COMP170
Discrete Mathematical Tools for Computer Science

Advanced Induction

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Discrete Math for Computer Science
K. Bogart, C. Stein and R.L. Drysdale
Section 4.5, pp. 189-193

Note: We have skipped section 4.4 of the textbook because the material it contains, especially the Master Theorem, will be taught in later classes, e.g., COMP271

More Advanced Induction

- Induction, as we've seen it so far, was about defining a statement $p(n)$, and then proving $p(n-1) \Rightarrow p(n)$ or $(p(1) \wedge p(2) \wedge \cdots \wedge p(n-1)) \Rightarrow p(n)$
- In "practice", in some real induction proofs, $p(n)$ might not be fullly defined *before* we start the proof and will only be fully described *during* the description of the proof
- In some cases it also helps to use a stronger induction hypothesis than the "natural" one.

We will illustrate these concepts with three example proofs:

Example 1 If $T(n) \leq 2T(n/2) + cn$ for some constant c, then $T(n) = O(n \log n)$.

Example 2 If $T(n) \leq T(n/3) + cn$ for some constant c, then $T(n) = O(n)$.

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Example 3 If $T(n) \le 4T(n/2) + cn$ for some constant c, then $T(n) = O(n^2)$.

Example 3 will illustrate what is meant by using a stronger induction hypothesis.

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$$\boxed{\begin{array}{l}\text{if } T(n) \leq 2T(n/2) + cn \text{ for some constant } c, \\ \quad \text{then } T(n) = O(n \log n).\end{array}}$$

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Our problem is that we do not know what k is
so we can't prove (*)

We want to prove that if, for all $n = 2^i$,

$T(n) \le 2T(n/2) + cn$ for some constant c,

$$\Rightarrow \quad \forall n > n_0, \quad T(n) \le kn\log n \qquad \text{(*)}$$

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Our proof will be by induction, but with a twist.

We will assume that we have a k for which (*) holds in the inductive hypothesis and then continue on to prove the inductive step.

$$\Big((*) \text{ True for } n = 2^{i-1}\Big) \Rightarrow \Big((*) \text{ True for } n = 2^i\Big)$$

While we are doing this, we will discover sufficient assumptions on k (and n_0) to ensure that k exists.

We want to prove that if, for all $n = 2^i$,
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In order to guarantee

$$T(n) \le kn \log n$$

we must have

$$-kn + cn \le 0.$$

We therefore make the final assumption:

$$\boxed{k \ge c.}$$

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We therefore conclude, *by the principle of mathematical induction*, that as long as all three of our assumptions are satisfied,
$\forall n > n_0 = 1 \;\; T(n) \le kn \log n$.
We have therefore **proved** that $T(n) = O(n \log n)$.

Our inductive hypothesis:
if $m = 2^j$ with $1 \le j < i$ then $T(m) \le km \log m$.

- Note that the inductive hypothesis (and associated inductive step) was not fully defined when we started the proof, since we didn't say what the value of k was.

- It was only while in the middle of the proof that we specified the value of k (by discovering the conditions on k that would allow the inductive step to work)

- After the fact, it is possible to write a more traditional inductive proof, in which the value of k is given, but this can be even more confusing.

A More Traditional Induction Proof

We want to prove that if, for all $n = 2^i$,

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$$\Rightarrow \quad \forall n > 1, \quad T(n) \le kn \log n$$

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(i) Since $\log 2 = 1$, $\quad T(2) = \frac{T(2)}{2} 2 \le k 2 \log 2$

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$\boxed{\text{And we are done!}}$

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Because the discussion in the first proof explained **why** we were making the choices we did, many people prefer the structure of the first proof to that of the second.

This type of inductive proof – in which conditions on the parameters are developed **during** the proof – is therefore iused quite often in books and articles.

Example 2: We now prove by induction that, for T defined on $n = 3^i$, $1 = 0, 1, 2, \ldots$

$$\boxed{\begin{array}{l}\text{if } T(n) \le T(n/3) + cn \text{ for some constant } c,\\ \quad \text{then } T(n) = O(n).\end{array}}$$

From definition of big O we need to show that

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(i) When $n = 1$, $\quad T(1) \le kn$ by definition

(ii) Let $n = 3^i$. Suppose $T(m) \le km$ for all $m = 3^j$ with $0 \le j < i$.

$$\begin{aligned} T(n) &\le T(n/3) + cn \\ &\le k(n/3) + cn \\ &= kn + (c - 2k/3)n. \\ &\le kn \end{aligned}$$

$\boxed{\text{And we are done!}}$

Example 3: We now prove by induction that, for T defined on $n = 2^i$, $1 = 0, 1, 2, \ldots$

$$\boxed{\text{if } T(n) \le 4T(n/2) + cn \text{ for some constant } c, \text{ then } T(n) = O(n^2).}$$

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$$k \ge T(1)$$

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for $m = 2^j$, $0 \leq j < i$,
$$T(m) \leq km^2$$
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Statement is **too weak** to be proved by induction.

To fix this, let's see if we can prove something that is actually *stronger* than we were originally trying to prove — namely, $T(n) \le k_1 n^2 - k_2 n$ for some positive constants k_1 and k_2.

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With these 2 assumptions we have proved *inductively* that $T(n) \le k_1 n^2 - k_2 n$ so $T(n) = O(n^2)$.

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- When we give an induction proof in this way, we are using a **stronger inductive hypothesis**.