COMP170
Discrete Mathematical Tools for Computer Science

Lecture 4

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 2.1, pp. 43-54

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b. 1877. d. 1947

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and

... then the great bulk of higher mathematics is **useless.** *Modern Geometry and algebra,* **the theory of numbers**, *the theory of aggregates and functions, relativity, quantum mechanics – no one of them stands the test much better than another, . . .*

If he could see the world now, G.H. Hardy would be spinning in his grave.

Number theory, introduced in this lecture, is the basis of modern coding theory.
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At one point, not long ago, the largest employer of mathematicians in the United States, and therefore probably the world, was the National Security Agency (NSA). The NSA is the largest spy agency in the US – bigger than the CIA – and has the responsibility for code design and breaking.

2.1 Cryptography and Modular Arithmetic

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- Arithmetic Modulo n

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  - Caesar Ciphers: Cryptography Using Addition $\bmod n$
  - Cryptography Using Multiplication $\bmod n$

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A quick review of the laws of arithmetic over the real numbers

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- The *commutative laws* for addition and multiplication

  $a + b = b + a; \quad ab = ba$

  Ex: $3 + 7.2 = 7.2 + 3;; \quad 3 \cdot 5 = 5 \cdot 3.$

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- The *associative laws* for addition and multiplication

  $a + (b + c) = (a + b) + c; \quad a(bc) = (ab)c$

  Ex: $5 + (3 + 7) = (5 + 3) + 7; \quad 5 \cdot (3 \cdot 7) = (5 \cdot 3) \cdot 7$

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- The *distributive law*

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  $a + (-a) = 0.$ Ex: $5 + (-5) = 0.$

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- Every number a has an *arithmetic inverse* $-a$ such that

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- Every number $a \neq 0$ has a *multiplicative inverse* a^{-1} s.t.

  $aa^{-1} = 1.$ Ex: $5 \cdot \frac{1}{5} = 1.$

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$25 \bmod 4 = 1$ because $25 = 4 \cdot 6 + 1$ and any other way of writing $25 = 4 \cdot q + r$ would have an r bigger than 1.

$-25 \bmod 4 = 3$ because $-25 = 4 \cdot (-7) + 3$ and any other way of writing $-25 = 4 \cdot q + r$ would have an r bigger than 3. (why?)

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Note: In general, except if $m = 0$,

$[(-m) \bmod n] = n - [m \bmod n]$ so

$[(-m) \bmod n] \neq [m \bmod n]$ unless

$m = n/2$

(Euclid's Division Theorem)
Let n be a *positive* integer. Then for *every* integer m, there exist *unique* integers q and r such that $m = nq + r$ and $0 \leq r < n$.

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This will be proven in next lecture.
It says that $m \bmod n$ is *uniquely* defined.

Arithmetic Modulo n

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Compute

$21 \bmod 9$

$38 \bmod 9$

$(21 \cdot 38) \bmod 9$

$(21 \bmod 9) \cdot (38 \bmod 9)$

$(21 + 38) \bmod 9$

$(21 \bmod 9) + (38 \bmod 9)$

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It looks as if $[(ab) \bmod n] = [(a \bmod n) \cdot (b \bmod n)]$
and $[(a+b) \bmod n] = [(a \bmod n) + (b \bmod n)]$

Is this true?

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Is this true? **No!** Try $a = 2$, $b = 8$, $n = 9$.

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So what is happening here?

True or false?

$$i \bmod n \;=\; (i + 2n) \bmod n?$$

True or false?

$i \bmod n \;=\; (i+2n) \bmod n$? $\qquad$ $i \bmod n \;=\; (i-3n) \bmod n$?

True or false?

$i \bmod n = (i + 2n) \bmod n$? $i \bmod n = (i - 3n) \bmod n$?

Both true, since adding multiples of n to i does not change the value of the *remainder*, $i \bmod n$.

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Lemma 2.2

$i \bmod n \ = \ (i+kn) \bmod n$ for all integers k.

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Both true, since adding multiples of n to i does not change the value of the *remainder*, $i \bmod n$.

Lemma 2.2

$i \bmod n = (i+kn) \bmod n$ for all integers k.

Proof:

- By Euclid's Division Theorem, $i = nq + r$ (*),
  for *unique* integers q and r, with $0 \leq r < n$.
- By (*) and definition of mod, $r = i \bmod n$.
- Adding kn to both sides, $i + kn = n(q+k) + r$ (**).
- From (**), Eucid's div thm and definition of mod,
  $r = (i+kn) \bmod n$, and we are done.

Lemma 2.3

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\begin{aligned}
(i+j) \bmod n &= (i+(j \bmod n)) \bmod n \\
&= ((i \bmod n)+j) \bmod n \\
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(i \cdot j) \bmod n &= (i \cdot (j \bmod n)) \bmod n \\
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We prove that item on left is equal to bottom item on right.
Proofs of all other equalities are very similar

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By *Euclid's Division Theorem*, for *unique* q_1 and q_2,

$i = (i \bmod n) + q_1 n$ and $j = (j \bmod n) + q_2 n$.

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By *Euclid's Division Theorem*, for *unique* q_1 and q_2,

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Adding these 2 equations together $\bmod n$ and using Lemma 2.2,

$$\begin{aligned}(i+j) \bmod n &= ((i \bmod n)+q_1 n+(j \bmod n)+q_2 n) \bmod n \\ &= ((i \bmod n)+(j \bmod n)+n(q_1+q_2)) \bmod n \\ &= ((i \bmod n)+(j \bmod n)) \bmod n.\end{aligned}$$

Definition:

Z_n is the set of integers $\{0, 1, \ldots, n-1\}$ with

addition $\bmod n$ $\quad i +_n j = (i+j) \bmod n$ and

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- If $x, y \in Z_n$, we use $x +_n y$ and $x \cdot_n y$ to perform algebraic operations on x, y.

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- If $x, y \in Z_n$, we use $x +_n y$ and $x \cdot_n y$ to perform algebraic operations on x, y.
- **Additive identity property**: $0 +_n i = i$.
  Multiplicative identity property: $1 \cdot_n i = i$.

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- **Additive identity property**: $0 +_n i = i$.
  Multiplicative identity property: $1 \cdot_n i = i$.
- $a -_n b$ denotes $a +_n (-b)$.

Theorem 2.4

Addition and multiplication $\mathrm{mod}\, n$ satisfy the **commutative**, **associative** and **distributive** laws.

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Addition and multiplication $\mod n$ satisfy the **commutative**, **associative** and **distributive** laws.

Proof: Commutativity of $+_n$ and $\cdot_n$ follows immediately from commutativity of ordinary addition and multiplication. We prove the associative law for addition in the following equations; the other laws follow similarly.

Theorem 2.4
Addition and multiplication $\mathrm{mod}\, n$ satisfy the **commutative**, **associative** and **distributive** laws.

$$a +_n (b +_n c) = (a + (b +_n c)) \text{ mod } n$$

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Associative law for ordinary sums.

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A **sender** is trying to send a message to a **receiver**. An **adversary** wants to steal the message.

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A **sender** is trying to send a message to a **receiver**. An **adversary** wants to steal the message.

Sensitive information is **encrypted** — modified in such a way that it should be only understandable by the receiver and **undecipherable** to the adversary.

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A *difficult* goal!

2.1 Cryptography and Modular Arithmetic

- Arithmetic Modulo n
- Introduction to Cryptography
- Private-Key Cryptography
  - Caesar Ciphers: Cryptography Using Addition $\bmod n$
  - Cryptography Using Multiplication $\bmod n$
- Public-Key Cryptography

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This cipher is named after the Roman emperor **Julius Caesar** (b. 100BC, d. 44BC). Caesar supposedly used this type of cipher (with a shift of 3) to communicate with his generals.

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Plaintext message: `ONE IF BY LAND AND TWO IF BY SEA.`

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Easy to implement using arithmetic mod 26.

Use 0 for `A`, 1 for `B`, $\ldots$.

Convert a message to a sequence of numbers.

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A: Replace n by $(n+2) \bmod 26$, $\quad\Rightarrow\quad$ `SEA` becomes $20\ 6\ 2$.

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Q: What if we shift every letter 13 places to the right?

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Use 0 for $\mathtt{A}$, 1 for $\mathtt{B}$, ...

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A Caesar cipher with shift s can easily be implemented on most computers by replacing each "letter" n with $(n+s) \bmod 26$. Most computer languages can easily convert between text and numbers, and provide predefined mod functions.

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  Easy: Replace n' by $(n' - s) \bmod 26$, A Caesar cipher with shift s can easily be implemented on most computers by replacing each “letter” n with $(n + s) \bmod 26$.

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- So, we've just seen how $+_n$ on Z_n (for $n = 26$) can be used to implement **encrypting** and **decrypting** Caesar ciphers.

A slightly different view

- A Caesar cipher has a private-key k
- To encode x, use the function $f_k(x) = x +_{26} k$
- To decode y, use the function $g_k(y) = y -_{26} k$
- Note that $g_k(y) = f_k^{-1}(y)$,
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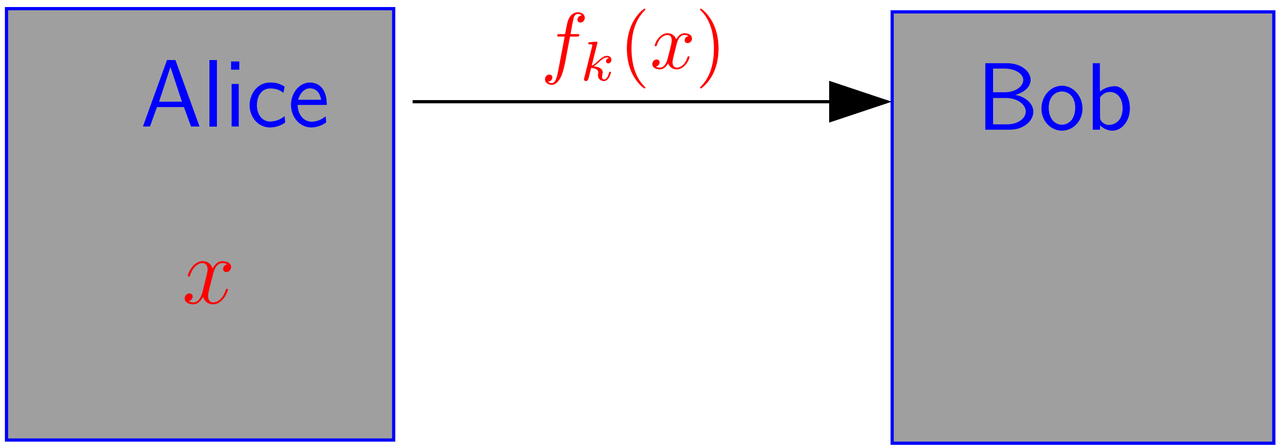
i) Alice has letter x

ii) She sends $f_k(x)$ to Bob



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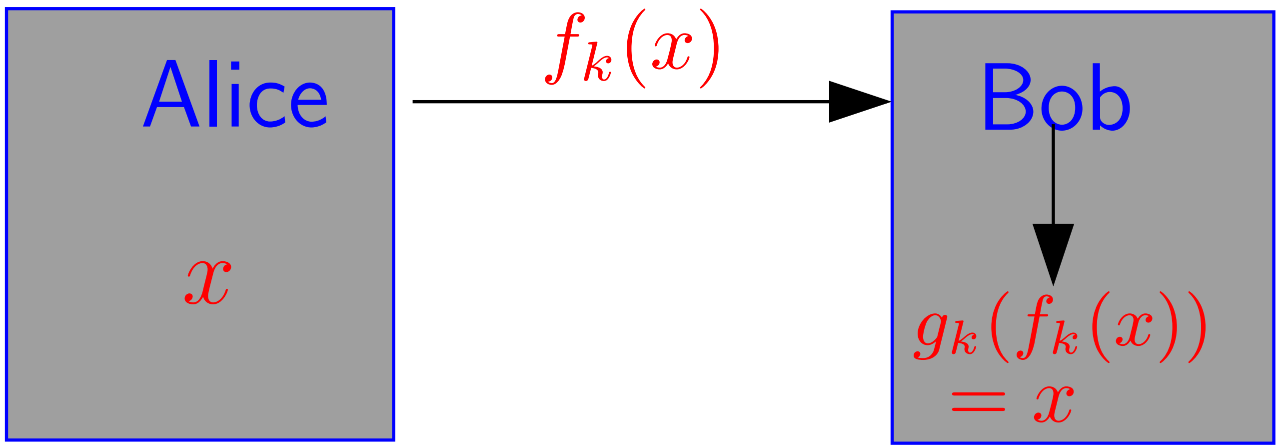
iii) Bob calculates
$x = g_k(f_k(x))$



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  Encrypt: $f_{a,n}(M) = a \cdot M \bmod n = a \cdot_n M$
  Decrypt: "Divide" $f(M)$ by a mod n.

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$g_{a,n}(f_{a,n}(M)) = M$ where $g_{a,n}(X) = a^{-1} \cdot_n X$

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Does division exist?

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Suppose, for some x, we had calculated $f(x) = a \cdot_n x$.
Does f^{-1} exist?

Consider the following three cases of a, x, n.
(a) $(a, x, n) = (4, 3, 12)$, (b) $(a, x, n) = (3, 6, 12)$,
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- You send the message $f(x) = 4 \cdot_{12} 3 = 0$.
- Recipient sees 0, and thinks, x could be 3, since $4 \cdot_{12} 3 = 0$.

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- But $4 \cdot_{12} 0 = 0$, $4 \cdot_{12} 6 = 0$, and $4 \cdot_{12} 9 = 0$ as well.
  So all $x \in \{0, 3, 6, 9\}$ could be possible values for x s.t. $f(x) = 0$.

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  So all $x \in \{0, 3, 6, 9\}$ could be possible values for x s.t. $f(x) = 0$.
- $\Rightarrow$ f^{-1} doesn't exist! Impossible to decrypt!

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- But $3 \cdot_{12} 2 = 6$, $3 \cdot_{12} 10 = 6$, as well.
  So all $x \in \{2, 6, 10\}$ could be possible values for x s.t. $f(x) = 6$.

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- You send the message $f(x) = 5 \cdot_{12} 7 = 11$.

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- You send the message $f(x) = 5 \cdot_{12} 7 = 11$.
- Recipient sees 11, and thinks, x could be 7, since $5 \cdot_{12} 7 = 11$.

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- $\Rightarrow$ Recepient could decrypt this messsage!

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- We just saw that in cases (a) and (b), there is an x' s.t. $f(x') = f(x)$ so recipient would not be able to decrypt message. This means that we can not use this $f(x)$ as an encoding function

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- We just saw that in cases (a) and (b), there is an x' s.t. $f(x') = f(x)$ so recipient would not be able to decrypt message. This means that we can not use this $f(x)$ as an encoding function
- In case (c), given x, recipient could uniquely calculate x so $f(x)$ might be a good encoding function.

What exactly does division $\bmod n$ mean?

Suppose, for some x, we had calculated $f(x) = a \cdot_n x$.
Does f^{-1} exist?

Consider the following three cases of a, x, n.
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- In case (c), given x, recipient could uniquely calculate x so $f(x)$ might be a good encoding function.
- $f(x)$ can be used as an encoding function
  when $f(x)$ has an inverse!

When does $f_{a,n}(x) = a \cdot_n x$ have an inverse?

$f_{a,n}(x) = a \cdot_n x$ has an inverse if and only if a and n are relatively prime, i.e., they have no common factors greater than 1.

In the next lecture we will see what this means and how to use it to define divsion in Z_n.

2.1 Cryptography and Modular Arithmetic

- Arithmetic Modulo n

- Introduction to Cryptography
- Private-Key Cryptography
  - Caesar Ciphers: Cryptography Using Addition $\bmod n$
  - Cryptography Using Multiplication $\bmod n$

- Public-Key Cryptography

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  No good answer to this

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- Motivation for Public-Key Cryptography

Public-Key Cryptosystems

- In **private-key cryptosystems** the sender and receiver *share* a private-key or codebook.
  The same key is used for encypting and decrypting.
  Implicit assumption: knowing how a message is encypted implies knowing how to decrypt it

- In **public-key cryptography** this is no longer true.
  Everybody has two keys; a **public key** and a **secret key**.

- **My public key:** Known by all. Used to send me a message
  My secret key: Known by only by me.
  Used to decrypt messages sent to me that were encrypted using my public key.

- **Important:** Even though everyone knows how messages sent to me were encrypted, only I can decrypt them.

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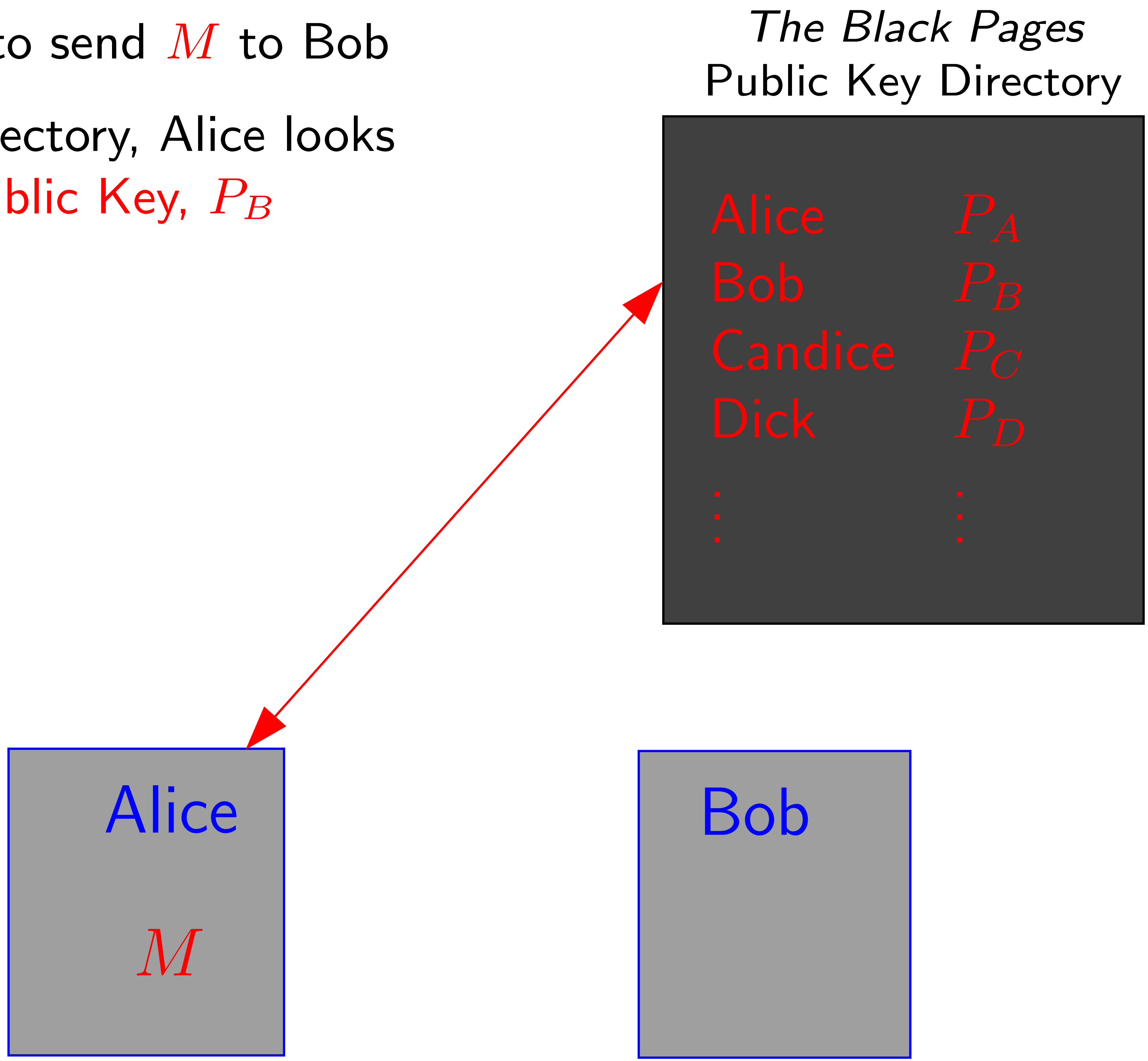
i) Alice wants to send M to Bob

The Black Pages
Public Key Directory

Alice	P_A
Bob	P_B
Candice	P_C
Dick	P_D
$\vdots$	$\vdots$

i) Alice wants to send M to Bob

ii) In public directory, Alice looks up Bob's Public Key, P_B



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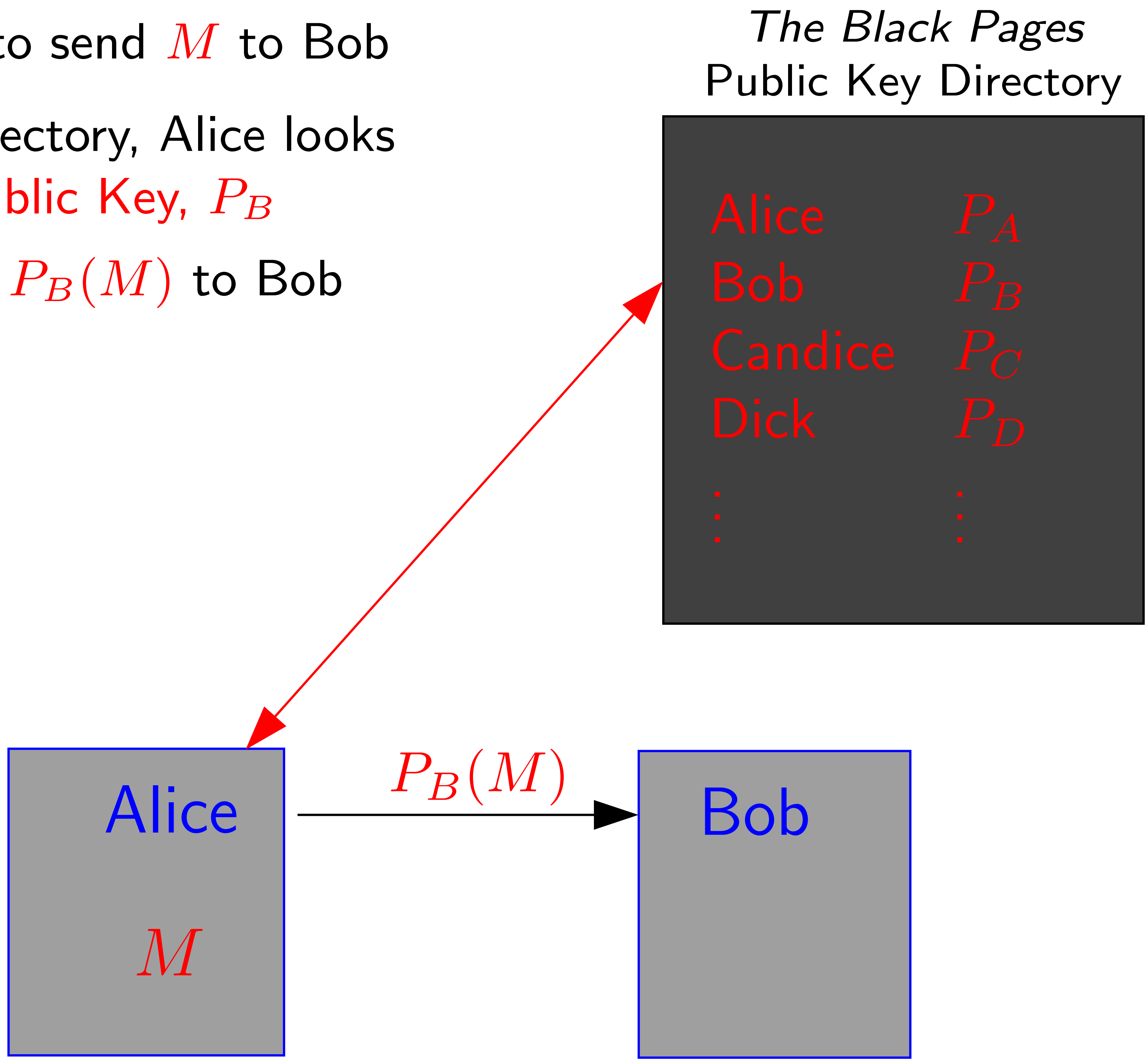
iii) Alice sends $P_B(M)$ to Bob

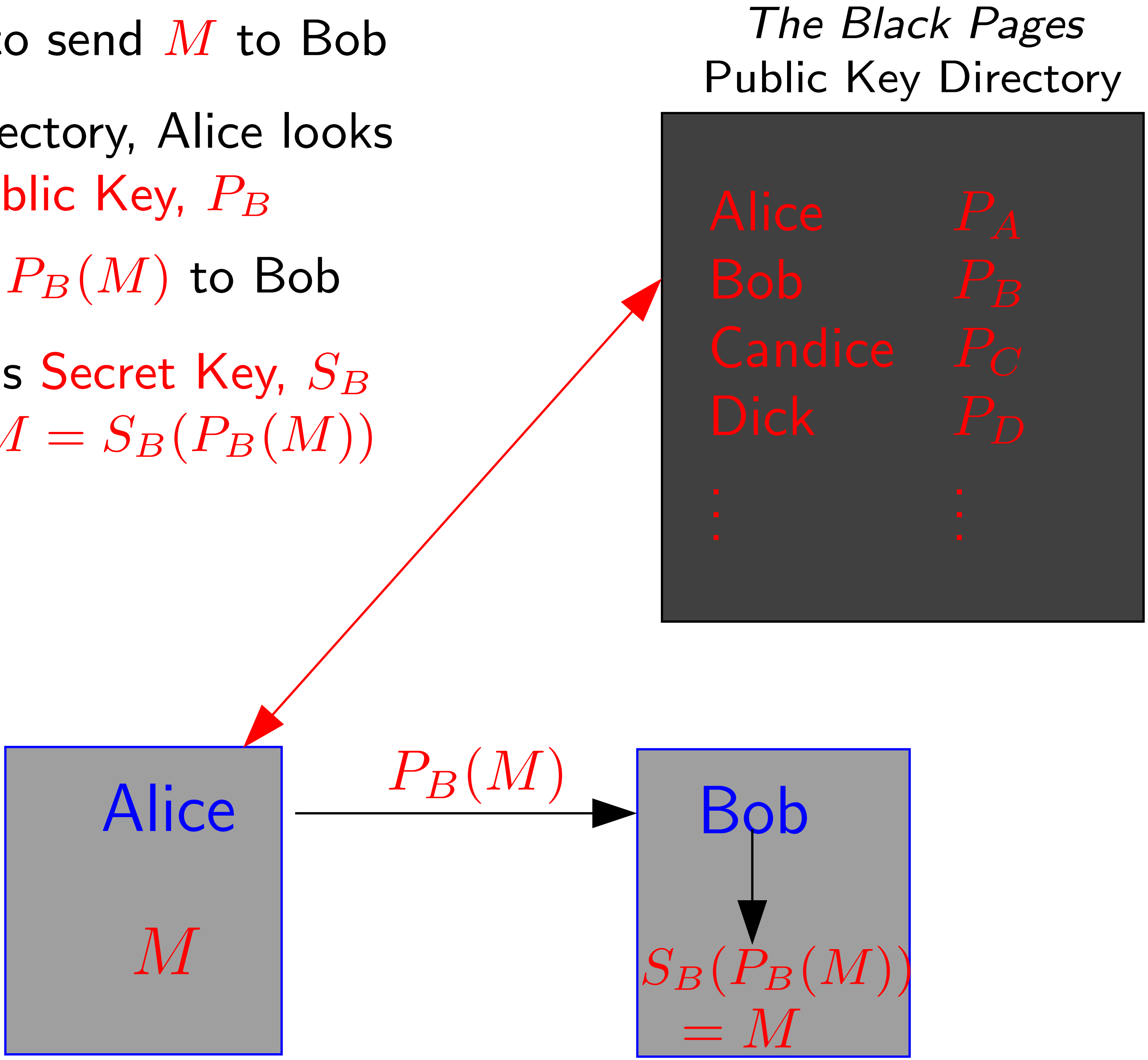


i) Alice wants to send M to Bob

ii) In public directory, Alice looks up Bob's Public Key, P_B

iii) Alice sends $P_B(M)$ to Bob

iv) Bob uses his Secret Key, S_B to decrypt $M = S_B(P_B(M))$



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- Public key available to anyone (including eavesdroppers).
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- Functions associated with KS_A, KP_A, KS_B,KP_B are S_A, P_A, S_B, and P_B. S_A and P_A are inverses; S_B and P_B are inverses; So, for any message M

$$M = S_A(P_A(M)) = P_A(S_A(M)),$$
$$M = S_B(P_B(M)) = P_B(S_B(M)).$$

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- In this case, $S_B(C) = 1000 - rev(C)$,
  so Bob can easily decode the message.
- Problem: this is *Not* secure, because *anyone* who knows **public key,** P_B**,** can figure out **secret key** S_B.

Challenge: In order for a public-key cryptosystem to work we must be able to find **public/secrete key pairs** such that

- Receiver Bob can easily calculate $S_B(X)$
- No one else knowing **public key,** P_B**,** will easily be able to figure our **secrete key,** S_B.

Constructing such **pubic/secret key pairs** sounds almost impossible. Surprisingly, in the mid 1970s, Rivest, Shamir and Adelman, figured out how to do this using simple modular arithmetic.

The result is the **RSA Public Key Cryptosystem**, which is the basis for most e-commerce. We will learn its details in the lecture following the next one