Determinant

(In GRE, $\det(A) = |A|$)

Def $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = ad - bc$

$A \in \mathbb{F}^{n \times n}$

removing i-th row, j-th column

$$\det(A) := \sum_{i=1}^{n} (-1)^{ij} \det(A^{ij}) \quad \text{for } 1 \le j \le n$$

Prop

- $\det(I_n) = 1$ I_n : $n \times n$ identity matrix
- $\det\begin{pmatrix} \vdots \\ A_j \\ \vdots \\ A_i \\ \vdots \end{pmatrix} = -\det(A)$
- $\det\begin{pmatrix} \vdots \\ cA_i \\ \vdots \end{pmatrix} = c\det(A)$
- $\det\begin{pmatrix} \vdots \\ A_i + cA_j \\ \vdots \end{pmatrix} = \det(A)$ $(i \neq j)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A) \neq 0 \iff A$ is non-singular.

Def $A \in \mathbb{F}^{n\times n}$ $\quad tr(A) := \sum_{i=1}^{n} a_{ii}$

Prop • $tr(cA+B) = c\,tr(A) + tr(B)$ $\quad$ ($tr: \mathbb{F}^{n\times n} \to \mathbb{F}$) lin. tf.

• $tr(AB) = tr(BA)$

Eigenvalues & eigenvectors

Def $A \in \mathbb{R}^{n\times n}$. An eigenvalue of A is $\lambda \in \mathbb{F}$ s.t.

$$Av = \lambda \cdot v$$

for some $v \in \mathbb{F}^n$, $v \neq 0$. v is called an eigenvector of λ

Suppose λ is an eigenvalue of A. Then

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

$$\Rightarrow \underbrace{\det(A - \lambda I)}_{\text{Polynomial of } \lambda.} = 0$$

Def Characteristic polynomial of A is

$$P_A(\lambda) := \det(A - \lambda I)$$

(rmk $P_A(A) = 0$)

Prop (Caley-Hamilton)

$A \in \mathbb{R}^{2\times 2}$, $\quad P_A(\lambda) = \lambda^2 - tr(A) + \det(A)$

Thm (Characterization of nonsingular matrices)

$A \in \mathbb{R}^{n \times n}$. TFAE

1) A is non-singular
2) A invertible
3) $rk(A) = n$
4) columns of A are lin. indep.
5) $null(A) = 0$ $\quad (N(A) = \{0\})$
6) T_A bijective (isomorphism)
7) $\det(A) \neq 0$
8) 0 is not an eigenvalue of A.

Diagonalization

Def $A, B \in \mathbb{R}^{n \times n}$. A and B are similar if $A = PBP^{-1}$ for some $P \in \mathbb{R}^{n \times n}$

Prop If $A \sim B$ (similar)

- $P_A(\lambda) = P_B(\lambda)$
- $\det(A) = \det(B)$
- $tr(A) = tr(B)$
- same eigenvalues.

Def $A \in \mathbb{R}^{n\times n}$ A is diagnalizable if A is similar to a diagonal matrix.

In this case $P = [v_1 \cdots v_n]$ where v_i's are lin. indep. eigenvectors of A.

Thm $A \in \mathbb{R}^{n\times n}$.

A is diagonalizable $\Leftrightarrow$ A has n lin. indep. eigen vectors.

e.g $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ A is not diagonalizable since the dimension of the eigenspace is 1.

Orthogonal matrices

Def (transpose) $(A^t)_{ij} = A_{ji}$

e.g $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^t = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Def (Standard inner product) $u, v \in \mathbb{F}^n$

$$\langle u, v \rangle = u^t \cdot v$$

rmk $\langle Au, v \rangle = \langle u, A^t v \rangle$ $A \in \mathbb{R}^{n\times n}$

Def $A \in \mathbb{R}^{n\times n}$ is an <u>orthogonal matrix</u> if

$$\langle Au, Av\rangle = \langle u, v\rangle$$

for any $u, v \in \mathbb{R}^n$

Prop If A is an orthogonal matrix,

- $A^tA = AA^t = I$

Def $A \in \mathbb{R}^{n\times n}$ A is a <u>symmetric matrix</u> if $A^t = A$.

Thm (Spectral thm) A symmetric matrix is diagonalizable

$$A = UDU^t$$

U^t → orthogonal matrix

<u>Complex matrices</u>

Recall the <u>conjugation</u> is $\overline{a+bi} = a-bi$

<u>Complex transpose</u> (adjoint) $A^* = \overline{A^t}$

<u>rmk</u> $\langle Au, v\rangle = \langle u, A^*v\rangle$ $A \in \mathbb{C}^{n\times n}$

Hermitian matrices (complex symmetric) $A^* = A$

Unitary matrices (complex orthogonal) $A^*A = AA^* = I$

normal matrices $A^*A = A^*A$