Dynamic Programming: The Rod Cutting Problem

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  1. Analyze the structure of an optimal solution
  2. Recursively define the value of an optimal solution
  3. Compute the value of an optimal solution (usually bottom-up)

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  - There are more options, but the maximum revenue is **10**
- In general, rod of length n can be cut in 2^{n-1} different ways, since we can choose cutting, or not cutting, at all distances i $(1 \leq i \leq n-1)$ from the left end

Optimal Solution

- We can calculate the maximum revenue r_n in terms of optimal revenues for shorter rods

$$r_n = \max(p_n,\ r_1 + r_{n-1},\ r_2 + r_{n-2},\ \ldots,\ r_{n-1} + r_1)$$

  - p_n if we do not cut at all
  - $r_1 + r_{n-1}$ if we take the sum of optimal revenues for 1 and $n-1$
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- Another approach. Set $r_0 = 0$ and

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

  - Cut a piece of length i, with remainder of length $n - i$
  - Only the remainder, and not the first piece, may be further divided

Recursive Top-down Implementation

Cut-Rod(p, n)

if $n = 0$ **then**
    return 0;
end
$q = -\infty$;
for $i = 1$ **to** n **do**
    $q = \max(q, p[i] + \text{Cut-Rod}(p, n - i))$;
end
return q;

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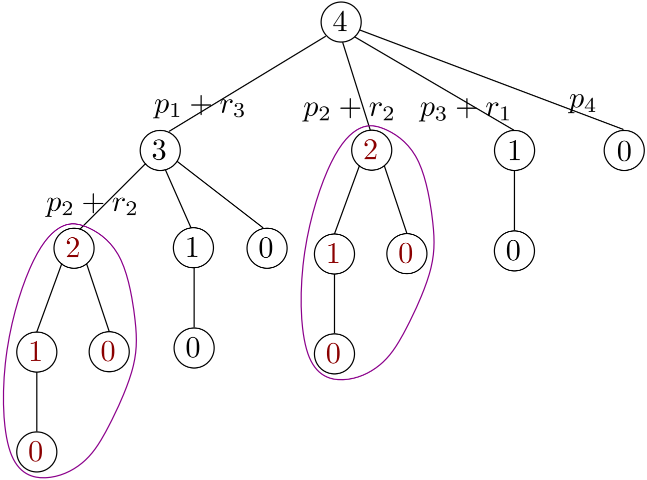
- Induction $\Rightarrow T(n) = 2^n$

- Algorithm calls same subproblem many times



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  - Don't wait until until subproblem is encountered.
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  - Combine solutions of small subproblems to solve larger ones

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i	1	2	3	4	. . . . . .	n
$\mathbf{r_i}$	p_1				. . . . . .	

DP Bottom-up Implementation

Bottom-Up-Cut-Rod(p, n)

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r[0] = 0; // Array r[0...n] stores the computed optimal values
for j = 1 to n do
    // Consider problems in increasing order of size
    q = −∞;
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  - To compute $r[j]$, the inner loop uses all values $r[0], r[1], \ldots, r[j-1]$ (i.e., $r[j-i]$ for $1 \le i \le j$)

Outputting the Cutting

- Algorithm only *computes* r_i. It does not output the cutting.
- Easy fix
  - When calculating $r_j = \max_{1 \le i \le j}(p_i + r_{j-i})$
    store value of i that achieved this max in new array $s[j]$.
  - This j is the size of last piece in the optimal cutting.
- After algorithm is finished, can reconstruct optimal cutting by unrolling the s_j.

Extended-Bottom-Up-Cut-Rod(p, n)

```
// Array s[0...n] stores the optimal size of the first piece to
   cut off
r[0] = 0; // Array r[0...n] stores the computed optimal values
for j = 1 to n do
    q = -∞;
    for i = 1 to j do
        // Solve problem of size j
        if q < p[i] + r[j - i] then
            q = p[i] + r[j - i];
            s[j] = i; // Store the size of the first piece
        end
    end
    r[j] = q;
end
while n > 0 do
    // Print sizes of pieces
    Print s[n];
    n = n - s[n];
end
```