EECS 16A Midterm 1 Review Session

Presented by <NAMES >(HKN)

Disclaimer

Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are posted at — on Piazza.

HKN Drop-In Tutoring

- These details should be edited

Systems of Equations and Gaussian Elimination

Vectors

Conceptually, a vector is a collection of numbers that each represent a variable. If there are n variables, then the vector is n-dimensional.

Example:

A point in 3D can be represented as (x, y, z) In vector form, this would be represented as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Matrices

- Collection of **vectors**
- 2D table for **storing data**
  - Systems of equations for imaging observations
- Notable/useful matrices
  - Identity matrix
  - Augmented matrices
  - Rotation matrix
  - Many others!

Augmented: $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -3 & 2 & 3-2 & 10 \end{array}\right]$

Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrix Transformations

Matrices are often used to perform transformations, especially in $\mathbb{R}^2$

Two important transformations:

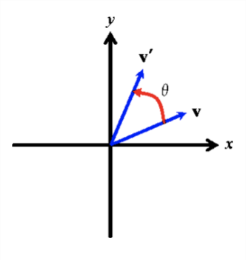

Figure 1: Rotation

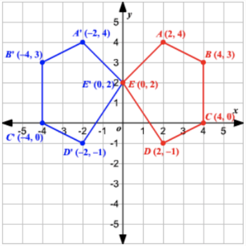

Figure 2: Reflection

Rotation Matrix

The rotation matrix rotates points by a specified angle, theta:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Use this matrix by plugging in desired rotation angle, then multiply with vector.

Note: Rotation matrices also preserve the length of a vector.

Example: Rotation Matrix that rotates vector by 90°

$$R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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Reflection Matrix

The reflection matrix reflects vectors across a line (Notice that such matrix also preserves the length of a vector)

Notable reflection matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection across x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

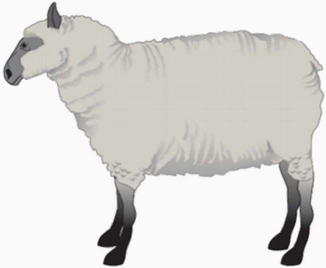
Reflection across y-axis

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflection across $y = x$

Matrix Transformations

All linear transformations can be expressed as a matrix



sheep

sheared sheep

Example 1

Question

What is the resulting vector after the following (non-subsequent) transformations are applied to the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

1. Rotate by 45°
2. Reflect across $y = x$

Example 1

Solution

What is the resulting vector after the following (non-subsequent) transformations are applied to the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

1. Rotate by 45°

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix}$$

2. Reflect across $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Determinants

Determinant of a 2x2 matrix:

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Also, an upper triangular matrix's determinant is the product of the diagonal:

$$det\left(\begin{bmatrix} a & * & * & \cdots \\ 0 & b & * & \cdots \\ 0 & 0 & c & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\right) = abc\cdots$$

Gaussian Elimination

Ultimate goal: Upper triangular form (**i.e** numbers below the diagonal are all 0)

Reminder of Motivation: Can work starting from the bottom row up in order to quickly calculate (from a computer's perspective) each variable's value.
For example, the last row has one variable and one value.

Gaussian Elimination

IDEA: Augmented matrix represents a system of equations where each row is an equation.
What are you allowed to do with equations to solve them?

1. **Row exchange** (same equations, different order)
2. **Scaling** (multiplying an equation by a scalar)
3. **Replace a row with the a linear combination of itself and another row** (intuitively, must include itself because otherwise there is "loss of information", like a deletion of one of the original rows)

GOAL:
Original Matrix $\rightarrow$ Row Echelon Form $\rightarrow$ Reduced Row Echelon Form

Problem

Use Gaussian Elimination to reduce the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

Gaussian Elimination Example

Solution

Use Gaussian Elimination to reduce the following system of equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \rightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow{R_2 - R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row implies 0 of three variables add up to -1. Therefore, there exists no solution.

Possible Outcomes from Gaussian Elimination

Possible Results	Row Picture	Column Picture	Properties of Matrix A
Unique solution	Equations intersect at exactly one point	**b** can be uniquely represented by the linear combination of the columns of **A**	**A** is invertible
Infinite solutions	Equations intersect along an infinite space (eg. the intersection is a line or a plane)	There are multiple ways of representing **b** in terms of the linear combination of the columns of **A**	**A** has linearly dependent columns
No solutions	Equations do not intersect	**b** is not in the span of the columns of **A**; **b** is not in the column space of **A**	Column space of **A** does not contain (span) the vector **b**