EECS 16A Midterm 1 Review Session

Presented by <NAMES >(HKN)

Disclaimer

Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are posted at — on Piazza.

HKN Drop-In Tutoring

- These details should be edited

Systems of Equations and Gaussian Elimination

Vectors

Conceptually, a vector is a collection of numbers that each represent a variable. If there are n variables, then the vector is n-dimensional.

Example:

A point in 3D can be represented as (x, y, z) In vector form, this would be represented as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Matrices

- Collection of **vectors**
- 2D table for **storing data**
  - Systems of equations for imaging observations
- Notable/useful matrices
  - Identity matrix
  - Augmented matrices
  - Rotation matrix
  - Many others!

Augmented: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -3 & 2 & 3-2 & 10 \end{array}\right]$

Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrix Transformations

Matrices are often used to perform transformations, especially in $\mathbb{R}^2$

Two important transformations:

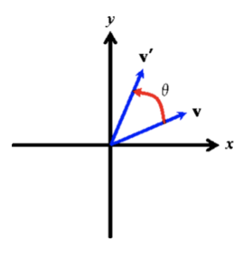

Figure 1: Rotation

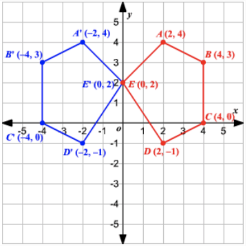

Figure 2: Reflection

Rotation Matrix

The rotation matrix rotates points by a specified angle, theta:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Use this matrix by plugging in desired rotation angle, then multiply with vector.

Note: Rotation matrices also preserve the length of a vector.

Example: Rotation Matrix that rotates vector by 90°

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation Matrix

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Example: Rotation Matrix that rotates vector by 90°

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Reflection Matrix

The reflection matrix reflects vectors across a line (Notice that such matrix also preserves the length of a vector)

Notable reflection matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection across x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

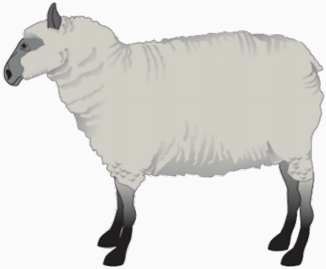
Reflection across y-axis

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflection across $y = x$

Matrix Transformations

All linear transformations can be expressed as a matrix



sheep

sheared sheep

Example 1

Question

What is the resulting vector after the following (non-subsequent) transformations are applied to the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

1. Rotate by 45°

2. Reflect across $y = x$

Example 1

Solution

What is the resulting vector after the following (non-subsequent) transformations are applied to the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

1. Rotate by 45°

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix}$$

2. Reflect across $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Determinants

Determinant of a 2x2 matrix:

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Also, an upper triangular matrix's determinant is the product of the diagonal:

$$det\left(\begin{bmatrix} a & * & * & \cdots \\ 0 & b & * & \cdots \\ 0 & 0 & c & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\right) = abc\cdots$$

Gaussian Elimination

Ultimate goal: Upper triangular form (**i.e** numbers below the diagonal are all 0)

Reminder of Motivation: Can work starting from the bottom row up in order to quickly calculate (from a computer's perspective) each variable's value.
For example, the last row has one variable and one value.

Gaussian Elimination

IDEA: Augmented matrix represents a system of equations where each row is an equation.
What are you allowed to do with equations to solve them?

1. **Row exchange** (same equations, different order)
2. **Scaling** (multiplying an equation by a scalar)
3. **Replace a row with the a linear combination of itself and another row** (intuitively, must include itself because otherwise there is "loss of information", like a deletion of one of the original rows)

GOAL:
Original Matrix $\rightarrow$ Row Echelon Form $\rightarrow$ Reduced Row Echelon Form

Gaussian Elimination Example

Problem

Use Gaussian Elimination to reduce the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

Solution

Use Gaussian Elimination to reduce the following system of equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row implies 0 of three variables add up to -1. Therefore, there exists no solution.

Possible Outcomes from Gaussian Elimination

Possible Results	Row Picture	Column Picture	Properties of Matrix A
Unique solution	Equations intersect at exactly one point	**b** can be uniquely represented by the linear combination of the columns of **A**	**A** is invertible
Infinite solutions	Equations intersect along an infinite space (eg. the intersection is a line or a plane)	There are multiple ways of representing **b** in terms of the linear combination of the columns of **A**	**A** has linearly dependent columns
No solutions	Equations do not intersect	**b** is not in the span of the columns of **A**; **b** is not in the column space of **A**	Column space of **A** does not contain (span) the vector **b**

Linear (In)dependence & Invertibility

Linear Combination

Core: Addition and Scaling

In context of vectors $v_1, v_2, \ldots, v_n$:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $a_1, a_2, \ldots, a_n$ are scalars

Taking a Step Back

Hopefully by now you have been able to recognize that the core of all linear algebra is linear combinations.
Think back to how one of the rules of row operations allows to a linear combination of rows as long as the original row being replaced is included!

Column Space

Let $v_1, v_2, \ldots, v_n$ be the columns of matrix V. The linear combination of the vectors is the **Column Space** of V.
This is also known as the **Range** of V, because when a matrix multiplies a vector or another matrix (which is just multiple vectors!), the result is a linear combination of all the columns. Hence, the linear combination of the columns is like the reach, or **Range** of the matrix.

Linear Dependence: Two Definitions

(i) A set of vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if there exist scalars $a_1, \ldots, a_n$ such that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ and not all a_i's are 0.

(ii) A set of vectors is linearly dependent if one of the vectors could be written as a linear combination of the rest of the vectors. $v_i = \sum_{j \neq i} a_j v_j$.

Linear Independence

Strictly speaking, a matrix is linearly independent if it is **not** linearly dependent.
In other words, for a set of linearly independent vectors, $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ implies that all $a_1 = 0, a_2 = 0, \cdots, a_n = 0$ (useful when doing proofs).

Linear (In)dependence

Conceptually, if you have 2 equations and 3 unknowns, you cannot solve it. Imagine if you have 3 equations but one equation was actually a linear combination of the first two. You still have the same "information" of 2 equations as your "basis". Many important implications.

Span

The **span** of a set of vectors $\{v_1, \ldots, v_n\}$ is the set of all linear combinations of $\{v_1, \ldots, v_n\}$.
In other words, if one vectors v_i in a set of n vectors lied in the span of the other vectors, then that set is considered linearly dependent.

Rank

The **rank** of a matrix A is the dimension of the column space of A.
This is the same as the number of linearly independent column vectors.
This is also the same as the number of linearly independent row vectors.

Relating it Back To Linear (In)dependence

Colloquially, a matrix A can be said to **span** n dimensions.
This means that matrix A has n linearly independent rows and n linearly independent columns.
This also means that matrix A is of **rank** n.

Span & Rank & (In)dependence

To figure out the **span** of a set of vectors, think of them as the column vectors of a matrix and Gaussian eliminate that matrix to RREF.
The number of non-zero rows is the **dimension** the vectors span.
If those vectors were actually the columns of a matrix, the dimension is also the **rank** of that matrix.
If the dimension is the same as the total number of vectors, then those vectors are **independent**.

Example of figuring out Span & Rank

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

SOL: Example of figuring out Span & Rank

$$\begin{bmatrix}1 & 2 & 1 & -4\\ 0 & -1 & 1 & -2\\ 1 & 4 & 1 & 2\end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix}1 & 2 & 1 & -4\\ 0 & -1 & 1 & -2\\ 0 & 2 & 0 & 6\end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow \frac{1}{2} R_3} \begin{bmatrix}1 & 2 & 1 & -4\\ 0 & 1 & 0 & 3\\ 0 & -1 & 1 & -2\end{bmatrix} \xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix}1 & 2 & 1 & -4\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 1\end{bmatrix}$$

There are 3 linearly independent rows!

Inverses

A matrix A is **invertible** if A is a square matrix of **rank** n where n is the number of rows and columns. (This means linearly independent rows/cols.)

If a matrix A is invertible, then there exists a matrix B such that $AB = BA = I$, where I is the identity matrix.

Normally notated as: A^{-1}

Inverse Matrix Properties

$$AA^{-1} = A^{-1}A = I$$

$$(A^{-1})^{-1} = A$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Finding the inverse

Construct an n x $2n$ augmented matrix:

$$[A \mid I]$$

Row reduce until the left becomes the identity; the right is the inverse

$$[A \mid I] \sim \ldots \sim [I \mid A^{-1}]$$

Cheat sheet: 2x2 Inverse

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse Example

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

SOL: Inverse Example

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Break!

Nullspace/Vector Space/Basis/Subspace

A set of elements closed under vector addition and scalar multiplication

Properties of a vector space V:

1. If u and v are vectors in V, then $u + v$ must be in V
2. If u is a vector in V and k is a real number, then ku must be in V

* Any linear combination of vectors in V is still in V (the span of a vector space is itself).

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w}\in\mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ for any $\vec{v},\vec{u}\in\mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0}\in\mathbb{V}$ such that $\vec{v}+\vec{0}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$.
  - Additive Inverse: For any $\vec{v}\in\mathbb{V}$, there exists $-\vec{v}\in\mathbb{V}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
  - Closure under vector addition: For any two vectors $\vec{v},\vec{u}\in\mathbb{V}$, their sum $\vec{v}+\vec{u}$ must also be in $\mathbb{V}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v})=(\alpha\beta)\vec{v}$ for any $\vec{v}\in\mathbb{V}$, $\alpha,\beta\in\mathbb{R}$.
  - Multiplicative Identity: There exists $1\in\mathbb{R}$ where $1\cdot\vec{v}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ for any $\alpha\in\mathbb{R}$ and $\vec{u},\vec{v}\in\mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta\in\mathbb{R}$ and $\vec{v}\in\mathbb{V}$.
  - Closure under scalar multiplication: For any vector $\vec{v}\in\mathbb{V}$ and scalar $\alpha\in\mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.

Practice: Vector Space or Not?

- 2D plane, (two vectors spanning $\mathbb{R}^2$)
- 5D space, (five vectors spanning $\mathbb{R}^5$)
- n-D space, (n vectors spanning $\mathbb{R}^n$)
- Line in $\mathbb{R}^2$
    - Intersecting origin
    - Not intersecting origin
- First and third quadrant of $\mathbb{R}^2$
- Plane in $\mathbb{R}^3$ intersecting origin
- $\{0\}$ (Just the zero vector)
- $\{v \in \mathbb{R}^n \mid v \neq 0\}$

Practice: Vector Space or Not?

- 2D plane, (two vectors spanning $\mathbb{R}^2$) (Yes)
- 5D space, (five vectors spanning $\mathbb{R}^5$) (Yes)
- n-D space, (n vectors spanning $\mathbb{R}^n$) (Yes)
- Line in $\mathbb{R}^2$ (It depends)
    - Intersecting origin (Yes)
    - Not intersecting origin (No)
- First and third quadrant of $\mathbb{R}^2$ (No)
- Plane in $\mathbb{R}^3$ intersecting origin (Yes)
- $\{0\}$ (Just the zero vector) (Yes)
- $\{v \in \mathbb{R}^n \mid v \neq 0\}$ (No)

Subspaces

A subspace is a subset of a vector space that is itself a vector space

Suppose we have a vector space V. A subset S of V is only a subspace if the following three properties are met:

1. The zero vector of V is in S
2. S is closed under vector addition. $u + v \in S$
3. S is closed under multiplication by scalars. $cu \in S$

(Same rules as before!)

Bases

A **basis** of a vector space is a **linearly independent** set of vectors that **span** the vector space.

1. All vectors in the set are **linearly independent**
2. All vectors in the vector space can be represented as a linear combination of the basis vectors (**span**).

Bases

A basis does not necessarily have to span $\mathbb{R}^n$ - it can span any vector space.

- A basis is a **minimum set of vectors** required to completely span a vector space.
  - e.g. Any basis of $\mathbb{R}^n$ contains **exactly** n vectors. In fact, an m-dimensional vector space must have m vectors in its basis.
- Bases are NOT unique! Why? How many are there?

Nullspace

The null space of a matrix (transformation) is the set of all solutions to the homogeneous equation $Ax = 0$. It is a subspace of $\mathbb{R}^n$.

Finding a null space:

1. Reduce to reduced row-echelon form, identify free variables
2. Write out system of equations, represent solutions in matrix form
3. Write out linear combination of vectors with free variables as coefficients
4. The vectors are the basis of the null space

Example of Solving Nullspace

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \xrightarrow{\text{Row Reduce}} \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Goal is finding all x where $Ax = 0$

A matrix will have the **same nullspace** as its row reduced form. This is because row reducing is the same as manipulating equations; the solution for the row reduced form is the same as the solution for the original matrix.

Example of Solving Nullspace

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \xrightarrow{\text{Row Reduce}} \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Goal is finding all x where $Ax = 0$

Let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$

Rewrite in system of equations:

$x_1 - 2x_2 - x_4 = 0 \implies x_1 = 2x_2 + x_4$

$x_3 + 2x_4 = 0 \implies x_3 = -2x_4$

$x_5 = 0$

Let $u = x_2$, $v = x_4$. Then:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2u + v \\ u \\ -2v \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} v$$

So the null space is the span of $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \end{bmatrix}^T$ because multiplying A by any linear combination of those two vectors equals 0.

Addition to Invertibility

If a matrix has a nontrivial nullspace (does not only contain the 0 vector), then it is not invertible. (Same as saying a matrix is invertible iff its nullspace is trivial i.e. only contains the 0 vector.) Intuitive explanation: If $Av = 0$ where $v \neq 0$, then the nullspace at least contains the span of v. There are then infinite solutions to $Au = 0$. So A^{-1} does not exist.

Transition Matrices / Page Rank

Transition Matrices

A **transition matrix** represents a directed graph of states and transitions.

- Edges of the graph represent what fraction of one state moves to the next
- Examples: Social networks, PageRank
- Fractional transitions (columns sum to ≤ 1)
- Entry ij in matrix means fraction of water from node j entering node i

Transition Matrices

What do we know about the system if all columns each sum to 1?

Less than 1? (Think about what physically happens if the system was water flows)

Greater than 1?

Transition Matrices

What do we know about the system if all columns each sum to 1?

Less than 1? (Think about what physically happens if the system was water flows)

Greater than 1?

Transition Matrices

What do we know about the system if all columns each sum to 1?

Total amount of "water" is **conserved**, no water is added or lost.

Less than 1? (Think about what physically happens if the system was water flows)

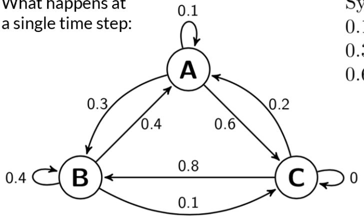
Total amount of "water" is **lost**.

Greater than 1?

Total amount of "water" **increases**.

Water Pump / Page Rank Example

What happens at a single time step:



System of equations:

$$0.1A + 0.4B + 0.2C = A$$
$$0.3A + 0.4B + 0.8C = B$$
$$0.6A + 0.1B + 0.0C = C$$

Rows: how much from each reservoir into reservoir
Cols: how much reservoir split up for the other reservoirs

Matrix representation:

How much from each reservoir goes into A →

$$\begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.8 \\ 0.6 & 0.1 & 0.0 \end{bmatrix}$$

How much reservoir A gives to each reservoir

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x}$$

Properties

- A matrix is uninvertible iff $\vec{0}$ is an eigenvalue (because there exists a vector $\vec{v}$ such that $A\vec{v} = \vec{0}$.
- A scalar times an eigenvector is still an eigenvector $(A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v}))$
- Eigenvectors with distinct eigenvalues are linearly independent (eigenvectors in the same span have the same eigenvalue)

Properties

$$A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

Quick Proof:

$A\vec{v} = \lambda\vec{v}$

$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$

$\vec{v} = \lambda A^{-1}\vec{v}$

$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$

Finding the e-vals and e-vecs

$A\vec{v} = \lambda\vec{v}$
$A\vec{v} - \lambda\vec{v} = \vec{0}$
$(A - \lambda I)\vec{v} = \vec{0}$
This means that for each eigenvalue, eigenvector $\vec{v}$ lies in the nullspace of $A - \lambda I$.
So any vector in the nullspace is valid (there are infinite eigenvectors per eigenvalue).
But how to find eigenvalues?
Remember that if $\vec{v}$ is non-trivial, $A - \lambda I$ is not invertible.
This means : $det(A - \lambda I) = 0$
So to solve for the eigenvalues, we solve this equation.

Walkthrough

Find the eigenvectors and eigenvalues of A:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det(A - \lambda I) = det(\begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix}$$

$$= (1 - \lambda)(1 - \lambda) - 2 \cdot 2 =$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$= -1, 3$$

$$\lambda_1 = -1$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = 3$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quick Aside: Change of Basis

A vector can be represented in a basis so long as it can be represented as a linear combination of all the vectors in that basis.
Normally, all vectors are represented in basis of **elementary vectors**.
In N-dimension, the elementary vectors are each column of the $N \times N$ identity matrix.

Ex: $$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

Quick Aside: Change of Basis

General change of basis in basis V:

x_v = representation of x in the basis of V

$a_1, a_2, \cdots a_n$ = coordinates of x in the basis of V, aka how much each vector in V composes x.

$$V = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$x = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}$$

$$x_v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$x = V x_v \text{ or } x_v = V^{-1} x$$

Diagonalization

$$A = V\Lambda V^{-1}$$

Exists if A is an $N \times N$ square matrix with N distinct eigenvalues.
What each matrix does:

1. Brings x into the basis of V (reminder: $x_v = V^{-1}x$)
2. Scales the components of x_v by the eigenvalues (first eigenvalue scales the first component of x_v, etc.)
3. Brings back into normal basis (reminder: $x = Vx_v$)

*reminder that the rightmost matrix is applied first

Diagonalization

Left: direct result of Ax

Right: result of $V\Lambda V^{-1}x$

Conclusion: equivalent

$$x = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \cdots + \alpha_n\vec{v_n}$$
$$Ax = A(\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \cdots + \alpha_n\vec{v_n})$$
$$= \alpha_1 A\vec{v_1} + \alpha_2 A\vec{v_2} + \cdots + \alpha_n A\vec{v_n}$$
$$= \alpha_1\lambda_1\vec{v_1} + \alpha_2\lambda_2\vec{v_2} + \cdots + \alpha_n\lambda_n\vec{v_n}$$

Recall that:

$$V^{-1}x = x_v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

So:

$$V\Lambda V^{-1}x = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1\alpha_1 \\ \lambda_2\alpha_2 \\ \vdots \\ \lambda_n\alpha_n \end{bmatrix}$$

$$= \alpha_1\lambda_1\vec{v_1} + \alpha_2\lambda_2\vec{v_2} + \cdots + \alpha_n\lambda_n\vec{v_n}$$

Example of Diagonalization

$$\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} = V \Lambda V^{-1}$$

Step 1: Find eigenvalues, aka solving $det(A - \lambda I) = 0$

$det(A - \lambda I) = det(\begin{bmatrix} 3 - \lambda & 3 \\ 1 & 5 - \lambda \end{bmatrix}$

$= (3 - \lambda)(5 - \lambda) - (3)(1)$

$= \lambda^2 - 8\lambda + 12$

$= (\lambda - 2)(\lambda - 6)$

So we get $\lambda_1 = 2, \lambda_2 = 6$, and $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

Note: swapping the order of the lambdas is totally valid.

Example of Diagonalization

Step 2: Find corresponding eigenvectors, aka solving $A - \lambda I = 0$ for each λ

$$A - \lambda_1 I = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

$$\vec{v_1} = (A - \lambda_1 I) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\vec{v_2} = (A - \lambda_2 I) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

Note: swapping the order of the vectors is also valid, as long as they pair with the corresponding lambda in the Lambda matrix

Example of Diagonalization

Step 3: Finding the inverse of V

$$V^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{(3)(1)-(1)(-1)} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Result : $\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

Steady State Values

Revisiting transition matrices, ex. PageRank, how do we know what an initial state might converge to?

Easy to tell what happens to a vector in its eigenbasis!

$x[t] = A^t x[0]$

$A^t = (V\Lambda V^{-1})(V\Lambda V^{-1})\cdots = V\Lambda^t V^{-1}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \\ & & & \lambda_n \end{bmatrix}, \Lambda^t = \begin{bmatrix} \lambda_1^t & 0 & 0 & \cdots \\ 0 & \lambda_2^t & 0 & \cdots \\ \vdots & \vdots & \ddots & \\ & & & \lambda_n^t \end{bmatrix}$$

Steady State Values

- If $|\lambda| < 1$ exists, then $x[t] = 0$ is possible *given that* $x[0]$ *consists of only eigenvectors with eigenvalues of magnitude less than one*
- If $|\lambda| = 1$ exists, then $x[t] = 0$ is possible *given that* $x[0]$ *consists of only eigenvectors with eigenvalues equal to one* *(*$\lambda = -1$ *oscillates)*
- If $|\lambda| > 1$ exists, then $x[t] = \inf$ is possible *given that* $x[0]$ *consists of only eigenvectors with eigenvalues of magnitude greater than one*

Thanks for coming!

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