#2

Normal equations $\qquad Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

$X^T X \hat{\beta} = X^T y \qquad Q = \sum_{i=1}^{n} \varepsilon_i^2$

a) $\varepsilon^T \varepsilon = \sum_{i=1}^{n} (y - X\hat{\beta})^T (y - X\hat{\beta})$

$$= y^T y - 2y^T X\hat{\beta} + \hat{\beta} X^T X \hat{\beta}$$

$$\frac{\partial \varepsilon^T \varepsilon}{\partial \beta} = -2X^T y + 2X^T X \hat{\beta} = 0$$

$$X^T y = X^T X \hat{\beta}$$

$$Q = \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i)^2$$

$$\frac{\partial Q}{\partial \beta_0} = -2 \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i) = 0$$

$$\sum_{i=1}^{n} Y_i - \beta_0 - \beta_1 X_i = 0$$

$$n\beta_0 = \sum_{i=1}^{n} Y_i - \beta_1 X_i$$

$$\beta_0 = \frac{1}{n} \sum_{i=1}^{n} Y_i - \beta_1 X_i$$

$$\beta_0 = \bar{Y} - \beta_1 \bar{X}$$

$$\sum_{i=1}^{n} x_i\left(y_i-\beta_0-\beta_1 x_i\right)=0$$

$$\sum_{i=1}^{n} x_i y_i-\beta_0 x_i-\beta_1 x_i^2=0$$

$$\sum_{i=1}^{n} x_i y_i-\beta_0 \sum_{i=1}^{n} x_i-\beta_1 \sum_{i=1}^{n} x_i^2=0$$

$$\sum_{i=1}^{n} x_i y_i-\left(\bar{y}-\beta_1 \bar{x}\right) \sum_{i=1}^{n} x_i-\beta_1 \sum_{i=1}^{n} x_i^2=0$$

$$\sum_{i=1}^{n} x_i y_i\left[\frac{1}{n} \sum_{i=1}^{n} y_i-\beta_1 \frac{1}{n} \sum_{i=1}^{n} x_i\right] \sum x_i-\beta_1 \sum_{i=0}^{n} x_i^2=0$$

$$\sum_{i=1}^{n} x_i y_i-\frac{1}{n} \sum_{i=1}^{n} y_i \sum x_i+\beta_1 \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i-\beta_1 \sum_{i=1}^{n} x_i^2=0$$

$$\sum_{i=1}^{n} x_i y_i-\frac{1}{n} \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i+\beta_1\left[\frac{1}{n} \sum x_i \sum x_i-\sum x_i^2\right]=0$$

$$\beta_1=\frac{\sum_{i=1}^{n} x_i y_i-\frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2-\frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i}=\frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)}{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}$$

#3

a) $y_i = \beta_0 + \beta_1(x_i - \bar{x}) + \varepsilon_i = \beta_0 + \beta_1 x_i - \beta_1\bar{x} + \varepsilon_i$

$$g(b) = \sum\left(y_i - \beta_0 - \beta_1(x_i - \bar{x})\right)^2$$

$$\frac{\partial g}{\partial b} = 0$$

$$g(b) = \sum(y_i - \beta_0 - \beta_1 x_i + \beta_1\bar{x})^2$$

$$\frac{\partial g}{\partial \beta_0} = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i + \beta_1\bar{x}) = 0$$

$$\sum_{i=1}^{n}(y_i - \beta_1 x_i) - n\beta_0 + n\beta_1\bar{x} = 0$$

$$n\beta_0 = \sum_{i=1}^{n}(y_i - \beta_1 x_i) + n\beta_1\bar{x}$$

$$\beta_0 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_1 x_i) + \beta_1\bar{x}$$

$$= \bar{y} - \beta_1\bar{x} + \beta_1\bar{x}$$

$$\beta_0 = \bar{y}$$

$$\frac{\partial g}{\partial b_1} = -2\sum_{i=1}^{n}(x_i - \bar{x})\left(y_i - \beta_0 - \beta_1(x_i - \bar{x})\right) = 0$$

$$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \beta_0 - \beta_1 x_i + \beta_1\bar{x}) = 0$$

$$\sum_{i=1}^{n}(x_i y_i - \beta_0 x_i - \beta_1 x_i^2 + \beta_1\bar{x}x_i - \bar{x}y_i + \bar{x}\beta_0 + \beta_1 x_i\bar{x} - \beta_1\bar{x}^2) = 0$$

$$\sum_{i=1}^{n}(x_i y_i - \beta_0 x_i - \beta_1 x_i^2 + \beta_1\bar{x}x_i - \bar{x}y_i + \beta_1 x_i\bar{x}) = (\beta_1\bar{x}^2 - \beta_0\bar{x})n$$

$$\frac{\partial g}{\partial \beta_1} = -2 \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \beta_0 - \beta_1 (x_i - \bar{x})) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - \beta_0 x_i - \beta_1 x_i^2 - \beta_1 \bar{x} x_i - \bar{x} y_i + \beta_0 \bar{x} + \beta_1 x_i \bar{x} - \beta_1 \bar{x}^2 = 0$$

$$\sum_{i=1}^{n} x_i y_i - \beta_0 x_i - \beta_1 x_i^2 - \cancel{\beta_1 \bar{x} x_i} - \bar{x} y_i + \cancel{\beta_1 x_i \bar{x}} = (\beta_1 \bar{x}^2 - \beta_0 \bar{x}) n$$

$$-\sum_{i=1}^{n} \beta_1 x_i^2 + \sum_{i=1}^{n} x_i y_i - \beta_0 x_i - \bar{x} y_i = n(\beta_1 \bar{x}^2 - \beta_0 \bar{x})$$

$$-\frac{1}{n} \sum_{i=1}^{n} \left[\beta_1 x_i^2 + x_i y_i\right] - \cancel{\beta_0 \bar{x}} - \bar{x}\bar{y} = \beta_1 \bar{x}^2 - \cancel{\beta_0 \bar{x}}$$

$$\beta_1 \overline{x^2} + \overline{xy} - \bar{x}\bar{y} = \beta_1 \bar{x}^2$$

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \bar{x} y_i = \beta_1 \bar{x}^2 - \beta_1 \overline{x^2}$$

$$\beta_1 = \frac{\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{x_i y_i}{n}}{\bar{x}^2 - \overline{x^2}}$$

#6

a) H is a $p \times p$ matrix ($n_{rows} = n_{columns} = p$)

b) $H^T = [X(X^TX)^{-1}X^T]^T = X[(X^TX)^{-1}]^T X^T$ $\qquad (ABC)^T = C^TB^TA^T$

$$= X[X^TX]^{-1}X^T = H$$

c) $H^2 = X(X^TX)^{-1}\underbrace{X^TX(X^TX)^{-1}}_{I}X^T$

$$= X(X^TX)^{-1}X^T = H$$

d) $tr(H) = tr(X(X^TX)^{-1}X^T)$

$$= tr(X^TX(X^TX)^{-1}) = tr(I)$$

since H is a $p \times p$ matrix

$tr(I) = p$

e)

f) $rank(H) = p$

g) $\hat{y} = y - \hat{\varepsilon} = y - (y - Hy) = y - y + Hy$

$\hat{y} = Hy$

h) $\hat{\varepsilon} = y - \hat{y} = y - Hy = (I - H)y$

i) $(I - H)^T = I^2 - H^T = I - H$

j) $(I - H)(I - H) = I^2 - H - H + H^2 = I^2 - 2H + H = I - H$ (since $I^2 = I$, $H^2 = H$)

k) $tr(I - H) = tr(I) - tr(H) = rank(I) - rank(H) = 0$