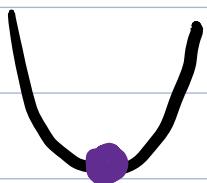


## Extreme value th

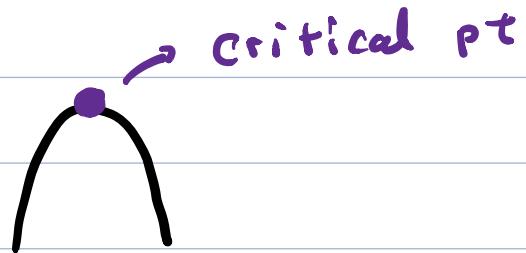
### single variable



local min

$$f'(x_0) = 0$$

$$f''(x_0) > 0$$



local max

$$f'(x_0) = 0$$

$$f''(x_0) < 0$$

### 2-variables

Critical point  $\nabla f(x_0, y_0) = (0, 0)$

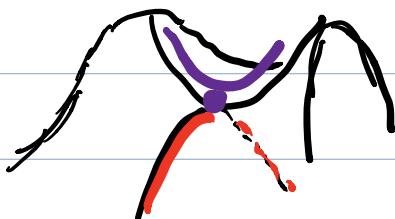
Hessian  $D = \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

at the critical pt  $(x_0, y_0)$ ,  $f(x_0, y_0)$  is

local min if  $f_{xx}(x_0, y_0) > 0$  &  $D > 0$

local max if  $f_{xx}(x_0, y_0) < 0$  &  $D > 0$

a Saddle point if  $D < 0$



- e.g
- $f(x, y) = x^2 + y^2 \rightarrow$  local min at  $(0, 0)$
  - $f(x, y) = -x^2 - y^2 \rightarrow$  local max at  $(0, 0)$
  - $f(x, y) = xy \rightarrow$  saddle pt at  $(0, 0)$

## Extreme values under constraints

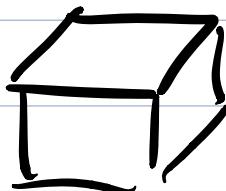
Thm (Lagrange multiplier method)

$f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$ , The min/max value of  $f(x, y, z)$  under constraints  $g(x, y, z) = C$  occurs at  $(x_0, y_0, z_0)$   
s.t.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

for some  $\lambda \in \mathbb{R}$

e.g A rectangular box is to be made from  $12\text{m}^2$  of card board. Find the maximum volume of the box



sol  $f(x, y, z) = xyz$

$$g(xyz) = 2xy + 2yz + 2zx = 12$$

$$\nabla f = (yz, zx, xy)$$

$$\nabla g = (2(y+z), 2(z+x), 2(x+y))$$

$$yz = 2\lambda(x+z) \quad \textcircled{1}$$

$$zx = 2\lambda(y+z) \quad \textcircled{2}$$

$$xy = 2\lambda(x+y) \quad \textcircled{3}$$

$$xy + yz + zx = 6 \quad \textcircled{4}$$

$$xyz = 2\lambda(xy+xz) \quad \textcircled{1} \cdot z - \textcircled{2} \cdot y$$

$$xyz = 2\lambda(yz+xy)$$

$$0 = 2\lambda z(x-y) \Rightarrow \lambda=0 \text{ or } \underline{\underline{z=0}} \text{ or } x=y=0.$$

$$\lambda=0 \Rightarrow \textcircled{1} : yz=6 \quad \times$$

$$\therefore x=y$$

$$\text{Similarly } \textcircled{2} \cdot y - \textcircled{3} \cdot z \quad \textcircled{1} \cdot x - \textcircled{3} \cdot z$$

$$\text{give } y=z, \quad x=z$$

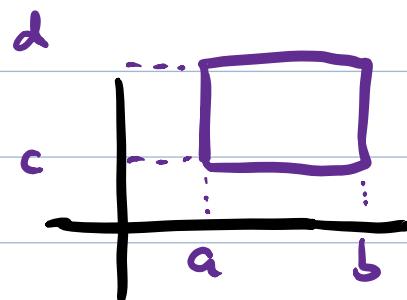
$$\therefore x=y=z$$

$$\textcircled{4} \Rightarrow 3x^2=6 \quad x=\sqrt{2},$$

$$f = xyz = 2\sqrt{2} \quad \square$$

## double integrals

e.g.  $\iint_R f(x,y) dx dy$ .



① (rectangular regions)

$$\iint_R f dx dy = \int_c^d \int_a^b f(x,y) dx dy$$

$$= \int_a^b \int_c^d f(x,y) dy dx$$

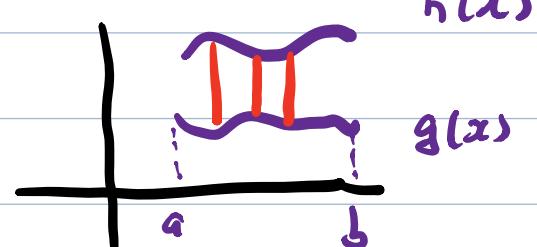
Integrate first

(consider x as a constant)

integrate next.

② (regions bounded by  $y = g(x)$ )  
 $y = h(x)$ )

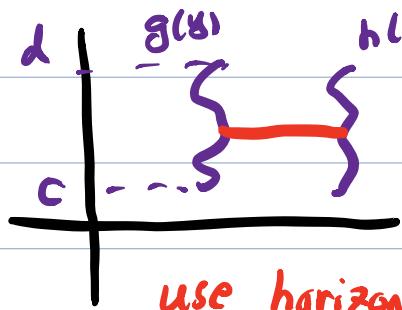
$$\iint_R f dx dy = \int_a^b \int_{g(x)}^{h(x)} f dy dx$$



use vertical lines

③ (regions bounded by  $x = g(y)$ )  
 $x = h(y)$ )

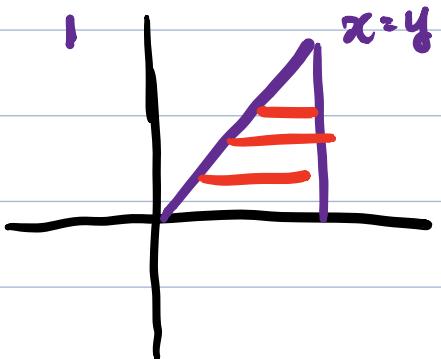
$$\iint_R f dA = \int_c^d \int_{g(y)}^{h(y)} f dx dy$$



use horizontal lines

$$\text{e.g. } \int_0^1 \int_y^1 e^{x^2} dx dy$$

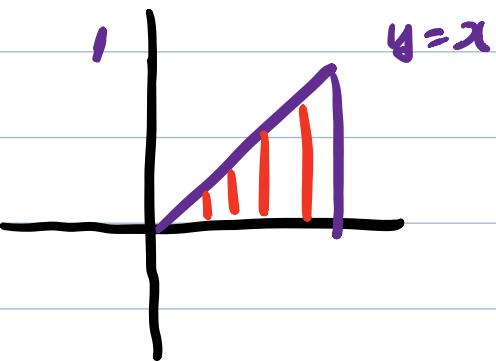
Q1 ① Sketch the region



② Change the order of integration

(use vertical lines)

$$0 \leq y \leq x \quad 0 \leq x \leq 1$$



$$\int_0^1 \int_0^x e^{x^2} dy dx$$

$$= \int_0^1 [ye^{x^2}]_{y=0}^x dx$$

$$= \int_0^1 xe^{x^2} dx$$

$$= \left[ \frac{e^{x^2}}{2} \right]_0^1 = \frac{e-1}{2}$$

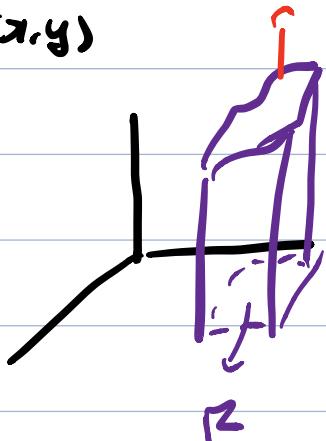
## Prop

① Area of a region  $R$ :  $\iint_R 1 \, dx \, dy$

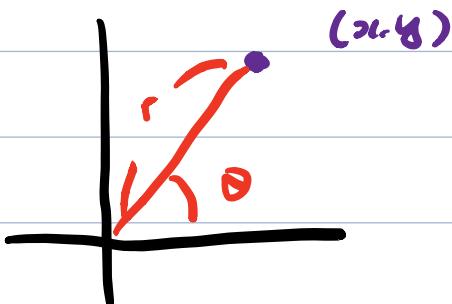
$$z = f(x, y)$$

② Volume of a region bounded by  $R$  and  $z = f(x, y)$

$$\iint_R f(x, y) \, dx \, dy.$$



## Polar Coord



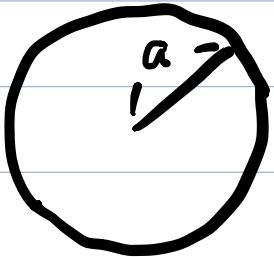
$$x \mapsto r \cos \theta$$

$$y \mapsto r \sin \theta$$

$$dx \, dy = r \, dr \, d\theta \quad *$$

$$\iint_R f(x, y) \, dx \, dy = \iint_{R'} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

e.g. Find area of a disk w/ radius  $a$



$$0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi$$

$$A = \iint_R 1 \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^a 1 \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{2} d\theta$$

$$= \pi a^2$$

## uv-sub (Change of variables)

$$u = x(u, v) \quad \text{differentiable, bijective}$$

$$v = y(u, v)$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Jacobian

$$\iint_R f(x, y) dx dy = \iint_{R'} f(x(u, v), y(u, v)) |J| du dv.$$

e.g (Polar coord)

$$x = r \cos \theta$$

$$(= x(r, \theta))$$

$$y = r \sin \theta$$

$$(= y(r, \theta))$$

$$J = \det \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\iint_R f(x, y) dx dy = \iint_{R'} f(r \cos \theta, r \sin \theta) r dr d\theta$$