

This small handout is a worked example to help clarify the difference between

- Sample spaces

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- Probability distributions on the sample spaces

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In particular, you will see two **different** probability distributions on the **same** sample space.

You will also see an example illustrating why the ***inclusion-exclusion formula*** is actually a statement about sets, and does not depend upon the actual weights of the probability distribution.

Consider the sample space of 3 coin flips (8 outcomes)

$$E = \{\text{first flip is Head}\} = \{HHT, HTT, HHH, HTH\}$$

$$F = \{\text{last flip is Head}\} = \{THH, TTH, HTH, HHH\}$$

$$E \cap F = \{\text{first \& last flips are Heads}\} = \{HHH, HTH\}$$

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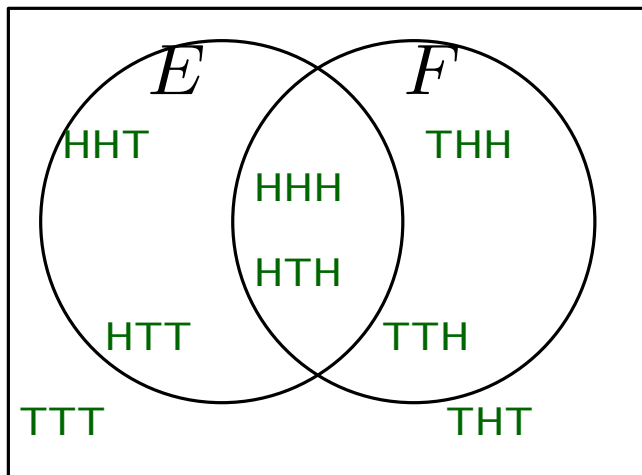
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$$|E| = 4$$

$$|F| = 4$$

$$|E \cap F| = 2$$

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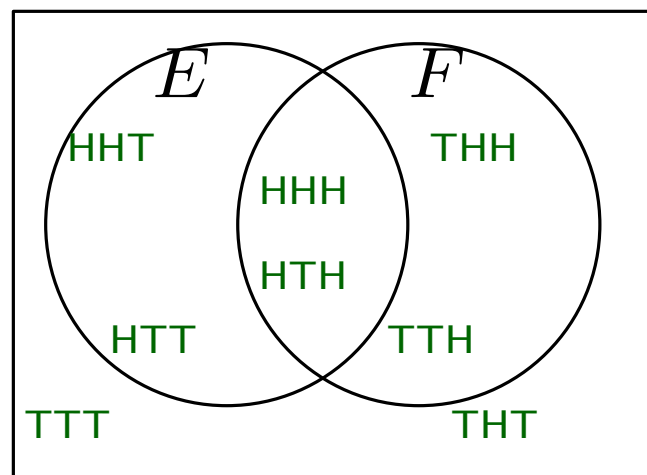
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On the left we first look
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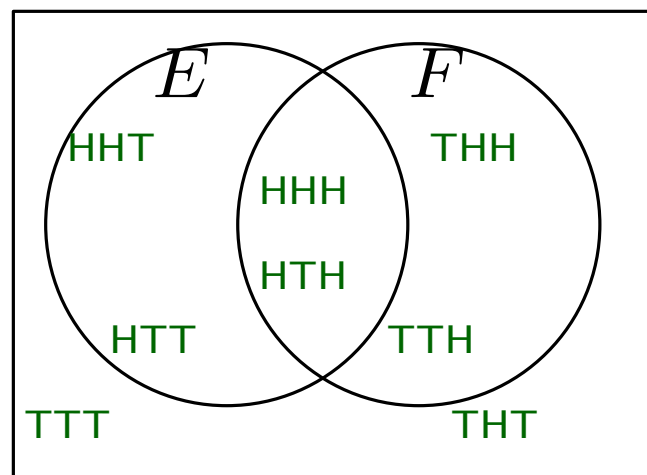
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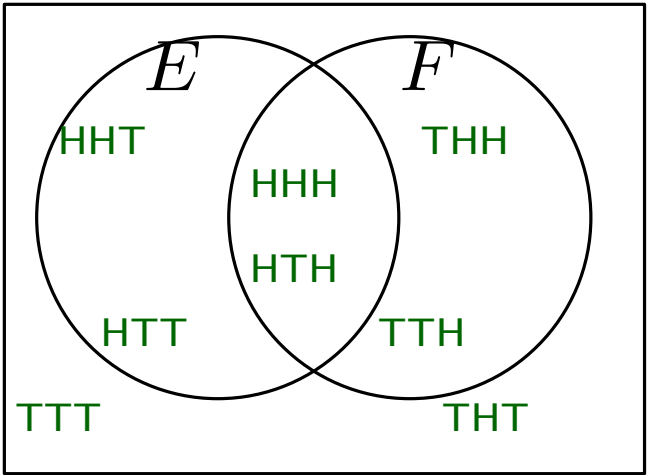
On the left we first look at the *size* of subsets.

Now, assume that the coin is fair so each outcome has the same probability, $\frac{1}{8}$, of occurring.

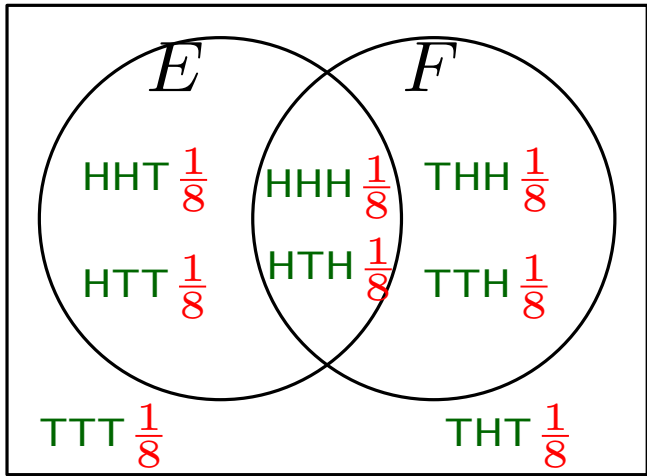
Consider the sample space of 3 coin flips (8 outcomes)

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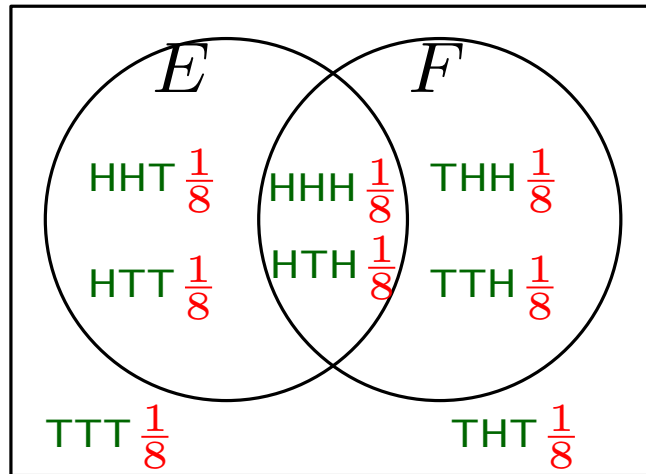
$|E| = 4$
 $|F| = 4$
 $|E \cap F| = 2$



$P(E) = 4 \frac{1}{8} = \frac{1}{2}$
 $P(F) = 4 \frac{1}{8} = \frac{1}{2}$
 $P(E \cap F) = 2 \frac{1}{8} = \frac{1}{4}$

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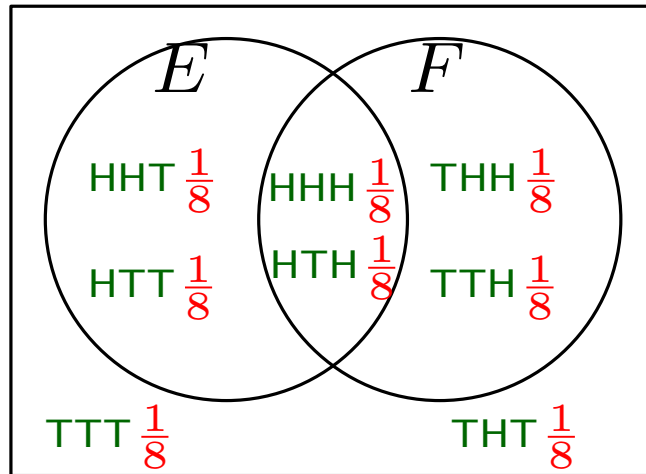
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We just saw the probabilities for fair coins.

This created a *uniform* distribution on the sample space.



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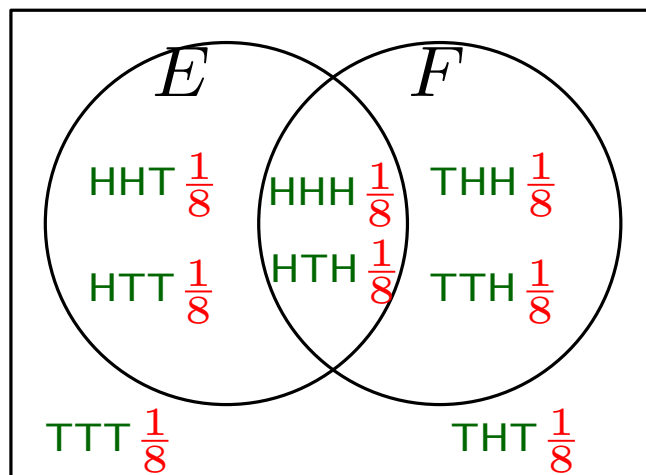
$$P(F) = 4 \frac{1}{8} = \frac{1}{2}$$

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Now assume that the coin is *not* fair. Instead $P(H) = \frac{1}{3}$. In this case we get a *different* probability distribution on the *same* sample space.



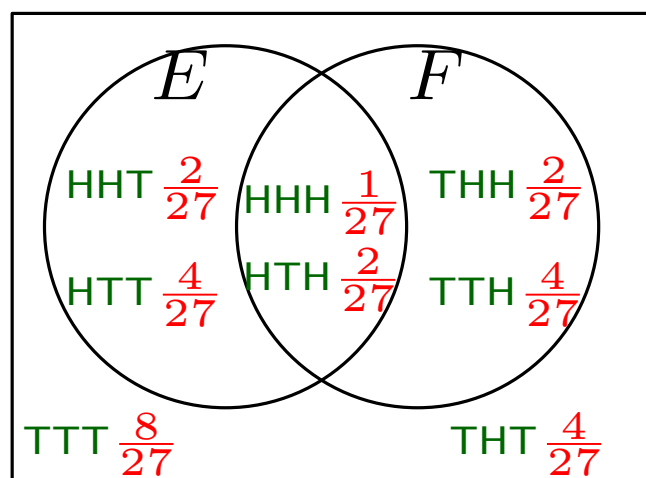
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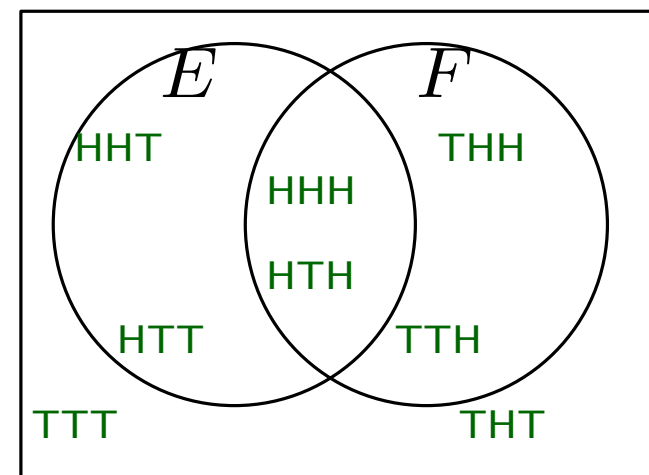
$$P(E) = \frac{1+2+2+4}{27} = \frac{1}{3}$$

$$P(F) = \frac{1+2+2+4}{27} = \frac{1}{3}$$

$$P(E \cap F) = \frac{1+2}{27} = \frac{1}{9}$$

Now assume that the coin is *not* fair. Instead $P(H) = \frac{1}{3}$. In this case we get a *different* probability distribution on the *same* sample space.

As a final note, we point out that the inclusion-exclusion formula is really a statement about the **sample space** and *does not depend* upon the actual weights of the **probability distribution**

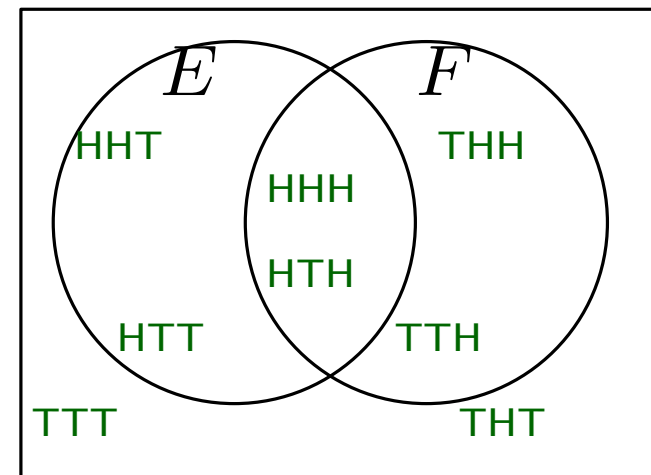


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As an example, consider the $n = 2$ case to the right.

Working through the sets, we see that, **independent of the actual probability weights,**

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

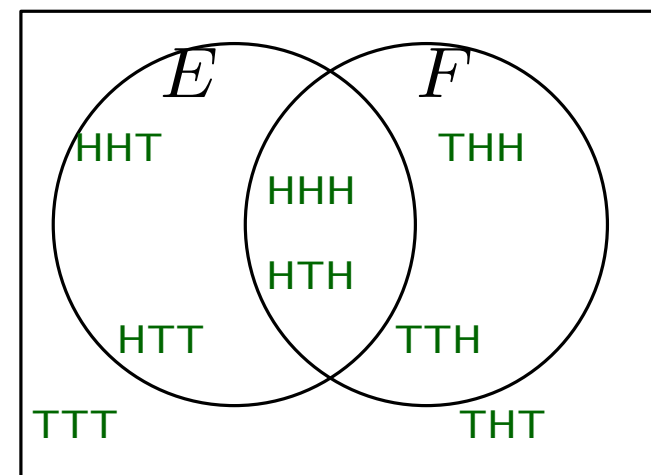


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$$P(E) = P(\{HHT\}) + P(\{HTT\}) + P(\{HHH\}) + P(\{HTH\})$$

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$$P(E \cap F) = P(\{HHH\}) + P(\{HTH\})$$

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