EECS 16B Midterm 1 Review Session

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Disclaimer

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Slides are also posted at @775 on Piazza.

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- HKN has office hours Monday through Friday from **1 PM - 5 PM** and **9 PM - 10 PM** on hkn.mu/ohqueue and in our Soda 345 and Cory 290 offices.
- The schedule of tutors can be found at hkn.mu/tutor

Differential Equations

Solving Differential Equations

Differentiation is linear!

$$\frac{d(c_1 x(t) + c_2 y(t))}{dt} = c_1 \frac{dx(t)}{dt} + c_2 \frac{dy(t)}{dt}$$

Solving systems of differential equations

Write the system in the following form:

$$\frac{dx(t)}{dt} = Ax(t)$$

Find the eigenvalues/eigenvectors of A and transform the system into the eigenbasis. If A has distinct eigenvalues, the general solution is:

$$x(t) = c_1 e^{\lambda_1 t} \vec{v_1} + c_2 e^{\lambda_2 t} \vec{v_2}$$

Then, change back to the standard basis.

Example: second-order differential equations

Solve the following differential equation:

$$\ddot{x} - \dot{x} - 2x = 0$$

$$x(0) = 2, \dot{x}(0) = 1$$

Example: second-order differential equations

$\ddot{x} - \dot{x} - 2x = 0,\ x(0) = 2, \dot{x}(0) = 1$

Solution:

Write in matrix form and find eigenvectors:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\lambda_1 = 2, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_2 = -1, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Use general form to solve:

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = c_1 e^{2t} v_1 + c_2 e^{-t} v_2$$

Example: second-order differential equations

$\ddot{x} - \dot{x} - 2x = 0,\ x(0) = 2, \dot{x}(0) = 1$

Solution:

Now plug in initial conditions:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = c_1 v_1 + c_2 v_2 = \begin{bmatrix} c_1 - c_2 \\ 2c_1 + c_2 \end{bmatrix}$$

Solving, we get $c_1 = 1, c_2 = -1$

so $x(t) = e^{2t} + e^{-t}$

Example: second-order differential equations

$\ddot{x} - \dot{x} - 2x = 0,\ x(0) = 2, \dot{x}(0) = 1$

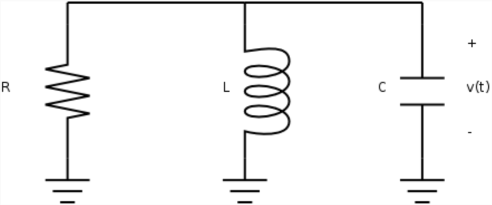
Solution:

Sanity check: compute derivatives and check that the original system is satisfied

$x(t) = e^{2t} + e^{-t}, \dot{x}(t) = 2e^{2t} - e^{-t}, \ddot{x}(t) = 4e^{2t} + e^{-t}$

$\implies \ddot{x} - \dot{x} - 2x = 0, x(0) = 2, \dot{x}(0) = 1$

Example: second-order differential equations

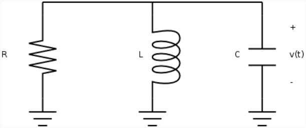


Given $x(t) = \begin{bmatrix} v(t) \\ i_L(t) \end{bmatrix}$

find matrix A such that $\frac{dx}{dt} = Ax$

Example: second-order differential equations

Direct all currents into ground.



$$i_R + i_L + i_C = 0$$

$$\frac{V}{R} + i_L = -C\frac{dV}{dt}$$

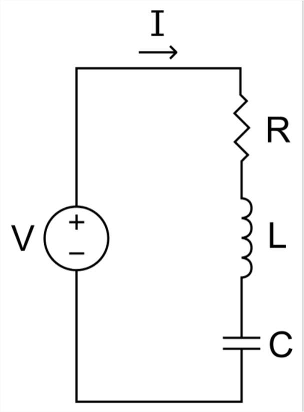
$$-\frac{1}{CR}V + -\frac{1}{C}i_L = \frac{dV}{dt}$$

Also

$L\frac{di_L}{dt} = V \implies A = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$

Example: second-order differential equations

At $t < 0$, the circuit is at steady state. At $t \geq 0$, the voltage source is set to 0. Find a differential equation for i_L for $t \geq 0$.



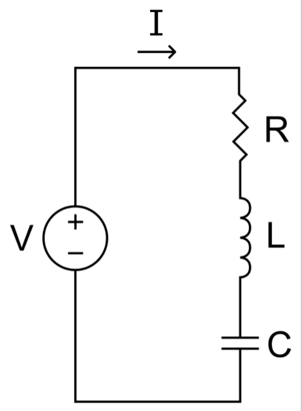
Example: second-order differential equations

Solution: use KCL/KVL to get

$$i_R = i_L = i_C, V_C + V_R + V_L = 0$$

$$\frac{V_R}{R} = i_L = C\frac{dV_C}{dt}, \frac{1}{C}i_L + R\frac{di_L}{dt} + L\frac{d^2 i_L}{dt} = 0$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{d^2 i_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} i_L \\ \frac{di_L}{dt} \end{bmatrix}$$



Example: second-order differential equations

If $R = 0$, $L = C = 1$, solve the differential equation:

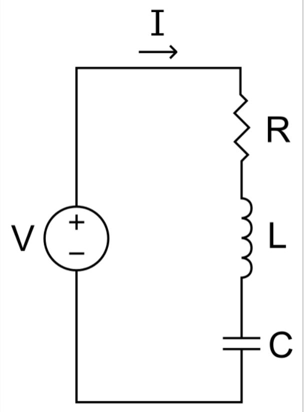
Initial conditions: $i_L(0) = 0$, $\frac{di_L}{dt}(0) = -V_c = -V$

Differential equation:

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{d^2 i_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} i_L \\ \frac{di_L}{dt} \end{bmatrix}$$

$$\lambda_1 = i, v_1 = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

$$\lambda_2 = -i, v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$



Example: second-order differential equations

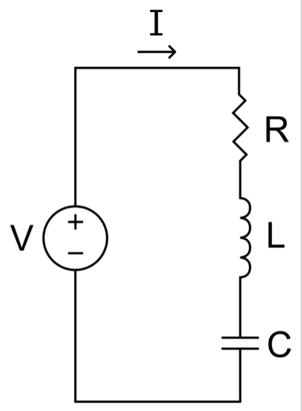
$$\begin{bmatrix} i_L \\ \frac{di_L}{dt} \end{bmatrix} = c_1 e^{it} v_1 + c_2 e^{-it} v_2$$

$$ic_1 + ic_2 = 0, -c_1 + c_2 = -V$$

$$\implies c_1 = \frac{V}{2}, c_2 = -\frac{V}{2}$$

so

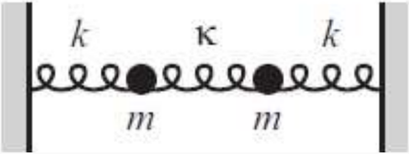
$$i_L = \frac{V}{2} i e^{it} - \frac{V}{2} e^{-it}$$



Example: 2 coupled diff eqs

Consider a system of 2 coupled harmonic oscillators, described by

$$m_i \ddot{x}_i = -(k + \kappa) x_i + \kappa x_j$$



Example: Coupled Oscillator

Write this system in matrix form

$$m_i\ddot{x}_i = -(k+\kappa)x_i + \kappa x_j$$

Example: Coupled Oscillator

Write this system in matrix form

$$m_i \ddot{x}_i = -(k+\kappa)x_i + \kappa x_j$$

$$\ddot{\vec{x}} = \begin{bmatrix} -\frac{k+\kappa}{m} & \frac{\kappa}{m} \\ \frac{\kappa}{m} & -\frac{k+\kappa}{m} \end{bmatrix} \vec{x}$$

Example: Coupled Oscillator

Now, diagonalize the system

$$\begin{bmatrix} -\frac{k+\kappa}{m} & \frac{\kappa}{m} \\ \frac{\kappa}{m} & -\frac{k+\kappa}{m} \end{bmatrix} = PDP'$$

P = ?

D = ?

Example: Coupled Oscillator

Now, diagonalize the system

$$\begin{bmatrix} -\frac{k+\kappa}{m} & \frac{\kappa}{m} \\ \frac{\kappa}{m} & -\frac{k+\kappa}{m} \end{bmatrix} = PDP'$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} k & 0 \\ 0 & 2\kappa + k \end{bmatrix}$$

Example: Coupled Oscillator

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} k & 0 \\ 0 & 2\kappa + k \end{bmatrix}$$

Since our original matrix was linearly taking a SECOND derivative in time, with only real eigenvalues, we expect solutions to be in the form of real sinusoids, optionally phaseshifted.

$$x(t) = A_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\sqrt{\frac{k}{m}} t + \phi_s) + A_f \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin(\sqrt{\frac{k + 2\kappa}{m}} t + \phi_f)$$

Change of Basis

Change of Basis

In the standard basis, we write vectors as a linear combination of the standard basis vectors.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Likewise, we can write our x vector as a linear combination of some other basis vectors. For example, if our V-basis has basis vectors $\vec{v_1}$ and $\vec{v_2}$:

$$\vec{x} = \tilde{x_1} \cdot \vec{v_1} + \tilde{x_2} \cdot \vec{v_2} = V\vec{\tilde{x}}$$

We can go from the V-basis to the standard basis by applying the V-matrix to x, and go from the standard basis to the V-basis by applying V^{-1}.

Change of Basis Diagram

- When you are doing change of basis for systems of equations, it is useful to use a diagram mapping transformations between basis
- Given T, to find T_a you would trace the path from u_a to v_a, applying each transformation to the left of the previous:
  $A \implies TA \implies A^{-1}TA$
  $T_a = A^{-1}TA$
- Step by step, we have
  $Au_a = u \implies TAu_a = v \implies A^{-1}TAu_a = v_a$

$$
\begin{tikzcd}
\vec{u} \arrow[r, mapsto, "T"] \arrow[d, shift right=2, "A^{-1}"'] & \vec{v} \arrow[d, shift left=2, "A^{-1}"] \\
\vec{u}_a \arrow[u, mapsto, shift right=2, "A"'] \arrow[r, mapsto, "T_a"'] & \vec{v}_a \arrow[u, mapsto, shift left=2, "A"]
\end{tikzcd}
$$

Diagonalization

Diagonalization

- The idea is that we want to change into a basis in which the system $A\vec{x} = \vec{y}$ is represented by a diagonal matrix. So how do we find such basis?
- Remember that for all eigenvalue-eigenvector pairs we have: $A\vec{v} = \lambda\vec{v}$
- Let's use our eigenvectors as our basis. Doing so we obtain:

$$\vec{x} = V\tilde{\vec{x}}, \vec{y} = V\tilde{\vec{y}}, \text{and} \Lambda\tilde{\vec{x}} = \tilde{\vec{y}}$$

  Where the upper-case lambda represents the diagonal matrix with eigenvalues on the diagonals.
- Transforming back to the standard basis, we get:

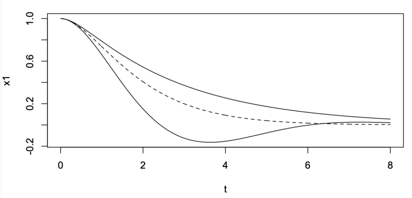
$$A = V\Lambda V^{-1}$$

Diagonalization Cont.

- Let's analyze this a bit further, why is it important to be able to do this?
- We see that diagonalizing the matrix makes the system much easier to solve, why?
- Also, we see that there is a "home state" for every system of linearly independent equations, i.e. the space in which the system's components are uncoupled.

What do we do with different types of eigenvalues?

Each interesting "case" of eigenvalues (real, imaginary, and repeated) is complementary to a case of resonance in 2d ODEs, so let's discuss them together:



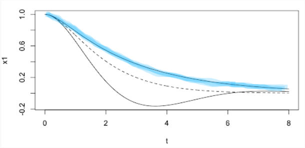
Overdamping: two real eigenvalues

- When a system has two real eigenvalues, it will have two real eigenvectors, so the solution will be a combination of exponentials in the form

$$ae^{\lambda_1 t} + be^{\lambda_2 t}$$

- For negative lambdas, this produces exponential decay.

At large t, greater eigenvalue dominates behavior, so how fast the system approaches to 0 is determined by the larger eigenvalue.



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Underdamping: imaginary or complex eigenvalues

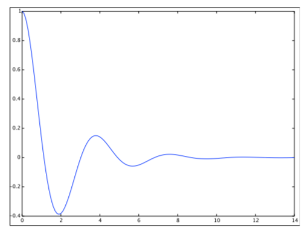
- You can still solve the system of differential equations as usual, but your solutions will be complex:

$$(a_1 j + b_1)e^{c_1 j + d_1} + (a_2 j + b_2)e^{c_2 j + d_2}$$

- You can rearrange terms and apply Euler's formula to write each term as a real exponential multiplied by a sinusoid

$$e^{d_1}(\alpha_1 sin(...) + \beta_1 cos(...)) + e^{d_2}(\alpha_2 sin(...) + \beta_2 cos(...))$$

If d_1 and d_2 are negative, then you can think of this as a sinusoid where the amplitude decays to 0



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Critical Damping: repeated eigenvalue

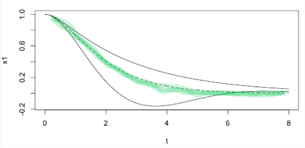
- You only get one linearly independent eigenvector.
- So, arbitrarily choose the second column of your V matrix (making sure it is linearly independent from the eigenvector v_1)
- Compute A_v using the change of basis diagram $A_v = V^{-1}AV$
- This gives you an upper-triangular matrix
- The bottom row of the matrix gives you an equation in one variable (only depends on the second component of x_v), which you can solve to get something in the form of $x_{v,2} = ae^{\lambda t}$

Critical Damping: repeated eigenvalue

- The top row of the matrix depends on both components of x_v, but you can plug in the value you got for $x_{v,2}$ and then solve for $x_{v,1}$
- If you solve this differential equation (which is a diff eq with a non-constant input*), you'll get an equation in the form:

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

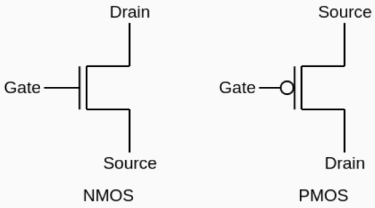
The critically damped case gives us the fastest-decaying exponential that doesn't oscillate



*You can solve this using the formula $x_p(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau$

CMOS Transistors and Logic

Transistors



Two varieties of MOSFETs: P-type and N-type

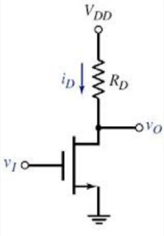
Any MOSFET has a characteristic threshold voltage V_{th}

NMOS "turns on" (connects drain to source) when $V_{GS} > V_{th}$

PMOS turns on when $V_{GS} < V_{th}$

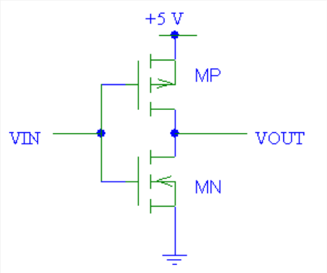
NMOS Logic

We can build an inverter (a circuit that flips a 1 to a 0 and vice versa) with a single N-type MOSFET!



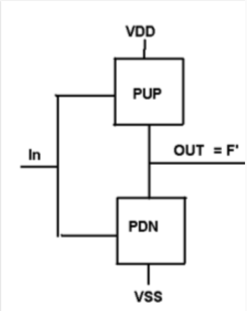
What is v_O when $v_I > V_{th}$? When $v_I < V_{th}$?
Key disadvantage: what is the power dissipated when $v_I > V_{th}$?
How can we rectify this?

CMOS Logic



Now, we have the same logical function as the NMOS inverter, but we're using more transistors and much less power is dissipated. Why?

CMOS Logic



This is broadly called Complementary Metal Oxide Semiconductor (CMOS) logic, using a Pull-Up Network of P-type and a Pull-Down Network of N-type MOSFETs.
Now we can build circuits that perform logical functions out of MOSFETs!

Transistors & CMOS Wrap-up

True or False:

1. Power (in the EE16B model) is dissipated in a CMOS circuit only when there is switching
2. NMOS devices turn on with a large VGS and off with low VGS
3. For a given choice of inputs in a CMOS inverter circuit, we can create a low resistance path from ground to high voltage

Transistors & CMOS Wrap-up

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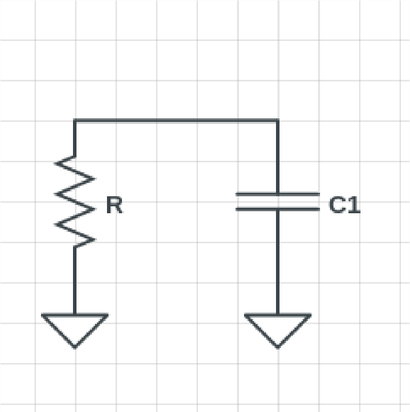
   True
3. For a given choice of inputs in a CMOS inverter circuit, we can create a low resistance path from ground to high voltage
   False

RC Circuits

RC Circuits

The capacitor and resistor in a NOT circuit form the most basic RC circuit:

Write down a differential equation describing the circuit below:



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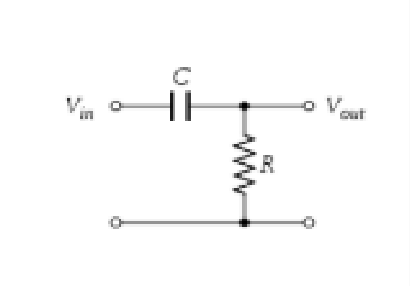

Solving the RC differential Equation

We have a differential equation describing V_{out} in terms of V_c.

$$\frac{dV_c}{dt} = -\frac{1}{RC} V_c$$

How do we actually solve it?

Solving the RC differential Equation

We have a differential equation describing V_{out} in terms of V_c.

$$\frac{dV_c}{dt} = -\frac{1}{RC} V_c$$

How do we actually solve it?

Think of the differential operation as a linear operator that scales V_c, since V_c is one of its eigenfunctions:

$$[\frac{d}{dt}]V_c = \lambda V_c$$

Solving the RC differential Equation

Which are the eigenfunctions of differentiation?

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$$Ae^{\lambda t}$$

Solving the RC differential Equation

Which are the eigenfunctions of differentiation?

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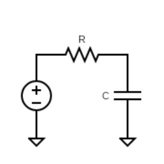
$$\frac{dV_c}{dt} = -\frac{1}{RC} V_c$$

The solution to our first order differential equations is therefore:

$$V_c(t) = V_c(0) e^{\frac{-1}{RC} t}$$

RC Differential Equation: Non-homogenous case

How do you solve a RC circuit with a voltage source?



Applying KCL at the top right node, along with Ohm's law and the capacitor relationship, we get:

$$\frac{dV_c}{dt} = \frac{1}{VC}(V_s - V_c)$$

We can't easily solve this equation, so we change variables to

$$x = V_c - V_s$$

RC Differential Equation: Non-homogenous case

Now, we have $\frac{dx}{dt} = -\frac{x}{RC}$

We already know how to solve this differential equation, and we get

$$x(t) = x_0 e^{-\frac{t}{RC}}$$

Finally, change back to the original variables by substituting V_c - V_s for x

$$V_c(t) = V_c(0)e^{-\frac{t}{RC}} + V_s(1 - e^{-\frac{t}{RC}})$$

Break Time!

Phasors

Phasors

- Phasors express the response of a circuit to a sinusoidal input
- Any real, periodic signal can be expresses as the sum of sinusoids - we can apply this technique very broadly!
- A phasor encodes information about amplitude and phase, but not frequency

$$v_s = A\cos(\omega t + \phi) \rightarrow \tilde{V}_s = \frac{1}{2}Ae^{j\phi}$$

Impedance

- Impedance (Z) is a generalized form of resistance - expresses a component's response to an alternating current
- Each common passive circuit element (resistor, capacitor, and inductor) has a characteristic impedance

$$Z_R = R$$

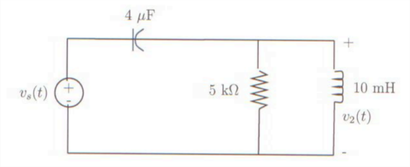
$$Z_C = \frac{1}{j\omega C}$$

$$Z_L = j\omega L$$

Phasor Analysis Procedure

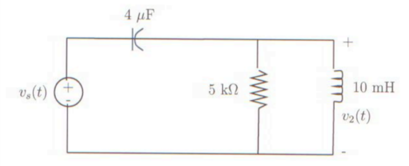
1. Express all time domain signals as cosines
2. Convert voltages, currents, and impedances to their phasor equivalents
3. Set up phasor equations and solve for unknowns
4. Transform back to time domain

Phasor Practice



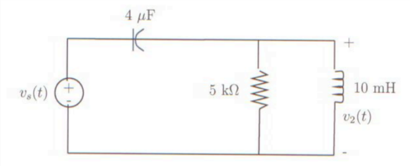
Let $v_s(t) = 5\cos(5000t + \frac{\pi}{4})$. Find $v_2(t)$.

Phasor Practice



1. Translate to phasor domain
   - $\omega = 5000$
   - $v_s(t) = 5\cos(5000t + \frac{\pi}{4}) \rightarrow \tilde{V}_s = 2.5e^{j\frac{\pi}{4}}$
   - $C = 4\mu F \rightarrow Z_C = \frac{1}{j(4*10^{-6})(5000)} = -j50$
   - $R = 5k\Omega \rightarrow Z_R = 5 * 10^3$
   - $L = 10mH \rightarrow Z_L = j(5000)(10 * 10^{-3}) = j50$

Phasor Practice

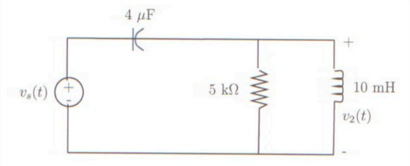


2. Solve for $\tilde{V}_2$
    - Set up as impedance divider:

$$\frac{\tilde{V}_s}{Z_C + (Z_R \| Z_L)} = \frac{\tilde{V}_2}{Z_R \| Z_L}$$

$$\tilde{V}_2 = \tilde{V}_s \frac{Z_R \| Z_L}{Z_C + (Z_R \| Z_L)} = \tilde{V}_s \frac{\frac{Z_R Z_L}{Z_R + Z_L}}{\frac{Z_R Z_L + Z_R Z_C + Z_L Z_C}{Z_R + Z_L}}$$

Phasor Practice

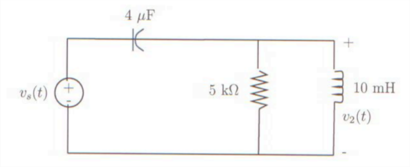


2. Solve for $\tilde{V}_2$

$$\tilde{V}_2 = \tilde{V}_s \frac{Z_R Z_L}{Z_R Z_L + Z_R Z_C + Z_L Z_C} = \tilde{V}_s \frac{j2.5 * 10^5}{j2.5 * 10^5 - j2.5 * 10^5 + 2500}$$

$$\tilde{V}_2 = j100\tilde{V}_s = 100\tilde{V}_s e^{j\frac{\pi}{2}}$$

Phasor Practice



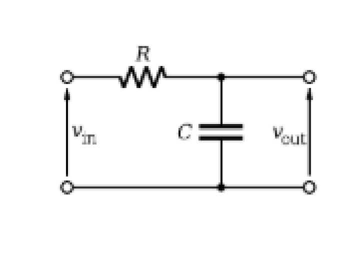
3. Find time-domain solution
   - $\tilde{V}_2 = 100e^{j\frac{\pi}{2}}\tilde{V}_s \rightarrow v_2(t) = 100v_s(t+\frac{\pi}{2}) = 500\cos(5000t+\frac{3\pi}{4})$

Transfer Function Review

A **transfer function** of a circuit or system describes the output response to an input excitation as a function of the angular frequency ω.

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)}$$

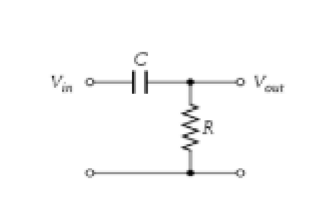
RC Circuits



$H(\omega) = \frac{1}{1+j\omega RC}$

Low pass:

$\lim_{\omega\to\infty} H(\omega) = 0, H(0) = 1$



$H(\omega) = \frac{j\omega RC}{1+j\omega RC}$

High pass:

$\lim_{\omega\to\infty} H(\omega) = 1, H(0) = 0$

Bode Plot Review

$V[dB] = 20\log_{10}(\frac{V}{V_0})$

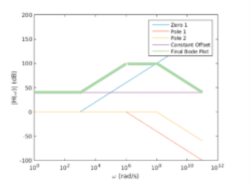
- Bode plots provide a way for us to easily visualize the output response of our system, depending on the input frequency ω.
- Because Bode plots are in log scale for ω, we are able to take advantage of the properties of logs.
    - $G = XY \implies G[dB] = X[dB] + Y[dB]$
    - $G = \frac{X}{Y} \implies G[dB] = X[dB] - Y[dB]$

Hence, we can break our transfer function H(ω) into a product of familiar transfer functions (simple poles, quadratic zeros, etc. - "functional forms"), graph them out individually, and then add them together on the graph.

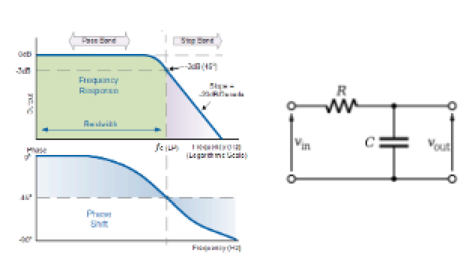
Bode Plot Review

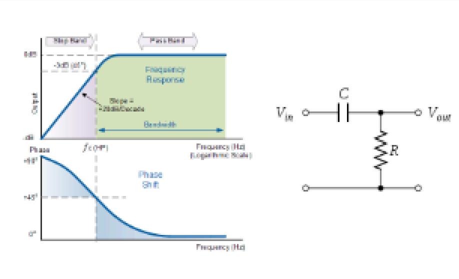
- We want to be able to plot the transfer function as a function of the frequency ω.
- However, it's a complex valued function so it's easier to have a plot for the magnitude and one for the phase.
- Because the magnitude plot is in log scale, and the phase plot is in semi-log scale we are able to take advantage of the properties of logs.
  - $\log(XY) = \log(X) + \log(Y)$
- Hence, we can break our transfer function H(ω) into a product of familiar transfer functions (simple poles, quadratic zeros, etc. - "functional forms"), graph them out individually, and then add them together on the graph.

Bode Plot Steps

1. Break transfer function into product of functional forms
   1.1 $H(\omega) = \frac{n(\omega)}{d(\omega)} = \frac{(j\omega)^n \alpha_n + (j\omega)^{n-1}\alpha_{n-1} + \ldots + j\omega\alpha_1 + \alpha_0}{(j\omega)^n \beta_n + (j\omega)^{n-1}\beta_{n-1} + \ldots + j\omega\beta_1 + \beta_0} = \kappa \frac{(j\frac{\omega}{\omega_{z1}}+1)(j\frac{\omega}{\omega_{z2}}+1)\ldots}{(j\frac{\omega}{\omega_{p1}}+1)(j\frac{\omega}{\omega_{p2}}+1)\ldots}$
2. Graph each transfer function individually
3. Add them up on the graph (thanks to log scale)
   3.1 

RC Circuits, revisited

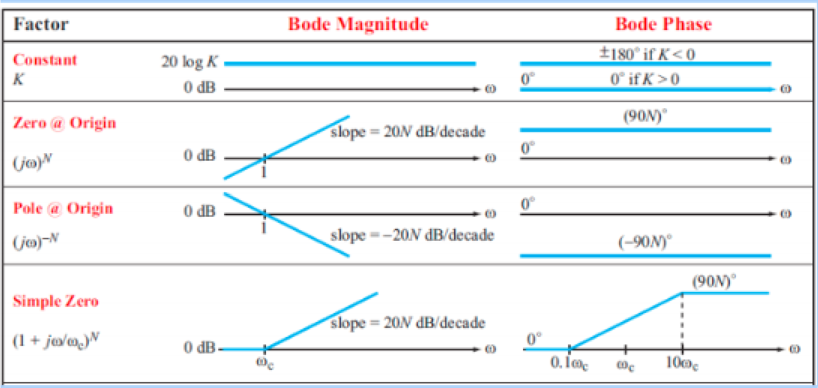




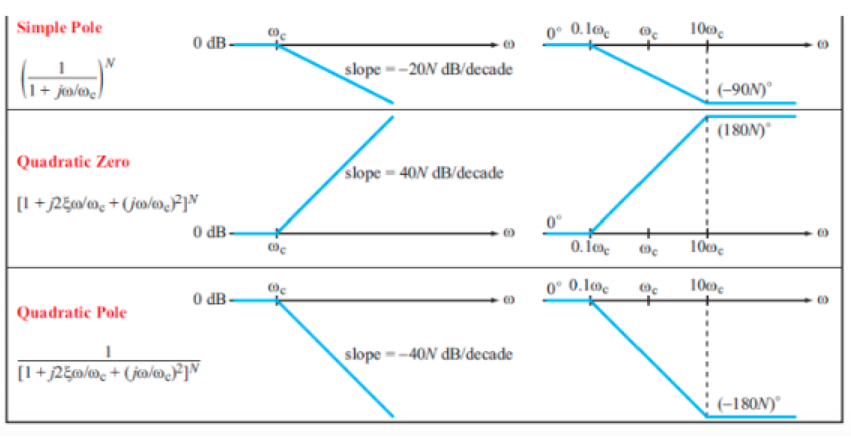
$$H(\omega) = \frac{1}{1+j\omega RC}$$

$$H(\omega) = \frac{j\omega RC}{1+j\omega RC}$$

Important Functional Forms



Important Functional Forms



Practice Problem

$$H(\omega) = \frac{\frac{j\omega}{10} + 100}{1 + \frac{(j\omega)^2}{10^14} + \frac{j\omega}{10^8} + \frac{j\omega}{10^6}}$$

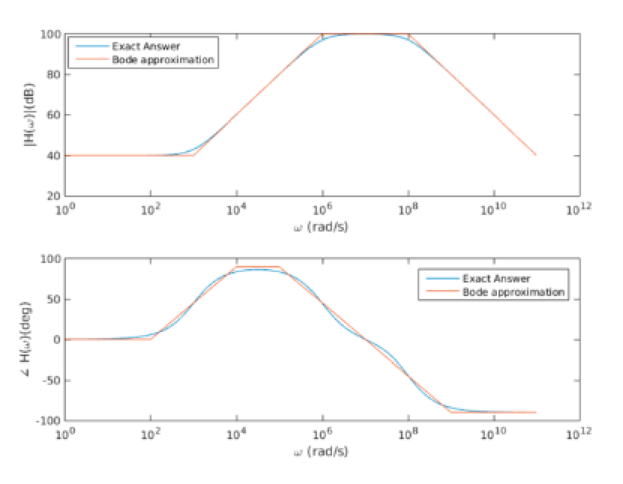
Draw the Bode plot for this transfer function.

$$H(\omega) = \frac{\frac{j\omega}{10} + 100}{1 + \frac{(j\omega)^2}{10^1 4} + \frac{j\omega}{10^8} + \frac{j\omega}{10^6}}$$

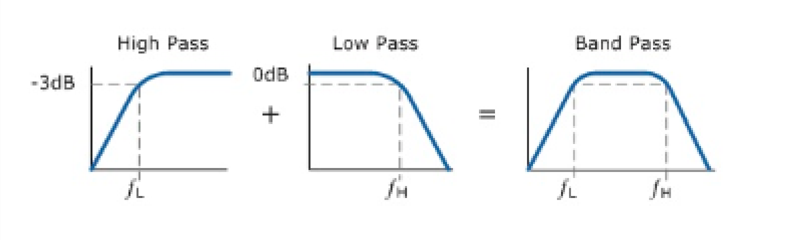
$$= \frac{100(\frac{j\omega}{10^3} + 1)}{(\frac{j\omega}{10^6} + 1)(\frac{j\omega}{10^8} + 1)}$$

Main idea: break our transfer function into the product of standard forms- a constant, one zero, and two poles

Final Result



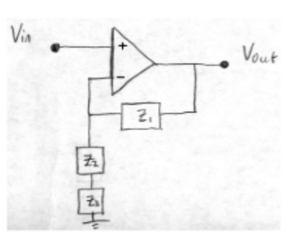
Common Filters and their Bode Plots



In the high pass filter, the frequencies greater than the corner frequency have a gain of 0dB (so they aren't changed), while frequencies less than the corner frequency have a gain ¡ 0dB (so they are multiplied by something less than 1).
In the lowpass filter, the opposite is true.

Design Problem

Design an active high pass filter with cutoff frequency of 1KHz and a gain of 1000.



- Hint 1: Find the transfer function of the following circuit.
- Hint 2: Which transfer function is a high pass filter?
  - $H_1(\omega) = 1 + \frac{j\omega R_1 C}{1+j\omega R_2 C}$
  - $H_2(\omega) = 1 + \frac{R_1 R_2}{1+j\omega R_1 C}$
- Hint 3: What is the impedance of a resistor in series with a capacitor?
- Hint 4: Replace the Z's with resistors/capacitors/wires/open

Design Problem: Hint 1

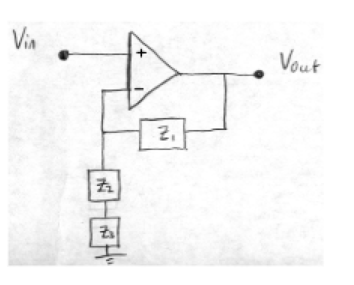
- Step 1: Apply the golden rules of Op. Amps. This means that $V^+ = V^-$.
- Step 2: By ohm's law, we can calculate the current from V- to ground.
  - $I = \frac{V_{in}}{Z_2 + Z_3}$
- Step 3: By Ohm's law, we can calculate the output voltage.
  - $V_{out} = V_{in}(1 + \frac{Z_1}{Z_2 + Z_3})$

Design Problem: Hint 2 + Hint 3

- The high pass filter transfer function is: $H_1(\omega) = 1 + \frac{j\omega R_1 C}{1+j\omega R_2 C}$
- To more easily separate the gain from the frequency filtering, we can approximate it as (at high frequencies): $H_1(\omega) = \frac{R_1}{R_2}(\frac{j\omega R_2 C}{1+j\omega R_2 C})$
- A resistor in series with a capacitor has impedances of: $R + \frac{1}{j\omega C} = \frac{j\omega RC+1}{j\omega C}$

Design Problem: Summary of Hints

Now, use all of the earlier hints to get a high pass filter!
(Cutoff Frequency is 1KHz, gain of 1000)



$$V_{out} = V_{in}(1 + \frac{Z_1}{Z_2 + Z_3})$$

$$H_1(\omega) = 1 + \frac{j\omega R_1 C}{1 + j\omega R_2 C}$$

$$R + \frac{1}{j\omega C} = \frac{j\omega RC + 1}{j\omega C}$$

Design Problem Solution

We implement the circuit such that it has the transfer function of :

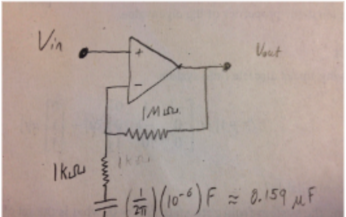
$$H_1(\omega) = 1 + \frac{j\omega R_1 C}{1 + j\omega R_2 C}$$

To meet our specifications:

$$\frac{R_1}{R_2} = 1000$$

$$R_2 C = \frac{1}{2\pi(1000)}$$

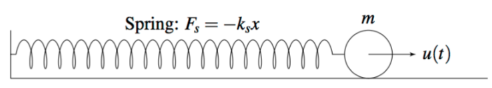
One Possible Solution:



State-Space Representations

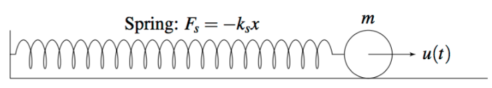
State Space Modeling: Example

Assume we have the following spring system:



State Space Modeling: Example

Assume we have the following spring system:



We can model the system as a linear continuous time state space model:

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$

$\vec{y}(t) = C\vec{x}(t)$

in which:

$\vec{x}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

State Space Modeling:

$$\begin{tikzcd} \text{original space:} & \vec{x}(t) \arrow[r, "A"] \arrow[d, bend right, "T^{-1}"'] & \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) \arrow[d, bend right, "T^{-1}"'] \\ \text{new space:} & \tilde{\vec{x}}(t) \arrow[r, "\tilde{A}"] \arrow[u, bend right, "T"'] & \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) \arrow[u, bend right, "T"'] \end{tikzcd}$$

Always apply operations to right side first

$$\tilde{A} = T^{-1}AT$$

$$\tilde{\vec{x}}(t) = T^{-1}\vec{x}(t)$$

State Space Modeling Procedure:

1. Set up differential equation of the form:
   $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$

State Space Modeling Procedure:

1. Set up differential equation of the form: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$
2. Find λ_i of A; let $\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

State Space Modeling Procedure:

1. Set up differential equation of the form:
   $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$
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3. Find eigenvectors $\vec{v_i}$ of A; let $T = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$

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4. Convert $\vec{x}(t)$ to $\tilde{\vec{x}}(t)$ using: $\tilde{\vec{x}}(t) = T^{-1}\vec{x}(t)$

State Space Modeling Procedure:

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5. Solve $\frac{d}{dt}\tilde{\vec{x}}(t) = \tilde{A}\tilde{\vec{x}}(t) + T\vec{b}u(t)$

State Space Modeling Procedure:

1. Set up differential equation of the form:
   $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$
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3. Find eigenvectors $\vec{v_i}$ of A; let $T = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$
4. Convert $\vec{x}(t)$ to $\tilde{\vec{x}}(t)$ using: $\tilde{\vec{x}}(t) = T^{-1}\vec{x}(t)$
5. Solve $\frac{d}{dt}\tilde{\vec{x}}(t) = \tilde{A}\tilde{\vec{x}}(t) + T\vec{b}u(t)$
6. Convert solution back to $\vec{x}(t)$

State Space Modeling:

Continuous time solution:

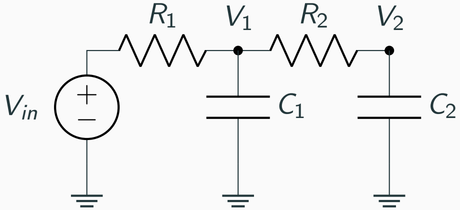
$$\tilde{x}(t) = e^{\lambda t}\tilde{x}(0) + \frac{e^{\lambda t} - 1}{\lambda} u(t) + w(t)$$

Discrete time solution:

$$x_d(i+1) = e^{\lambda \Delta} x_d(i) + \frac{e^{\lambda \Delta} - 1}{\lambda} u(i) + w(i)$$

State Space Modeling Example:

Given the following circuit:



in which $R_1 = 2\,\Omega$, $R_2 = \frac{8}{3}\Omega$, $C_1 = 1\,\mathrm{C}$, $C_2 = \frac{3}{2}\mathrm{C}$
solve equations for V_1 and V_2

Discretization

Discretization: Q1

Note: this section follows hw8 q1 almost exactly. Suppose we have a scalar system

$$\frac{d}{dt}x(t) = \alpha x + \vec{\beta}^T \vec{u}(t)$$

and we apply a constant input $\vec{u}_n$ for times $t \in [nT, (n+1)T)$ for some $T > 0$. Given $x(nT)$ solve the differential equation

Discretization: Q1 Sol

From $t = nT$ to $t = (n+1)T$, $\vec{\beta}^T \vec{u}$ is a constant scalar. Thus, we can solve this like a normal differential equation. Let $x = x' - \frac{\vec{\beta}^T \vec{u}}{\alpha}$.

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \alpha(x' - \frac{\vec{\beta}^T \vec{u}}{\alpha}) + \vec{\beta}^T \vec{u}(t) \\
&= \alpha x' \\
x' &= Ae^{\alpha(x-nT)} \\
x + \frac{\vec{\beta}^T \vec{u}}{\alpha} &= Ae^{\alpha(x-nT)} \\
x &= Ae^{\alpha(x-nT)} - \frac{\vec{\beta}^T \vec{u}}{\alpha}
\end{aligned}
$$

Discretization: Q1 Sol Continued

At which point we can use our initial condition to get

$$
\begin{aligned}
x(nT) &= A - \frac{\vec{\beta}^T \vec{u}}{\alpha} \\
A &= x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha} \\
x &= \left(x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha} \right) e^{\alpha(t-nT)} - \frac{\vec{\beta}^T \vec{u}}{\alpha}
\end{aligned}
$$

Discretization: Q2

Using the differential equation derived from question 1, create a discrete-time system to model the continuous time. In other words, if $x[n] = x(nT), \vec{u}[n] = \vec{u}(nT)$, find a relation such that

$$x[n+1] = A_d x[n] + B_d \vec{u}[n]$$

Discretization: Q2 Sol

We can solve the previous solution for $x((n+1)T)$

$$x((n+1)T) = \left(x(nT) + \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha} \right) e^{\alpha((n+1)T - nT)} - \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha}$$

$$x[n+1] = e^{\alpha T} x[n] + \frac{e^{\alpha T} - 1}{\alpha} \vec{\beta}^T \vec{u}[n]$$

We see that $A_d = e^{\alpha T}, B_d = ((e^{\alpha T} - 1)/\alpha)\vec{\beta}^T$

Discretization: Q3

Instead of a scalar, we instead have a diagonal matrix A such that

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$$

Discretize this system in the same was as Q2.

Discretiziation: Q3 Sol

Expanding the original system out line-by-line gives

$$\frac{d}{dt}x_i = a_i x_i + b_i \vec{u}_i$$

where x_i is the ith variable of $\vec{x}$, a_i is the diagonal entry of A, and b_i is the row of B.

Discretization: Generic Matrix

Math not shown, but we can perform a change of basis from our original space to our diagonal space, and then apply the results of the previous part.

Controls

Reviewing State Space

Discrete Time State Space Model:

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]$$

Where $\vec{x}[\cdot]$ as the state vector, $u[\cdot]$ as the input vector.

Controllability

Goal: Modify $x(t)$ to be in any state we desire.

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Expand out $x[t]$ in terms of the initial state and all inputs,

$$\vec{x}(t) = A^t\vec{x}(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + ABu(t-2) + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{\triangleq R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-2) \\ u(t-1) \end{bmatrix}$$

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Given the initial condition, $x(0)$ the output of the system can be expressed in terms of the solely our inputs!

What states can we change $x(t)$ to?

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{\triangleq R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-2) \\ u(t-1) \end{bmatrix}$$

The $\mathrm{Col}(R_t)$ determines the subspace $\vec{u}(t)$ can map to.

In order to control the state to any vector in $\mathbb{R}^n$, $\mathrm{Col}(R_t) = R^n$, or it must be full rank.

i.e. The system is Controllable if and only if

$$\operatorname{rank} R_n = \operatorname{rank} \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} = n$$

Stability, Observability, and Controllability

Stability, Observability, Controllability:

given:

$\vec{x}(i+1) = A\vec{x}(i) + Bu(i)$

$\vec{y}(i) = C\vec{x}(i)$

in which:

$\vec{x}$ is our state,

$\vec{u}$ is our input,

$\vec{y}$ is what we can observe:

Stability (Discrete time):

Discrete time model:
if $|\lambda_i| < 1$ for all λ_i of A, system is stable
intuition: if any $|\lambda_i| >= 1$, state vector is increasing each time step will be infinitely magnified over time

Stability (Continuous time):

Continuous time model:
if the real parts of all eigenvalues of A are strictly negative, system is stable
intuition: if real part of eigenvalue is positive, state vector is increasing over time and will be infinitely magnified over time

Controllability:

if $\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$ spans R^n, then system is controllable

Feedback:

if system is controllable, we can set: $u(t) = K\vec{x}(t)$
plugging in, we get: $\vec{x}(t+1) = (A+BK)\vec{x}(t)$
we can find the eigenvalues of (A + BK) to check for stability

Observability:

if $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ spans R^n, system is observable

intuition: if observability matrix is full rank, it is invertible, and we can retrieve all the past states without loss of information

Stability, Controllability, Observability Example:

given the following system:

$$\vec{x}[t+1] = \begin{bmatrix} -5 & 0 \\ 7 & 6 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

$$\vec{y}[t] = \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x}[t]$$

Stability Check:

$\lambda = 6, -5$

Stability Check:

$\lambda = 6, -5$

System is unstable

Controllability Check:

$$\begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -10 & 2 \\ 8 & -1 \end{bmatrix} \text{ which spans } R^n$$

Controllability Check:

$$\begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -10 & 2 \\ 8 & -1 \end{bmatrix} \text{ which spans } R^n$$

System is controllable

Observability Check:

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \text{ which spans } R^n$$

Observability Check:

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix} \text{ which spans } R^n$$

System is observable