Sorting and Searching

Lecture 2: Priority Queues, Heaps, and Heapsort

Priority Queue: Motivating Example

3 jobs have been submitted to a printer in the order A, B, C.
Consider the printing pool at this moment.

Sizes: Job A — 100 pages
Job B — 10 pages
Job C — 1 page

Average finish time with FIFO service:

$$(100+110+111)\ /\ 3 = 107 \text{ time units}$$

Average finish time for shortest-job-first service:

$$(1+11+111)\ /\ 3 = 41 \text{ time units}$$

Priority Queue: Motivating Example

- The elements in the queue are printing jobs, each with the associated number of pages that serves as its priority
- Processing the shortest job first corresponds to extracting the smallest element from the queue
- Insert new printing jobs as they arrive

A queue capable of supporting two operations: Insert and Extract-Min?

Priority Queue

Priority queue is an abstract data structure that supports two operations

- Insert: inserts the new element into the queue
- Extract-Min: removes and returns the smallest element from the queue

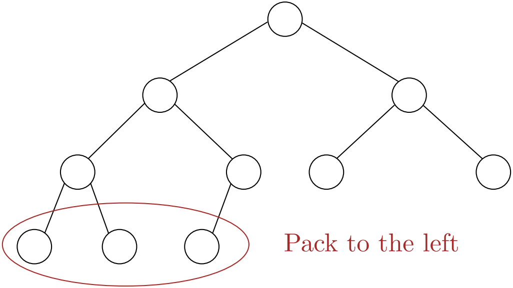
Possible Implementations

- unsorted list + a pointer to the smallest element
  - Insert in $O(1)$ time
  - Extract-Min in $O(n)$ time, since it requires a linear scan to find the new minimum
- sorted array
  - Insert in $O(n)$ time
  - Extract-Min in $O(1)$ time
- sorted doubly linked list
  - Insert in $O(n)$ time
  - Extract-Min in $O(1)$ time

Question

Is there any data structure that supports both these priority queue operations in $O(\log n)$ time?

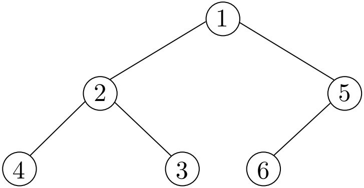
(Binary) Heap



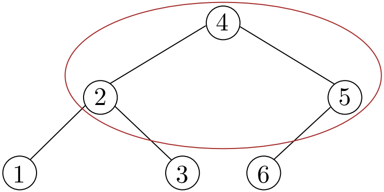
Heaps are "almost complete binary trees"

- All levels are full except possibly the lowest level
- If the lowest level is not full, then nodes must be packed to the left

Heap-order Property



A min-heap



Not a heap

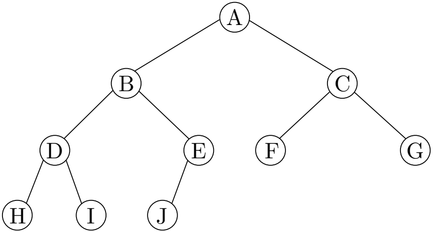
Heap-order property:

The value of a node is at least the value of its parent — Min-heap

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are n elements in the heap)
  - Insert in $O(\log n)$ time
  - Extract-Min in $O(\log n)$ time
- Structure properties
  - A heap of height h has between 2^h to $2^{h+1} - 1$ nodes. Thus, an n-element heap has height $\Theta(\log n)$.
  - The structure is so regular, it can be represented in an array and no links are necessary !!!

Array Implementation of Heap

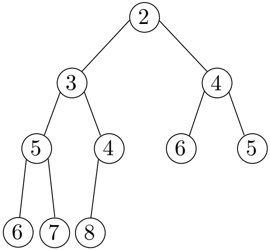


1	2	3	4	5	6	7	8	9	10
A	B	C	D	E	F	G	H	I	J

- The root is in array position 1
- For any element in array position i
  - The left child is in position $2i$
  - The right child is in position $2i + 1$
  - The parent is in position $\lfloor i/2 \rfloor$
- We will draw the heaps as trees, with the understanding that an actual implementation will use simple arrays

Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



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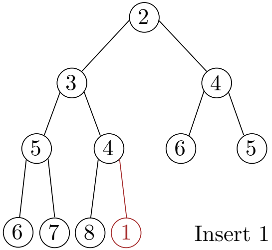

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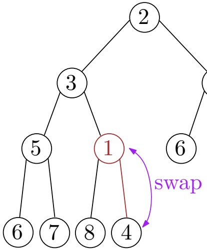

Percolate up to maintain the min-heap property

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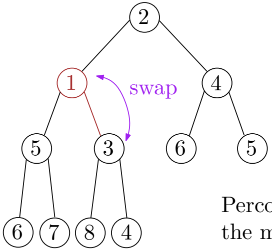

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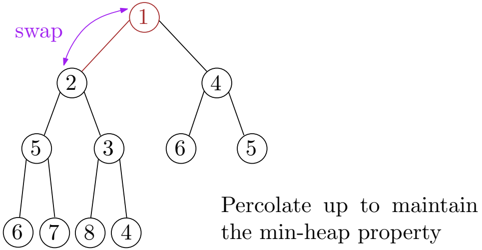
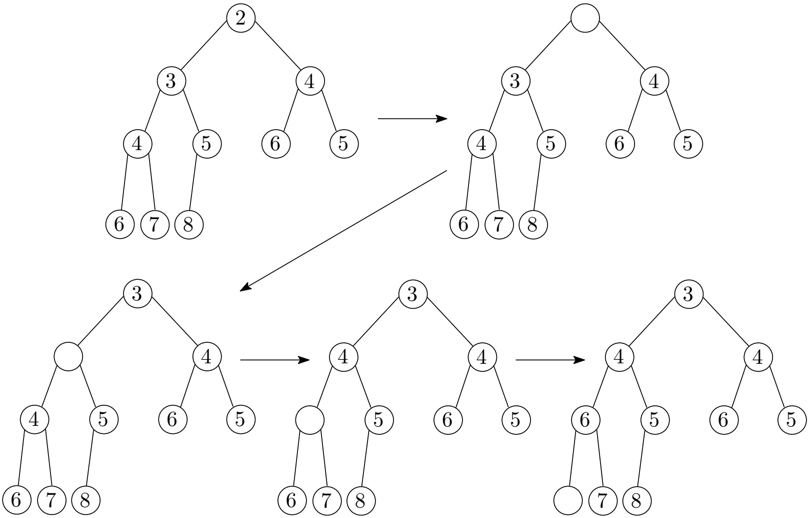


- Correctness: after each swap, the min-heap property is satisfied for the subtree rooted at the new element
- Time complexity $= O(\text{height}) = O(\log n)$

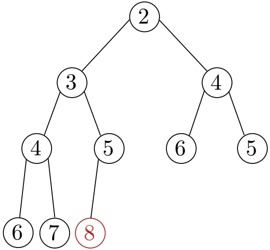
Extract-Min: First Attempt



Min-heap property preserved, but completeness not preserved!

Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



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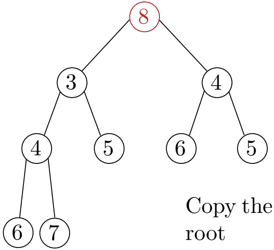

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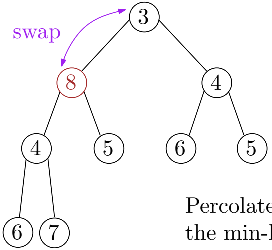

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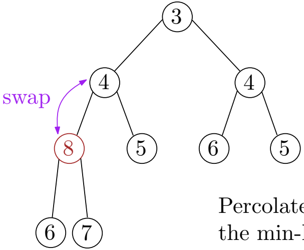

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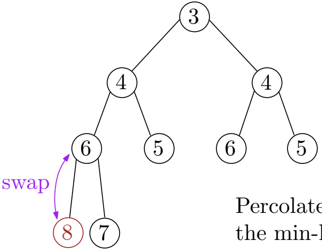

- Correctness: after each swap, the min-heap property is satisfied for all nodes except the node containing the element (with respect to its children)
- Time complexity $= O(\text{height}) = O(\log n)$

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap
  - insert n elements one by one
    $\Longrightarrow O(n \log n)$
    (A more clever approach can do this in $O(n)$ time.)
- Perform n Extract-Min operations
  - the elements are extracted in sorted order
  - each Extract-Min operation takes $O(\log n)$ time
    $\Longrightarrow O(n \log n)$
- Total time complexity: $O(n \log n)$

Summary

- A *Priority queue* is an abstract data structure that supports two operations: Insert and Extract-Min.
- If priority queues are implemented using heaps, then these two operations are supported in $O(\log n)$ time.
- Heapsort takes $O(n \log n)$ time, which is as efficient as merge sort and quicksort.

New Operation

Sometimes priority queues need to support another operation called Decrease-Key

- Decrease-Key: decreases the value of one specified element

- Decrease-Key is used in later algorithms, e.g., in Dijkstra's algorithm for finding Shortest Path Trees

Question

How can heaps be modified to support Decrease-Key in $O(\log n)$ time?

Going Further

- Original algorithm due to Williams in *Communications of the Association for Computing Machinery*, (**7**)(6), 1964.
- For some algorithms, there are other desirable Priority Queue operations, e.g., *Delete* an arbitrary item and *Meld*ing or taking the union of two priority queues
- There is a tradeoff between the costs of the various operations. Depending upon where the data structure is used, different priority queues might be better.
- Most famous variants are *Binomial Heaps* and *Fibonacci Heaps*