# COMP170 Discrete Mathematical Tools for Computer Science

Lecture 3

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Discrete Math for Computer Science K. Bogart, C. Stein and R.L. Drysdale Section 1.3, pp. 19-26

Pascal's Triangle

- Pascal's Triangle
- A Proof using the Sum Principle

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- The Binomial Theorem

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- The Binomial Theorem
- Labeling and Trinomial Coefficients

• 
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
 is the number of  $k$ -element subsets of an  $n$ -element set.

- $\binom{n}{0} = 1$  only one set of size 0.
- $\binom{n}{n} = 1$  only one set of size n.
- $\binom{n}{k} = \binom{n}{n-k}$  Obvious from equation. Can you think of a simple bijection that explains this?

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

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 Use Sum Principle Let  $P = \text{set of all subsets of } \{1,2,\ldots,n\}$  
$$S_i = \text{set of all } i \text{ subsets of } \{1,2,\ldots,n\}$$

### Use Sum Principle

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$$P = \text{set of all subsets of } \{1,2,\ldots,n\}$$

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### Use Sum Principle

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Let  $L = L_1 L_2 \dots L_n$  be a list of size n from  $\{0, 1\}$ If  $\mathcal{L} = \text{set of all such lists} \Rightarrow |\mathcal{L}| = 2^n$ 

There is a *bijection* (next page) between  $\mathcal L$  and P so  $|P|=2^n$  and we are done.

Define the following function  $f:\mathcal{L}\to P$  If  $L\in\mathcal{L}$  then f(L) is the set  $S\subseteq\{1,2,\ldots,n\}$  defined by  $i\in S \iff L_i=1$ 

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Ex: n = 5  $f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset.$ 

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Ex: 
$$n = 5$$
  $f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset.$ 

Note: L is sometimes called the incidence vector or membership vector associated with L

$$P = \left\{ \begin{array}{cccc} \{1\} & \{1,2\} \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{2\} & \{1,4\} \{2,3\} & \{1,2,4\} \\ \{3\} & \{2,4\} \{3,4\} & \{1,3,4\} \\ \{4\} & & \{2,3,4\} \end{array} \right\}$$

$$P = \begin{cases} \{1\} & \{1,2\} \{1,3\} & \{1,2,3\} & \{1,2,3,4\} \\ \{2\} & \{1,4\} \{2,3\} & \{1,2,4\} \\ \{3\} & \{2,4\} \{3,4\} & \{1,3,4\} \\ \{2,3,4\} & \{2,3,4\} & S_2, \end{cases}$$

$$P = \{S_0, S_1, S_2, S_3, S_4 \}$$

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$$|S_0| = {4 \choose 0}, |S_1| = {4 \choose 1}, |S_2| = {4 \choose 2}, |S_3| = {4 \choose 3}, |S_4| = {4 \choose 4}$$

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$$|P| = |S_0| + |S_1| + |S_2| + |S_3| + |S_4|$$

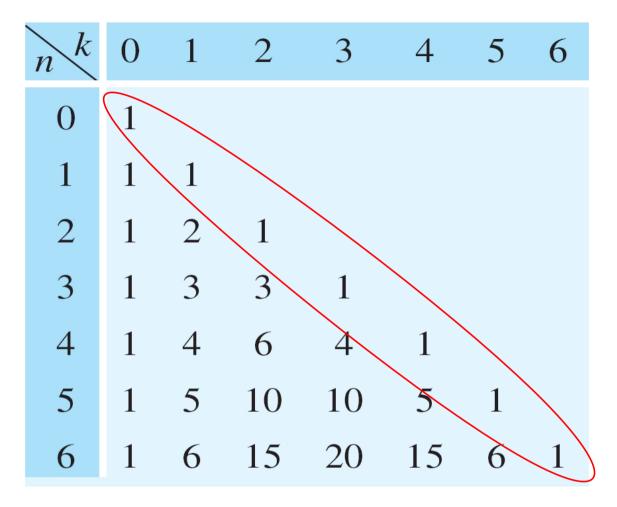
$$= {4 \choose 0} + {4 \choose 1} + {4 \choose 2} + {4 \choose 3} + {4 \choose 4}$$

$$= 2^4 = 16$$

$n^{k}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

$n^{k}$			2				
0 1 2 3 4 5 6	$\sqrt{1}$						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

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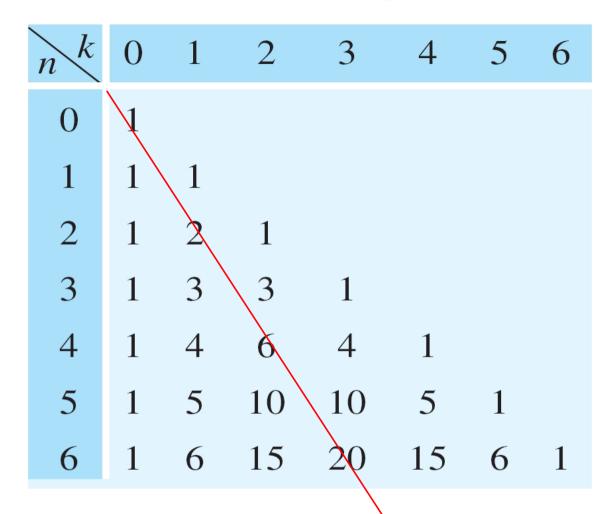
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(will see why in homework)



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Sum of items on  $n^{th}$  row is  $2^n$ 

# Pascal's Triangle

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Take the table

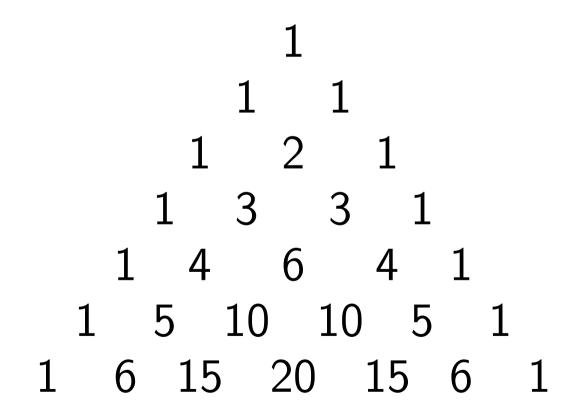
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0	1						
1	1	1					
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3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle



What is the next row in the table?

```
1 2 1
     3 3 1
   4 6 4
  5 10 10 5
   15 20 15 6 1
7 21 35 35 21 7 1
```

What is the next row in the table?

1 2 1 1 3 3 1 4 6 4 5 10 10 5 6 15 20 15 6 1 7 21 35 35 21 7 1

### Pascal relationship

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

1 2 1 4 6 4 5 10 10 5 1 6) 15 20 15 6 1 21 35 35 21 7 1

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4 6 4 5 10 10 5 6 15 1 7 21 35

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1 2 1 4 6 4 5 10 10 5 1 6 15 20 (15) 1 7 21 35 35

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1 2 1 4 6 4 5 10 10 5 1 6 15 20 15 1 7 21 35 35 21

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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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A purely *algebraic* proof (manipulating formulas) is possible.

In discrete mathematics, though, we prefer to derive intuitive explanations. In this case, that would involve interpreting Pascal's relationship as a statement describing *relationships among sets*.

# A Proof Using the Sum Principle

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From Theorem 1.2 and

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we know  $\binom{n}{k}$  is the number of k-element subsets of an n-element set.

Therefore, each term (left and right) in

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

represents the number of subsets of a particular size chosen from an appropriately sized set.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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Example: n = 5, k = 2

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Set  $S_1$  of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Set  $S_1$  of 2-subsets of S can be partitioned into 2 disjoint parts.  $S_2$  the 2 subsets that contain E and  $S_3$ , the set of 2-subsets that do not contain E.

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 $S_2$  is equivalent to choosing 1 item out of  $\{A, B, C, D\}$ :  $|S_2| = \binom{4}{1}$ 

 $S_3$  chooses 2 items out of  $\{A,B,C,D\}$ :  $|S_3|=\binom{4}{2}$ 

Sum Principle: 
$$\binom{5}{2} = |S_1| = |S_2| + |S_3| = \binom{4}{1} + \binom{4}{2}$$

If n and k are integers satisfying 0 < k < n, then

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Suppose 
$$S = \{x_1, x_2, ..., x_n\}$$
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**Proof:** Apply sum principle.

Partition set of k-element subsets of an n-element set into *disjoint union* of two other disjoint sets.

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.

Let  $S_1$  be  $\binom{n}{k}$ -element set of all k-element subsets.

To apply sum principle, partition  $S_1$  into  $S_2$  and  $S_3$ .

Let  $S_2$  be set of k-element subsets that contain  $x_n$ .

Let  $S_3$  be set of k-element subsets that don't contain  $x_n$ 

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Let  $S_2$  be set of k-element subsets that contain  $x_n$ . Let  $S_3$  be set of k-element subsets that don't contain  $x_n$ 

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$$\Rightarrow \binom{n}{k} = |S_1| = |S_2| + |S_3| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

## Blaise Pascal

- Born 1623; Died 1662
- French Mathematician
- A Founder of Probability Theory
- Inventor of one of the first (the 2nd?) mechanical calculating machines
- Pascal Programming Language named for him



$$(x+y) = \begin{pmatrix} 1\\0 \end{pmatrix} x + \begin{pmatrix} 1\\1 \end{pmatrix} y$$

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$$(x+y)^{3} = x^{3} + 3x^{2}y + 3xy^{2} + y^{3}$$
$$= {3 \choose 0}x^{3} + {3 \choose 1}x^{2}y + {3 \choose 2}xy^{2} + {3 \choose 3}y^{3}$$

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## The Binomial Theorem

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### Theorem 1.4 (Binomial Theorem)

For any integer  $n \geq 0$ ,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

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or, in summation notation,

$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

Example:  $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$ 

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$$(x+y)(x+y)(x+y) = [x(x+y) + y(x+y)](x+y) = (xx + yx + xy + yy)(x+y) = (xx + yx + xy + yy)x + (xx + yx + xy + yy)y = xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy.$$

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$$(x + y)(x + y)(x + y)$$
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=  $(xx + yx + xy + yy)(x + y)$   
=  $(xx + yx + xy + yy)x + (xx + yx + xy + yy)y$   
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Each *monomial* term in the final result is of form  $x^{3-i}y^i$  and is the product of – one blue, one red, and one green.

For each color we can choose either an x or a y.

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Each *monomial* term in the final result is of form  $x^{3-i}y^i$  and is the product of – one blue, one red, and one green.

For each color we can choose either an x or a y.

Coefficient of  $x^{3-i}y^i$  is # of ways of choosing i y's from three colors  $= \binom{3}{i}$ 

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Aternatively, can think of the monomial as *lists* where each item of the list is either x or y.

```
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```

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Thus, product of n binomials,  $(x + y)^n$ , is sum of  $2^n$  monomials

Each monomial is a length-n list of x's and y's.

In each list, the *i*th entry comes from the *i*th binomial factor.

Number of lists that have a y in k places is thus the number of ways to select k binomial factors to contribute a y to our list.

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Therefore, the coefficient of  $x^{n-k}y^k$ , is  $\binom{n}{k}$ .

By applying the binomial theorem

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Now set x = y = 1. This gives

$$2^{n} = (1+1)^{n} = \sum_{i=0}^{n} \binom{n}{i}$$

# Labelling and Trinomial Coefficients

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- Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., green, and  $k_3=n-k_1-k_2$  labels of a third kind, e.g., orange, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to n objects

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There are  $\binom{n}{k}$  ways to choose the items with red labels. The other n-k items will then get the green labels So this is just  $\binom{n}{k}$ 

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There are  $\binom{n}{k_1}$  ways to choose the red items

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Using the product principle the total number of labellings is

$$\binom{n}{k_1} \binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$

When  $k_1 + k_2 + k_3 = n$ , we call

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Note that this slightly modifies the notation for binomial coefficients. If we really wanted the notation to be consistent (which we don't) we could write the binomial coefficient  $\binom{n}{k}$  as

$$\begin{pmatrix} n \\ k & (n-k) \end{pmatrix}$$

### We really just saw that the Trinomial Coefficient

$$\begin{pmatrix} n \\ k_1 & k_2 & k_3 \end{pmatrix}$$

is the number of ways to partition a set of size n into three subsets (where order of the subsets does not count) of sizes  $k_1$ ,  $k_2$  and  $k_3$ .

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$$(x + y + z)(x + y + z)(x + y + z)(x + y + z) = xxxx + xxxy + xxxz + xxyx + \dots + zzzy + zzzz.$$

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After opening the parentheses and multiplying, there will be, in total,  $3^4 = 81$  different monomial terms (lists)

Each term, (after rewriting using commutativity), is in the form  $x^{k_1}y^{k_2}z^{k_3}$  where  $k_1+k_2+k_3=4$ 

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The coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  is exactly the number of ways of writing a list of size 4 with  $k_1$  x's,  $k_2$  y's, and  $k_3$  z's such that Writing a not  $k_1 + k_2 + k_3 = 4$ , which is  $\begin{pmatrix} 4 \\ k_1 & k_2 & k_3 \end{pmatrix}$ 

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