GRE Math Subject Prep Course: Differential Equations

July 7, 2021

1. (Chapter 4 Prob 1)[1] Let $y = f(x)$ be the solution of the equation

$$\frac{dy}{dx} = \frac{x^2}{x^2 + 1}$$

such that $y = 0$ when $x = 0$. What is the value of $f(1)$?

(A) $\mathbb{Q}$, the set of rational numbers

(B) $\mathbb{Z}$, the set of all integers

(C) $\mathbb{Q}^c$, the set of all irrational numbers

(D) $\mathbb{Z}^+$, the set of positive integers

(E) $\{(a, b) : a \in \mathbb{Z}, b \in \mathbb{Z}\}$.

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2. (Week 4 Prob 10)[2] Find all solutions of the equation $yy'' - 2(y')^2 = 0$ which pass through $x = 1$, $y = 1$.

(A) T and $S \cap T$ (B) S and $S \cup T$ (C) T and $S^c \cup T$

(D) Both (A) and (B) (E) Both (A) and (C)

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3. (Chapter 4 Prob 5) If a is a positive constant, let $y = f(x)$ be the solution of the equation

$$y''' - ay'' + a^2y' - a^3y = 0$$

such that $f(0) = 1$, $f'(0) = 0$, and $f''(0) = a^2$. How many positive values of x satisfies the equation $f(x) = 0$?

(A) $(-\infty, \infty)$ (B) $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (C) $[0, \infty)$

(D) $(-\infty, 0]$ (E) the empty set

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4. (Week 4 Prob 3) A tank initially contains a salt solution of 3 grams of salt dissolved in 100 liters of water. A salt solution containing 0.02 grams of salt per liter is pumped into the tank at 4 liters per minute. The tank is also draining at 4 liters per minute. Assuming the mixing is instantaneous, how many grams of salt are in the tank after 100 minutes?

[1] The problems with "Chapter *" are taken from "Cracking the GRE Mathematics Test", 4th Edition.
[2] The problems with "Week *" are taken from Christian Parkinson's GRE preparation course

(A) $\{(x, y, z) : 0 < x \leq 3,\ 0 < y \leq 3,\ 0 < z \leq 3\}$

(B) $\{(x, y, z) : x < 2,\ y < 2,\ z < 2\}$

(C) $\{(x, y, z) : |x| + |y| + |z| \leq 5\}$

(D) $\{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0\}$

(E) $\{(x, y, z) : 2 \leq x^2 + y^2 + z^2 < 8\}$

5. (Week 4 Prob 10) Which of the following are linear subspaces of the continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

I. $\{f : f \text{ is twice differentiable and } f''(x) - 2f'(x) + 3f(x) = 0 \text{ for all } x\}$

II. $\{g : g \text{ is twice differentiable and } g''(x) = 3g'(x) \text{ for all } x\}$

III. $\{h : h \text{ is twice differentiable and } h''(x) = h(x) + 1 \text{ for all } x\}$

(A) 0 (B) infimum of S (C) supremum of S

(D) $-\infty$ (E) does not exist

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6. (Practice Prob 44)[3] If y is a real-valued function defined on the real line and satisfying the initial value problem

$$\begin{aligned} y' + xy &= x \\ y(0) &= -1. \end{aligned}$$

Then $\lim_{x \to -\infty} y(x) =$

(A) B is closed.

(B) B is not open.

(C) b is a limit point of B.

(D) No sequence in B converges to b.

(E) There is an open interval containing b but containing no point of B.

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7. (Week 4 Prob 4) Find the solution of $xdy + (y - xe^x)dx = 0$ which passes through the point $(1, 0)$.

(A) $\{x_n\}$ contains infinitely many convergent subsequences

(B) $\{x_n\}$ contains convergent subsequences with different limits.

(C) $\{y_n = \min_{k \leq n} x_k\}$ is convergent.

(D) All the above

(E) (A) and (C) only

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[3]The problems with "Practice Book" are taken from the mathematics test practice book by ETS, which can be found at http://www.ets.org/Media/Tests/GRE/pdf/Math.pdf

8. (Chapter 4 Prob 7) Let $y = f(x)$ be the solution of the equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin x$$

such that $f(\pi) = 1$. What is the value of $f(\frac{1}{2}\pi)$?

I. S is a connected subset of $\mathbb{R}$.
II. S is an open subset of $\mathbb{R}$.
III. S is a bounded subset of $\mathbb{R}$.

(A) I only (B) I and II only (C) I and III only

(D) II and III only (E) I, II and III

9. (Week 4 Prob 6) Find the general solution of $y''' - 3y'' + 3y' - y = 0$.

(A) $f'(x)$ is Riemann integrable

(B) $f''(x)$ exists

(C) $f'(x)$ is continuous

(D) $f(x)$ may be unbounded

(E) $f(x)$ is uniformly continuous on the interval

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10. (Chapter 4 Prob 8) Let $y = f(x)$ be the solution of the equation

$$\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}$$

such that $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) = -1$. What is $f(x)$?

(A) f is constant.

(B) f is monotone.

(C) $\{f_n\}$ converges uniformly to f.

(D) Both (A) and (B).

(E) Both (B) and (C).

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11. (Chapter 4 Prob 6) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable and integrable function. The integral curve of the differential equation

$$[y + g(x)]dx + [x - g(y)]dy = 0$$

that passes through the point $(1, 1)$ must also pass through which of the following points?

I. There is a constant $C > 0$ such that $|f(x) - f(y)| \leq C$ for all x and y in $[0, 1]$.

II. There is a constant $D > 0$ such that $|f(x) - f(y)| \leq 1$ for all x and y in $[0, 1]$ that satisfy $|x - y| \leq D$.

III. There is a constant $E > 0$ such that $|f(x) - f(y)| \leq E|x - y|$ for all x and y in $[0, 1]$.

(A) I only (B) III only (C) I and II only

(D) II and III only (E) I, II and III

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12. (Chapter 4 Prob 2) A population of bacteria grows at a rate proportional to the number present. After two hours, the population has tripled. After two more hours elapse, the population will have increased by a factor of k. What is the value of k?

(A) 1/2 (B) 0 (C) 1/3

(D) 1 (E) none of the above

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13. (Chapter 4 Prob 10) Given the following differential equations has an integrating factor of the form $\mu(x, y) = x^m y^n$, determine its general solution.

$$(3xy^2 - 5y)dx + (2x^2y - 3x)dy = 0$$

(A) 1/2 (B) 0 (C) 1/3

(D) 1 (E) none of the above

Answer: CBAC CCDC EBCB

(In the handout in class, 4.(C) has a typo: "$|x+y+z| \leq 5$" should change to "$|x|+|y|+|z| \leq 5$", which is corrected in this file.)