# Sorting: Lower Bounds and Linear Time

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# Lower Bound for Sorting

- All sorting algorithms we seen so far are based on comparing elements
  - E.g., Insertion sort, Selection sort, Mergesort, Heapsort and Quicksort
- Insertion sort, Selection sort and Quicksort have worst-case running times $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

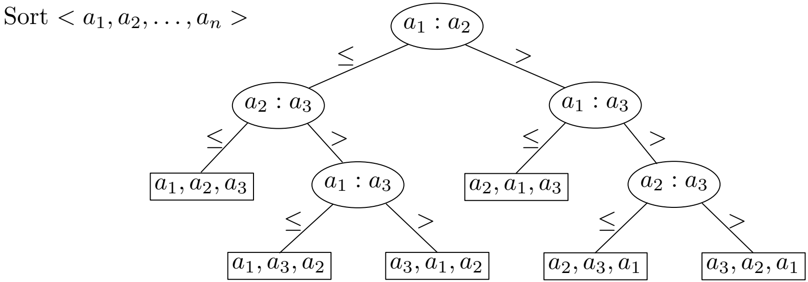
**Question**

Can we do better?

**Goal**

We will prove that any comparison-based sorting algorithm has a worst-case running time $\Omega(n \log n)$.

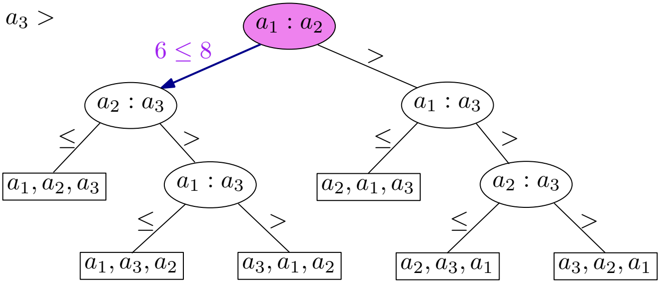
# Decision-tree Example




- Each internal node is labeled $a_i : a_j$ for $\{1, 2, \ldots, n\}$
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

# Decision-tree Example

Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:

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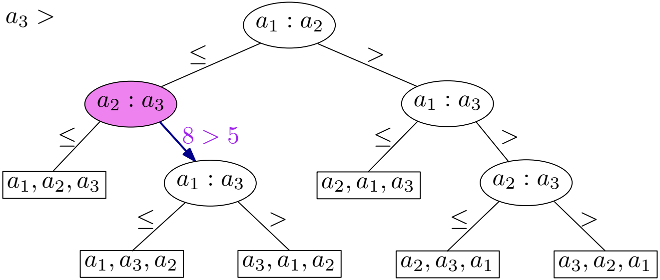


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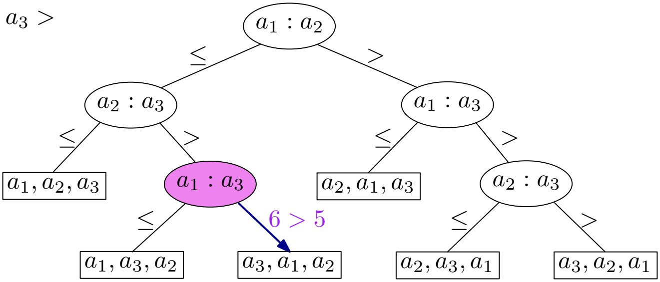


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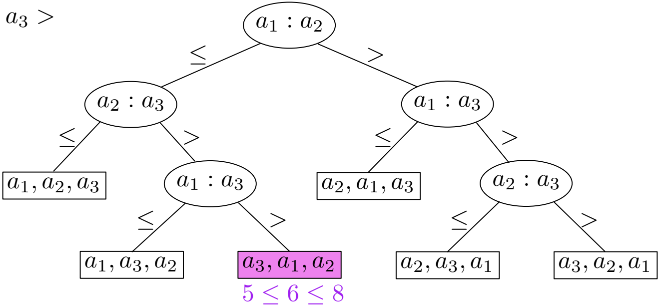


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# Decision-tree Model

A decision tree can model the execution of **any** comparison-based sorting algorithm

- One tree for each input size $n$
- Worst-case running time $=$ height of tree

# Lower Bound for Sorting

## Theorem

*Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.*

## Proof.

- A decision tree to sort $n$ elements must have at least $n!$ leaves, since each of the $n!$ orderings is a possible answer.
- A binary tree of height $h$ has at most $2^h$ leaves
- Thus, $n! \leq 2^h$
  $\Rightarrow h \geq \log n! = \Omega(n \log n)$ (Stirling's approximation)

$\square$

## Corollary

*Heapsort and merge sort are asymptotically optimal comparison-based sorting algorithms.*

# Lower Bound for Average Running Time of Sorting

- We just proved that worst case number of comparisons used is $\Omega(n \log n)$
- Suppose that each of the $n!$ input permutations is equally likely. What can be said about the average case running time?

*Note: Average is taken by adding up individual running time of algorithm on each possible input and dividing total by $n!$.*

**Theorem**

*When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.*

# Lower Bound for Average Running Time of Sorting

## Theorem

*When all input permutations are equally likely, any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons on average.*

## Proof.

- The *External Path Length (EPL)* of a tree is the sum over all leaves of the tree, of the length of the paths from the root to the leaves.
- Average number of comparisons used by a sorting algorithm is EPL of its associated comparison tree divided by $n!$.
- The EPL of a binary tree with $m$ leaves is at least $m \log_2 m + O(m)$.
- The comparison tree has $m = n!$ leaves
  $\Rightarrow$ its external path length is $n! \log_2 n! + O(n!)$
  $\Rightarrow$ average number of comparisons used is $\log_2 n! + O(1)$.
- We already saw $\log_2 n! = \Omega(n \log n)$.

$\square$

# Can we do better?

Are there sorting algorithms which are not comparison-based?
Can they beat the $\Omega(n \log n)$ lower bound?

- Counting sort
  - Assumes items are in set $\{1, 2, \ldots, k\}$.
  - Is a *stable* sort (defined soon).
- Radix sort
  - Assumes items are stored in fixed size words using finite alphabet

# Counting Sort

## Counting-sort($A$, $B$, $k$)

**Input**: $A[1 \ldots n]$, where $A[j] \in \{1, 2, \ldots, k\}$
**Output**: $B[1 \ldots n]$, sorted
let $C[1 \ldots k]$ be a new array;
**for** $i \leftarrow 1$ **to** $k$ **do**
    $C[i] \leftarrow 0$;
**end**
**for** $j \leftarrow 1$ **to** $n$ **do**
    $C[A[j]] \leftarrow C[A[j]] + 1$; // $C[i] = |\{\text{key} = i\}|$
**end**
**for** $i \leftarrow 2$ **to** $k$ **do**
    $C[i] \leftarrow C[i] + C[i-1]$; // $C[i] = |\{\text{key} \leq i\}|$
**end**
**for** $j \leftarrow n$ **to** $1$ **do**
    $B[C[A[j]]] \leftarrow A[j]$;
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**end**

# Example: Counting Sort

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $A$ | 4 | 2 | 1 | 4 | 2 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $C$ | | | | |

| | | | | | |
|---|---|---|---|---|---|
| $B$ | | | | | |

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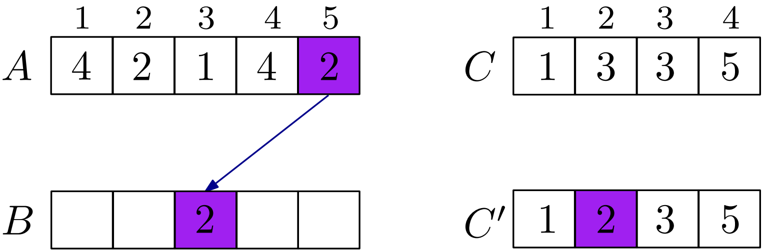


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**for** $i \leftarrow 1$ **to** $k$ **do**
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**end**

Total: $O(n + k)$

# Running Time

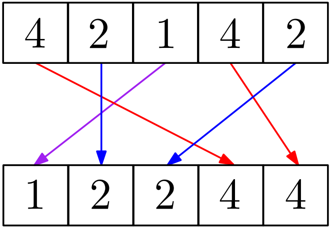
If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.
- Note that counting sort is *not* a comparison-based sorting algorithm.
- In fact, it makes no comparisons at all!

# Stable Sorting

Counting sort is a stable sort

- it preserves the input order among equal elements.




**Exercise**

What other sorts have this property?

# Radix Sort

- Sort on *least significant* digit first using stable sort

| | | | | |
|---|---|---|---|---|
| 2329 | 2720 | 2720 | 2329 | 2329 |
| 5457 | 5355 | 2329 | 5355 | 2720 |
| 3657 | 3436 | 3436 | 3436 | 3436 |
| 5839 | 5457 | 5839 | 5457 | 3657 |
| 3436 | 3657 | 5355 | 3657 | 5355 |
| 2720 | 2329 | 5457 | 2720 | 5457 |
| 5355 | 5839 | 3657 | 5839 | 5839 |

# Radix Sort: Correctness

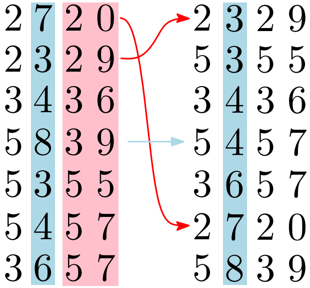
*Induction on digit position*

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit $i$

| | | |
|---|---|---|
| 2 7 2 0 | | 2 3 2 9 |
| 2 3 2 9 | | 5 3 5 5 |
| 3 4 3 6 | | 3 4 3 6 |
| 5 8 3 9 | → | 5 4 5 7 |
| 5 3 5 5 | | 3 6 5 7 |
| 5 4 5 7 | | 2 7 2 0 |
| 3 6 5 7 | | 5 8 3 9 |

*Induction on digit position*

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit $i$
    - Two numbers that differ on digit $i$ are correctly sorted by their low-order $i$ digits



# Radix Sort: Correctness

*Induction on digit position*

- Assume that the numbers are sorted by their low-order $i-1$ digits
- Sort on digit $i$
  - Two numbers that differ on digit $i$ are correctly sorted by their low-order $i$ digits
  - Two numbers equal on digit $i$ are put in the same order as the input $\Rightarrow$ correctly sorted by their low-order $i$ digits

| | |
|---|---|
| 2 7 2 0 | 2 3 2 9 |
| 2 3 2 9 | 5 3 5 5 |
| 3 4 3 6 | 3 4 3 6 |
| 5 8 3 9 | 5 4 5 7 |
| 5 3 5 5 | 3 6 5 7 |
| 5 4 5 7 | 2 7 2 0 |
| 3 6 5 7 | 5 8 3 9 |

# Radix Sort: Running Time & Application

**Lemma**

*Given $n$ $d$-digit numbers in which each digit can take on up to $k$ possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.*

Application:
Sorting numbers in the range from 0 to $n^b - 1$, where $b$ is a constant

- $b \log n$ bits for each number
- each number can be viewed as having $O(b)$ digits of $\log n$ bits each
- running time is $O(d(n+k)) = O(b(n+2^{\log n})) = O(bn)$
- since $b$ is a constant, the running time is $O(n)$.