Explicit MPC

Based on **Explicit Model-Predictive Control** by **Alberto Bemporad**

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Outline

- Explicit & Implicit definition
- Implicit MPC vs Explicit MPC
- Explicit MPC
  - Advantages
  - Multi-parametric programming
  - Piecewise Affine functions
  - Methodology
  - Limitations
  - Applications
- Case Study

MPC

- **Definition:** An MPC uses linear plant, disturbance, and noise models to estimate the controller state and predict future plant outputs.
- **Objective:** Using the predicted plant outputs, the controller solves a quadratic programming optimization problem to determine optimal manipulated variable adjustments.

  which has 2 approaches:

  - Implicit MPC
  - Explicit MPC

Explicit & Implicit definition

- **Implicit MPC**: Traditional approach, in which MPC requires the solution of a quadratic program (QP) online to compute the control action, often restricting its applicability to slow processes.
- **Explicit MPC:** Explicit MPC completely removes the need for on-line solvers by precomputing the control law off-line, so that online operations reduce to a simple function evaluation.

Implicit MPC vs Explicit MPC

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Implicit MPC vs Explicit MPC

- The differences between two can be summarized as following:

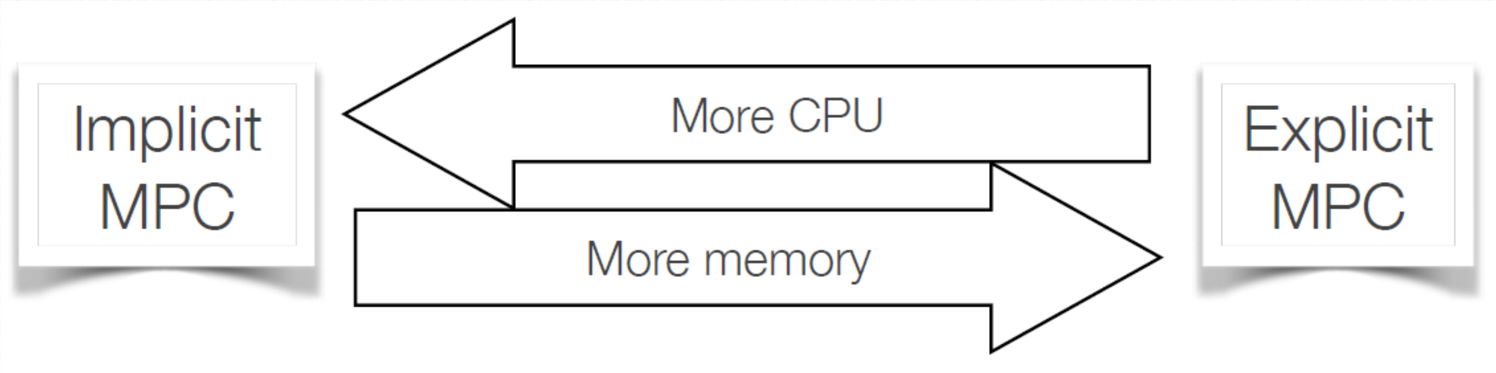
	Implicit MPC	Explicit MPC
Problem size	Large	Small
KKT solver	Online matrix inverse	Off-line factorization
Active set update	Heuristics	Sequential
Memory storage	Small	Large

	Vars	Constr.	States
Small	≤ 6	15	8
Large	≈ 500	3000	2000

Implicit MPC vs Explicit MPC

In Implicit MPC, there involves online solvers. Hence, requires larger processors.

Whereas, Explicit MPC need not process online but store a number of off-line algorithms. Thus, greater storage capacity is required



Implicit MPC vs Explicit MPC

	Implicit MPC (active-set method)	Explicit MPC (incremental regionless)
Memory storage	*H, F, G, w, E*	*G, w, E* primal optimizer, dual optimizer
KKT solver	Matrix inversions	Matrix/vector multiplication
Active set update	Identical algorithm	

Explicit MPC Simulink Block

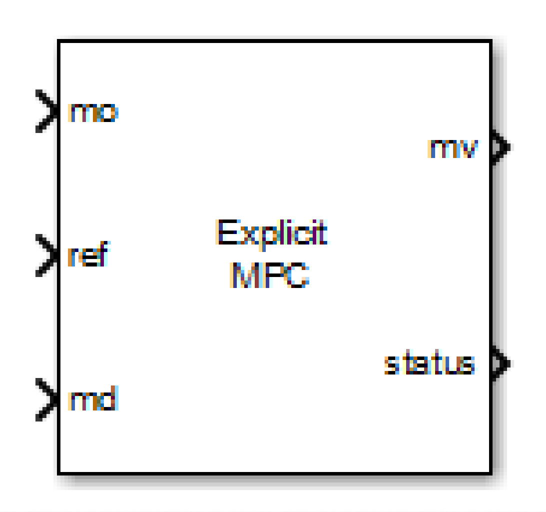
The Explicit MPC Controller block uses the following input signals:

- Measured plant outputs (mo)
- Reference or setpoint (ref)
- Measured plant disturbance (md), if any

Measured plant outputs are obtained from Implicit approach.

Hence, to create an **Explicit MPC controller**:

- first **Traditional (Implicit) MPC controller** is designed.
- then an **Explicit MPC Controller** is generated based on this traditional controller design.



Agenda

- Explicit & Implicit definition
- Implicit MPC vs Explicit MPC
- Explicit MPC
  - **Advantages**
  - Multi-parametric programming (Linear and Quadratic)
  - Piecewise Affine functions
  - Methodology (MPLP and MPQP
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Advantages of Explicit MPC

- It can be noted that Explicit MPC:
  - Does off-line and simpler computations.
  - Requires fewer run-time computations, thus useful for system with low sample rates.
  - Requires processors of less computational capacity.

MPC Problem

- In practical, an MPC should have a QP solution method:
  - embedded in the control hardware
  - the method must be fast enough to provide a solution within short sampling intervals
  - require simple hardware, limited memory to store the data defining the optimization problem and the code implementing the algorithm itself
  - a simple program code
  - and a good **worst-case** estimates of the execution time to meet real-time system requirements

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Multi-parametric programming

- MPC takes a different approach to meet the above requirements, where Multi-Parametric Quadratic Programming is proposed to pre-solve the QP off-line, therefore converting the MPC law into a continuous and **Piecewise-Affine Function** of the parameter vector (Bemporad et al. 2002b).

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Definition 1 *We say a function* $L : \Re^m \rightarrow \Re^n$ *is linear if (1) for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\Re^m$, $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$, *and (2) for any vector* $\mathbf{x}$ *in* $\Re^m$ *and scalar* a, $L(a\mathbf{x}) = aL(\mathbf{x})$.

Definition 4 *We say a function* $A : \Re^m \rightarrow \Re^n$ *is affine if there is a linear function* $L : \Re^m \rightarrow \Re^n$ *and a vector* $\mathbf{b}$ *in* $\Re^n$ *such that* $A(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b}$ *for all* $\mathbf{x}$ *in* $\Re^m$.

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} \{||Qx_k||_2 + ||Ru_k||_2\} + ||Px_N||_2$$
$$x_{k+1} = Ax_k + Bu_k,\ k = 0,\ldots,N-1$$
$$x_0 = x(t)$$
$$H_u u_k \le k_u,\ k = 0,\ldots,N-1$$
$$H_x x_k \le k_x,\ k = 0,\ldots,N-1$$
$$H_N x_N \le k_N$$

PWA model

$$\min_{z} \sum_{k=0}^{N-1} \{||Qx_k||_2 + ||Ru_k||_2\} + ||Px_N||_2$$
$$x_{k+1} = f_{PWA}(x_k, u_k),\ k = 0,\ldots,N-1$$
$$x_0 = x(t)$$
$$H_u u_k \le k_u,\ k = 0,\ldots,N-1$$
$$H_x x_k \le k_x,\ k = 0,\ldots,N-1$$
$$H_N x_N \le k_N$$

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MPQP

Algorithm 1

1 Let $X \subseteq \mathbb{R}^n$ be the set of parameters (states);
2 execute **partition**(X);
3 for all regions where $z($
a convex set, compute s
4 **end**.

procedure partition(Y
1.1 **let** $x_0 \in Y$ and ϵ th
1.2 **if** $\epsilon \leq 0$ **then exit**;
1.3 For $x = x_0$, compu
the QP (13);
1.4 Determine the set of active constraints when $z = z_0$, $x = x_0$, and build $\tilde{G}$, $\tilde{W}$, $\tilde{S}$;
1.5 If $r = \operatorname{rank} \tilde{G}$ is less than the number ℓ of rows of $\tilde{G}$, take a subset of r linearly independent rows, and redefine $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ accordingly;
1.6 Determine $\tilde{\lambda}(x)$, $z(x)$ from (17) and (18);
1.7 Characterize the CR from (19) and (20);
1.8 Define and partition the rest of the region as in Theorem 3;
1.9 For each new sub-region R_i, **partition**(R_i);
end procedure.

$$\boxed{Hz + G'\lambda = 0,\ \lambda \in \mathbb{R}^q,}$$

$$\lambda_i(G^i z - W^i - S^i x) = 0,\ i = 1, \ldots, q,$$

$$\lambda \geq 0,$$

$$Gz \leq W + Sx,$$

$$\begin{aligned} &\max_{x,z,\epsilon}\ \epsilon \\ &\text{subj. to}\ \ T^i x + \epsilon \|T^i\| \leq Z^i \\ &\qquad\qquad Gz - Sx \leq W \end{aligned} \tag{14}$$

$$\begin{aligned} \min_{U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}} &\Big\{ J(U, x(t)) = x'_{t+N_y|t} P x_{t+N_y|t} \\ &+ \sum_{k=0}^{N_y-1} \left[x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k} \right] \\ \text{subj. to}\quad & y_{\min} \leq y_{t+k|t} \leq y_{\max},\ k = 1, \ldots, N_c \\ & u_{\min} \leq u_{t+k} \leq u_{\max},\ k = 0, 1, \ldots, N_c \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k},\ k \geq 0 \\ & y_{t+k|t} = C x_{t+k|t},\ k \geq 0 \\ & u_{t+k} = K x_{t+k|t},\ N_u \leq k < N_y \end{aligned}$$

$$z = -H^{-1} G' \lambda$$

$$\begin{aligned} V_z(x) = \min_z \quad & \tfrac{1}{2} z' H z \\ \text{subj. to}\ & Gz \leq W + Sx(t), \end{aligned} \tag{13}$$

MPQP

Algorithm 1

1 Let $X \subseteq \mathbb{R}^n$ be the set of parameters (states);
2 execute **partition**(X);
3 for all regions where $z($
a convex set, compute s
4 **end**.

procedure partition(
1.1 **let** $x_0 \in Y$ and ϵ th
1.2 **if** $\epsilon \leq 0$ **then exit**;
1.3 For $x = x_0$, compu
the QP (13);
1.4 Determine the set of active constraints when $z = z_0$, $x = x_0$, and build $\tilde{G}$, $\tilde{W}$, $\tilde{S}$;
1.5 If $r = \operatorname{rank} \tilde{G}$ is less than the number ℓ of rows of $\tilde{G}$, take a subset of r linearly independent rows, and redefine $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ accordingly;
1.6 Determine $\tilde{\lambda}(x)$, $z(x)$ from (17) and (18);
1.7 Characterize the CR from (19) and (20);
1.8 Define and partition the rest of the region as in Theorem 3;
1.9 For each new sub-region R_i, **partition**(R_i);
end procedure.

$$Hz + G'\lambda = 0,\ \lambda \in \mathbb{R}^q,$$

$$\boxed{\lambda_i(G^i z - W^i - S^i x) = 0,\ i = 1, \ldots, q,}$$

$$\lambda \geq 0,$$

$$Gz \leq W + Sx,$$

For inactive constraints, $\breve{\lambda} = 0$.

$$\tilde{\lambda} = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \qquad (17)$$

$$z = -H^{-1}G'\lambda$$

$$z = H^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \qquad (18)$$

MPQP

Algorithm 1

1 Let $X \subseteq \mathbb{R}^n$ be the set of parameters (states);
2 execute **partition**(X);
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a convex set, compute su
4 **end.**

procedure partition$(Y$
1.1 **let** $x_0 \in Y$ and ϵ th
1.2 **if** $\epsilon \leq 0$ **then exit**;
1.3 For $x = x_0$, compu
the QP (13);
1.4 Determine the set of activ
$x = x_0$, and build $\tilde{G}$, $\tilde{W}$, $\tilde{S}$
1.5 If r = rank $\tilde{G}$ is less than
$\tilde{G}$, take a subset of r linea
redefine $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ accordingly;
1.6 Determine $\tilde{\lambda}(x)$, $z(x)$ from (17) and (18);
1.7 Characterize the CR from (19) and (20);
1.8 Define and partition the rest of the region as in Theorem 3;
1.9 For each new sub-region R_i, **partition**(R_i);
end procedure.

$$Hz + G'\lambda = 0, \;\; \lambda \in \mathbb{R}^q,$$

$$\lambda_i(G^i z - W^i - S^i x) = 0, \;\; i = 1, \ldots, q,$$

$$\boxed{\lambda \geq 0,}$$

$$Gz \leq W + Sx,$$

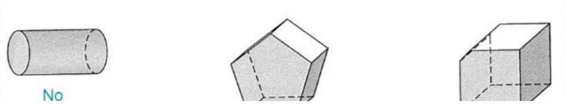
$$V_z(x) = \min_z \;\; \tfrac{1}{2} z' H z$$

$$\text{subj. to } Gz \leq W + Sx(t), \qquad (13)$$

$$z = -H^{-1} G' \lambda$$

$$G H^{-1} \tilde{G}' (\tilde{G} H^{-1} \tilde{G}')^{-1} (\tilde{W} + \tilde{S}x) \leq W + Sx \qquad (19)$$

$$\tilde{\lambda} = -(\tilde{G} H^{-1} \tilde{G}')^{-1} (\tilde{W} + \tilde{S}x) \qquad (17)$$



MP

Algorithm 1

1 Let $X \subseteq \mathbb{R}^n$ be the set of parameters (states);
2 execute **partition**(X);
3 for all regions where $z(x)$ is the same and whose union is a convex set, compute such a union as in [9];
4 **end.**

procedure partition(Y)
1.1 **let** $x_0 \in Y$ and ϵ the solution to the LP (14);
1.2 **if** $\epsilon \leq 0$ **then exit**; (no full dimensional CR is in Y)
1.3 For $x = x_0$, compute the optimal solution (z_0, λ_0) of the QP (13);
1.4 Determine the set of active constraints when $z = z_0$, $x = x_0$, and build $\tilde{G}$, $\tilde{W}$, $\tilde{S}$;
1.5 If $r = \operatorname{rank} \tilde{G}$ is less than the number ℓ of rows of $\tilde{G}$, take a subset of r linearly independent rows, and redefine $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ accordingly;
1.6 Determine $\tilde{\lambda}(x)$, $z(x)$ from (17) and (18);
1.7 Characterize the CR from (19) and (20);
1.8 Define and partition the rest of the region as in Theorem 3;
1.9 For each new sub-region R_i, **partition**(R_i);
end procedure.

Theorem 3
$\{x \in Y : Ax$
Also let

$$R_i = \Big\{ x$$

$$u = \begin{cases}
\begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix} x & \text{if } \begin{bmatrix} -5.9220 & -6.8883 \\ 5.9220 & 6.8883 \\ -1.5379 & 6.8291 \\ 1.5379 & -6.8291 \end{bmatrix} x \leq \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix} \\
 & \text{(Region \#1)} \\
2.0000 & \text{if } \begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix} \\
 & \text{(Region \#2, \#4)} \\
2.0000 & \text{if } \begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix} \\
 & \text{(Region \#3)} \\
-2.0000 & \text{if } \begin{bmatrix} -0.1259 & -0.0922 \\ -0.0679 & 0.0924 \end{bmatrix} x \leq \begin{bmatrix} -0.0519 \\ -0.0524 \end{bmatrix} \\
 & \text{(Region \#5)} \\
\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x + 0.6423 & \text{if } \begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix} \\
 & \text{(Region \#6)} \\
-2.0000 & \text{if } \begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix} \\
 & \text{(Region \#7, \#8)} \\
\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x - 0.6423 & \text{if } \begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix} \\
 & \text{(Region \#9)}
\end{cases}$$

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Limitations

- Though Explicit MPC is faster and uses off-line approach it is capable of handling smaller systems (generally with variables <6).
- Requires large storage units.
- Sequential and less dynamic.

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Applications

- In literature, it can be seen that Explicit MPC techniques have addressed several industrial problems.
- Explicit MPCs are reported to be most suitable for fast-sampling problems (in the order of 1-50 ms) and relatively small size (1-2 manipulated inputs, 5-10 parameters) such as **automotive domain and electrical power converters**.

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Conclusion

- Explicit MPC is a powerful tool to convert an MPC design into an equivalent offline control law that has simpler implementations.
- Implementation of the explicit MPC for solving the QP problem depends on available CPU time, data memory, and program memory and other practical considerations.
- Explicit MPC approach remains convenient for relatively small and offline problems (such as one or two command inputs, short control and constraint horizons, up to ten states).
- For larger problems, and/or problems that are linear time varying, online QP solution methods tailored to conventional Implicit MPC may be preferable.

References

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Simulation Results