ISTANBUL TECHNICAL UNIVERSITY

KOM 518E

MODEL PREDICTIVE CONTROL

MIDTERM

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Question 1: Consider the system with a transfer function given by,

$$G(z) = \frac{0.2713z^{-3}}{1 - 0.8351z^{-1}}$$

Obtain impulse response model of the system. Perform DMC design for a prediction horizon 10 and control horizon 5. Make simulations for $\lambda = 1, \alpha = 0.7$ and $\lambda = 0.1, \alpha = 0$. Compare the results.

First, impulse response coefficients can be found with impz matlab command and then step response coefficients are obtained through iterative sum from impulse responses:

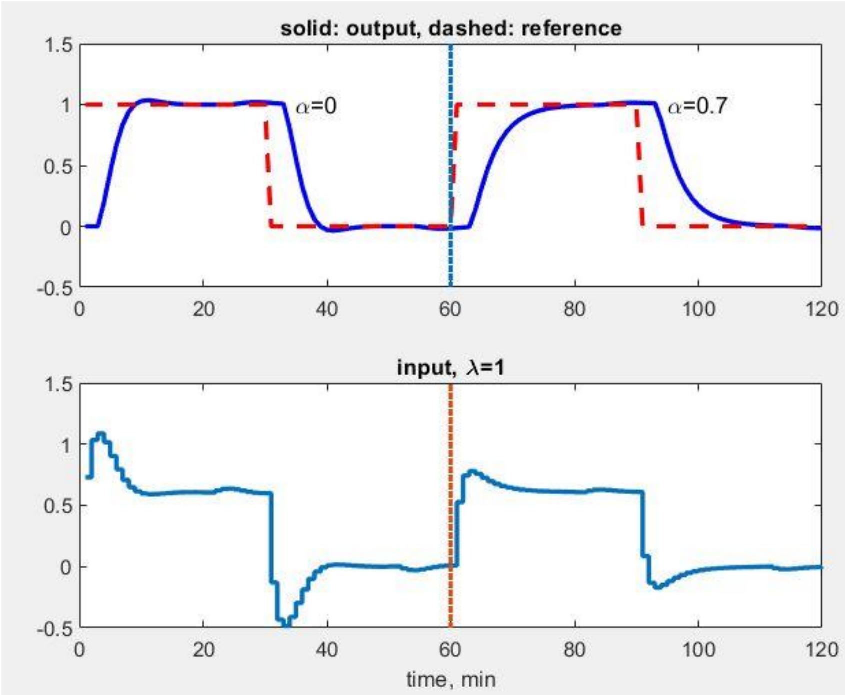
```
num=[0 0 0 0.2713]
denum=[1 -0.8351 0 0]
%Obtain the impulse response
impresponse= impz(num,denum,30)

% The Step Response(sr) from the impulse response
p.sr=zeros(30,1);
for i=1:30
    for j=1:i
    p.sr(i)=p.sr(i)+impresponse(j)
    end
end
```

Considering a prediction horizon of 10 and a control horizon of 5, the dynamic matrix is obtained from the coefficients of the step response and is given by:

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.271 & 0 & 0 & 0 & 0 \\ 0.498 & 0.271 & 0 & 0 & 0 \\ 0.687 & 0.498 & 0.271 & 0 & 0 \\ 0.845 & 0.687 & 0.498 & 0.271 & 0 \\ 0.977 & 0.845 & 0.687 & 0.498 & 0.271 \\ 1.087 & 0.977 & 0.845 & 0.687 & 0.498 \\ 1.179 & 1.087 & 0.977 & 0.845 & 0.687 \\ 1.256 & 1.179 & 1.087 & 0.977 & 0.845 \end{bmatrix}$$

Taking λ = 1, matrix (GTG + λI) − 1GT is calculated and therefore the control law is given by the product of the first row of this matrix (K) times the vector that contains the difference between the reference trajectory and the free response

δu(t) = K(w − f) with K = [0 0 0.1465 0.1836 0.1640 0.1224 0.0780 0.0410 0.0101 − 0.0157].



Here it can be seen that a small value of α makes the system response faster with a slight oscillation, while a small value of λ gives bigger control actions.

Question 3. A plant with 1 input, 1 output and 1 sate has the model

$$x(k+1) = 0.9x(k) + 0.5u(k), \qquad y(k) = x(k)$$

Predictive control is to be used with Q=1, R=0, Np=30, Nu=2. A constant step disturbance at the plant input is assumed (that is, u(k) is replacedby u(k)+d(k)).

a) Show how the plant model can be augmented to incorporate this disturbance model.
b) Design a state observer with a deatbeat response for the plant with this disturbance.
c) Simulate the response of the predictive controller to an input step disturbance when your observer is used.Compare the response to that obtained when the default DMC observer is used.
d) Assign the observer pole to z=0.5. Simulate the response of the predictive controller to an input step disturbance when your observer is used.
e) How does the observer design affect the set-point response?

a) Augmenting the plant model to incorporate disturbance model in state space form yields;

$$\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(k)$$

$$[y(k)] = [1 \quad 0] \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}$$

b) Observer equation A-LC forms

$$[A - L * C] = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} L_x \\ L_d \end{bmatrix} [1 \quad 0] = \begin{bmatrix} 0.9 - L_x & 0.5 \\ -L_d & 1 \end{bmatrix}$$

For deadbeat observer, both eigenvalues of this matrix should be zero, so both determinants and trace should be zero:

$$Determinant\text{—}\!\!\rightarrow (0.9 - L_x) * 1 - (-L_d * 0.5) = 0$$
$$Trace\text{—}\!\!\rightarrow (1.9 - L_x) = 0$$

Here it is found that L_x =1.9 and L_d= 2.

c) Augmented A matrix of the plant with state vector,

$\breve{A} = \begin{bmatrix} \begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [0.9 \quad 0.5] & 1 \end{bmatrix}$ and corresponding $\breve{C} = [0 \quad 0 \quad 1]$.

Observer gain matrix L having 3 elements and

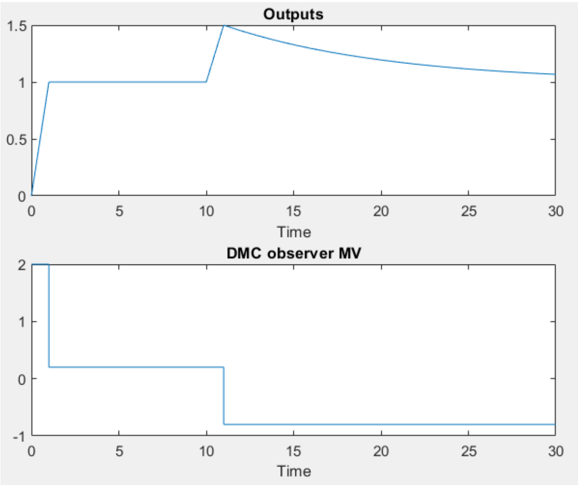
$$[\breve{A} - L * \breve{C}] = \begin{bmatrix} \begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [0.9 \quad 0.5] & 1 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} [0 \quad 0 \quad 1] = \begin{bmatrix} 0.9 & 0.5 & -L_1 \\ 0 & 1 & -L_2 \\ 0.9 & 0.5 & 1 - L_3 \end{bmatrix}$$

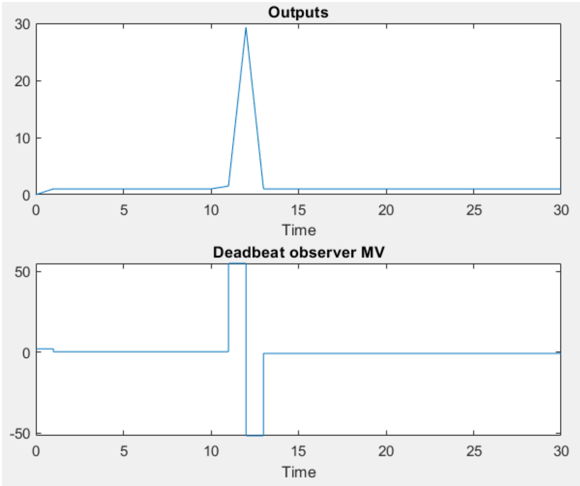
The characteristic polynomial of observer gain matrix

$$det[\lambda I - \widetilde{(A - L * \breve{C})}] = \lambda^3 + (L_3 - 2.9)\lambda^2 + (2.8 + 0.9L_1 + 0.5L_2 - 1.9L_3)\lambda$$

For a deadbeat observer, all the roots of this polynomial be at zero, so setting the coefficients to zero will yield $L_1 = 1.9, L_2 = 2, L_3 = 2.9$.

Below Figures 1 and 2 show the results using the DMC and deadbeat observer, respectively. As expected, with the deadbeat observer the controller completely compensates for the disturbance after two steps (because the plant with disturbance model has two states), but large input changes occur and the peak deviation of the output from its set-point is large. On the other hand, the controller with the default (DMC) observer takes longer to correct for the disturbance, but the peak deviation of the output is much smaller, and much less violent input changes are made.





d)

e) The observer makes no difference to the set-point response, as can be seen from the two sets of plots, which are identical until the disturbance occurs. This can be checked further by defining various trajectories for the set-point vector r in the simulations.

Question 6. Consider unstable the system

$$x(k+1) = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

with the input constraints

$$-1 \leq u(k) \leq 1, k = 0, \dots, N-1$$

and the state constraints

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}, k = 0, \dots, N-1$$

Formulate the corresponding QP problem for N_p=3, N_u=2, Q=I and R=0.1. (Namely including the derivation determine H, f, A, b in the MATLAB function U=QUADPROG(H,f,A,b) to solve min 0.5*U'*H*U + f'*U subject to: A*U <= b).

Here; it can be seen that $A = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

From discrete state space model, below equation can be written considering Np=3 and Nu=2 as

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ A^2 \\ A^3 \end{bmatrix}}_{F} x_k + \underbrace{\begin{bmatrix} B & 0 & 0 \\ AB & B & 0 \\ A^2B & AB+B & 0 \end{bmatrix}}_{H_x} \begin{bmatrix} u_k \\ u_{k+1} \\ u_{k+2} \end{bmatrix}$$

If we rewrite in short form as ;

$$\xrightarrow{x_k} = F * x_k + H_x * \xrightarrow{u_{k-1}}$$

Putting above equation into state constraints given in the question and define states in terms of input (u)

$$\underline{x} \leq x \leq \overline{x}$$

$$\underline{x} \leq F * x_k + H_x * \xrightarrow{u_{k-1}} \leq \overline{x}$$

$$\underline{x} - F * x \leq H_x * U \leq \overline{x} - F * x$$

Hence,

$$\begin{bmatrix} H_x \\ -H_x \end{bmatrix} U \leq \begin{bmatrix} \overline{X} - F * x \\ -\underline{X} + F * x \end{bmatrix}$$

Input constraints can be defined as

$$\underline{U} \leq U \leq \overline{U}$$

$$\begin{bmatrix} I \\ -I \end{bmatrix} U \leq \begin{bmatrix} \overline{U} \\ -\underline{U} \end{bmatrix}$$

Thus, R and C matrices are written as;

$$\underbrace{\begin{bmatrix} H_x \\ -H_x \\ I \\ -I \end{bmatrix}}_{A} U \leq \underbrace{\begin{bmatrix} \overline{X} - F * x \\ -\underline{X} + F * x \\ \overline{U} \\ -\underline{U} \end{bmatrix}}_{b}$$

H and f matrices are formed as;

$$H = 2 * (H'_x * Q * H_x) + R$$

$$f = 2 * x' * F' * Q * H_x$$

And given to quadprog in Matlab as U=quadprog(H,f,A,b).

Note: Initial conditions for state is defined as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in order to run Q6.m file.

Question 7. (Use MATLAB MPC toolbox) Consider the linearized model of Cessna Citation Aircraft given below.

Linearized continous-time model: (at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

- Input:elevator angle
- States: x_1:angle of attack, x_2:pitch angle, x_3:pich rate, x_4:altitude
- Outputs:pitch angle and altitude
- Constraints:elevator angle $\mp 0.262 rad (\mp 15°)$, elevator rate $\mp 0.524 rad/s$ $(60°/s)$, pitch angle $\mp 0.349 rad (\mp 39°)$

Consider LQR and Linear MPC problems with quadratic cost appended below.

LQR

$$J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_t^T Q x_t + u_k^T R u_k$$

$$\text{s.t.} \ x_{k+1} = A x_k + B u_k$$

$$x_0 = x(t)$$

Assume: $Q = Q^T \succeq 0, \ R = R^T \succ 0$

MPC

$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} {x_k}^T Q x_k + {u_k}^T R u_k$$

$$\text{s.t.} \ x_{k+1} = A x_k + B u_k$$

$$x_k \in \mathcal{X}, \ u_k \in \mathcal{U}$$

$$x_0 = x(t)$$

a) Consider infinite horizon LQR design, take sampling period as 0.25s, $Q = I$ and $R = 10$. Make simulations for initial condition $x_0 = [0,0,0,10]$ and at the desired altitute $x_4 = 0$. Afterwards, saturate the LQR controller with $|u_k|<0.262$ and verify that applying LQR and saturating controller can lead instability.

b) Consider linear MPC design with $Q = I$, $R = 10$ and $N = 10$ considering the input constraint $|u|<0.262$ (in this step don't consider the rate constrate and state constraints) using MATLAB MPC toolbox. Give simulation results for initial condition $x_0 = [0,0,0,10]$ and at the desired altitute $x_4 = 0$ and comment on results.

c) Now add the rate constraint $|\dot{u}_k|<0.524$ (in discrete-time $|u_k - u_{k-1}|<0.524*Ts$) to step b) and give simulation results for same initial condition and desired altitude and comment on results. Give also simulation results for initial condition $x_0 = [0,0,0,100]$.

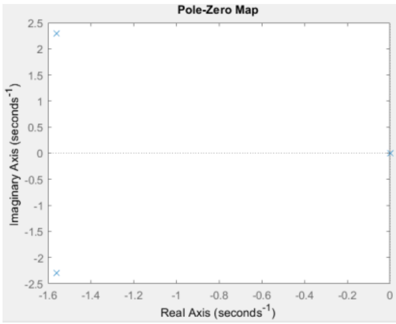
d) Add the state constraint $|x_2|<0.349$ on pitch angle for costumer comfort to step c). Give simulation results for $x_0 = [0,0,0,100]$.

e) Consider step d) and decrease horizon to $N = 4$ and show that decrease in prediciton horizion may cause loss of stability.

f) Consider step e) and add terminal cost and terminal constraint to MPC formulation and show that the stability is ensured.

a) First looking at open loop system poles, we see open loop system is unstable.

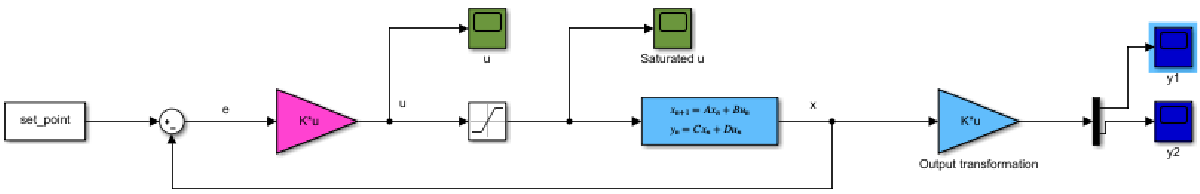
```
The open loop system is unstable!
>> OL_poles= pole(sys)

OL_poles =

   0.0000 + 0.0000i
   0.0000 + 0.0000i
  -1.5594 + 2.2900i
  -1.5594 - 2.2900i
```



Closed loop system is found to be asymptotically stable as eigenvalues are inside unit circle(discrete time).

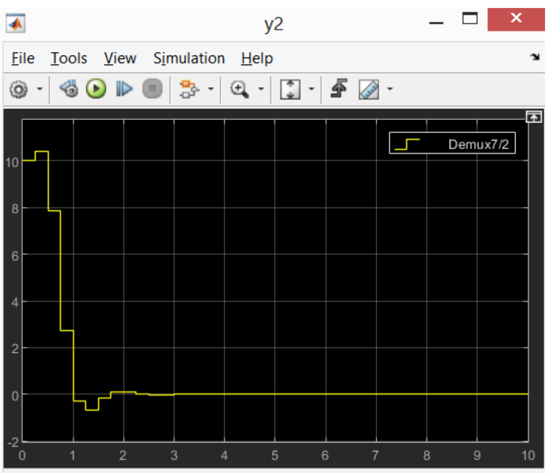
```
The assumptions for the asymptotical stability of LQR are verified
Closed loop matrix A-BK eignevalues = 0.21461-0.47754i        0.21461+0.47754i        0.26444-0.099147i        0.26444+0.099147i
Eigenvalues inside unit circle -> closed loop system is AS.
```

LQR controller structure established in Simulink after necessary coefficients are calculated within Matlab

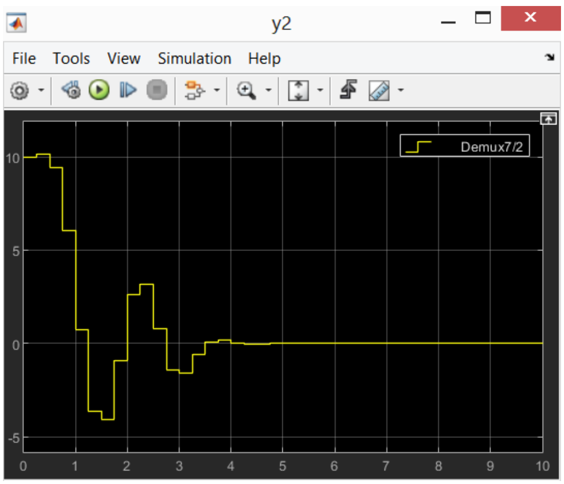


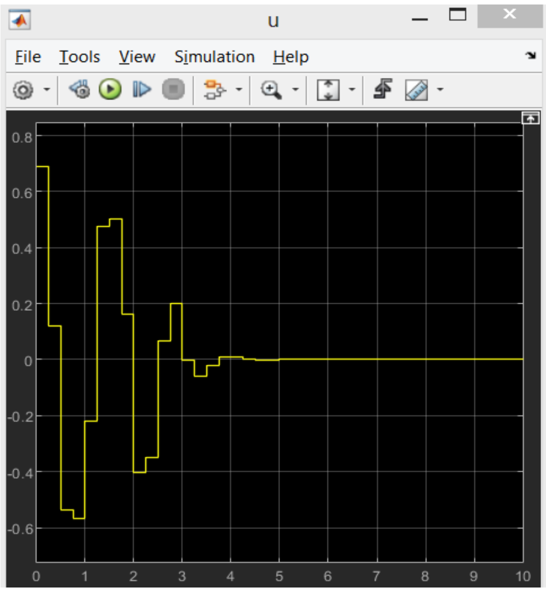
Here coefficients of LQR is given to the system with Gain (K) and input saturation is applied through Simulink “Saturation” block.

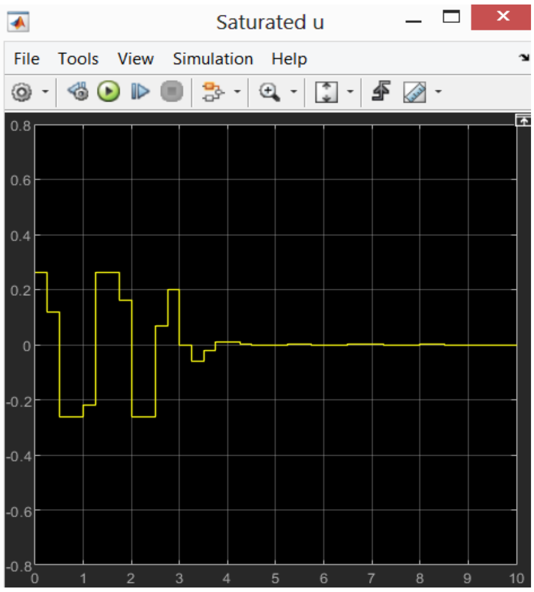
Before the saturation



After the saturation:

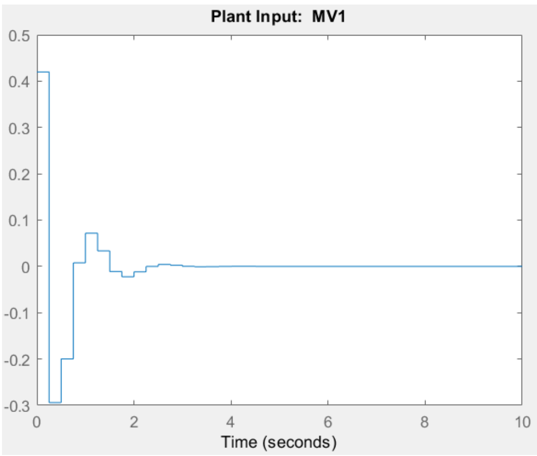




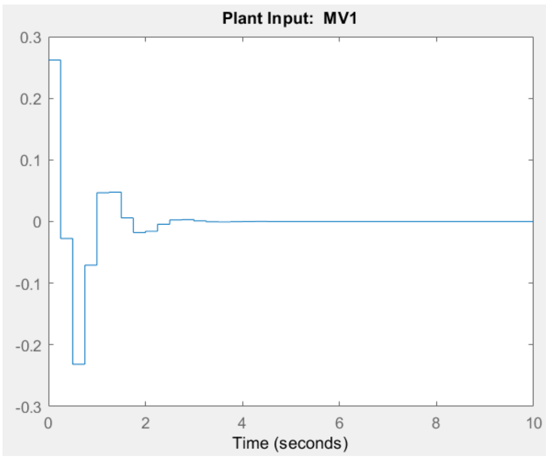


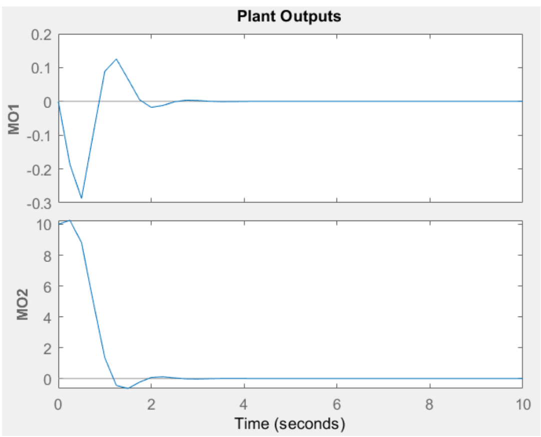
b) Linear MPC design is constructed with Q= I , R=10 and N=10 using MPC Toolbox.

Before the saturation



At 100 m :

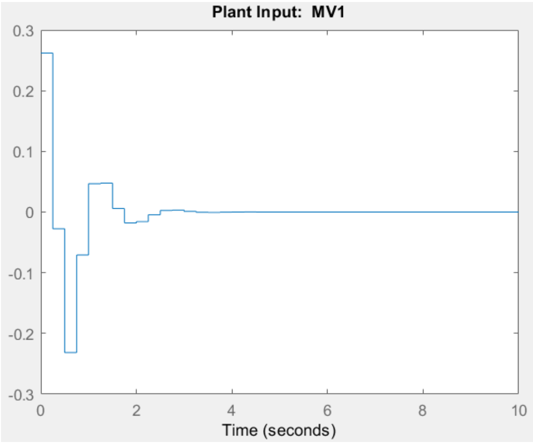


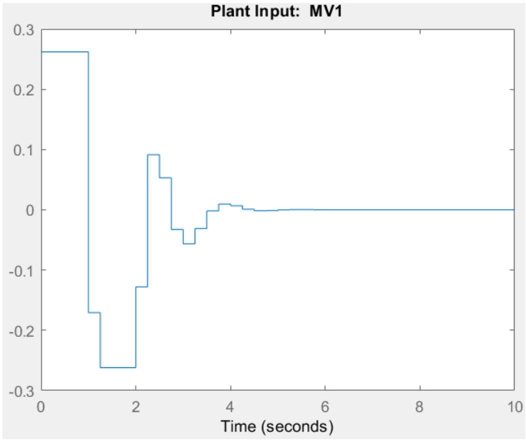


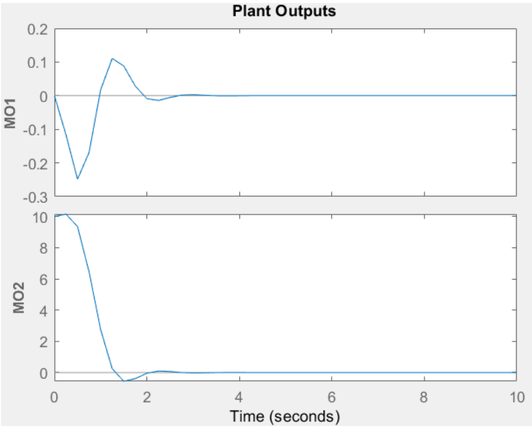
c)

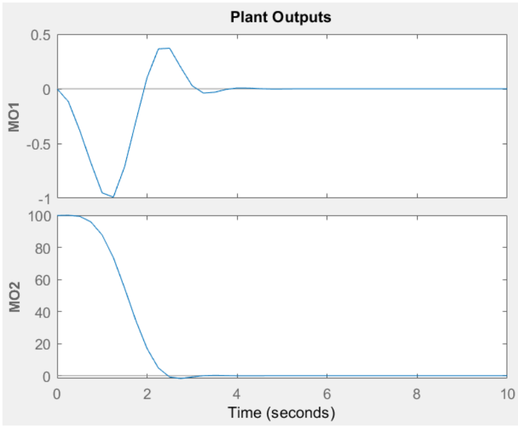
At 10 m with Input rate constraints

At 100 m with Input rate constraints









Question 5

We have the system model stated below in equation (5.1)

$$y(k) - 1.5y(k-1) + 0.56y(k-2) = 0.6u(k-1) - 0.9u(k-2) \qquad (5.1)$$

In order to apply GPC, we have to convert equation (5.1) to the generalized form below,

$$y(k)A(z^{-1}) = B(z^{-1})u(k-1) \qquad (5.2)$$

Rearranging equation (5.1) according to the generalized formulation we obtain our model as,

$$y(k)(1 - 1.5z^{-1} + 0.56z^{-2}) = u(k-1)(0.9 - 0.6z^{-1}) \qquad (5.3)$$

By observation, we can directly obtain the values of $A(z^{-1})$ and $B(z^{-1})$ as,

$$\tilde{A}(z^{-1}) = A * (1 - z^{-1}) = (1 - 2.5z^{-1} + 2.06z^{-2} - 0.56z^{-3}) \ (5.4)$$

$$B(z^{-1}) = u(k-1)(0.9 - 0.6z^{-1})$$

The generalized GPC equation is stated below,

$$y = Gu + F(z^{-1})y(k) + G'(z^{-1})u(k-1) \qquad (5.5)$$

The previous and future values and split in the equations and represented by G and f respectively,

$$f = F(z^{-1})y(k) + G'(z^{-1})u(k-1) \qquad (5.5)$$

$$y = Gu + f$$

Since our $\mathrm{Np} = 10$, we have to find the $E_1, E_2, \dots, E_{10}$ together with $F_1, F_2, \dots, F_{10}$. And using these we should find the $G_1, G_2, \dots, G_{10}$.

Applying the recursion technique, we find the values of $E_1, E_2, \dots, E_{10}$ together with $F_1, F_2, \dots, F_{10}$ by dividing 1 over $\tilde{A}$:

$$E_{j+1} = E_j(z^{-1}) + f_{j,0}(z^{-j}) \qquad (5.7)$$

$$f_{j+1,i} = f_{j,i+1} - f_{j,0}\, a_1$$

$$E_1 = 1$$

$$E_2 = 1 + 2.5z^{-1}$$

$$E_3 = 1 + 1.5z^{-1} + 4.19z^{-2}$$

…

and

$F_1 = 2.5z^{-1} - 2.06z^{-2} \ - 0.56z^{-3}$

$F_2 = 4.19z^{-2} - 4.59z^{-3} + 1.4z^{-4}$

…

The values of G are obtained as,

$G_j(z^{-1}) = E_j(z^{-1})B(z^{-1})$ (5.8)

Since Nu=2, beyond u(k+1) should be added to coefficient of u(k+1)

Now we can form the matrix form as,

$$\begin{bmatrix} y(k+1\,/\,k) \\ y(k+2\,/\,k) \\ y(k+3\,/\,k) \\ \vdots \\ y(k+10/k) \end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ 1.65 & 0.9 \\ 2.271 & 1.65+0.9 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} + \begin{bmatrix} -0.6 \\ -1.5 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} u(k-1) + \begin{bmatrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{bmatrix} y(k) + \begin{bmatrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{bmatrix} y(k-1) + \begin{bmatrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{bmatrix} y(k-2) \quad (5.9)$$

Rearranging the expression, one can show that;

$y = Gu + f$ (5.10)

We can calculate the U values as by considering

$U = (G^T G + \lambda I)^{-1} G^T (W - f)$ (5.11)

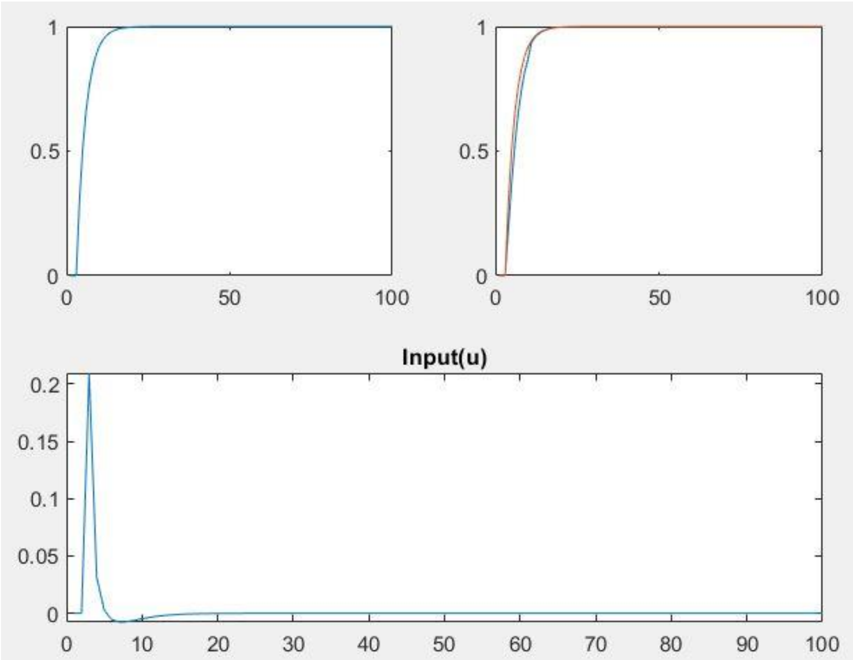


Figure 1: Input-output @ λ=0.8 and α=0.7

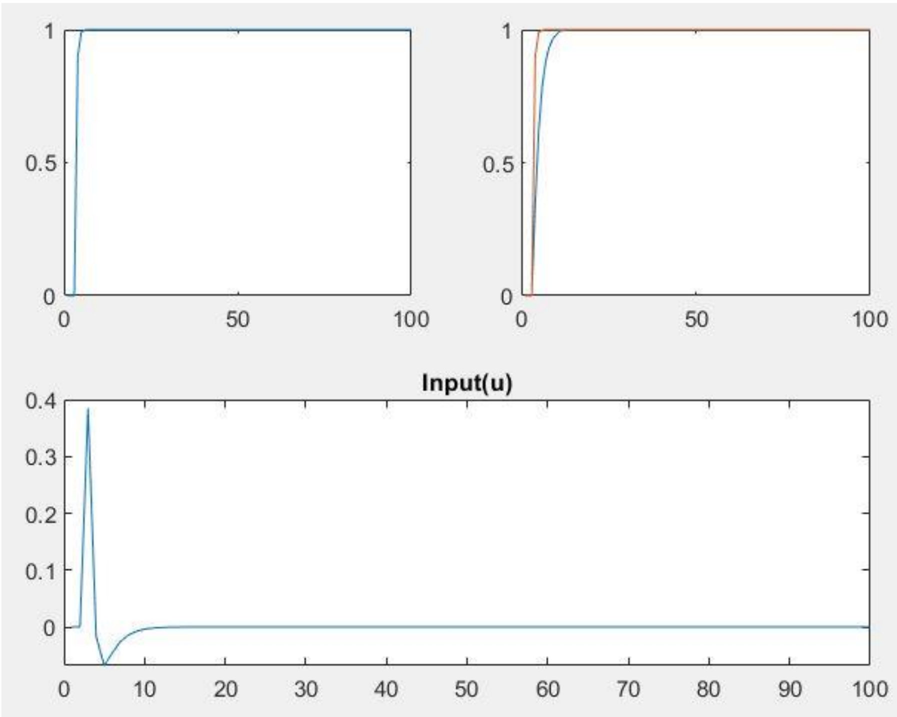


Figure 1: Input-output @ λ=0.8 and α=0.1

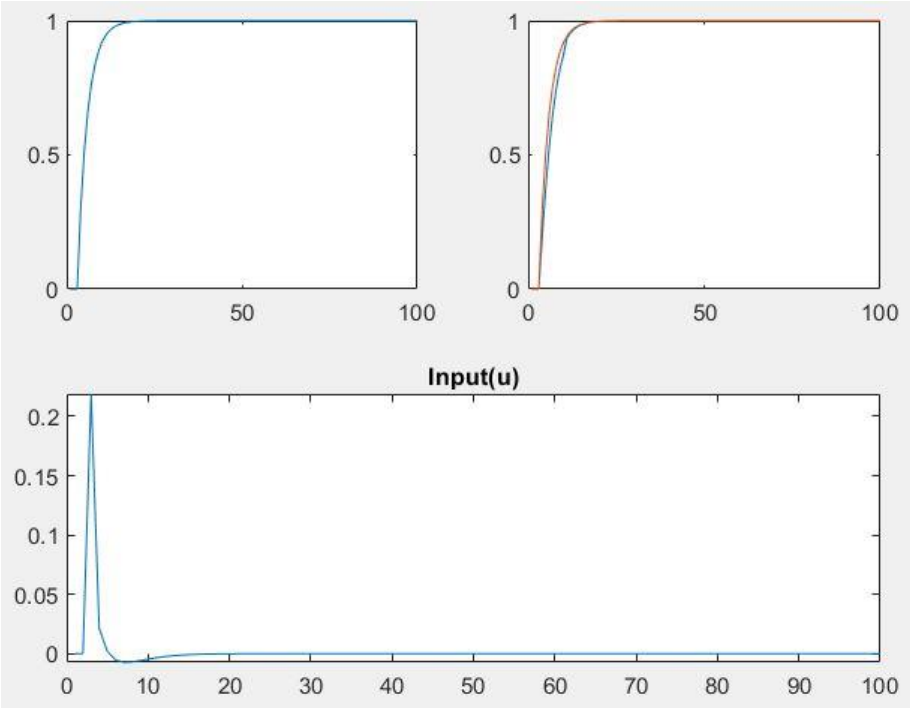


Figure 1: Input-output @ λ=0.1 and α=0.7

Conclusion: Here it can be seen that a small value of α makes the system response faster with a slight oscillation, while a small value of lambda gives bigger control actions.