Problem 1. Choose a random variable X of which probability density function is

$$\rho = \begin{cases} \int_1^\infty (x\log(1+x))^{-2}\,dx/(x\log(1+x))^2 & x > 1, \\ 0 & x < 1. \end{cases}$$

Now $\mathbb{E}(X) = \int_1^\infty t\,\rho(t)\,dt < \infty$. However, $\mathbb{E}(X\log(1+X)) = \int_1^\infty t\log(1+t)\rho(t)\,dt = \infty$. $\square$

Problem 2. The probability density function is

$$\rho = \begin{cases} 1/3 & x \in [2,4] \cup [8,9], \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\mathbb{E}(X) = \int t\,\rho\,dt = 29/6$, $\mathbb{E}(X^2) = \int t^2\,\rho\,dt = 91/3$.
Thus $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 153/3 = 51$. $\square$

Problem 3. By continuity, $\rho_X(t) \geq \epsilon > 0$ for $|t| < \delta$. Thus

$$\mathbb{E}(\frac{1}{|X|}) = \int_{-\infty}^{\infty} \frac{\rho_X(t)}{|t|}\,dt \geq \int_{-\delta}^{\delta} \frac{\epsilon}{|t|}\,dt = \infty. \quad \square$$

Problem 4. Suppose X is not ∞ a.e. Then there is $c > 0$ such that

$$\mathbb{P}\{X < c\} > 0.$$

Suppose $A := \{X < c\}$. We claim that $A \subseteq \bigcap\{\xi_n < c\}$ for all but finitely many n. Suppose not. Then there is $x \in A$ such that for infinite subset of $\mathbb{N}$, $x \notin \{\xi_n < c\}$, which implies that $\limsup \xi_n(x) > c$. Therefore, $\mathbb{P}\{\bigcap\{\xi_n < c\}\} > 0$ for all but finitely many n. Since ξ_i are independent, $\mathbb{P}\{\bigcap\{\xi_n < c\}\} = \prod \mathbb{P}\{\xi_n < c\}$. Thus $\lim \mathbb{P}\{\xi_n < c\} = 1$. Thus $X < c$ almost everywhere.
Now suppose $\mathbb{P}\{X < c'\} > 0$ for some $c' < c$. By the same argument above, we have $X < c'$ almost everywhere, which is a contradiction. Thus $X = c$ almost everywhere.

Problem 5. $X_n := \sum_{i=1}^n \xi_i$. Then $\{X_n \leq \frac{\epsilon^2 n}{100}\} \subseteq \{|\{i \leq n \mid \xi_i \geq \epsilon\}| \leq \frac{\epsilon n}{100}\}$

Problem 6. X is uniformly distributed on the unit circle. Thus we have

$$F_X(a.b) = (\text{length of arc contatined in (x,y)} \ \leq \ \text{(a,b)})/2\pi. \quad \square$$

Problem 7. The probability density functions are

$$\rho_1 = \begin{cases} 1/2 & t \in [1,3] \\ 0 \quad \text{otherwise} \end{cases}$$

$$\rho_2 = \begin{cases} 1/2 & t \in [1,2] \cup [4,5] \\ 0 \quad \text{otherwise} \end{cases}$$

Let $Z := X_1 + X_2$. Since X_1 and X_2 are independent, $\mathbb{P}\{Z = z\} = \sum_t \mathbb{P}\{X_1 = t \;\&\; X_2 = z - t\} = \sum_t \mathbb{P}\{X_1 = t\}\mathbb{P}\{X_2 = z - t\}$. Thus,

$$\begin{aligned} \rho_Z(z) &= \int_{-\infty}^{\infty} \rho_1(t)\rho_2(z-t)\,dt \\ &= \int_1^3 \frac{1}{2}\rho_2(z-t)\,dt \end{aligned}$$

Now ρ_2 is nonzero if $1 \le z - t \le 2$ or $4 \le z - t \le 5$. Thus we have

$$\rho_Z(z) = \begin{cases} 0 & z \le 2 \text{ or } z \ge 8 \\ \int_1^{z-1} \frac{1}{4}\,dt = \frac{z-2}{4} & 2 \le z \le 3 \\ \int_1^2 \frac{1}{4}\,dt = \frac{1}{4} & 3 \le z \le 4 \\ \int_{z-3}^2 \frac{1}{4}\,dt = \frac{5-z}{4} & 4 \le z \le 5 \\ \int_4^{z-1} \frac{1}{4}\,dt = \frac{z-5}{z} & 5 \le z \le 6 \\ \int_4^5 \frac{1}{4}\,dt = \frac{1}{4} & 6 \le z \le 7 \\ \int_{z-3}^5 \frac{1}{4}\,dt = \frac{8-z}{4} & 7 \le z \le 8 \quad \square \end{cases}$$