HKN ECE 210 Exam 3 Review Session

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Topics

- Fourier Transform
- Signal Energy and Bandwidth
- LTI System Response with Fourier Transform
- Modulation, AM, Coherent Demodulation
- Impulse Response and Convolution
- Sampling and Analog Reconstruction
- LTIC and BIBO Stability

What it does is that it decomposes a signal into many sinusoids & using Fourier Transform, we change our domain to frequency domain then see the contribution of each sinusiod.

Fourier Transform

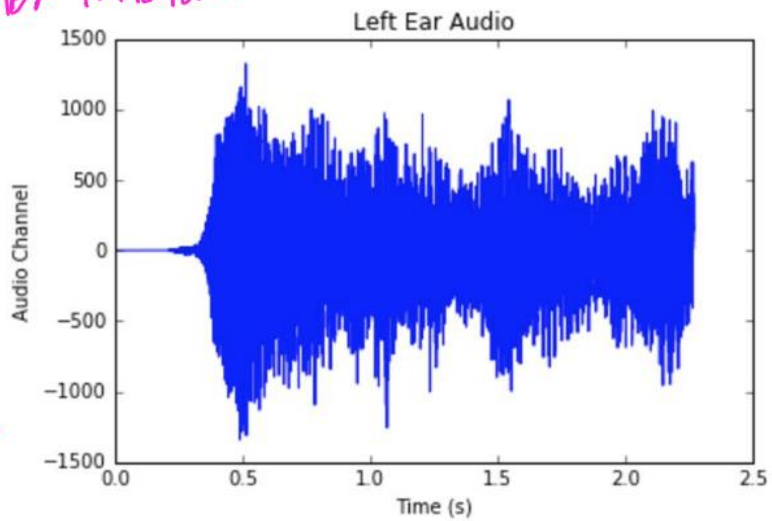
- The Fourier Transform of a signal shows the frequency content of that signal
  - In other words, we can see how much energy is contained at each frequency for that signal
  - ~~This is a big deal!~~
  - This is the biggest deal!

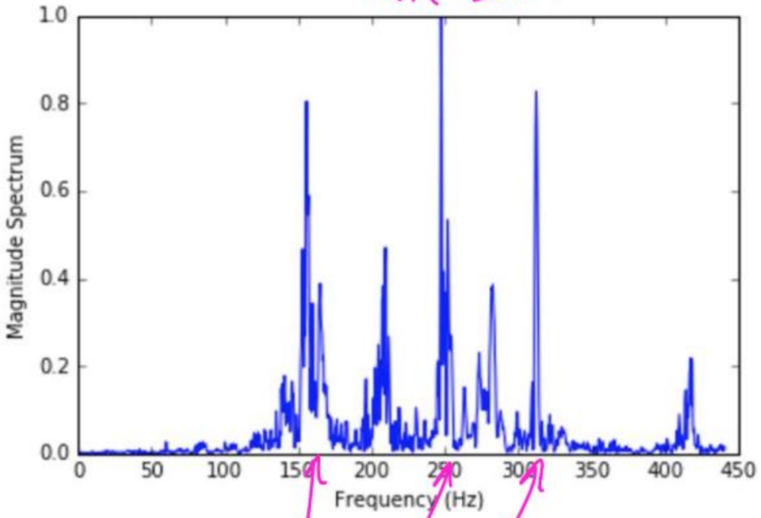
Fourier Transform pairs

- $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$ Forier Transform
- $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$

inverse fourier transform

here we have an audio file of signal.





160 250 310

highest energy content in the signal

Important Signals for Fourier Transform

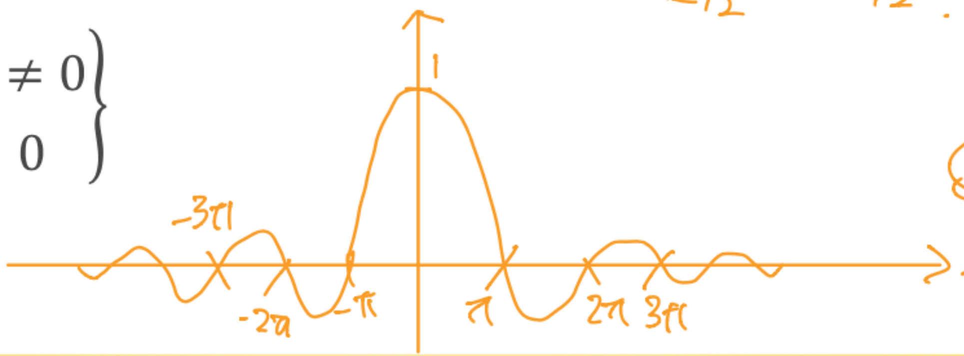
- $rect\left(\frac{t}{T}\right) = \begin{cases} 1, for\ |t| < \frac{T}{2} \\ 0, for\ |t| > \frac{T}{2} \end{cases}$

width / base of the rect

- $u(t) = \begin{cases} 0, for\ t < 0 \\ 1, for\ t > 0 \end{cases}$

- $\Delta\left(\frac{t}{T}\right) = \begin{cases} 1 + \frac{2t}{T}, for \frac{-T}{2} < t < 0 \\ 1 - \frac{2t}{T}, for\ 0 < t < \frac{T}{2} \end{cases}$

- $sinc(t) = \begin{cases} \frac{\sin(t)}{t}, for\ t \neq 0 \\ 1, for\ t = 0 \end{cases}$



AND they have corresponding Fourier Transform pairs given in formula sheet

① Peak (By using L'Hopital's rule)

② Zero crossings (when sin(t)=0)

③ Decreasing amplitude as t ↑s.

Fourier Transform Tips

- Convolution in the time domain is multiplication in the frequency domain
  - $f(t) * g(t) \overset{\mathcal{F}}{\leftrightarrow} F(\omega)G(\omega)$
- Conversely, multiplication in the time domain is convolution in the frequency domain
  - $f(t)g(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi} F(\omega) * G(\omega)$

$e^{-4}(e^{-2t+2})$

- Scaling your signal can force properties to appear; typically time delay
  - Ex: $e^{-2(t+1)}u(t-1) \rightarrow e^{-4}e^{-2(t-1)}u(t-1)$ ← all the terms that has t need to be in the same form before applying property
- The properties really do matter! Take the time to acquaint yourself with them.

① Scaling ② Hermitian ③ Shifting

- Remember that the Fourier Transform is linear, so you can express a spectrum as the sum of easier spectra
  - Ex: Staircase function
- Magnitude Spectrum is even symmetric, Phase Spectrum is odd symmetric for real valued signals

due to Hermitian Property

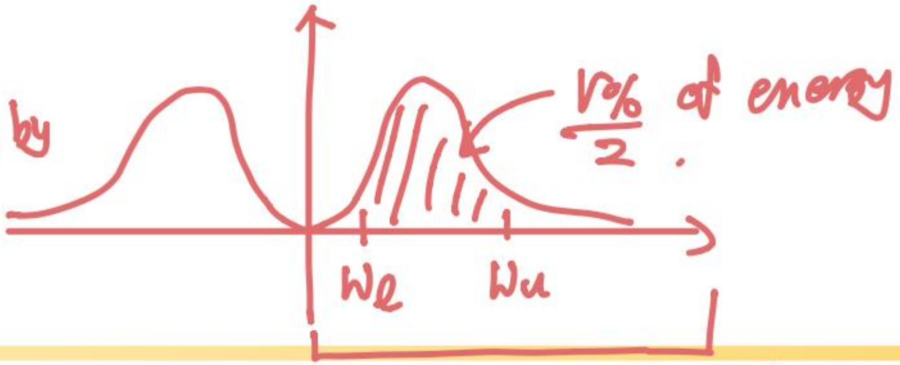
Signal Energy and Bandwidth

Parseval's Theorem

- $Energy = W = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

  time domain   frequency domain

  - Energy signals can be either low-pass or band-pass signals
    - Why not high-pass? Not band limited $\Rightarrow \infty$ energy
- Bandwidth for Low-pass Signals
  - 3dB BW → for $\omega \geq 0$.
    - $\frac{|F(\Omega)|^2}{|F(0)|^2} = \frac{1}{2}$
  - r% BW Find bandwidth so that energy contained within it is r% of the entire energy.
    - $\frac{1}{2\pi}\int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega = rW$
- Bandwidth for Band-pass signals
  - r% BW
    - $\frac{1}{2\pi}\int_{\omega_l}^{\omega_u} |F(\omega)|^2\, d\omega = \frac{rW}{2}$, $\Omega = \omega_u - \omega_l$ bandwidth given by



only look at this half

Problem 1 FA19

(a) The signal $f(t)$ has Fourier transform $F(\omega)$. Determine the inverse Fourier transform of the following three signals. Leave your answers in terms of $f(t)$.

iii. (6 pts) $F_3(\omega) = F(200)e^{-j600}\delta(\omega - 200) - F(-200)e^{j600}\delta(\omega + 200)$

a iii) When you see delta function next to another function ^ think about Sampling property

$F(200)e^{-j600}\delta(\omega-200)$ $\quad$ $F(-200)e^{j600}\delta(\omega+200)$

F(w) Sampled @ w=200 rad/s $\quad$ sampled @ w=-200

$F(\omega)e^{-j3\omega}\delta(\omega-200) - F(\omega)e^{-j3\omega}\delta(\omega+200)$

Using $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$

multiplication in frequency = convolution in time

$F(\omega)e^{-j3\omega}[\delta(\omega-200)-\delta(\omega+200)]$

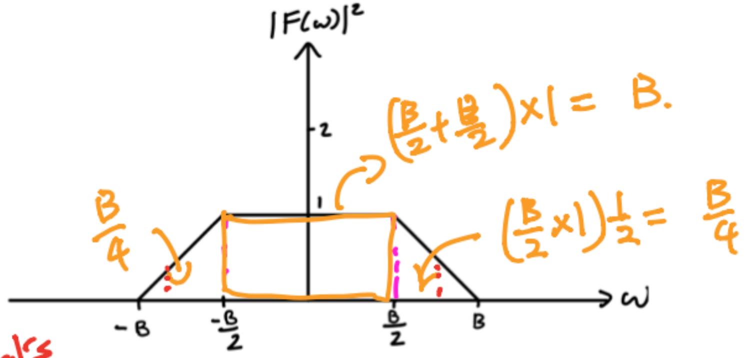
$f(t-3) * \left[\frac{e^{j200t}}{2\pi} - \frac{e^{-j200t}}{2\pi}\right] \Rightarrow f(t-3) * \frac{j\sin(200t)}{\pi}$ #

using

Euler's formula

Problem 2 FA20

(c) Given the energy spectrum below:



Use Parseval's eqn in frequency domain

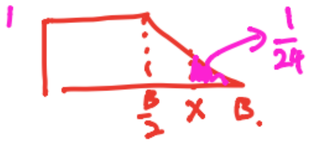
i. (4 pts) Determine the signal energy, W, in terms of B.

$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi}\left[B + 2\cdot\frac{B}{4}\right].$

$= \frac{3}{4\pi}B.$

W = $\frac{3}{4\pi}B$ J.

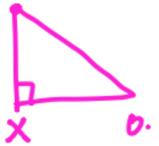
ii. (4 pts) What is the bandwidth, Ω, needed to capture 11/12 of the energy in this signal, expressed in terms of B?



Energy of small Δ = Area of $\Delta \cdot 2 \cdot \frac{1}{2\pi}$.

$\frac{1}{12}\cdot\frac{3}{4\pi}B$ = Area of $\Delta \cdot \pi$

Area of $\Delta = \frac{B}{16.} = \frac{1}{2}\left(\frac{B}{4}\times\frac{1}{2}\right)$



$y = \frac{2}{B}x + 2.$

$\frac{1}{2}(w \times h).$

$\frac{1}{2}\cdot\frac{B}{4}$

$\frac{B}{8}$, => half of that

$\frac{B}{16}$

Ω = $\frac{3B}{4}$

Quadratic eqn, find x and subtract it from B.

Be careful: x will have 2 values => chose one that is less than B

LTI System Response using Fourier Transform

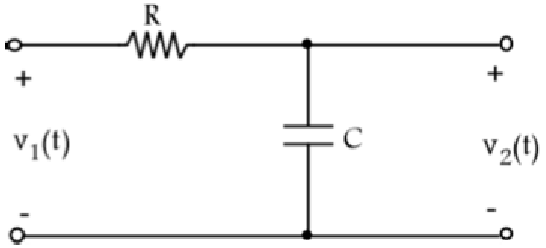
- Given the following LTI system:

$$f(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t)$$

- $Y(\omega) = F(\omega)H(\omega)$
- Computationally speaking, and in future courses, we prefer to use Fourier Transforms instead of convolution to evaluate an LTI system
- Why?
  - So much faster: $O(nlogn)\ vs.\ O(n^2)$

Problem 4 FA17

4. (25 pts) For the circuit below, $R = 10\text{K}\Omega$ and $C = 1\mu F$, determine:



(a) The differential equation relating the input x (voltage $v_1(t)$) to output y (voltage $v_2(t)$).

(b) The frequency response, $H(\omega) =$______________________________.

(d) The response to $v_1(t) = 5u(t)$V. $v_2(t) =$ ______________________________.

Modulation, AM Radio, Coherent Demodulation

- Modulation Property
  - $f(t) \leftrightarrow F(\omega), f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}[F(\omega - \omega_o) + F(\omega + \omega_o)]$
- In general, for Amplitude Modulation in communications, we modulate with cosine in order to shift our frequency spectrum into different frequency bands
- Coherent Demodulation refers to the process of modulating a signal in order to make it band-pass, then modulating it back to the original low-pass baseband before low-pass filtering in order to recover the original signal
- Envelope detection is the process of Full-wave Rectification (absolute value), then low-pass filtering in order to extract the signal

Problem 2 SP20

(b) [11 pts] Suppose $f_1(t) \leftrightarrow 4\text{rect}(\frac{\omega}{2})$, $f_2(t) \leftrightarrow 4\Delta(\frac{\omega}{2})$, $r_1(t) = f_1(t)\cos(10t)$, $r_2(t) = f_2(t)\cos((16+\frac{2\alpha}{10})t)$ and $y(t) = [r_1(t)+r_2(t)]\cos((13+\frac{\alpha}{10})t)$. Sketch the Fourier transforms of $r_1(t)$, $r_2(t)$, and $y(t)$ for $-20 < \omega < 20$. Carefully label important points (both horizontal and vertical).________ **sketches** ?

(c) [6 pts] Suppose we want to use the equations in part (b) as a simplified model to illustrate the image station problem in AM communications, where $r_1(t)$ and $r_2(t)$ represent AM signals broadcast by two different stations. Identify the local oscillator frequency ω_{LO} and the center frequency ω_{IF} of the IF band used in the model. Explain your answers.

.________ $\omega_{\mathbf{LO}}$, $\omega_{\mathbf{IF}} = ?$

2 Sp '20

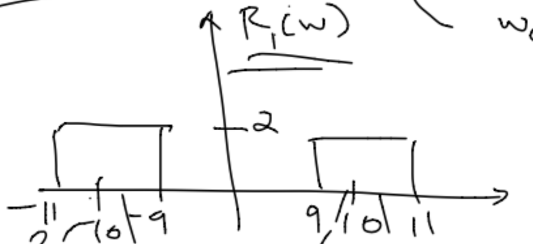
- $f_1(t) \longleftrightarrow 4\,\text{rect}\left(\frac{w}{2}\right)$
- $f_2(t) \longleftrightarrow 4\,\Delta\left(\frac{w}{2}\right)$
- $r_1(t) = f_1(t)\cos(10t)$
- $r_2(t) = f_2(t)\cos\left(\left(16+\frac{2\alpha}{10}\right)t\right)$
- $y(t) = \boxed{r_1(t)+r_2(t)}\cos\left(\left(13+\frac{\alpha}{10}\right)t\right)$

Using $\alpha=6$, $-20 < w < 20$

$$\boxed{\text{REMEMBER: } f(t)\cos(w_0 t) \Leftrightarrow \frac{1}{2}\left[F(w-w_0)+F(w+w_0)\right]}$$

$w_0 = 10$ rad/s

i)

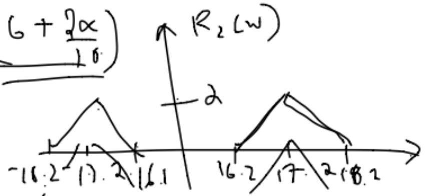


$\frac{1}{2}(4)\left[\text{rect}\left(\frac{w}{2}-10\right)+\text{rect}\left(\frac{w}{2}+10\right)\right]$

ii) $w_0 = \left(16+\frac{2\alpha}{10}\right)$

$\alpha = 6$

$\boxed{17.2}$



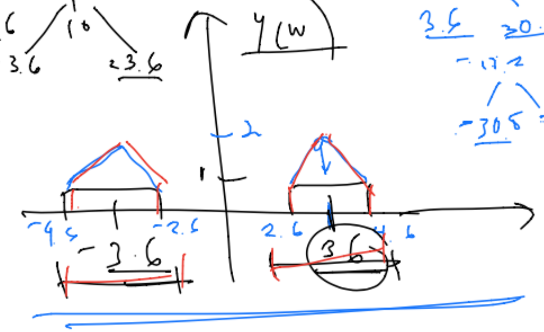
$\frac{1}{2}(4)\left[\Delta\left(\frac{w}{2}-w_0\right)+\Delta\left(\frac{w}{2}+w_0\right)\right]$

iii) $w_0 = 13+\frac{\alpha}{10} \rightarrow 13.6$

-1
10

3.6 10
-23.6 -3.6 23.6

17.2
3.6 30.8
-17.2
-30.8 -3.6



c) w_{LO}, w_{IF}

$w_{LO} = 13+\frac{\alpha}{10} = \boxed{13.6 \text{ rad/s}}$

$+w_{IF}$ 2 signals

carrier image station

>12

center frequency

$w_{IF} = 3+\frac{\alpha}{10}$

$\boxed{3.6 \text{ rad/s}}$

Impulse Response and Convolution

- Convolution
  - $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$
  - We "flip and shift" one signal and evaluate the integral of the product of the two signals at any value of t (our delay for the shift)
- Representing LTI Systems
  - $y(t) = x(t) * h(t), where\ h(t)\ is\ the\ \boldsymbol{impulse\ response}\ of\ the\ system$
- Impulse Response is the system output to a $\delta(t)$ input
- Graphical convolution helps to visualize the process of flipping and shifting

Helpful Properties for Convolution

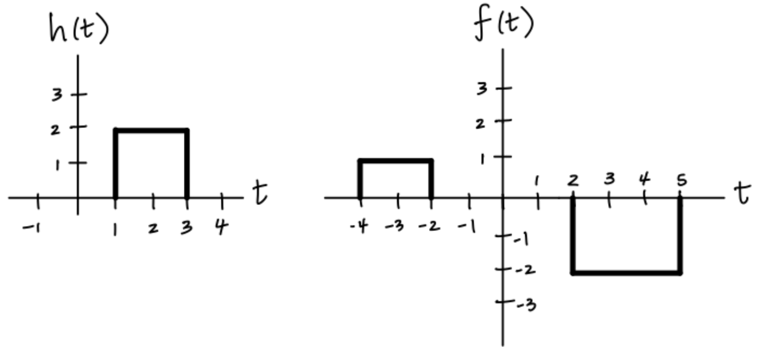
- Derivative
  - $h(t) * f(t) = y(t) \rightarrow \frac{d}{dt}h(t) * f(t) = h(t) * \frac{d}{dt}f(t) = \frac{d}{dt}y(t)$
  - Use of Derivative property: Finding the impulse response from the unit-step response
    - $If\ y(t) = u(t) * h(t), then\ \frac{d}{dt}y(t) = \frac{d}{dt}u(t) * h(t) = \delta(t) * h(t) = h(t)$
- Start Point
  - If the two signals have start points at $t_1 and\ t_2$, then the start point of their convolution will be at $t_1 + t_2$
- End Point
  - Similarly for the end points, if the two signals have end points at $t_1 and\ t_2$, then the end point of their convolution will be at $t_1 + t_2$
- Width
  - From the above two properties, we can see that if the two signals have widths W_1 and W_2, then the width of their convolution will be $W_1 + W_2$

The Impulse Function $\boldsymbol{\delta(t)}$

- The impulse function is the limit of $\frac{1}{T} rect\left(\frac{t}{T}\right)$ $as\ T \to 0$
  - Infinitesimal Width
  - Infinite Height
  - Of course, it integrates to 1. $(0 * \infty = 1)$
- Sifting
  - $\int_a^b \delta(t - t_o) f(t) dt = \begin{cases} f(t_0), t_0 \in [a, b] \\ 0, \quad\quad\quad else \end{cases}$
- Sampling
  - $f(t)\delta(t - t_o) = f(t_o)\delta(t - t_o)$
- Unit-step derivative
  - $\frac{du}{dt} = \delta(t)$

Problem 1 FA20

1. Consider the functions $h(t)$ and $f(t)$ shown below. Let $y(t) = h(t) * f(t)$.



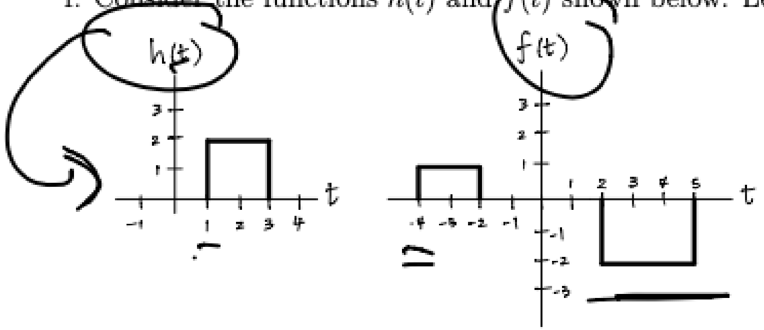
(a) (2 pts) What is the earliest time t_0 for which $y(t) \neq 0$?

(b) (4 pts) What is the minimum value of $y(t)$, denoted by y_{min}, and at what time(s) t_{min} does $y(t_{min}) = y_{min}$?

(d) (6 pts) What is $y(-2)$?

(e) (10 pts) Plot $y(t)$ over the range $-10 < t < 10$. Be sure to label both the vertical and horizontal axes.

1. Consider the functions $h(t)$ and $f(t)$ shown below. Let $y(t) = h(t) * f(t)$.



$$\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

(a) (2 pts) What is the earliest time t_0 for which $y(t) \neq 0$?

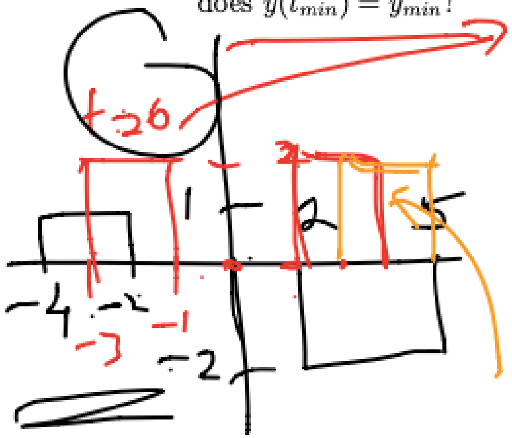
$t_0 \to y(t)$?

start time $h(t)$ + start time of $f(t)$

$t_0 = 1 + (-4) = -3$

$t_0 =$ $-3s$

(b) (4 pts) What is the minimum value of $y(t)$, denoted by y_{min}, and at what time(s) t_{min} does $y(t_{min}) = y_{min}$?



$t = 0 \to y$

$t \quad t_0$

$$\int_3^5 (2)(-2) = -8 \qquad t\,(5,6)$$

$$\int_4^6 (2)(-2) = -8$$

$y_{min} =$ -8

t_{min} : $5 < t < 6$

(d) (6 pts) What is $y(-2)$?

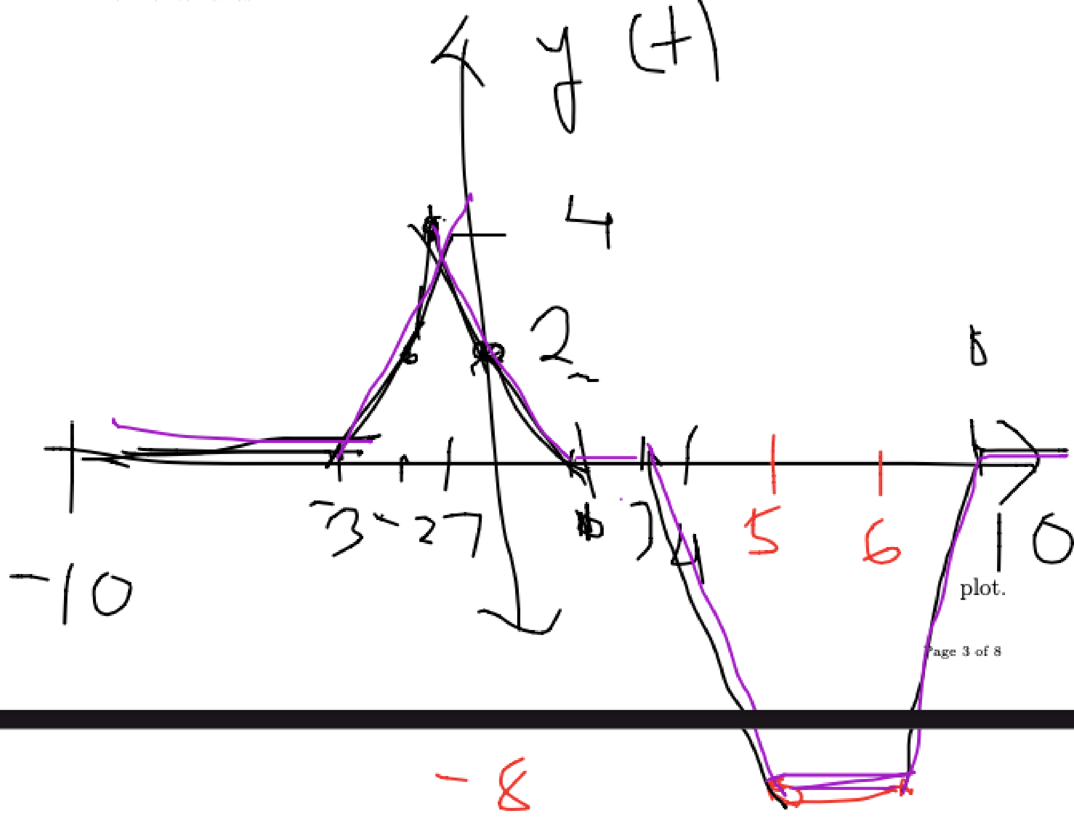
$t = -2$

$$\int_{-4}^{-2} 2(1)$$

2

$y(-2) =$ 2

(e) (10 pts) Plot $y(t)$ over the range $-10 < t < 10$. Be sure to label both the vertical and horizontal axes.



plot.

Problem 3 FA20

(b) [5 pts] Find the impulse response $h(t)$ of the LTI system having the following unit-step response:

$$y(t) = t^2 u(t)$$

Sampling and Analog Reconstruction

- If we have an original analog signal $f(t)$
- Our digital samples of the signal are obtained through sampling property as:
  - $f[n] = f(nT)$ where T is our sampling period; this is Analog to Digital (A/D) conversion
  - This results in infinitely many copies of the original signal's Fourier Transform spaced by $\frac{2\pi}{T}$
- We must make sure to satisfy Nyquist Criterion:
  - $T < \frac{1}{2B}$ or $f_s > 2B$, B = Bandwidth (in Hz)
- Following A/D conversion, we perform D/A conversion, then low-pass filter our signal in order to obtain our original signal according to the following relation

$$f(t) = \sum_n f_n sinc\left(\frac{\pi}{T}(t - nT)\right)$$

- For a more complete explanation, take ECE 310!

Sampling and Analog Reconstruction (cont'd)

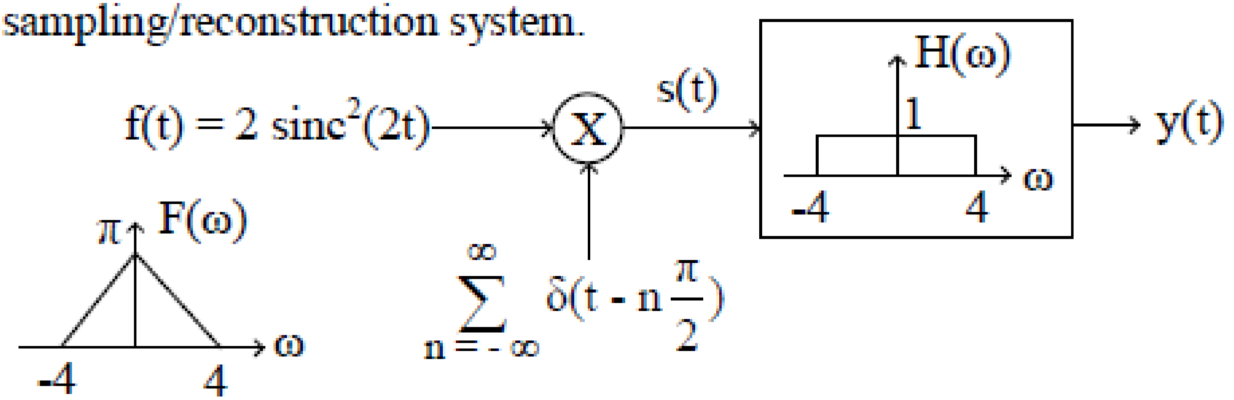
- The complete mathematical relationship between the continuous frequency spectrum and the discrete time (sampled) spectrum in analog frequency

$$F_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F\left(\omega + \frac{2\pi k}{T}\right)$$

- The nuances of this representation will be explored and clarified in ECE 310!

Problem 4 SP14

) Consider this sampling/reconstruction system.



i) Circle the correct answer and explain it below.

f(t) is: UNDERSAMPLED / OVERSAMPLED / SAMPLED AT NYQUIST RATE

Explanation:

ii) Sketch $S(\omega)$ and $Y(\omega)$ on the axes below.

iii) Determine y(t)

BIBO Stability

- A system is BIBO stable if for any bounded input, we obtain a bounded output
- Two ways to check BIBO stability
- By the definition:
  - $If\ |f(t)| \leq \alpha < \infty, then\ |y(t)| \leq \beta < \infty, \forall\ t$
- By Absolute Integrability
  - $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

LTIC

- LTIC stands for Linearity Time-Invariance and Causality
- Linearity
  - Satisfy Homogeneity and Additivity
  - Can be summarized by Superposition
    - If $x_1(t) \rightarrow y_1(t)\ and\ x_2(t) \rightarrow\ y_2(t),\ then\ ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
- Shift Invariance
  - If $x(t) \rightarrow y(t)\ then\ x(t - t_o) \rightarrow y(t - t_o)\ \forall\ t_0\ and\ x(t)$
- Causality
  - Output cannot depend on future input values
  - $h[n] = 0$ for $n < 0$.

Problem 4 SP 20

[12 pts] A system has the input-output relation $y(t) = \text{rect}(t-\alpha-1) * f^2(t+1)$. Determine whether or not the system is BIBO stable and/or causal. Explain your reasoning.

. ______________________________ **BIBO, causal ?**

Note: alpha is a positive integer