COMP170
Discrete Mathematical Tools for Computer Science

Independence

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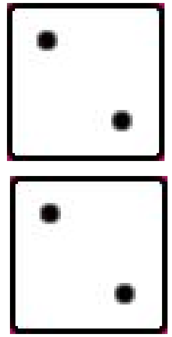
Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 5.3, pp. 236-247

Conditional Probability and Independence

- Conditional Probability
- Independence
- Independent Trials Processes

Conditional Probability

Suppose we've thrown two fair dice. The probability of seeing "double-twos" is $\frac{1}{36}$.



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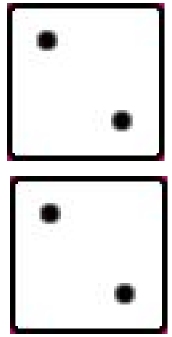

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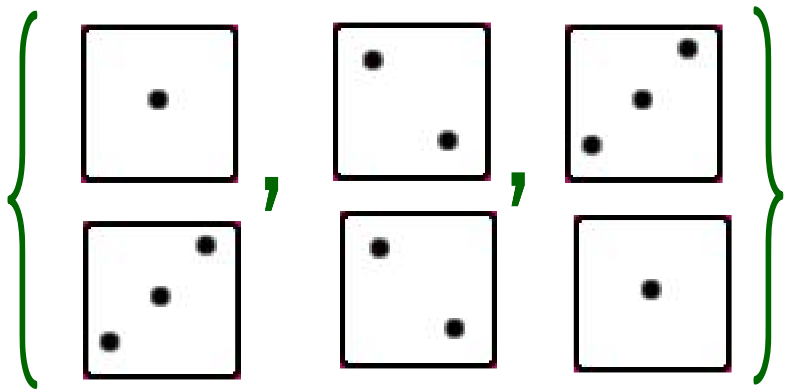

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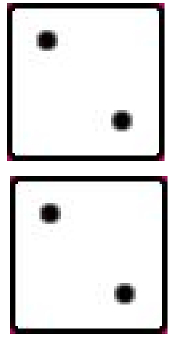

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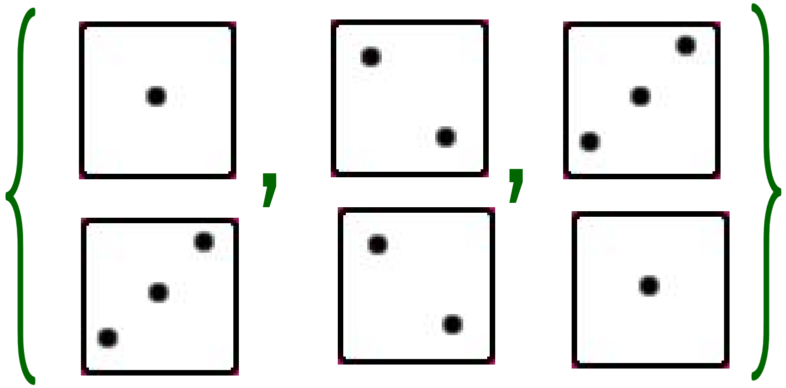

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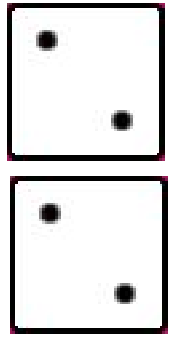

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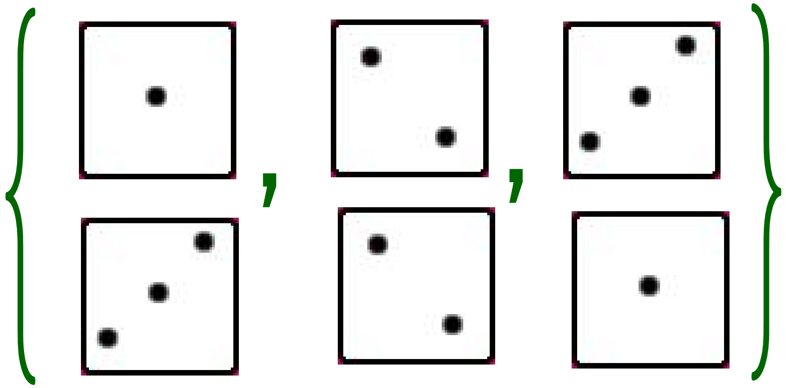

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This lecture formalizes this intuition.

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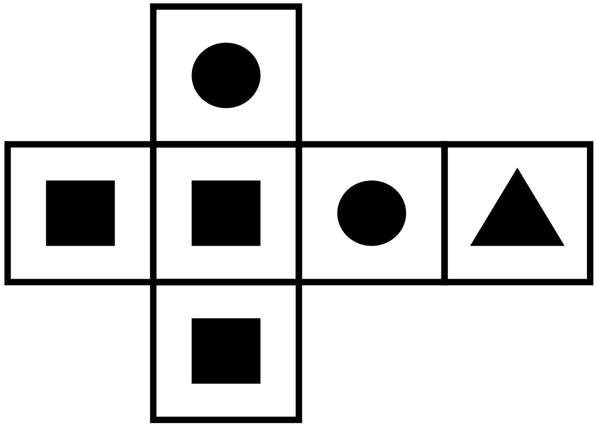

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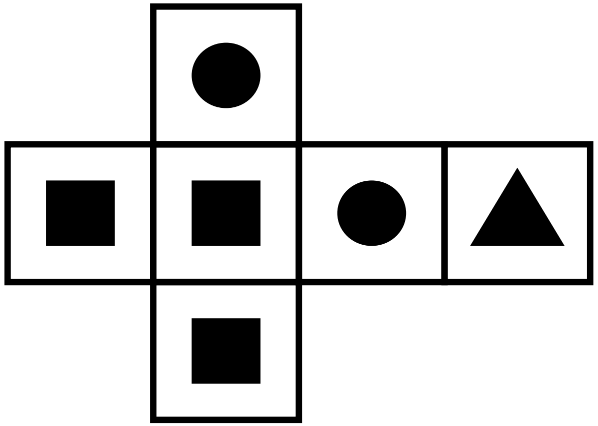

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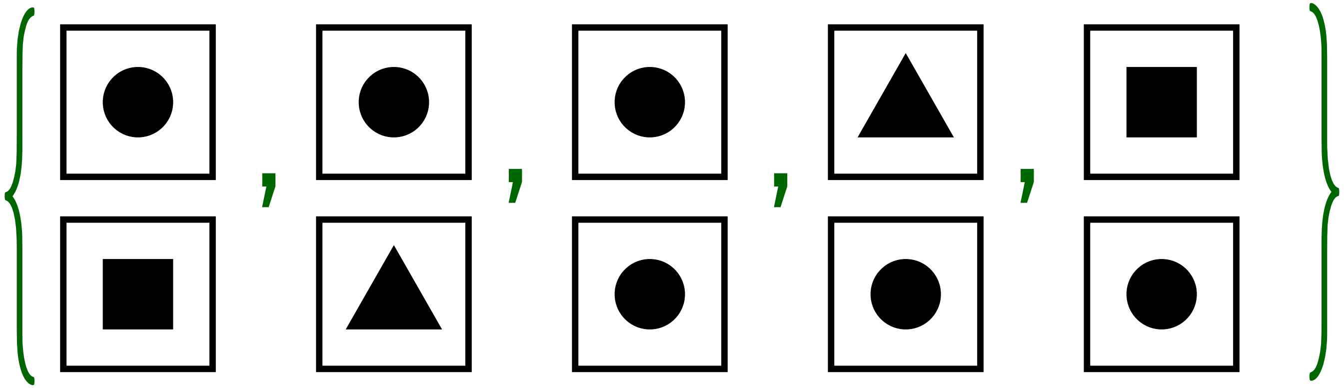

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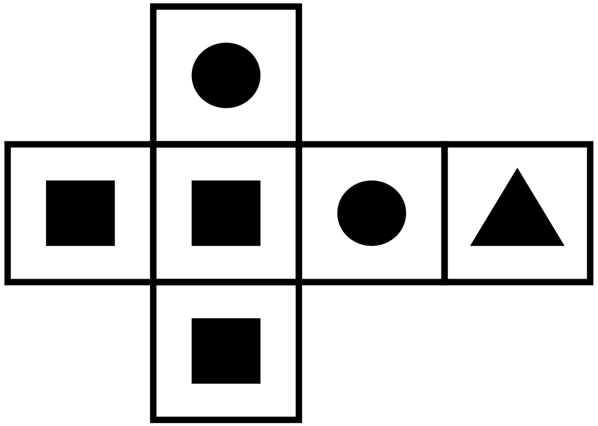

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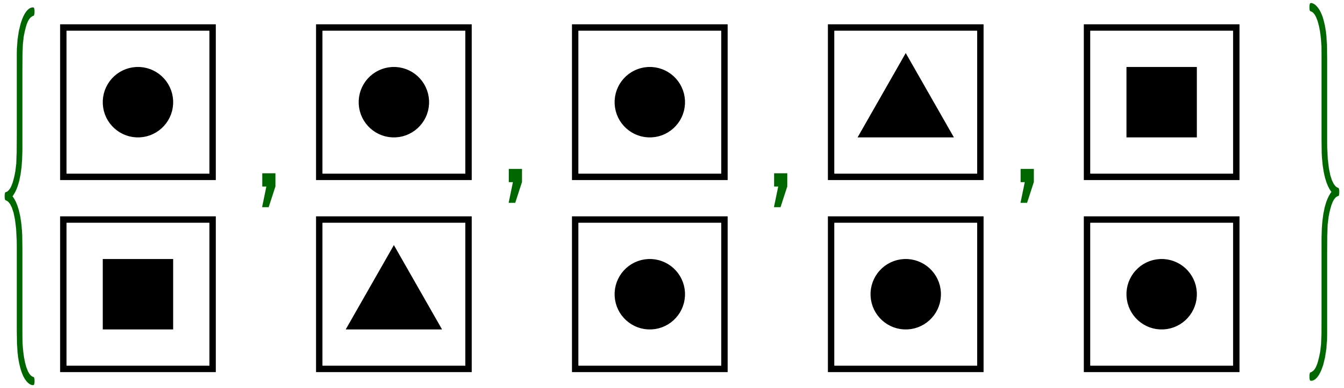

Applying **principle of inclusion and exclusion**: probability of seeing a circle on at least one top when we roll the dice is

$$\frac{1}{3} + \frac{1}{3} - \frac{1}{9} = \frac{5}{9}$$

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$p + 4p + 9p = 1$ or $p = \frac{1}{14}$, and

$P(\text{two circles if both tops are the same}) = 4p = \frac{2}{7}$.

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WARNING

There are situations where our intuitive idea of probability does not always agree with what the rules of probability give us!

Rolling our two unusual dice

Original sample space with probabilities

$$\begin{array}{ccccccccc} \{\texttt{TT}, & \texttt{TC}, & \texttt{TS}, & \texttt{CT}, & \texttt{CC}, & \texttt{CS}, & \texttt{ST}, & \texttt{SC}, & \texttt{SS}\}. \\ \frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{18} & \frac{1}{9} & \frac{1}{6} & \frac{1}{12} & \frac{1}{6} & \frac{1}{4} \end{array}$$

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Given this, what probability should we assign event of seeing a circle ($\texttt{CC}$)?

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new probabilities will preserve ratios and sum to 1.

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We now capture this reasoning process in a formula!

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Note: This definition doesn't make sense when $P(F) = 0$.
In this case we define $P(E|F) = E$.
This makes sense, since if event F can not occur then it occuring gives us no information
(since this can't happen).

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$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/9}{1/6} = \frac{2}{3}.$$

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The answer might surprise you!

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Use conditional probabilities:

$$\begin{aligned} P(R) &= P(R \cap K) + P\left(R \cap \overline{K}\right) \\ &= P(R|K)P(K) + P\left(R|\overline{K}\right) P\left(\overline{K}\right) \\ &= 1 \cdot .8 + .5 \cdot .2 = .9. \end{aligned}$$

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Which implies that she wil get a 90% on the exam!

Conditional Probability and Independence

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$$\begin{aligned} P(E|F) &= P(\text{total sum is odd} \mid \text{red die is odd}) \\ &= P(\text{green die is even}). \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned}$$

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Thus, by definition of independence, "total sum is odd" and "red dice shows an odd number of dots" are independent.

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Case 1: F is empty.

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E is independent of F if and only if $P(E \cap F) = P(E)P(F)$

Proof:

Case 1: F is empty.

Then $P(E) = P(E|F)$ so E is independent of F.

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Also $P(E)P(F) = 0 = P(E \cap F)$.

So in this case,
E is independent of F and $P(E \cap F) = P(E)P(F)$.

Case 2: F is nonempty.

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So in this case as well,
E is independent of F if and only if $P(E \cap F) = P(E)P(F)$.

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Suppose E and F are events in a sample space. Then

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E is independent of F if and only if F is independent of E.

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E is independent of F if and only if F is independent of E.

Example:
We already saw, when throwing a red, green die
that "total sum is odd" is independent of "red die is odd"
$\Rightarrow$ "red die is odd" is independent of "total sum is odd".

Coin Flipping

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When flipping a coin twice, we think of second outcome as being independent of first.

Coin Flipping

When flipping a coin twice, we think of second outcome as being independent of first.

Does definition of independence capture this intuitive idea? Let's compute relevant probabilities to see if it does!

Flipping a coin twice

Flipping a coin twice

Sample space with their probabilities

$$\{\texttt{HH},\ \texttt{HT},\ \texttt{TH},\ \texttt{TT}\}.$$

$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}$$

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$$P(\texttt{H}\ \text{first}) = 1/4 + 1/4 = 1/2$$

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$$\frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4}$$

$$P(\texttt{H}\text{ first}) = 1/4 + 1/4 = 1/2$$

$$P(\texttt{H}\text{ second}) = 1/4 + 1/4 = 1/2$$

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$$P(\texttt{H}\text{ first})P(\texttt{H}\text{ second}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(\texttt{H}\text{ first and }\texttt{H}\text{ second}).$$

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By Theorem 5.5, "H second" is independent of "H first".

Flipping a coin twice

Sample space with their probabilities $\{\mathtt{HH},\ \mathtt{HT},\ \mathtt{TH},\ \mathtt{TT}\}$.

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$$P(\mathtt{H}\text{ first}) = 1/4 + 1/4 = 1/2$$

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By Theorem 5.5, "H second" is independent of "H first".

Similarly

"T second" is independent of "T first".

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Example:

Recall the sample space and probability weights we used when discussing hashing.

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We will now show that the two events
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- If $i \neq j$,

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- This is the product of the probabilities that

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- This is the product of the probabilities that

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- Therefore, these two events are independent.

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independent when $i = j$?

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If $i = j$, probability of

"i hashes to r and j hashes to q"

is 0, unless $r = q$, in which case it is 1.

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independent when $i = j$?

If $i = j$, probability of

“i hashes to r and j hashes to q”

is 0, unless $r = q$, in which case it is 1.

Thus, these two events are **not** independent.

Conditional Probability and Independence

- Conditional Probability

- Independence

- Independent Trials Processes

Independent Trials Processes

So far, we've considered *static* sample sets.
That is, we assumed that our sample space contains all possible outcomes that can happen. Many problems, though, are modelled using dynamic processes.

For example, we flip coins one-by-one. After flipping 5 coins, we might do something, and then flip the 6^{th}. Our intuition is that the sixth flip should be independent of the outcomes of the first five.

As another example, we don't hash n keys all at once. We usually hash the first key, then the second, then the third, etc.. Our intuition is that the hashing of the k^{th} key should also be independent of the hashing of the first $(k-1)$ keys.

We formalize this idea with the introduction of
Independent Trials Processes.

Examples:
Coin Flipping and Hashing

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The Process Proceeds in Stages:

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ex: $S_i = \{\mathtt{H},\mathtt{T}\}, 1 \le i \le n$.

Examples:

Coin Flipping and Hashing

The Process Proceeds in Stages:

x_i: outcome at stage i. ex: $x_i = \mathtt{H}$.

S_i : set of possible outcomes of stage i.

ex: $S_i = \{\mathtt{H},\mathtt{T}\}, 1 \le i \le n$.

A process that occurs in stages is called an **independent trials process** if

$$P(x_i = a_i | x_1 = a_1, \ldots, x_{i-1} = a_{i-1}) = P(x_i = a_i)$$

for each sequence $a_1, a_2, \ldots, a_n$, with $a_i \in S_i$.

Formally, let E_i be the event that $x_i = a_i$. Then

$$P(x_i = a_i | x_1 = a_1, \ldots, x_{i-1} = a_{i-1}) = P(x_i = a_i)$$

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$$P(E_i | E_1 \cap E_2 \cap \ldots \cap E_{i-1}) = P(E_i).$$

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In words:

An independent trials process has the property that outcome of stage i is independent of outcomes of stages 1 through $i-1$.

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In words:
An independent trials process has the property that outcome of stage i is independent of outcomes of stages 1 through $i-1$.

By product principle for independent probabilities (Theorem 5.5), this is equivalent to

$$P(E_1 \cap E_2 \cap \ldots \cap E_{i-1} \cap E_i) = P(E_1 \cap E_2 \cap \ldots \cap E_{i-1}) P(E_i).$$

Theorem 5.7 In an independent trials process, the probability of a sequence $a_1, a_2, \ldots, a_n$ of outcomes is

$$P(\{a_1\})\, P(\{a_2\}) \cdots P(\{a_n\}).$$

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Proof:

Apply mathematical induction and use

$$P(E_1 \cap E_2 \cap \ldots \cap E_{i-1} \cap E_i) = P(E_1 \cap E_2 \cap \ldots \cap E_{i-1})P(E_i).$$

Relation of independent trials

Relation of independent trials to coin flipping:

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Sample space consists of sequences of n `H`'s and `T`'s.

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Probability of "$\texttt{H}$ on the ith flip, given a particular sequence on the first $i-1$ flips", is

$$\frac{2^{n-(i-1)-1}}{2^{n-(i-1)}} = \frac{1}{2}$$

.

Then "$\texttt{H}$ (or $\texttt{T}$) on ith flip" is independent of "$\texttt{H}$ (or $\texttt{T}$) on each of first $i-1$ flips".

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$$P\left(\begin{array}{l}\text{key } i \text{ hashes to } r \qquad\qquad\qquad \text{and} \\ \text{keys } 1 \text{ through } i-1 \text{ hash to } q_1, q_2, \ldots, q_{i-1}\end{array}\right)$$

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$$P(\text{keys } 1 \text{ through } i-1 \text{ hash to } q_1, q_2, \ldots, q_{i-1}) = \frac{k^{n-(i-1)}}{k^n} = k^{1-i}.$$

By definition of conditional probability,

$$P\left(\begin{array}{l} \text{key } i \text{ hashes to } r \\ \quad \mid \text{ keys } 1 \text{ through } i-1 \text{ hash to } q_1, q_2, \ldots, q_{i-1} \end{array} \right)$$

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Since this is equal to

$$P(\text{key } i \text{ hashes to } r\ ) = \frac{k^{n-1}}{k^n} = k^{-1}$$

our model of hashing is an independent trials process.

Suppose we draw a card from a standard deck of 52 cards, replace it, draw another card, and continue for a total of ten draws.

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Suppose we draw a card from a standard deck of 52 cards, replace it, draw another card, and continue for a total of ten draws.
Is this an independent trials process?

Yes.

Because the probability that we draw a given card at one stage does *not* depend on the cards we drawn in earlier stages.

Suppose we draw a card from a standard deck of 52 cards, discard it (i.e., we do not replace it), draw another card, and continue for a total of ten draws.

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In the second draw, we have only 51.

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In particular, we do not have the same cards to draw from on the second draw as on the first.

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In the second draw, we have only 51.

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So, the probabilities for each possible outcome on the second draw depend on the outcome of the first draw.

Example:

Draw two cards.
What is the probability that you are holding two aces?

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(1) Drawing with replacement (first case):

$$\frac{4^2}{52^2} = \frac{1}{13^2} \approx .0059$$

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What is the probability that you are holding two aces?

(1) Drawing with replacement (first case):

$$\frac{4^2}{52^2} = \frac{1}{13^2} \approx .0059$$

(2) Drawing without replacement (second case):

$$\frac{4 \cdot 3}{52 \cdot 51} = \frac{3}{13 \cdot 51} \approx .0045$$

More Examples:

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Suppose we flip n coins and want to calculate the probability that *at least one coin shows a H*. One way to do this would be to use the inclusion-exclusion principle. Now that we know that coin tosses are independent trials, though, another easier way is as follows:

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Suppose we flip n coins and want to calculate the probability that *at least one coin shows a H*. One way to do this would be to use the inclusion-exclusion principle. Now that we know that coin tosses are independent trials, though, another easier way is as follows:

Let E_i be the event of "$\mathtt{T}$ on ith flip"

$$Pr(E_i) = \frac{1}{2}$$

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So, the probability that *all coins show a T is*

$$Pr(E_1 \cap E_2 \cap \cdots \cap E_n) = Pr(E_1) \cdot Pr(E_2) \cdots Pr(E_n) = \frac{1}{2^n}$$

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and the probability that *at least one coin shows an H* is

$$1 - \frac{1}{2^n}$$