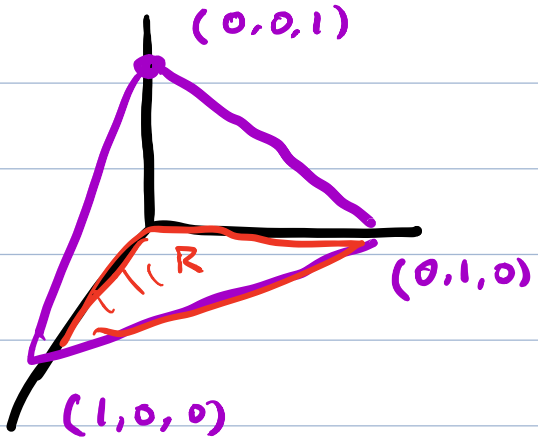
triple integrals

$\iiint_D f \, dx\,dy\,dz$

e.g Find the volume of a region bounded by $x+y+z=1$ in the first octant.



$z = 1-x-y$

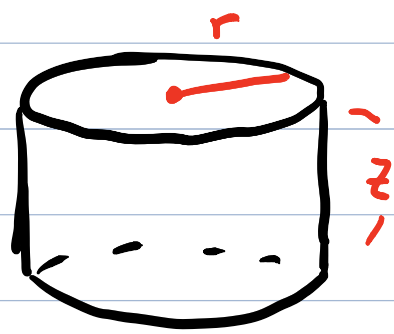
$y = 1-x$] R

$0 \le x \le 1$

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, dz\,dy\,dx$$

$$= 1/6$$

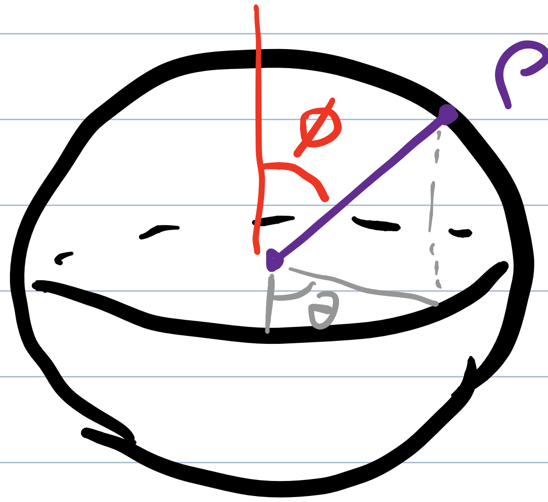
cylindrical coord



$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$dx\,dy\,dz = r\,dz\,dr\,d\theta.$

Spherical coord



(ρ, ϕ, θ)

$0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi$

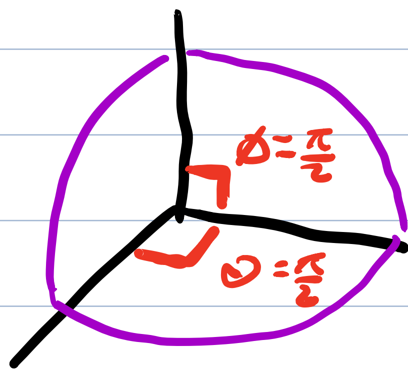
$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

e.g. Find the volume of region enclosed by $x^2+y^2+z^2=1$ in the first octant



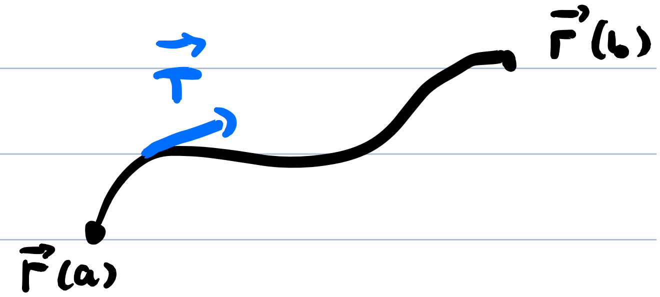
$0 \le \rho \le 1 \quad 0 \le \phi \le \frac{\pi}{2} \quad 0 \le \theta \le \frac{\pi}{2}$

$V = \iiint_D 1 \, dV$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \frac{\pi}{6}$

line integrals



$f: \mathbb{R}^3 \to \mathbb{R}$ function

$\int_C f\, dr = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\, dt$

$C: \vec{r}(t) = (x(t), y(t), z(t))$

$a \le t \le b$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

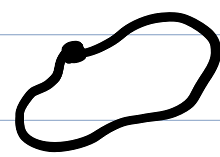
$\vec{F} = (M, N, P)$

$\int_C \vec{F} \cdot dr = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$\left(= \int_C M\,dx + N\,dy + P\,dz \right)$

* If C is a closed curve ($\vec{r}(a) = \vec{r}(b)$), we write

$\oint_C f\, dr \quad \oint \vec{F} \cdot dr$



Physical interpretation :

Work done by force $\vec{F}$ along C

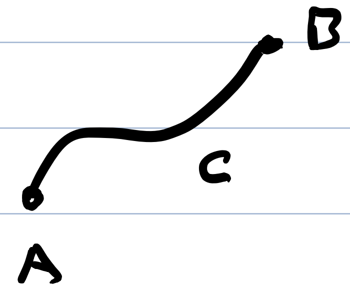
Conservative vector fields

$\vec{F} = \nabla f$ for some $f: \mathbb{R}^3 \to \mathbb{R}$.

Thm (Fund thm of multivariable Calculus)

If $\vec{F}$ is a conservative vector field ($\vec{F} = \nabla f$) and C : a path from A to B

$$\int_C \vec{F} \cdot dr = f(B) - f(A)$$



→ If C is closed $\int_C \vec{F} \cdot dr = 0$

If C_1 and C_2 have the same endpoints

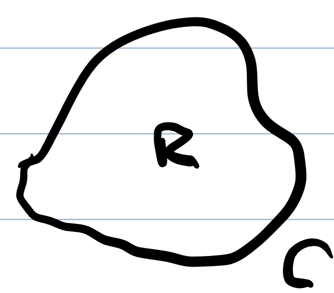
$$\int_{C_1} \vec{F} \cdot dr = \int_{C_2} \vec{F} \cdot dr$$

Green thm (line integrals ⟷ double integrals)

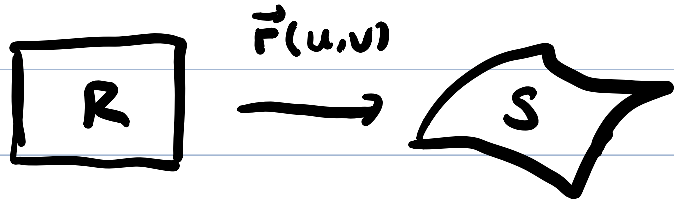
Thm $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ vector field

C : simple closed curve enclosing a region R

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$



Surface integrals



$S: \vec{F}(u,v) = (f(u,v), g(u,v), h(u,v))$

$$\iint_S f \, d\sigma = \iint_R f \, |\vec{r_u} \times \vec{r_v}| \, du \, dv$$

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_R \vec{F} \cdot (\vec{r_u} \times \vec{r_v}) \, du \, dv.$$

↳ normal vector of the surface.

Thm (Stoke's thm)

$$\iint_S (\nabla \times \vec{F}) \cdot d\sigma = \int_C \vec{F} \, dr$$

Thm (divergence thm)

$$\iint_S \vec{F} \cdot n \, d\sigma = \iiint_D \nabla \cdot \vec{F} \, dV$$