GRE
- ODE
- Linear
- exact
- non-exact (integrating factors, homogeneous)

Def (ODE) An <u>ordinary differential equation</u> is an equation of the form

$$F(x, y, y', \cdots, y^{(n)}) = 0$$

for $F : \mathbb{R}^{n+2} \to \mathbb{R}$

Def (linearity) An ODE

$$F(x, y, \cdots, y^{(n)}) = 0$$

is <u>linear</u> if $F(x, -) : \mathbb{R}^{n+1} \to \mathbb{R}$ is a linear map.

We can write a linear ODE of the form

$$a_n(x)\, y^{(n)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) + f(x) = 0$$

for $a_i : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$

- n is called an <u>order (degree)</u> of linear ODE
- Solutions to an ODE are often called <u>integral curves</u>

Def (Exact equations) Let $F, M, N : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions. Suppose

$$\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y).$$

Then an ODE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called an <u>exact equation</u>

(also written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$
)

Def (IVP) An <u>initial value problem</u> is a differential equation along with an appropriate number of initial conditions (order -1)

e.g. $4y'' + yy' + e^x = 0 \qquad y(0) = 1 \quad y'(0) = 2$

<u>Separable equations</u>

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)\,dy = f(x)\,dx$$

$$\int g(y)\,dy = \int f(x)\,dx.$$

e.g $\frac{dy}{dx} = \frac{x}{e^y}$ $y(0) = 0$

$$\int e^y dy = \int x\,dx$$

$$e^y = \frac{1}{2}x^2 + C$$

$$y = \log\left(\frac{1}{2}x^2 + C\right)$$

$y(0) = \log C = 0$ $C = 1$

Homogeneous equations

$f(x,y)$ is homogenous of deg n if

$$f(tx, ty) = t^n f(x,y)$$

$M(x,y)\,dx + N(x,y)\,dy = 0$
is homogeneous if M, N are homogeneous functions of the same degree.

e.g $(x^2+y^2)\,dx - 2xy\,dy = 0$

$y = tx \Rightarrow (x^2 + t^2x^2)\,dx - 2x^2t\,dy = 0$

$$\Rightarrow \quad (1+t^2)\,dx - 2t\,d(tx) = 0$$

$$\Rightarrow \quad (1-t^2)\,dx - 2tx\,dt = 0$$

$$\Rightarrow \quad \frac{1}{x}\,dx = \frac{2t}{1-t^2}\,dt$$

$$\Rightarrow \quad \ln|x| = -\ln|1-t^2| + \ln c$$

$$\Rightarrow \quad |x| = \frac{c}{|1-t^2|} = \frac{c}{|1-(y/x)^2|} \qquad c>0$$

$$\Rightarrow \quad x^2 - y^2 = cx \qquad c \neq 0$$

<u>Exact equations</u>

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$F_x = M(x,y) \qquad F_y = N(x,y)$

sol: $F(x,y) = c$

<u>e.g</u> $(1-2xy)\,dx + (4y^3 - x^2)\,dy = 0$

$F = \int M\,dx = \int 1-2xy\,dx = x - x^2y + g(y)$

$F_y = -x^2 + g'(y) = N = 4y^3 - x^2$

$g(y) = \int 4y^3\,dy = y^4$

$\therefore F(x,y) = x - x^2y + y^4 = C.$

Integrating factors

$\mu(x)$ or $\eta(y)$

$$\text{Not exact} \xrightarrow[\text{factor}]{\times\text{ integrating}} \text{exact}$$

- $\dfrac{(M_y - N_x)}{N} = f(x) \Rightarrow \mu(x) = e^{\int f(x)\,dx}$
- $-\dfrac{M_y - N_x}{M} = g(y) \Rightarrow \eta(y) = e^{\int g(y)\,dy}$

e.g. $(xy + x - 1)\,dx + x^2\,dy = 0$

$$\frac{M_y - N_x}{N} = -\frac{1}{x} \Rightarrow \mu(x) = e^{\int -\frac{1}{x}\,dx}$$

$$= \frac{1}{x}$$

$\Rightarrow \quad (y+1-\frac{1}{x})\,dx + x\,dy = 0$

$F = \int y+1-\frac{1}{x}\,dx = xy+x-\ln|x| + h(y)$

$F_y = N = x + h'(y) = x$

$F(x,y) = xy+x-\ln|x| = C.$

1st order ODE

- $\frac{dy}{dx} + P(x)\cdot y = Q(x)$

$\mu(x) = e^{\int P\,dx}$

$\mu y' + \mu P y = \mu Q$

$\Rightarrow (\mu y)' = \mu Q$ $\quad\quad (\mu' y = P\mu y)$

$\Rightarrow \quad y = \frac{1}{\mu}\int \mu Q\,dx$

e.g $\quad \frac{dy}{dx} = 5x - \frac{3y}{x}, \quad\quad y(1)=2$

$\mu = e^{\int \frac{3}{x}dx} = x^3$

$x^3 y = \int 5x^4 dx = x^5 + c$

$y = x^2 + \frac{c}{x^3}$

$y(1) = 1 + c \qquad c = 1.$

High order linear ODE w/ const Coeff

$ay'' + by' + cy = d(x) \quad — (*)$

$y = y_p + y_h$

y_p : particular solution of $(*)$

y_h : general solutions of $ay'' + by' + cy = 0$

Characteristic eq : $at^2 + bt + c = 0 \quad — (*)$

1. roots of $(*)$ are real and distinct :

$t = m_1, m_2$

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

2. roots are real and identical : $t = m$

$y_h = c_1 e^{mx} + c_2 x e^{mx}$

3. roots are not real $t = \alpha \pm \beta i$

$$y_h = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$