Infinite series

$(a_n): a_1, a_2, \cdots$ sequence

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$ Partial sum

$S_n = a_1 + \cdots + a_n$

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n$$

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$$ Geometric series

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r} \quad \text{if} \quad r \neq 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if} \quad |r| < 1$$

diverges if $|r| \geq 1$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$ harmonic series

Prop. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

pf $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$$\geq 2^0+\underbrace{2^{-1}+2^{-1}}_{2}+\underbrace{2^{-2}+2^{-2}+2^{-2}+2^{-2}}_{2^2}+\cdots+\underbrace{2^{-m}+\cdots+2^{-m}}_{2^m}$$

$= m \qquad (m=\lfloor \log_2 n \rfloor)$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n} \geq \lim_{n\to\infty} \log_2 n = \infty$

$\sum \frac{1}{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$ p-series

converges if $p>1$
diverges if $p\leq 1$

Convergence test

1. The comparison test

$0\leq a_n\leq b_n$ for $n\geq N$ any large integer

$\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges
$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

2. The ratio test

$$a_n \geq 0 \quad \text{for} \quad n \geq N$$

$$L := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$L < 1 \Rightarrow \sum a_n$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

3. The root test

$$a_n \geq 0 \quad \text{for} \quad n \geq N$$

$$L := \lim_{n\to\infty} \sqrt[n]{a_n}$$

$L < 1 \Rightarrow \sum a_n$ converges

$L > 1 \Rightarrow \sum a_n$ diverges

4. The integral test

$f(x) \geq 0$, monotonically decreasing for $x \geq N$

$a_n := f(n)$ for $n \in \mathbb{N}$

$\sum a_n$ converges $\Leftrightarrow \int_1^\infty f(x)\,dx$ converges

$\sum a_n$ diverges $\Leftrightarrow \int_1^\infty f(x)\,dx$ diverges

5. The alternating series test

$a_n \geq 0$, monotonically decreasing for $n \geq N$.

$\lim_{n\to\infty} a_n = 0 \quad \Rightarrow \quad \sum (-1)^{n+1} a_n$ converges

* If $\sum |a_n|$ converges, we say the series converges absolutely.

6. The absolute convergence test

$\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges.

e.g 1) $\sum \frac{1}{n^p}$ 2) $\sum \frac{\sin^2 n}{n^2+n}$ 3) $\sum \frac{n^{2021}}{n!}$

4) $\sum \frac{n^3}{(\ln 3)^n}$

1) $f(x) = \frac{1}{x^p}$

$\int_1^\infty x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p} \right]_1^\infty$ converges if $p > 1$, diverges if $p \leq 1$

By the integral test,

$\sum \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \leq 1$

2) $\frac{\sin^2 n}{n^2+n} \lesssim \frac{1}{n^2}$

By the comparison test,

$\sum \frac{\sin^2 n}{n^2+n}$ converges

3) $a_n = \frac{n^{2021}}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{n+1}{n}\right)^{2021}$$

$$= 0.$$

$\sum \frac{n^{2021}}{n!}$ converges by the ratio test

4) $a_n = \frac{n^3}{(\ln 3)^n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n^3}}{\ln 3} = \frac{1}{\ln 3} < 1$$

(exercise : show $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$)

$\sum \frac{n^3}{(\ln 3)^n}$ converges by the root test.