Greedy Algorithms: The Fractional Knapsack

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THE DEPARTMENT OF COMPUTER SCIENCE
計算機科學系 HKUST

Outline

- Introduction
- The Knapsack problem.
- A greedy algorithm for the fractional knapsack problem
- Correctness

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- Final output is an optimal solution.
- Greedy algorithms don't always yield optimal solutions but, when they do, they're usually the simplest and most efficient algorithms available.

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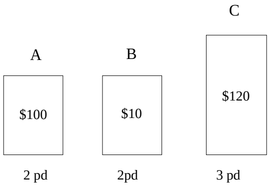
The Knapsack Problem...



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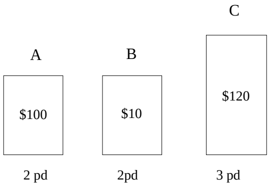

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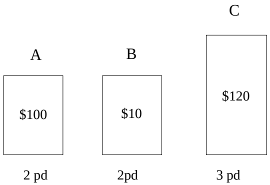

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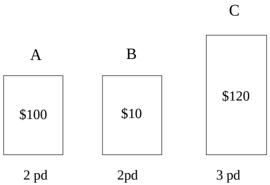

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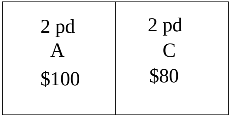

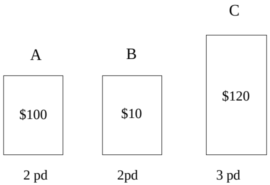

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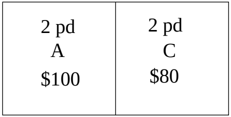

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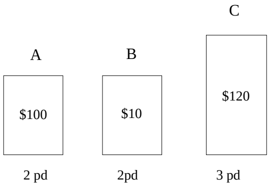

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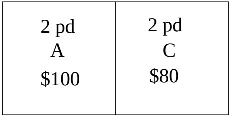

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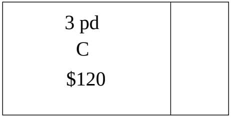

The Fractional Knapsack Problem: Formal Definition

- Given K and a set of n items:

weight	w_1	w_2	$\ldots$	w_n
value	v_1	v_2	$\ldots$	v_n

- Find: $0 \le x_i \le 1$, $i = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} x_i w_i \le K$$

and the following is maximized:

$$\sum_{i=1}^{n} x_i v_i$$

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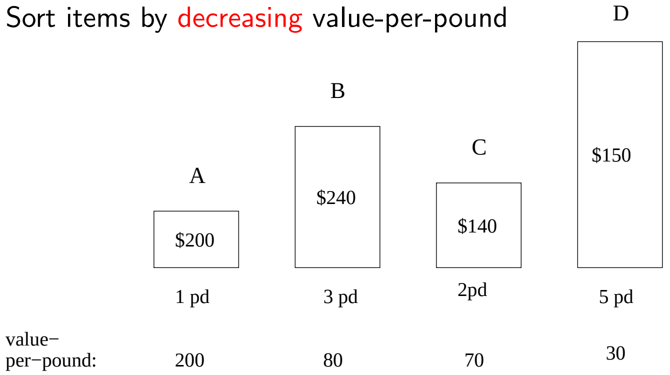

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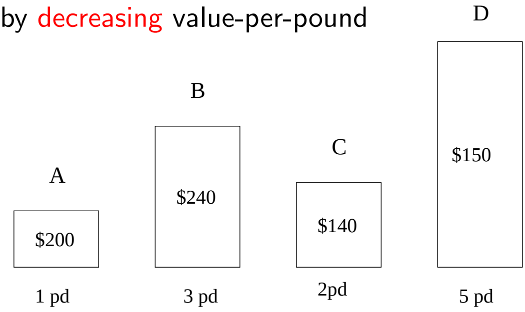

value−per−pound:	200	80	70	30

If knapsack holds $K = 5$ pd, solution is:

1	pd	A
3	pd	B
1	pd	C

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Running time: $O(n \log n)$.

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- We claim that the total value for this set of items is the optimal value.

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- By definition, solution G takes a greater amount of item i than solution O (because the greedy solution always takes as much as it can). Let $x = x_i - y_i$.

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Therefore G is also optimal!

The 0-1 Knapsack Problem does not have a greedy solution!

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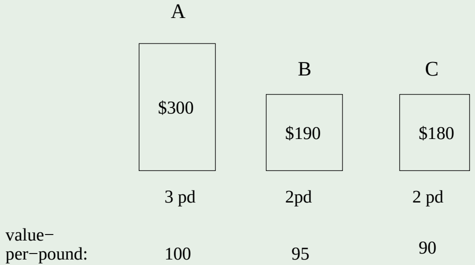

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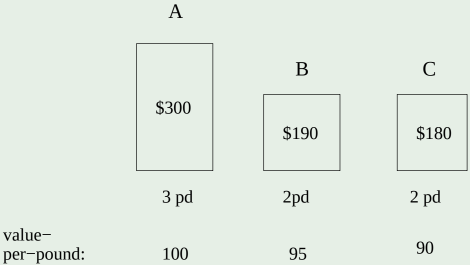

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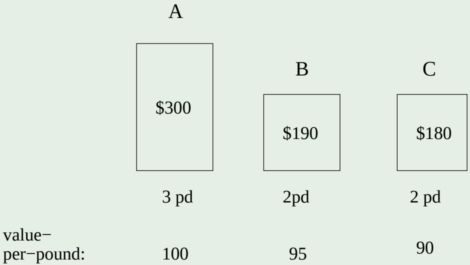

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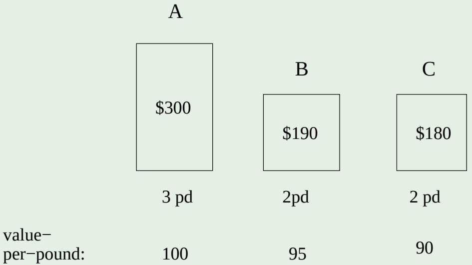

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Suppose we tried to prove the greedy algorithm for 0-1 knapsack problem **does** construct an optimal solution. If we follow exactly the same argument as in the fractional knapsack problem where does the proof fail?