Solutions

1. **Determine the phasor of the sinusoidal signal $f(t) = 10\sin\left(\frac{\pi}{4}t - \frac{3\pi}{4}\right)$? Express it both in polar and rectangular.**

   To solve this question, let's first transform the sine function to a cosine function. To do so, we can subtract $\frac{\pi}{2}$, or 90°, from the phase:

   $$f(t) = 10\sin\left(\frac{\pi}{4}t - \frac{3\pi}{4}\right) = 10\cos\left(\frac{\pi}{4}t - \frac{3\pi}{4} - \frac{\pi}{2}\right) = 10\cos\left(\frac{\pi}{4}t - \frac{5\pi}{4}\right)$$

   From this form, we get that the polar form of this function is $10e^{-\frac{5\pi}{4}j}$. Expressing this in rectangular form, we have:

   $$10\left(\frac{-\sqrt{2}}{2}\right) + 10j\left(\frac{\sqrt{2}}{2}\right) = -5\sqrt{2} + 5\sqrt{2}j$$

2. **Determine the phasor of the sinusoidal signal $f(t) = 4\cos\left(5t + \frac{\pi}{4}\right) + 12\cos\left(t + \frac{7\pi}{4}\right)$? Express it either polar or rectangular form.**

   To solve this, we must first note that the two terms have **different frequencies** ($\omega_1 = 5$ rad/s and $\omega_2 = 1$ rad/s). Phasors are frequency-specific representations; therefore, we cannot add these two terms into a single phasor representation. We must determine the phasor for each component separately.

   Component 1 ($\omega = 5$):

   $$\mathbf{F_1} = 4\angle\frac{\pi}{4} = 4e^{j\pi/4}$$

   Rectangular:

   $$4\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 2\sqrt{2} + j2\sqrt{2}$$

   Component 2 ($\omega = 1$):

   $$\mathbf{F_2} = 12\angle\frac{7\pi}{4} = 12e^{j7\pi/4}$$

   Rectangular:

   $$12\left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right) = 6\sqrt{2} - j6\sqrt{2}$$

   Thus, the signal is represented by the superposition of these two distinct phasors.

3. **Given an LTI system with a frequency response $H(\omega) = \frac{2}{12+j\omega}$ and an input of $f(t) = 7 + \sin\left(5t + \frac{\pi}{2}\right) + \cos\left(12t + \frac{\pi}{4}\right)$, determine the output $y(t)$.**

   To solve this problem, we use the property of superposition and determine the output for each frequency component separately.

Part 1 (DC, $\omega = 0$): The input is 7 (which is $7\cos(0t)$).

$$H(0) = \frac{2}{12 + j(0)} = \frac{1}{6}$$

The output is the input magnitude times the DC gain:

$$y_1(t) = 7 \cdot H(0) = 7 \cdot \frac{1}{6} = \frac{7}{6}$$

Part 2 ($\omega = 5$): The input is $\sin\left(5t + \frac{\pi}{2}\right) = \cos(5t)$. Evaluate frequency response at $\omega = 5$:

$$H(5) = \frac{2}{12 + 5j}$$

Magnitude:

$$|H(5)| = \frac{2}{\sqrt{12^2 + 5^2}} = \frac{2}{\sqrt{144 + 25}} = \frac{2}{13}$$

Phase:

$$\angle H(5) = -\arctan\left(\frac{5}{12}\right)$$

Output component:

$$y_2(t) = \frac{2}{13}\cos\left(5t - \arctan\left(\frac{5}{12}\right)\right)$$

Part 3 ($\omega = 12$): The input is $\cos\left(12t + \frac{\pi}{4}\right)$. Evaluate frequency response at $\omega = 12$:

$$H(12) = \frac{2}{12 + 12j} = \frac{1}{6(1 + j)}$$

Magnitude:

$$|H(12)| = \frac{1}{6\sqrt{1^2 + 1^2}} = \frac{1}{6\sqrt{2}} = \frac{\sqrt{2}}{12}$$

Phase of H(12):

$$\angle H(12) = -\arctan\left(\frac{12}{12}\right) = -\frac{\pi}{4}$$

Total phase = Input phase + System phase = $\frac{\pi}{4} - \frac{\pi}{4} = 0$. Output component:

$$y_3(t) = \frac{\sqrt{2}}{12}\cos(12t)$$

Total Output: Combining all parts:

$$y(t) = \frac{7}{6} + \frac{2}{13}\cos\left(5t - \arctan\left(\frac{5}{12}\right)\right) + \frac{\sqrt{2}}{12}\cos(12t)$$

4. **Consider the RC circuit with $R = 2\ \Omega$, $C = 1$ F, and input $f(t) = 4e^{-t}u(t)$ with $v(0^-) = 1$.**

To solve this problem, we first write KCL at the capacitor node. The current through the resistor plus the current into the capacitor must sum to zero:

$$\frac{v(t) - f(t)}{R} + C\frac{dv(t)}{dt} = 0$$

Plugging in $R = 2$ and $C = 1$:

$$\frac{v(t) - 4e^{-t}}{2} + \frac{dv(t)}{dt} = 0$$

Multiplying by 2 gives the differential equation:

$$2\frac{dv(t)}{dt} + v(t) = 4e^{-t}$$

Solving this ODE, we rewrite it as:

$$\frac{dv(t)}{dt} + \frac{1}{2}v(t) = 2e^{-t}$$

Using the integrating factor $e^{t/2}$, we get:

$$\frac{d}{dt}\left(v(t)e^{t/2}\right) = 2e^{-t/2}$$

Integrating both sides:

$$v(t)e^{t/2} = -4e^{-t/2} + K$$

Solving for $v(t)$:

$$v(t) = -4e^{-t} + Ke^{-t/2}$$

Applying the initial condition $v(0) = 1$:

$$1 = -4 + K \quad \Rightarrow \quad K = 5$$

Thus, the full solution is:

$$v(t) = -4e^{-t} + 5e^{-t/2}$$