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You will also see an example illustrating why the *inclusion-exclusion formula* is actually a statement about sets, and does not depend upon the actual weights of the probability distribution.

Consider the sample space of 3 coin flips (8 outcomes)

$$
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E &= \{\text{first flip is Head}\} && = \{HHT, HTT, HHH, HTH\} \\
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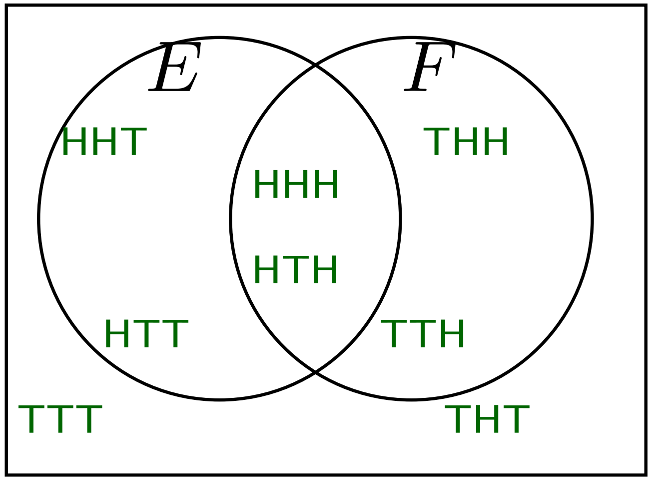

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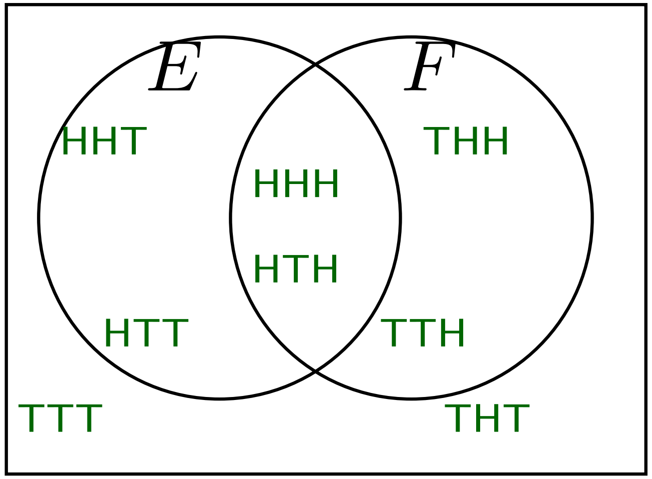

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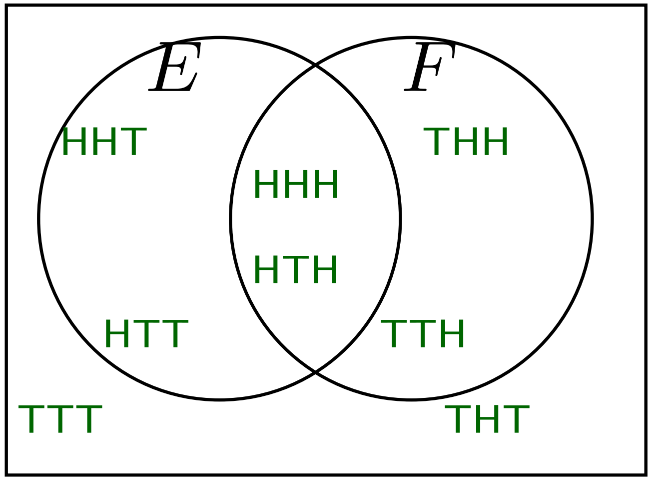

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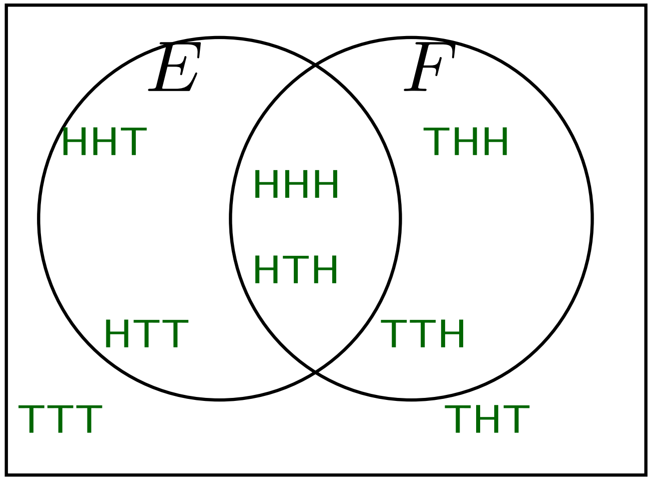

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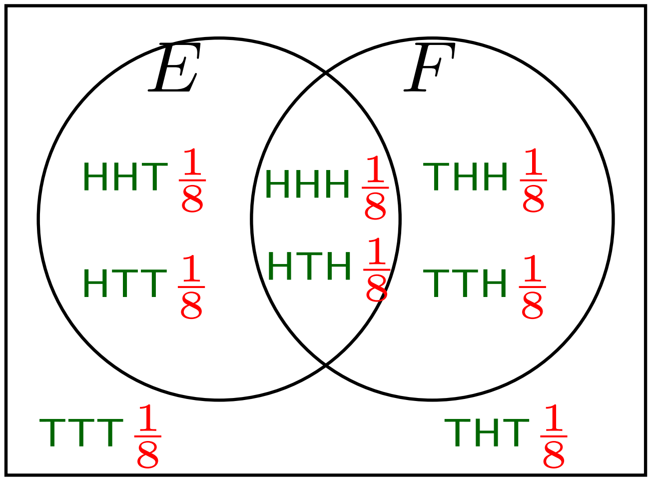

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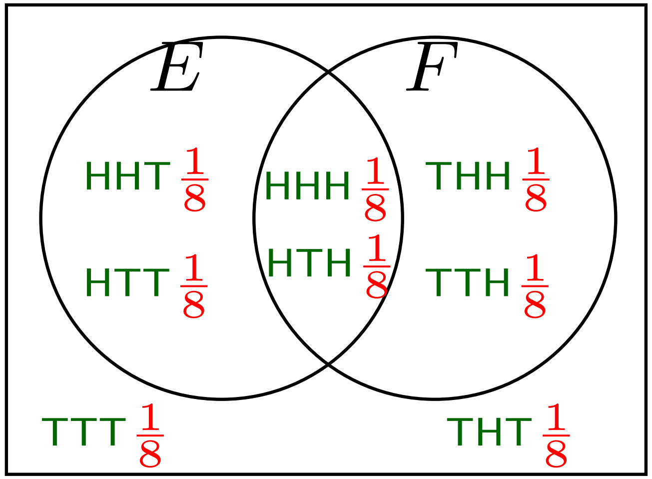

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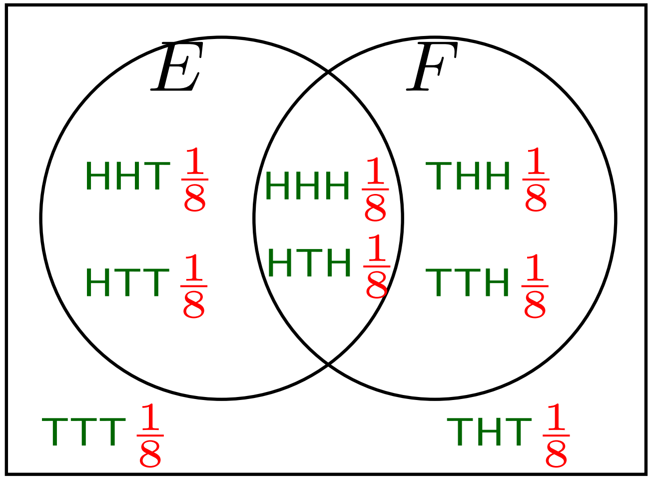
This created a *uniform* distribution on the sample space.



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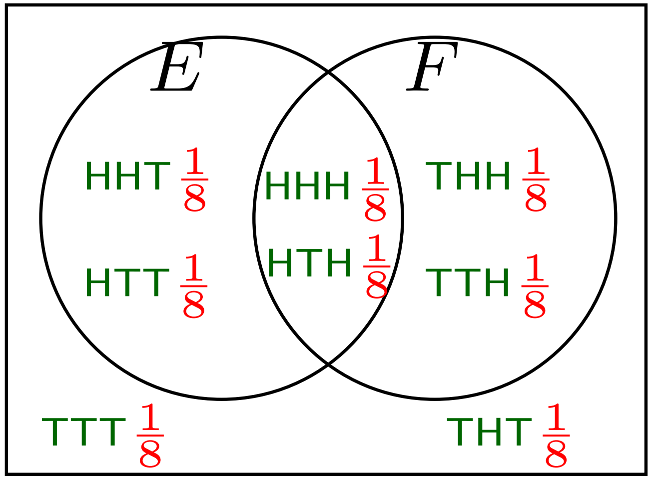

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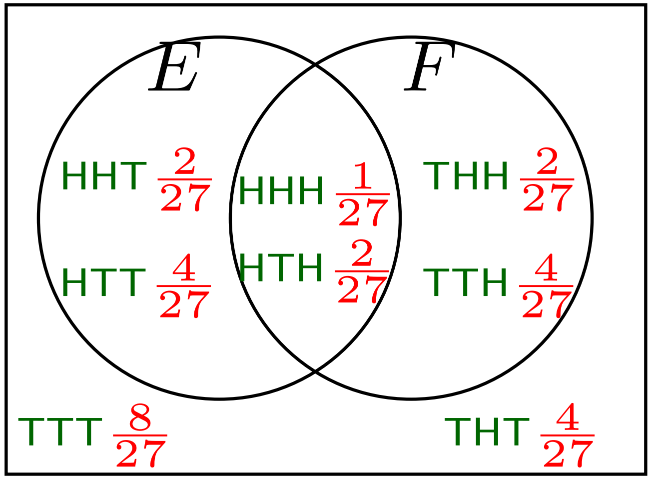

$$P(E) = \frac{1+2+2+4}{27} = \frac{1}{3}$$
$$P(F) = \frac{1+2+2+4}{27} = \frac{1}{3}$$
$$P(E \cap F) = \frac{1+2}{27} = \frac{1}{9}$$

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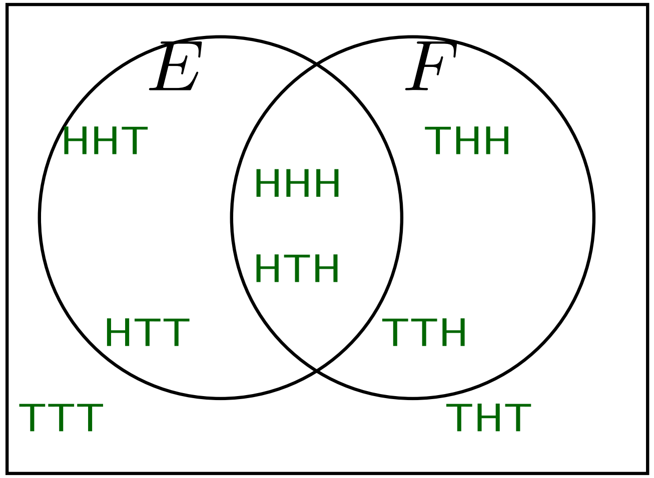
As a final note, we point out that the inclusion-exclusion formula is really a statement about the sample space and *does not depend* upon the actual weights of the probability distribution



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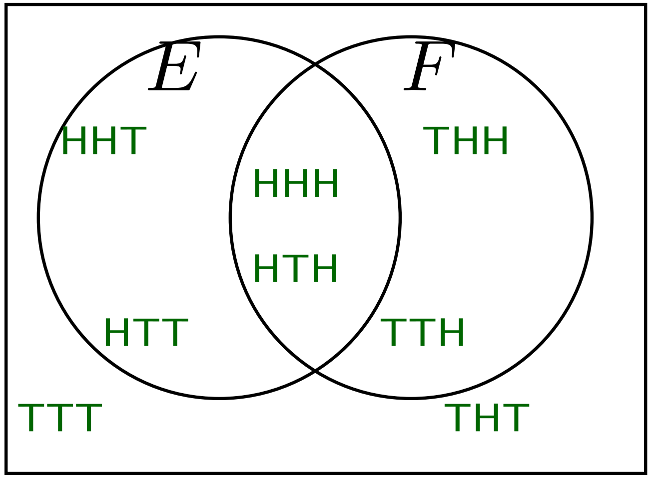

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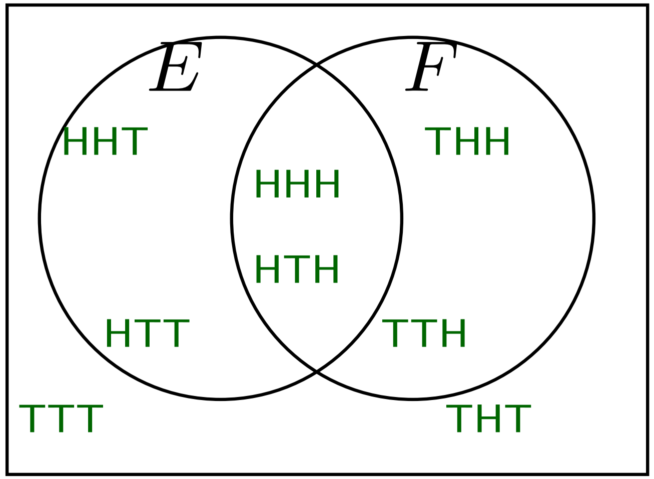

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