Introduction to Graph Algorithms

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THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Graphs

- Extremely useful tool in modeling problems

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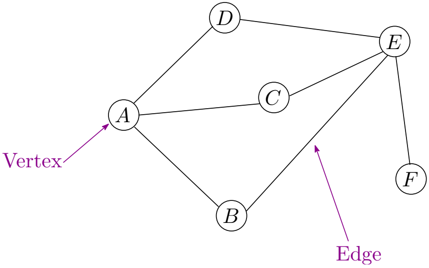
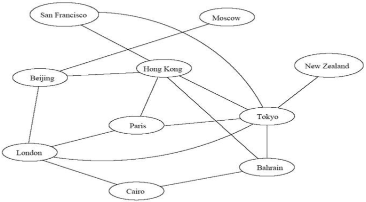
- Consist of:
  - Vertices
  - Edges



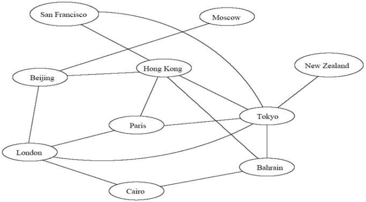
Vertices can be considered as "sites" or locations.

Edges represent connections.

Graph Application



Air flight system



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- Each vertex represents a city

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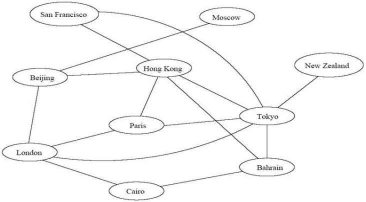

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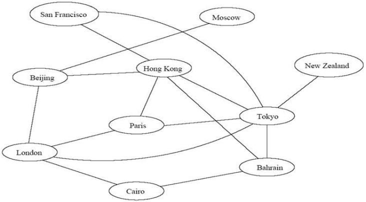

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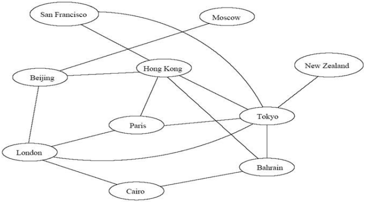

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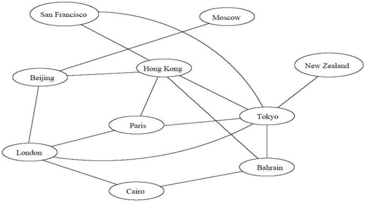

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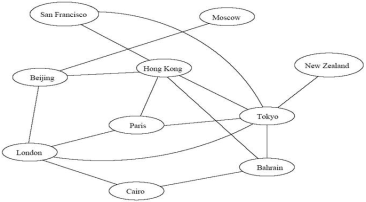
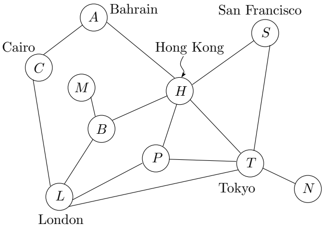


original graph



simplified graph

- Each vertex represents a city
- Each edge represents a direct flight between two cities
- A query on direct flight = a query on whether an edge exists
- A query on how to get to a location = does a path exist from A to B
- We can even associate costs/time to edges (weighted graphs), then ask "what is the cheapest/fastest path from A to B"

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- Hundreds of interesting computational problems defined on graphs
- We will sample a few basic ones

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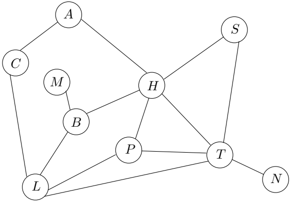

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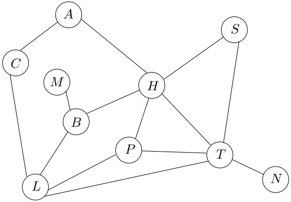

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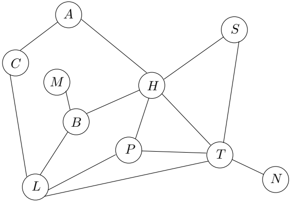

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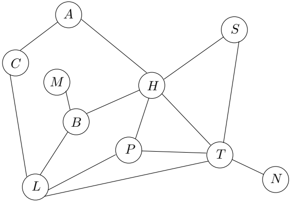

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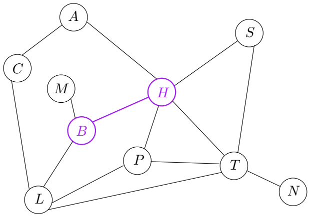
- For directed graph, we distinguish between edge (u, v) and edge (v, u); for undirected graph, no such distinction is made.



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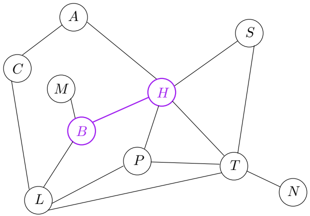
Terminology



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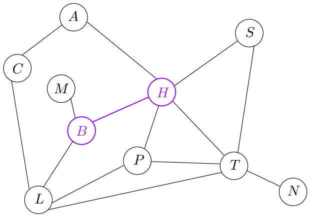

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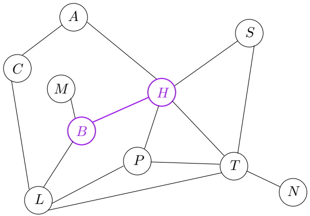

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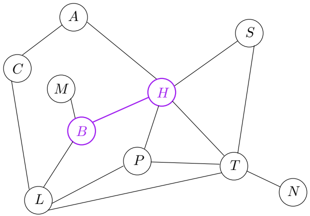

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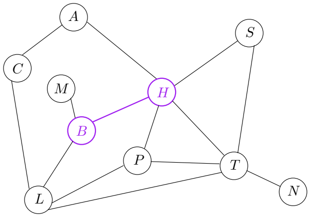

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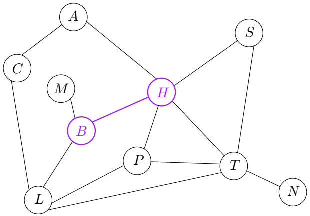

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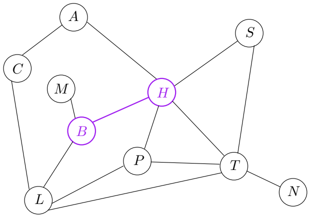

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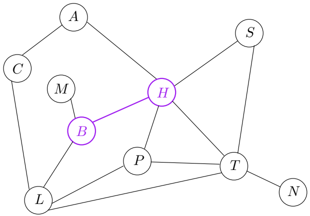

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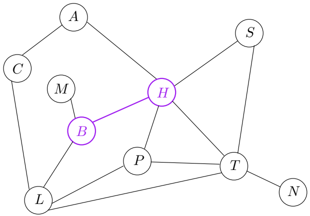

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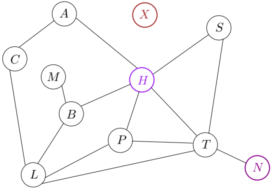

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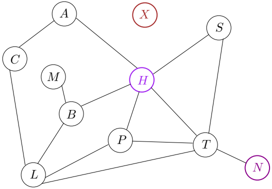

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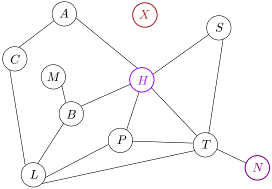

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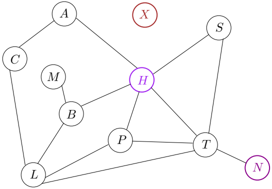

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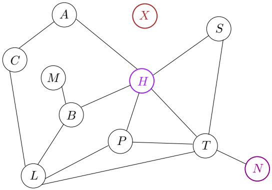

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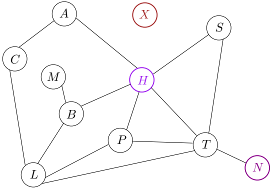

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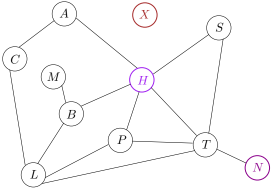

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Lemma

$$\sum_{v \in V} degree(v) = 2|E|.$$

Proof.

An edge $e = (u, v)$ in a graph contributes one to degree(u) and contributes one to degree(v). $\square$

Path

A path in a graph is a sequence $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ of vertices such that $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$

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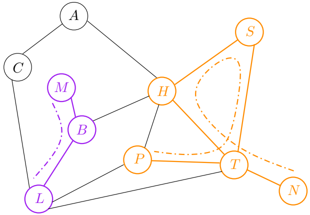
- There is a path from v_0 to v_k
- Length of a path = # of edges on the path
- Path contains the vertices $v_0, v_1, \ldots, v_k$ and the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$
- For any $0 \leq i \leq j \leq k$, $\langle v_i, v_{i+1}, \ldots, v_j \rangle$ is its subpath
- If there is a path p from u to v, v is said to be reachable from u
- A path is simple if all vertices in the path are distinct



- $\langle L, B, M \rangle$ is a path
  - length is 2
  - $\langle B, M \rangle$ is its subpath
  - M is reachable from L
  - a simple path
- $\langle N, T, H, S, T, P \rangle$ is a path
  - length is 5
  - $\langle T, H, S \rangle$ is its subpath
  - P is reachable from N
  - not a simple path

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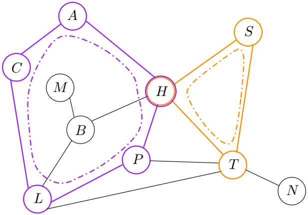

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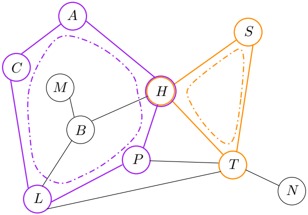

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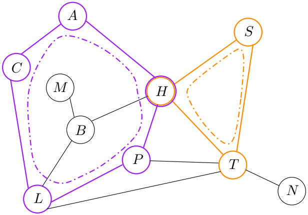

- $\langle T, S, H, T \rangle$ is a simple cycle
- $\langle A, C, L, P, H, A \rangle$ is a simple cycle

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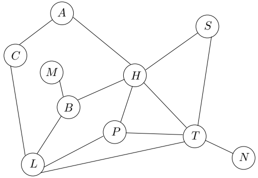
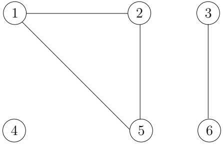
- connected graph
- one connected component $\{A, B, C, H, L, M, N, P, S, T\}$



- disconnected graph
- 3 connected components
  - $\{1, 2, 5\}$
  - $\{3, 6\}$
  - $\{4\}$



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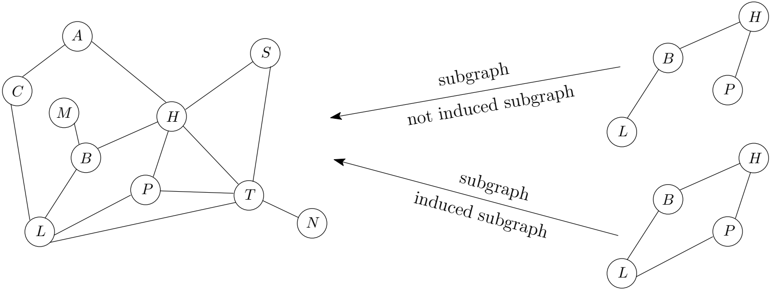

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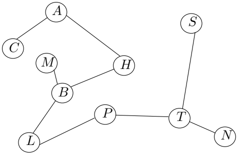
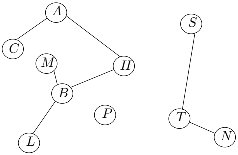
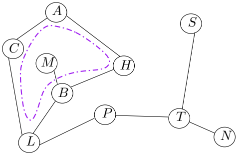


A tree



A forest



neither a tree nor a forest

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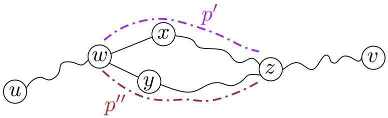

- Proof by contradiction
- Suppose that vertices u and v are connected by two distinct simple paths p_1 and p_2, as shown in the above figure

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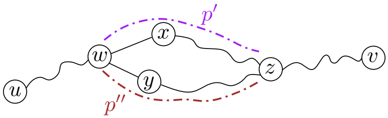

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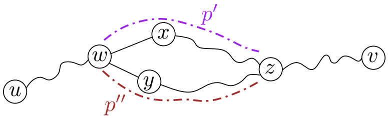

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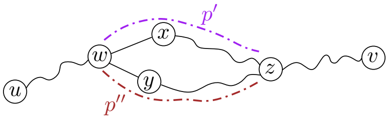

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  - p' is the subpath of p_1 from w through x to z
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  - The path obtained by concatenating p' and the reverse of p'' is a cycle, which yields the contradiction!

Proof

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- If (u, v) is deleted from G, there is no path from u to v, and hence its removal disconnects G

Proof

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    - Each component satisfies (3), or else G would not satisfy (3)
    - Thus, by induction, the number of edges in 2 components combined is $|V| - 2$
    - Adding in the removed edge yields $|E| = |V| - 1$

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- If $k + 1 < |V|$, we can continue, defining G_{k+2} in the same manner, and so forth, until we obtain $G_n = (V_n, E_n)$, where $n = |V|$, $V_n = V$, and $|E_n| = |V_n| = |V|$

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- Since G_n is a subgraph of G, we have $E_n \subseteq E$, and hence $|E| \geq |V|$, which contradicts the assumption that $|E| = |V| - 1$

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- Suppose that G is acyclic and that $|E| = |V| - 1$
- Let k be the number of connected components of G
- Each connected component is a free tree by definition, and since (1) implies (5), the sum of all edges in all connected components of G is $|V| - k$

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- Thus, there is a path from u to v, and since u and v were chosen arbitrarily, G is connected