COMP 170 Discrete Mathematical Tools for CS
2005 Fall Semester – Comments Sept 13, 2005

In the supplementary review session on Tuesday Sept 13, 2005, some students wanted to know the following:

a) What is the number of surjections (onto functions) from S_m to S_n?

b) What is the value of $\sum_{i=0}^{n} i^2$?

Here are the answers. Please note that this material is only being provided to answer some students' questions and *will not be on any exam.*

a) Unlike the number of one-to-one functions (which is easy, since the number of one-to-one functions is just the number of m-element permutations of $\{1, \ldots n\}$ which is $n^{\underline{m}}$ if $m \leq n$ and 0 if $m > n$) the number of surjections is a hard problem. Or rather, it is not simple to write it out easily in terms of functions that we know.

In order to answer this question we need to introduce a *new type* of number, $S(n, k)$, called a *Stirling number of the second kind.*

$S(n, k)$ is defined to be *the The number of ways of partitioning a set of n elements into m nonempty sets.*

The number of surjections is then just

$$m!S(n, m).$$

While there is no simple formula for $S(n, k)$, there is quite a lot known about these numbers (they even have their own version of a Pascal triangle type of identity). For more on Stirling numbers of the second kind, please see Chapter 6 of the class reference book (it's on reserve in the library) *Graham, R. L.; Knuth, D. E.; and Patashnik, O, Concrete Mathematics: A Foundation for Computer Science.* Alternatively, check out the web site
http://mathworld.wolfram.com/StirlingNumberoftheSecondKind.html

(b) Let $F_0(n) = \sum_{i=1}^{n} 1 = n$; $F_1(n) = \sum_{i=1}^{n} i = n(n+1)/2$. We want to calculate $F_2(n) = \sum_{i=1}^{n} i^2$.

The first thing to notice is that

$$n^3 = \sum_{i=1}^{n} \left[i^3 - (i-1)^3 \right] \tag{1}$$

(This is what we sometimes call a *telescoping sum*.)

The next thing to notice is that

$$\begin{aligned}
\sum_{i=1}^{n} \left[i^3 - (i-1)^3 \right] &= \sum_{i=1}^{n} \left[3i^2 - 3i + 1 \right] && (2)\\
&= 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 && (3)\\
&= 3F_2(n) - 3F_1(n) + F_0(n) && (4)\\
&= 3F_2(n) - 3n(n+1)/2 + n && (5)\\
&= 3F_2(n) - \frac{3n^2+n}{2} && (6)
\end{aligned}$$

Solving for $F_2(n)$ gives

$$\begin{aligned}
F_2(n) &= \frac{1}{3}\left[n^3 + \frac{3n^2+n}{2} \right] && (7)\\
&= \frac{2n^3+3n^2+n}{6} && (8)\\
&= \frac{n(2n+1)(n+1)}{6} && (9)
\end{aligned}$$

Note that this technique can actually be iterated to find $F_j(n) = \sum_{i=1}^{n} i^j$. As an example, let $F_3(n) = \sum_{i=1}^{n} i^3$.

$$n^4 = \sum_{i=1}^{n} \left[i^4 - (i-1)^4 \right] \tag{10}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} \left[i^4 - (i-1)^4 \right] &= \sum_{i=1}^{n} \left[4i^3 - 6i^2 + 4i - 1 \right] && (11)\\
&= 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 && (12)\\
&= 4F_3(n) - 6F_2(n) + +4F_1(n) - F_0(n) && (13)\\
&= 4F_3(n) - n(2n+1)(n+1) + 2n(n+1) - n && (14)\\
&= 4F_3(n) - (2n^3 + n^2) && (15)
\end{aligned}$$

Solving for $F_3(n)$ gives

$$F_3(n) = \frac{1}{4}\left[n^4 + 2n^3 + n^2\right] \tag{16}$$

$$= \frac{n^2(n+1)^2}{4} \tag{17}$$