Huffman Coding

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Outline

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- Coding and Decoding
- The optimal source coding problem
- Huffman coding: A greedy algorithm
- Correctness

Encoding

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  - a code is a set of codewords.
  - e.g., $\{000, 001, 010, 011, 100, 101\}$
    and
    $\{0, 101, 100, 111, 1101, 1100\}$

  are codes over the binary alphabet $\Sigma = \{0, 1\}$.

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- Goal is to save space!

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$\{a = 0,\ b = 110,\ c = 01,\ d = 111\}$ is *not* a prefix code.
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We are therefore interested in finding *good* (best compression) prefix-free codes.

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Huffman Coding Problem

Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that minimizes the number of bits

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needed to encode a message of $\sum_{i=1}^{n} f(a_i)$ characters, where

- c_i is the codeword for encoding a_i, and
- $L(c_i)$ is the length of the codeword c_i.

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Remark: We will see later that this is the optimum (lowest cost) prefix code.

Correspondence between Binary Trees and Prefix Codes

1-1 correspondence between leaves and characters.

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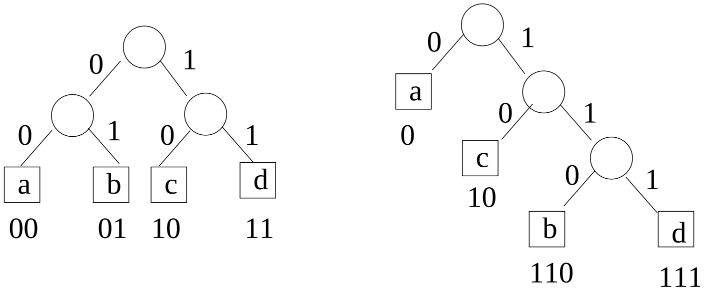

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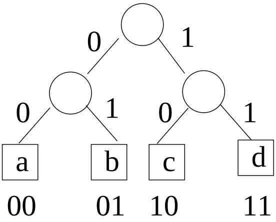

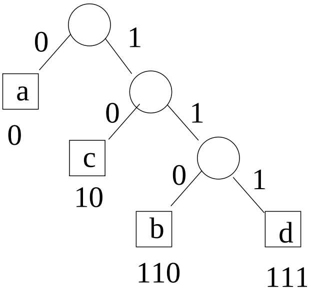

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- The binary string on a path from the root to a leaf is the codeword associated with the character at the leaf.

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The Huffman encoding problem is equivalent to the minimum-weight external pathlength problem.

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- It encodes the optimum (minimum-cost) prefix code for the given frequency distribution.

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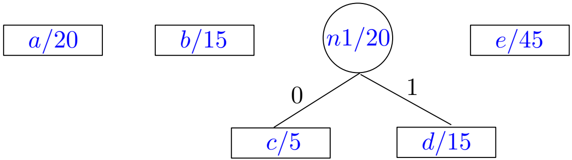

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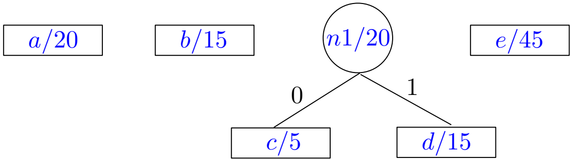

Now have $S = \{a/20,\ b/15,\ \mathbf{n1}/\mathbf{20},\ e/45\}$.

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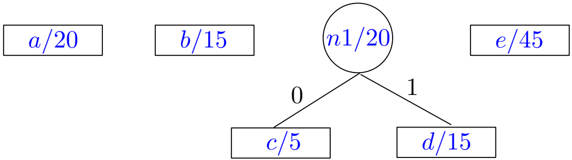

   Now have $S = \{a/20,\ b/15,\ \mathbf{n1/20},\ e/45\}$.

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Example of Huffman Coding

Let $S = \{a/20, b/15, c/5, d/15, e/45\}$ be the original character set S alphabet and its corresponding frequency distribution.

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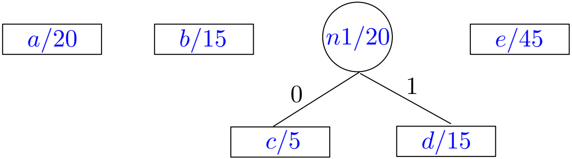

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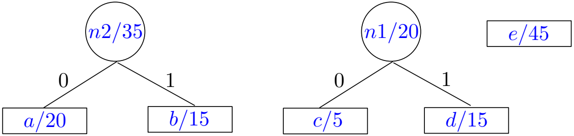

Now have $S = \{n2/35, n1/20, e/45\}$.

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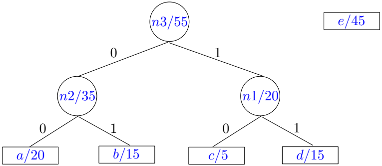

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Example of Huffman Coding – Continued

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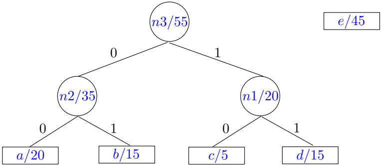

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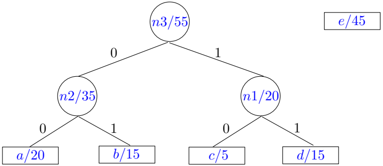

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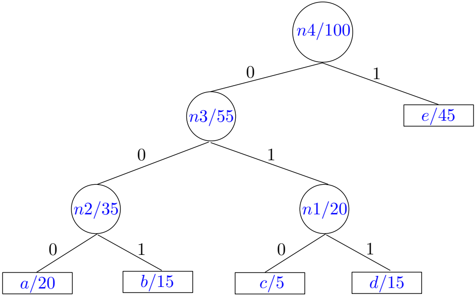

The Huffman code is:
$a = 000$, $b = 001$,
$c = 010$, $d = 011$,
$e = 1$.

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Running time is $O(n \log n)$, as each priority queue operation takes time $O(\log n)$.

Outline

- Coding and Decoding
- The optimal source coding problem
- Huffman coding: A greedy algorithm
- Correctness

Lemma 1

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An optimal prefix code tree must be "full", i.e., every internal node has exactly two children.

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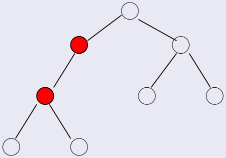
If some internal node had only one child,



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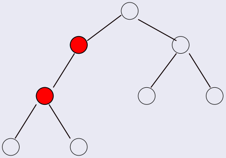

then we could simply get rid of this node and replace it with its unique child. This would decrease the total cost of the encoding. □

Lemma (2)

Let T be prefix code tree and T' the tree obtained by swapping two leaves x and b in T. If,

$$f(x) \le f(b), \quad \text{and} \quad d(x) \le d(b)$$

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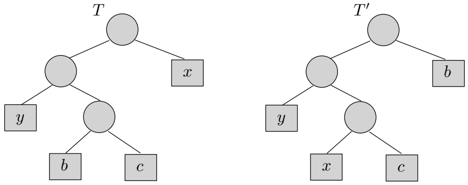

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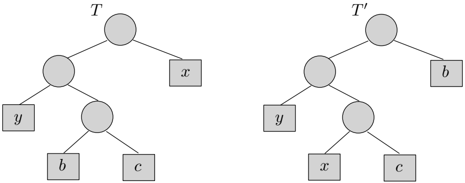

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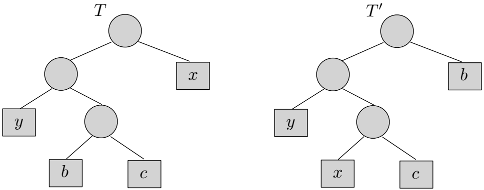

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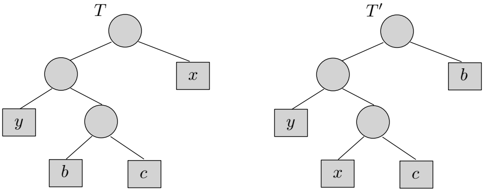

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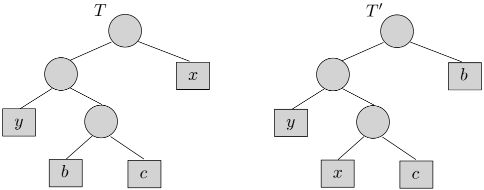

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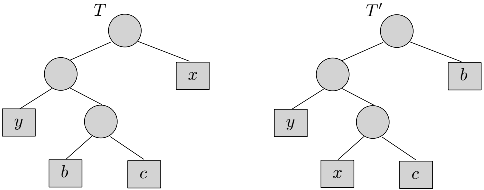

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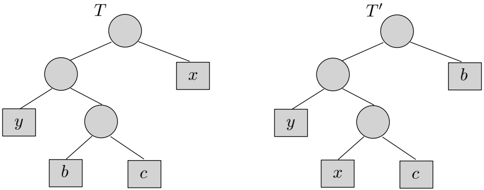

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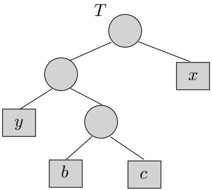

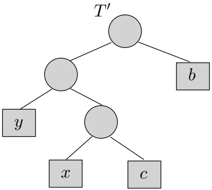

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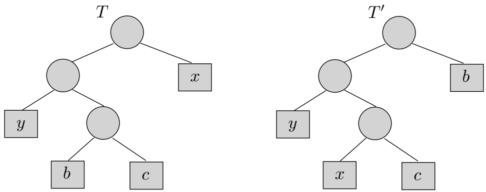

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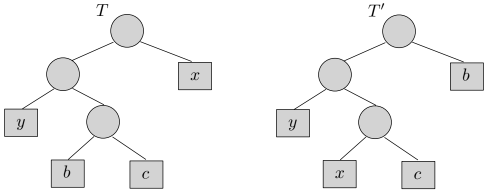

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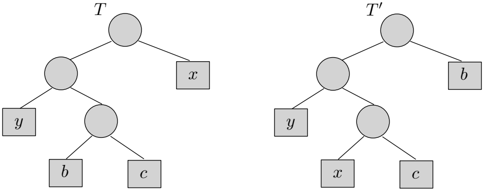

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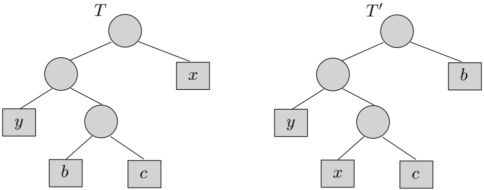

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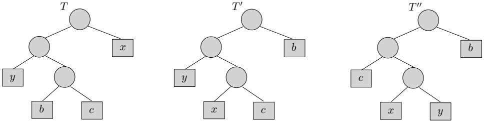

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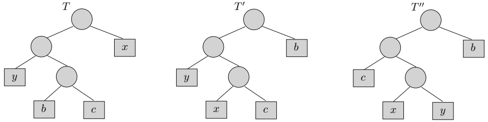

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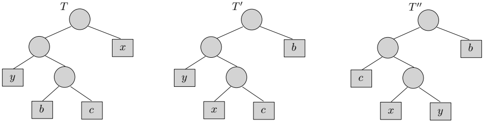

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- (If necessary) swap x with b and swap y with c.
- Proof follows from Lemma 2.

Lemma 4

Lemma (4)

- *Let T be a prefix code tree and x, y two sibling leaves.*
- *Let T' be obtained from T by removing x and y, naming the parent z, and setting $f(z) = f(x) + f(y)$*
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- **Induction step**: Consider the case of n characters:
  - Let H be the tree produced by the Huffman's algorithm.

Huffman Codes are Optimal

Theorem

The prefix code tree (hence the code) produced by the Huffman coding algorithm is optimal.

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    (iii) setting $f(z) = f(x) + f(y)$
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  - H' is the tree produced by Huffman's algorithm for S'.
    - By the induction hypothesis, H' is optimal for S'.
    - By Lemma 4, $\mathbf{B(H)} = \mathbf{B(H')} + \mathbf{f(x)} + \mathbf{f(y)}$.

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  - Therefore, H must be optimal!