Functions

Cunsheng Ding

HKUST, Hong Kong

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Contents

What is a Function?

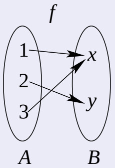
Definition 1

1. A <u>function</u> from a set A to a set B is a binary relation f from A to B with the property, for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$. In this case, we write $f(a) = b$.
2. A is called the <u>domain</u> of f, and B is called the <u>codomain</u> of f. The <u>range</u> of f is defined as

$$\mathrm{Range}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$

Example 2

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Then $f = \{(1, x), (2, y), (3, x)\}$ is a function from A to B. The arrow diagram is given on the right-hand side.



Comments on the Definition of Functions

1. For every $a \in A$, $f(a)$ must be defined.
2. For every $a \in A$, $f(a)$ must be in B, the codomain.
3. For every $a \in A$, $f(a)$ must be unique.

Example 3

Let $A = \{1,2,3,4\}$ and $B = \{x,y\}$. The binary relation $f = \{(1,x),(2,y),(3,x)\}$ is not a function, as $f(4)$ is not defined.

Example 4

Let $A = B = \{1,2,3,4\}$. Define $f(x) = x+1$. Then f is not a function, as $f(4) = 5 \notin B$.

Example 5

Let $A = \{1,2,3\}$ and $B = \{\Delta,\Gamma\}$. Define a binary relation g as

$$g = \{(1,\Delta),(1,\Gamma),(2,\Delta),(3,\Delta)\}$$

Then g is not a function, as $g(1)$ is not unique.

Descriptions of Functions

Remarks

1. Functions are also called <u>mappings</u>.
2. Let f be a function from A to B. $f(a)$ is called the <u>image</u> of a.
3. $(a,b) \in f$ means that $b = f(a)$. In this case, a is called the <u>preimage</u> of b with respect to f.
4. Write $f : A \to B$ to mean that f is a function from A to B.
   $f(a) = b$ means that $f : a \mapsto b$.

Ways to describe functions

1. In terms of ordered pairs.

$$f = \{(\ \ ), (\ \ ), \ldots, (\ \ )\}$$

2. Using arrow diagram.
3. Using "$\mapsto$".

$$\begin{aligned} f \quad : \quad & x_1 \mapsto y_1 \\ & x_2 \mapsto y_2 \\ & \vdots \\ & x_n \mapsto y_n \end{aligned}$$

4. Using mathematical formulas.

$$f(x) = x^2 + x - 6$$

Equality of Two Functions

Definition 6

Two functions f and g are said equal iff they have the same domain and codomain and $f(a) = g(a)$ for each a in the domain.

Example 7

Define functions f and g from $\mathbb{R}$ to $\mathbb{R}$ by the formulas: for all $x \in \mathbb{R}$,

$$f(x) = 2x \text{ and } g(x) = \frac{2x^3 + 2x}{x^2 + 1}$$

Show that $f = g$.

Proof.

We need to prove that $f(x) - g(x) = 0$ for all $x \in \mathbb{R}$. Note that for all $x \in \mathbb{R}$,

$$f(x) - g(x) = 0/(x^2 + 1) = 0.$$

$\square$

One-to-one Functions (1)

Definition 8

A function $f : A \to B$ is <u>one-to-one</u> or <u>injective</u> iff

$$f(a_1) = f(a_2) \text{ implies that } a_1 = a_2$$

Example 9

Let $A = B = \mathbb{Z}$ and define

$$f(a) = 2a \text{ for all } a \in A$$

Then f is a one-to-one function.

Proof.

Note that $f(a) - f(b) = 2(a - b)$. Hence $f(a) = f(b)$ if and only if $a = b$. By definition, f is one-to-one. $\square$

One-to-one Functions (2)

Question 1

Let A and B be two finite sets with m and n elements, respectively, where m and n are positive integers with $m \leq n$. What is the total number of one-to-one functions from A to B?

Onto Functions

Definition 10

A function $f : A \to B$ is <u>onto</u> or <u>surjective</u> if $\mathrm{Range}(f) = B$; ie iff

$$b \in B \text{ means that } b = f(a) \text{ for some } a \in A$$

Example 11

Let $A = B = \mathbb{R}$. Define $f(a) = 4a - 3$. Then f is onto.

Proof.

For any $b \in \mathbb{R}$, we need to find an element $a \in \mathbb{R}$ such that

$$f(a) = b \text{ iff } 4a - 3 = b \text{ iff } a = \frac{b+3}{4}.$$

Hence for any $b \in B$ there is an $a \in A$ such that $f(a) = b$. □

Onto Functions (2)

Recall of definition

A function $f : A \to B$ is <u>onto</u> or <u>surjective</u> if $\mathrm{Range}(f) = B$; ie iff

$$b \in B \text{ means that } b = f(a) \text{ for some } a \in A$$

Example 12

Let $A = B = \mathbb{R}$. Define $f(x) = x^2$. Then f is not onto.

Proof.

Let $b = -1 \in B$. Clearly, there is no $a \in A$ such that $f(a) = a^2 = -1 = b$. By definition, f is not onto. □

Any Relationship between One-to-one and Onto Functions?

Answer

No.

Example 13

One-to-one, but not onto: let $A = B = \mathbb{Z}$ and define $f(x) = 2x$.

Example 14

Onto, but not one-to-one: let $A = \mathbb{Z}$, $B = \{0, 1\}$ and define $f(x) = x \bmod 2$.

Example 15

Onto and one-to-one: let $A = B = \mathbb{Z}$ and define $f(x) = x - 10$.

One-to-one Correspondences

Definition 16

A function f is called a <u>one-to-one correspondence</u> or <u>bijection</u> if it is both one-to-one and onto.

Example 17

Let $A = B = \mathbb{R}$. Define $f(x) = 101x + 1$. Then f is a bijection.

Proof.

It is easy to prove that it is both onto and one-to-one. $\square$

Functions of More Arguments

Definition 18

Recall that a function $f : A \to B$ is a special binary relation from A to B. If $A = A_1 \times A_2 \times \cdots A_n$, we say that f is a function of n arguments.

Example 19

$f(n, m) = 2n + 3m$ is a function of two arguments from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

The Inverse of Functions (1)

Proposition 20

Let $f : A \to B$ be a bijection. Then the <u>inverse relation</u> f^{-1} is a function from B to A.

Proof.

Recall

$$f^{-1} = \{(b, a) \mid (a, b) \in f\}$$

Since f is onto, for any $b \in B$, there is at least on $a \in A$ such that $f(a) = b$. Since f is one-to-one, there is only one such $a \in A$. Hence for any $b \in B$, there is only one $a \in A$ such that $(b, a) \in f^{-1}$. Therefore f^{-1} is a function from B to A. $\square$

The Inverse of Functions (2)

Definition 21

Let $f : A \to B$ be a bijection. The inverse relation f^{-1} is called the <u>inverse function</u> of f.

Example 22

Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, then

$$f = \{(1, x), (2, y), (3, z), (4, t)\}$$

is a bijection from A to B. And

$$f^{-1} = \{(x, 1), (y, 2), (z, 3), (t, 4)\}$$

is the inverse of f.

The Composition of Functions (1)

Definition 23

If $f : B \to A$ and $g : B \to C$ are functions, then the composition of f and g is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(a) = g(f(a)), \ \forall a \in A$$

Example 24

If f and g are the functions $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 3$, $g(x) = x^2 + 1$, then both $g \circ f$ and $f \circ g$ are defined. We have

$$(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 1$$

and

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) - 3$$

The Composition of Functions (2)

Remarks

1. The composition of functions is the same as that of binary relations.
2. Even if both $f \circ g$ and $g \circ f$ are defined, $f \circ g$ may equal to $g \circ f$. See Example 24

The Composition of Functions (3)

Proposition 25

The composition of functions is an associative operation on functions.

Proof.

Let

$$h : A \to B, \quad g : B \to C, \quad f : C \to D$$

be functions. We want to prove that

$$(f \circ g) \circ h = f \circ (g \circ h).$$

By definition,

$$((f \circ g) \circ h)(a) = (f \circ g)(h(a)) = f(g(h(a)))$$
$$(f \circ (g \circ h))(a) = f((g \circ h)(a)) = f(g(h(a))).$$

The desired conclusion then follows. □

The Composition of Functions (4)

Definition 26

Let A be any set. The identity function on A, denoted by i_A is defined by

$$i_A(a) = a, \ \forall a \in A$$

The Composition of Functions (5)

Proposition 27

If $f : A \to A$ is any function and i_A denotes the identity function on A, then $f \circ i_A = i_A \circ f$.

Proof.

On one hand, for any $a \in A$ we have

$$(f \circ i_A)(a) = f(i_A(a)) = f(a).$$

On the other hand, for any $a \in A$ we have

$$(i_A \circ f)(a) = i_A(f(a)) = f(a).$$

The desired conclusion then follows from the definition of the equality of two functions. $\square$

The Composition of Functions (6)

Proposition 28

Functions $f : A \to B$ and $g : B \to A$ are inverses iff

$$g \circ f = i_A \text{ and } f \circ g = i_B$$

i.e. iff

$$g(f(a)) = a \text{ and } f(g(b)) = b$$

for all $a \in A$ and $b \in B$.

Proof.

Left as an exercise. □

The Composition of Functions (7)

Example 29

Show that the function $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = \frac{1}{x}$ is the inverse of itself.

Proof.

$$(f \circ f)(a) = f\left(\frac{1}{a}\right) = a, \ \forall a \in A.$$

The conclusion then follows from Proposition 28. $\square$