

ECE 313 Midterm 2 Solutions

HKN Members

Answer to Q1: Pramod's Great Escape

Part A: The Rocket

We have 500 engines, numbered 0 through 499. Pramod will not successfully escape if at least one engine fails. Let F_m be the event that engine m fails, and let M be the event that Pramod will not successfully escape. Then $P\{M\} = P\{F_0 \cup F_1 \cup \dots \cup F_{499}\}$, and $P\{F_m\} = (\frac{1}{10^3})(\frac{9}{10})^m$.

Because the probability of any given engine failing is small ($\leq \frac{1}{10^3}$), it is appropriate use the union bound to estimate $P\{M\} \approx P\{F_0\} + P\{F_1\} + \dots + P\{F_{499}\}$. This gives:

$$P\{M\} \approx (\frac{1}{10^3})(\frac{9}{10})^0 + (\frac{1}{10^3})(\frac{9}{10})^1 + \dots + (\frac{1}{10^3})(\frac{9}{10})^{499}$$

$$P\{M\} \approx \sum_{m=0}^{499} (\frac{1}{10^3})(\frac{9}{10})^m$$

$$P\{M\} \approx \sum_{m=0}^{\infty} (\frac{1}{10^3})(\frac{9}{10})^m - \sum_{m=500}^{\infty} (\frac{1}{10^3})(\frac{9}{10})^m$$

Because $(\frac{9}{10})^{500} \approx 10^{-23}$, the $\sum_{m=500}^{\infty} (\frac{1}{10^3})(\frac{9}{10})^m$ term is insignificant in this calculation.

$$P\{M\} \approx \sum_{m=0}^{\infty} (\frac{1}{10^3})(\frac{9}{10})^m$$

Because $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$, we have:

$$P\{M\} \approx \frac{\frac{1}{10^3}}{1 - \frac{9}{10}}$$

$$P\{M\} \approx \frac{1}{100}$$

So the likelihood that Pramod will not successfully escape is approximately **0.01**.

For the second part of this problem, we need to estimate the order of magnitude of the error of this estimate.

We wanted to find $P\{F_0 \cup F_1 \cup \dots \cup F_{499}\}$, but we actually calculated $P\{F_0\} + P\{F_1\} + \dots + P\{F_{499}\}$.

Let E we the error of our calculation. Because the probability of more than two engines failing is orders of magnitude less than the probability of two engines failing, we can approximate this error as $E = P\{F_0F_1\} + P\{F_0F_2\} + \dots + P\{F_0F_{499}\} + P\{F_1F_2\} + P\{F_1F_3\} + \dots + P\{F_1F_{499}\} + \dots + P\{F_{498}F_{499}\}$. This gives $E = \sum_{m=1}^{499} P\{F_0F_m\} + \sum_{m=2}^{499} P\{F_1F_m\} + \dots + \sum_{m=499}^{499} P\{F_{498}F_m\}$.

$$\sum_{m=1}^{499} P\{F_0F_m\} = \sum_{m=1}^{499} (\frac{1}{10^3})(\frac{9}{10})^0(\frac{1}{10^3})(\frac{9}{10})^m = \sum_{m=1}^{499} (\frac{1}{10^6})(\frac{9}{10})^m$$

Since the $m \geq 500$ terms are negligible, we can approximate this as an infinite sum:

$$\sum_{m=1}^{499} P\{F_0F_m\} = \sum_{m=1}^{\infty} (\frac{1}{10^6})(\frac{9}{10})^m$$

$$\sum_{m=1}^{499} P\{F_0F_m\} = \sum_{m=0}^{\infty} (\frac{1}{10^6})(\frac{9}{10})^m - (\frac{1}{10^6})(\frac{9}{10})^0$$

$$\sum_{m=1}^{499} P\{F_0F_m\} = \frac{\frac{1}{10^6}}{1 - \frac{9}{10}} - (\frac{1}{10^6})$$

$$\sum_{m=1}^{499} P\{F_0F_m\} = 9 * 10^{-6}$$

Using the same method, we can find $\sum_{m=2}^{499} P\{F_1F_m\} = 7.29 * 10^{-6}$

The later summations in the equation for E will decrease towards zero, and the sum of first two summations is on the order of magnitude of 10^{-5} . Therefore, the order of magnitude of the error is 10^{-5} .

Part B: The Asteroids

The question has a fixed-rate continuous time event, i.e a Poisson process. The question being asked involves the k th instance of this event, which can be modelled by an Erlang distribution, so using its CDF:

$$P = 1 - \sum_{n=0}^{k-1} \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Inserting $\lambda = 1$, $k = 500$, and $t = 500$

$$P = 1 - \sum_{n=0}^{499} \frac{e^{-500} (500)^n}{n!}$$

This is a rather unfortunate sum that we would rather not solve. However, recall that the question does not ask for precision past 1%. We can therefore re-frame the question like this:

Given a random variable X , which is defined as the sum of 500 exponential distributions with rate $\lambda = 1$, what is the likelihood that $X > 500$?

Which is a trivial CLT problem because in this case $E[X] = 500$, meaning that $P \approx 0.5$. Note that this only holds because the mean and median are equal in Gaussian distributions, which is not the case in the original Erlang distribution.

Answer to Q2: Captain's Treasure

Part A: The Shipwreck

Notice how since Captain Chatterjee has no idea where the treasure is, it could be anywhere in that 30km zone, making this a uniform distribution. Which means it does not matter where he pays the surveyor to work, and the probability of success will always be $p = \frac{1}{3}$

Part B: The Mines

Since the mines appear at a fixed rate in distance, we can model this situation as an exponential distribution over distance. As such, the rate $\lambda = 3$ mines per km. Using $P(X \leq x) = 1 - e^{-\lambda x}$, the probability that the next mine is between 100m and 200m is the following: $P(0.1 \leq X \leq 0.2) = P(X \leq 0.2) - P(X \leq 0.1) = (1 - e^{-0.6}) - (1 - e^{-0.3}) = 0.192$.

Answer to Q3: Not Fine Fines

Part A: The Payment

This can be modeled using a binomial distribution, with $n = 2000000$ and $p = 0.5$. For large n , we can approximate it by a normal distribution due to the Central Limit Theorem. Its mean would be $np = 0.5 * 2000000 = 1000000$ and its standard deviation would be $\sigma = \sqrt{n * p * (1 - p)} = \sqrt{2000000 * 0.5 * 0.5} = \sqrt{500000} = 707.1$. For calculating the probability of less than 500000 lands heads-up, we do the following: $P\left\{\frac{X-1000000}{707.1} \leq \frac{500000-1000000}{707.1}\right\} = \Phi\left(\frac{500000-1000000}{707.1}\right) = \Phi(-707.1) = Q(707.1) = 0$. $Q(707.1)$ is so small that the probability of less than 500000 heads-up would be very small, essentially 0.

Part B: The Casino

This question can be modeled as a Poisson distribution of $\lambda = 0.012$ because it has a low p of 0.001 and $n = 12$. To win a gazillion dollars in 12 hours, he needs to succeed 3 times. First, we can calculate the probability of winning less than three times, $P(X < 3)$, by calculating the sum of the likelihood of winning 0, 1, 2 times:

$$\begin{aligned}P(X = 0) &= \lambda^0 * \frac{e^{-\lambda}}{0!} = e^{-\lambda} = e^{-0.012} \approx 0.988 \\P(X = 1) &= \lambda^1 * \frac{e^{-\lambda}}{1!} = \lambda * e^{-\lambda} = 0.012 * e^{-0.012} \approx 0.0119 \\P(X = 2) &= \lambda^2 * \frac{e^{-\lambda}}{2!} = \lambda^2 * \frac{e^{-\lambda}}{2} = 0.012^2 * \frac{e^{-0.012}}{2} \approx 0.000071 \\P(X < 3) &= 0.988 + 0.0119 + 0.000071 = 0.999971\end{aligned}$$

Therefore, we can get the possibility of succeeding more than or equal to 3 times:

$$P(X \geq 3) = 1 - P(X < 3) = 0.000029$$

Part C: The Last Stand

We model it as an exponential distribution with parameter λ . To get the maximum likelihood (ML) estimate of λ , we want to maximize the likelihood function $\lambda e^{-\lambda t}$ for $t = 10$, since the town goes bankrupt in 10 days.

By taking the derivative to find the maximum, we have:

$$\frac{d}{d\lambda} (\lambda e^{-\lambda t}) = (1 - \lambda t)e^{-\lambda t} = 0,$$

which gives us

$$\lambda_{\text{ML}} = \frac{1}{t} = 0.1.$$

Answer to Q4: ML Estimators

Part A:

Most of the work of this problem is knowing how to deal with functions of random variables. Start by setting up the relation:

$$P\{X = 7\} = P\{3Y - 2 = 7\} = P\{Y = 3\}.$$

Now you just need to use the \hat{p}_{ML} equation for geometric distributions, $\hat{p}_{\text{ML}} = \frac{1}{k}$. This equation is derived in the textbook by taking the derivative of the geometric distribution with respect to p . This gives you a final answer of $\hat{p}_{\text{ML}} = \frac{1}{3}$.

Part B:

Since all values in the set have an equal probability, the pmf of x is

$$P_x(n) = \begin{cases} \frac{1}{n}, & 1, 3, 5, \dots, 2n + 1 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

The probability is minimized when n is as small as possible. This means n should be the smallest value for which the set is still valid. Since 13 is observed, it must be in the set, so the smallest possible value of n is 6.

Answer to Q5: Hypothesis Testing

Part A:

Both hypotheses can be modeled using the binomial distribution. Using the binomial distribution formula:

$$P(x) = \binom{n}{x} \cdot (p)^x \cdot (1 - p)^{(n-x)} \quad (2)$$

The ML decision table is shown below.

Hypothesis	0	1	2	3	4
H_1	0.3164	0.421875	0.2109375	0.046875	0.00390625
H_0	0.0625	0.25	0.375	0.25	0.0625

The ML decision rule is underlined in the table.

Part B:

By convention, $P_{\text{false alarm}}$ is defined by the conditional probability

$$P_{\text{false alarm}} = P(\text{declare } H_1 \text{ true} \mid H_0) \quad (3)$$

and P_{miss} is defined by the conditional probability

$$P_{\text{miss}} = P(\text{declare } H_0 \text{ true} \mid H_1) \quad (4)$$

$P_{\text{false alarm}}$ can be found by taking the sum of the non underlined elements in the H_0 row.

$$P_{\text{false alarm}} = 0.0625 + 0.25 = 0.3125.$$

Similarly, P_{miss} is the sum of the non underlined elements in the H_1 row.

$$P_{\text{miss}} = 0.2109 + 0.0469 + 0.0039 = 0.2617$$

Finally, p_e is found through

$$p_e = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}}$$

$$p_e = \frac{1}{3} * 0.3125 + \frac{2}{3} * 0.2617 = 0.2786$$