

COMP170

# Discrete Mathematical Tools for Computer Science

Dealing with floors and ceilings in  
divide-and-conquer recurrences

We have seen that when  $n$  is a power of 2.

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases} \quad (*)$$

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Note that, when  $n$  is not a power of 2, a D&C recurrence will split  $n$  into  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ . Eq (\*) then becomes

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases} \quad (**)$$

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*When  $n$  is a power of 2 then  $(**)$  is defined by  $(*)$ .*

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Let  $m = 2^{i+1}$  be the smallest power of 2  $\geq n$ . Since the interval  $[n, 2n - 1]$  contains a power of 2 we have  $m < 2n$ . So,

$$\begin{aligned} T(n) &\leq T(m) \\ &= m(1 + \log_2 m) \\ &\leq 2n(1 + \log_2 2n) \\ &= 2n(2 + \log_2 n) \end{aligned}$$

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This gives us an *upper bound*.

On the other hand,  $m/2 = 2^i \leq n < m$ . So,

$$\begin{aligned} T(n) &\geq T\left(\frac{m}{2}\right) \\ &= \frac{m}{2} \left(1 + \log_2 \frac{m}{2}\right) \\ &> \frac{n}{2} \left(1 + \log_2 \frac{n}{2}\right) \\ &= \frac{n}{2} \log_2 n \end{aligned}$$



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This gives us a *lower bound*.

We have just seen that if  $T(n)$  is defined by

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then (assuming that Theorem 1 is true)

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*So, getting rid of the condition that  $n$  be a power of 2 and adding the floors and ceilings didn't really change much. The approach we have seen can, with a bit more work, be made into a general technique for getting rid of floors and ceilings*

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We will actually prove the *stronger* statement

## **Theorem 2**

For any positive integer  $n$ ,  $T(n) < T(n + 1)$

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## Theorem 2

For any positive integer  $n$ ,  $T(n) < T(n + 1)$

Proof: (by strong induction)

**Basis:**  $T(2) = 2 * T(1) + 2 = 4 > T(1)$ .

## Theorem 2

For any positive integer  $n$ ,  $T(n) < T(n + 1)$

**Hypothesis:** Let  $n > 2$ .

Assume that for all  $m < n$ ,  $T(m) < T(m + 1)$ .

**Step:** There are two possibilities for  $n$ :

(i)  $n$  is even: Then, for some  $m < n$ ,  $n = 2m$ ,

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We are now finished since this immediately implies (why?)

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