1. Limit of functions

Def $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$. The limit of $f(x)$ as x approaches x_0 exists if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$ for some fixed $L \in \mathbb{R}$. We denote by

$$\lim_{x \to x_0} f(x) = L$$

The limit of $f(x)$ as x approaches ∞ exists if for any $\varepsilon > 0$, there exists $M > 0$ s.t. $x > M \Rightarrow |f(x) - L| < \varepsilon$ for some fixed $L \in \mathbb{R}$, we denote by

$$\lim_{x \to \infty} f(x) = L$$

Thm (Sandwich thm) $f, g, h : \mathbb{R} \to \mathbb{R}$ and $g(x) \leq f(x) \leq h(x)$.

If $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} f(x) = L$.

e.g. $\lim_{x \to \infty} \frac{\sin x}{x}$.

$-1 \leq \sin x \leq 1$ $\quad\quad$ $-\frac{1}{x} \leq \frac{\sin x}{x} < \frac{1}{x}$.

$\therefore \lim_{x \to \infty} \frac{\sin x}{x} = 0$. $\square$

Prop $\lim_{x \to x_0} f(x) = L_1$ $\quad \lim_{x \to x_0} g(x) = L_2$

- $\lim_{x \to x_0} f(x) \pm g(x) = L_1 \pm L_2$
- $\lim_{x \to x_0} f(x) \cdot g(x) = L_1 L_2$
- $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ (if $L_2 \neq 0$)
- $\lim_{x \to x_0} c f(x) = c L_1$

Def The <u>right-hand limit</u> of $f(x)$ as x approaches x_0 exists if for any $\varepsilon > 0$, there exists $\delta > 0$ s.t

$0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < L$

for some fixed $L \in \mathbb{R}$. We denote by

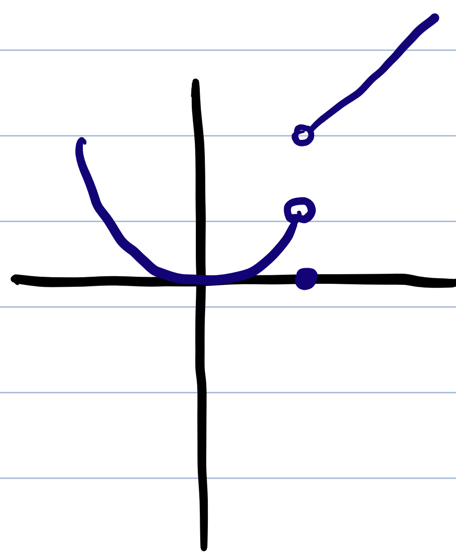
$$\lim_{x \to a+} f(x) = L$$

The <u>left-hand limit</u> of $f(x)$ as x approaches x_0 is defined similarly and denoted by

$$\lim_{x \to a-} f(x) = L.$$

e.g.

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x+1 & \text{if } x > 1 \\ 0 & \text{if } x = 1 \end{cases}$$

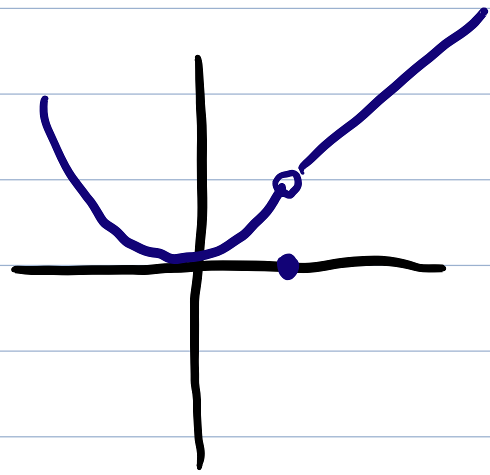


$\lim_{x \to 1-} f(x) = 1$ $\quad \lim_{x \to 1+} f(x) = 2.$

2. Continuous functions

Def f is <u>continuous</u> at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

e.g $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x & \text{if } x > 1 \\ 0 & \text{if } x = 1 \end{cases}$



$\lim_{x \to 1} f(x) = 1 \neq f(1)$.

f is not continuous at 1.

Prop Suppose f, g are continuous at x_0.
Then the following functions are also cont. at x_0

- $f \pm g$
- $f \cdot g$
- f / g (if $g(x_0) \neq 0$)
- $f \circ g$ (if f is continuous at $f(g(x_0))$)

e.g $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & x \neq 1 \\ a & x = 1 \end{cases}$ f is continuous when $a = ?$

$$\lim_{x\to 1} f = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

f is continuous at 1 if $a=2$.

<u>e.g</u> for $x \in (0,1)$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \quad (n>0,\ \gcd(m,n)=1) \end{cases}$$

f is continuous at

A) all rational numbers B) all irrational numbers

C) 0 D) no where.

<u>sol</u> We will show $\lim_{x\to x_0} f(x) = 0$ for any $x_0 \in \mathbb{R}$.

Then the answer is (B)

$|x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Choose $\varepsilon = \frac{1}{N}$. Then there there are fewer than

$1+2+\cdots+(N-1) = \frac{N(N-1)}{2}$ rational numbers in

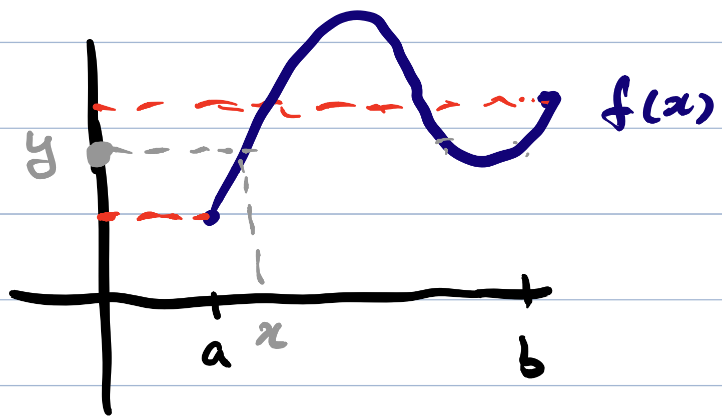
$(0,1)$ and of the form $\frac{m}{n}$ with $n < N$.

$\left(1, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \cdots, \frac{1}{N-1}, \cdots, \frac{N-1}{N-1}\right)$

so Choose δ to be the minimum number among

$|1-x_0|, |\frac{1}{2}-x_0|, \cdots, |\frac{1}{N-1}-x_0|, \cdots, |\frac{N-1}{N-1}-x_0|$.

Thm (Intermediate value thm) $f : [a,b] \to \mathbb{R}$ continuous.
Then for any y between $f(a)$ and $f(b)$, there is $x \in [a,b]$ such that $f(x) = y$.



e.g $f(x) = x^3 - 3x + 1$. Check there is a root in $[1,2]$.

$f(1) = -2 \quad f(2) = 1$. By IVT, there is $x \in [0,1]$ satisfying $f(x) = 0$.

3. Derivatives

Def $f : \mathbb{R} \to \mathbb{R}$. f is differentiable at x_0 if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{exists.}$$

We donate this limit by

$$f'(x_0), \quad \frac{df}{dx}(x_0).$$

Prop If f is differentiable at x_0, then f is continuous at x_0.

Prop (Derivatives table)

- $(x^n)' = n x^{n-1}$
- $(e^x)' = e^x$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\ln x)' = \frac{1}{x}$ for $x > 0$

$\vdots$

(check your textbook)

Prop f, g are differentiable at x_0.

- $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
- $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- $(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$ (if f is diffb at $g(x_0)$)
- $(f/g)'(x_0) = \frac{f'g - fg'}{g^2}(x_0)$

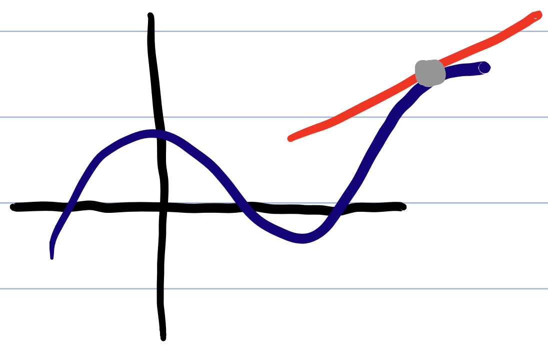
- If f is invertible near x_0, then

$$(f^{-1})'(x_0) = \frac{1}{f'(x_0)}$$

Def (tangent lines) $f: \mathbb{R} \to \mathbb{R}$ differentiable at $x_0 \in \mathbb{R}$.

The tangent line to the graph $y = f(x)$ at x_0 is given by

$$y = f(x_0) + \underbrace{f'(x_0)}_{\text{slope of tangent}} (x - x_0).$$



e.g Find the derivative of $f(x) = \dfrac{\ln(\sin x)}{\cos x}$

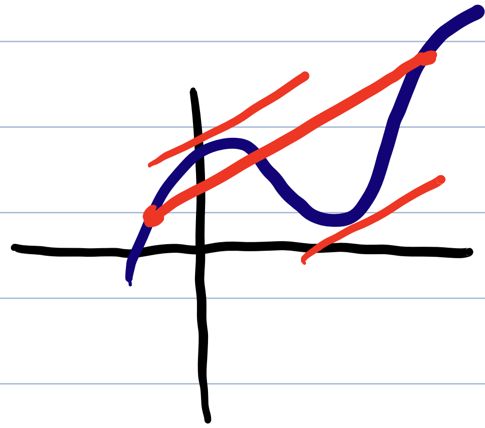
$$(\cos x)\, f(x) = \ln(\sin x)$$

$$\Rightarrow \quad (-\sin x)\, f(x) + (\cos x)\, f'(x) = \frac{\cos x}{\sin x}$$

$$\Rightarrow \quad f'(x) = \frac{1}{\sin x} + \frac{(\sin x)\ln(\cos x)}{\cos^2 x}$$

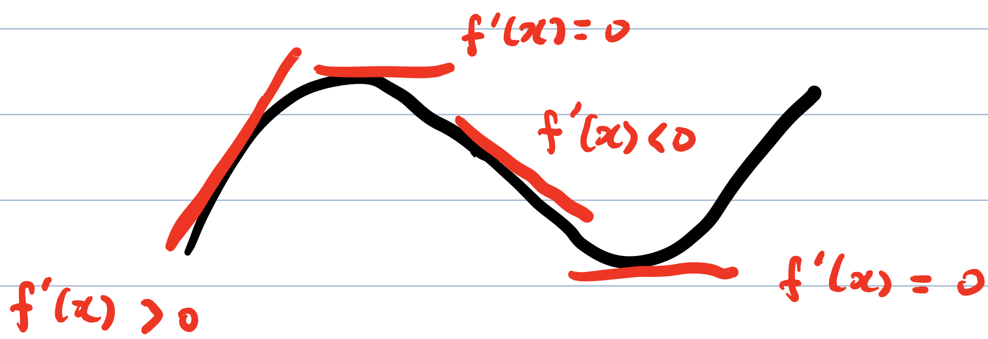
Thm (Mean value thm) $f: [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Prop $f : \mathbb{R} \to \mathbb{R}$ differentiable

- $f'(x) > 0 \Rightarrow f$ is increasing at x
- $f'(x) < 0 \Rightarrow f$ is decreasing at x
- extreme values (local min / max) occurs at a point x s.t. $f'(x) = 0$



Thm (l'Hopital's Rule) $f, g : \mathbb{R} \to \mathbb{R}$ diffb w/ $g'(x) \neq 0$ near x_0. Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \begin{cases} 0 \\ \pm\infty \end{cases}$

Then if $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ exists, we have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

e.g. • $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\cos x}{1} = 1$

$\lim\limits_{x \to x_0} f(x) = f(\lim\limits_{x \to x_0} x)$

Continuity

• $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = C$

$$\ln C = \ln\left(\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x\right) = \lim_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^x$$

$$= \lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x}$$

$$= \lim_{x \to \infty} \frac{-1/x^2}{-1/x^2\left(1 + \frac{1}{x}\right)} = 1$$

$\therefore C = e$