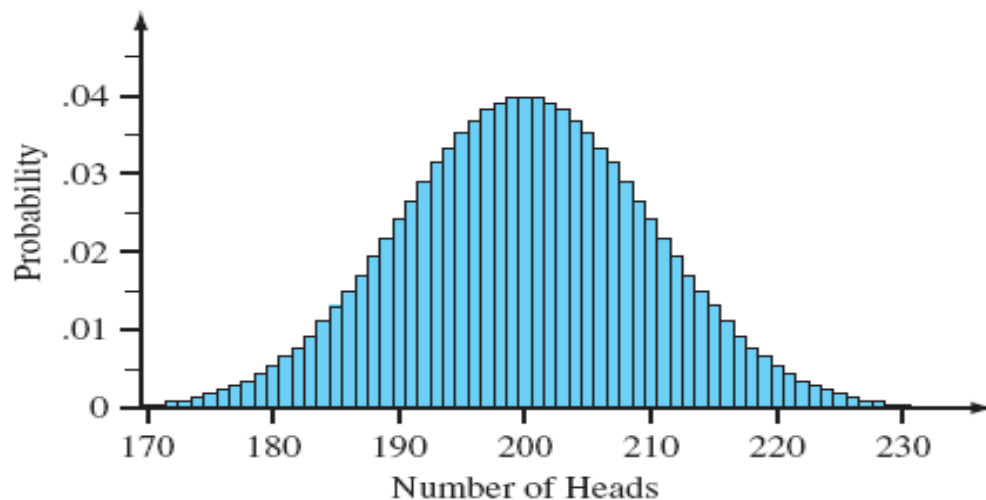
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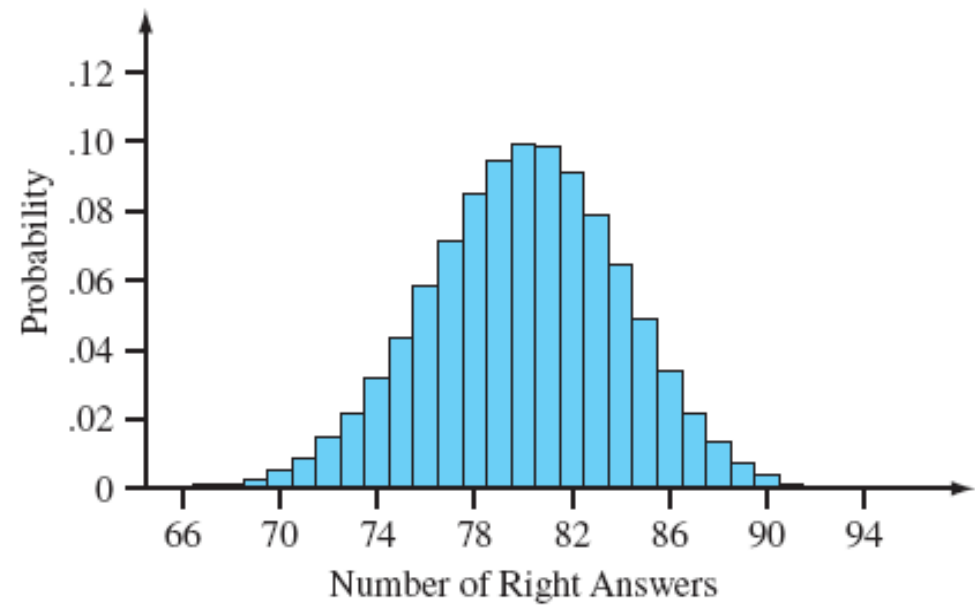
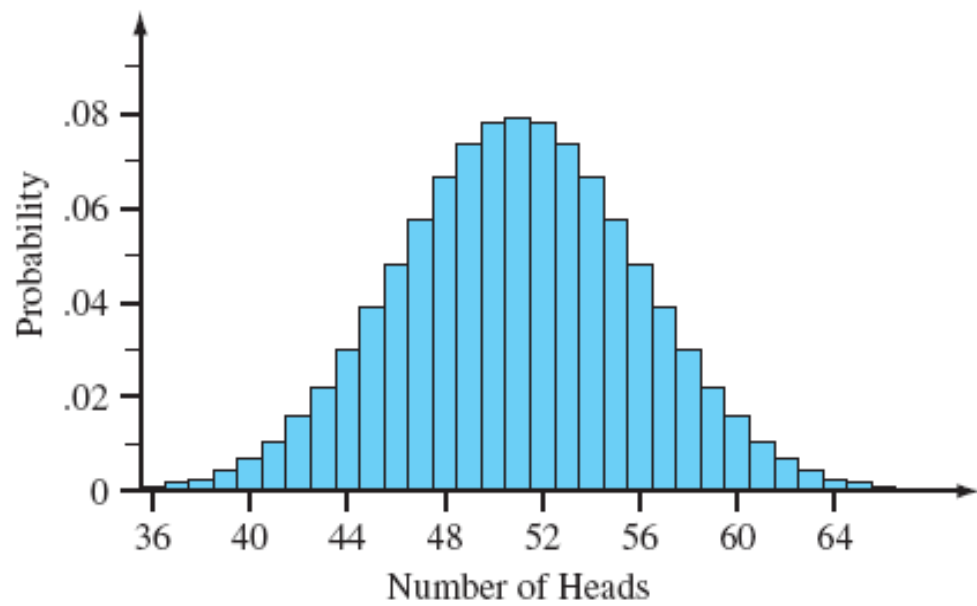
How many values to see essentially all results in 400 trials.

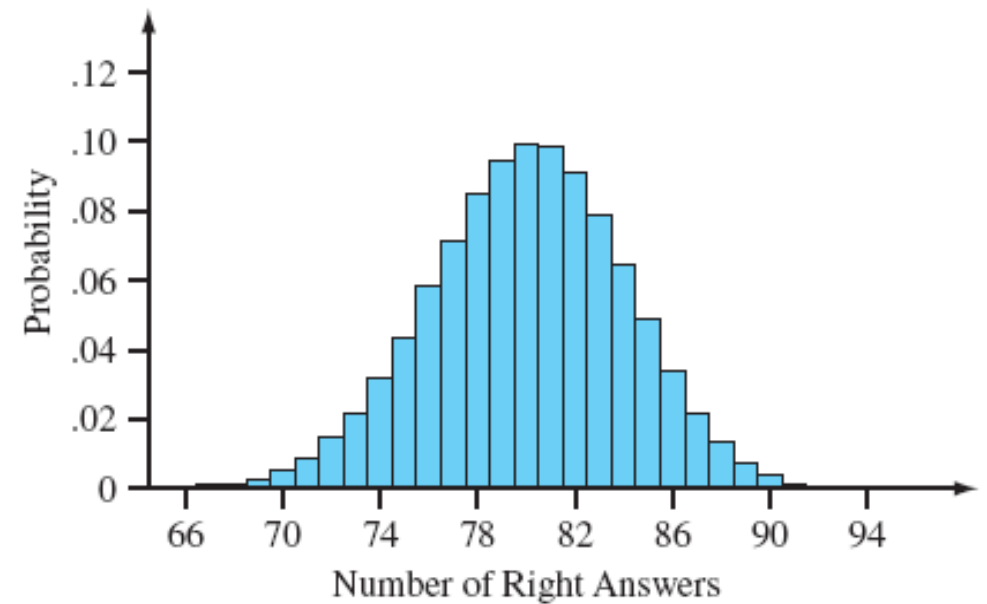
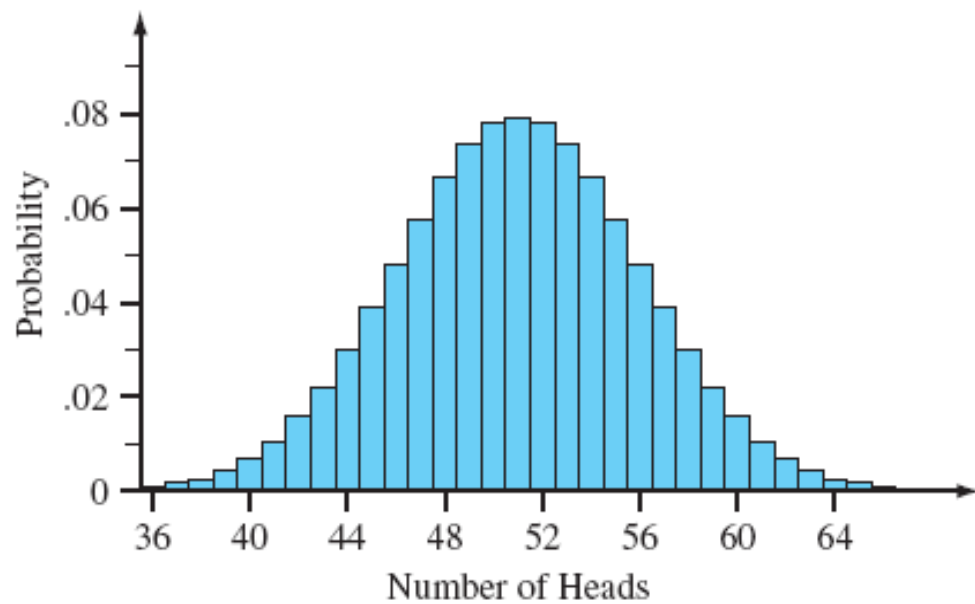
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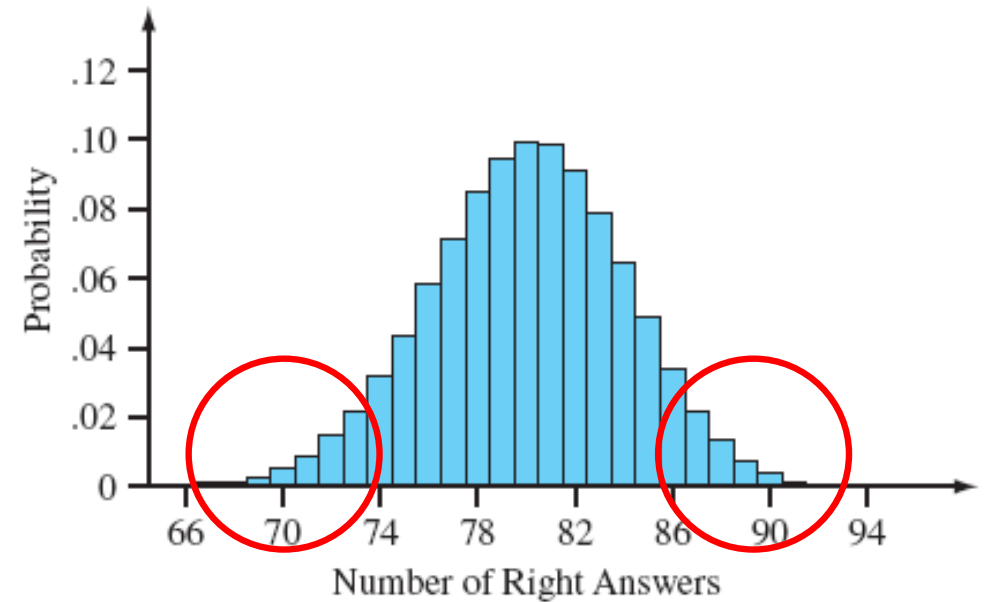
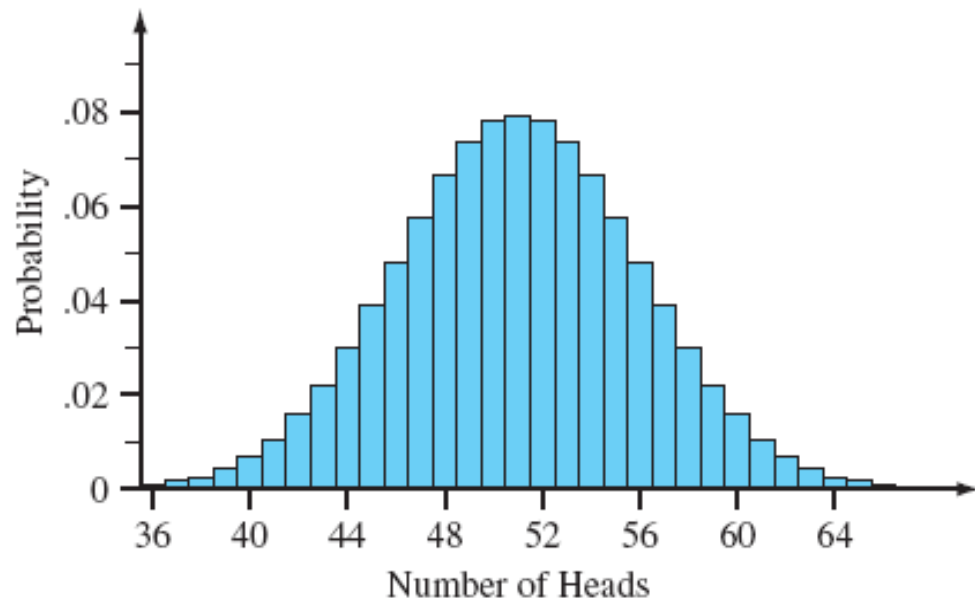
Only about 60 values.





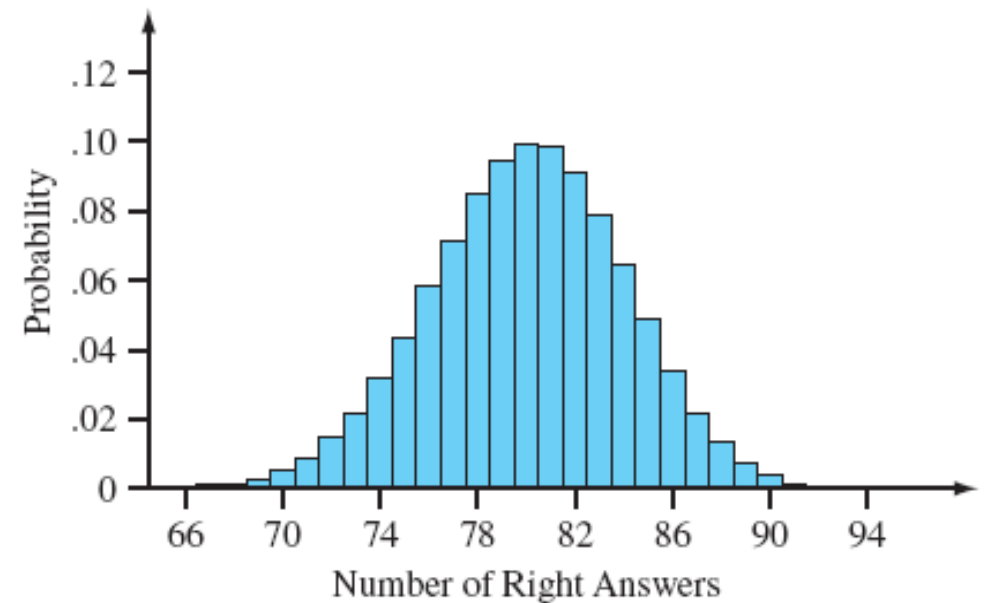
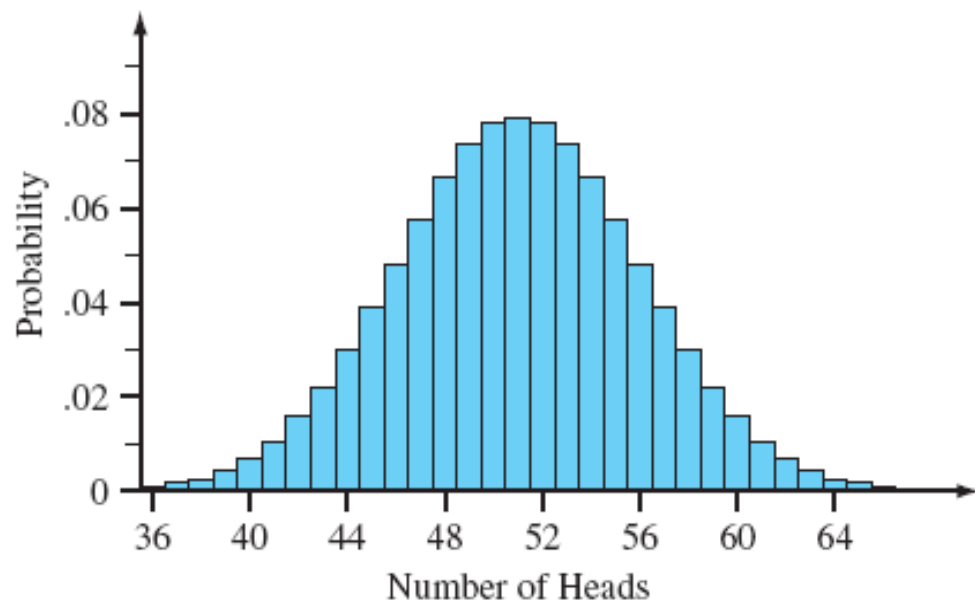


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We want an algebraic way to measure the difference between a random variable and its expected value.

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When we think of $E(X)$ as a random variable,

it has a constant value traditionally denoted by μ .

By Lemma 5.26, we have that $E(E(x)) = E(\mu) = \mu = E(x)$.

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Our next attempt will be to look at $E(Y^2)$.

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$$\begin{aligned} V(x) &= E((X - 2)^2) \\ &= (0 - 2)^2 \cdot \frac{1}{16} + (1 - 2)^2 \cdot \frac{1}{4} + (2 - 2)^2 \cdot \frac{3}{8} \\ &\quad + (3 - 2)^2 \cdot \frac{1}{4} + (4 - 2)^2 \cdot \frac{1}{16} = 1 \end{aligned}$$

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- Calculating variances from scratch is very time consuming. Many R.V.s, X , such as the binomial distribution, can actually be built as the sum of simpler R.Vs, i.e.,
$$X = \sum_{i=1}^n X_i.$$
- Is there any way of constructing $V(X)$ from the $V(X_i)$?

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We already saw that the variance for 4 coin flips is 1, which is 4 times the variance for one coin flip.

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Getting correct answer to a question has probability .8.

Let's compute variance for number of right answers
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Is there a relationship between these two variances?

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1 question:

$$V(X) = .2(0 - .8)^2 + .8(1 - .8)^2 = .16.$$

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5 questions:

$$\begin{aligned} V(X) = & 4^2 \cdot (.2)^5 + 3^2 \cdot 5 \cdot (.2)^4 \cdot (.8) + 2^2 \cdot 10 \cdot (.2)^3 \cdot (.8)^2 \\ & + 1^2 \cdot 10 \cdot (.2)^2 \cdot (.8)^3 + 0^2 \cdot 5 \cdot (.2)^1 \cdot (.8)^4 + 1^2 \cdot (.8)^5 = .8. \end{aligned}$$

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Result is five times variance for one question.





Withdraw one coin.

What is expected amount of money we withdraw?

What is the variance?



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Expected amount of money for one draw is 3 HKD.



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$$V(X) = .5(1 - 3)^2 + .5(5 - 3)^2 = 4.$$

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What are $E(X_1)$ and $V(X_1)$? $E(X_2)$ and $V(X_2)$?

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Now let $X = X_1 + X_2$ be the *total* amount withdrawn.

Since both coins are withdrawn, $X = 6$ so

$$E(X) = 6 \text{ and } V(X) = 0.$$

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Let X_1 be amount withdrawn on first draw and X_2 amount withdrawn on second.

What are $E(X_1)$ and $V(X_1)$? $E(X_2)$ and $V(X_2)$?

$$E(X_1) = 3 \qquad V(X_1) = 4$$

$$E(X_2) = 3 \qquad V(X_2) = 4$$

Now let $X = X_1 + X_2$ be the *total* amount withdrawn.

Since both coins are withdrawn, $X = 6$ so

$$E(X) = 6 \text{ and } V(X) = 0.$$

$$\Rightarrow \boxed{V(X_1 + X_2) \neq V(X_1) + V(X_2)}$$

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However, the other two exercises suggest that such a result might be true for a sum of variances in **independent trials processes**.

Before continuing, we need to introduce concept of **Independent Random Variables**

(as opposed to the Independent events that we have already seen).

Random variables X and Y are **independent** when “ X has value x ” is independent of “ Y has value y ”, regardless of choice of x and y .

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Example: Roll two dice. X is the amount rolled on the first die, Y the amount on the second. They are independent because, for every $1 \leq i, j \leq 6$,

$$P((X = i) \wedge (Y = j)) = \frac{1}{36} = P(X = i) \cdot P(Y = j)$$

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$$E(XY) = E(X) \cdot E(Y)$$

Lemma 5.28

If X and Y are independent random variables on sample space S with values x_1, x_2, \dots, x_k and y_1, y_2, \dots, y_m , respectively, then $E(XY) = E(X)E(Y)$.

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$$E(X)E(Y) = \sum_{i=1}^k x_i P(X = x_i) \sum_{j=1}^m y_j P(Y = y_j)$$

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Example

Flip two fair coins and observe whether they come up **H** or **T**.
Define the two random variables X_1, X_2 by

$$\begin{array}{c} \text{Result of coin 1} \\ \hline X = \begin{cases} 1 & \text{if } \mathbf{H} \\ 0 & \text{if } \mathbf{T} \end{cases} \end{array}$$

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Note that $XZ = 0$ (why)? and $E(Z) = \frac{1}{2}$

$$\Rightarrow E(XZ) = 0 \neq \frac{1}{4} = E(X)E(Z)$$

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$$\Rightarrow E\left([X - E(X)] [Y - E(Y)]\right) = 0$$

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Variance for 400 flips is 100.

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By Theorem 5.29,

$$V(X) = V(X_1) + V(X_2) + \cdots V(X_n) = np(1 - p)$$

Returning to our previous histograms (illustrating coin flip and answer distributions) we see that when number of trials grew by a factor of 4, spread observed in histograms grew by factor of two.

Theorem on previous page tells us that when number of trials grows by 4, **Variance** grows by 4 as well.

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This “suggests” that a natural measure of spread “might” be the **square root** of the variance.

This quantity, is called the **standard deviation of RV** X and is usually denoted by $\sigma(X) = \sqrt{V(X)}$, or sometimes just by σ .

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For 25 flips, Variance = $25 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{25}{4} \Rightarrow \sigma = \frac{5}{2}$.

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For 25 flips, Variance = $25 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{25}{4} \Rightarrow \sigma = \frac{5}{2}$.

So, ± 3 standard deviations from expected value is a range of 15 points, which is, again, what we observed.

Central Limit Theorem

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s.d. is standard deviation

Assume a relatively large number of
independent trials with two outcomes.

Percentage of results within 1 s.d. of mean is about 68%;
percentage within 2 s.d.s of mean is about 95.5%;
percentage within 3 s.d.s of mean is about 99.7%.

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Example:

If $a = -1.5$ and $b = 2$, then theorem tells us an approximate probability that sum is between 1.5 standard deviations less than its expected value and 2 standard deviations more than its expected value.

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The distribution given by

$$P(a \leq X \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx$$

is called the **normal distribution**.

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So, we want an n such that $2\sqrt{n}/2 = .01(.5n)$.

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Therefore, need to flip a coin 40,000 times to be 95% sure that number of heads will be within 1% of expected value of 20,000.