Exotic symplectic caps

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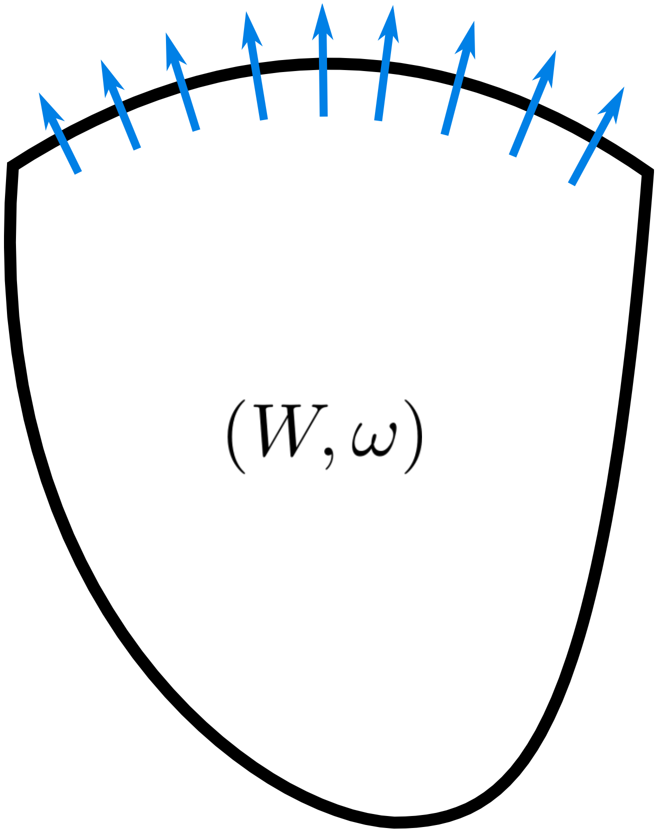
4-manifolds with boundary

- Which 3-manifold bounds a 4-manifold with exotic smooth structures?
- e.g. S^3, $\Sigma(2,3,6n-1)$, S^1-bundles over surfaces, etc.

Symplectic manifolds with boundary

- Symplectic fillings
- Weak symplectic fillings
- Symplectic caps

Symplectic fillings



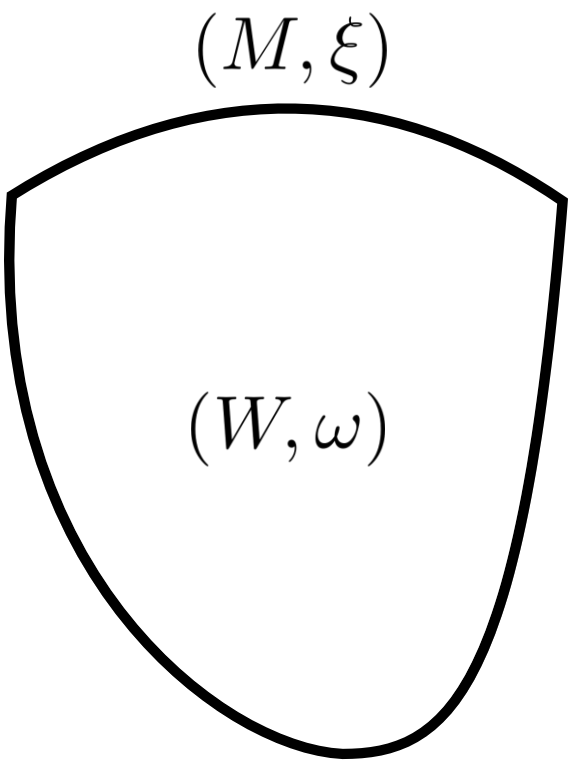
There is a vector field *v* near boundary pointing outward that dilates ω

$$\mathcal{L}_v \omega = \omega$$

e.g. $(B^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$

$$v = \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right)$$

Weak symplectic fillings

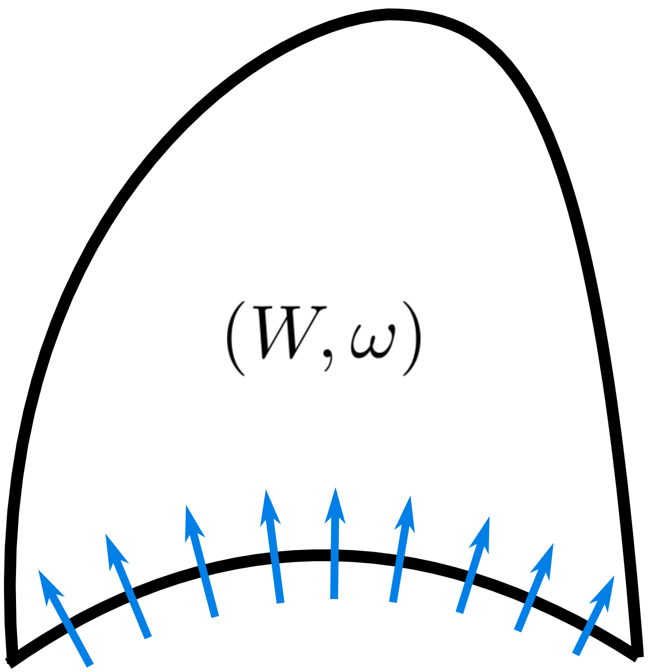


If M admits a contact structure ξ satisfying

$$\omega|_{\xi} > 0$$

(Eliashberg, Gromov) (M,ξ) is tight.

Symplectic caps



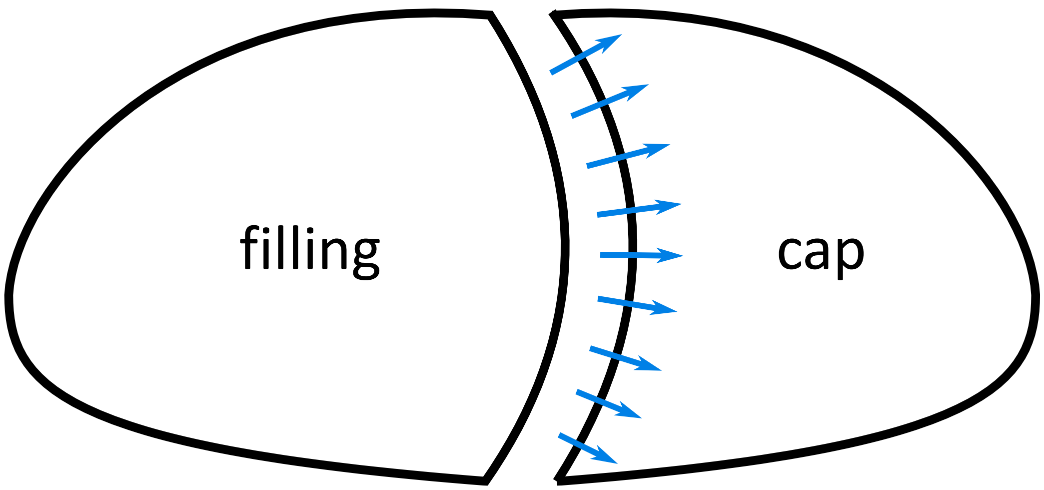
There is a vector field *v* near boundary pointing inward that dilates ω

$$\mathcal{L}_v \omega = \omega$$

(Etnyre-Honda) Every contact 3-manifold admits infinitely many symplectic caps

Gluing theorem

Theorem (Eliashberg, Etnyre) Every weak symplectic filling can be embedded into a closed symplectic 4-manifold.



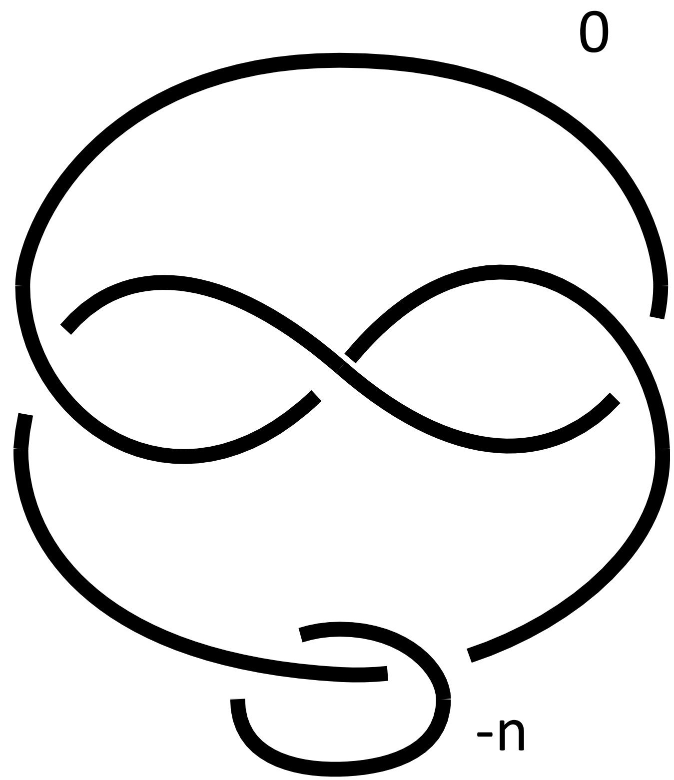
Exotic 4-manifolds

- Strategy: construct a family of homeomorphic manifolds and distinguish them by invariants.
- Invariants: Seiberg-Witten, Heegaard Floer, Monopole Floer, relative genus function, etc.
- Constructions: Log transformation, rational blow-down, Fintushel-Stern surgery, Luttinger surgery, etc.

Fintushel-Stern surgery

- Cusp neighborhood.
  - Neighborhood of a singular fiber in an elliptic fibration.

- Remove an embedded torus in a cusp neighborhood.
- Glue $(S^3 \setminus K) \times S^1$.
- Keeps the intersection form



Gompf nucleus

Fintushel-Stern surgery

Theorem (Fintushel-Stern) Suppose X is a smooth 4-manifold with $b_2^+ > 1$ containing a cusp neighborhood. If X_K is the result of the knot surgery, then

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t_{[T]}^2).$$

4-manifolds with boundary

- Which 3-manifold bounds a 4-manifold with exotic smooth structures?

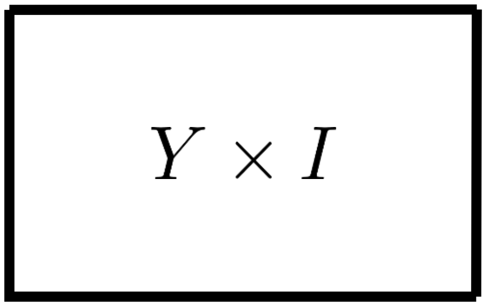
Exotic symplectic caps

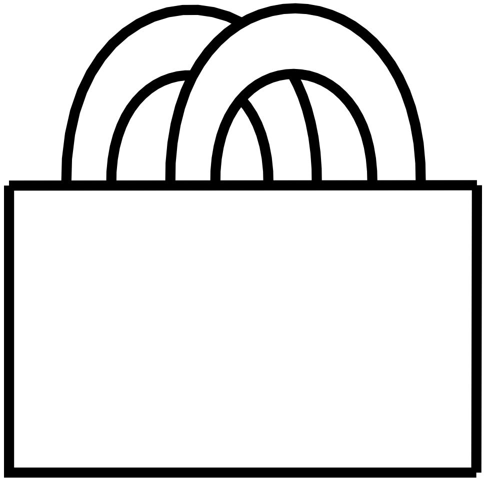
Theorem (E-M-M). For any weakly fillable contact 3-manifolds, there are infinitely many simply-connected exotic symplectic caps.

- Yasui proved the same argument using a relative genus function.

Exotic symplectic caps

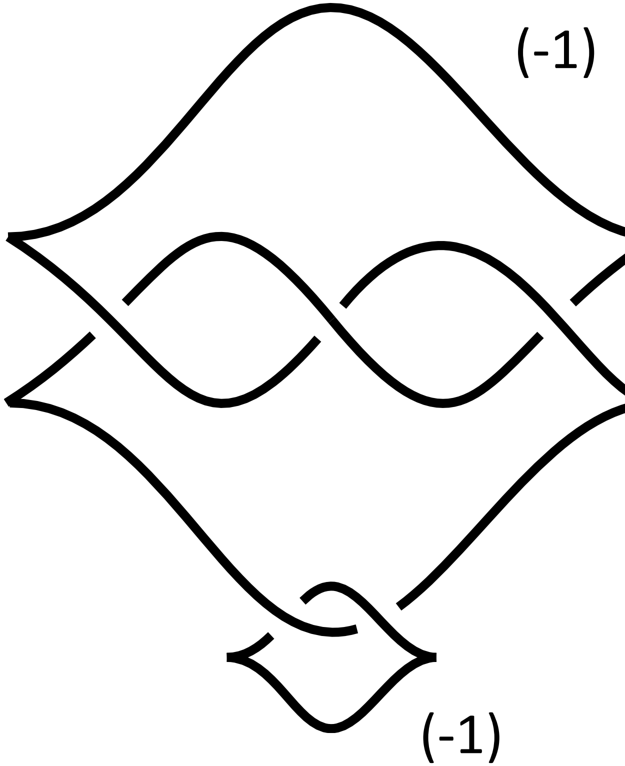
- Construct a Stein cobordism $Y \times I$
- Attach Weinstein 2-handles and kill the fundamental group





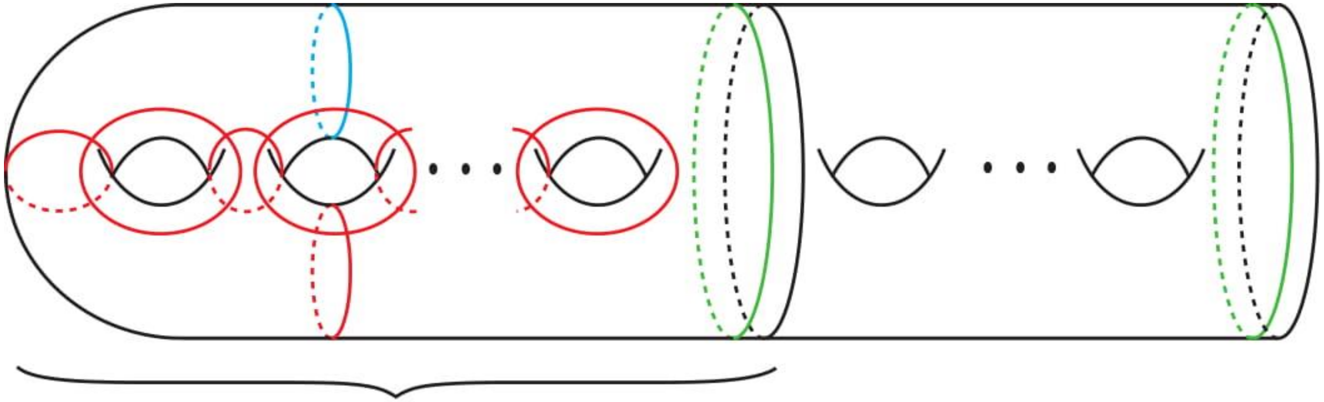
Exotic symplectic caps

- Embed a Gompf nucleus



Exotic symplectic caps

- Attach more Weinstein 2-handles and obtain S^1-bundle over a surface.
- Cap it off by D^2-bundle over a surface (with a positive Euler number).



Exotic symplectic caps

- (Eliashberg, Etnyre) Embed the cap into a closed symplectic 4-manifold.

- Perform the Fintushel-Stern surgery using fibered knots.

- Theorem (Boyer) There are finitely many homeomorphism types of compact simply-connected 4-manifolds with boundary Y and intersection pairing $(\mathbb{Z}^n, L)$.

Exotic symplectic caps

- Obtain an infinite family of homeomorphic 4-manifolds with boundary.

- Theorem (Taubes) If X is a closed symplectic 4-manifold, then $\mathcal{SW}(X) \neq 0$

- Taubes + Fintushel-Stern + more: different smooth structures rel boundary.

Remark

- Using non-monic polynomial, there are infinitely many smooth structures which does not admit a symplectic structure.

- Absolute exotic smooth structures by using a relative genus function.

Further direction

- Large b_2. Can make it small?
- Every closed orientable 3-manifold admits exotic fillings?
- Which 3-manifolds can be embedded into Stein domains or closed symplectic manifolds?
- Every prime manifold is weakly fillable one of its orientation?
- Which 3-manifolds bound a 4-manifold with non-vanishing mixed invariant in HF?

Thank you!