#1 Model A

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad E(y_i) = \bar{y} = \beta_0 + \beta_1 \bar{x}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad \hat{\beta}_1 = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2}$$

$$y_i - \bar{y} = \beta_0 + \beta_1 x_i + \varepsilon_i - \bar{y}$$
$$= \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i + \varepsilon_i - \bar{y}$$
$$= \hat{\beta}_1 (x_i - \bar{x}) + \varepsilon_i$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(\hat{\beta}_1 (x_i - \bar{x}) + \varepsilon_i)}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n} \beta_1 (x_i - \bar{x})^2 + \varepsilon_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$= \beta_1 + \frac{\sum_{i=1}^{n} (x_i - \bar{x}) \varepsilon_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$E(\hat{\beta}_1) = \beta_1 + E\left(\frac{\Sigma (x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2} \varepsilon_i \right) = \beta_1 \quad \text{since } E(\varepsilon_i) = 0$$

$$Var(\hat{\beta}_1) = Var(\beta_1) + Var\left(\frac{\Sigma (x_i - \bar{x})}{\Sigma (x_i - \bar{x})} \varepsilon_i \right) = \left(\frac{\Sigma (x_i - \bar{x})}{\Sigma (x_i - \bar{x})^2} \right)^2 Var\, \varepsilon_i$$

$$= \frac{SS_{xx}}{SS_{xx}^2} \sigma^2 = \frac{\sigma^2}{SS_{xx}}$$

$$\hat{\beta}_0 = \beta_0 + \beta_1 \bar{x} - \hat{\beta}_1 \bar{x} \qquad E(\hat{\beta}_0) = \beta_0 + \beta_1 \bar{x} - \bar{x}\, E(\hat{\beta}_1) = \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x}$$
$$= \beta_0$$

$$Var(\hat{\beta}_0) = Var(\hat{\beta}_1 \bar{x}) = \bar{x}^2\, Var(\hat{\beta}_1) = \frac{\bar{x}^2 \sigma^2}{SS_{xx}}$$

#1 Model B

$$y_i = \beta_1 x_i + \varepsilon_i$$

$$\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$E(\hat{\beta}_1) = E\left(\frac{\sum\left[x_i(\beta_1 x_i + \varepsilon_i)\right]}{\sum x_i^2}\right)$$

$$= E\left(\frac{\sum(\beta_1 x_i^2 + x_i \varepsilon_i)}{\sum x_i^2}\right) = E\left(\beta_1 + \frac{\sum x_i \varepsilon_i}{\sum x_i^2}\right)$$

$$= \beta_1 + E\left(\frac{\sum x_i}{\sum x_i^2}\right) E(\varepsilon_i) = \beta_1$$

$$Var(\hat{\beta}_1) = Var\left(\frac{\sum x_i \varepsilon_i}{\sum x_i^2}\right) = \frac{\sum x_i^2 \, Var(\varepsilon_i)}{\left(\sum x_i^2\right)^2} = \frac{Var(\varepsilon_i)}{\sum x_i^2}$$

$$= \frac{\sigma^2}{n \overline{X^2}}$$

Model C

$$E(\hat{\beta}_0) = E(\bar{y}) = E\left(\frac{1}{n} \Sigma y_i\right)$$

$$= \frac{1}{n} E\left(\Sigma \beta_0 + \varepsilon_i\right)$$

$$= \frac{1}{n} \Sigma E(\beta_0) + \frac{1}{n} \Sigma E(\varepsilon_i)$$

$$= \beta_0 + 0$$

$$= \beta_0$$

$$Var(\hat{\beta}_0) = Var\left(\frac{1}{n} \Sigma y_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var\, y_i$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var\left(\beta_0 + \varepsilon_i\right)$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} Var\, \beta_0 + Var\, \varepsilon_i\right]$$

$$= \frac{n}{n^2}\left(0 + \sigma^2\right)$$

$$= \frac{\sigma^2}{n}$$

2. $Cov(\hat{\beta}_1, \bar{Y})$

$$= E(\hat{\beta}_1 \bar{Y}) - E(\hat{\beta}_1)E(\bar{Y}) = E\left[(\hat{\beta}_1 - \beta_1)(\bar{Y} - E(\bar{Y}))\right]$$

$$= \bar{Y}E(\hat{\beta}_1) - \bar{Y}E(\hat{\beta}_1)$$

$$= 0$$

$Cov(\hat{\beta}_1, \hat{\beta}_0)$

$$= E(\hat{\beta}_1 \hat{\beta}_0) - E(\hat{\beta}_0)E(\hat{\beta}_1) = E\left[(\hat{\beta}_1 - \beta_1)(\hat{\beta}_0 - \beta_0)\right]$$

$$\hat{\beta}_0 - \beta_0 = \beta_1 \bar{x} - \hat{\beta}_1 \bar{x} = -\bar{x}(\hat{\beta}_1 - \beta_1)$$

$$Cov(\hat{\beta}_1, \hat{\beta}_0) = E\left[(\hat{\beta}_1 - \beta_1)(-\bar{x}(\hat{\beta}_1 - \beta_1))\right]$$

$$= E\left[-\bar{x}(\hat{\beta}_1 - \beta_1)^2\right]$$

$$= -\bar{x}\,E\left[(\hat{\beta}_1 - \beta_1)^2\right] = -\bar{x}\,Var(\hat{\beta}_1)$$

$$= \frac{-\bar{x}\sigma^2}{SS_{xx}}$$

#3

Model B

3a) $y_i = \beta_1 x_i + \varepsilon_i$

$$\hat{\beta}_2 = \frac{\bar{y}}{\bar{x}} \qquad \hat{\beta}_3 = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}$$

$$E(\hat{\beta}_2) = E\left(\frac{\frac{1}{n}\sum_{i=1}^{n} y_i}{\bar{x}}\right) = E\left(\frac{\sum_{i=1}^{n} \beta_1 x_i + \varepsilon_i}{\sum_{i=1}^{n} x_i}\right)$$

$$= E\left(\beta_1 + \frac{\varepsilon_i}{n\bar{x}}\right) = E(\beta_1) + E\left(\frac{\varepsilon_i}{n\bar{x}}\right) = \beta_1 + 0$$

$E(\hat{\beta}_2) = \beta_1$, thus is an unbiased estimator

3b) $$\hat{\beta}_2 = \frac{\frac{1}{n}\sum_{i=1}^{n} y_i}{\bar{x}} = \sum_{i=1}^{n} \frac{1}{n\bar{x}} y_i$$

$$c_i = \frac{1}{n\bar{x}}$$

3.c) $Var(\hat{\beta}_2) = Var\left(\frac{\bar{y}}{\bar{x}}\right) = 0$

3.d) $Var(\hat{\beta}_3)$ would be a function of $Var\left(\frac{y_i}{x_i}\right)$. Since $Var(\hat{\beta}_2)$ is zero, then $Var(\hat{\beta}_3)$ would be larger since it is nonzero.

3e) $E(\hat{\beta}_3) = E\left(\frac{1}{n}\Sigma\frac{y_i}{x_i}\right)$

$$= \frac{1}{n}\Sigma E\left(\frac{y_i}{x_i}\right) = \frac{1}{n}\Sigma E\left(\frac{\beta_1 x_i + \varepsilon_i}{x_i}\right)$$

$$= \frac{1}{n}\Sigma E\left(\beta_1 + \frac{\varepsilon_i}{x_i}\right) = \frac{1}{n}\Sigma E(\beta_1) = \beta_1$$

$\hat{\beta}_3$ is unbiased.

3f) $\hat{\beta}_3 = \Sigma \frac{1}{nx_i} y_i$

$$c_i = \frac{1}{nx_i}$$

3g) $Var(\hat{\beta}_3) = Var\left(\frac{1}{n}\Sigma\frac{y_i}{x_i}\right)$

$$= \frac{1}{n^2}\Sigma Var\left(\frac{y_i}{x_i}\right) = \frac{1}{n^2}\Sigma Var\left(\frac{\beta_1 x_i + \varepsilon_i}{x_i}\right)$$

$$= \frac{1}{n^2} Var\left(\Sigma\left(\frac{\varepsilon_i}{x_i}\right)\right) = \frac{1}{n^2}\Sigma\frac{1}{x_i^2} Var(\varepsilon_i) = \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\frac{1}{x_i}$$

3h) $Var(\hat{\beta}_3)$ is smaller than $Var(\hat{\beta}_1)$ due to the $\frac{1}{n^2}$ term