Problem 1.1. We first claim that if $A \subset B$, then $\mu(A \setminus B) = \mu(A) - \mu(B)$. It is clear from the equality

$$\mu(A) = \mu((A \setminus B) \cup B) = \mu(A \setminus B) + \mu(B).$$

Now we have

$$\begin{aligned} \mu(\bigcup_{i=1} A_i) &= \sum_{i=1} \mu(A_{i+1} \setminus A_i) \\ &= \sum_{i=1} (\mu(A_{i+1}) - \mu(A_i)) \\ &= \lim_{i\to\infty} \mu(A_i). \quad \square \end{aligned}$$

Problem 1.2. We will verify the 3 conditions.

- $\phi, \Omega \in S$: trivial.
- $A, B \in S \Rightarrow A \cap B \in S$:

$$\bigcup_{i=1}^{n} A_i \cap \bigcup_{j=1}^{m} B_j = \bigcup_{i,j=1}^{n,m} (A_i \cap B_j).$$

- $A \in S \Rightarrow A^c \in S$: trivial.
  Also any field containing S clearly has the property that any finite union of elements in S should be in the field. $\square$

Problem 1.3. Let $n \geq 1$, Show that the following set is a semi-field.

$$S := \{\prod_{i=1}^{n} (a_i, b_i] : -\infty \leq a_1, ..., a_n, b_1, ..., b_n \leq \infty\}.$$

Solution. We will verify the 3 conditions.

- $\phi, \Omega \in S$: trivial.
- $A, B \in S \Rightarrow A \cap B \in S$: trivial.
- $A \in S \Rightarrow A^c$ is a finite union of sets in S: $(a_i, b_i]^c = (-\infty, a_i] \cup (b_i, \infty]$. $\square$

Problem 2.1. Let μ_1 be the counting measure on $\mathbb{R}$ and $\mu_2(A) = \infty$ except for $A = \phi$.
Since

$$\bigcap_{n=1}^{\infty} (a - 1/n, a] = \{a\}$$

is in the Borel σ-field, the two measures do not coincide.
The proof of **Corollary 2.5.** doesn't work since $\mathcal{L}$ is not a λ-system (consider a complement of $R \setminus \{p\}$). $\square$

Problem 2.2. Let $\mathcal{C} = \{\{1,2\},\{2,3\}\}$. Clearly $\sigma(\mathcal{C}) = 2^{\{1,2,3,4\}}$. Notice that $\mathcal{C}$ is not a π-system. Now define μ_1, μ_2 as follows.

$$\mu_1(\{i\}) = 1/4,$$
$$\mu_2(\{1\}) = \mu_2(\{3\}) = 1/2,\ \mu_2(\{2\}) = \mu_2(\{4\}) = 0$$

Problem 2.3. $\mathcal{L} = \{\phi, [-1,0], [0,1], (-\infty,-1) \cup (0,\infty), (-\infty,0) \cup (1,\infty), \mathbb{R}\}$ is a λ-system. However, it's not a σ-field. (since $[-1,0]$ and $[0,1]$ intersect)

Problem 5.1. Easy to check from the definition.