

COMP170

Discrete Mathematical Tools for Computer Science

Lecture 13 *Version 2: Last updated, Dec 8, 2005*

Discrete Math for Computer Science

K. Bogart, C. Stein and R.L. Drysdale

Section 4.5, pp. 189-193

*Note: We have skipped section 4.4 of the textbook because the material it contains, especially the **Master Theorem**, will be taught in later classes, e.g., COMP271*

More Advanced Induction

- Induction, as we've seen it so far, was about defining a statement $p(n)$, and then proving $p(n-1) \Rightarrow p(n)$ or $(p(1) \wedge p(2) \wedge \cdots \wedge p(n-1)) \Rightarrow p(n)$
- In “practice”, in some real induction proofs, $p(n)$ might not be fully defined *before* we start the proof and will only be fully described *during* the description of the proof
- In some cases it also helps to use a **stronger** induction hypothesis than the “natural” one.

We will illustrate these concepts with three example proofs:

Example 1 If $T(n) \leq 2T(n/2) + cn$ for some constant c ,
then $T(n) = O(n \log n)$.

Example 2 If $T(n) \leq T(n/3) + cn$ for some constant c ,
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Example 3 If $T(n) \leq 4T(n/2) + cn$ for some constant c ,
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Example 3 will illustrate what is meant by using a stronger
induction hypothesis.

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Our problem is that we do not know what k is
so we can't prove (*)

We want to prove that if, for all $n = 2^i$,

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Our proof will be by induction, but with a twist.

We will assume that we have a k for which $(*)$ holds in the inductive hypothesis and then continue on to prove the inductive step.

$$\left((*) \text{ True for } n = 2^{i-1} \right) \Rightarrow \left((*) \text{ True for } n = 2^i \right)$$

While we are doing this, we will discover sufficient assumptions on k (and n_0) to ensure that k exists.

We want to prove that if, for all $n = 2^i$,

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In order to guarantee

$$T(n) \leq kn \log n$$

we must have

$$-kn + cn \leq 0.$$

We therefore make the
final assumption:

$$k \geq c.$$

We have just shown that if, for all $n = 2^i$,

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We therefore conclude, *by the principle of mathematical induction*,
that as long as all three of our assumptions are satisfied,

$$\forall n > n_0 = 1 \quad T(n) \leq kn \log n.$$

We have therefore **proved** that $T(n) = O(n \log n)$.

Our inductive hypothesis:

if $m = 2^j$ with $1 \leq j < i$ then $T(m) \leq km \log m$.

- Note that the inductive hypothesis (and associated inductive step) was **not** fully defined when we started the proof, since we didn't say what the value of k was.
- It was only while in the middle of the proof that we specified the value of k (by discovering the conditions on k that would allow the inductive step to work)
- After the fact, it is possible to write a more traditional inductive proof, in which the value of k is given, but this can be even more confusing.

A More Traditional Induction Proof

We want to prove that if, for all $n = 2^i$,

$$T(n) \leq 2T(n/2) + cn \text{ for some constant } c,$$

$$\Rightarrow \forall n > 1, \quad T(n) \leq kn \log n$$

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(i) Since $\log 2 = 1$, $T(2) = \frac{T(2)}{2} 2 \leq k 2 \log 2$

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Because the discussion in the first proof explained **why** we were making the choices we did, many people prefer the structure of the first proof to that of the second.

This type of inductive proof – in which conditions on the parameters are developed **during** the proof – is therefore used quite often in books and articles.

Example 2: We now prove by induction that, for T defined on $n = 3^i$, $i = 0, 1, 2, \dots$

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From definition of big O we need to show that

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Statement is **too weak** to be proved by induction.

To fix this, let's see if we can prove something that is actually *stronger* than we were originally trying to prove — namely,

$$T(n) \leq k_1 n^2 - k_2 n \text{ for some positive constants } k_1 \text{ and } k_2.$$

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With these 2 assumptions we have proved *inductively* that
 $T(n) \leq k_1n^2 - k_2n$ so $T(n) = O(n^2)$.

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- When we give an induction proof in this way, we are
using a **stronger inductive hypothesis**.