# Exotic 4-manifolds with boundary

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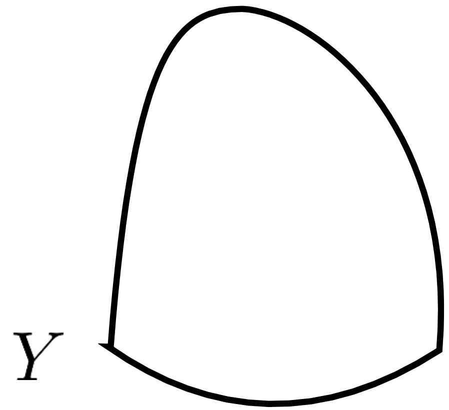
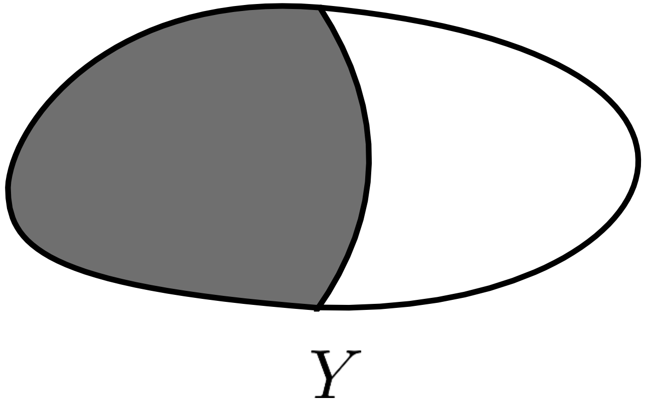
# 4-manifolds with boundary

- Goal: studying smooth structures on 4-manifolds with boundary.

- Try to figure out which 3-manifolds bound a compact 4-manifold with infinitely many smooth structures.

# Strategy

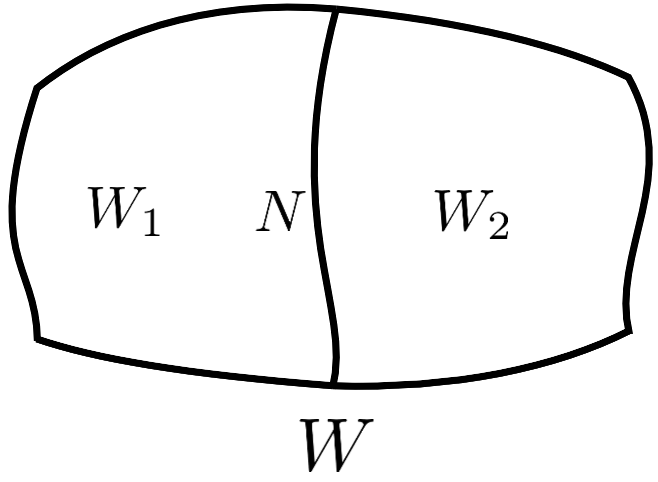



- Find/build a 4-manifold which $Y$ bounds (embeds).
- Modify the 4-manifold keeping the homeomorphism type.
- Use smooth invariants (SW, HF, genus, …) to distinguish them.

# Heegaard Floer invariants

- $W$ is a 4-manifold with $b_2^+ > 1$
- An admissible cut is a separating hypersurface $N$ satisfying




$$b_2^+(W_1), b_2^+(W_2) \geq 1$$

$$\delta : H^1(N) \to H^2(W, \partial W) \text{ vanishes}$$

$$F^{mix}_{W,s} := F^+_{W_2, s|_{W_2}} \circ \tau \circ F^-_{W_1, s|_{W_1}}$$

# Heegaard Floer invariants

- $X$ is a 4-manifold with connected boundary
- $b = (b_1, ..., b_n)$ is a basis of $H^2(X, \partial X)$
- (Juhasz-Zemke) Ozsvath-Szabo polynomial is defined as

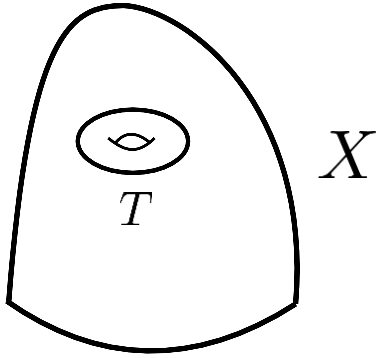
$$\Phi_{X;b} := \sum_{s \in \mathrm{Spin}^c(X)} F^{mix}_{X,s}(\theta^-) \cdot z_1^{\langle i_*(s-s_0) \cup b_1, [X,\partial X] \rangle} \cdots z_n^{\langle i_*(s-s_0) \cup b_n, [X,\partial X] \rangle}$$

# Concordance surgery

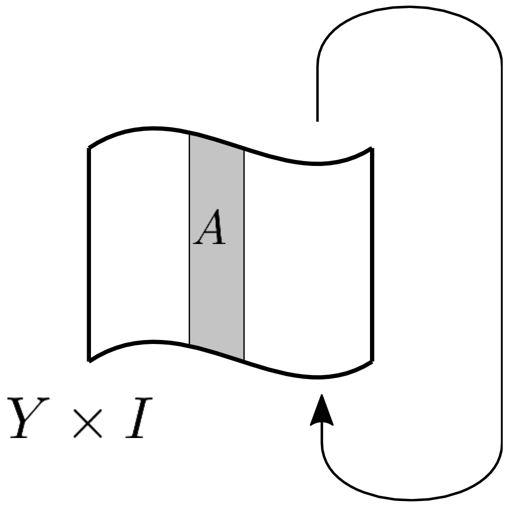
- A generalization of Fintushel-Stern knot surgery.

# Concordance surgery

- Let $X$ be a 4-manifold containing an embedded torus $T$ with a trivial normal bundle.




- Consider a homology sphere $Y$ and a self concordance $C = (Y \times I, A, K)$. Glue the ends of $(Y \times I, A)$ to obtain $(Y \times S^1, T')$.



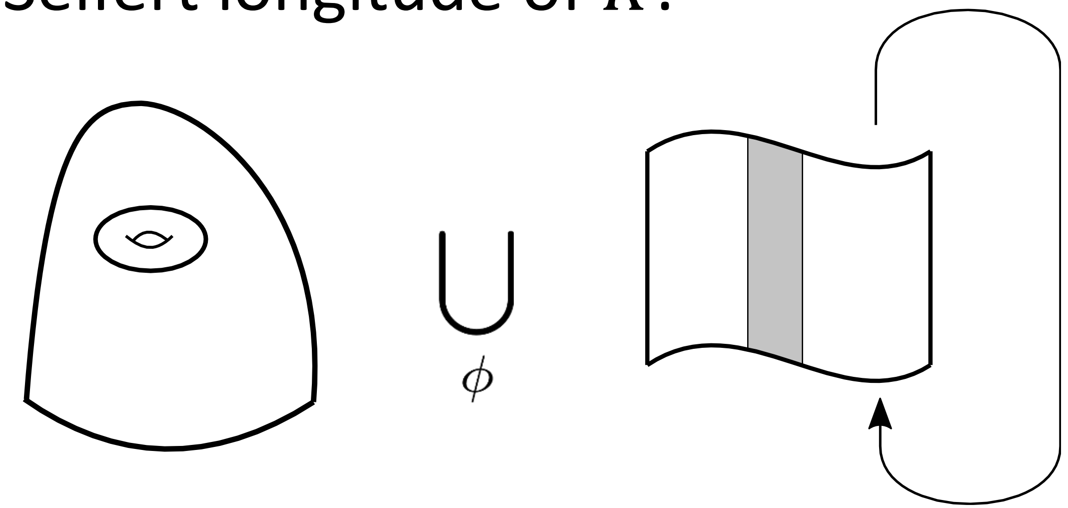
# Concordance surgery

- We obtain

$$X_{\mathcal{C}} := X \setminus N(T) \cup_\phi Y \times S^1 \setminus N(T')$$

where $\phi : \partial(X \setminus N_T) \to \partial W_{\mathcal{C}}$ is a diffeomorphism sending $\partial D^2$

- in $N(T)$ to the Seifert longitude of $K$.



# Graded Lefschetz number

(Juhász-Maregon) For a self-concordance $\mathcal{C} = (Y \times I, A, K)$

$$\widehat{F}_{\mathcal{C},i} : \widehat{HFK}(Y, K, i) \to \widehat{HFK}(Y, K, i)$$

$$\mathrm{Lef}_{\mathcal{C}}(z) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\widehat{F}_{\mathcal{C},i}) \cdot z^i$$

# Concordance surgery formula

- (Juhasz-Zemke) If $X$ is a closed 4-manifold containing a homologically non-trivial torus with trivial normal bundle,

$$\Phi_{X_\mathcal{C};b} = \mathrm{Lef}_\mathcal{C}(z) \cdot \Phi_{X;b}$$

- (Etnyre-M-Mukherjee) If $X$ is a 4-manifold with boundary

$$\Phi_{X_\mathcal{C};b} = \mathrm{Lef}_\mathcal{C}(z) \cdot \Phi_{X;b}$$

- If $\mathrm{Lef}_\mathcal{C}(z)$ and $\mathrm{Lef}_{\mathcal{C}'}(z)$ are different and $\Phi_{X;b} \neq 0$, then $X_\mathcal{C}$ is not diffeomorphic to $X_{\mathcal{C}'}$

# Main Theorem

A closed oriented 3-manifold $Y$ admits infinitely many simply-connected exotic fillings if

1) $Y$ admits a contact structure with non-vanishing contact invariant in $HF^+(Y)$, or
2) $Y$ is a rational homology 3-sphere embedding into a closed definite 4-manifold as a separating hypersurface.

# Proof of (2)

- Suppose $Y$ embeds into a closed negative definite 4-manifold $W$.

- For any Spin$^c$ structure on $W$, the map

$$F^+_{W,s} : HF^+(S^3, t) \to HF^+(S^3, t)$$

  is surjective.

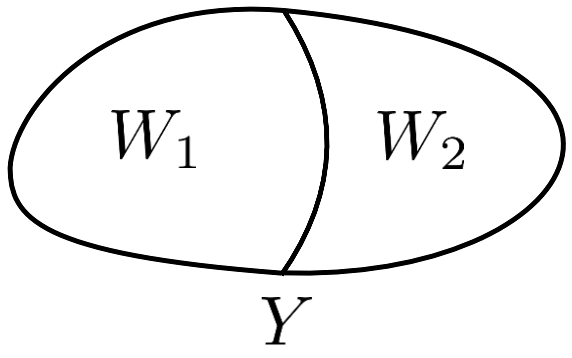
# Proof of (2)

- Cut $W$ along $Y$ into $W_1$ cup $W_2$. Then the map

$$F^+_{W_1,s|_{W_1}} : HF^+(S^3,t) \to HF^+(Y,s|_Y)$$

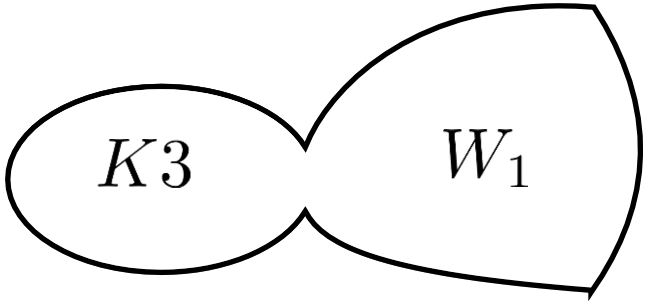
satisfies

$$F^+_{W_1,s|_{W_1}}(\theta^+) \neq 0$$



# Proof of (2)

- Consider a 4-manifold $X = K3\#W_1$




- Since $F^{mix}_{K3,s}(\theta^-) = \theta^+$, we have

$$F^{mix}_{K3\#W_1}(\theta^-) \neq 0$$

- Now apply the surgery formula on the torus in K3 surface.

# Questions

- Given a 3-manifold $Y$ and an element $\eta \in HF^+(Y)$, is there a 4-manifold $X$ such that $F_{X,s}^{mix}(\theta^-) = \eta$?

- Given a 3-manifold $Y$, is there any 4-manifold $X$ such that $F_{X,s}^{mix}(\theta^-) \neq 0$?

Thank you!